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Interpolation in Weighted Spaces of Holomorphic Functions

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Doctoral dissertation written
under the supervision of
prof. dr hab. Paweł Domański

Poznań 2011

Research supported by Polish Ministry of Science and Higher Education grant "Interpolacja w wagowych przestrzeniach funkcji holomorficznych" N N201 526938.

Acknowledgements

I am particularly grateful to my supervisor, prof. Paweł Domański, for suggesting the topic of this dissertation and for hundreds of hours of explaining, inspiring, and encouraging. His help in the process of writing this paper was inestimable. Many defects and inaccuracies have been removed only due to his perfectionism. I am also indebted to Xavier Massaneda and Joaquim Ortega-Cerdá for their immediate and comprehensive answers to my emails with mathematical questions. My sincere thanks go to Myriam Ounaïes for presenting the proof of the theorem about unions of interpolating varieties to me, and allowing me to publish it in this dissertation.

I greatly appreciate the help of my sister Agnieszka, whose English is far better than mine, and who found and corrected language errors in this text. I would also like to thank my parents. This paper would have never been finished without their support and unabated faith in me. Furthermore, I would like to express my gratitude to my parents-in-law for their help whenever financial situation became harder.

The years of writing this dissertation were not easy for my family, especially for my wife Anna. The lack of time and restless nights were an ordeal. I owe a great debt of gratitude to her for supporting me in my worse moments and taking care of the household duties.

Contents

Introduction

The aim of this dissertation is to give geometric characterisations of interpolating sequences for weighted spaces of entire functions related to spaces of ultradistributions. In particular, we present such a characterisation for Fourier-Laplace transform images of so-called Roumieu ultradistributions (see Theorem 5.3.2) and analyse in depth conditions appearing in this result (see Sections 4.1 and 4.2). We also relate our result to the known characterisation of interpolating sequences for the Beurling ultradistributions. Before going into more detailed presentation of the contents of this paper, we shall write some words about background and motivation.

Background

At the beginning of the twentieth century M. Gevrey discovered that solutions of the heat equation

$$\partial_t u(x,t) = \Delta_x u(x,t), \quad (t \in \mathbb{R}, x \in \mathbb{R}^n),$$

known to be not analytic in general, but always infinitely differentiable, admit a better smoothness than arbitrary C^∞ functions (see [Gev18]). He proved that for a solution u, for any compact set $K \subset R^{n+1}$ it holds

$$|\partial^\alpha u(t,x)| \leq C^{|\alpha|+1}(\alpha!)^2, \quad ((t,x) \in K)$$

for some constant $C > 0$ and every multiindex $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$. Here $\partial^\alpha = \partial_t^{\alpha_1}\partial_{x_1}^{\alpha_2}\ldots\partial_{x_n}^{\alpha_{n+1}}$, $|\alpha| = \alpha_1 + \ldots + \alpha_{n+1}$ and $\alpha! = \alpha_1!\cdots\alpha_{n+1}!$. This fact inspired Gevrey to introduce a scale measuring smoothness of infinitely differentiable functions - for an open set $\Omega \subset \mathbb{R}^{n+1}$ and $s \geq 1$

$$G_s(\Omega) = \left\{ f \in C^\infty(\Omega) | \ \forall\, K \Subset \Omega \ \exists\, h > 0 : \sup_{x\in K} \sup_{\alpha \in \mathbb{N}_0^{n+1}} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|}(\alpha!)^s} < \infty \right\}.$$

For $s = 1$ the elements of this space are exactly the real analytic functions on Ω. The general idea behind this scale is that one weakens Cauchy's inequalities. Since always $A(\Omega) \subset G_s(\Omega) \subset C^\infty(\Omega)$, the elements of $G_s(\Omega)$ are called ultradifferentiable functions.

Another result which initiated the study of ultradifferentiable functions, was obtained by E. Borel in 1895 (see [Bor95]). He proved that for any sequence $(a_n)_{n\in\mathbb{N}_0}$ of complex numbers there exists a function f infinitely differentiable on a neighbourhood of 0 and satisfying

$$\frac{f^{(n)}(0)}{n!} = a_n \quad (n \in \mathbb{N}_0).$$

In other words, Taylor coefficients of a C^∞ function can be arbitrary. This is just the opposite to the case of real analytic functions, where Taylor coefficients in a point determine a function completely. This observation led to a question - how Cauchy's inequalities can be relaxed without the loss of this uniqueness property. E. Borel provided an example of a C^∞ function with its Taylor series divergent at a point, but uniquely determined by its Taylor coefficients in that point. One can study this problem more systematically replacing the terms $(\alpha!)^s$ in the definition of the Gevrey classes by an arbitrary sequence $(M_{|\alpha|})$, and asking whether the obtained class does not contain non-trivial functions with compact support. Such classes have been later called quasianalytic. In 1921 A. Denjoy (see [Den21]) found some sufficient conditions for such a class to be quasianalytic, and in 1926 T. Carleman (see [Car26]) characterised quasianalytic classes of ultradifferentiable functions completely in terms of conditions involving the sequence $(M_{|\alpha|})$ (comp. [Rud87, Theorem 19.11]).

In 1936, in the paper [Sob36], S. Sobolev introduced generalised functions - distributions, and used them to study differential equations. The great advantage of distributions stems from the fact that they provide a way of differentiating continuous or just locally integrable functions. Distributions became very popular and widely used in the fifties after more systematic works of L. Schwartz ([Sch50], [Sch51]) had been published. It was natural then to ask about an appropriate theory of distributions for ultradifferentiable functions. The answer came in 1960 with the paper [Rou60] of C. Roumieu. He studied distributions on the spaces $G_s(\Omega)$. Nowadays, these are called Gevrey ultradistributions. Moreover, Roumieu studied also the expanded scale of ultradifferentiability with the terms $(\alpha!)^s$ replaced by an arbitrary sequence $(M_{|\alpha|})$, and developed a theory of ultradistributions also in this general case. This approach was later thoroughly investigated by H. Komatsu (see [Kom73], [Kom77]).

A. Beurling found and announced in 1961 (see [Beu61]) a new way of defining ultradifferentiable functions. He used the observation that a continuous function with compact support is infinitely differentiable if and only if its Fourier transform satisfies

$$\int_{\mathbb{R}^n} \widehat{\varphi}(\xi) e^{h \ln(1+|\xi|)} \, \mathrm{d}\xi < \infty$$

for every $h > 0$. Beurling replaced the function $\ln(1+|\xi|)$ by an arbitrary larger weight ω obtaining some subclasses of C^∞. Not much later, G. Björck took up the idea and developed a theory of ultradistributions on these spaces (see [Bjö66]). Topological properties of spaces of these ultradistributions have been later studied by many authors (see for instance [BMT90], [BM01], [BD07]). In 1990 Braun, Meise and Taylor proved that ultradifferentiable functions defined by weights can also be characterised by imposing growth conditions on their derivatives (see [BMT90]), and therefore they deserve such a name. Ultradifferentiable functions and ultradistributions have been extensively used in the study of differential operators (see [Rod93] for the case of Gevrey classes, [MTV96], [BMV94], [Rös97], [Lan94] for other classes).

One may observe that one of the differences between Gevrey/Roumieu's and Beurling's approaches was the quantifier before the h constant - existential in the first case and universal in the second. But there are no obstacles to consider the other quantifier in both cases. And indeed, such classes have been later investigated equally intensively as the original ones. Nowadays, both sequential and functional classes with the universal quantifier bear the name of Beurling, while classes with the existential quantifier are named after Roumieu.

The original theorem of Paley and Wiener states that $f \in L^2[-M, M]$ if and only if the Fourier transform of f,

$$\widehat{f}(z) = \int_{-M}^{M} f(x) e^{-ixz} \, \mathrm{dx} \quad (z \in \mathbb{C}),$$

is an entire function of exponential type, i.e.,

$$|\widehat{f}(z)| \leq C e^{M|z|} \quad (z \in \mathbb{C})$$

for some $C > 0$, and $\widehat{f} \in L^2(\mathbb{R})$. This theorem allows complex analysis tools to be used in the study of properties of the function f via its Fourier transform. For this reason many theorems of this type (called Paley-Wiener type theorems) have later been proved for various spaces of functions, in particular, for ultradifferentiable functions with compact support (see [Bjö66], [BMT90]). Moreover, Braun, Meise and Taylor proved in [BMT90], with the use of functional analysis methods, certain isomorphisms between spaces of ultradifferentiable functions with compact support and spaces of entire functions with growth conditions.

The idea of transforming problems concerning functions of a real variable to problems for entire functions, can also be applied to ultradistributions with compact support. To make this possible one defines the Fourier-Laplace transform for ultradistributions by the formula

$$\widehat{\mu}(z) := \langle \mu_x, e^{-ixz} \rangle$$

for an ultradistribution u with compact support and $z \in \mathbb{C}$. Just like in the function case, $\widehat{\mu}(z)$ is an entire function and satisfies a certain growth condition. Furthermore, isomorphisms between spaces of ultradistributions and some weighted spaces of entire functions with growth conditions can be shown (see [BMT90], [Rös97], [HM07]).

A particular field which benefits greatly from the Fourier-Laplace transform theory and Paley-Wiener type theorems is the theory of convolution and linear partial differential operators. Consider an equation

$$\mu * f = g$$

where μ is a distribution, g is an infinitely differentiable function with compact support (the input data), and f is a solution we are looking for. Then applying the Fourier-Laplace transform one obtains an equivalent equation

$$\widehat{\mu} \cdot \widehat{f} = \widehat{g}$$

involving only entire functions and multiplication. Then to obtain a solution one just needs to divide $\widehat{g}$ by $\widehat{\mu}$. Of course, it is not always possible, but this problem is in general much simpler to deal with than solving the original equation.

Problems of interpolation arose in the theory of complex functions and for many years were studied independently of the theory of differential and convolution equations. In general form one may express an interpolation problem as follows. Suppose that we have a function space E on a domain Ω and a sequence space S. Then we ask for a characterisation of all discrete sets $\Lambda \subset \Omega$ satisfying that for every sequence of values $(w_\lambda)_{\lambda \in \Lambda} \in S$ there exists a function $f \in E$ such that

$$f(\lambda) = w_\lambda, \quad (\lambda \in \Lambda).$$

In 1958 L. Carleson published a solution to this problem for $E = H^\infty(D)$, the space of bounded holomorphic functions on the unit disc, and $S = l^\infty$ (see [Car58]). In 1961 Shapiro and Shields solved the problem completely for other Hardy spaces $E = H^p(D)$ and $S = l^p$ (see [SS61]). Later this problem has been considered in many other spaces like Bergman, Bloch, Paley-Wiener spaces (see [Sei04]), Bernstein algebra (see [MOC09]), Hörmander algebras (see [BT79], [Squ81], [Squ83],

[BL95], [BLV95], [MOCO03], [Oun03], [Oun07], [Oun08]), as well as many others (see for instance [Mas98], [OCS99], [HM00], [MMOC03], [HMNT04]). For us, the most interesting results are these devoted to Hörmander algebras A_p, which consist of entire functions with growth conditions given by a weight function p. In this case the problem of interpolation requires a characterisation of so-called interpolating varieties. A multiplicity variety $\{(\lambda, m_\lambda)|\ \lambda \in \Lambda\}$, where λ runs through a discrete set $\Lambda \subset \mathbb{C}$, m_λ are positive integers, is called an interpolating variety for A_p if for any doubly indexed sequence of complex numbers $\{v_{\lambda,l}\}$ with suitable growth there exists $f \in A_p$ satisfying

$$\frac{f^{(l)}(\lambda)}{l!} = v_{\lambda,l}$$

for any $\lambda \in \Lambda$ and $0 \leq l < m_\lambda$. Particularly desired are geometric descriptions. The most notable results of this kind have been obtained for radial weights by Berenstein and Li [BL95] for A_p algebras and by Berenstein, Li and Vidras [BLV95] for A_p^0 - Hörmander algebras of minimal type.

From our point of view, the most important geometric characterisation of interpolating varieties came with the paper of Massaneda, Ortega-Cerdá, and Ounaïes [MOCO03], where the weight was of the form $p(z) = |\operatorname{Im} z| + \omega(z)$. Under certain conditions on ω the space A_p is then isomorphic, by a Paley-Wiener type theorem, to the space of ultradistributions of Beurling type with compact support on the real line. This is the point where the study of convolution operators connects with interpolation problems.

As we have seen, on the side of Fourier-Laplace transforms of ultradifferentiable functions or ultradistributions a convolution operator T_μ becomes just a multiplication operator M_μ which multiplies by an entire function $\hat{\mu}$. The image of this operator consists of functions in $A_{(\omega)} := A_{|\operatorname{Im} z|+\omega(z)}$ vanishing at zeroes of $\hat{\mu}$ (in some cases all such functions). If the set of zeroes is an interpolating variety then $A_{(\omega)}/\operatorname{Im} M_{\hat{\mu}}$ can be identified with the space of sequences with suitable growth, which is in turn isomorphic to the dual of the kernel of T_μ. This idea lies behind results in [Mei89], [FM90], [BM08], [Mey97], [Lan94], [BM90], [BMV90], [MT88], where sequential descriptions of kernels of convolution operators are given, surjectivity characterisations of convolution operators are obtained, or existence of right inverses for convolution operators is established. This methodology was developed in [Mei85], [MT87a], [MT87b], [Bra87] and it is connected with the study of ideals in Hörmander algebras.

Results

The geometric characterisation of interpolating varieties for $A_{(\omega)}$, i.e., the space of Fourier-Laplace transforms of ultradistributions of Beurling type with compact support $\widehat{\mathscr{E}'}_{(\omega)}$, due to Massaneda, Ortega-Cerdá, and Ounaïes [MOCO03], was stated for weights ω which are subadditive and non-quasianalytic. In this dissertation we show that the assumption of subadditivity can be replaced by a weaker condition - $\omega(2t) \leq C\omega(t)$ for some $C > 0$ and all t big enough. Moreover, we show that most of their proof works equally well in the quasianalytic case and this way we find sufficient conditions for interpolation in this case (see Theorem 5.2.17). Another aim of this dissertation is an analysis of geometric conditions introduced in [MOCO03]. There are three types of such conditions - imposing estimates on the growth of the Nevanlinna counting function N, conditions involving the Poisson balayage of a certain measure related to the mutliplicity variety, and Carleson type conditions involving Blaschke products (similar to the famous Carleson condition from the paper [Car58]). We explain the meaning of the conditions of the first kind by expressing them in a nearly equivalent more straightforward form (see Propositions 4.1.10 and 4.1.12). Further, we introduce new conditions with Poisson balayage, and provide a thorough analysis of dependencies between all considered geometric conditions (see Sections 4.2 and 5.4). Moreover, we show that also these new conditions can be used instead of conditions from [MOCO03] to characterise interpolating varieties in the non-quasianalytic Beurling case (see Theorem 5.2.1).

The problem of interpolation can also be considered for the space $A_{\{\omega\}}$ of Fourier-Laplace transforms of ultradistributions of Roumieu type with compact support $\widehat{\mathscr{E}'}_{\{\omega\}}$. This problem has never been studied earlier. First, we give an analogue in the Roumieu case (see Lemmas 2.1.2 and 2.1.4) of the earlier known uniform interpolation lemma for the Beurling case (see [BL95, Lemma 3.3]). The methods used for the proof are rather standard.

The main result of this dissertation is a geometric characterisation of interpolating varieties for the spaces $A_{\{\omega\}}$ for non-quasianalytic weights (see Theorem 5.3.2). The geometric conditions used in this theorem are similar in spirit to those for the Beurling case. In this part we use considerably our previous analysis of these properties (see Sections 4.1 and 4.2).

In the quasianalytic Roumieu case we give sufficient conditions for interpolation (see Theorem 5.3.1). It is worth noting that this result covers the case of the space of Fourier-Laplace transforms of the analytic functionals on the real line $A'(\mathbb{R})$. This space plays in turn a primary role in the definition and development of the so-called hyperfunctions (see [Sat59], [Sat60]).

Furthermore, we describe the relation between interpolation in the Beurling case and in the Roumieu case. More precisely, for non-quasianalytic weights we show a highly non-obvious fact, that a multiplicity variety X is interpolating for $A_{\{\omega\}}$ if and only if it is interpolating for some $A_{(\sigma)}$ where σ is a weight satisfying $\sigma = o(\omega)$ (see Theorem 5.3.2). For quasianalytic weights we prove only the implication from right to left (see Corollary 3.3.1).

At the end, we derive several consequences of geometric descriptions of interpolating varieties in the Roumieu case, mostly for non-quasianalytic weights. We prove an analogue of the analytic characterisation given by Berenstein and Li [BL95] for the Beurling case (see Theorem 5.4.6). We show that finite unions of interpolating varieties are interpolating under some mild conditions (see Theorem 5.4.5, and compare [Oun03] or Theorem 2.3.6). We prove that if a variety is $A_{\{\sigma\}}$ interpolating then it is $A_{\{\omega\}}$ interpolating for every weight ω satisfying $\sigma = O(\omega)$ (see Corollary 5.4.7 and compare Corollary 3.1.2). Finally, we show that the Carleson type conditions introduced in Section 4.2 are not necessary for interpolation in the quasianalytic case (both Beurling and Roumieu).

Structure of the dissertation

The paper is structured as follows. In Sections 1.2, 1.3 we define classes of ultradifferentiable functions, ultradistributions and weighted algebras of entire functions. In Section 1.4 we introduce weighted spaces of sequences which are a natural choice in our setting for the interpolation problem. In Section 1.5 we study properties of weight functions. First, we give a standard definition, and then we derive several consequences. The most important result of this section is Lemma 1.5.16 proved by Braun, Meise and Taylor in [BMT90]. This lemma provides a way of finding a weight σ smaller than a given weight ω and bigger than an arbitrary function q, in the sense that $q = o(\sigma)$, $\sigma = o(\omega)$. It allows to transform problems in the Roumieu case to problems in the Beurling case. We will frequently make use of this lemma. In Section 1.6 we describe a relation between the sequential and the functional approach of defining ultradifferentiable functions and state that these two ways in many cases give the same classes. Then we mention some properties of spaces of ultradifferentiable functions and spaces of ultradistributions. We present also Paley-Wiener type theorems. In Section 1.7 we prove basic properties of the weighted algebras of entire functions and in Section 1.8 properties of the sequence spaces. Particularly important is Section 1.9, where we define the main objects of our study - interpolating varieties. To describe their geometric properties we will need Nevanlinna counting functions introduced in Section 1.10. The whole Chapter 1 is mostly a survey of known definitions and facts.

In Chapter 2 we study basic properties of interpolating varieties. The most important in this part are the results concerning uniform interpolation (Section 2.1). In Section 2.2 we give some elementary properties of interpolating varieties. In particular, we prove that shift of an interpolating variety is again interpolating, and that multiplicities of an interpolating variety admit certain estimates on their growth. In Section 2.3 we present a never published Ounaïes' theorem about finite unions of interpolating varieties. This result was proved for all multiplicities equal one in [Oun03].

In Chapter 3 we start with the analytic characterisations of interpolating varieties in the Beurling case due to Berenstein and Li [BL95, Theorem 3.1]. Then in Section 3.2 we give a new partial result of this type for the Roumieu case. We end this chapter by proving that an interpolating variety for $A_{(\sigma)}$ is interpolating for $A_{\{\omega\}}$ for any weight ω satisfying $\sigma = O(\omega)$.

In Chapter 4 we study geometric conditions for multiplicity varieties. In Section 4.1 we introduce notion of sparsity of multiplicity varieties. It imposes estimates on the growth of the Nevanlinna function N associated with a given multiplicity variety. The Beurling version of this notion was known. We introduce a suitable Roumieu version. We provide a new analysis of both notions together with some other conditions. The most interesting results of this section are Proposition 4.1.3, which relates Roumieu sparsity to Beurling sparsity, and Propositions 4.1.10, 4.1.12 giving necessary and sufficient conditions for sparsity. In Section 4.2 we introduce and investigate conditions involving the Poisson balayage of certain measures and conditions of Carleson type. These conditions are related to condition (b) and Carleson type condition of Massaneda, Ortega-Cerdá, and Ounaïes (see Theorem 1 and Remark 6 in [MOCO03]). However, our approach is much more extensive and covers both Beurling and Roumieu cases. All these conditions can be considered as geometric as they use quantities dependent only on points and multiplicities of a given variety. In Corollary 4.2.13 and Proposition 4.2.14 we provide a relation between conditions involving Poisson balayage and Carleson type conditions, while in Propositions 4.2.3 and 4.2.4 we connect Beurling and Roumieu cases.

Chapter 5 is the core of this dissertation and concerns geometric characterisation of interpolating varieties. In Section 5.1 we prove, in both the Beurling and the Roumieu case, the necessity of earlier introduced geometric conditions for mutliplicity varieties to be interpolating. The most important in this section are Theorems 5.1.1, 5.1.3, 5.1.4, and 5.1.5.

In Section 5.2 we deal with sufficiency in the Beurling case (see Theorem 5.2.17), and then we obtain a characterisation of interpolating varieties for the non-quasi-analytic case (see Theorem 5.2.1). This result extends the known characterisation of Massaneda, Ortega-Cerdá, and Ounaïes (see [MOCO03]) to non-subadditive weights, and it also adds new characterising conditions involving Poisson balayage. Moreover, we show that the same conditions are sufficient also in the quasianalytic case.

In Section 5.3 we deal with sufficiency in the Roumieu case (see Theorem 5.3.1), also for $A'(\mathbb{R})$ the space of real analytic functionals. We give sufficient conditions for interpolation in both quasianalytic and non-quasianalytic cases. We finish this section with a geometric characterisation of interpolating varieties in the non-quasianalytic Roumieu case (see Theorem 5.3.2). In particular, we use for the characterisation the Carleson type conditions. We also obtain a relation between Beurling and Roumieu interpolating varieties. This section is short, because all the work needed has been accomplished in Sections 4.1, 4.2, and 5.2.

Finally, in Section 5.4 we show some new consequences of geometric descriptions of interpolating varieties. The most important are the analytic characterisation of interpolating varieties (see Theorem 5.4.6), the theorem about finite unions of interpolating varieties (see Theorem 5.4.5), and the monotonicity of interpolation with respect to weights (see Corollary 5.4.7).

For unexplained notions from functional analysis we refer the reader to the book [MV97], for notions from complex analysis and potential theory we refer to [BG91], [Con78], [Con95], [Ran95]. The dissertation is written in *British* English.

Chapter 1

Preliminaries

1.1 Notation

We introduce the following notation:

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ - the set of natural numbers,
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,
- $\mathbb{Z}$ - the set of integer numbers,
- $\mathbb{R}$ - the set of real numbers,
- $\mathbb{C}$ - the set of complex numbers,
- $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ - the Riemann sphere,
- $\lfloor x \rfloor$ - the biggest integer n satisfying $n \geq x$ for $x \in \mathbb{R}$,
- $\operatorname{Re} z$ - the real part of $z \in \mathbb{C}$,
- $\operatorname{Im} z$ - the imaginary part of $z \in \mathbb{C}$,
- $D(z, r) = \{w \in \mathbb{C} |\ |w - z| < r\}$ - an open disc of radius $r > 0$ centred at $z \in \mathbb{C}$,
- $\bar{D}(z, r) = \{w \in \mathbb{C} |\ |w - z| \leq r\}$ - a closed disc of radius $r > 0$ centred at $z \in \mathbb{C}$,
- $R(z, r_1, r_2) = \{w \in \mathbb{C} |\ r_1 < |w - z| < r_2\}$ - an open annulus of inner radius $r_1 > 0$ and outer radius $r_2 > r_1$ centred at $z \in \mathbb{C}$,
- $\mathbb{H}_+ = \{z \in \mathbb{C} |\ \operatorname{Im} z > 0\}$ - the upper half-plane,

- $\mathbb{H}_- = \{z \in \mathbb{C} | \ \operatorname{Im} z > 0\}$ - the lower half-plane,
- $\delta_{\cdot,\cdot}$ - the Kronecker delta, the function $\delta_{\cdot,\cdot}\colon X \times X \to \{0,1\}$ defined on a set X given by the formula

$$\delta_{xy} = \begin{cases} 1 & \text{for } x = y, \\ 0 & \text{for } x \neq y, \end{cases}$$

  for $x, y \in X$,
- χ_A - the characteristic function of a set $A \subset X$ given by the formula

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \in X \setminus A, \end{cases}$$

- dx - the Lebesgue measure on the real line,
- dz - the two dimensional Lebesgue measure,
- $\langle T, \varphi \rangle$ - a symbol defined by the formula $\langle T, \varphi \rangle := T(\varphi)$ for a distribution T and a function φ in its domain,
- δ_z - the Dirac's delta in $z \in \mathbb{C}$, the distribution defined by the formula $\langle \delta_z, \varphi \rangle := \varphi(z)$ for a function φ,
- $f \lesssim g$ - a relation between functions $f, g\colon X \to \mathbb{R}$ defined on some set X holding if and only if $f(x) \leq Cg(x)$ for some constant $C > 0$ and every $x \in X$,
- $f \simeq g \Leftrightarrow f \lesssim g$ and $g \lesssim f$,
- $f = O(g)$ - a relation between functions $f, g\colon [0, \infty) \to \mathbb{R}$ holding if and only if

$$\exists\, C > 0 \ \exists\, x_0 \in [0, \infty) \ \forall\, x \geq x_0\colon \quad f(x) \leq Cg(x),$$

- $f = o(g)$ - a relation between functions $f, g\colon [0, \infty) \to \mathbb{R}$ holding if and only if

$$\forall\, \epsilon > 0 \ \exists\, x_0 \in [0, \infty) \ \forall\, x \geq x_0\colon \quad f(x) \leq \epsilon g(x),$$

- $C^\infty(\Omega)$ - the space of all infinitely differentiable functions on an open set $\Omega \subset \mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$),

- $C_c^\infty(\Omega)$ - the space of all infinitely differentiable functions with compact support contained in an open set $\Omega \subset \mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$), also called test functions or cut-off functions,

- $H(\Omega)$ - the space of all holomorphic functions on $\Omega \subset \mathbb{C}$,

- $A(\Omega)$ - the space of all real analytic functions on $\Omega \subset \mathbb{R}$,

- $\operatorname{proj}_{n\in\mathbb{N}} X_n$ - the topological projective limit of a sequence of locally convex spaces X_n,

- $\operatorname{ind}_{n\in\mathbb{N}} X_n$ - the locally convex inductive limit of a sequence of locally convex spaces X_n.

1.2 Ultradifferentiable functions and ultradistributions

There are two ways of defining classes of ultradifferentiable functions. In terms of sequences and in terms of so-called weights, functions giving regularity conditions. These approaches are not equivalent, but in a whole class of examples they give the same functions (see [BMM07]). For a thorough study of these spaces in the sequential approach we refer the reader to [Kom73], for the functional approach we refer to [BMT90], and also [Bjö66], [BM01], [Rös97], [BD07].

Definition 1.2.1. For a sequence $(M_n)_{n\in\mathbb{N}_0} \subset (0,\infty)$ and an open set $\Omega \subset \mathbb{R}$ we define the class of ultradifferentiable functions of Beurling type as follows

$$\mathscr{E}_{(M_n)}(\Omega) := \left\{ f \in C^\infty(\Omega) \,\middle|\, \forall\, K \Subset \Omega \;\forall\, h > 0 : \sup_{\substack{x\in K\\ n\in\mathbb{N}_0}} \frac{|f^{(n)}(x)|}{h^n M_n} < \infty \right\}$$

and of Roumieu type in the following way

$$\mathscr{E}_{\{M_n\}}(\Omega) := \left\{ f \in C^\infty(\Omega) \,\middle|\, \forall\, K \Subset \Omega \;\exists\, h > 0 : \sup_{\substack{x\in K\\ n\in\mathbb{N}_0}} \frac{|f^{(n)}(x)|}{h^n M_n} < \infty \right\}.$$

Both these definitions contain a natural candidate for a topology. Denote

$$\|f\|_{K,h} := \sup_{\substack{x\in K\\ n\in\mathbb{N}_0}} \frac{|f^{(n)}(x)|}{h^n M_n}, \qquad K \Subset \Omega, h > 0.$$

Then the topologies are given by the representations

$$\mathscr{E}_{(M_n)}(\Omega) = \operatorname{proj}_{K \Subset \Omega} \operatorname{proj}_{h>0} \mathscr{E}_{[M_n],K,h},$$
$$\mathscr{E}_{\{M_n\}}(\Omega) = \operatorname{proj}_{K \Subset \Omega} \operatorname{ind}_{h>0} \mathscr{E}_{[M_n],K,h}$$

where

$$\mathscr{E}_{[M_n],K,h} := \{f \in C^\infty(K) |\ \|f\|_{K,h} < \infty\}$$

is a normed space. A little word is needed to explain what the symbol $C^\infty(K)$ denotes. We may assume that the compact set K has a dense interior and then we can consider $C^\infty(K)$ as the space of all infinitely differentiable functions on $Int\, K$ with all derivatives extending continuously to the boundary of K. More generally, one can consider elements of $C^\infty(K)$ as Whitney jets on K, but this notion will not be used later. We do not give any details concerning properties of spaces $\mathscr{E}_{(M_p)}(\Omega)$ and $\mathscr{E}_{\{M_p\}}(\Omega)$ as these will not be important for the main subject of this dissertation.

Definition 1.2.2. Let Ω be an open subset of $\mathbb{R}$. We say that a class of functions $E \subset C^\infty(\Omega)$ is quasianalytic if it does not contain any nontrivial function with compact support. Otherwise, it is called non-quasianalytic.

Example 1.2.3. The most known spaces of ultradifferentiable functions defined by sequences are the Gevrey classes (in the book [Rod93] of L. Rodino the reader will find a deep study concerning these classes). For $s > 1$ one defines

$$\begin{aligned} G_s(\mathbb{R}) &:= \mathscr{E}_{\{(n!)^s\}}(\mathbb{R}) \\ &= \left\{ f \in C^\infty(\mathbb{R}) |\ \forall\, M \in \mathbb{N}\ \exists\, h > 0 : \sup_{x \in [-M,M]} \sup_{n \in \mathbb{N}_0} \frac{|f^{(n)}(x)|}{h^n (n!)^s} < \infty \right\}. \end{aligned}$$

The space $G_s(\mathbb{R})$ is non-quasianalytic for every $s > 1$.

For $s = 1$ the class $G_1(\mathbb{R})$ is the space of real analytic functions on the real line (see [Rud87, Theorem 19.9]). This is an example of a quasianalytic class.

To define ultradifferentiable functions given by a function $\omega\colon [0, \infty) \to [0, \infty)$ we need to introduce another function associated with ω. Let $\varphi\colon \mathbb{R} \to [0, \infty)$ be given by $\varphi(t) := \omega(e^t)$ and its Young conjugate by

$$\varphi^*(x) := \sup\{xy - \varphi(y) : y > 0\}$$

for $x \geq 0$. The function φ^* is an extended function and can take the value ∞.

Definition 1.2.4. The space of ω-ultradifferentiable functions of Beurling type is defined in the following way

$$\mathscr{E}_{(\omega)}(\Omega) := \left\{ f \in C^\infty(\Omega) |\ \forall\, K \Subset \Omega\ \ \forall\, m \in \mathbb{N} : \sup_{\substack{x \in K \\ n \in \mathbb{N}_0}} |f^{(n)}(x)| \exp\Big(-m\, \varphi^* \Big(\frac{n}{m} \Big) \Big) < \infty \right\},$$

and the space of ω-ultradifferentiable functions of Roumieu type as follows

$$\mathscr{E}_{\{\omega\}}(\Omega) := \left\{ f \in C^\infty(\Omega) |\ \forall\, K \Subset \Omega\ \exists\, \epsilon > 0 : \sup_{\substack{x\in K \\ n\in\mathbb{N}_0}} |f^{(n)}(x)| \exp\!\Big(-\epsilon\, \varphi^*\Big(\frac{n}{\epsilon}\Big)\Big) < \infty \right\}.$$

This definition is quite complex, but, as we will see later, the duals of these spaces are more tractable. The topologies of these spaces can be given analogously as in the sequential case. We define

$$\|f\|_{K,p} := \sup_{\substack{x\in K \\ n\in\mathbb{N}_0}} |f^{(n)}(x)| \exp\Big(-p\, \varphi^*\Big(\frac{n}{p}\Big)\Big), \qquad K \Subset \Omega, p > 0.$$

Then

$$\mathscr{E}_{(\omega)}(\Omega) = \operatorname{proj}_{K\Subset\Omega} \operatorname{proj}_{m\in\mathbb{N}} \mathscr{E}_{[\omega],K,m},$$
$$\mathscr{E}_{\{\omega\}}(\Omega) = \operatorname{proj}_{K\Subset\Omega} \operatorname{ind}_{m\in\mathbb{N}} \mathscr{E}_{[\omega],K,\frac{1}{m}}$$

where

$$\mathscr{E}_{[\omega],K,p} := \{ f \in C^\infty(K) |\ \|f\|_{K,p} < \infty \}$$

is a normed spaces. To prove any useful properties of these spaces we will need certain assumptions imposed on the function ω. We will study conditions for weights in Section 1.5, and in Section 1.6 we will give more detailed information about the spaces $\mathscr{E}_{(\omega)}(\Omega)$ and $\mathscr{E}_{\{\omega\}}(\Omega)$.

Example 1.2.5. Let Ω be an open subset of $\mathbb{R}$. For $\omega(t) = \ln(1+t)$ we have $\varphi^*(x) = \infty$ for x big enough, thus

$$\mathscr{E}_{(\omega)}(\Omega) = C^\infty(\Omega).$$

For $\omega(t) = t$ we have

$$\mathscr{E}_{\{\omega\}}(\Omega) = A(\Omega).$$

For $0 < q < 1$ and $\omega(t) = t^q$ it holds

$$\mathscr{E}_{\{\omega\}}(\mathbb{R}) = G_{\frac{1}{q}}(\mathbb{R}).$$

To draw the complete picture, we add that one considers also ultradifferentiable functions with compact support. Then the elements of their strong duals are called ultradistributions. This reflects the standard definition of distributions. Finally, taking strong duals of $\mathscr{E}_{(M_n)}(\Omega)$, $\mathscr{E}_{\{M_n\}}(\Omega)$, $\mathscr{E}_{(\omega)}(\Omega)$ and $\mathscr{E}_{\{\omega\}}(\Omega)$ we obtain spaces of ultradistributions with compact support of Beurling or Roumieu type, respectively. These spaces will serve as a framework for all our studies.

1.3 Weighted algebras of entire functions

Independently of ultradistributions we can define closely related spaces of entire functions with growth conditions (comp. Section 1.6).

Definition 1.3.1. Let $\omega\colon [0,\infty) \to [0,\infty)$. For a function $f \in H(\mathbb{C})$ and a constant $M \in \mathbb{N}$ we define

$$\|f\|_M := \sup_{z\in\mathbb{C}} |f(z)| e^{-M(|\operatorname{Im} z| + \omega(z))},$$

and for a function $f \in H(\mathbb{C})$ and two constants $M \in \mathbb{N}$, $m \in \mathbb{N}$

$$\|f\|_{M,m} := \sup_{z\in\mathbb{C}} |f(z)| e^{-M|\operatorname{Im} z| - \frac{1}{m}\omega(z)}.$$

Then we introduce weighted algebras of entire functions of Beurling type in the following way

$$A_{(\omega)} := \big\{ f \in H(\mathbb{C}) |\ \exists\, M \in \mathbb{N} : \|f\|_M < \infty \big\}$$

and of Roumieu type as follows

$$A_{\{\omega\}} := \big\{ f \in H(\mathbb{C}) |\ \exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N} : \|f\|_{M,m} < \infty \big\}.$$

It can be immediately seen that $A_{(\omega)}$ and $A_{\{\omega\}}$ are indeed algebras.

Proposition 1.3.2. *$A_{(\omega)}$ and $A_{\{\omega\}}$ are unital algebras with the pointwise multiplication.*

Proof. Let $f, g \in A_{\{\omega\}}$. Then for some constants $M_1, M_2 > 0$, every $m \in \mathbb{N}$ and some B_m, C_m we have

$$|(f \cdot g)(z)| \le e^{B_m + C_m + (M_1+M_2)|\operatorname{Im} z| + \frac{2}{m}\omega(z)}.$$

Hence $f \cdot g \in A_{\{\omega\}}$. Moreover, the function $f \equiv 1$ belongs to $A_{\{\omega\}}$.

The proof for $A_{(\omega)}$ is similar. □

The topology of $A_{(\omega)}$ is given by the following representation

$$A_{(\omega)} = \operatorname{ind}_{M\in\mathbb{N}} A_{(\omega),M}$$

where

$$A_{(\omega),M} := \big\{ f \in H(\mathbb{C}) |\ \|f\|_M < \infty \big\}$$

are normed spaces.

In the case of $A_{\{\omega\}}$ algebras the topology is given by the representation

$$A_{\{\omega\}} = \operatorname{ind}_{M\in\mathbb{N}} \operatorname{proj}_{m\in\mathbb{N}} A_{\{\omega\},M,m}$$

where

$$A_{\{\omega\},M,m} := \big\{ f \in H(\mathbb{C}) |\ \|f\|_{M,m} < \infty \big\}$$

are normed spaces.

The algebras $A_{(\omega)}$ are a special case of so-called Hörmander algebras, where the weight $|\mathrm{Im}\, z| + \omega(z)$ is replaced by an arbitrary function $p\colon \mathbb{C} \to [0,\infty)$. These algebras were introduced by L. Hörmander in [Hör90], [Hör67] and used to study the Cauchy-Riemann equations in $\mathbb{C}^n$. Further information about these spaces can be found for instance in [Mei85] or [BG95]. The spaces $A_{\{\omega\}}$ were studied for instance in the paper of Meise [Mei89].

Example 1.3.3. For $\rho > 0$ and $p(z) = |z|^\rho$, A_p is the space of entire functions of finite type and order ρ. For $p(z) = \ln(1+|z|^2)$, $A_p = \mathbb{C}[z]$ where $\mathbb{C}[z]$ is the space of complex polynomials.

Interesting examples of the spaces $A_{(\omega)}$ and $A_{\{\omega\}}$ will be presented in Section 1.6.

We end this section with an elementary observation.

Proposition 1.3.4. *Let $\omega, \sigma\colon [0,\infty) \to [0,\infty)$ satisfy $\omega = O(\sigma)$ and $\sigma = O(\omega)$. Then*

$$A_{(\omega)} = A_{(\sigma)} \qquad \textit{and} \qquad A_{\{\omega\}} = A_{\{\sigma\}}.$$

In the sequel we will use the notation $A_{[\omega]}$ to indicate that a statement applies to both Beurling and Roumieu cases. Further information about the spaces $A_{[\omega]}$ will be given in Sections 1.6 and 1.7.

1.4 Weighted spaces of sequences

In this section we introduce spaces of sequences with growth conditions, which will be appropriate for the study of the interpolation problem in our framework. To do that we need the following notion.

Definition 1.4.1 (Multiplicity variety)**.** Let $\Lambda \subset \mathbb{C}$ be discrete. Given a sequence (m_λ) of natural numbers we define multiplicity variety as $\{(\lambda, m_\lambda) |\ \lambda \in \Lambda\}$.

The spaces of sequences are defined as follows.

Definition 1.4.2. For $\omega\colon [0,\infty) \to [0,\infty)$, a multiplicity variety $X = \{(\lambda, m_\lambda) | \lambda \in \Lambda\}$ and a doubly-indexed sequence of complex numbers $(v_{\lambda,l})_{\lambda\in\Lambda, 0\le l<m_\lambda}$ we define

$$\|v\|_M := \sup_{\lambda\in\Lambda} \sum_{l=0}^{m_\lambda-1} |v_{\lambda,l}| e^{-M(|\operatorname{Im}\lambda|+\omega(\lambda))}$$

and

$$\|v\|_{M,m} := \sup_{\lambda\in\Lambda} \sum_{l=0}^{m_\lambda-1} |v_{\lambda,l}| e^{-M|\operatorname{Im}\lambda|-\frac{1}{m}\omega(\lambda)}$$

where $M, m \in \mathbb{N}$. Then the sequence space of the Beurling type is defined in the following way

$$S_{(\omega)}(X) := \left\{(v_{\lambda,l})_{\lambda\in\Lambda,0\le l<m_\lambda} | \ \exists\, M \in \mathbb{N} : \|v\|_M < \infty\right\}$$

and of the Roumieu type as follows

$$S_{\{\omega\}}(X) := \left\{(v_{\lambda,l})_{\lambda\in\Lambda,0\le l<m_\lambda} | \ \exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N} : \|v\|_{M,m} < \infty\right\}.$$

The topology of $S_{(\omega)}(X)$ is given by the following representation

$$S_{(\omega)}(X) = \operatorname{ind}_{M\in\mathbb{N}} S_{(\omega),M}$$

where

$$S_{(\omega),M} := \left\{(v_{\lambda,l})_{\lambda\in\Lambda,0\le l<m_\lambda} | \ \|v\|_M < \infty\right\}$$

are normed space.

The topology of $S_{\{\omega\}}(X)$ is given by the following representation

$$S_{\{\omega\}}(X) = \operatorname{ind}_{M\in\mathbb{N}} \operatorname{proj}_{m\in\mathbb{N}} S_{\{\omega\},M,m}$$

where

$$S_{\{\omega\},M,m} := \left\{(v_{\lambda,l})_{\lambda\in\Lambda,0\le l<m_\lambda} | \ \|v\|_{M,m} < \infty\right\}$$

are normed space.

The spaces $S_{(\omega)}(X)$ and $S_{\{\omega\}}(X)$ were studied in a greater generality in the papers of Meise [Mei85] and [Mei89], respectively. In the notation used in these papers

$$S_{(\omega)}(X) = k^\infty(A, (E_\lambda)_{\lambda\in\Lambda})$$

where A is a Köthe matrix consisted of elements $a_{\lambda,M} = e^{-M|\mathrm{Im}\,\lambda| - M\omega(\lambda)}$ $(\lambda \in \Lambda, M \in \mathbb{N})$, and

$$E_\lambda = \Big(\mathbb{C}^{m_\lambda}, \|x\| := \sum_{l=0}^{m_\lambda - 1} |x_l|\Big).$$

In turn

$$S_{\{\omega\}}(X) = K\big((\alpha_\lambda)_{\lambda\in\Lambda}, (\beta_\lambda)_{\lambda\in\Lambda}, (E_\lambda)_{\lambda\in\Lambda}\big)$$

where $\alpha_\lambda = |\mathrm{Im}\,\lambda|$, $\beta_\lambda = \omega(\lambda)$ and E_λ is defined as in the previous case.

As in the function spaces case we have the following relations.

Proposition 1.4.3. *Let $X = \{(\lambda, m_\lambda) |\ \lambda \in \Lambda\}$ be a multiplicity variety, and assume that $\omega, \sigma\colon [0,\infty) \to [0,\infty)$ satisfy $\omega = O(\sigma)$ and $\sigma = O(\omega)$. Then*

$$S_{(\omega)} = S_{(\sigma)} \qquad \textit{and} \qquad S_{\{\omega\}} = S_{\{\sigma\}}.$$

We use the same notation for norms in both sequential and function cases, but it cannot lead to any misunderstanding as they are applied to different objects. In the sequel we will write $S_{[\omega]}(X)$ if a statement applies to both Beurling and Roumieu cases. Further information about the spaces $S_{[\omega]}(X)$ will be given in section 1.8.

1.5 Weight functions

Without additional assumptions we cannot go any farther. In this section we present standard assumptions imposed on weights, the functions ω appearing in the previous sections. We use a small modification of the notation proposed in [BMM07].

Definition 1.5.1 (Weight function)**.** A function $\omega\colon [0,\infty) \to [0,\infty)$ is called a *Beurling weight* if it is continuous, increasing and satisfies the following conditions

(α) $\omega(2t) = O(\omega(t))$, (β^*) $\omega(t) = o(t)$,

(δ) $\varphi\colon t \to \omega(e^t)$ is convex, (γ) $\ln t = O(\omega(t))$.

A function $\omega\colon [0,\infty) \to [0,\infty)$ is called a *Roumieu weight* if it is continuous, increasing, satisfies (α), (δ) and the following two conditions

(β) $\omega(t) = O(t)$, (γ^*) $\ln t = o(\omega(t))$.

We call ω a *weight* if it is either a Beurling weight or Roumieu weight. If it additionally satisfies

$$(Q) \qquad \int_0^\infty \frac{\omega(t)}{1+t^2}\,\mathrm{dt} = \infty$$

then we call it quasianalytic and non-quasianalytic otherwise.

Conditions (β) and (γ) provide natural boundaries for our framework. We have seen in Example 1.2.5 that $\mathscr{E}_{(\ln(1+t))}(\mathbb{R}) = C^\infty(\mathbb{R})$ and $\mathscr{E}_{\{\omega\}}(\mathbb{R}) = A(\mathbb{R})$. Hence weights satisfying $\ln(t) = O(\omega(t))$ and $\omega(t) = O(t)$ provide an entire hierarchy of smoothness of C^∞ functions. Condition (α) is a regularity condition and sometimes was replaced by a stronger condition, subadditivity of the weight (see the definition below). A function satisfying (α) is usually called doubling in the literature. The importance of condition (δ) is revealed in Lemma 1.5.3 below. Finally, as the name suggests, for quasianalytic weights ω the spaces $\mathscr{E}_{(\omega)}$ and $\mathscr{E}_{\{\omega\}}$ are quasianalytic, and for non-quasinalytic weights they are non-quasianalytic. This holds by the Denjoy-Carleman theorem (see [BMT90, Corollary 2.5, Corollary 2.6]).

In the sequel we will just write *weight* when a statement applies to Beurling and Roumieu weights.

Example 1.5.2. Quasianalytic weights:

- $\omega(t) = t$,
- $\omega(t) = \frac{t}{\ln(1+t)}$,
- $\omega(t) = \frac{t}{\ln^q(1+t)}$ for $0 < q \leq 1$.

Non-quasianalytic weights:

- $\omega(t) = \ln(1+t)$,
- $\omega(t) = t^q$ for $0 < q < 1$,
- $\omega(t) = \ln^s(1+t)$ for $s \geq 1$.

We extend every weight radially to the entire complex plane by the formula

$$\tilde{\omega}\colon \mathbb{C} \to [0,\infty), \quad \tilde{\omega}(z) = \omega(z).$$

By abuse of notation we will denote it ω. The following lemma asserts that it is always subharmonic.

Lemma 1.5.3 ([Ran95, Theorem 2.6.6])**.** *Let $\omega\colon [0,\infty) \to \mathbb{R}$. Then ω is increasing and the map $x \mapsto \omega(e^x)$ convex if and only if the function $\tilde{\omega}$ is subharmonic.*

This property will allow us to use the $\bar{\partial}$ technique of Hörmander with weighted estimates.

Definition 1.5.4. A function $\omega\colon [0,\infty) \to [0,\infty)$ is called *subadditive* if it satisfies

$$\omega(x+y) \leq \omega(x) + \omega(y)$$

for all $x, y \in [0,\infty)$. It is called *weakly subadditive* if it satisfies

$$\exists\, C > 0 \ \forall\, x, y \in [0,\infty) : \omega(x+y) \leq C(\omega(x) + \omega(y) + 1).$$

From the standard assumptions imposed on weights we can derive additional useful statements. The following one comes from [BMT90].

Lemma 1.5.5. *Every weight is weakly subadditive.*

Proof. Let ω be a weight. By (α) for some $C > 0$ for all $x, y \in [0, \infty)$

$$\omega(x+y) \leq \omega(2\max(x,y)) \leq C(\omega(\max(x,y)) + 1) \leq C(\omega(x) + \omega(y) + 1). \qquad \square$$

The following property will be frequently used later on.

Lemma 1.5.6. *Every weight ω satisfies the following condition:*

$$\exists\, C > 0 \ \forall\, z, \xi \in \mathbb{C}, r > 0 \colon |z - \xi| \leq r \Rightarrow \omega(z) \leq C(\omega(\xi) + r + 1).$$

Proof. Using Lemma 1.5.5, the continuity of ω and condition (β)

$$\omega(z) \leq \omega(|\xi| + r) \leq C(\omega(|\xi|) + \omega(r) + 1) \leq C(\omega(|\xi|) + Ar + B). \qquad \square$$

In the next lemma we formulate other properties of weights.

Lemma 1.5.7. *Let ω be a weight and $p(z) = |\operatorname{Im} z| + \omega(z)$. Then the following conditions are satisfied:*

(1) $\forall\, c > 0 \ \exists\, C, D > 0 \ \forall\, z \in \mathbb{C} \ \forall\, \xi \in D(z, c\,p(z))$

$$p(\xi) \leq Cp(z) + D,$$

(2) $\exists\, \epsilon > 0 \ \exists\, C, D > 0 \ \forall\, z \in \mathbb{C} \ \forall\, \xi \in D(z, \epsilon\, p(z))$

$$p(z) \leq Cp(\xi) + D.$$

Proof.
(1) Let $c > 0, z \in \mathbb{C}$ be arbitrary and $\xi \in D(z, cp(z))$. Then using Lemma 1.5.6 and condition (β)

$$\begin{aligned} p(\xi) = |\operatorname{Im} \xi| + \omega(\xi) &\leq |\operatorname{Im} z| + |\xi - z| + C(\omega(z) + \omega(\xi - z) + 1) \\ &\leq |\operatorname{Im} z| + cp(z) + C(\omega(z) + A|\xi - z| + B) \\ &\leq (c+1)p(z) + C(p(z) + Acp(z) + B) \\ &= (c + 1 + C + CAc)p(z) + CB \end{aligned}$$

for some constants $A, B, C > 0$.
(2) Let $\epsilon > 0, z \in \mathbb{C}$ be arbitrary and $\xi \in D(z, \epsilon p(z))$. Then, again using Lemma 1.5.6 and condition (β),

$$\begin{aligned} p(z) = |\operatorname{Im} z| + \omega(z) &\leq |\operatorname{Im} \xi| + |z - \xi| + C(\omega(\xi) + \omega(z - \xi) + 1) \\ &\leq p(\xi) + \epsilon p(z) + C(p(\xi) + A|z - \xi| + B) \\ &\leq p(\xi) + \epsilon p(z) + Cp(\xi) + CA\epsilon p(z) + CB \end{aligned}$$

for some constants $A, B, C > 0$. Hence

$$p(z) - \epsilon p(z) - CA\epsilon p(z) \leq (C+1)p(\xi) + CB.$$

Assuming $\epsilon < \frac{1}{CA+1}$ we obtain

$$p(z) \leq \frac{(C+1)p(\xi)}{1-\epsilon(CA+1)} + \frac{CB}{1-\epsilon(CA+1)}. \qquad \square$$

Remark 1.5.8. Lemma 1.5.7 is also true if we replace the function p by ω. But the version with p will more convenient later.

We can give a slightly more precise estimate than in Lemma 1.5.7 (1). The following lemma shows that weights are somehow stable on long intervals.

Lemma 1.5.9. *Let ω be a Beurling weight and denote by C the weak subadditivity constant of ω. Then for every constant $c > 0$ there exist $t_0 \in [0, \infty)$ such that for every $t \geq t_0$*

$$\frac{1}{3C} \leq \frac{\omega(x)}{\omega(t)} \leq 3C$$

for every $x \in (t - c\omega(t), t + c\omega(t))$.

Proof. Let $t_0 \in [0, \infty)$ be such that for every $t \geq t_0$

$$3c\omega(t) \leq t \quad \text{and} \quad \omega(t - c\omega(t)) \geq 1.$$

This is possible since $\omega(t) = o(t)$. Then by the weak subadditivity of the weight we have

$$\begin{aligned}\omega(t - c\omega(t)) \leq \omega(x) \leq \omega(t + c\omega(t)) &\leq C(\omega(t - c\omega(t)) + \omega(2c\omega(t)) + 1) \\ &\leq 3C\omega(t - c\omega(t)).\end{aligned}$$

Finally,

$$\frac{1}{3C} = \frac{\omega(t - c\omega(t))}{3C\omega(t - c\omega(t))} \leq \frac{\omega(t - c\omega(t))}{\omega(t + c\omega(t))} \leq \frac{\omega(x)}{\omega(t)} \leq \frac{\omega(x)}{\omega(t - c\omega(t))} \leq 3C. \qquad \square$$

Remark 1.5.10. Assuming that ω is subadditive one can prove Lemma 1.5.9 with $C = 2$ (see [MOCO03, Property (g)]).

A similar in spirit result is true when we replace the interval $(t - c\omega(t), t + c\omega(t))$ by $(t - \epsilon t, t + \epsilon t)$ for small $\epsilon > 0$.

Lemma 1.5.11. *Let ω be a weight and denote by C the weak subadditivity constant of ω. Then for every constant $\epsilon \in (0, \frac{1}{3}]$ there exist $t_0 \in [0, \infty)$ such that for every $t \geq t_0$*

$$\frac{1}{3C} \leq \frac{\omega(x)}{\omega(t)} \leq 3C$$

for every $x \in (t - \epsilon t, t + \epsilon t)$.

Proof. Let $t_0 \in [0, \infty)$ be such that for every $t > t_0$

$$\omega(t - \epsilon t) \geq 1.$$

Then by the weak subadditivity of the weight we have

$$\begin{aligned}\omega(t-\epsilon t) \leq \omega(x) \leq \omega(t+\epsilon t) &\leq C(\omega(t-\epsilon t) + \omega(2\epsilon t) + 1) \\ &\leq 3C\omega(t-\epsilon t).\end{aligned}$$

Finally,

$$\frac{1}{3C} = \frac{\omega(t-\epsilon t)}{3C\omega(t-\epsilon t)} \leq \frac{\omega(t-\epsilon t)}{\omega(t+\epsilon t)} \leq \frac{\omega(x)}{\omega(t)} \leq \frac{\omega(x)}{\omega(t-\epsilon t)} \leq 3C. \qquad \square$$

In fact, Lemma 1.5.9 could be derived as a corollary to Lemma 1.5.11.

A less general version of the following lemma for subadditive weights was proved by Björck in [Bjö66].

Proposition 1.5.12. *Every increasing function $\omega\colon [0, \infty) \to [0, \infty)$ not having property (Q) (quasianalyticity) satisfies*

$$\omega(t) = o\Big(\frac{t}{\ln t}\Big).$$

In particular, every non-quasianalytic weight satisfies this condition.

Proof. First, we will show that $\omega(t) = o(t)$. Assume to the contrary that for some $\epsilon > 0$ there exists a sequence (t_n) tending to infinity such that $\omega(t_n) \geq \epsilon t_n$ for every $n \in \mathbb{N}$. Then

$$\infty > \int_1^\infty \frac{\omega(t)}{t^2} \geq \sum_{n=1}^\infty \epsilon t_n \int_{t_n}^{t_{n+1}} \frac{1}{t^2} = \epsilon \sum_{n=n_0}^\infty \Big(1 - \frac{t_n}{t_{n+1}}\Big).$$

By [Rud87, Theorem 15.5] we obtain that $0 < \prod_{n=1}^\infty \frac{t_n}{t_{n+1}}$, but we can calculate that this product is 0. A contradiction.

Now, we will construct a new sequence (t_n). Let $t_0 \in [0, \infty)$ be such that $\omega(t_0) > 0$. Then for some $n_0 \in \mathbb{N}$ we have $\omega(t_0) \geq \frac{t_0}{n_0}$. Since $\omega(t) = o(t)$ there must be $t_{n_0} > t_0$ such that $\omega(t_{n_0}) = \frac{t}{n_0}$. Assume that we have an increasing sequence $t_{n_0}, t_{n_0+1}, \ldots, t_n$ such that $\omega(t_k) = \frac{t_k}{k}$ for $k = n_0, \ldots, n$. As $\omega(t_n) > \frac{t_n}{n+1}$ and $\omega(t) = o(t)$ we can find another point $t_{n+1} > t_n$ in which $\omega(t_{n+1}) = \frac{t_{n+1}}{n+1}$. We can go on with this process. Since ω is increasing and for every fixed $t \in [0, \infty)$, $\frac{t}{n}$ tends to 0 with n tending to infinity, t_n must diverge to infinity. Then we obtain

$$\infty > \int_1^\infty \frac{\omega(t)}{t^2} \geq \sum_{n=n_0}^\infty \frac{t_n}{n} \int_{t_n}^{t_{n+1}} \frac{1}{t^2} = \sum_{n=n_0}^\infty \frac{1}{n}\Big(1 - \frac{t_n}{t_{n+1}}\Big).$$

This yields $\lim_{n\to\infty}\frac{t_{n+1}}{t_n}=1$. Hence for every $\epsilon>0$ we have $t_{n+1}\le(1+\epsilon)t_n$ for n big enough. Using this recursively we obtain

$$\forall\,\epsilon>0\ \exists\,n_\epsilon\in\mathbb{N}\ \forall\,n\ge n_\epsilon:\quad t_n\le(1+\epsilon)^{n-n_\epsilon}t_{n_\epsilon}.$$

Then

$$\frac{\ln t_n}{n}\le\frac{\ln\left((1+\epsilon)^{n-n_\epsilon}t_{n_\epsilon}\right)}{n}=\frac{n-n_\epsilon}{n}\ln(1+\epsilon)+\frac{\ln(t_{n_\epsilon})}{n}\le\ln(1+\epsilon)+\frac{\ln(t_{n_\epsilon})}{n}.$$

Choosing ϵ small enough we can make this expression arbitrarily close to zero for all n big enough. Hence $\lim_{n\to\infty}\frac{\ln t_n}{n}=0$. Once again using $\lim_{n\to\infty}\frac{t_{n+1}}{t_n}=1$ we obtain that

$$\lim_{n\to\infty}\frac{t_n}{t_{n-1}}\frac{\ln t_{n-1}}{n}=0.$$

Let $\epsilon>0$ be arbitrary. Then for n big enough and all $t\in[t_{n-1},t_n]$ we get

$$\omega(t)\le\omega(t_n)=\frac{t_n}{n}\le\epsilon\frac{t_{n-1}}{\ln t_{n-1}}\le\epsilon\frac{t}{\ln t}.\qquad\square$$

The next lemma allows us to assume, when needed, that a weight is infinitely differentiable. It was proved by Braun, Meise and Taylor in [BMT90] (see the proof of [BMT90, Lemma 1.7]) for non-quasianalytic weights. However, its proof applies equally well to quasianalytic weights. For the sake of completeness we give here a detailed proof. We will use the notation $\varphi'_+(x)$ to denote the right derivative of a function φ in x.

Lemma 1.5.13. *For every function $\omega\colon[0,\infty)\to[0,\infty)$ continuous, increasing and satisfying (α) there is a C^∞ function $\nu\colon[0,\infty)\to[0,\infty)$ with the same properties and satisfying that for every $\delta>0$ there exists $C>0$ such that for all $t\ge\delta$*

$$\omega(t)\le\nu(t)\le C\omega(t)\tag{1.1}$$

and for every $A>1$

$$\limsup_{t\to\infty}\frac{\nu(At)}{\nu(t)}\le\limsup_{t\to\infty}\frac{\omega(At)}{\omega(t)}.\tag{1.2}$$

If $\varphi(x)=\omega(e^x)$ is convex then ν could be chosen such that $\psi(x)=\nu(e^x)$ is convex as well. The same holds for strict convexity. Moreover, the convexity of φ and the condition $\varphi'_+(x)\xrightarrow{t\to\infty}\infty$ imply that ν could be chosen with $\psi'(x)\xrightarrow{t\to\infty}\infty$. Finally, $\omega(0)=0$ then we can make ν to satisfy $\nu(0)=0$.

Remark 1.5.14. The right derivative φ'_+ exists in every point, since φ is convex. Notice that for ω and ν as in the lemma we have $A_{[\omega]} = A_{[\nu]}$ and $S_{[\omega]}(X) = S_{[\nu]}(X)$. We recall that square brackets stand for both Beurling and Roumieu cases. Furthermore, by (1.1) conditions (β), (γ), $(\gamma *)$ and (Q) are carried from ω to ν. Taking (1.2) into account we see that if ω was a weight then ν would be so as well. Quasianalyticity would also be preserved then.

Proof. Denote $\varphi(x) = \omega(e^x)$. Choose a C^∞ function $\chi \geq 0$ with $\operatorname{supp} \chi \subset (0, \ln 2)$ and $\int_{\mathbb{R}} \chi \, \mathrm{dm} = 1$, and define

$$\psi(x) = \int_{\mathbb{R}} \varphi(s + x)\chi(s) \, \mathrm{ds} \quad \text{and} \quad \nu(t) = \psi(\ln t).$$

It is a convolution of continuous and C^∞ function hence it is of class C^∞. By (α) and the continuity of ω there exists $C > 0$ such that $\omega(2t) \leq C\omega(t)$ for $t \geq \delta$. Then

$$\varphi(x + \ln 2) = \omega(e^{x+\ln 2}) = \omega(2e^x) \leq C\omega(e^x) = C\varphi(x)$$

for $x \geq \ln \delta$. Recall that φ is increasing and $\operatorname{supp} \chi \subset (0, \ln 2)$. Thus

$$\varphi(x) = \int_{\mathbb{R}} \varphi(x)\chi(s) \, \mathrm{ds} \leq \int_{\mathbb{R}} \varphi(x + s)\chi(s) \, \mathrm{ds} = \psi(x)$$

and

$$\psi(x) = \int_0^{\ln 2} \varphi(x + s)\chi(s) \, \mathrm{ds} \leq \int_0^{\ln 2} \varphi(x + \ln 2)\chi(s) \, \mathrm{ds} \leq C \int_{\mathbb{R}} \varphi(x)\chi(s) \, \mathrm{ds} = C\varphi(x)$$

for $x \geq \ln \delta$. Therefore for $t \geq \delta$

$$\omega(t) = \varphi(\ln t) \leq \psi(\ln t) = \nu(t)$$

and

$$\nu(t) = \psi(\ln t) \leq C\varphi(\ln t) = \omega(t).$$

Let $A > 1$ be given. Using (α), maybe several times, we obtain B such that for all t big enough

$$\omega(At) \leq B\omega(t).$$

Denote $a = \ln A$. Then

$$\psi(x + a) = \int_{\mathbb{R}} \varphi(s + x + a)\chi(s) \, \mathrm{ds} \leq \int_{\mathbb{R}} B\varphi(x + s)\chi(s) \, \mathrm{ds} = B\psi(x).$$

This gives (1.2) and we can conclude (α) for ν taking $A = 2$.

Suppose now that φ is convex. Then for $\alpha + \beta = 1$

$$\begin{aligned}\psi(\alpha x + \beta y) &= \int_{\mathbb{R}} \varphi(\alpha x + \beta y + s)\chi(s)\,\mathrm{ds} \\ &= \int_{\mathbb{R}} \varphi(\alpha(x+s) + \beta(y+s))\chi(s)\,\mathrm{ds} \\ &\leq \int_{\mathbb{R}} (\alpha\varphi(x+s) + \beta\varphi(y+s))\chi(s)\,\mathrm{ds} \\ &= \alpha\psi(x) + \beta\psi(y).\end{aligned}$$

Thus $\psi(x) = \nu(e^x)$ is convex. Assume that additionally $\varphi'_+(x) \xrightarrow{t\to\infty} \infty$. As φ is convex, φ'_+ is increasing and we obtain

$$\psi'(x) = \int_0^{\ln 2} \varphi'_+(x+s)\chi(s)\,\mathrm{ds} \geq \int_0^{\ln 2} \varphi'_+(x)\chi(s)\,\mathrm{ds} = \varphi'_+(x)$$

and the assertion follows. Finally, by the definition of ψ we see that $\lim_{x\to-\infty}\psi(x) = \lim_{x\to-\infty}\varphi(x)$ hence $\omega(0) = 0$ implies $\nu(0) = 0$. $\square$

Before stating an important result about weights we need an additional technical lemma.

Lemma 1.5.15. *Let $\omega\colon [0,\infty) \to [0,\infty)$ be an increasing, continuous function with the map $\varphi\colon \mathbb{R} \to [0,\infty)$, $\varphi(x) = \omega(e^x)$ convex. Then $\ln t = o(\omega(t))$ if and only if $\varphi'_+(x) \xrightarrow{x\to\infty} \infty$.*

Proof. ($\Rightarrow$) The condition $\ln t = o(\omega(t))$ implies

$$\frac{\ln t}{\varphi(\ln t)} \xrightarrow{t\to\infty} 0 \quad \text{and} \quad \frac{\varphi(x)}{x} \xrightarrow{x\to\infty} \infty \tag{1.3}$$

Further, for $x \in \mathbb{R}$ and $h > 0$

$$\frac{\frac{\varphi(x+h)}{x+h} - \frac{\varphi(x)}{x}}{h} = \frac{1}{x+h}\,\frac{\varphi(x+h) - \varphi(x)}{h} - \frac{\varphi(x)}{x(x+h)}.$$

Hence the right derivative of $\frac{\varphi(x)}{x}$ exists, since there exists $\varphi'_+(x)$. We will show that condition (1.3) gives that for every $n \in \mathbb{N}$ there is $x_n > n$ such that

$$\Big(\frac{\varphi(x_n)}{x_n}\Big)'_+ \geq 0. \tag{1.4}$$

Indeed, if it was not the case we would have that for some $n \in \mathbb{N}$ and all $x \geq n$

$$\left(\frac{\varphi(x)}{x}\right)'_+ < 0.$$

This would imply that $\varphi(x)/x$ was bounded above, which contradicts (1.3). This proves (1.4). Further,

$$0 \leq \left(\frac{\varphi(x_n)}{x_n}\right)'_+ = \frac{\varphi'_+(x_n)x_n - \varphi(x_n)}{x_n^2},$$

thus

$$\varphi'_+(x_n) \geq \frac{\varphi(x_n)}{x_n}.$$

Since the right-hand side of this inequality tends to infinity with n and φ'_+ is non-decreasing by convexity of φ, we get $\varphi'_+(x) \xrightarrow{x\to\infty} \infty$.

($\Leftarrow$) We have $\varphi(x) = \omega(e^x)$ hence for $x \in \mathbb{R}$ and $h > 0$

$$\begin{aligned}\frac{\omega(e^x + e^x(e^h - 1)) - \omega(e^x)}{e^x(e^h - 1)} &= \frac{\omega(e^{x+h}) - \omega(e^x)}{e^x(e^h - 1)} \\ &= \frac{\varphi(x+h) - \varphi(x)}{h} \frac{h}{e^x(e^h - 1)} \\ &\xrightarrow{h\to 0_+} \frac{\varphi'_+(x)}{e^x}.\end{aligned}$$

Therefore $\varphi'_+(x) = \omega'_+(e^x)e^x$ for every $x \in \mathbb{R}$ and $\varphi'_+(\ln t) = \omega'_+(t)t$ for every $t > 0$. Since $\varphi'_+(\ln t)$ tends to infinity with t, we get that for every $n \in \mathbb{N}$ there exists $t_n > 0$ such for all $t \geq t_n$

$$\omega'_+(t) \geq \frac{n}{t}.$$

Further, using Proposition 1.6.1 of [NP06] we obtain

$$\begin{aligned}\omega(t) - \omega(t_n) = \varphi(\ln t) - \varphi(\ln t_n) &= \int_{\ln t_n}^{\ln t} \varphi'_+(s)\,\mathrm{ds} = \int_{\ln t_n}^{\ln t} \omega'_+(e^s)e^s\,\mathrm{ds} \\ &= \int_{t_n}^{t} \omega'_+(s)\,\mathrm{ds} \geq \int_{t_n}^{t} \frac{n}{s}\,\mathrm{ds} = n(\ln t - \ln t_n).\end{aligned}$$

For t big enough we have $\ln t - \ln t_n \geq \frac{1}{2}\ln t$ thus

$$n \leq \frac{\omega(t) - \omega(t_n)}{\ln t - \ln t_n} \leq \frac{\omega(t)}{\ln t - \ln t_n} \leq \frac{2\omega(t)}{\ln t}.$$

Therefore

$$\frac{\omega(t)}{\ln t} \geq \frac{n}{2}$$

provided t is big enough. This means $\frac{\omega(t)}{\ln t} \xrightarrow{t\to\infty} \infty$. $\square$

The following lemma comes from [BMT90, Lemma 1.7]. It was originally stated for non-quasianalytic weights, but [BMT90, Remark 1.8(1)] extends its applicability to quasianalytic weights as well. For the sake of completeness we present here its proof. This lemma is of great importance in translating problems from the Roumieu case to the Beurling case.

Lemma 1.5.16. *Let ω be a Roumieu weight and $q\colon [0,\infty) \to [0,\infty)$ be any function with $q(t) = o(\omega(t))$. Then there is a Roumieu weight ν such that*

(i) $q(t) = o(\nu(t))$,

(ii) $\nu(t) = o(\omega(t))$,

(iii) for every $A > 1$*:*

$$\limsup_{t\to\infty} \frac{\nu(At)}{\nu(t)} \le \limsup_{t\to\infty} \frac{\omega(At)}{\omega(t)}.$$

If $\omega(0) = 0$ or it is strictly increasing then ν could be chosen with the same property.

Proof. By Lemma 1.5.13 we may assume that $\omega \in C^\infty$. We put $x_1 = y_1 = z_1 = 0$ and define inductively x_n, y_n, z_n with

$$x_n > y_{n-1} + n, \tag{1.5}$$

$$q(e^x) \le \frac{\varphi(x)}{n^2} \quad \text{for all } x \ge x_n, \tag{1.6}$$

$$\varphi(x_n) \ge n \sum_{i=1}^{n-1} \varphi(z_i), \tag{1.7}$$

$$\varphi'(y_n) = \frac{n}{n-1}\,\varphi'(x_n), \tag{1.8}$$

$$\varphi(z_n) = n\varphi(x_n) - (n-1)\varphi(y_n) + n(y_n - x_n)\varphi'(x_n), \tag{1.9}$$

where $\varphi(x) = \omega(e^x)$. First, we choose x_n. We are able to fulfil (1.6) since $q(t) = o(\omega(t))$, (1.7) is possible because φ is increasing. Then we choose y_n satisfying (1.8). It is possible since $o(\ln t) = \omega(t)$ and by Lemma 1.5.15, $\varphi'(x) \xrightarrow{t\to\infty} \infty$. Then we can choose z_n satisfying (1.9). We claim that

$$x_n \le z_n \le y_n. \tag{1.10}$$

From (1.8) it follows that $x_n \le y_n$. From (1.9) and (1.8) we get

$$\begin{aligned}\frac{\varphi(z_n) - \varphi(x_n)}{y_n - x_n} &= -(n-1)\frac{\varphi(y_n) - \varphi(x_n)}{y_n - x_n} + n\varphi'(x_n)\\ &= (n-1)\Big(\varphi'(y_n) - \frac{\varphi(y_n) - \varphi(x_n)}{y_n - x_n}\Big),\end{aligned}$$

which is nonnegative because φ' is nondecreasing. On the other hand, (1.9) also implies

$$\frac{\varphi(y_n)-\varphi(z_n)}{y_n-x_n} = n\frac{\varphi(y_n)-\varphi(x_n)}{y_n-x_n} - n\varphi'(x_n) \geq 0.$$

We define $\psi\colon \mathbb{R} \to [0,\infty)$ by

$$\psi(x) = \begin{cases} \frac{1}{n-1}\varphi(x_n) + \sum_{i=1}^{n-2} \frac{1}{i(i+1)}\varphi(z_{i+1}) + \frac{x-x_n}{n-1}\varphi'(x_n) & \text{for } x_n \leq x \leq y_n, \\ \frac{1}{n}\varphi(x) + \sum_{i=1}^{n-1} \frac{1}{i(i+1)}\varphi(z_{i+1}) & \text{for } y_n \leq x < x_{n+1}, \\ \varphi(x) & \text{for } x < 0. \end{cases}$$

This function is affine on intervals $[x_n, y_n]$ and on intervals $[y_n, x_{n+1}]$ it is the function φ leaned more and more. All the constants appearing in the definition of ψ are just to make it continuously differentiable. In fact, conditions (1.8) and (1.9) give that $\psi \in C^1$. Moreover, it is convex since it consists only of linear parts and of dilated and shifted parts of φ.

We define

$$\nu(t) = \psi(\ln t)$$

for $t \in [0,\infty)$. We think of $\nu(0)$ as the limit of ψ at $-\infty$. It is again a C^1 function.

We claim

$$\psi(x) \geq \frac{1}{n}\varphi(x) \quad \text{for } x \in [x_n, x_{n+1}], n \geq 2. \tag{1.11}$$

This is obvious for $x \in [y_n, x_{n+1}]$. For $x \in [x_n, y_n]$ we get from (1.8) and the convexity of φ

$$\psi(x) \geq \frac{1}{n-1}\varphi(x_n) + \frac{x-x_n}{n-1}\varphi'(x_n) \geq \frac{1}{n}\varphi(x_n) + \frac{x-x_n}{n}\varphi'(y_n) \geq \frac{1}{n}\varphi(x).$$

Then by (1.6) and (1.11) for $t \geq e^{x_n}$

$$q(t) \leq \frac{\varphi(\ln t)}{n^2} \leq \frac{1}{n}\psi(\ln t) = \frac{1}{n}\nu(t),$$

which proves (i).

For the proof of (ii), we first consider the case $y_n \leq x \leq x_{n+1}$. We apply (1.7) and (1.10),

$$\begin{aligned} \frac{\psi(x)}{\varphi(x)} &= \frac{1}{n} + \sum_{i=1}^{n-1} \frac{1}{i(i+1)} \frac{\varphi(z_{i+1})}{\varphi(x)} \\ &= \frac{1}{n} + \sum_{i=1}^{n-2} \frac{1}{i(i+1)} \frac{\varphi(z_{i+1})}{\varphi(x)} + \frac{1}{n(n-1)} \frac{\varphi(z_n)}{\varphi(x)} \\ &\leq \frac{1}{n} + \sum_{i=1}^{n-1} \frac{\varphi(z_i)}{\varphi(x)} + \frac{1}{n(n-1)} \\ &\leq \frac{1}{n} + \frac{1}{n} + \frac{1}{n(n-1)}. \end{aligned}$$

Now let $x_n \le x \le y_n$. Using the convexity of φ and (1.7) we obtain

$$\begin{aligned}\frac{\psi(x)}{\varphi(x)} &= \frac{1}{n-1}\frac{\varphi(x_n)}{\varphi(x)} + \sum_{i=1}^{n-2}\frac{1}{i(i+1)}\frac{\varphi(z_{i+1})}{\varphi(x)} + \frac{x-x_n}{n-1}\frac{\varphi'(x_n)}{\varphi(x)} \\ &\le \frac{1}{n-1}\frac{\varphi(x_n)}{\varphi(x)} + \frac{x-x_n}{n-1}\frac{\varphi'(x_n)}{\varphi(x)} + \sum_{i=1}^{n-2}\frac{\varphi(z_{i+1})}{\varphi(x)} \\ &\le \frac{1}{n-1}\frac{\varphi(x_n)}{\varphi(x)} + \frac{x-x_n}{n-1}\frac{\varphi'(x_n)}{\varphi(x)} + \sum_{i=2}^{n-1}\frac{\varphi(z_i)}{\varphi(x_n)} \le \frac{1}{n-1}\frac{\varphi(x)}{\varphi(x)} + \frac{1}{n}.\end{aligned}$$

To prove (iii) we will estimate $\psi(x+a)-\psi(x)$ in terms of $\psi(x)$. Here $a = \ln A$. Choose

$$B > \limsup_{t\to\infty}\frac{\omega(At)}{\omega(t)}$$

and $n > a$ so large that for all $x \ge x_n$ we have $\varphi(x+a) \le B\varphi(x)$. We have to consider five cases depending into which intervals x and $x+a$ fall. Because of (1.5) and $n > a$ there are only five possibilities:

1. $x_n \le x, x+a < y_n$,
2. $x_n \le x < y_n \le x+a < x_{n+1}$,
3. $y_n \le x, x+a < x_{n+1}$,
4. $y_n \le x < x_{n+1} \le x+a < y_{n+1}$,
5. $y_n \le x < x_{n+1} \le y_{n+1} \le x+a < x_{n+2}$.

Case 1: We use (1.11),

$$\begin{aligned}\psi(x+a)-\psi(x) &= \frac{a}{n-1}\varphi'(x_n) \le \frac{a}{n-1}\frac{\varphi(x_n+a)-\varphi(x_n)}{a} \\ &\le \frac{B-1}{n-1}\varphi(x_n) \le (B-1)\frac{n}{n-1}\psi(x).\end{aligned}$$

Case 2: First, we apply (1.9),

$$\begin{aligned}\psi(x+a)-\psi(x) &= \frac{1}{n}\varphi(x+a) + \frac{1}{n(n-1)}\varphi(z_n) - \frac{1}{n-1}\varphi(x_n) - \frac{x-x_n}{n-1}\varphi'(x_n) \\ &= \frac{1}{n}\varphi(x+a) - \frac{1}{n}\varphi(y_n) + \frac{y_n-x}{n-1}\varphi'(x_n).\end{aligned}$$

Then, using

$$\varphi'(x_n) \le \frac{\varphi(y_n)-\varphi(x)}{y_n-x},$$

which is true by convexity of φ, and referring to (1.11) at the end, we obtain

$$\begin{aligned}\psi(x+a)-\psi(x) &\leq \frac{1}{n}\varphi(x+a)-\frac{1}{n}\varphi(y_n)+\frac{1}{n-1}(\varphi(y_n)-\varphi(x)) \\ &\leq \frac{1}{n-1}\varphi(x+a)-\frac{1}{n-1}\varphi(y_n)+\frac{1}{n-1}(\varphi(y_n)-\varphi(x)) \\ &= \frac{1}{n-1}(\varphi(x+a)-\varphi(x)) \leq \frac{B-1}{n-1}\varphi(x) \leq (B-1)\frac{n}{n-1}\psi(x).\end{aligned}$$

Case 3:

$$\psi(x+a)-\psi(x) = \frac{1}{n}(\varphi(x+a)-\varphi(x)) \leq \frac{B-1}{n}\varphi(x) \leq (B-1)\psi(x).$$

Case 4: By convexity of φ we have

$$\varphi'(x_{n+1}) \leq \frac{\varphi(x+a)-\varphi(x_{n+1})}{x+a-x_{n+1}},$$

hence

$$\begin{aligned}\psi(x+a)-\psi(x) &= \frac{1}{n}\varphi(x_{n+1})+\frac{x+a-x_{n+1}}{n}\varphi'(x_{n+1})-\frac{1}{n}\varphi(x) \\ &\leq \frac{1}{n}(\varphi(x+a)-\varphi(x)) \leq \frac{B-1}{n}\varphi(x) \leq (B-1)\psi(x).\end{aligned}$$

Case 5:

$$\begin{aligned}\psi(x+a)-\psi(x) &= \frac{1}{n+1}\varphi(x+a)+\frac{1}{n(n+1)}\varphi(z_{n+1})-\frac{1}{n}\varphi(x) \\ &\leq \frac{1}{n+1}\varphi(x+a)+\frac{1}{n(n+1)}\varphi(x+a)-\frac{1}{n}\varphi(x) \\ &= \frac{1}{n}(\varphi(x+a)-\varphi(x)) \leq \frac{B-1}{n}\varphi(x) \leq (B-1)\psi(x).\end{aligned}$$

Now (iii) implies (α) for ν. We have $\ln t = o(\omega)$ hence q can always be chosen with $\ln t = o(q(t))$ pushing ν to satisfy this property as well. □

1.6 Relation between spaces of ultradistributions and weighted algebras

In this section we will say how the sequential approach (the spaces $\mathscr{E}_{(M_n)}(\Omega)$ and $\mathscr{E}_{\{M_n\}}(\Omega)$) can be reduced to the functional approach (the spaces $\mathscr{E}_{(\omega)}(\Omega)$ and $\mathscr{E}_{\{\omega\}}(\Omega)$) under certain assumptions on the sequence (M_n). This part will be based on the paper [BMM07]. Further, we will present properties of the spaces $\mathscr{E}_{(\omega)}(\Omega)$ and $\mathscr{E}_{\{\omega\}}(\Omega)$ and their strong duals. Then we will show the way the duals relate to weighted algebras of entire functions. This part will be based on three papers [BMT90], [Rös97] and [HM07]. The first one covers non-quasianalytic Beurling and Roumieu cases, the second one concerns the quasianalytic Roumieu case, the third one covers all cases, in particular the missing quasianalytic Beurling case.

Theorem 1.6.1 ([BMM07, Theorem 14])**.** *Let $(M_n)_{n\in\mathbb{N}_0}$ be a sequence of positive numbers such that*

$$\exists\, A, K > 0 \;\forall\, n \in \mathbb{N}_0\colon \quad M_n \le AK^n \min_{0\le k\le n} M_k M_{n-k}$$

and

$$\exists\, k \in \mathbb{N}\colon \quad \liminf_{n\to\infty} \frac{m_{kn}}{m_n} > 1$$

where $m_n = \frac{M_n}{M_{n-1}}$ $(n \in \mathbb{N})$. Then $\omega\colon [0,\infty) \to [0,\infty)$ defined by

$$\omega(t) = \sup_{n\in\mathbb{N}_0} \ln \frac{t^n}{M_n} \quad \text{for} \quad t > 0, \qquad \omega(0) = 0$$

is a weight and it holds

$$\mathscr{E}_{[M_n]}(\Omega) = \mathscr{E}_{[\omega]}(\Omega)$$

for any open set $\Omega \subset \mathbb{C}$.

Now we can turn to properties of $\mathscr{E}_{(M_n)}(\Omega)$ and $\mathscr{E}_{\{M_n\}}(\Omega)$. The following proposition can be proved by standard arguments.

Proposition 1.6.2. *Let ω be a weight and Ω an open subset of $\mathbb{R}$. Then $\mathscr{E}_{[\omega],K,m} = \{f \in C^\infty(K)|\ \|f\|_{K,m} < \infty\}$ is a Banach space.*

We recall that

$$\|f\|_{K,p} := \sup_{\substack{x\in K\\ n\in\mathbb{N}_0}} |f^{(n)}(x)| \exp\left(-p\,\varphi^*\left(\frac{n}{p}\right)\right)$$

where $K \Subset \Omega$, $p > 0$.

The following proposition was proved in [BMT90, Proposition 4.9] for non-quasianalytic weights. For quasianalytic weights it was mentioned as a remark on page 366 in [HM07].

Proposition 1.6.3. *Let ω be a Beurling weight and Ω an open subset of $\mathbb{R}$. Then $\mathscr{E}_{(\omega)}(\Omega)$ is a nuclear Fréchet space.*

Using [MV97, Proposition 25.20] we can deduce the following proposition.

Proposition 1.6.4. *Let ω be a Beurling weight and Ω an open subset of $\mathbb{R}$. Then $\mathscr{E}'_{(\omega)}(\Omega)$ is a (DFN) space, in particular it is Hausdorff, complete, ultra-bornological.*

For the Roumieu case we have the following.

Proposition 1.6.5 ([BMT90, Proposition 4.9]). *Let ω be a Roumieu non-quasianalytic weight and Ω an open subset of $\mathbb{R}$. Then $\mathscr{E}_{\{\omega\}}(\Omega)$ is nuclear, complete and reflexive.*

For arbitrary weights, as is written in [HM07] on page 373 the following statement is true.

Proposition 1.6.6. *Let ω be a Roumieu weight and Ω an open subset of $\mathbb{R}$. Then $\mathscr{E}_{\{\omega\}}(\Omega)$ is a complete Schwartz space.*

Using [MV97, Proposition 24.23] we can obtain the following about the strong dual of $\mathscr{E}_{\{\omega\}}(\Omega)$.

Proposition 1.6.7. *Let ω be a Roumieu weight and Ω an open subset of $\mathbb{R}$. Then $\mathscr{E}'_{\{\omega\}}(\Omega)$ is ultra-bornological.*

In the proof of [HM07, Theorem 3.7] the following statement is shown.

Proposition 1.6.8. *Let ω be a Roumieu weight and Ω an open subset of $\mathbb{R}$. Then $\mathscr{E}'_{\{\omega\}}(\Omega)$ is an (LF)-space.*

Now we are going to introduce the Fourier-Laplace transform for ultradistributions.

Definition 1.6.9 (Fourier-Laplace transform). Let ω be a weight and Ω an open subset of $\mathbb{R}$. For $z \in \mathbb{C}$ let $f_z \colon \mathbb{R} \to \mathbb{C}$ be given by

$$f_z(x) := e^{-ixz}$$

for $x \in \mathbb{R}$. We define the Fourier-Laplace transform $\widehat{\mu}$ of $\mu \in \mathscr{E}'_{[\omega]}(\Omega)$ by the formula

$$\widehat{\mu}(z) := \langle \mu, f_z \rangle$$

for $z \in \mathbb{C}$. Finally, we define the Fourier-Laplace transform $\mathcal{F} \colon \mathscr{E}'_{[\omega]}(\Omega) \to H(\mathbb{C})$ by

$$\mathcal{F}(u) := \widehat{u}.$$

For this definition to be correct, one needs that for any $z \in \mathbb{C}$, $f_z \in \mathscr{E}_{[\omega]}(\Omega)$ and that $\widehat{\mu}$ is an entire function. For an argument that this is the case we refer the reader to [BMT90, Definition 7.1].

The Fourier-Laplace transform gives a way of changing ultradistributions to entire functions and vice versa. The precise relation is described in the following two Paley-Wiener type theorems.

Theorem 1.6.10 ([HM07, Theorem 3.6]). *Let ω be a Beurling weight. Then*

$$\mathcal{F}\colon \mathscr{E}'_{(\omega)}(\mathbb{R}) \to A_{(\omega)}$$

is a linear topological isomorphism.

Theorem 1.6.11 ([HM07, Theorem 3.7]). *Let ω be a Roumieu weight. Then*

$$\mathcal{F}\colon \mathscr{E}'_{\{\omega\}}(\mathbb{R}) \to A_{\{\omega\}}$$

is a linear topological isomorphism.

Now we can give several examples of the spaces $A_{(\omega)}$ and $A_{\{\omega\}}$ which are especially important.

Example 1.6.12.

- $A_{(\ln(1+t^2))} = \mathcal{F}(\mathscr{E}'(\mathbb{R}))$, where $\mathscr{E}'(\mathbb{R})$ is the space of distributions with compact support on the real line,
- for $0 < q < 1$, $A_{\{t^q\}} = \mathcal{F}\big(G'_{\frac{1}{q}}(\mathbb{R})\big)$, where $G'_{\frac{1}{q}}(\mathbb{R})$ are the Gevrey ultradistributions,
- $A_{\{t\}} = \mathcal{F}(A'(\mathbb{R}))$, where $A'(\mathbb{R})$ is the space of real analytic functionals.

1.7 Properties of the weighted algebras of entire functions

In this section we will study properties of the spaces $A_{(\omega)}$ and $A_{\{\omega\}}$. All the facts presented here are known, but their proofs are usually omitted. We will write them for the sake of completeness. Using Theorems 1.6.10 and 1.6.11 we could derive many properties from the previous section. But it is not a proper way, since some of these properties are used to prove that the Fourier-Laplace transform is an isomorphism.

The following two propositions show an another important consequence of conditions (γ), (γ^*) of weights.

Proposition 1.7.1. *Let ω be a Beurling weight. Then $A_{(\omega)}$ contains all polynomials.*

Proof. By property (γ) of the weight

$$|z| = e^{\ln|z|} \leq e^{C\omega(z)+C}$$

for some $C > 0$ and all $z \in \mathbb{C}$. As $A_{(\omega)}$ is an algebra, this completes the proof. □

Proposition 1.7.2. *Let ω be a Roumieu weight. Then $A_{\{\omega\}}$ contains all polynomials.*

Proof. By property (γ^*) of the weight for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that $\ln|z| \leq C_m + \frac{1}{m}\omega(z)$ for all $z \in \mathbb{C}$. Hence

$$|z| = e^{\ln|z|} \leq e^{C_m + \frac{1}{m}\omega(z)}.$$

As $A_{\{\omega\}}$ is an algebra, this completes the proof. □

It is seen from the proofs given above that conditions (γ), (γ^*) are also necessary for the polynomials to belong to $A_{(\omega)}$, $A_{\{\omega\}}$, respectively.

The next proposition shows a consequence of the regularity condition (α) for weights.

Proposition 1.7.3. *Let ω be a weight. Then $A_{[\omega]}$ is closed under differentiation and translation.*

Proof. By Lemma 1.5.6 we have

$$\omega(\xi) \leq C(\omega(z) + r + 1)$$

for some $C > 0$, all $r > 0$ and all $z, \xi \in \mathbb{C}$ satisfying $|z - \xi| \leq r$. Assume that $f \in A_{(\omega)}$. Using Cauchy's inequality

$$|f'(z)| \leq \sup_{\xi\colon |z-\xi|\leq 1} |f(\xi)| \leq \sup_{\xi\colon |z-\xi|\leq 1} e^{M+M|\operatorname{Im}\xi|+M\omega(\xi)} \leq e^{2M+2CM+M|\operatorname{Im} z|+CM\omega(z)}$$

for some constant $M > 0$. Hence $f' \in A_{(\omega)}$. For $f \in A_{\{\omega\}}$ we have that there exists $M \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $C_m > 0$ for which

$$|f(z)| \leq e^{C_m + M|\operatorname{Im} z| + \frac{1}{m}\omega(z)}$$

for all $z \in \mathbb{C}$. Hence

$$|f'(z)| \leq \sup_{\xi\colon |z-\xi|\leq 1} |f(\xi)| \leq \sup_{\xi\colon |z-\xi|\leq 1} e^{C_m+M|\operatorname{Im}\xi|+\frac{1}{m}\omega(\xi)} \leq e^{C_m+M+\frac{2C}{m}+M|\operatorname{Im} z|+\frac{C}{m}\omega(z)}.$$

This means that $f' \in A_{\{\omega\}}$.

Similarly, using Lemma 1.5.6, we obtain that for any given $\eta \in \mathbb{C}$ and $f \in A_{[\omega]}$ it holds $f(\cdot - \eta) \in A_{[\omega]}$. □

Proposition 1.7.4. *Let ω be a weight. Then $A_{(\omega),M}$ and $A_{\{\omega\},M,m}$ are Banach spaces for any $M, m \in \mathbb{N}$.*

Proof. Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $A_{(\omega),M}$. Then

$$|f_n(z) - f_k(z)| \le \epsilon e^{M|\operatorname{Im} z| + M\omega(z)}$$

for any given $\epsilon > 0$, all $z \in \mathbb{C}$ and all $n, k \in \mathbb{N}$ big enough. Since $e^{M|\operatorname{Im} z|+M\omega(z)}$ is bounded above on any compact subset of $\mathbb{C}$, $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $H(\mathbb{C})$. Hence this sequence converges locally uniformly to some $f \in H(\mathbb{C})$. We have

$$\forall\, \epsilon > 0\ \exists\, k_0 \in \mathbb{N}\ \forall\, z \in \mathbb{C}\ \forall\, n, k \ge k_0\colon \quad |f_n(z) - f_k(z)|\, e^{-M|\operatorname{Im} z| - M\omega(z)} \le \epsilon.$$

Therefore

$$\forall\, \epsilon > 0\ \exists\, k_0 \in \mathbb{N}\ \forall\, z \in \mathbb{C}\ \forall\, k \ge k_0\colon \quad |f(z) - f_k(z)|\, e^{-M|\operatorname{Im} z| - M\omega(z)} \le \epsilon,$$

which means that $(f_k)_{k\in\mathbb{N}}$ converges to f in $A_{(\omega),M}$. Finally,

$$\|f\|_M \le \|f - f_k\|_M + \|f_k\|_M < \infty$$

for $k \in \mathbb{N}$ big enough.

The proof for $A_{\{\omega\},M,m}$ is analogous. □

Now we can state final results about topologies of $A_{(\omega)}$ and $A_{\{\omega\}}$.

Proposition 1.7.5. *Let ω be a Beurling weight. Then $A_{(\omega)}$ is an (LB)-space.*

Proof. We need to show that the inductive system $(j_M\colon A_{(\omega),M} \to A_{(\omega)})$ is an imbedding spectrum. But this is immediate since $\|\cdot\|_{M+1} \le \|\cdot\|_M$ for every $M \in \mathbb{N}$. Furthermore, every $A_{(\omega),M}$ embeds continuously into $H(\mathbb{C})$, which is Hausdorff. Hence the inductive topology of $A_{(\omega)}$ is Hausdorff as well. □

Remark 1.7.6. From Proposition 1.6.4 and Theorem 1.6.10 we see that $A_{(\omega)}$ is a (DFN)-space.

Proposition 1.7.7. *Let ω be a Roumieu weight. Then for every $M \in \mathbb{N}$*

$$A_{\{\omega\},M} := \left\{ f \in H(\mathbb{C}) \,|\, \forall\, m \in \mathbb{N}\colon \|f\|_{M,m} < \infty \right\}$$

is a Fréchet space, and $A_{\{\omega\}}$ is an (LF)-space.

Proof. Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $A_{\{\omega\},M}$. Then for every $\epsilon > 0$ and every $m \in \mathbb{N}$

$$|f_n(z) - f_k(z)| \le \epsilon e^{M|\operatorname{Im} z| + \frac{1}{m}\omega(z)}$$

for all $z \in \mathbb{C}$ and $n, k \in \mathbb{N}$ big enough. Since $e^{M|\operatorname{Im} z|+\frac{1}{m}\omega(z)}$ is bounded above on any compact subset of $\mathbb{C}$, $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $H(\mathbb{C})$. Hence this sequence converges locally uniformly to some $f \in H(\mathbb{C})$. We have

$$\forall\, \epsilon > 0\ \forall\, m \in \mathbb{N}\ \exists\, k_0 \in \mathbb{N}\ \forall\, z \in \mathbb{C}\ \forall\, n, k \geq k_0\colon\ |f_n(z) - f_k(z)|\, e^{-M|\operatorname{Im} z|-\frac{1}{m}\omega(z)} \leq \epsilon.$$

Therefore

$$\forall\, \epsilon > 0\ \forall\, m \in \mathbb{N}\ \exists\, k_0 \in \mathbb{N}\ \forall\, z \in \mathbb{C}\ \forall\, k \geq k_0\colon \quad |f(z) - f_k(z)|\, e^{-M|\operatorname{Im} z|-\frac{1}{m}\omega(z)} \leq \epsilon,$$

which means that $(f_k)_{k\in\mathbb{N}}$ converges to f in $A_{(\omega),M}$. Finally, for every $m \in \mathbb{N}$

$$\|f\|_{M,m} \leq \|f - f_k\|_{M,m} + \|f_k\|_{M,m} < \infty$$

for $k \in \mathbb{N}$ big enough.

Next we need to show that the inductive system $(j_M\colon A_{\{\omega\},M} \to A_{(\omega)})$ is an imbedding spectrum. This follows from $\|\cdot\|_{M+1,m} \leq \|\cdot\|_{M,m}$ which holds for every $M \in \mathbb{N}$ and $m \in \mathbb{N}$. Finally, every $A_{\{\omega\},M}$ embeds continuously into $H(\mathbb{C})$, which is Hausdorff. Hence the inductive topology of $A_{\{\omega\}}$ is Hausdorff as well. □

Remark 1.7.8. One can show that $A_{\{\omega\}}$ is nuclear whenever $\omega = o(t)$ (see [Mei89, Proposition 1.3]).

An important property of the spaces $A_{(\omega)}$ and $A_{\{\omega\}}$ is that the norms $\|\cdot\|_M$ and $\|\cdot\|_{M,m}$ can be replaced by certain integral norms. This will allow us to use Hörmander's theorem for solving $\bar{\partial}$ equation in the context of these spaces. To be precise we introduce the following notation (comp. [BG95, page 110]).

Definition 1.7.9. Let Ω be an open subset of $\mathbb{C}$ and $\omega\colon [0,\infty) \to [0,\infty)$. Then we define

$$W_{(\omega)}(\Omega) := \Big\{ f\colon \Omega \to \mathbb{C} |\ f \text{ measurable },\exists\, M \in \mathbb{N}\colon \int_\Omega |f(z)|^2 e^{-M|\operatorname{Im} z|-M\omega(z)}\, \mathrm{dz} < \infty \Big\}$$

and

$$W_{(\omega)}(\Omega) := \Big\{ f\colon \Omega \to \mathbb{C} |\ f \text{ measurable },\exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\colon \int_\Omega |f(z)|^2 e^{-M|\operatorname{Im} z|-\frac{1}{m}\omega(z)}\, \mathrm{dz} < \infty \Big\}.$$

Further, we define

$$C^\infty_{(\omega)}(\Omega) := \big\{ f \in C^\infty(\Omega) |\ \exists\, M \in \mathbb{N} : \|f\|_M < \infty \big\}$$

and

$$C^\infty_{\{\omega\}}(\Omega) := \big\{ f \in C^\infty(\Omega) |\ \exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N} : \|f\|_{M,m} < \infty \big\}$$

where $\|\cdot\|_M$ and $\|\cdot\|_{M,m}$ are the weighted norms introduced in Section 1.3.

The following two propositions show an another important consequence of conditions (γ) and (γ^*) of weights.

Proposition 1.7.10. *Let ω be a Beurling weight. Then*

$$A_{(\omega)} = W_{(\omega)}(\mathbb{C}) \cap H(\mathbb{C}) = C^\infty_{(\omega)}(\mathbb{C}) \cap H(\mathbb{C}) \quad \textit{and} \quad C^\infty_{(\omega)}(\mathbb{C}) \subset W_{(\omega)}(\mathbb{C}) \cap C^\infty(\mathbb{C}).$$

Proof. Denote $p(z) = |\operatorname{Im} z| + \omega(z)$. We have to show that for any continuous function f on $\mathbb{C}$,

$$|f(z)| \le e^{M+Mp(z)}$$

for some constant $M \in \mathbb{N}$ and all $z \in \mathbb{C}$ implies

$$\int_{\mathbb{C}} |f(z)|^2 e^{-Mp(z)}\,\mathrm{dz} < \infty$$

for some $M \in \mathbb{N}$, and that the converse holds for holomorphic functions.
($\Rightarrow$) By (γ) there exists $C > 0$ such that $\ln(1+|z|^2) \le C\omega(z) + C$ for all $z \in \mathbb{C}$. Take any $A > 2M + 2C$. Then

$$\begin{aligned}\int_{\mathbb{C}} |f(z)|^2 e^{-Ap(z)}\,\mathrm{dz} &\le e^{2M} \int_{\mathbb{C}} e^{2Mp(z)} e^{-Ap(z)}\,\mathrm{dz} \le e^{2M} \int_{\mathbb{C}} e^{-2Cp(z)}\,\mathrm{dz} \\ &\le e^{2M} \int_{\mathbb{C}} e^{-2\ln(1+|z^2|)+2C}\,\mathrm{dz} = e^{2M+2C} \int_{\mathbb{C}} \frac{1}{(1+|z|^2)^2}\,\mathrm{dz} < \infty.\end{aligned}$$

($\Leftarrow$) We assume that f is holomorphic. Using Lemma 1.5.6 we obtain constants $C, D > 0$ such that

$$p(z) \ge Cp(\xi) - D$$

whenever $|z - \xi| \le 1$. Since $|f|^2$ is subharmonic on $\mathbb{C}$, we have for $M > 0$ and $z \in \mathbb{C}$

$$\begin{aligned}|f(z)|^2 e^{-2Mp(z)} &\le \frac{1}{\pi} \int_{D(z,1)} |f(\xi)|^2 e^{-2Mp(z)}\,\mathrm{d}\xi \le \frac{1}{\pi} \int_{D(z,1)} |f(\xi)|^2 e^{-2MCp(\xi)+2MD}\,\mathrm{d}\xi \\ &\le \frac{e^{2MD}}{\pi} \int_{\mathbb{C}} |f(\xi)|^2 e^{-2MCp(\xi)}\,\mathrm{d}\xi.\end{aligned}$$

Choosing M big enough we obtain that $|f(z)|e^{-Mp(z)} \le A$ for some $A > 0$ and all $z \in \mathbb{C}$. □

Proposition 1.7.11. *Let ω be a Roumieu weight. Then*

$$A_{\{\omega\}} = W_{\{\omega\}}(\mathbb{C}) \cap H(\mathbb{C}) = C^\infty_{\{\omega\}}(\mathbb{C}) \cap H(\mathbb{C}) \quad \textit{and} \quad C^\infty_{\{\omega\}}(\mathbb{C}) \subset W_{\{\omega\}}(\mathbb{C}) \cap C^\infty(\mathbb{C}).$$

Proof. We have to show that for any continuous function f on $\mathbb{C}$,

$$\exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, C_m > 0\ \forall\, z \in \mathbb{C}\colon \quad |f(z)| \le e^{C_m + M|\operatorname{Im} z| + \frac{1}{m}\omega(z)}$$

implies

$$\exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\colon \quad \int_{\mathbb{C}} |f(z)|^2 e^{-M|\operatorname{Im} z| - \frac{1}{m}\omega(z)}\, \mathrm{dz} < \infty,$$

and that the converse is true for holomorphic functions.

($\Rightarrow$) By (γ^*) for all $m \in \mathbb{N}$ there exists $A_m > 0$ such that $\ln(1+|z|^2) \le A_m + \frac{1}{m}\omega(z)$ for all $z \in \mathbb{C}$. Then

$$\begin{aligned}
\int_{\mathbb{C}} |f(z)|^2 e^{-2M|\operatorname{Im} z| - \frac{4}{m}\omega(z)}\, \mathrm{dz} &= \int_{\mathbb{C}} \left(|f(z)| e^{-M|\operatorname{Im} z| - \frac{1}{m}\omega(z)}\right)^2 e^{-\frac{2}{m}\omega(z)}\, \mathrm{dz} \\
&\le e^{2C_m} \int_{\mathbb{C}} e^{-\frac{2}{m}\omega(z)}\, \mathrm{dz} \\
&\le e^{2C_m + 2A_m} \int_{\mathbb{C}} \frac{1}{(1+|z|^2)^2}\, \mathrm{dz} < \infty.
\end{aligned}$$

($\Leftarrow$) We assume that f is holomorphic. Using Lemma 1.5.6 we obtain constants $C, D > 0$ such that

$$\omega(z) \ge C\omega(\xi) - D$$

whenever $|z - \xi| \le 1$. Since $|f|^2$ is subharmonic on $\mathbb{C}$, we have for $M > 0$, $m \in \mathbb{N}$ and $z \in \mathbb{C}$

$$\begin{aligned}
|f(z)|^2 e^{-2M|\operatorname{Im} z| - \frac{2}{m}\omega(z)} &\le \frac{1}{\pi} \int_{D(z,1)} |f(\xi)|^2 e^{-2M|\operatorname{Im} z| - \frac{2}{m}\omega(z)}\, \mathrm{d}\xi \\
&\le \frac{1}{\pi} \int_{D(z,1)} |f(\xi)|^2 e^{-2M|\operatorname{Im} \xi| - \frac{2C}{m}\omega(\xi) + 2M + \frac{D}{m}}\, \mathrm{d}\xi \\
&\le \frac{e^{2M + \frac{D}{m}}}{\pi} \int_{\mathbb{C}} |f(\xi)|^2 e^{-2M|\operatorname{Im} \xi| - \frac{2C}{m}\omega(\xi)}\, \mathrm{d}\xi.
\end{aligned}$$

By the assumption there exists $M > 0$ such that for every $m \in \mathbb{N}$ the last integral is finite, which completes the proof. □

Finally, we give a relation between Beurling algebras and Roumieu algebras (comp. [BMT90, Proposition 7.6], [AJO10, Corollary 4.6]).

Proposition 1.7.12. *In the algebraic sense*

$$A_{\{\omega\}} = \bigcup_{\sigma = o(\omega)} A_{(\sigma)}.$$

Proof. Let $f \in A_{\{\omega\}}$. Then $|f(z)| \leq e^{C_m + M|\operatorname{Im} z| + \frac{1}{m}\omega(z)}$. Using Lemma 1.5.16 for $q(t) = \inf_{m\in\mathbb{N}}(C_m + \frac{1}{m}\omega(t))$ and ω we obtain a weight $\sigma = o(\omega)$ such that $f \in A_{(\sigma)}$. The converse inclusion follows from the observation that if $\sigma = o(\omega)$ then for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that $\sigma(t) \leq C_m + \frac{1}{m}\omega(t)$ for all $t \in [0, \infty)$. □

1.8 Properties of the weighted spaces of sequences

In this section we study properties of the sequence spaces introduced in Section 1.4.

The algebras $S_{[\omega]}(X)$ are invariant under translation of the variety X. The following lemma makes this statement precise.

Proposition 1.8.1. *Let ω be a weight, $X = \{(\lambda, m_\lambda) |\ \lambda \in \Lambda\}$ and $Y = \{(\lambda + \eta, m_\lambda) |\ \lambda \in \Lambda\}$ for some $\eta \in \mathbb{C}$. Then $S_{[\omega]}(X) = S_{[\omega]}(Y)$.*

In fact, we will show more, but only the version above will be important later.

Proposition 1.8.2. *Let ω be a weight, $X = \{(\lambda, m_\lambda) |\ \lambda \in \Lambda\}$, $(\eta_\lambda)_{\lambda\in\Lambda}$ be a bounded sequence of complex numbers, and $Y = \{(\lambda + \eta_\lambda, m_\lambda) |\ \lambda \in \Lambda\}$. Then $S_{[\omega]}(X) = S_{[\omega]}(Y)$.*

Remark 1.8.3. We will use the term *bounded perturbation* for the geometric operation described above. Then Proposition 1.8.2 can be rephrased as follows: the sequence spaces $S_{[\omega]}(X)$ are invariant under bounded perturbations of the variety.

Proof. It is sufficient to show only one inclusion, for instance $S_{[\omega]}(Y) \subset S_{[\omega]}(X)$. The other one will follow from this. Denote $K = \sup_{\lambda\in\Lambda}|\eta_\lambda|$. Take any sequence $v = \{v_{\lambda,l} |\ \lambda \in \Lambda,\ 0 \leq l < m_\lambda\} \in S_{[\omega]}(Y)$.

In the Beurling case we have

$$\exists\, M > 0\ \forall\, \lambda \in \Lambda : \sum_{l=0}^{m_\lambda - 1} |v_{\lambda,l}| \leq e^{M + M|\operatorname{Im}(\lambda+\eta_\lambda)| + M\omega(\lambda+\eta_\lambda)}.$$

Using Lemma 1.5.6 we obtain for some $C > 0$ and all $\lambda \in \Lambda$

$$\sum_{l=0}^{m_\lambda - 1} |v_{\lambda,l}| \leq e^{M + MK + C(K+1)M + M|\operatorname{Im}\lambda| + CM\omega(\lambda)}.$$

In the Roumieu case we have

$$\exists\, M > 0\ \forall\, m > 0\ \exists\, C_m > 0\ \forall\, \lambda \in \Lambda : \sum_{l=0}^{m_\lambda - 1} |v_{\lambda,l}| \leq C_m e^{M|\operatorname{Im}(\lambda+\eta_\lambda)| + \frac{1}{m}\omega(\lambda+\eta_\lambda)}.$$

Using Lemma 1.5.6 we obtain for some $C > 0$ and all $\lambda \in \Lambda$

$$\sum_{l=0}^{m_\lambda - 1} |v_{\lambda,l}| \leq C_m e^{MK + \frac{C(K+1)}{m} + M|\operatorname{Im}\lambda| + \frac{C}{m}\omega(\lambda)}. \qquad \square$$

Proposition 1.8.4. *Let ω be a weight and $X = \{(\lambda, m_\lambda)|\ \lambda \in \Lambda\}$ a multiplicity variety. Then $S_{(\omega),M}(X)$ and $S_{\{\omega\},M,m}(X)$ are Banach spaces for any $M, m \in \mathbb{N}$.*

Proof. Let $(v^{(k)})_{k\in\mathbb{N}}$ be a Cauchy sequence in $S_{(\omega),M}(X)$. Then

$$\sum_{l=0}^{m_\lambda - 1} |v^{(n)}_{\lambda,l} - v^{(k)}_{\lambda,l}| \leq \epsilon e^{M|\operatorname{Im}\lambda| + M\omega(\lambda)}$$

for any given $\epsilon > 0$, all $\lambda \in \Lambda$ and all $n, k \in \mathbb{N}$ big enough. Hence for every $\lambda \in \Lambda$, $0 \leq l < m_\lambda$ there exists $\lim_{k\to\infty} v^{(k)}_{\lambda,l} =: v_{\lambda,l}$. We have

$$\forall\, \epsilon > 0\ \exists\, k_0 \in \mathbb{N}\ \forall\, \lambda \in \Lambda\ \forall\, n, k \geq k_0\colon \quad \sum_{l=0}^{m_\lambda - 1} |v^{(n)}_{\lambda,l} - v^{(k)}_{\lambda,l}|\, e^{-M|\operatorname{Im}\lambda| - M\omega(\lambda)} \leq \epsilon.$$

Therefore

$$\forall\, \epsilon > 0\ \exists\, k_0 \in \mathbb{N}\ \forall\, \lambda \in \Lambda\ \forall\, k \geq k_0\colon \quad \sum_{l=0}^{m_\lambda - 1} |v_{\lambda,l} - v^{(k)}_{\lambda,l}|\, e^{-M|\operatorname{Im}\lambda| - M\omega(\lambda)} \leq \epsilon,$$

which means that $(v^{(k)})_{k\in\mathbb{N}}$ converges to $v := \{v_{\lambda,l}|\ \lambda \in \Lambda, 0 \leq l < m_\lambda\}$ in $S_{(\omega),M}(X)$. Finally,

$$\|v\|_M \leq \|v - v^{(k)}\|_M + \|v^{(k)}\|_M < \infty$$

for $k \in \mathbb{N}$ big enough.

The proof for $S_{\{\omega\},M,m}(X)$ is analogous. $\square$

Now we can say more about topologies of $S_{(\omega)}(X)$ and $S_{\{\omega\}}(X)$.

Proposition 1.8.5. *Let ω be a weight and X a multiplicity variety. Then $S_{(\omega)}(X)$ is an (LB)-space.*

Proof. Since $\|\cdot\|_{M+1} \leq \|\cdot\|_M$ for every $M \in \mathbb{N}$, we have that the inductive system $(j_M\colon S_{(\omega),M}(X) \to S_{(\omega)})(X)$ is an imbedding spectrum. Furthermore, every $S_{(\omega),M}(X)$ embeds continuously into the space of all sequences with pointwise convergence, which is Hausdorff. Hence the inductive topology of $S_{(\omega)}(X)$ is Hausdorff as well. $\square$

Remark 1.8.6. From [Mei85, Proposition 1.3] it follows that $S_{(\omega)}(X)$ is nuclear if and only if

$$m_\lambda \leq M|\operatorname{Im}\lambda| + M\omega(\lambda)$$

for some $M > 0$ and every $\lambda \in \Lambda$.

Proposition 1.8.7. *Let ω be a weight and X a multiplicity variety. Then for every $M \in \mathbb{N}$*

$$S_{\{\omega\},M}(X) := \big\{v = \{v_{\lambda,l} |\ \lambda \in \Lambda,\ 0 \leq l < m_\lambda\} |\ \forall\, m \in \mathbb{N}\colon \|f\|_{M,m} < \infty\big\}$$

is a Fréchet space, and $S_{\{\omega\}}(X)$ is an (LF)-space.

Proof. Let $(v^{(k)})_{k\in\mathbb{N}}$ be a Cauchy sequence in $S_{\{\omega\},M}(X)$. Then for every $\epsilon > 0$ and every $m \in \mathbb{N}$

$$\sum_{l=0}^{m_\lambda - 1} |v^{(n)}_{\lambda,l} - v^{(k)}_{\lambda,l}| \leq \epsilon e^{M|\mathrm{Im}\,\lambda| + \frac{1}{m}\omega(\lambda)}$$

for all $\lambda \in \Lambda$ and all $n, k \in \mathbb{N}$ big enough. Hence for every $\lambda \in \Lambda$, $0 \leq l < m_\lambda$ there exists $\lim_{k\to\infty} v^{(k)}_{\lambda,l} =: v_{\lambda,l}$. We have

$$\forall\, \epsilon > 0\ \ \forall\, m \in \mathbb{N}\ \exists\, k_0 \in \mathbb{N}\ \ \forall\, \lambda \in \Lambda\ \ \forall\, n, k \geq k_0$$
$$\sum_{l=0}^{m_\lambda - 1} |v^{(n)}_{\lambda,l} - v^{(k)}_{\lambda,l}|\, e^{-M|\mathrm{Im}\,\lambda| - \frac{1}{m}\omega(\lambda)} \leq \epsilon.$$

Therefore

$$\forall\, \epsilon > 0\ \ \forall\, m \in \mathbb{N}\ \exists\, k_0 \in \mathbb{N}\ \ \forall\, \lambda \in \Lambda\ \ \forall\, k \geq k_0$$
$$\sum_{l=0}^{m_\lambda - 1} |v_{\lambda,l} - v^{(k)}_{\lambda,l}|\, e^{-M|\mathrm{Im}\,\lambda| - \frac{1}{m}\omega(\lambda)} \leq \epsilon.$$

which means that $(v^{(k)})_{k\in\mathbb{N}}$ converges to $v := \{v_{\lambda,l} |\ \lambda \in \Lambda, 0 \leq l < m_\lambda\}$ in $S_{(\omega),M}(X)$. Finally, for every $m \in \mathbb{N}$

$$\|v\|_{M,m} \leq \|v - v^{(k)}\|_{M,m} + \|v^{(k)}\|_{M,m} < \infty$$

for $k \in \mathbb{N}$ big enough.

Next we need to show that the inductive system $(j_M\colon S_{\{\omega\},M}(X) \to S_{(\omega)})(X)$ is an imbedding spectrum. This follows from $\|\cdot\|_{M+1,m} \leq \|\cdot\|_{M,m}$ which holds for every $M \in \mathbb{N}$ and $m \in \mathbb{N}$. Finally, every $S_{\{\omega\},M}(X)$ embeds continuously into the space of all sequences with pointwise convergence, which is Hausdorff. Hence the inductive topology of $S_{\{\omega\}}(X)$ is Hausdorff as well. □

Remark 1.8.8. From [Mei89, Proposition 1.6] it follows that $S_{\{\omega\}}(X)$ is the strong dual of a complete Schwartz space.

As in the function case we have the following relation.

Proposition 1.8.9. *In the algebraic sense*

$$S_{\{\omega\}}(X) = \bigcup_{\sigma=o(\omega)} S_{(\sigma)}(X).$$

Proof. Let $v \in S_{\{\omega\}}$. Then $|f(z)| \leq e^{C_m+M|\operatorname{Im} z|+\frac{1}{m}\omega(z)}$. Using Lemma 1.5.16 for $q(t) = \inf_{m\in\mathbb{N}}(C_m + \frac{1}{m}\omega(t))$ and ω we obtain a weight $\sigma = o(\omega)$ such that $f \in A_{(\sigma)}$. The converse inclusion follows from the observation that if $\sigma = o(\omega)$ then for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that $\sigma(t) \leq C_m + \frac{1}{m}\omega(t)$ for all $t \in [0, \infty)$. The proof of the second equality is analogous. □

1.9 Interpolation problem

The main topic of our study is the interpolation problem. It will be expressed with the use of a so-called restriction operator. But before we state the definition we need the following lemma.

Lemma 1.9.1. *Let ω be a weight. Then*

1. *For every $f \in A_{(\omega)}$ the following condition is satisfied:*

$$\exists\, M \in \mathbb{N}\ \forall\, z \in \mathbb{C}: \quad \sum_{l=0}^{\infty}\left|\frac{f^{(l)}(z)}{l!}\right| \leq e^{M+M|\operatorname{Im} z|+M\omega(z)}.$$

2. *For every $f \in A_{\{\omega\}}$ the following condition is satisfied:*

$$\exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, C_m > 0\ \forall\, z \in \mathbb{C}: \quad \sum_{l=0}^{\infty}\left|\frac{f^{(l)}(z)}{l!}\right| \leq C_m e^{M|\operatorname{Im} z|+\frac{1}{n}\omega(z)}.$$

Proof. Using Cauchy's inequalities for $R = 2$ we obtain

$$\sum_{l=0}^{\infty}\left|\frac{f^{(l)}(z)}{l!}\right| \leq \sum_{l=0}^{\infty} \frac{\sup_{\xi\in\partial D(z,2)}|f(\xi)|}{2^n}$$
$$= 2 \sup_{\xi\in\partial D(z,2)} |f(\xi)|.$$

Now we are going to use Lemma 1.5.6. In the Beurling case we have then for some constants $M, C > 0$ and all $z \in \mathbb{C}$

$$\sum_{l=0}^{\infty}\left|\frac{f^{(l)}(z)}{l!}\right| \leq \sup_{\xi\in\partial D(z,2)} e^{M+M|\operatorname{Im}\xi|+M\omega(\xi)} \leq e^{2M+MC+M|\operatorname{Im} z|+MC\omega(z)}.$$

While, in the Roumieu case

$$\exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, C_m > 0\ \forall\, z \in \mathbb{C}:$$
$$\sum_{l=0}^{\infty}\left|\frac{f^{(l)}(z)}{l!}\right| \leq 2C_m \sup_{\xi\in\partial D(z,2)} e^{M|\operatorname{Im}\xi|+\frac{1}{m}\omega(\xi)} \leq 2C_m e^{2M+\frac{C}{m}+M|\operatorname{Im} z|+\frac{C}{m}\omega(z)}.$$ □

Definition 1.9.2 (Restriction operator)**.** Let $X = \{(\lambda, m_\lambda)|\ \lambda \in \Lambda\}$ be a multiplicity variety and ω a weight. We define

$$R\colon A_{[\omega]} \to S_{[\omega]}(X), \qquad f \mapsto \left\{\frac{f^{(l)}(\lambda)}{l!}\right\}_{\lambda\in\Lambda,\, 0\le l<m_\lambda}$$

In both Beurling and Roumieu cases this operator is continuous. For the convenience of the reader we recall how one checks the continuity of an operator between (LF)-spaces (see [MV97, Proposition 24.7 and Theorem 24.33]).

Proposition 1.9.3. *Let $(E_N)_{N\in\mathbb{N}}$ and $(F_M)_{M\in\mathbb{N}}$ be families of Fréchet spaces. Let $(j_N\colon E_N \to E)_{N\in\mathbb{N}}$, $(l_M\colon F_M \to F)_{M\in\mathbb{N}}$ be imbedding spectra and assume that the inductive topologies on E and F are Hausdorff. Then a linear operator $A\colon E \to F$ is continuous if and only if for every $N \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that $A(E_N) \subset F_M$ and $A|_{E_N}\colon E_N \to F_M$ is continuous.*

Proposition 1.9.4. *Let $X = \{(\lambda, m_\lambda)|\ \lambda \in \Lambda\}$ be a multiplicity variety and ω a Beurling weight. Then the restriction operator $R\colon A_{(\omega)} \to S_{(\omega)}(X)$ is continuous.*

Proof. Denote $p(z) = |\operatorname{Im} z| + \omega(z)$. From Lemma 1.5.6 we have for some $C, D > 0$

$$p(z) \ge Cp(\xi) - D$$

provided $|z - \xi| \le 2$.

Let $N \in \mathbb{N}$ be given and assume that $f \in A_{(\omega),N}$. From Cauchy's inequalities and the above inequality we obtain

$$\begin{aligned}\|R(f)\|_M &= \sup_{\lambda\in\Lambda}\sum_{l=0}^{m_\lambda-1}\left|\frac{f^{(l)}(\lambda)}{l!}\right| e^{-Mp(\lambda)} \le \sup_{z\in\mathbb{C}}\sum_{l=0}^{\infty}\left|\frac{f^{(l)}(z)}{l!}\right| e^{-Mp(z)}\\ &\le \sup_{z\in\mathbb{C}}\sum_{l=0}^{\infty}\frac{\sup_{\xi\in\partial D(z,2)}|f(\xi)|}{2^l}\, e^{-Mp(z)} \le 2\sup_{z\in\mathbb{C}}\sup_{\xi\in\partial D(z,2)}|f(\xi)|\, e^{-MCp(\xi)+MD}\\ &= 2e^{MD}\sup_{z\in\mathbb{C}}|f(z)|\, e^{-MCp(z)} = A\|f\|_N\end{aligned}$$

if M and A were chosen to be $M = \frac{N}{C}$ and $A = 2e^{MD}$. □

Proposition 1.9.5. *Let $X = \{(\lambda, m_\lambda)|\ \lambda \in \Lambda\}$ be a multiplicity variety and ω a Roumieu weight. Then the restriction operator $R\colon A_{\{\omega\}} \to S_{\{\omega\}}(X)$ is continuous.*

Proof. We have to show that

$$\forall\, N \in \mathbb{N}\ \exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, n \in \mathbb{N}, C > 0\ \forall\, f \in A_{\{\omega\},N}$$
$$\|R(f)\|_{M,m} \le C\|f\|_{N,n}.$$

From Lemma 1.5.6 we have for some $C, D > 0$

$$\omega(z) \geq C\omega(\xi) - D$$

provided $|z - \xi| \leq 2$.

Let $N \in \mathbb{N}$ be given and assume that $f \in A_N$. For given $M, m \in \mathbb{N}$ we obtain from Cauchy's inequalities and the above inequality

$$\begin{aligned}
\|R(f)\|_{M,m} &= \sup_{\lambda\in\Lambda} \sum_{l=0}^{m_\lambda-1} \left|\frac{f^{(l)}(\lambda)}{l!}\right| e^{-M|\operatorname{Im}\lambda| - \frac{1}{m}\omega(\lambda)} \\
&\leq \sup_{z\in\mathbb{C}} \sum_{l=0}^{\infty} \left|\frac{f^{(l)}(z)}{l!}\right| e^{-M|\operatorname{Im} z| - \frac{1}{m}\omega(z)} \\
&\leq \sup_{z\in\mathbb{C}} \sum_{l=0}^{\infty} \frac{\sup_{\xi\in\partial D(z,2)}|f(\xi)|}{2^l} e^{-M|\operatorname{Im} z| - \frac{1}{m}\omega(z)} \\
&\leq 2 \sup_{z\in\mathbb{C}} \sup_{\xi\in\partial D(z,2)} |f(\xi)|\, e^{2M + \frac{D}{m} - M|\operatorname{Im}\xi| - \frac{C}{m}\omega(\xi)} \\
&= 2e^{2M+\frac{D}{m}} \sup_{z\in\mathbb{C}} |f(z)|\, e^{-M|\operatorname{Im} z| - \frac{C}{m}\omega(z)}.
\end{aligned}$$

Therefore for all $N \in \mathbb{N}$ there exists $M = N$ such that for all $m \in \mathbb{N}$ there exist $n \geq \frac{m}{C}$, $C = 2e^{2M+\frac{D}{m}}$ such that

$$\|R(f)\|_{M,m} \leq C\|f\|_{N,n}. \qquad \square$$

Now we can define the most important notion of this dissertation.

Definition 1.9.6 (Interpolating variety)**.** Let $X = \{(\lambda, m_\lambda)|\ \lambda \in \Lambda\}$ be a multiplicity variety and ω be a weight. We call X an interpolating variety for $A_{[\omega]}$ if the restriction operator R is surjective onto $S_{[\omega]}(X)$.

Example 1.9.7. The only examples of interpolating varieties one can give without deep studies of the topic are finite sets. In that case one can interpolate any sequence of values by a polynomial. For more sophisticated examples we will have to wait until Chapter 5.

1.10 Counting functions

In order to express geometric properties of interpolating varieties we will use Nevanlinna counting functions. They were introduced by R. Nevanlinna and successfully used in the study of value distribution of holomorphic functions as well as in the relation between growth of entire functions and distribution of their zeroes. There is a huge literature concerning these topics. We mention just several positions [Nev70], [Hay64], [Lev80], [Lev96], [CY01].

Definition 1.10.1 (Nevanlinna counting functions)**.** Let $X = \{(\lambda, m_\lambda) | \lambda \in \Lambda\}$ be a multiplicity variety. Define

$$n(z, r, X) = \sum_{\lambda \in \bar{D}(z,r) \cap \Lambda} m_\lambda$$

and

$$N(z, r, X) = \int_0^r \frac{n(z, t, X) - n(z, 0, X)}{t} \,\mathrm{dt} + n(z, 0, X) \ln r.$$

The argument X will be omitted whenever it does not lead to any misunderstanding.

The function n counts the number of points (with multiplicities) of a variety in a disc of radius r centred at z. The N function is more sensitive. Its value depends on the number of points in that disc but also on the density of these points. Hence its value in one given r carries more information about geometric properties of a variety than the value of n in the given r. Moreover, whenever X is a zero variety of a holomorphic function (the set of zeroes of the function with multiplicities), the corresponding N function appears in the Jensen's formula (see [Rud87, Theorem 15.18], [Lev96, Section 2.3]). This is the main reason of its usefulness. The form of the function N encountered more often in the Jensen's formula is given in the following proposition.

Proposition 1.10.2. *Let X be a multiplicity variety. Then for all $z \in \mathbb{C}$ and $r > 0$*

$$N(z, r, X) = \sum_{\lambda\colon 0 < |\lambda - z| \leq r} m_\lambda \ln \frac{r}{|z - \lambda|} + n(z, 0, X) \ln r.$$

Proof. Denote

$$n_\lambda(z, r) = \begin{cases} 0 & \text{for } 0 \leq r < |z - \lambda|, \\ m_\lambda & \text{for } r \geq |z - \lambda|. \end{cases}$$

For any $z \in \mathbb{C}$, $r > 0$ we have

$$\begin{aligned}
N(z,r) &= \int_0^r \frac{n(z,t)-n(z,0)}{t}\,\mathrm{dt} + n(z,0)\ln r \\
&= \int_0^r \frac{\sum_{\lambda\in\Lambda} n_\lambda(z,t) - n(z,0)}{t}\,\mathrm{dt} + n(z,0)\ln r \\
&= \int_0^r \frac{\sum_{\lambda\colon 0<|\lambda-z|} n_\lambda(z,t)}{t}\,\mathrm{dt} + n(z,0)\ln r \\
&= \sum_{\lambda\colon 0<|\lambda-z|\le r} \int_0^r \frac{n_\lambda(z,t)}{t}\,\mathrm{dt} + n(z,0)\ln r \\
&= \sum_{\lambda\colon 0<|\lambda-z|\le r} \int_{|z-\lambda|}^r \frac{m_\lambda}{t}\,\mathrm{dt} + n(z,0)\ln r \\
&= \sum_{\lambda\colon 0<|\lambda-z|\le r} m_\lambda \ln\frac{r}{|z-\lambda|} + n(z,0)\ln r. \qquad \square
\end{aligned}$$

In the following lemma we give a strict relation between functions N and n.

Lemma 1.10.3. *Let X be a multiplicity variety. Then for all $z \in \mathbb{C}$ and $r > 0$*

$$N(z,r,X) = n(z,r,X)\ln r - \sum_{\lambda\colon 0<|\lambda-z|\le r} m_\lambda \ln|z-\lambda|.$$

Proof. Follows immediately from Proposition 1.10.2. □

Another relation is given by the following inequality.

Lemma 1.10.4. *Let X be a multiplicity variety. Then for all $z \in \mathbb{C}$ and $r > 0$*

$$n(z,r,X) \le N(z,er,X) - n(z,0,X)\ln r.$$

Proof. We have

$$\begin{aligned}
N(z,er) &= \int_0^{er} \frac{n(z,t)-n(z,0)}{t}\,\mathrm{dt} + n(z,0)\ln er \\
&\ge \int_r^{er} \frac{n(z,t)-n(z,0)}{t}\,\mathrm{dt} + n(z,0)\ln er \\
&\ge \int_r^{er} \frac{n(z,r)-n(z,0)}{t}\,\mathrm{dt} + n(z,0)\ln er \\
&= (n(z,r)-n(z,0))\ln e + n(z,0)\ln er \\
&= n(z,r) + n(z,0)\ln r. \qquad \square
\end{aligned}$$

Chapter 2

Properties of interpolating varieties

2.1 Uniform interpolation

A standard feature of interpolation is that it can be performed in a uniform way. The following lemma was proved in the form of Lemma 2.1.3 in [BL95, Lemma 3.3]. We present a more general form which is in fact the essence of the open mapping theorem - if the restriction map $R\colon A_{(\omega)} \to S_{(\omega)}(X)$ or $R\colon A_{\{\omega\}} \to S_{\{\omega\}}(X)$ is surjective then by [MV97, Theorem 24.30] it is open. We use a more sophisticated version of the open mapping theorem since the spaces are neither Banach nor Fréchet.

Lemma 2.1.1 (Uniform Interpolation in $A_{(\omega)}$). *Let ω be a Beurling weight and X a multiplicity variety. If the restriction operator $R\colon A_{(\omega)} \to S_{(\omega)}(X)$ is surjective then for every $N \in \mathbb{N}$ there are $M \in \mathbb{N}$ and $r > 0$ such that*

$$B_N(0,1) \subset R(B_M(0,r))$$

where $B_M(z,r)$ denotes a ball in the norm $\|\cdot\|_M$ in the suitable space centred at z with radius r.

We omit the proof of this lemma as it is just a simpler version of the proof of the following result for the Roumieu case.

Lemma 2.1.2 (Uniform Interpolation in $A_{\{\omega\}}$). *Let ω be a Roumieu weight and X a multiplicity variety. If $R\colon A_{\{\omega\}} \to S_{\{\omega\}}(X)$ is surjective then*

$$\forall\, N \in \mathbb{N}\ \exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, n \in \mathbb{N}, r > 0$$
$$B_{N,n}(0,1) \subset R(B_{M,m}(0,r))$$

where $B_{M,m}(z,r)$ denotes a ball in the norm $\|\cdot\|_{M,m}$ in the suitable space centred at z with radius r.

Proof. We recall that $A_{\{\omega\}}$ and $S_{\{\omega\}}(X)$ are (LF)-spaces.

As R is surjective we have $S_{\{\omega\}}(X) = \bigcup_{M\in\mathbb{N}} R(A_{\{\omega\},M})$. By Grothendieck's factorization theorem [MV97, Theorem 24.33] for every $N \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that $S_{\{\omega\},N} \subset R(A_{\{\omega\},M})$. Denote by j the inclusion from $S_{\{\omega\},N}$ to $R(A_{\{\omega\},M})$. Consider on $R(A_{\{\omega\},M})$ the topology τ induced from $A_{\{\omega\},M}$ by R (the quotient topology). Define $j_M\colon A_{\{\omega\},M} \to A_{\{\omega\}}$ to be the inclusion. Then

$$R_M = j_M \circ R\colon A_{\{\omega\},M} \to (R(A_{\{\omega\},M}), \tau) \quad \text{and} \quad j\colon S_{\{\omega\},N} \to (R(A_{\{\omega\},M}), \tau)$$

are continuous by the closed graph theorem [MV97, 24.31]. Hence for $B_{M,m}(0,1) = \{f \in H(\mathbb{C})\colon \|f\|_{M,m} < 1\}$, the set $j^{-1}(R(B_{M,m}(0,1)))$ is a 0-neighbourhood in $S_{\{\omega\},N}$. This completes the proof. □

The standard form of the uniform interpolation for the Beurling case, which one can find in the literature, can now be obtained as a corollary to Lemma 2.1.1.

Lemma 2.1.3. *If $X = \{(\lambda, m_\lambda)|\ \lambda \in \Lambda\}$ is an interpolating variety for $A_{(\omega)}$ then there are $M \in \mathbb{N}$, $C > 0$ such that for all $\lambda \in \Lambda$, $0 \le l < m_\lambda$ there is a function $f_{\lambda,l} \in A_{(\omega)}$ with the following Taylor coefficients*

$$\frac{f_{\lambda,l}^{(k)}(\eta)}{k!} = \begin{cases} 1 & \text{if } \eta = \lambda \text{ and } k = l, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\eta \in \Lambda$, $0 \le k < m_\eta$ and

$$\|f_{\lambda,l}\|_M \le C.$$

Proof. For every $\lambda \in \Lambda$, $0 \le l < m_\lambda$ and any $N \in \mathbb{N}$ we have

$$(\delta_{\eta,\lambda}\delta_{k,l})_{\eta\in\Lambda,0\le k<m_\eta} \in S_{(\omega)}(X)$$

and

$$\|(\delta_{\eta,\lambda}\delta_{k,l})_{\eta\in\Lambda,0\le k<m_\eta}\|_N = e^{-N(|\operatorname{Im}\lambda|+\omega(\lambda))} < 1.$$

Therefore Lemma 2.1.1 yields the assertion. □

We can obtain a similar lemma for the Roumieu case.

Lemma 2.1.4. *If $X = \{(\lambda, m_\lambda)|\ \lambda \in \Lambda\}$ is an interpolating variety for $A_{\{\omega\}}$ then there exists $M > 0$ such that for all $m \in \mathbb{N}$ there exists $C_m > 0$ such that for all $\lambda \in \Lambda, 0 \le l < m_\lambda$ there is a function $f_{\lambda,l} \in A_{\{\omega\}}$ satisfying*

$$\frac{f_{\lambda,l}^{(k)}(\eta)}{k!} = \begin{cases} 1 & \text{if } \eta = \lambda \text{ and } k = l, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\eta \in \Lambda$, $0 \leq k < m_\eta$ *and*

$$\|f_{\lambda,l}\|_{M,m} \leq C_m.$$

Proof. For every $\lambda \in \Lambda$, $0 \leq l < m_\lambda$ and any $N \in \mathbb{N}$, $n \in \mathbb{N}$ we have

$$(\delta_{\eta,\lambda}\delta_{k,l})_{\eta\in\Lambda,0\leq k<m_\eta} \in S_{\{\omega\}}(X)$$

and

$$\|(\delta_{\eta,\lambda}\delta_{k,l})_{\eta\in\Lambda,0\leq k<m_\eta}\|_{N,n} = e^{-N|\operatorname{Im}\lambda|-\frac{1}{n}\omega(\lambda)} < 1$$

Therefore Lemma 2.1.2 yields the assertion. □

2.2 Elementary properties

Definition 2.2.1 (Subvariety)**.** Let $X = \{(\lambda, m_\lambda)|\ \lambda \in \Lambda\}$ be a multiplicity variety. We say that $Y = \{(\eta, \tilde{m}_\eta)|\ \eta \in \mathcal{K}\}$ is a subvariety of X if $\mathcal{K} \subset \Lambda$ and $\tilde{m}_\eta \leq m_\eta$ for every $\eta \in \mathcal{K}$. We write $Y \subset X$.

The following proposition follows immediately from the definitions.

Proposition 2.2.2. *Let* ω *be a weight. Assume that* Y *is a subvariety of* X*. If* X *is interpolating for* $A_{[\omega]}$ *then* Y *is interpolating for* $A_{[\omega]}$ *as well.*

Another property of interpolating varieties is that they can be shifted by a constant vector.

Proposition 2.2.3 (Shift of an interpolating variety)**.** *Let* ω *be a weight,* $X = \{(\lambda, m_\lambda)|\ \lambda \in \Lambda\}$ *an interpolating variety for* $A_{[\omega]}$ *and* $\eta \in \mathbb{C}$ *an arbitrary point. Then* $Y = \{(\lambda + \eta, m_\lambda)|\ \lambda \in \Lambda\}$ *is interpolating for* $A_{[\omega]}$.

Proof. Let $v = \{v_{\lambda+\eta,l}|\ \lambda \in \Lambda,\ 0 \leq l < m_\lambda\} \in A_{[\omega]}(Y)$. Then $v \in A_{[\omega]}(X)$ from Proposition 1.8.1. As X is interpolating, we can find $f \in A_{[\omega]}$ interpolating v on X. Finally, $f(\cdot - \eta)$ interpolates v on Y and from Lemma 1.7.3, $f(\cdot - \eta) \in A_{[\omega]}$. □

One may ask if the essence of Proposition 2.2.3 lies in the fact that all points of the variety are shifted not too far from their original positions. More precisely, if, for a sequence $(\eta_\lambda)_{\lambda\in\Lambda} \subset \mathbb{C}$ such that $\sup_{\lambda\in\Lambda}|\eta_\lambda| < \infty$, $Y = \{(\lambda + \eta_\lambda, m_\lambda)|\ \lambda \in \Lambda\}$ is interpolating whenever $X = \{(\lambda, m_\lambda)|\ \lambda \in \Lambda\}$ is interpolating. Or in other words, whether interpolating varieties are stable under bounded perturbation. Proposition 1.8.2 gives us the possibility to make the first step of the proof given above. But the answer is no (see Section 5.4). Another operation that is not allowed with interpolating varieties is rotation (see Section 5.4). To see this we will need a geometric characterisation of interpolating varieties which will be given in Chapter 5.

It turns out that the multiplicities of an interpolating variety cannot grow too fast. The first assertion of the following proposition was proved by Squires [Squ83, Theorem 1].

Proposition 2.2.4. *1. If X is an interpolating variety for $A_{(\omega)}$ then*

$$\exists\, M \in \mathbb{N}, C > 0\ \forall\, \lambda \in \Lambda : m_\lambda \leq C + M|\operatorname{Im}\lambda| + M\omega(\lambda).$$

2. If X is an interpolating variety for $A_{\{\omega\}}$ then

$$\exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, C_m > 0\ \forall\, \lambda \in \Lambda : m_\lambda \leq C_m + M|\operatorname{Im}\lambda| + \frac{1}{m}\omega(\lambda).$$

Proof. We will treat both cases simultaneously. Let $f_\lambda := f_{\lambda, m_\lambda - 1}$ be the functions given by Lemma 2.1.3 in the Beurling case and by Lemma 2.1.4 in the Roumieu case. An application of Cauchy's formula gives

$$\begin{aligned} 1 = \frac{f_\lambda^{(m_\lambda - 1)}(\lambda)}{(m_\lambda - 1)!} &= \frac{1}{2\pi i} \int\limits_{\partial D(\lambda, e)} \frac{f_\lambda(\xi)}{(\xi - \lambda)^{m_\lambda}}\ \mathrm{d}\xi \\ &= \frac{1}{2\pi i} \int\limits_0^{2\pi} \frac{f_\lambda(\lambda + e\, e^{i\theta})}{(e\, e^{i\theta})^{m_\lambda}}\, e\, i\, e^{i\theta}\ \mathrm{d}\theta \\ &= \frac{1}{2\pi e^{m_\lambda - 1}} \int\limits_0^{2\pi} \frac{f_\lambda(\lambda + e\, e^{i\theta})}{(e^{i\theta})^{m_\lambda - 1}}\ \mathrm{d}\theta. \end{aligned}$$

Therefore

$$1 \leq \frac{1}{2\pi e^{m_\lambda - 1}} \int\limits_0^{2\pi} |f_\lambda(\lambda + e\, e^{i\theta})|\ \mathrm{d}\theta \leq \frac{1}{e^{m_\lambda - 1}} \sup_{z \in \partial D(0, e)} |f_\lambda(\lambda + z)|$$

and

$$m_\lambda \leq 1 + \ln\big(\sup_{z \in \partial D(0,e)} |f_\lambda(\lambda + z)|\big).$$

Now we will use the estimates for the modulus of f_λ. In the Beurling case we have that

$$\exists\, M \in \mathbb{N}, C > 0\ \forall\, \lambda \in \Lambda : m_\lambda \leq 1 + \ln\big(\sup_{z \in \partial D(0,e)} C e^{M|\operatorname{Im}(\lambda + z)| + M\omega(\lambda + z)}\big).$$

Whereas, in the Roumieu case

$$\exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, C_m > 0\ \forall\, \lambda \in \Lambda :$$
$$m_\lambda \leq 1 + \ln\big(\sup_{z \in \partial D(0,e)} C_m e^{M|\mathrm{Im}(\lambda+z)| + \frac{1}{m}\omega(\lambda+z)}\big).$$

The use of Lemma 1.5.6 completes the proof. □

The following theorem was proved by Berenstein and Li [BL95, Pages 13-14].

Theorem 2.2.5. *Let ω be a weight. If $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ is interpolating for $A_{(\omega)}$ then for any $n \in \mathbb{N}$ the variety $X = \{(\lambda, n \cdot m_\lambda)\}_{\lambda\in\Lambda}$ (we multiply all multiplicities by n) is also interpolating for $A_{(\omega)}$.*

2.3 Union of Beurling interpolating varieties

In this section we will study unions of interpolating varieties in the Beurling case. The Roumieu case will have to wait until we have more powerful tools. We start with the definition of the union of multiplicity varieties. We will use the notation $p(z) = |\mathrm{Im}\, z| + \omega(z)$ for a weight function ω. Furthermore, $X = \{(\lambda, \widetilde{m}_\lambda)\}_{\lambda\in\Lambda}$ and $Y = \{(\eta, \widehat{m}_\eta)\}_{\eta\in\mathcal{K}}$ will denote two multiplicity varieties. Whenever it does not lead to any misunderstanding we will just write m_λ and m_η for these multiplicities.

Definition 2.3.1 (Union of multiplicity varieties)**.** We define

$$X \cup Y = \{(\tau, m_\tau)\}_{\tau\in\Lambda\cup\mathcal{K}}$$

where

$$m_\tau = \begin{cases} \widetilde{m}_\tau & \text{if } \tau \in \Lambda \text{ and } \tau \notin \mathcal{K},\\ \widehat{m}_\tau & \text{if } \tau \notin \Lambda \text{ and } \tau \in \mathcal{K},\\ \max(\widetilde{m}_\tau, \widehat{m}_\tau) & \text{if } \tau \in \Lambda \cap \mathcal{K}. \end{cases}$$

The following notion is crucial when considering unions of interpolating varieties in the Beurling case.

Definition 2.3.2. Let ω be a weight. Denote $p(z) = |\mathrm{Im}\, z| + \omega(z)$. We say that $X = \{(\lambda, m_\lambda) |\ \lambda \in \Lambda\}$ is *weakly* (ω)*-separated* (or just *weakly separated*) if there exist $\delta, C > 0$ such that for

$$\delta_\lambda = \delta e^{-C\frac{p(\lambda)}{m_\lambda}}$$

the disks $D(\lambda, \delta_\lambda)$ are pairwise disjoint for all $\lambda \in \Lambda$. Numbers δ_λ are called separation radii.

The next lemma was proved in [BL94, Lemma 3.1] for the case $m_\lambda = 1$ for all $\lambda \in \Lambda$. Small adjustments give the result for arbitrary multiplicities.

Lemma 2.3.3. *Let ω be a weight. Denote $p(z) = |\operatorname{Im} z| + \omega(z)$. If a multiplicity variety $X = \{(\lambda, m_\lambda) |\ \lambda \in \Lambda\}$ is weakly separated then there exists $M > 0$ such that*

$$\sum_{\lambda\in\Lambda} m_\lambda e^{-Mp(\lambda)} < \infty.$$

Proof. By Proposition 2.2.4 we have

$$m_\lambda \le e^{m_\lambda} \le Ce^{Cp(\lambda)}$$

for some $C > 0$ and every $\lambda \in \Lambda$. Hence it is enough to prove that

$$\sum_{\lambda\in\Lambda} e^{-Mp(\lambda)} < \infty$$

for some $M > 0$. Further, we may assume that $0 \notin \Lambda$.

Let δ_λ be like in the definition of weak separation. Denote $|D_\lambda| = \int_{D_\lambda} \mathrm{dz}$. Then

$$|D_\lambda| = \pi\delta_\lambda^2 = \pi\delta^2 e^{-2C\frac{p(\lambda)}{m_\lambda}} \ge \pi\delta^2 e^{-2Cp(\lambda)}.$$

By (γ) there is $D > 0$ such that $\ln|z| \le Dp(z) + D$ for all $z \in \mathbb{C}$. Using Lemma 1.5.6 we obtain constants $A, B > 0$ such that

$$Ap(z) - B \le p(\lambda)$$

for all $z \in \mathbb{C}$, $\lambda \in \Lambda$ satisfying $|z - \lambda| \le 1$. Then for $M > 2C$ we get

$$\begin{aligned}
\sum_{\lambda\in\Lambda} e^{-Mp(\lambda)} &= \sum_{\lambda\in\Lambda} \frac{1}{|D_\lambda|} \int_{D_\lambda} e^{-Mp(\lambda)}\,\mathrm{dz} \le \frac{1}{\pi\delta^2} \sum_{\lambda\in\Lambda} \int_{D_\lambda} e^{(2C-M)p(\lambda)}\,\mathrm{dz} \\
&\le \frac{e^{B(M-2C)}}{\pi\delta^2} \sum_{\lambda\in\Lambda} \int_{D_\lambda} e^{(2C-M)Ap(z)}\,\mathrm{dz} \le \frac{e^{B(M-2C)}}{\pi} \int_{\mathbb{C}} e^{(2C-M)Ap(z)}\,\mathrm{dz} \\
&\le \frac{e^{(B+A)(M-2C)}}{\pi\delta^2} \int_{\mathbb{C}} |z|^{\frac{(2C-M)A}{D}}\,\mathrm{dz}.
\end{aligned}$$

If we choose M such that $\frac{(2C-M)A}{D} < -\frac{3}{2}$ this expression will be finite. $\square$

In the paper [Oun03, Lemma II.1, Theorem II.1] M. Ounaïes proved Lemmas 2.3.4, 2.3.5 and Theorem 2.3.6 for all multiplicities equal one. Although the proof for arbitrary multiplicities is also due to Ounaïes, she has never published it. It was communicated to the author through a private correspondence.

Lemma 2.3.4. *Let ω be a weight and denote $p(z) = |\operatorname{Im} z| + \omega(z)$. Assume that $X = \{(\lambda, m_\lambda)|\ \lambda \in \Lambda\}$, $Y = \{(\eta, m_\eta)|\ \eta \in \mathcal{K}\}$ are disjoint (i.e., $\Lambda \cap \mathcal{K} = \emptyset$), interpolating for $A_{(\omega)}$, and $X \cup Y$ is weakly separated. For $\lambda \in \Lambda$ define a sequence*

$$v^\lambda := (v^\lambda_{\eta,l})_{\eta\in\mathcal{K},0\le l<m_\eta}, \quad v^\lambda_{\eta,l} := (-1)^l \binom{m_\lambda + l - 1}{l} \frac{1}{(\eta-\lambda)^{m_\lambda+l}}.$$

Then $v^\lambda \in A_{(\omega)}(Y)$ for every $\lambda \in \Lambda$ and

$$\|v^\lambda\|^Y_M \le e^{Cp(\lambda)}$$

for some $C, M > 0$ independent of λ.

We write Y in the symbol $\|v^\lambda\|^Y_M$ to indicate that this norm is taken with respect to Y.

Proof. For $\lambda \in \Lambda$ and $\eta \in \mathcal{K}$ we have

$$\sum_{l=0}^{m_\eta-1} |v^\lambda_{\eta,l}| = \frac{1}{(\eta-\lambda)^{m_\lambda}} \sum_{l=0}^{m_\eta-1} \binom{m_\lambda + l - 1}{l} \frac{1}{(\eta-\lambda)^{l}}.$$

Since

$$\binom{k+n}{n} \le \binom{k+n+1}{n+1}$$

for all $k, n \in \mathbb{N}_0$ and

$$\binom{n}{k} \le \binom{n+1}{k}, \quad \binom{n}{k} \le 2^n$$

for $k, n \in \mathbb{N}$, we get

$$\binom{m_\lambda + l - 1}{l} \le \binom{m_\lambda + m_\eta}{m_\eta} \le 2^{m_\eta+m_\lambda} < e^{m_\eta+m_\lambda}.$$

Further, from the weak separation of $X \cup Y$ for some $C > 0$ and $\delta > 0$ we have

$$|\lambda - \eta| \ge \delta e^{-C\frac{p(\lambda)}{m_\lambda}}$$

for all $\lambda \in \Lambda$ and $\eta \in \mathcal{K}$. Then

$$\frac{1}{|\eta-\lambda|^{m_\lambda}} \le \frac{1}{\delta^{m_\lambda}} e^{Cm_\lambda\frac{p(\lambda)}{m_\lambda}} = e^{-m_\lambda \ln\delta + Cp(\lambda)}$$

and

$$\frac{1}{|\eta-\lambda|^{l}} \le \frac{1}{\delta^{l}} e^{Cl\frac{p(\eta)}{m_\eta}} = e^{-l\ln\delta + C\frac{l}{m_\eta}p(\eta)} \le e^{-m_\eta \ln\delta + Cp(\eta)}.$$

Putting this all together and using Proposition 2.2.4 we obtain

$$\begin{aligned}\sum_{l=0}^{m_\eta-1}|v^\lambda_{\eta,l}| &= \frac{1}{|\eta-\lambda|^{m_\lambda}}\sum_{l=0}^{m_\eta-1}\binom{m_\lambda+l-1}{l}\frac{1}{|\eta-\lambda|^l}\\ &\le e^{-m_\lambda\ln\delta+Cp(\lambda)}\,m_\eta\,e^{m_\eta+m_\lambda}e^{-m_\eta\ln\delta+Cp(\eta)}\\ &\le e^{Ap(\lambda)}e^{Mp(\eta)}\end{aligned}$$

for some big enough constants $A, M > 0$ independent of λ and η. Thus

$$\|v_\lambda\|_M^Y \le e^{Ap(\lambda)}. \qquad \square$$

Lemma 2.3.5. *Let ω be a weight and denote $p(z) = |\operatorname{Im} z| + \omega(z)$. Assume that $X = \{(\lambda, m_\lambda)|\ \lambda\in\Lambda\}$, $Y = \{(\eta, m_\eta)|\ \eta\in\mathcal{K}\}$ are disjoint, interpolating for $A_{(\omega)}$, and $X\cup Y$ is weakly separated. Then there are two constants $A, M > 0$ such that for every $\lambda\in\Lambda$ there exists $g_\lambda\in A_{(\omega)}$ satisfying the following conditions*

$$\begin{aligned}&(1)\quad \frac{g_\lambda^{(l)}(\eta)}{l!} = 0 \qquad \textit{for all } \eta\in\mathcal{K},\ 0\le l< m_\eta,\\ &(2)\quad \frac{g_\lambda^{(l)}(\lambda)}{l!} = \delta_{l0} \qquad \textit{for all } 0\le l< m_\lambda,\\ &(3)\quad |g_\lambda(z)|\le Ae^{Mp(\lambda)}e^{Mp(z)} \qquad \textit{for all } z\in\mathbb{C}.\end{aligned}$$

Proof. Let v^λ be sequences as in Lemma 2.3.4. Then for some $M, C > 0$ and every $\lambda\in\Lambda$

$$\|v^\lambda e^{-Cp(\lambda)}\|_M \le 1.$$

By Lemma 2.1.1 used for Y there exist constants $A, N > 0$ and functions $h_\lambda\in A_{(\omega)}$ ($\lambda\in\Lambda$) satisfying

$$\frac{h_\lambda^{(l)}(\eta)}{l!} = v^\lambda_{\eta,l} \qquad \text{for } \eta\in\mathcal{K},\ 0\le l< m_\eta$$

and

$$\|h_\lambda\|_N \le Ae^{Cp(\lambda)}.$$

From Theorem 2.2.5 we know that $\{(\lambda, 2m_\lambda)\}_{\lambda\in\Lambda}$ is also interpolating. Therefore, by Lemma 2.1.3 there exist functions $f_\lambda\in A_{(\omega)}$ such that

$$\frac{f_\lambda^{(l)}(\lambda)}{l!} = \begin{cases}1 & \text{if } l = m_\lambda,\\ 0 & \text{if } 0\le l< m_\lambda \text{ or } m_\lambda< l< 2m_\lambda,\end{cases}$$

and for some $A_1, N_1 > 0$ independent of λ, $\|f_\lambda\|_{N_1} \le A_1$. Next, for $\lambda \in \Lambda$ define

$$g_\lambda(z) = (1 - (z-\lambda)^{m_\lambda} h_\lambda(z)) \frac{f_\lambda(z)}{(z-\lambda)^{m_\lambda}}.$$

Since the value and all derivatives of f_λ in λ up to the order $m_\lambda - 1$ are 0 hence $\frac{f_\lambda(z)}{(z-\lambda)^{m_\lambda}}$ is an entire function. Moreover, for $|z-\lambda| > 1$

$$\frac{|f_\lambda(z)|}{|z-\lambda|^{m_\lambda}} \le A_1 e^{N_1 p(z)}.$$

From the maximum modulus principle for every $\lambda \in \Lambda$ there is z_λ such that $|z_\lambda - \lambda| = 1$ and

$$\sup_{|z-\lambda|\le 1} \frac{|f_\lambda(z)|}{|z-\lambda|^{m_\lambda}} = \frac{|f_\lambda(z_\lambda)|}{|z_\lambda-\lambda|^{m_\lambda}}.$$

This, the previous inequality and Lemma 1.5.6 yield

$$\frac{|f_\lambda(z)|}{|z-\lambda|^{m_\lambda}} \le A_1 \max(e^{N_1 p(z_\lambda)}, e^{N_1 p(z)}) \le A_2 e^{N_2 p(\lambda)} e^{N_1 p(z)}$$

for every $z \in \mathbb{C}$ and for some constants $A_2, N_2 > 0$ independent of λ. Therefore,

$$\begin{aligned} |g_\lambda(z)| &\le \frac{|f_\lambda(z)|}{|z-\lambda|^{m_\lambda}} + |f_\lambda(z)||h_\lambda(z)| \le A_2 e^{N_2 p(\lambda)} e^{N_1 p(z)} + A_1 e^{N_1 p(z)} A e^{C p(\lambda)} e^{N p(z)} \\ &\le 2\max(A_2, AA_1)\, e^{\max(N_2, C) p(\lambda)}\, e^{(N_1+N)p(z)}. \end{aligned}$$

Thus we have shown (3).

Denote

$$F_\lambda(z) = \frac{f_\lambda(z)}{(z-\lambda)^{m_\lambda}}.$$

Then

$$F_\lambda^{(l)}(\lambda) = \begin{cases} 1 & \text{if } l = 0, \\ 0 & \text{if } 1 \le l < m_\lambda. \end{cases}$$

Now we are going to calculate the values and derivatives of g_λ. For $\lambda \in \Lambda$ and $0 \le l < m_\lambda$ we have

$$g_\lambda^{(l)}(\lambda) = \sum_{k=0}^{l} \binom{l}{k} (1 - (z-\lambda)^{m_\lambda} h_\lambda(z))^{(k)}\Big|_{z=\lambda} F_\lambda^{(l-k)}(\lambda).$$

If $1 < k < m_\lambda$ then

$$(1 - (z-\lambda)^{m_\lambda} h_\lambda(z))^{(k)}|_{z=\lambda} = \sum_{j=0}^{k} \binom{k}{j} \frac{m_\lambda!}{(m_\lambda - j)!} \Big((z-\lambda)^{m_\lambda - j} h_\lambda^{(k-j)}(z) \Big)\Big|_{z=\lambda} = 0.$$

Thus

$$g_\lambda(\lambda) = (1 - (z-\lambda)^{m_\lambda} h_\lambda(z))\Big|_{z=\lambda} F_\lambda(\lambda) = 1$$

and for $l > 0$

$$g_\lambda^{(l)}(\lambda) = \binom{l}{0}(1 - (z-\lambda)^{m_\lambda} h_\lambda(z))\Big|_{z=\lambda} F_\lambda^{(l)}(\lambda) = F_\lambda^{(l)}(\lambda) = 0.$$

This proves (2).

On the other hand, for $\eta \in \mathcal{K}$ and $0 \le l < m_\eta$

$$g_\lambda^{(l)}(\eta) = \sum_{k=0}^{l} \binom{l}{k} \left(\frac{1}{(z-\lambda)^{m_\lambda}} - h_\lambda(z) \right)^{(k)} \Big|_{z=\eta} f_\lambda^{(l-k)}(\eta).$$

But

$$\begin{aligned} \left(\frac{1}{(z-\lambda)^{m_\lambda}} \right)^{(k)} \Big|_{z=\eta} &= (-1)^k \frac{(m_\lambda + k - 1)!}{(m_\lambda - 1)!} \frac{1}{(\eta - \lambda)^{m_\lambda + k}} \\ &= (-1)^k \binom{m_\lambda + k - 1}{k} \frac{1}{(\eta-\lambda)^{m_\lambda + k}}\, k! = v_{\eta,k}^{\lambda}\, k!, \end{aligned}$$

and we have chosen functions h_λ such that

$$h_\lambda^{(k)}(\eta) = v_{\eta,k}^{\lambda}\, k!$$

for any $0 \le k < m_\eta$. This gives (1) and completes the proof. □

Theorem 2.3.6. *Let ω be a weight. Assume that $X = \{(\lambda, \widetilde{m}_\lambda) |\ \lambda \in \Lambda\}$, $Y = \{(\eta, \widehat{m}_\eta) |\ \eta \in \mathcal{K}\}$ are interpolating for $A_{(\omega)}$ and $X \cup Y$ is weakly separated. Then $X \cup Y$ is interpolating for $A_{(\omega)}$.*

Proof. First, we replace X by

$$X' := \big\{(\tau, m_\tau) |\ \tau \in \Lambda \text{ and } m_\tau = \widetilde{m}_\tau \text{ and } (\tau \in \mathcal{K} \Rightarrow \widetilde{m}_\tau \ge \widehat{m}_\tau)\big\}$$

and Y by

$$Y' := \big\{(\tau, m_\tau) |\ \tau \in \mathcal{K} \text{ and } m_\tau = \widehat{m}_\tau \text{ and } (\tau \in \Lambda \Rightarrow \widetilde{m}_\tau < \widehat{m}_\tau)\big\}.$$

Then $X' \subset X$ and $Y' \subset Y$, hence X' and Y' are interpolating. Moreover, X' and Y' are disjoint and $X' \cup Y' = X \cup Y$. Therefore without loss of generality we may assume in the rest of the proof that X and Y are disjoint. We may also write m_λ for $\widetilde{m}_\lambda$ and m_η for $\widehat{m}_\eta$ as it cannot lead to any misunderstanding anymore.

Denote $p(z) = |\operatorname{Im} z| + \omega(z)$. Let $v = (v_{\lambda,l})_{\lambda \in \Lambda \cup \mathcal{K},\, 0 \le l < m_\lambda} \in S_{(\omega)}(X \cup Y)$. Then we can split v into two parts

$$(v_{\lambda,l})_{\lambda \in \Lambda \cup \mathcal{K},\, 0 \le l < m_\lambda} = (x_{\lambda,l})_{\lambda \in \Lambda, 0 \le l < m_\lambda} \cup (y_{\eta,l})_{\eta \in \mathcal{K}, 0 \le l < m_\eta}.$$

Then $(x_{\lambda,l})_{\lambda\in\Lambda,0\leq l<m_\lambda} \in S_{(\omega)}(X)$ and $(y_{\eta,l})_{\eta\in\mathcal{K},0\leq l<m_\eta} \in S_{(\omega)}(Y)$. We have to find $f \in A_{(\omega)}$ such that

$$\frac{f^{(l)}(\lambda)}{l!} = x_{\lambda,l} \quad \text{for } \lambda \in \Lambda,\ 0 \leq l < m_\lambda$$

and

$$\frac{f^{(l)}(\eta)}{l!} = y_{\eta,l} \quad \text{for } \eta \in \mathcal{K},\ 0 \leq l < m_\eta.$$

To do this we will construct two functions $f_1, f_2 \in A_{(\omega)}$ satisfying

$$\begin{aligned} \frac{f_1^{(l)}(\lambda)}{l!} &= x_{\lambda,l} \quad \text{for } \lambda \in \Lambda,\ 0 \leq l < m_\lambda, \\ \frac{f_1^{(l)}(\eta)}{l!} &= 0 \quad \text{for } \eta \in \mathcal{K},\ 0 \leq l < m_\eta, \end{aligned}$$

and

$$\begin{aligned} \frac{f_2^{(l)}(\lambda)}{l!} &= 0 \quad \text{for } \lambda \in \Lambda,\ 0 \leq l < m_\lambda, \\ \frac{f_2^{(l)}(\eta)}{l!} &= y_{\eta,l} \quad \text{for } \eta \in \mathcal{K},\ 0 \leq l < m_\eta, \end{aligned}$$

and then take $f = f_1 + f_2$. However, we will only show the construction of f_1. Function f_2 can be obtained in a similar way.

By Lemma 2.3.3 there exists $M_1 > 0$ such that

$$\sum_{\lambda\in\Lambda} e^{-M_1 p(\lambda)} < \infty. \tag{2.1}$$

Take arbitrary $M > 0$. Then for all $\lambda \in \Lambda$, $0 \leq k < m_\lambda$ it holds

$$\|(\delta_{\lambda\lambda'}\delta_{kl}e^{Mp(\lambda)})_{\lambda'\in\Lambda,0\leq l<m_{\lambda'}}\|_M = 1.$$

By Lemma 2.1.1 used for X there are $A, N > 0$ such that for all $\lambda \in \Lambda$, $0 \leq k < m_\lambda$ there are functions $f_{\lambda,k} \in A_{(\omega)}$ with the following Taylor coefficients

$$\frac{f_{\lambda,k}^{(l)}(\lambda')}{l!} = \delta_{\lambda\lambda'}\delta_{kl}e^{Mp(\lambda)} \qquad \text{for all } \lambda' \in \Lambda \text{ and } 0 \leq l < m_{\lambda'} \tag{2.2}$$

and satisfying

$$|f_{\lambda,k}(z)| \leq Ae^{Np(z)} \tag{2.3}$$

for all $z \in \mathbb{C}$.

By Lemma 2.3.5 there are $A_1, N_1 > 0$ such that for every $\lambda \in \Lambda$ there is $g_\lambda \in A_{(\omega)}$ with

$$\frac{g_\lambda^{(l)}(\eta)}{l!} = 0 \quad \text{for all } \eta \in \mathcal{K},\ 0 \le l < m_\eta \tag{2.4}$$

$$\frac{g_\lambda^{(l)}(\lambda)}{l!} = \delta_{l0} \quad \text{for all } 0 \le l < m_\lambda \tag{2.5}$$

$$|g_\lambda(z)| \le A_1 e^{N_1 p(\lambda)} e^{N_1 p(z)} \quad \text{for all } z \in \mathbb{C}. \tag{2.6}$$

Since $\{x_{\lambda,l}\}_\Lambda \in S_{(\omega)}(X)$ there are $A_2, N_2 > 0$ such that

$$\sum_{l=0}^{m_\lambda - 1} |x_{\lambda,l}| \le A_2 e^{N_2 p(\lambda)} \tag{2.7}$$

for all $\lambda \in \Lambda$.

We set

$$f_1(z) = \sum_{\lambda \in \Lambda} \sum_{k=0}^{m_\lambda - 1} g_\lambda(z) f_{\lambda,k}(z) e^{-Mp(\lambda)} x_{\lambda,k}. \tag{2.8}$$

For every $\lambda \in \Lambda$

$$\sum_{k=0}^{m_\lambda - 1} g_\lambda(z) f_{\lambda,k}(z) e^{-Mp(\lambda)} x_{\lambda,k}.$$

is an entire function. Furthermore, by (2.3), (2.6) and (2.7)

$$\begin{aligned}\left| \sum_{k=0}^{m_\lambda - 1} \left(g_\lambda(z) f_{\lambda,k}(z) e^{-Mp(\lambda)} x_{\lambda,k} \right) \right| &\le A_3 e^{(N+N_1)p(z)} e^{(N_1 - M)p(\lambda)} \sum_{k=0}^{m_\lambda - 1} |x_{\lambda,k}| \\ &\le A_4 e^{(N+N_1)p(z)} e^{(N_1 + N_2 - M)p(\lambda)}\end{aligned}$$

for some $A_3, A_4 > 0$. Choose M such that $N_1 + N_2 - M \le -M_1$, where M_1 is from (2.1). Then the terms of the series (2.8) are uniformly bounded by the terms of some convergent series on any compact set of the complex plane. Hence f_1 is a well defined, entire function. The above estimate shows, moreover, that $f_1 \in A_{(\omega)}$. It remains to evaluate the values and derivatives of f_1 on Λ and $\mathcal{K}$. For $\lambda \in \Lambda$, $0 \le l < m_\lambda$ using (2.2) and (2.5) consecutively, we obtain

$$\begin{aligned}\frac{f_1^{(l)}(\lambda)}{l!} &= \sum_{\lambda' \in \Lambda} \sum_{k=0}^{m_{\lambda'} - 1} \sum_{j=0}^{l} \binom{l}{j} g_{\lambda'}^{(l-j)}(\lambda) f_{\lambda',k}^{(j)}(\lambda) e^{-Mp(\lambda')} x_{\lambda',k} \\ &= \sum_{\lambda' \in \Lambda} \sum_{k=0}^{m_{\lambda'} - 1} \sum_{j=0}^{l} \binom{l}{j} g_{\lambda'}^{(l-j)}(\lambda) \delta_{\lambda',\lambda} \delta_{k,j} e^{Mp(\lambda')} e^{-Mp(\lambda')} x_{\lambda',k} \\ &= \sum_{k=0}^{l} \binom{l}{k} g_\lambda^{(l-k)}(\lambda) x_{\lambda,k} = x_{\lambda,l}.\end{aligned}$$

On the other hand, (2.4) gives immediately that

$$\frac{f_1^{(l)}(\eta)}{l!} = \sum_{\lambda' \in \Lambda} \sum_{k=0}^{m_{\lambda'}-1} \sum_{j=0}^{l} g_{\lambda'}^{(l-j)}(\eta) f_{\lambda',k}^{(j)}(\eta) e^{-Mp(\lambda')} x_{\lambda',k} = 0$$

for $\eta \in \mathcal{K}$, $0 \leq l < m_\eta$. $\square$

Chapter 3

Analytic description of interpolating varieties

3.1 The Beurling case

A priori the definition of interpolating variety does not suggest any easy and explicit way to check whether a given multiplicity variety is interpolating. One would have to take an arbitrary sequence of values in the appropriate sequence space and try to figure out if there exists a function interpolating it. This does not seem to be easy even for simple examples of varieties. One needs a better criterion. In this chapter we give a description of interpolating varieties in analytic terms. The information if a multiplicity variety is interpolating is actually hidden in the existence of just one special function. The following theorem covers the Beurling case and was published in [BL95, Theorem 3.1]. Earlier partial results can be found in [BT79, Theorem 4], [Squ81]. As it is far from our final aim, we do not give its proof. In the sequel we will use the notation $Z(f)$ for the variety of all zeroes with multiplicities of a holomorphic function f.

Theorem 3.1.1. *Let $X = \{(\lambda, m_\lambda) |\ \lambda \in \Lambda\}$ be a multiplicity variety and ω be a weight. Variety X is interpolating for $A_{(\omega)}$ if and only if there exists $f \in A_{(\omega)}$ with $X \subset Z(f)$ and satisfying that for some $\epsilon > 0$ and $M \in \mathbb{N}$ for all $\lambda \in \Lambda$*

$$\frac{|f^{(m_\lambda)}(\lambda)|}{m_\lambda!} \geq \epsilon e^{-M|\operatorname{Im}\lambda| - M\omega(\lambda)}. \tag{3.1}$$

This theorem provides an easier way than the direct method of checking whether a given multiplicity variety is interpolating. But still it is not explicit enough. For more plain criterion we have to wait until Chapter 5. Nevertheless, we can derive a very useful consequence from Theorem 3.1.1.

Corollary 3.1.2. *Let ν be a weight and X be an interpolating variety for $A_{(\nu)}$. Then X is interpolating for $A_{(\omega)}$ for every weight ω satisfying $\nu = O(\omega)$.*

Proof. It is enough to notice that the bigger the weight the easier to fulfil condition (3.1). □

3.2 The Roumieu case

In the Roumieu case we have just a partial result of the analytic kind. A complete analytic characterisation will be obtained in Section 5.4 (see Theorem 5.4.6) for non-quasianalytic weights with the use of geometric descriptions of interpolating varieties.

Theorem 3.2.1. *Let ω be a weight and let $f \in A_{\{\omega\}}$ with $X \subset Z(f)$. If for some $M \in \mathbb{N}$, for all $m \in \mathbb{N}$ there exists $C_m \in \mathbb{R}$ such that for all $\lambda \in \Lambda$*

$$\frac{|f^{(m_\lambda)}(\lambda)|}{m_\lambda!} \geq e^{-C_m - M|\operatorname{Im} z| - \frac{1}{m}\,\omega(z)}$$

then X is an interpolating variety for $A_{\{\omega\}}$.

Proof. By the assumption

$$\exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, C_m \in \mathbb{R}\ \forall\, \lambda \in \Lambda :$$
$$\ln\left(\frac{m_\lambda!}{|f^{(m_\lambda)}(\lambda)|}\right) \leq C_m + M|\operatorname{Im}\lambda| + \frac{1}{m}\,\omega(\lambda)$$

and

$$\exists\, M_1 \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, B_m \in \mathbb{R}\ \forall\, \lambda \in \Lambda :$$
$$\ln|f(z)| \leq B_m + M_1|\operatorname{Im} z| + \frac{1}{m}\,\omega(z).$$

Take any sequence $v \in S_{\{\omega\}}(X)$. We have then

$$\exists\, M_2 \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, A_m \in \mathbb{R}\ \forall\, \lambda \in \Lambda :$$
$$\ln\left(\sum_{l=0}^{m_\lambda - 1} |v_{\lambda,l}|\right) \leq A_m + M_2|\operatorname{Im}\lambda| + \frac{1}{m}\omega(\lambda).$$

We can take $D_m = \max(A_m, B_m, C_m)$ obtaining the same constants in all three inequalities. Denote $q(z) = \inf_{m\in\mathbb{N}} \left(D_m + \frac{1}{m}\,\omega(z)\right)$. Then $q(t) = o(\omega(t))$ as $t \to \infty$. Moreover,

$$\ln\left(\frac{m_\lambda!}{|f^{(m_\lambda)}(\lambda)|}\right) \leq M|\operatorname{Im}\lambda| + q(\lambda) \tag{3.2}$$

for all $\lambda \in \Lambda$,

$$|f(z)| \le e^{M_1|\operatorname{Im} z|+q(z)} \tag{3.3}$$

for all $z \in \mathbb{C}$ and

$$\sum_{l=0}^{m_\lambda -1} |v_{\lambda,l}| \le e^{M_2|\operatorname{Im} \lambda|+q(\lambda)} \tag{3.4}$$

for all $\lambda \in \Lambda$. We can use lemma 1.5.16 for q and ω obtaining a weight ν such that $q(t) = o(\nu(t))$ and $\nu(t) = o(\omega(t)$ as $t \to \infty$. Then by (3.3), $f \in A_{(\nu)}$ and by (3.4), $v \in S_{(\nu)}(X)$. Moreover, condition (3.2) allows us to apply Theorem 3.1.1. Thus X is $A_{(\nu)}$ interpolating and there exists $F \in A_{(\nu)}$ with $R(F) = v$. Proposition 1.7.12 implies finally that $F \in A_{\{\omega\}}$. □

3.3 Relation between Beurling and Roumieu interpolating varieties

Propositions 1.7.12 and 1.8.9 provide us with a relation connecting Beurling and Roumieu weighted algebras of functions and sequences. One may then expect similar results for interpolating varieties. The following lemma gives just a partial answer to this problem. For a complete relation we have to wait until we obtain in Chapter 5 a more straightforward description of interpolating varieties in terms of geometric conditions (see 5.3.2).

Corollary 3.3.1. *Let ν be a weight and assume that X is interpolating for $A_{(\nu)}$. Then X is interpolating for $A_{\{\omega\}}$ for any weight ω satisfying $\nu = o(\omega)$.*

Proof. Let $v \in S_{\{\omega\}}(X)$. Proposition 1.8.9 yields $v \in S_{(\mu)}(X)$ for some $\mu = o(\omega)$. Since $\max(\nu, \mu) = o(\omega)$ we can use Lemma 1.5.16 to obtain another weight σ between $\max(\nu, \mu)$ and ω. Then by Corollary 3.1.2, X is $A_{(\sigma)}$ interpolating. Moreover, $v \in S_{(\sigma)}(X)$ hence we obtain $f \in A_{(\sigma)}$ satisfying $R(f) = v$. One more application of Proposition 1.7.12 completes the proof. □

Chapter 4

Geometric conditions

4.1 Sparse varieties

In this chapter we will study geometric conditions for multiplicity varieties. One way of describing geometric properties of multiplicity varieties, mentioned already in Section 1.10, is the use of the Nevanlinna counting function N. Conditions imposing bounds on its growth were successfully used in the study of interpolation problems in many spaces of entire functions (see [Squ83], [BL95], [BLV95], [HM00], [MOCO03]), [Oun07], [Oun08], [MOC09]). In this section we introduce several conditions of this type, we express their meaning in different, more readable forms and give some consequences which will be useful later.

We recall that the Nevanlinna counting function for a variety $X = \{(\lambda, m_\lambda) | \lambda \in \Lambda\}$ is given by the formula

$$N(z, r, X) = \int_0^r \frac{n(z, t, X) - n(z, 0, X)}{t} \,\mathrm{dt} + n(z, 0, X) \ln r$$

where $n(z, r, X)$ is the number of points of X with multiplicities in the closed disc $\bar{D}(z, r)$.

Definition 4.1.1. Let ω be a weight. Denote $p(z) = |\mathrm{Im}\, z| + \omega(z)$. We say that a multiplicity variety $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ is (ω)-sparse if

$$\exists\, C > 0 \;\forall\, \lambda \in \Lambda : \quad N(\lambda, p(\lambda), X) \leq Cp(\lambda).$$

It is called $\{\omega\}$-sparse if

$$\exists\, M \in \mathbb{N} \;\forall\, m \in \mathbb{N} \;\exists\, C_m \in \mathbb{R} \;\forall\, r > 0, \lambda \in \Lambda$$
$$N(\lambda, r, X) \leq C_m + M|\mathrm{Im}\, \lambda| + \frac{1}{m}\,\omega(\lambda) + Mr.$$

Right from the notation one might see that (ω)-sparsity corresponds to the Beurling case. This condition was introduced by Massaneda, Ortega-Cerdá, and Ounaïes in [MOCO03] and was called there condition (a). $\{\omega\}$-sparsity is a counterpart of (ω)-sparsity introduced to fit into the framework of the Roumieu case. One can see that (ω)-sparsity and $\{\omega\}$-sparsity are not so similar as one could expect. Looking at the form of $\{\omega\}$-sparsity one would rather define (ω)-sparsity in the following form.

Definition 4.1.2. Let ω be a weight and denote $p(z) = |\operatorname{Im} z| + \omega(z)$. We say that a multiplicity variety X is $(\omega)^*$-sparse if

$$\exists\, C > 0 \;\forall\, \lambda \in \Lambda \;\forall\, r \geq p(\lambda) : \quad N(\lambda, r, X) \leq Cr.$$

We see that $(\omega)^*$-sparsity is a priori stronger than (ω)-sparsity. Although they are equivalent in some cases, as we will see later, neither is their equivalency in general proved nor is a counterexample known. Furthermore, we will show that (ω)-sparsity along with some other condition is enough to describe interpolating varieties in the Beurling case. But it cannot be transformed directly to be useful also in the Roumieu case. It seems that the closest useful form is the one we have proposed as $\{\omega\}$-sparsity.

In the following lemma we give a relation between $(\omega)^*$-sparsity and $\{\omega\}$-sparsity.

Proposition 4.1.3. *Let ω be a Roumieu weight. A multiplicity variety X is $\{\omega\}$-sparse if and only if there exists a Beurling weight $\sigma = o(\omega)$ such that X is $(\sigma)^*$-sparse.*

Proof. Assume that X is $\{\omega\}$-sparse. Define $q(z) = \inf_{m\in\mathbb{N}}(C_m + \frac{1}{m}\omega(z))$. Then $q(t) = o(\omega(t))$ when $t \to \infty$. By Lemma 1.5.16 there exists a weight σ satisfying $q = o(\sigma)$ and $\sigma = o(\omega)$. Then by $\{\omega\}$-sparsity of X for some $M \in \mathbb{N}$

$$N(\lambda, r) \leq M|\operatorname{Im} \lambda| + q(\lambda) + Mr$$

for all $\lambda \in \Lambda$. Since $q = o(\sigma)$ we have $q(z) \leq C + \sigma(z)$ for some $C > 0$ and every $z \in \mathbb{C}$. This proves $(\sigma)^*$-sparsity of X.

Assume now that X is $(\sigma)^*$-sparse for some weight $\sigma = o(\omega)$. Then for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that $\sigma(t) \leq C_m + \frac{1}{m}\omega(t)$ for every $t \in [0, \infty)$. For $r > 0$ we obtain

$$\begin{aligned} N(\lambda, r) &\leq N(\lambda, |\operatorname{Im} \lambda| + \sigma(\lambda) + r) \leq C(|\operatorname{Im} \lambda| + \sigma(\lambda) + r) \\ &\leq C|\operatorname{Im} \lambda| + CC_m + C\frac{1}{m}\omega(\lambda) + Cr. \end{aligned}$$

□

Due to Proposition 4.1.3 all results obtained for the Beurling case provide information about the Roumieu case as well. This will be seen throughout this section. In several situations we will even omit the corresponding result or notion for the Roumieu case as it can be easily deduced and will not be used in the remaining part of this dissertation.

The meaning of sparsity is not visible immediately. We will show some consequences of it, which are more straightforward. Later we will prove that these consequences almost characterise sparse varieties. This will facilitate us to give examples of sparse varieties.

Definition 4.1.4. We say that a multiplicity variety X is (ω)-*separated* (or just *separated*) if

$$\prod_{\lambda' :\, 0<|\lambda'-\lambda|\leq 1} |\lambda - \lambda'|^{m_{\lambda'}} \geq \delta e^{-Cp(\lambda)}$$

for some $\delta, C > 0$ and all $\lambda \in \Lambda$.

As one may expect (ω)-separation implies weak (ω)-separation introduced in Definition 2.3.2.

Proposition 4.1.5. *Let ω be a weight. If a multiplicity variety X is (ω)-separated then it is weakly (ω)-separated.*

Proof. Consider points $\lambda, \lambda' \in \Lambda$ with $p(\lambda) \geq 1$ and $|\lambda'| > 1$. By Lemma 1.5.6 there exists a constant $C > 0$ such that

$$p(\lambda') \leq Cp(\lambda) + C \leq 2Cp(\lambda)$$

provided $|\lambda - \lambda'| \leq 1$. Hence

$$|\lambda - \lambda'|^{m_\lambda} \geq \prod_{\lambda :\, 0<|\lambda-\lambda'|\leq 1} |\lambda - \lambda'|^{m_\lambda} \geq e^{-Cp(\lambda')} \geq e^{-2Cp(\lambda)}.$$

Assume now that $\lambda, \lambda' \in \Lambda$ satisfy $p(\lambda) < 1$ and $|\lambda - \lambda'| \leq 1$. By property (γ) of ω, there are only finitely many such pairs. Therefore we can choose $\delta > 0$ such that

$$|\lambda - \lambda'|^{m_\lambda} \geq \delta \geq \delta e^{-2Cp(\lambda)} \quad \text{and} \quad |\lambda - \lambda'|^{m_{\lambda'}} \geq \delta \geq \delta e^{-2Cp(\lambda')}$$

for all such λ, λ'. ☐

The following example shows that (ω)-separation is strictly stronger than weak (ω)-separation.

Example 4.1.6. Let ω be an arbitrary weight. For every $\epsilon > 0$ small enough it holds

$$\frac{1}{2} \leq \frac{\sin x}{x} \leq 2 \tag{4.1}$$

for every $x \in [0, \epsilon]$.

Consider the set $Y = \{2mi |\ m \in \mathbb{N}\} \subset \mathbb{C}$. Around each point $2mi$ of Y we draw a regular polygon with $n = 2^m$ sides inscribed in a circle of radius $\frac{1}{2}$ centred at $2mi$. It is known that the side length of such a polygon is

$$d_1 = s = \sin \frac{\pi}{n}$$

and distances between vertices are

$$d_k = \sin \frac{k}{n}\pi \quad (k = 1, \ldots, n-1).$$

Let X consist of all the vertices of these polygons with multiplicities equal 1. Then for m big enough and λ, λ' belonging to the polygon around $2mi$

$$|\lambda - \lambda'| \geq \sin \frac{\pi}{n} \geq \frac{\pi}{2n} = \frac{\pi}{2} e^{-m \ln 2} \geq e^{-2m-\omega(2mi)} = e^{-p(2mi)}.$$

As $p(2mi)$ is proportional to $p(\lambda)$ with constants not depending on m, hence X is weakly (ω)-separated.

On the other hand, let $\epsilon > 0$ be such that (4.1) is satisfied. Then for λ belonging to the polygon around $2mi$

$$\prod_{\lambda' \in D'(\lambda,1)} |\lambda - \lambda'|^{m_{\lambda'}} = \prod_{k=1}^{n-1} d_k \leq \prod_{k=1}^{\lfloor \frac{\epsilon n}{\pi} \rfloor} \sin \frac{k}{n}\pi \leq \prod_{k=1}^{\lfloor \frac{\epsilon n}{\pi} \rfloor} \frac{k}{n}\pi = \Big(\frac{\pi}{n}\Big)^{\lfloor \frac{\epsilon n}{\pi} \rfloor} \Big\lfloor \frac{\epsilon n}{\pi} \Big\rfloor !$$

By Stirling's formula

$$\begin{aligned}\Big(\frac{\pi}{n}\Big)^{\lfloor \frac{\epsilon n}{\pi} \rfloor} \Big\lfloor \frac{\epsilon n}{\pi} \Big\rfloor ! &\leq e \Big(\frac{\pi}{n}\Big)^{\lfloor \frac{\epsilon n}{\pi} \rfloor} \sqrt{2\pi \Big\lfloor \frac{\epsilon n}{\pi} \Big\rfloor} \Big(\frac{\lfloor \frac{\epsilon n}{\pi} \rfloor}{e}\Big)^{\lfloor \frac{\epsilon n}{\pi} \rfloor} \leq \frac{en}{\pi} \Big(\frac{\pi}{n}\Big)^{\frac{\epsilon n}{\pi}} \sqrt{2\epsilon n} \Big(\frac{\epsilon n}{\pi e}\Big)^{\frac{\epsilon n}{\pi}} \\ &= \frac{e\sqrt{2\epsilon}}{\pi} n\sqrt{n} \Big(\frac{\epsilon}{e}\Big)^{\frac{\epsilon n}{\pi}} = Cn\sqrt{n} e^{-Dn},\end{aligned}$$

where $C = \frac{e\sqrt{2\epsilon}}{\pi}$ and $D = \frac{\epsilon}{\pi} \ln \frac{e}{\epsilon}$. We have that for any constant $A > 0$

$$C(2\sqrt{2})^m e^{Am} < e^{D2^m}$$

for m big enough. Therefore

$$\prod_{\lambda' \in D'(\lambda,1)} |\lambda - \lambda'|^{m_{\lambda'}} < e^{-Am}$$

for such m. Further, by property (β) of ω, for some $B \geq 1$ it holds $\omega(t) \leq Bt$ for t big enough. Hence for arbitrary $A > 0$ and m big enough

$$\prod_{\lambda' \in D'(\lambda,1)} |\lambda - \lambda'|^{m_{\lambda'}} < e^{-Am} = e^{-\frac{A}{2}m - \frac{A}{4B}2Bm} \leq e^{-\frac{A}{4}|\mathrm{Im}\, 2mi| - \frac{A}{4B}\omega(2mi)} \leq e^{-\frac{A}{4B}p(2mi)}.$$

As $p(2mi)$ is proportional to $p(\lambda)$ and this holds for arbitrary $A > 0$, X is not (ω)-separated.

The trick described in the next lemma will be used often in the sequel.

Lemma 4.1.7. *Let ω be a weight and X a multiplicity variety. Denote $p(z) = |\mathrm{Im}\, z| + \omega(z)$. Then the set $\Lambda_1 := \Lambda \cap \{z \in \mathbb{C} |\ p(z) \leq 1\}$ is finite, and for every $A > 0$ there exists $B > 0$ such that*

$$A \leq Bp(\lambda)$$

for every $\lambda \in \Lambda \setminus \{0\}$.

Proof. As ω is continuous and satisfies (γ), the set $K := \{z \in \mathbb{C} |\ p(z) \leq 1\}$ is compact. Hence, as Λ is discrete, $K \cap \Lambda$ is finite. Further, as ω is increasing, it holds $p(z) = 0 \Leftrightarrow z = 0$. Therefore

$$C := \inf_{\lambda \in \Lambda \setminus \{0\}} p(\lambda) > 0.$$

Then for any $\lambda \in \Lambda \setminus \{0\}$

$$A = \frac{A}{C} \inf_{\lambda' \in \Lambda \setminus \{0\}} p(\lambda') \leq \frac{A}{C} p(\lambda),$$

which yields the assertion for $B = \frac{A}{C}$. □

In the following lemma we give a description of the separation property in terms of an estimate for the function N.

Proposition 4.1.8. *Let ω be a weight. A multiplicity variety X is (ω)-separated if and only if*

$$N(\lambda, \min(1, p(\lambda)), X) \leq Cp(\lambda)$$

for some $C > 0$ and every $\lambda \in \Lambda$.

Proof. Using Proposition 1.10.2 we obtain

$$N(\lambda, 1) = -\sum_{\lambda' :\, 0 < |\lambda' - \lambda| \leq 1} \ln|\lambda - \lambda'|^{m_{\lambda'}} = -\ln \prod_{\lambda' :\, 0 < |\lambda' - \lambda| \leq 1} |\lambda - \lambda'|^{m_{\lambda'}} \qquad (4.2)$$

for every $\lambda \in \Lambda$.

($\Rightarrow$) We have

$$\prod_{\lambda' \colon 0<|\lambda'-\lambda|\leq 1} |\lambda - \lambda'|^{m_{\lambda'}} \geq \delta e^{-Cp(\lambda)}$$

for some $\delta, C > 0$ and every $\lambda \in \Lambda$. Then using (4.2) we obtain

$$N(\lambda, \min(1, p(\lambda))) \leq N(\lambda, 1) \leq Cp(\lambda) + \ln \frac{1}{\delta}.$$

Applying Lemma 4.1.7 we get the assertion for all $\lambda \in \Lambda \setminus \{0\}$. If $\lambda = 0 \in \Lambda$ then $N(\lambda, \min(1, p(\lambda))) = N(\lambda, 0) = -\infty$, and the assertion is satisfied trivially.

($\Leftarrow$) Take any $\lambda \in \Lambda$ satisfying $p(\lambda) \geq 1$. Then by (4.2) and the assumption we have

$$\prod_{\lambda' \colon 0<|\lambda'-\lambda|\leq 1} |\lambda - \lambda'|^{m_{\lambda'}} = e^{-N(\lambda,1)} \geq e^{-Cp(\lambda)}$$

for some $C > 0$ independent of λ. Further, by Lemma 4.1.7 the set $\Lambda_1 := \Lambda \cap \{z \in \mathbb{C} | \ p(z) \leq 1\}$ is finite. Hence there exists $\delta > 0$ such that

$$\prod_{\lambda' \colon 0<|\lambda'-\lambda|\leq 1} |\lambda - \lambda'|^{m_{\lambda'}} \geq \delta \geq \delta e^{-Cp(\lambda)}$$

for every $\lambda \in \Lambda_1$. □

Remark 4.1.9. In [MOCO03, Lemma 7(i)] Massaneda, Ortega-Cerdá, and Ounaïes proved that (ω)-sparsity implies weak (ω)-separation.

In the following proposition we show that (ω)-separation is a consequence of (ω)-sparsity. Moreover, we give estimates on the growth of the Nevanlinna function n that follow from (ω)-sparsity. This provides more straightforward information about geometric properties of sparse varieties.

Proposition 4.1.10. *Let ω be a weight. Suppose that $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ is (ω)-sparse. Then X is (ω)-separated and for every ϵ satisfying $0 < \epsilon < 1$ there exists $C > 0$ such that the following conditions are satisfied for every $\lambda \in \Lambda$ with $p(\lambda) > 1$*

(1)

$$n(\lambda, 1) \leq C \frac{p(\lambda)}{\ln p(\lambda)},$$

(2) for $r \in [1, p(\lambda)^{1-\epsilon}]$

$$n(\lambda, r) \leq C\Big(r + \frac{p(\lambda)}{\ln p(\lambda)}\Big),$$

(3) for $r \in [p(\lambda)^{1-\epsilon}, \epsilon p(\lambda)]$

$$n(\lambda, r) \le C \frac{p(\lambda)}{\ln \frac{p(\lambda)}{r}},$$

(4) for $r \in [\epsilon p(\lambda), (1-\epsilon)p(\lambda)]$

$$n(\lambda, r) \le Cp(\lambda).$$

If X *is* $(\omega)^*$*-sparse then additionally*

(4) for* $r \in [\max(1, \epsilon p(\lambda)), \infty)$,

$$n(\lambda, r) \le Cr.$$

for every $\lambda \in \Lambda$.

Proof. By (ω)-sparsity

$$N(\lambda, \min(1, p(\lambda))) \le N(\lambda, p(\lambda)) \le Cp(\lambda)$$

for some $C > 0$ and every $\lambda \in \Lambda$. Hence (ω)-separation of X follows from Proposition 4.1.8.

Now let λ be such that $p(\lambda) \ge 1$. Then (ω)-sparsity implies

$$\begin{aligned} Cp(\lambda) &\ge \int_0^{p(\lambda)} \frac{n(\lambda,t) - n(\lambda,0)}{t}\,\mathrm{dt} + n(\lambda,0)\ln p(\lambda) \\ &= \int_0^{1} \frac{n(\lambda,t) - n(\lambda,0)}{t}\,\mathrm{dt} + \int_1^{p(\lambda)} \frac{n(\lambda,t) - n(\lambda,0)}{t}\,\mathrm{dt} + n(\lambda,0)\ln p(\lambda) \\ &= \int_0^{1} \frac{n(\lambda,t) - n(\lambda,0)}{t}\,\mathrm{dt} + \int_1^{p(\lambda)} \frac{n(\lambda,t)}{t}\,\mathrm{dt}. \end{aligned}$$

Hence

$$\int_1^{p(\lambda)} \frac{n(\lambda,t)}{t}\,\mathrm{dt} \le Cp(\lambda).$$

For $r \in [1, p(\lambda)]$ we obtain then

$$Cp(\lambda) \ge \int_r^{p(\lambda)} \frac{n(\lambda,t)}{t}\,\mathrm{dt} \ge \int_r^{p(\lambda)} \frac{n(\lambda,r)}{t}\,\mathrm{dt} = n(\lambda,r)\ln\frac{p(\lambda)}{r}.$$

Hence

$$n(\lambda, r) \leq C \frac{p(\lambda)}{\ln \frac{p(\lambda)}{r}} \tag{4.3}$$

for every $\lambda \in \Lambda$ satisfying $p(\lambda) > 1$ and every $r \in [1, p(\lambda)]$. This proves (3) for any given $\epsilon > 0$.

Now we can prove (4). Let $\epsilon > 0$. Take $\lambda \in \Lambda$ satisfying $p(\lambda) > 1$, and $r \in [\epsilon p(\lambda), (1-\epsilon)p(\lambda)]$. If $(1-\epsilon)p(\lambda) \geq 1$ then by (4.3)

$$n(\lambda, r) \leq n(\lambda, (1-\epsilon)p(\lambda)) \leq C \frac{p(\lambda)}{\ln \frac{p(\lambda)}{(1-\epsilon)p(\lambda)}} = \frac{C}{\ln \frac{1}{1-\epsilon}} p(\lambda).$$

On the other hand, there are only finitely many $\lambda \in \Lambda$ satisfying $1 < p(\lambda) < \frac{1}{1-\epsilon}$. Hence

$$D := \sup_{\lambda \in \Lambda:\ 1 < p(\lambda) < \frac{1}{1-\epsilon}} \frac{1}{\ln p(\lambda)} < \infty.$$

Therefore by (4.3)

$$n(\lambda, r) \leq n(\lambda, 1) \leq C \frac{p(\lambda)}{\ln p(\lambda)} \leq CDp(\lambda)$$

for $\lambda \in \Lambda$ such that $1 < p(\lambda) < \frac{1}{1-\epsilon}$ and $r \in [\epsilon p(\lambda), (1-\epsilon)p(\lambda)]$. This gives (4).

Now we are going to prove (2) with the use of (4.3). Let $\epsilon > 0$ be given. Take $D > \max(\frac{e^2 C}{4}, \frac{C}{\epsilon})$. We want to show that

$$f_\lambda(r) := C \frac{p(\lambda)}{\ln \frac{p(\lambda)}{r}} - Dr - D \frac{p(\lambda)}{\ln p(\lambda)} < 0 \tag{4.4}$$

for every $\lambda \in \Lambda$ satisfying $p(\lambda) > 1$ and every $r \in [1, p(\lambda)^{1-\epsilon}]$. We have

$$f'_\lambda(r) = \frac{Cp(\lambda)}{r \ln^2 \frac{p(\lambda)}{r}} - D$$

and

$$f'_\lambda(1) = \frac{Cp(\lambda)}{\ln^2 p(\lambda)} - D.$$

Observe that

$$h(x) := f'_\lambda\Big(\frac{p(\lambda)}{e^x}\Big) = \frac{Ce^x}{x^2} - D$$

is strictly increasing for $x \geq 2$. Moreover, $h(2) = \frac{Ce^2}{4} - D < 0$. Therefore there exists $x_0 > 2$ such that $h(x_0) = 0$, $h(x) < 0$ for every $x \in [2, x_0)$ and $h(x) > 0$ for every $x \in (x_0, \infty)$. We can define then a subset of Λ as follows

$$\Lambda' := \Big\{ \lambda \in \Lambda \Big| \ \frac{Cp(\lambda)}{\ln^2 p(\lambda)} > D,\ p(\lambda)^{1-\epsilon} < \frac{p(\lambda)}{e^{x_0}} \text{ and } p(\lambda) > 1 \Big\}.$$

As $\frac{x}{\ln^2 x}$ is increasing for $x \geq e^2$, the set $\Lambda \setminus \Lambda'$ is finite. Moreover, for $\lambda \in \Lambda'$ it holds $f'_\lambda(1) > 0$. Therefore, for such λ and $r \in [1, \frac{p(\lambda)}{e^{x_0}})$, $\ln \frac{p(\lambda)}{r} > x_0$ and thus

$$f'_\lambda(r) = h\Big(\ln \frac{p(\lambda)}{r}\Big) > 0.$$

This yields

$$\begin{aligned} f_\lambda(r) < f_\lambda(p(\lambda)^{1-\epsilon}) &= C\frac{p(\lambda)}{\ln p(\lambda)^\epsilon} - Dp(\lambda)^{1-\epsilon} - D\frac{p(\lambda)}{\ln p(\lambda)} \\ &= \Big(\frac{C}{\epsilon} - D\Big)\frac{p(\lambda)}{\ln p(\lambda)} - Dp(\lambda)^{1-\epsilon} < 0 \end{aligned}$$

for every $\lambda \in \Lambda'$ and $r \in [1, p(\lambda)^{1-\epsilon}] \subset [1, \frac{p(\lambda)}{e^{x_0}})$. Thus we have proved (4.4) for $\lambda \in \Lambda'$. Further, as $\Lambda \setminus \Lambda'$ is finite

$$M := \sup_{\lambda \in \Lambda \setminus \Lambda'} \sup_{r \in [1, p(\lambda)^{1-\epsilon}]} C\frac{p(\lambda)}{\ln \frac{p(\lambda)}{r}} < \infty$$

and

$$C\frac{p(\lambda)}{\ln \frac{p(\lambda)}{r}} \leq M \leq Mr + M\frac{p(\lambda)}{\ln p(\lambda)}$$

for every $\lambda \in \Lambda \setminus \Lambda'$ and every $r \in [1, p(\lambda)^{1-\epsilon}]$. Therefore we have proved (4.4) for some $D \geq \max(M, \frac{e^2 C}{4}, \frac{C}{\epsilon})$, for every $\lambda \in \Lambda$ satisfying $p(\lambda) > 1$ and every $r \in [1, p(\lambda)^{1-\epsilon}]$. Using (4.3) we obtain condition (2).

Now we turn to the proof of (1). For $\lambda \in \Lambda$ with $p(\lambda) > 1$ by (2) we obtain

$$n(\lambda, 1) \leq C + C\frac{p(\lambda)}{\ln p(\lambda)}.$$

But

$$\delta := \min\Big(1, \inf_{\lambda \in \Lambda:\, p(\lambda) > 1} \frac{p(\lambda)}{\ln p(\lambda)}\Big) > 0.$$

Hence

$$n(\lambda, 1) \leq C + C\frac{p(\lambda)}{\ln p(\lambda)} \leq \frac{C}{\delta}\delta + \frac{C}{\delta}\frac{p(\lambda)}{\ln p(\lambda)} \leq \frac{2C}{\delta}\frac{p(\lambda)}{\ln p(\lambda)}$$

for every $p(\lambda) > 1$. This proves (1).

Finally, Lemma 1.10.4 gives $n(\lambda, r) \leq N(\lambda, er)$ for every $\lambda \in \Lambda$ and $r \geq 1$. Hence for $r \in [\max(1, \epsilon p(\lambda)), \infty)$ we have $\frac{e}{\epsilon} r \geq p(\lambda)$ and

$$n(\lambda, r) \leq N(\lambda, er) \leq N\Big(\lambda, \frac{e}{\epsilon} r\Big) \leq \frac{Ce}{\epsilon} r.$$

This proves (4*) and completes the proof. □

Remark 4.1.11. Using Propositions 4.1.10 and 4.1.5 we can write the following series of implications:

$$(\omega)^*\text{-}sparsity \Rightarrow (\omega)\text{-}sparsity \Rightarrow (\omega)\text{-}separation \Rightarrow weak\ (\omega)\text{-}separation.$$

The converse of Proposition 4.1.10 is unfortunately not true. The estimate

$$n(\lambda, r) \leq C\frac{p(\lambda)}{\ln \frac{p(\lambda)}{r}}$$

for $r \in [p(\lambda)^{1-\epsilon}, \epsilon p(\lambda)]$ is too large, and we need to replace it by

$$n(\lambda, r) \leq C\Big(r + \frac{p(\lambda)}{\ln p(\lambda)}\Big)$$

for such r.

Proposition 4.1.12. *Let ω be a weight. Let $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ be (ω)-separated and assume that for some $0 < \epsilon < 1$, $C > 0$, and $R > \frac{1}{\epsilon}$, for every $\lambda \in \Lambda$ with $p(\lambda) > R$,*

(1) for $r \in [1, \epsilon p(\lambda)]$

$$n(\lambda, r) \leq C\Big(r + \frac{p(\lambda)}{\ln p(\lambda)}\Big),$$

(2) for $r \in [\epsilon p(\lambda), p(\lambda)]$

$$n(\lambda, r) \leq Cp(\lambda).$$

Then X is (ω)-sparse.

If additionally

(2) for $r \in [\max(1, \epsilon p(\lambda)), \infty)$*

$$n(\lambda, r) \leq Cr$$

for every $\lambda \in \Lambda$, then X is $(\omega)^*$-sparse.

Proof. Let $\lambda \in \Lambda$ satisfying $p(\lambda) > R$ be given. By Proposition 4.1.8 we have $N(\lambda, 1) \leq Cp(\lambda)$. Using (1) we obtain

$$\int_1^{\epsilon p(\lambda)} \frac{n(\lambda, t)}{t}\,\mathrm{dt} \leq \int_1^{\epsilon p(\lambda)} C\,\mathrm{dt} + C\frac{p(\lambda)}{\ln p(\lambda)}\int_1^{\epsilon p(\lambda)} \frac{1}{t}\,\mathrm{dt}$$
$$= C(\epsilon p(\lambda) - 1) + Cp(\lambda)\frac{\ln \epsilon p(\lambda)}{\ln p(\lambda)} \leq 2Cp(\lambda).$$

Next, by (2)

$$\int_{\epsilon p(\lambda)}^{p(\lambda)} \frac{n(\lambda,t)}{t}\,\mathrm{dt} \le Cp(\lambda)\ln\frac{1}{\epsilon}.$$

Therefore

$$N(\lambda,p(\lambda)) = N(\lambda,1) + \int_{1}^{\epsilon p(\lambda)} \frac{n(\lambda,t)}{t}\,\mathrm{dt} + \int_{\epsilon p(\lambda)}^{p(\lambda)} \frac{n(\lambda,t)}{t}\,\mathrm{dt} \le \Big(3+\ln\frac{1}{\epsilon}\Big)Cp(\lambda).$$

On the other hand, the set of $\lambda \in \Lambda$ satisfying $p(\lambda) \le R$. Hence

$$A := \sup_{\lambda\in\Lambda:\, p(\lambda)\le R} N(\lambda,R) < \infty. \tag{4.5}$$

Therefore by Lemma 4.1.7 for some $B > 0$

$$N(\lambda,p(\lambda)) \le N(\lambda,R) \le A \le Bp(\lambda)$$

for every $\lambda \in \Lambda$ satisfying $0 < p(\lambda) \le R$. Finally, if $\lambda = 0 \in \Lambda$ then $N(\lambda,p(\lambda)) = N(\lambda,0) = -\infty < p(\lambda)$. This completes the proof of (ω)-sparsity.

If additionally (2^*) is satisfied then

$$\int_{\epsilon p(\lambda)}^{r} \frac{n(\lambda,t)}{t}\,\mathrm{dt} \le C(r-\epsilon p(\lambda)) \le Cr \tag{4.6}$$

for every $\lambda \in \Lambda$ and $r \in [\max(1,\epsilon p(\lambda)),\infty)$. Hence for $\lambda \in \Lambda$ satisfying $p(\lambda) > R$

$$N(\lambda,r) = N(\lambda,1) + \int_{1}^{\epsilon p(\lambda)} \frac{n(\lambda,t)}{t}\,\mathrm{dt} + \int_{\epsilon p(\lambda)}^{r} \frac{n(\lambda,t)}{t}\,\mathrm{dt} \le 3Cp(\lambda) + Cr \le 4Cr$$

for some $C > 0$ and every $r \ge p(\lambda)$.

Further, there are only finitely many points $\lambda \in \Lambda$ such that $p(\lambda) \le R$. Moreover, for every $\lambda \in \Lambda$, $N(\lambda,r)$ is increasing with respect to r, and for $r > 0$ small enough it holds $N(\lambda,r) < 0$. Hence

$$B := \sup_{\lambda\in\Lambda:\, p(\lambda)\le R}\ \sup_{0<r\le R} \frac{N(\lambda,r)}{r} < \infty.$$

Thus for every $\lambda \in \Lambda$ satisfying $p(\lambda) \le R$ and every $r \in [p(\lambda),R]$

$$N(\lambda,r) \le Br.$$

Finally, for $\lambda \in \Lambda$ with $p(\lambda) \le R$ and for $r > R$ we obtain by (4.5) and (4.6)

$$N(\lambda,r) = N(\lambda,R) + \int_{R}^{r} \frac{n(\lambda,t)}{t}\,\mathrm{dt} \le A + Cr \le (A+C)r.$$

This completes the proof of $(\omega)^*$-sparsity. $\square$

Now we want to say more about geometric operations which preserve or do not preserve sparsity. Recall that Lemma 2.2.3 tells us that a shift of an interpolating variety is again interpolating. One may ask if a shift of a sparse variety is again sparse. It is the case for $(\omega)^*$-sparsity and $\{\omega\}$-sparsity. For (ω)-sparsity the answer in general case is not known to the author.

Proposition 4.1.13. *Let ω be a weight. Suppose that X is $(\omega)^*$-sparse. Then for any fixed $\eta \in \mathbb{C}$, $Y = \{(\lambda + \eta, m_\lambda)\}_{\lambda\in\Lambda}$ is $(\omega)^*$-sparse as well. The same holds for $\{\omega\}$-sparsity.*

Proof. Observe that

$$N(\lambda + \eta, r, Y) = N(\lambda, r, X).$$

Assume that X is $(\omega)^*$-sparse and denote $p(z) = |\operatorname{Im} z| + \omega(z)$. Take $r \geq p(\lambda+\eta)$. If $r \geq p(\lambda)$ then

$$N(\lambda + \eta, r, Y) \leq Cr$$

for some $C > 0$ and any $\lambda \in \Lambda$. Otherwise, for the case $p(\lambda + \eta) \leq r < p(\lambda)$, using Lemma 1.5.6 we get

$$N(\lambda + \eta, r, Y) \leq N(\lambda, p(\lambda), X) \leq Cp(\lambda) \leq D(p(\lambda + \eta) + |\eta| + 1)$$

for some $C, D > 0$. Using Lemma 4.1.7 we obtain a constant $B > 0$ such that for every $\lambda \in \Lambda \setminus \{-\eta\}$

$$N(\lambda + \eta, r, Y) \leq Bp(\lambda + \eta) \leq Br.$$

Finally, as Λ is discrete,

$$d := \inf_{\lambda\in\Lambda\setminus\{-\eta\}} |\lambda + \eta| > 0,$$

and if $-\eta \in \Lambda$ then just by the definition of the function N we have

$$N(0, r, Y) \leq 0 < r$$

for $0 \leq r \leq d$. Furthermore, for some constant $A > 0$ and every $d \leq r < p(-\eta)$

$$N(0, r, Y) \leq N(-\eta, p(-\eta), X) \leq Cp(-\eta) \leq Ar.$$

This completes the proof of $(\omega)^*$-sparsity of Y.

Assume now that X is $\{\omega\}$-sparse. Then

$$\exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, C_m > 0\ \ \forall\, \lambda \in \Lambda, r > 0$$

$$\begin{aligned} N(\lambda + \eta, r, Y) &= N(\lambda, r, X) \leq C_m + M|\operatorname{Im} \lambda| + \frac{1}{m}\omega(\lambda) + Mr \\ &\leq C_m + \left(M + \frac{D}{m}\right)|\eta| + \frac{D}{m} + M|\operatorname{Im}(\lambda + \eta)| + \frac{D}{m}\omega(\lambda + \eta) + Mr \end{aligned}$$

for some $D > 0$, again by Lemma 1.5.6. □

The following example shows that a bounded perturbation (see Section 2.2) of an (ω)-sparse variety need not to even be weakly (ω)-separated.

Example 4.1.14. By Proposition 4.1.12 the variety $X = \{(m,1)|\ m \in \mathbb{N}\}$ is (ω)-sparse for any weight ω. Define a sequence $(\eta_m)_{m\in\mathbb{N}}$ by

$$\eta_m = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ e^{-\omega^2(m)} - 1 & \text{if } m \text{ is even,} \end{cases}$$

for $m \in \mathbb{N}$. Then let $Y = \{(m + \eta_m, 1)|\ m \in \mathbb{N}\}$. We see that $\sup_{m\in\mathbb{N}}|\eta_m| = 1$ and Y arises as a perturbation of X by (η_m). Denoting $\lambda_m = m + \eta_m$ we obtain

$$|\lambda_{2m} - \lambda_{2m-1}| = \left|\left(2m + \left(e^{-\omega^2(2m)} - 1\right)\right) - (2m-1)\right| = e^{-\omega^2(2m)}.$$

But for any $C > 0$ there exists $m \in \mathbb{N}$ such that $C < \omega(2m)$ and thus

$$e^{-\omega^2(2m)} < e^{-C\omega(2m)},$$

which proves that Y is not weakly (ω)-separated. In particular, Y is not (ω)-sparse.

Another interesting operation can be performed by changing a point λ of a variety with multiplicity m_λ to m_λ points close to λ with multiplicities equal one. We will use the term *diffusion* for such an operation. Of course, any diffusion applied to finitely many points of a sparse variety will not break down its sparsity. The following example shows that this is no longer true when one applies diffusion to infinitely many points. Again, the property which is not preserved is weak separation.

Example 4.1.15. We take $X = \{(2m, 2)|\ m \in \mathbb{N}\}$. This variety is weakly (ω)-separated and also, by Proposition 4.1.12, (ω)-sparse for any weight ω. Then the variety Y from Example 4.1.14 can be thought of as a diffusion of X. But Y is not weakly (ω)-separated. Moreover, we see that the smaller the changes are, the worse.

The last operation we will deal with is rotation. Again, we will show that weak separation is not preserved, and thus sparsity is not preserved either. But we have seen that the density of points of a variety was the only problem in the case of perturbation and diffusion. In the case of rotation also the local quantity of points can be an obstacle. This may seem surprising as rotation does not change distances between points. The reason lies in the non-radiality of the function $p(z) = |\operatorname{Im} z| + \omega(z)$. The following example uses proximity of points to break separation down.

Example 4.1.16. Let ω be a Beurling weight and denote $p(z) = |\operatorname{Im} z| + \omega(z)$. Define

$$X := \left\{(2mi, 1)|\ m \in \mathbb{N}\right\} \cup \left\{\left((2m + e^{-m})i, 1\right)|\ m \in \mathbb{N}\right\}.$$

Then

$$|(2m+e^{-m})i - 2mi| = e^{-m} \geq e^{-(|\operatorname{Im} 2mi| + \omega(2mi))} = e^{-p(2mi)},$$

which proves that X is weakly (ω)-separated. Moreover, $n(\lambda, r) \leq r + 2$ for every $\lambda \in X$ and $r > 0$. Hence X is (ω)-sparse by Proposition 4.1.12.

Now define

$$Y := (-i)X = \{(2m, 1) |\ m \in \mathbb{N}\} \cup \{\big(2m + e^{-m}, 1\big) |\ m \in \mathbb{N}\}.$$

This time, by property (β^*) of the weight, we have that for each $C > 0$ there exists $m \in \mathbb{N}$ such that $e^{-m} < e^{-C\omega(2m)} = e^{-Cp(2m)}$. Hence Y is not weakly (ω)-separated and therefore not (ω)-sparse.

The following example shows that local quantities of points of a variety also cause a problem for sparsity in the case of rotation.

Example 4.1.17. Now consider $\omega(t) = \sqrt{t}$ and

$$X = \Big\{\Big(3^m i, \frac{3^m}{m}\Big) |\ m \in \mathbb{N}\Big\}.$$

For $r \in [0, 3^m - 3^{m-1})$ it holds

$$n(3^m i, r) = \frac{3^m}{m} = \frac{|\operatorname{Im} 3^m i|}{\ln |\operatorname{Im} 3^m i|} \leq \frac{p(3^m i)}{\ln p(3^m i)}.$$

Further, for $r \in [3^m - 3^{m-1}, 3^m]$

$$\begin{aligned} n(3^m i, r) &= \sum_{k\colon\, 3^m - r \leq 3^k \leq 3^m} \frac{3^k}{k} \leq \sum_{k\colon\, 1 \leq k \leq m} 3^k = \frac{3}{2}(3^{m+1} - 1) \\ &\leq 6 \cdot 3^m = 9(3^m - 3^{m-1}) \leq 9r. \end{aligned}$$

Hence by Proposition 4.1.12, X is (ω)-sparse.

But for

$$Y := (-i)X = \Big\{\Big(3^m, \frac{3^m}{m}\Big) |\ m \in \mathbb{N}\Big\}$$

we have

$$n\Big(3^m, \frac{p(3^m)}{2}\Big) = n\Big(3^m, \frac{3^{\frac{m}{2}}}{2}\Big) \geq \frac{3^m}{m} > C 3^{\frac{m}{2}} = C\omega(3^m) = Cp(3^m)$$

for any $C > 0$ provided m is big enough. Hence Y is not (ω)-sparse as the necessary condition (4) of Proposition 4.1.10 for $\epsilon = \frac{1}{2}$ is not satisfied.

Due to Proposition 4.1.3 all the examples of this section may be used to show that also Roumieu sparsity is not invariant under perturbation, diffusion and rotation.

We end this section with some important properties of sparse varieties which will be used later. The following proposition provides information about multiplicities of a sparse variety. This result is similar to Proposition 2.2.4.

Proposition 4.1.18. *Let ω be a weight and denote $p(z) = |\operatorname{Im} z| + \omega(z)$. If X is (ω)-sparse then there exists a constant $C > 0$ such that*

$$m_\lambda \le C\frac{p(\lambda)}{\ln p(\lambda)}$$

for $\lambda \in \Lambda$ with $p(\lambda) > 1$ and

$$m_\lambda \le Cp(\lambda)$$

for every $\lambda \in \Lambda \setminus \{0\}$.

Proof. From condition (1) of Proposition 4.1.10 we obtain

$$m_\lambda \le n(\lambda, 1) \le C\frac{p(\lambda)}{\ln p(\lambda)}$$

for every $\lambda \in \Lambda$ with $p(\lambda) > 1$. This yields

$$m_\lambda \le n(\lambda, 1) \le Cp(\lambda)$$

for every $\lambda \in \Lambda$ with $p(\lambda) \ge e$. On the other hand, there are only finitely many $\lambda \in \Lambda$ such that $p(\lambda) < e$. Hence

$$\sup_{\lambda\in\Lambda\colon p(\lambda)<e} m_\lambda < \infty.$$

We can complete the proof with the use of Lemma 4.1.7. □

A similar result is true for the Roumieu case.

Proposition 4.1.19. *Let ω be a Roumieu weight. If X is $\{\omega\}$-sparse then*

$$\exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, C_m > 0\ \forall\, \lambda \in \Lambda\colon \quad m_\lambda \le C_m + M|\operatorname{Im}\lambda| + \frac{1}{m}\omega(\lambda).$$

Proof. By Proposition 4.1.3 there exists weight $\sigma = o(\omega)$ such that X is (σ)-sparse. Then Proposition 4.1.18 yields

$$m_\lambda \le C(|\operatorname{Im}\lambda| + \omega(\lambda))$$

for some $C > 0$ and every $\lambda \in \Lambda \setminus \{0\}$. Since for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that $\sigma(t) \le C_m + \frac{1}{m}\omega(t)$ for every $t \ge 0$, the assertion follows. □

The next lemma extends estimates for the Nevanlinna function n to all points of the complex plane. This result is technical and will be used later.

Lemma 4.1.20 ([MOCO03, Lemma 7(ii)])**.** *Let ω be a weight and denote $p(z) = |\operatorname{Im} z| + \omega(z)$. Assume that X is (ω)-sparse and $0 \notin \Lambda$. Then*

$$n(z, \epsilon p(z)) \leq Cp(z)$$

for some $\epsilon, C > 0$ and all $z \in \mathbb{C}$.

Proof. For $z \in \Lambda$ the assertion follows from Proposition 4.1.10 (4). Let $z \notin \Lambda$. If there is no $\lambda \in D(z, \epsilon p(z))$, then the estimate is trivially fulfilled. Consider the set

$$G_\epsilon = \{z \in \mathbb{C} | \, \exists \, \lambda \in \Lambda : \lambda \in D(z, \epsilon p(z))\}.$$

We are going to show that $\inf_{z \in G_\epsilon} |z| > 0$ if ϵ is chosen small enough. By property (β) of the weight we have $p(z) \leq C|z| + D$ for some constants $C, D > 0$. Choose $r > 0$ and $\epsilon > 0$ such that $(1 + \epsilon C)r + \epsilon D < \inf_{\lambda \in \Lambda} |\lambda|$. Then for $z \in D(0, r)$ and $w \in D(z, \epsilon p(z))$ we have

$$\begin{aligned} |w| &\leq |z| + \epsilon p(z) \leq |z| + \epsilon(C|z| + D) = (1 + \epsilon C)|z| + \epsilon D \\ &< (1 + \epsilon C)r + \epsilon D < \inf_{\lambda \in \Lambda} |\lambda| \end{aligned}$$

hence $w \notin \Lambda$. We have proved that $\inf_{z \in G_\epsilon} |z| > 0$.

Suppose now that $\lambda \in D(z, \epsilon p(z))$. By condition (1) of Lemma 1.5.7 we get $p(\lambda) \leq Cp(z) + D$ and by condition (2) of this lemma (possibly for smaller ϵ), $p(z) \leq Cp(\lambda) + D$. Since $\inf_{z \in G} |z| > 0$, we can adjust the constant $C > 0$ so that $p(\lambda) \leq Cp(z)$ and $p(z) \leq Cp(\lambda)$, where the constant does not depend on λ, z or ϵ. Once again we decrease ϵ to have $2\epsilon C \leq 1 - \epsilon$.

Since $|z - \lambda| < \epsilon p(z)$ we obtain

$$n(z, \epsilon p(z)) \leq n(\lambda, 2\epsilon p(z)) \leq n(\lambda, 2\epsilon C p(\lambda)) \leq n(\lambda, (1 - \epsilon)p(\lambda)).$$

Finally, by condition (4) of Proposition 4.1.10

$$n(z, \epsilon p(z)) \leq n(\lambda, (1 - \epsilon)p(\lambda)) \leq Cp(\lambda) \leq C^2 p(z). \qquad \square$$

4.2 Conditions (B)

In the paper [Car58, Theorem 3] L. Carleson proved that every bounded sequence of values $(w_\lambda)_{\lambda \in \Lambda}$ can be interpolated by a function $f \in H^\infty(D(0, 1))$ on a sequence of points $\Lambda \subset D(0, 1)$, i.e.,

$$f(\lambda) = w_\lambda \quad \text{for all } \lambda \in \Lambda$$

if and only if Λ satisfies

$$\prod_{\lambda'\neq\lambda}\left|\frac{\lambda-\lambda'}{1-\lambda\bar{\lambda}'}\right|\geq\delta \tag{4.7}$$

for some $\delta>0$ and all $\lambda\in\Lambda$. Condition (4.7) was later called Carleson condition or uniform separation of Λ. Recall that the Blaschke product of a sequence $\Lambda\subset D(0,1)$ is defined by

$$B(z)=\prod_{\lambda}\frac{|\lambda|}{\lambda}\frac{z-\lambda}{1-z\bar{\lambda}}.$$

This means that in the Carleson condition we estimate reduced Blaschke products

$$B_\lambda(z)=\prod_{\lambda'\neq\lambda}\frac{|\lambda'|}{\lambda'}\frac{z-\lambda'}{1-z\bar{\lambda}'}$$

in points $\lambda\in\Lambda$. Such conditions will also turn out to be useful in the description of interpolating varieties for the spaces $A_{(\omega)}$ and $A_{\{\omega\}}$. However, Blaschke products for the upper and the lower half-planes will be more appropriate. In the following, by $\mathbb{H}_+$ we will denote the upper half-plane and by $\mathbb{H}_-$ the lower half-plane. We will use the notation $\mathbb{H}_*$ if we want to indicate that a statement applies to both $\mathbb{H}_+$ and $\mathbb{H}_-$. Recall that the Blaschke product for the upper half-plane for a variety $X=\{(\lambda,m_\lambda)\}_{\lambda\in\Lambda}\subset\mathbb{H}_+$ is defined by

$$B(z)=\left(\frac{z-i}{z+i}\right)^m\prod_{\lambda\in\Lambda}\left(\frac{|\lambda^2+1|}{\lambda^2+1}\cdot\frac{z-\lambda}{z-\bar{\lambda}}\right)^{m_\lambda}$$

where m is the multiplicity of i in Λ, and the Blaschke condition assuring that this formula defines a holomorphic function is

$$\sum_{\lambda\in\Lambda}m_\lambda\frac{|\operatorname{Im}\lambda|}{1+|\lambda|^2}<\infty. \tag{4.8}$$

We will also use the notation

$$B_{\lambda'}(z)=\left(\frac{z-i}{z+i}\right)^m\prod_{\lambda\in\Lambda\setminus\{\lambda'\}}\left(\frac{|\lambda^2+1|}{\lambda^2+1}\cdot\frac{z-\lambda}{z-\bar{\lambda}}\right)^{m_\lambda}$$

for $\lambda'\in\Lambda$.

Definition 4.2.1. Let ω be a weight and $p(z)=|\operatorname{Im}z|+\omega(z)$. We say that a multiplicity variety $X=\{(\lambda,m_\lambda)\}_{\lambda\in\Lambda}$ satisfies (ω)-*Carleson condition* for $\mathbb{H}_*$ if

$$|B_\lambda(\lambda)|=\prod_{\lambda'\in\Lambda\cap\mathbb{H}_*\setminus\{\lambda\}}\left|\frac{\lambda-\lambda'}{\lambda-\bar{\lambda}'}\right|^{m_{\lambda'}}\geq e^{-Cp(\lambda)}$$

for some $C > 0$ and every $\lambda \in \Lambda \cap \mathbb{H}_*$. We say that it satisfies $\{\omega\}$-*Carleson condition* for $\mathbb{H}_*$ if

$$\exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, C_m > 0\ \forall\, \lambda \in \Lambda \cap \mathbb{H}_*$$
$$|B_\lambda(\lambda)| = \prod_{\lambda' \in \Lambda \cap \mathbb{H}_* \setminus \{\lambda\}} \left| \frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'} \right|^{m_{\lambda'}} \geq e^{-C_m - M|\mathrm{Im}\,\lambda| - \frac{1}{m}\omega(\lambda)}.$$

(ω)-Carleson condition was introduced in the paper [MOCO03] (see Remark 6). Conditions of similar type were used to describe free interpolating sequences for the Nevanlinna class - holomorphic functions on the unit disc admitting a harmonic majorant (see [HMNT04]).

In the paper [MOCO03] the authors introduced also the following condition for a multiplicity variety X,

$$\sup_{x \in \mathbb{R}} \sum_{\lambda:\ |\mathrm{Im}\,\lambda| > \omega(\lambda)} m_\lambda \frac{|\mathrm{Im}\,\lambda|}{|x - \lambda|^2} < \infty. \tag{4.9}$$

In Section 5.2 we will see that it is useful in the description of interpolating varieties in the Beurling case. We will show that it could be replaced by a more general condition. Moreover, the new condition could be transformed to work also in the Roumieu case.

Definition 4.2.2. Let ω be a weight. We say that a multiplicity variety $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ satisfies $(B(\omega))$ if

$$\exists\, C > 0\ \forall\, U \subset \mathbb{H}_+, \mathbb{H}_-\ \forall\, x \in \mathbb{R}$$
$$\sum_{\lambda \in U \cap \Lambda} m_\lambda \frac{|\mathrm{Im}\,\lambda|}{|x - \lambda|^2} \leq C \frac{|\mathrm{Im}\,\lambda_x| + \omega(\lambda_x)}{|\mathrm{Im}\,\lambda_x|}$$

where $\lambda_x \in U \cap \Lambda$ is such that $d(x, \lambda_x) = d(x, U \cap \Lambda)$. We say that it satisfies $(B\{\omega\})$ if

$$\exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, C_m \in \mathbb{R}\ \forall\, U \subset \mathbb{H}_+, \mathbb{H}_-\ \forall\, x \in \mathbb{R}$$
$$\sum_{\lambda \in U \cap \Lambda} m_\lambda \frac{|\mathrm{Im}\,\lambda|}{|x - \lambda|^2} \leq \frac{C_m + M|\mathrm{Im}\,\lambda_x| + \frac{1}{m}\,\omega(\lambda_x)}{|\mathrm{Im}\,\lambda_x|}$$

where $\lambda_x \in U \cap \Lambda$ is such that $d(x, \lambda_x) = d(x, U \cap \Lambda)$.

Conditions $(B(\omega))$ and $(B\{\omega\})$ are strictly related to conditions from Definition 4.2.1. The rest of this section will be mainly devoted to studying this relation. Consequently, we use the convention that all conditions with round brackets are intended for the Beurling case and conditions with curly brackets for the Roumieu case.

In Proposition 4.1.3 we gave a connection between $(\omega)^*$-sparsity and $\{\omega\}$-sparsity. One may expect a similar result for the weighted Carleson conditions and properties $(B(\omega))$ and $(B\{\omega\})$. Propositions 4.2.3 and 4.2.4 provide such a relation.

Proposition 4.2.3. *Let ω be a Roumieu weight. A multiplicity variety $X \subset \mathbb{H}_*$ satisfies $\{\omega\}$-Carleson condition for $\mathbb{H}_*$ if and only if there exists a Beurling weight $\sigma = o(\omega)$ such that X satisfies (σ)-Carleson condition.*

Proof. Assume that X satisfies $\{\omega\}$-Carleson condition for $\mathbb{H}_*$. Let constants C_m be given by the definition of $\{\omega\}$-Carleson condition. Define $q(z) = \inf_{m\in\mathbb{N}}(C_m + \frac{1}{m}\omega(z))$. Then $q(t) = o(\omega(t))$. By Lemma 1.5.16 there exists a weight σ satisfying $q = o(\sigma)$ and $\sigma = o(\omega)$. Then by the assumption for some $M \in \mathbb{N}$ for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that

$$|B_\lambda(\lambda)| = \prod_{\lambda' \in \Lambda\cap\mathbb{H}_*\setminus\{\lambda\}} \left|\frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'}\right|^{m_{\lambda'}} \geq e^{-C_m - M|\operatorname{Im}\lambda| - \frac{1}{m}\omega(\lambda)},$$

for every $\lambda \in \Lambda \cap \mathbb{H}_*$. This yields

$$|B_\lambda(\lambda)| \geq e^{-M|\operatorname{Im}\lambda| - q(\lambda)} \geq e^{-M|\operatorname{Im}\lambda| - \sigma(\lambda)}.$$

Assume now that X satisfies (σ)-Carleson condition for $\mathbb{H}_*$ for some weight $\sigma = o(\omega)$. Then for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that $\sigma(t) \leq C_m + \frac{1}{m}\omega(t)$ for every $t \in [0, \infty)$. Then for every $\lambda \in \Lambda \cap \mathbb{H}_*$

$$|B_\lambda(\lambda)| \geq e^{-C|\operatorname{Im}\lambda| - C\sigma(\lambda)} \geq e^{-CC_m - C|\operatorname{Im}\lambda| - \frac{C}{m}\omega(\lambda)}.$$ □

Proposition 4.2.4. *Let ω be a Roumieu weight. A multiplicity variety X satisfies $(B\{\omega\})$ if and only if there exists a Beurling weight $\sigma = o(\omega)$ such that X satisfies $(B(\sigma))$.*

Proof. The proof goes analogously as the proof of Proposition 4.2.4. □

The next three strictly technical lemmas provide important inequalities between the terms from Definitions 4.2.1 and 4.2.2. The proofs of the first two are based on the proof of [MOCO03, Proposition 5].

Lemma 4.2.5. *For arbitrary points $z, w \in \mathbb{H}_*$*

$$\frac{|\operatorname{Im} z||\operatorname{Im} w|}{|z - \bar{w}|^2} \leq \frac{1}{2}\ln\left|\frac{z - \bar{w}}{z - w}\right|.$$

Proof. For arbitrary $z, w \in \mathbb{C}$ we have

$$|z - \bar{w}|^2 - |z - w|^2 = 4\operatorname{Im} z \operatorname{Im} w.$$

Along with $1 - t \leq \ln t^{-1}$ for $t \in (0, 1)$ this yields

$$\frac{|\operatorname{Im} z||\operatorname{Im} w|}{|z - \bar{w}|^2} = \frac{1}{4}\left(1 - \frac{|z - w|^2}{|z - \bar{w}|^2}\right) \leq \frac{1}{2}\ln\left|\frac{z - \bar{w}}{z - w}\right|.$$ □

Lemma 4.2.6. *Let $U \subset \mathbb{H}_*$ and X be a multiplicity variety. Then for every $x \in \mathbb{R}$*

$$\sum_{\lambda \in \Lambda \cap U} m_\lambda \frac{|\operatorname{Im} \lambda|}{|x-\lambda|^2} \le \frac{m_{\lambda_x}}{|\operatorname{Im} \lambda_x|} + \frac{2}{|\operatorname{Im} \lambda_x|} \sum_{\substack{\lambda \in \Lambda \cap U \\ \lambda \neq \lambda_x}} m_\lambda \ln \left| \frac{\lambda_x - \bar{\lambda}}{\lambda_x - \lambda} \right|$$

where $\lambda_x \in \Lambda \cap U$ is such that $d(x, \lambda_x) = d(x, \Lambda \cap U)$.

Proof. Given $x \in \mathbb{R}$ let $\lambda_x \in U \cap \Lambda$ be the closest element to x. Then for all $\lambda \in U \cap \Lambda$

$$|\lambda_x - \bar{\lambda}| \le |\lambda_x - x| + |x - \bar{\lambda}| \le |\lambda - x| + |x - \bar{\lambda}| = 2|x - \lambda|.$$

Therefore

$$\sum_{\lambda \in U \cap \Lambda} m_\lambda \frac{|\operatorname{Im} \lambda|}{|x-\lambda|^2} \le 4 \sum_{\lambda \in U \cap \Lambda} m_\lambda \frac{|\operatorname{Im} \lambda|}{|\lambda_x - \bar{\lambda}|^2}.$$

Using Lemma 4.2.5, we obtain

$$\begin{aligned} \sum_{\lambda \in U \cap \Lambda} m_\lambda \frac{|\operatorname{Im} \lambda|}{|x-\lambda|^2} &\le 4 \sum_{\lambda \in U \cap \Lambda} m_\lambda \frac{|\operatorname{Im} \lambda|}{|\lambda_x - \bar{\lambda}|^2} \\ &= \frac{m_{\lambda_x}}{|\operatorname{Im} \lambda_x|} + 4 \sum_{\substack{\lambda \in U \cap \Lambda \\ \lambda \neq \lambda_x}} m_\lambda \frac{|\operatorname{Im} \lambda|}{|\lambda_x - \bar{\lambda}|^2} \\ &\le \frac{m_{\lambda_x}}{|\operatorname{Im} \lambda_x|} + \frac{2}{|\operatorname{Im} \lambda_x|} \sum_{\substack{\lambda \in U \cap \Lambda \\ \lambda \neq \lambda_x}} m_\lambda \ln \left| \frac{\lambda_x - \bar{\lambda}}{\lambda_x - \lambda} \right|. \end{aligned}$$

□

Lemma 4.2.7. *Let $U \in \mathbb{H}_*$, X be a multiplicity variety contained in $\mathbb{H}_*$ and $\lambda \in \Lambda$ be fixed. Let η be any constant satisfying $0 < \eta < 1$. Denote $\Lambda' = \Lambda \cap U \cap \{z \in \mathbb{C} |\ \left|\frac{z-\lambda}{z-\bar{\lambda}}\right| \ge \eta\}$. Then*

$$\sum_{\lambda' \in \Lambda'} m_{\lambda'} \ln \left| \frac{\lambda - \bar{\lambda}'}{\lambda - \lambda'} \right| \le C |\operatorname{Im} \lambda| \sum_{\lambda' \in \Lambda'} m_{\lambda'} \frac{|\operatorname{Im} \lambda'|}{|\operatorname{Re} \lambda - \bar{\lambda}'|^2}$$

for some $C > 0$ which does not depend on λ.

Remark 4.2.8. The expression

$$\Lambda' = \Lambda \cap U \cap \left\{ z \in \mathbb{C} \middle|\ \left| \frac{z-\lambda}{z-\bar{\lambda}} \right| \ge \eta \right\}$$

just means that we take only these points of Λ which belong to U and do not belong to the pseudo-hyperbolic disc

$$D_H(\lambda, \eta) := \left\{ z \in \mathbb{C} \middle|\ \left| \frac{z-\lambda}{z-\bar{\lambda}} \right| < \eta \right\} = D\left(\operatorname{Re} \lambda + i \frac{1+\eta^2}{1-\eta^2} \operatorname{Im} \lambda, \frac{2\eta}{1-\eta^2} |\operatorname{Im} \lambda| \right).$$

Proof. We will use the inequality $\ln \frac{1}{t} \leq C(1-t)$, which is true for some constant $C > 0$ and all t satisfying $\eta < t < 1$. We obtain

$$
\begin{aligned}
\sum_{\lambda' \in \Lambda'} m_{\lambda'} \ln \left| \frac{\lambda - \bar{\lambda}'}{\lambda - \lambda'} \right| &\leq C \sum_{\lambda' \in \Lambda'} m_{\lambda'} \frac{|\lambda - \bar{\lambda}'| - |\lambda - \lambda'|}{|\lambda - \bar{\lambda}'|} \\
&= C \sum_{\lambda' \in \Lambda'} m_{\lambda'} \frac{|\lambda - \bar{\lambda}'|^2 - |\lambda - \lambda'||\lambda - \bar{\lambda}'|}{|\lambda - \bar{\lambda}'|^2} \\
&\leq C \sum_{\lambda' \in \Lambda'} m_{\lambda'} \frac{|\lambda - \bar{\lambda}'|^2 - |\lambda - \lambda'|^2}{|\lambda - \bar{\lambda}'|^2} \\
&= C \sum_{\lambda' \in \Lambda'} m_{\lambda'} \frac{4|\operatorname{Im} \lambda||\operatorname{Im} \lambda'|}{|\lambda - \bar{\lambda}'|^2} \\
&\leq 4C|\operatorname{Im} \lambda| \sum_{\lambda' \in \Lambda'} m_{\lambda'} \frac{|\operatorname{Im} \lambda'|}{|\operatorname{Re} \lambda - \bar{\lambda}'|^2}.
\end{aligned}
$$

□

In the next proposition we show that all the conditions we have introduced in Definitions 4.2.1 and 4.2.2 imply the Blaschke conditions (4.8) for the lower and the upper half-plane.

Proposition 4.2.9. *Assume that a multiplicity variety $X \subset \mathbb{H}_*$ satisfies one of the following conditions:*

(1) X satisfies (ω)-Carleson condition for $\mathbb{H}_*$,

(2) X satisfies $\{\omega\}$-Carleson condition for $\mathbb{H}_*$,

(3) X satisfies $(B(\omega))$,

(4) X satisfies $(B\{\omega\})$.

Then

$$
\sum_{\lambda \in \Lambda} m_\lambda \frac{|\operatorname{Im} \lambda|}{1 + |\lambda|^2} < \infty.
$$

Proof. In cases (1) and (2) we have that for some λ

$$
\prod_{\lambda' \in \Lambda \setminus \{\lambda\}} \left| \frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'} \right|^{m_{\lambda'}} > 0.
$$

This yields

$$
\sum_{\lambda' \in \Lambda \setminus \{\lambda\}} m_{\lambda'} \ln \left| \frac{\lambda - \bar{\lambda}'}{\lambda - \lambda'} \right| < \infty.
$$

It holds

$$\frac{|\lambda-\bar{\lambda}'|^2}{1+|\lambda'|^2}\to 1$$

when $|\lambda'|\to\infty$. Hence for $|\lambda'|\geq R$ with R big enough $|\lambda-\bar{\lambda}'|^2\leq 2(1+|\lambda'|^2)$. By the use of Lemma 4.2.5 we obtain then

$$\sum_{|\lambda'|\geq R} m_{\lambda'}\frac{|\operatorname{Im}\lambda'|}{1+|\lambda'|^2}\leq 2\sum_{|\lambda'|\geq R} m_{\lambda'}\frac{|\operatorname{Im}\lambda'|}{|\lambda-\bar{\lambda}'|^2}\leq \frac{1}{|\operatorname{Im}\lambda|}\sum_{|\lambda'|\geq R} m_{\lambda'}\ln\left|\frac{\lambda-\bar{\lambda}'}{\lambda-\lambda'}\right|<\infty.$$

There are only finitely many λ' with $\lambda'<R$, hence the proof for cases (1) and (2) is complete.

The proof for cases (3) and (4) is immediate, since we can take $U=\mathbb{H}_*$ and $x=0$. □

Now, we turn to the crucial part of this section, to the relation between the weighted Carleson conditions and conditions (B). The most important are Corollary 4.2.13, and Propositions 4.2.14, 4.2.16.

Proposition 4.2.10. *Let ω be a weight and $p(z)=|\operatorname{Im}z|+\omega(z)$. Assume that $X\subset\mathbb{H}_*$ is a multiplicity variety satisfying*

$$\exists\, C>0\ \forall\,\lambda\in\Lambda: \quad m_\lambda\leq Cp(\lambda).$$

If X satisfies (ω)-Carleson condition for $\mathbb{H}_*$ then it satisfies $(B(\omega))$.

Proof. By (ω)-Carleson condition we have

$$\sum_{\lambda\in\Lambda\setminus\{\lambda'\}} m_{\lambda'}\ln\left|\frac{\lambda'-\bar{\lambda}}{\lambda'-\lambda}\right|\leq Cp(\lambda') \tag{4.10}$$

for some $C>0$ and every $\lambda'\in\Lambda$. Let $U\subset\mathbb{H}_*$ be given. By Lemma 4.2.6 for every $x\in\mathbb{R}$

$$\sum_{\lambda\in\Lambda\cap U} m_\lambda\frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2}\leq\frac{m_{\lambda_x}}{|\operatorname{Im}\lambda_x|}+\frac{2}{|\operatorname{Im}\lambda_x|}\sum_{\substack{\lambda\in\Lambda\cap U\\ \lambda\neq\lambda_x}} m_\lambda\ln\left|\frac{\lambda_x-\bar{\lambda}}{\lambda_x-\lambda}\right|$$

where $\lambda_x\in\Lambda\cap U$ is such that $d(x,\lambda_x)=d(x,\Lambda\cap U)$. Using (4.10) and the estimate for multiplicities, we obtain

$$\sum_{\lambda\in\Lambda\cap U} m_\lambda\frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2}\leq C\frac{|\operatorname{Im}\lambda_x|+\omega(|\lambda_x|)}{|\operatorname{Im}\lambda_x|}$$

for $x\in\mathbb{R}$ and some $C>0$. □

Proposition 4.2.11. *Let ω be a weight and $p(z) = |\operatorname{Im} z| + \omega(z)$. Assume that $X \subset \mathbb{H}_*$ is a multiplicity variety satisfying*

$$\exists\, C > 0 \ \forall\, \lambda \in \Lambda : \quad N(\lambda, 2p(\lambda), X) \le Cp(\lambda). \tag{4.11}$$

If X satisfies $(B(\omega))$ then it satisfies (ω)-Carleson condition for $\mathbb{H}_*$.

Remark 4.2.12. $(\omega)^*$-sparsity implies (4.11), and (4.11) implies (ω)-sparsity.

Proof. We have to show that

$$\sum_{\lambda' \in \Lambda \setminus \{\lambda\}} m_{\lambda'} \ln\left|\frac{\lambda - \bar\lambda'}{\lambda - \lambda'}\right| \le Cp(\lambda)$$

for some $C > 0$ and all $\lambda \in \Lambda$. Let $\lambda \in \Lambda$ be given. Denote $U = \mathbb{H}_* \setminus \bar D(\operatorname{Re}\lambda, |\operatorname{Im}\lambda|)$ and $\Lambda' = \Lambda \cap U$. For $\eta > 0$ small enough $D_H(\lambda, \eta) \cap \Lambda = \{\lambda\}$ hence

$$(U \setminus D_H(\lambda, \eta)) \cap \Lambda = U \cap \Lambda = \Lambda'.$$

Then by Lemma 4.2.7

$$\sum_{\lambda' \in \Lambda'} m_{\lambda'} \ln\left|\frac{\lambda - \bar\lambda'}{\lambda - \lambda'}\right| \le C|\operatorname{Im}\lambda| \sum_{\lambda' \in \Lambda'} m_{\lambda'} \frac{|\operatorname{Im}\lambda'|}{|\operatorname{Re}\lambda - \bar\lambda'|^2}$$

for some $C > 0$, which does not depend on λ. Point λ is the closest point of $\Lambda' \cup \{\lambda\}$ to $\operatorname{Re}\lambda$, thus using $(B(\omega))$

$$\begin{aligned}\sum_{\lambda' \in \Lambda'} m_{\lambda'} \ln\left|\frac{\lambda - \bar\lambda'}{\lambda - \lambda'}\right| &\le C|\operatorname{Im}\lambda| \sum_{\lambda' \in \Lambda' \cup \{\lambda\}} m_{\lambda'} \frac{|\operatorname{Im}\lambda'|}{|\operatorname{Re}\lambda - \bar\lambda'|^2} \\ &\le D|\operatorname{Im}\lambda| \frac{|\operatorname{Im}\lambda| + \omega(\lambda)}{|\operatorname{Im}\lambda|} = Dp(\lambda)\end{aligned}$$

for some $D > 0$, which does not depend on λ.

Let now $\Lambda'' = (\Lambda \setminus \{\lambda\}) \cap \bar D(\operatorname{Re}\lambda, |\operatorname{Im}\lambda|)$. It remains to prove that

$$\sum_{\lambda' \in \Lambda''} m_{\lambda'} \ln\left|\frac{\lambda - \bar\lambda'}{\lambda - \lambda'}\right| \le Cp(\lambda) \tag{4.12}$$

for some $C > 0$ independent of λ. We have

$$\sum_{\lambda' \in \Lambda''} m_{\lambda'} \ln\left|\frac{\lambda - \bar\lambda'}{\lambda - \lambda'}\right| \le \sum_{\lambda' \in \Lambda''} m_{\lambda'} \ln \frac{2|\operatorname{Im}\lambda|}{|\lambda - \lambda'|} \le \sum_{\lambda' \in \Lambda''} m_{\lambda'} \ln \frac{2p(\lambda)}{|\lambda - \lambda'|}.$$

It holds $\Lambda'' \subset \Lambda''' := (\Lambda \setminus \{\lambda\}) \cap \bar D(\lambda, 2p(\lambda))$. Using Proposition 1.10.2 we obtain

$$\sum_{\lambda' \in \Lambda''} m_{\lambda'} \ln\left|\frac{\lambda - \bar\lambda'}{\lambda - \lambda'}\right| \le \sum_{\lambda' \in \Lambda'''} m_{\lambda'} \ln \frac{2p(\lambda)}{|\lambda - \lambda'|} \le N(\lambda, \max(1, 2p(\lambda))).$$

Then, by (4.11), for some $C > 0$

$$\sum_{\lambda' \in \Lambda''} m_{\lambda'} \ln \left| \frac{\lambda - \bar{\lambda}'}{\lambda - \lambda'} \right| \leq N(\lambda, \max(1, 2p(\lambda))) \leq C \max(1, 2p(\lambda)),$$

provided λ satisfies $p(\lambda) \geq \frac{1}{2}$. Furthermore, there are only finitely many $\lambda \in \Lambda$ such that $p(\lambda) < \frac{1}{2}$. Hence

$$\sup_{\lambda \in \Lambda \colon p(\lambda) < \frac{1}{2}} N(\lambda, \max(1, 2p(\lambda))) < \infty.$$

Since $0 \notin \Lambda$, we can use Lemma 4.1.7 to complete the proof. □

We can join Propositions 4.2.10 and 4.2.11 together to obtain the following corollary.

Corollary 4.2.13. *Let ω be a weight and $X \subset \mathbb{H}_*$ an $(\omega)^*$-sparse multiplicity variety. Then X satisfies (ω)-Carleson condition for $\mathbb{H}_*$ if and only if X satisfies $(B(\omega))$. In particular, if $X \subset \mathbb{C}$ is an $(\omega)^*$-sparse multiplicity variety then X satisfies (ω)-Carleson conditions for both $\mathbb{H}_+$ and $\mathbb{H}_-$ if and only if X satisfies $(B(\omega))$.*

Proof. $(\Rightarrow)$ By Proposition 4.1.18 the assumption on multiplicities required in Proposition 4.2.10 is satisfied, and we can use this Proposition to obtain the assertion.

$(\Leftarrow)$ Condition (4.11) in Proposition 4.2.11 is satisfied, since X is $(\omega)^*$-sparse. Therefore we may use this Proposition to obtain the assertion. □

Applying Propositions 4.2.3 and 4.2.4 to the foregoing result we can obtain an analogous equivalency for the Roumieu case.

Proposition 4.2.14. *Let ω be a Roumieu weight and $X \subset \mathbb{H}_*$ an $\{\omega\}$-sparse multiplicity variety. Then X satisfies $\{\omega\}$-Carleson condition if and only if X satisfies $(B\{\omega\})$. In particular, if $X \subset \mathbb{C}$ is an $\{\omega\}^*$ sparse multiplicity variety then X satisfies $\{\omega\}$-Carleson conditions for both $\mathbb{H}_+$ and $\mathbb{H}_-$ if and only if X satisfies $(B\{\omega\})$.*

Proof. By Proposition 4.1.3 there exists a Beurling weight $\sigma = o(\omega)$ such that X is $(\sigma)^*$-sparse.

$(\Rightarrow)$ Assume that X satisfies $\{\omega\}$-Carleson condition. By Proposition 4.2.3 there exists a Beurling weight $\nu = o(\omega)$ such that X satisfies (ν)-Carleson condition. Then $\mu = \max(\sigma, \nu)$ is again a Beurling weight satisfying $\mu = o(\omega)$. Moreover, X is $(\mu)^*$-sparse and satisfies (μ)-Carleson condition. Corollary 4.2.13 gives then $(B(\mu))$ for X and finally Proposition 4.2.4 asserts that X satisfies $(B\{\omega\})$.

$(\Leftarrow)$ The proof of this implication is analogous. □

In Propositions 4.2.15 and 4.2.16 we show how condition (4.9) is contained in the theory we have developed in this section.

Proposition 4.2.15. *Let ω be a weight and X be a multiplicity variety contained in the set $\{z \in \mathbb{C} |\ |\operatorname{Im} z| > h\omega(z)\}$ for some $h > 0$. Then X satisfies*

$$\sup_{x\in\mathbb{R}} \sum_{\lambda:\ |\operatorname{Im}\lambda| > h\omega(\lambda)} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} < \infty$$

if and only if X satisfies $(B(\omega))$.

Proof. $(\Rightarrow)$ Denote

$$M := \sup_{x\in\mathbb{R}} \sum_{\lambda:\ |\operatorname{Im}\lambda| > h\omega(\lambda)} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2}.$$

Let $U \subset \mathbb{H}_*$ be arbitrary. Then for every $x \in \mathbb{R}$

$$\sum_{\lambda\in\Lambda\cap U} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} \le \sup_{x\in\mathbb{R}} \sum_{\lambda:\ |\operatorname{Im}\lambda| > h\omega(\lambda)} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} = M \le M \frac{|\operatorname{Im}\lambda_x| + \omega(\lambda_x)}{|\operatorname{Im}\lambda_x|}$$

where $\lambda_x \in \Lambda \cap U$ is such that $d(x, \lambda_x) = d(x, \Lambda \cap U)$. Therefore X satisfies $(B(\omega))$.

$(\Leftarrow)$ Using $(B(\omega))$ for $U = \{z \in \mathbb{C} |\ |\operatorname{Im} z| > h\omega(z)\} \cap \mathbb{H}_+$ we obtain for some $M > 0$

$$\sup_{x\in\mathbb{R}} \sum_{\lambda:\ \operatorname{Im}\lambda > h\omega(\lambda)} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} \le \sup_{x\in\mathbb{R}} M \frac{|\operatorname{Im}\lambda_x| + \omega(\lambda_x)}{|\operatorname{Im}\lambda_x|} \le M\Big(1 + \frac{1}{h}\Big) < \infty.$$

We proceed analogously for $\mathbb{H}_-$. ☐

Proposition 4.2.16. *Let ω be a weight, $p(z) = |\operatorname{Im} z| + \omega(z)$, and X a multiplicity variety contained in the set $\{z \in \mathbb{C} |\ |\operatorname{Im} z| > h\omega(z)\} \cap \mathbb{H}_*$ for some $h > 0$. Further, assume that X satisfies $m_\lambda \le Cp(\lambda)$ for some $C > 0$ and all $\lambda \in \Lambda$. Then X satisfies*

$$\sup_{x\in\mathbb{R}} \sum_{\lambda:\ |\operatorname{Im}\lambda| > h\omega(\lambda)} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} < \infty \tag{4.13}$$

if and only if X satisfies (ω)-Carleson condition for $\mathbb{H}_*$.

Proof. $(\Rightarrow)$ We have to show that

$$\sum_{\lambda'\in\Lambda\setminus\{\lambda\}} m_{\lambda'} \ln\left|\frac{\lambda-\bar{\lambda}'}{\lambda-\lambda'}\right| \le Cp(\lambda)$$

for some $C > 0$ and all $\lambda \in \Lambda$. Let $\lambda \in \Lambda$ be given. Denote $U = \{z \in \mathbb{C} |\ |\operatorname{Im} z| > h\omega(z)\} \cap \mathbb{H}_*$ and $\Lambda' = \Lambda \setminus \{\lambda\}$. For $\eta > 0$ small enough $D_H(\lambda, \eta) \cap \Lambda = \{\lambda\}$ hence

$$(U \setminus D_H(\lambda, \eta)) \cap \Lambda = \Lambda \setminus \{\lambda\} = \Lambda'.$$

Therefore, by Lemma 4.2.7, for some $C > 0$

$$\sum_{\lambda'\in\Lambda'} m_{\lambda'} \ln\left|\frac{\lambda-\bar{\lambda}'}{\lambda-\lambda'}\right| \leq C|\operatorname{Im}\lambda| \sum_{\lambda'\in\Lambda'} m_{\lambda'} \frac{|\operatorname{Im}\lambda'|}{|\operatorname{Re}\lambda-\bar{\lambda}'|^2}$$
$$\leq C|\operatorname{Im}\lambda| \sup_{x\in\mathbb{R}} \sum_{\lambda'\in\Lambda'} m_{\lambda'} \frac{|\operatorname{Im}\lambda'|}{|x-\bar{\lambda}'|^2},$$

which, by (4.13), is smaller than $Dp(\lambda)$ for some $D > 0$ independent of λ.

($\Leftarrow$) Because of the assumption on multiplicities we may use Proposition 4.2.10 to obtain that X satisfies $(B(\omega))$. Then we can complete the proof with the use of Proposition 4.2.15. □

Now we are going to study the stability of conditions (B) and the weighted Carleson conditions under translation, perturbation, diffusion and rotation (see definitions in Section 2.2). As these properties are strictly bound to half-planes we should restrict all operations to those which keep a variety inside a half-plane. Indeed, in the general case one could consider varieties with many points on the real line. Every such a variety satisfies conditions (B) and the weighted Carleson conditions. But then any operations moving these points into one half-plane will break these properties down. But even points near the real line can cause a problem as is shown in the following examples.

Example 4.2.17. Let $X = \{(\lambda_n, 1) | \, n \in \mathbb{N}\}$ for some points $\lambda_n \in \mathbb{H}_*$. Then for $x \in \mathbb{R}$

$$\sum_{n\in\mathbb{N}} m_{\lambda_n} \frac{|\operatorname{Im}\lambda_n|}{|x-\lambda_n|^2} = \sum_{n\in\mathbb{N}} \frac{|\operatorname{Im}\lambda_n|}{(x-\operatorname{Re}\lambda_n)^2 + |\operatorname{Im}\lambda_n|^2}.$$

Now we can choose $\operatorname{Im}\lambda_n = \frac{1}{n}$ and $\operatorname{Re}\lambda_n$ in such a way that the series

$$\sum_{n\in\mathbb{N}} \frac{1}{n((x-\operatorname{Re}\lambda_n)^2 + \frac{1}{n^2})} \tag{4.14}$$

be bounded by $M + \frac{M}{|\operatorname{Im}\lambda_x|}$ for $x \in \mathbb{R}$, and the series

$$\sum_{n\in\mathbb{N}} \frac{1}{(x-\operatorname{Re}\lambda_n)^2}$$

be divergent for some $x \in \mathbb{R}$. We could take for instance $\operatorname{Re}\lambda_n = \sqrt{n}$. One could show then by the use of the integral test for convergence that the series (4.14) is indeed bounded by $M + \frac{M}{|\operatorname{Im}\lambda_x|}$ for $x \in \mathbb{R}$. We omit here technical details. Then X satisfies $(B(\omega))$ and $(B\{\omega\})$ for any weight ω. On the other hand,

$$\sum_{n\in\mathbb{N}} \frac{|\operatorname{Im}(\lambda_n+i)|}{(\operatorname{Re}\lambda_n)^2 + |(\operatorname{Im}\lambda_n+i)|^2} \geq \sum_{n\in\mathbb{N}} \frac{1}{n+(1+\frac{1}{n})^2} = \infty,$$

which shows that $X+i$ satisfies neither $(B(\omega))$ nor $(B\{\omega\})$. We see that X and $X+i$ are contained in $\mathbb{H}_+$, hence even a translation leaving a variety in the same half-plane can break conditions (B) down. Furthermore, $X+i$ may be considered as a bounded perturbation of X. This shows that conditions (B) are also not invariant under bounded perturbations. Finally, $i \cdot X$ does not satisfy $(B(\omega))$ nor $(B\{\omega\})$ for any weight ω, hence also rotation do not preserve conditions (B).

For rotation we can give even a simpler example.

Example 4.2.18. Let $X = \{(n+i, 1) |\, n \in \mathbb{N}\}$. Then by the integral test for convergence

$$\sum_{n\in\mathbb{N}} \frac{1}{(x-n)^2+1} \leq C$$

for some $C > 0$ and all $x \in \mathbb{R}$. This implies that X satisfies $(B(\omega))$ and $(B\{\omega\})$ for every weight ω. But for $i \cdot X = \{(-1+ni, 1) |\, n \in \mathbb{N}\}$ the series in conditions (B) diverges for every $x \in \mathbb{R}$ and sets U big enough.

The situation is different with diffusion. We do not give any precise results, only an idea. Looking at the form of the series

$$\sum_{\lambda\in\Lambda\cap U} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} \tag{4.15}$$

we see that if every point λ with multiplicity m_λ is divided into m_λ points, say $\eta_{\lambda,i}$, close to λ, the term $\frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2}$ will be proportional to $\frac{|\operatorname{Im}\eta_{\lambda,i}|}{|x-\eta_{\lambda,i}|^2}$ and the series (4.15) will be proportional to the series made of points $\eta_{\lambda,i}$ with multiplicities equal 1, whenever all the proportions are made uniformly with respect to $\lambda \in \Lambda$ and $x \in \mathbb{R}$.

Similarly, in general, the weighted Carleson conditions are not invariant in general under translation, perturbation and rotation, but behave well under small diffusion.

We end this section with a short study of the relation between sparsity and the weighted Carleson conditions. There is no easy dependence. However, looking at the form of the function N given in Proposition 1.10.2

$$N(\lambda, r) = \sum_{\lambda' :\, 0<|\lambda-\lambda'|\leq r} m_{\lambda'} \ln \frac{r}{|\lambda-\lambda'|} + m_\lambda \ln r$$

and the series obtained from the weighted Carleson conditions

$$\sum_{\lambda'\neq\lambda} m_{\lambda'} \ln \left|\frac{\lambda-\bar{\lambda}'}{\lambda-\lambda'}\right|$$

one can see a promising similarity. These formulae suggest that up to some extent they contain the same information about a variety. We will present here several facts shedding some light on this problem. In Chapter 5 our knowledge will go far beyond that.

Proposition 4.2.19. *Let $X \subset \mathbb{H}_*$ be a multiplicity variety. Then for every $\lambda \in \Lambda$*

$$\sum_{\lambda' :\, 0<|\lambda-\lambda'|\le r} m_{\lambda'} \ln\left|\frac{\lambda-\bar{\lambda}'}{\lambda-\lambda'}\right| \le N(\lambda, 3r)$$

for $r \ge |\operatorname{Im}\lambda|$.

Proof. It is enough to see that for λ, λ' satisfying $0 < |\lambda - \lambda'| \le r$ it holds

$$|\lambda - \bar{\lambda}'| \le |\lambda - \lambda'| + 2|\operatorname{Im}\lambda| \le 3r.$$

Then

$$\begin{aligned}\sum_{\lambda' :\, 0<|\lambda-\lambda'|\le r} m_{\lambda'} \ln\left|\frac{\lambda-\bar{\lambda}'}{\lambda-\lambda'}\right| &\le \sum_{\lambda' :\, 0<|\lambda-\lambda'|\le r} m_{\lambda'} \ln\frac{3r}{|\lambda-\lambda'|} \\ &\le \sum_{\lambda' :\, 0<|\lambda-\lambda'|\le 3r} m_{\lambda'} \ln\frac{3r}{|\lambda-\lambda'|} \le N(\lambda, 3r).\end{aligned}$$

□

Proposition 4.2.20. *Let $X \subset \mathbb{H}_*$ be a multiplicity variety. Then for every $\lambda \in \Lambda$*

$$\sum_{\lambda' :\, 0<|\lambda-\lambda'|\le r} m_{\lambda'} \ln\frac{r}{|\lambda-\lambda'|} \le \sum_{\lambda' :\, 0<|\lambda-\lambda'|\le r} m_{\lambda'} \ln\left|\frac{\lambda-\bar{\lambda}'}{\lambda-\lambda'}\right|$$

for $r \le |\operatorname{Im}\lambda|$.

Proof. This is trivial, since $r \le |\operatorname{Im}\lambda| \le |\lambda - \bar{\lambda}'|$ as λ and $\bar{\lambda}'$ are in different half-planes. □

Corollary 4.2.21. *Let ω be a weight and let $X \subset \{z \in \mathbb{C}|\ |\operatorname{Im} z| \ge 1\} \cap \mathbb{H}_*$ be a multiplicity variety satisfying (ω)-Carleson condition for $\mathbb{H}_*$. Then X is (ω)-separated.*

Proof. By the previous result for every $\lambda \in \Lambda$ we have

$$N(\lambda, 1) = \sum_{\lambda' :\, 0<|\lambda-\lambda'|\le 1} m_{\lambda'} \ln\frac{r}{|\lambda-\lambda'|} \le \sum_{\lambda' :\, 0<|\lambda-\lambda'|\le 1} m_{\lambda'} \ln\left|\frac{\lambda-\bar{\lambda}'}{\lambda-\lambda'}\right|.$$

The use of (ω)-Carleson condition and Proposition 4.1.8 gives the assertion. □

As we have seen in Proposition 4.2.9 conditions (B) and the weighted Carleson conditions imply Blaschke condition for the upper and the lower half-plane. But there are sparse varieties which are not Blaschke. Thus sparsity cannot imply conditions (B) or the weighted Carleson conditions in general.

Example 4.2.22. Let $X = \{(ni, 1)|\ n \in \mathbb{N}\}$. By Proposition 4.1.12 this variety is (ω)-sparse for any weight ω. But it does not satisfy the Blaschke condition for the upper half-plane.

Every variety contained in the real line satisfies $(B(\omega))$, $(B\{\omega\})$ and the weighted Carleson conditions for any weight ω. This way we can give trivial examples of not sparse varieties satisfying conditions (B). The following example is less trivial.

Example 4.2.23. Let $\omega(t)=\sqrt{t}$ and $X=\{(2^n+i,2^{\frac{n}{2}})|\ n\in\mathbb{N}\}$. Denote $\lambda_n=2^n+i$, $m_n=2^{\frac{n}{2}}$. Then for $k\in\mathbb{N}$

$$\begin{aligned}\sum_{n\neq k} m_n \ln\left|\frac{\lambda_k-\bar{\lambda}_n}{\lambda_k-\lambda_n}\right| &= \sum_{n\neq k} 2^{\frac{n}{2}}\ln\left|\frac{2^k-2^n+2i}{2^k-2^n}\right| \leq \sum_{n\neq k} 2^{\frac{n}{2}}\ln\left(1+\frac{2}{|2^k-2^n|}\right)\\ &\leq 2\sum_{n\neq k}\frac{2^{\frac{n}{2}}}{|2^k-2^n|} = 2\sum_{n<k}\frac{2^{\frac{n}{2}}}{2^k-2^n}+2\sum_{n>k}\frac{2^{\frac{n}{2}}}{2^n-2^k}\\ &\leq 2\sum_{n<k}\frac{2^{\frac{n}{2}}}{2^n}+2\sum_{n>k}\frac{2^{\frac{n}{2}}}{2^n(1-\frac{1}{2^{n-k}})} \leq 2\sum_{n<k}\frac{2^{\frac{n}{2}}}{2^n}+4\sum_{n>k}\frac{2^{\frac{n}{2}}}{2^n}\\ &\leq 4\sum_{n\in\mathbb{N}}\left(\frac{1}{\sqrt{2}}\right)^n = \frac{4}{\sqrt{2}-1}\leq C\omega(\lambda_k)\end{aligned}$$

for $C>0$ big enough. Hence X satisfies (ω)-Carleson condition ($\{\omega\}$-Carleson condition as well), but it does not satisfy the growth condition for multiplicities given by Proposition 4.1.18 (or Proposition 4.1.19 for the Roumieu case), and therefore it is not (ω)-sparse.

Taking into account Proposition 4.2.10 we see that X satisfies also $(B(\omega))$.

Chapter 5

Geometric description of interpolating varieties

5.1 Necessary conditions

In this chapter we turn to the most desired form of description of interpolating varieties - in terms of geometric conditions. We will start with the necessity of conditions introduced in the Chapter 4. Then, in the next section, we will deal with their sufficiency. In this section we will show that every interpolating variety for $A_{(\omega)}$ is $(\omega)^*$-sparse and every interpolating variety for $A_{\{\omega\}}$ is $\{\omega\}$-sparse. These results hold for all weights. Furthermore, we will show that for non-quasianalytic weights every interpolating variety for $A_{(\omega)}$ satisfies (ω)-Carleson condition and $(B(\omega))$, and every interpolating variety for $A_{\{\omega\}}$ satisfies $\{\omega\}$-Carleson condition and $(B\{\omega\})$.

The proof of part (1) of the following theorem is due to W. A. Squires [Squ83, Theorem 1].

Theorem 5.1.1. *Let ω be a weight and X a multiplicity variety.*

(1) If X is interpolating for $A_{(\omega)}$ then X is $(\omega)^*$-sparse.

(2) If X is interpolating for $A_{\{\omega\}}$ then X is $\{\omega\}$-sparse.

Proof. We will treat both cases simultaneously. Let $f_\lambda := f_{\lambda,m_\lambda-1}$ be the functions given by Lemma 2.1.3 in the Beurling case and by Lemma 2.1.4 in the Roumieu case. The functions f_λ interpolate the sequences $(\delta_{\eta,\lambda}\delta_{l,m_\lambda-1})_{\eta\in\Lambda,0\le l<m_\eta}$ and admit uniform estimates on the norms which we will recall later in the proof. For every $\lambda\in\Lambda$ define

$$g_\lambda(z) = \frac{f_\lambda(z)}{(z-\lambda)^{m_\lambda-1}}.$$

Then $g_\lambda(\lambda) = 1$. By the Jensen's formula [Rud87, Theorem 15.18] applied to g_λ in the disc $D(\lambda, r)$ we obtain

$$\sum_{i=1}^{N} \ln\Big(\frac{r}{|\alpha_i - \lambda|}\Big) + m_\lambda \ln r = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln|f_\lambda(\lambda + re^{i\theta})| \, \mathrm{d}\theta + \ln r. \tag{5.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_N$ are the zeroes of g_λ which lie in $D(\lambda, r)$ counted according to their multiplicities. But every point λ' of Λ not equal λ contained in $D(\lambda, r)$ is a zero of g_λ with multiplicity at least $m_{\lambda'}$, hence

$$\begin{aligned}
\sum_{i=1}^{N} \ln\Big(\frac{r}{|\alpha_i - \lambda|}\Big) + m_\lambda \ln r &= \int_0^r \frac{n(\lambda, t, Z(g_\lambda))}{t} \, \mathrm{d}t + m_\lambda \ln r \\
&\geq \int_0^r \frac{n(\lambda, t, X) - n(\lambda, 0, X)}{t} \, \mathrm{d}t + m_\lambda \ln r \\
&= N(\lambda, r, X).
\end{aligned}$$

From (5.1) we get then

$$N(\lambda, r, X) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln|f_\lambda(\lambda + re^{i\theta})| \, \mathrm{d}\theta + \ln r.$$

In the Beurling case by the assumption on the norms of f_λ there exists $M \in \mathbb{N}$ and $C > 0$ such that $\|f_\lambda\|_M \leq C$, hence

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln|f_\lambda(\lambda + re^{i\theta})| \, \mathrm{d}\theta &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln|Ce^{M|\mathrm{Im}(\lambda + re^{i\theta})| + M\omega(\lambda + re^{i\theta})}| \, \mathrm{d}\theta \\
&\leq \ln C + M \sup_{z \in \partial D(\lambda, r)} (|\mathrm{Im}\, z| + \omega(z)).
\end{aligned}$$

Therefore, using Lemma 1.5.6 we get

$$N(\lambda, r, X) \leq \ln C + D + M|\mathrm{Im}\, \lambda| + DM\omega(\lambda) + \ln r + Mr + DMr$$

for some $D > 0$, all $\lambda \in \Lambda$ and $r > 0$. Then by Lemma 4.1.7

$$N(\lambda, |\mathrm{Im}\, \lambda| + \omega(\lambda), X) \leq A + A|\mathrm{Im}\, \lambda| + A\omega(\lambda) \leq B(|\mathrm{Im}\, \lambda| + \omega(\lambda))$$

for some $A, B > 0$ and every $\lambda \in \Lambda \setminus \{0\}$. If $\lambda = 0 \in \Lambda$ then

$$N(\lambda, |\mathrm{Im}\, \lambda| + \omega(\lambda), X) = N(\lambda, 0, X) = -\infty < B(|\mathrm{Im}\, \lambda| + \omega(\lambda)).$$

In the Roumieu case the proof goes similarly. By the assumption on the norms of f_λ there exists $M \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $C_m > 0$ such that $\|f_\lambda\|_{M,m} \le C_m$, hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|f_\lambda(\lambda + re^{i\theta})|\, \mathrm{d}\theta \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|C_m e^{M|\mathrm{Im}(\lambda+re^{i\theta})|+\frac{1}{m}\omega(\lambda+re^{i\theta})}|\, \mathrm{d}\theta$$

$$\le \ln C_m + \sup_{z\in\partial D(\lambda,r)} \Big(M|\mathrm{Im}\, z| + \frac{1}{m}\omega(z)\Big).$$

Therefore, using Lemma 1.5.6 we get

$$N(\lambda, r, X) \le \ln C_m + C + M|\mathrm{Im}\,\lambda| + \frac{C}{m}\omega(\lambda) + \ln r + Mr + \frac{C}{m}r$$

for all $\lambda \in \Lambda$ and $r > 0$. This completes the proof of (2). □

Now we will work on the necessity of the weighted Carleson conditions and conditions (B) for a multiplicity variety to be interpolating. A very important part of the proof is a construction of a non-zero holomorphic function with a prescribed growth. This will be done in the next lemma. Its proof relies essentially on the non-quasianalyticity of the weight. This result in a little weaker form was proved in [MOCO03, Proof of Proposition 5] with the use of [Bjö66, Lemma 1.3.11]. We will use [BMT90, Lemma 2.2].

Lemma 5.1.2. *Let ω be a non-quasianalytic weight. There exist a holomorphic function $H\colon \mathbb{H}_* \to \mathbb{C}$ with $H(z) \ne 0$ for all $z \in \mathbb{H}_*$ and such that*

$$\frac{1}{4}\omega(z) \le \ln|H(z)| \le C(\omega(z) + |\mathrm{Im}\, z| + 1)$$

for some $C > 0$ and all $z \in \mathbb{H}_*$.

Proof. By [BMT90, Lemma 2.2] the harmonic extension $P_\omega\colon \mathbb{H}_* \to [0,\infty)$ of $\omega(|t|)$ given by the formula

$$P_\omega(x+iy) = \frac{|y|}{\pi}\int_{\mathbb{R}} \frac{\omega(|t|)}{(x-t)^2+y^2}\ \mathrm{d}t$$

satisfies

1. $P_\omega(x+iy) \ge \frac{1}{4}\omega(|x+iy|)$ for all $x+iy \in \mathbb{H}_*$,

2. there exists $C > 0$ such that $P_\omega(x+iy) \le C(\omega(x)+|y|+1)$ for all $x+iy \in \mathbb{H}_*$.

Define $H(z) = e^{P_\omega(z)+i\tilde{P_\omega}(z)}$ where $\tilde{P_\omega}$ is a harmonic conjugate of P_ω. Then H is holomorphic and $H(z) \ne 0$ for all $z \in \mathbb{H}_*$. Since $\ln|H(z)| = P_\omega(z)$ this completes the proof. □

Having this function we are able to prove the following theorem due to Massaneda, Ortega-Cerdá, and Ounaïes.

Theorem 5.1.3 ([MOCO03, Proposition 5])**.** *Let ω be a non-quasianalytic weight. If X is an interpolating variety for $A_{(\omega)}$ then it satisfies (ω)-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$.*

Proof. We will only write the proof for $\mathbb{H}_+$. The proof for $\mathbb{H}_-$ is similar. For $\lambda' \in \Lambda \cap \mathbb{H}_+$ let $f_{\lambda'} = f_{\lambda',0}$ be the functions given by Lemma 2.1.3 and H be the function from Lemma 5.1.2. Define

$$h_{\lambda'}(z) = \frac{f_{\lambda'}(z)e^{iMz}}{H^M(z)}$$

with constant $M \in \mathbb{N}$ to be chosen later. Since this function is holomorphic in the upper half-plane, $\ln|h_{\lambda'}(\cdot)|$ is subharmonic there. Moreover, for some $C > 0$ and $N \in \mathbb{N}$

$$\begin{aligned} |h_{\lambda'}(z)| = \left|\frac{f_{\lambda'}(z)e^{iMz}}{H(z)}\right| &\le Ce^{N|\operatorname{Im} z| + N\omega(z)}e^{-M|\operatorname{Im} z|}e^{-M\ln|H(z)|} \\ &\le Ce^{N|\operatorname{Im} z| + N\omega(z)}e^{-M|\operatorname{Im} z|}e^{-\frac{M}{4}\omega(z)} \\ &= Ce^{(N-M)|\operatorname{Im} z| + (N-\frac{M}{4})\omega(z)} \end{aligned}$$

and

$$\begin{aligned} |h_{\lambda'}(\lambda')| &= e^{-M|\operatorname{Im}\lambda'| - M\ln|H(\lambda')|} \\ &\ge e^{-M|\operatorname{Im}\lambda'| - M(C\omega(\lambda') + C|\operatorname{Im}\lambda'| + C)} \\ &= e^{-MC - M(C+1)|\operatorname{Im}\lambda'| - MC\omega(\lambda')}. \end{aligned}$$

Fix M satisfying $M \ge 4N$. Then $h_{\lambda'}$ is a bounded holomorphic function and thus $\ln|h_{\lambda'}(\cdot)|$ admits a least harmonic majorant $k_{\lambda'}$. Applying the Poisson-Jensen formula for subharmonic functions in the upper half-plane for $\ln|h_{\lambda'}(\cdot)|$ (see [Ran95, Theorem 4.5.4]) we obtain

$$\ln|h_{\lambda'}(\lambda')| = k_{\lambda'}(\lambda') - \sum_{\substack{z \in \mathbb{H}_+ \\ h_{\lambda'}(z) = 0}} m_z \ln\left|\frac{\lambda' - \bar{z}}{\lambda' - z}\right|.$$

Therefore

$$\sum_{\substack{\lambda \in \Lambda \cap \mathbb{H}_+ \\ \lambda \ne \lambda'}} m_\lambda \ln\left|\frac{\lambda' - \bar{\lambda}}{\lambda' - \lambda}\right| \le k_{\lambda'}(\lambda') - \ln|h_{\lambda'}(\lambda')|.$$

Since $k_{\lambda'}(\lambda') < D$ for some constant $D > 0$, we obtain

$$\sum_{\substack{\lambda\in\Lambda\cap\mathbb{H}_+\\ \lambda\neq\lambda'}} m_\lambda \ln\left|\frac{\lambda'-\bar{\lambda}}{\lambda'-\lambda}\right| \leq D + MC + M(C+1)|\operatorname{Im}\lambda'| + MC\omega(\lambda').$$

We complete the proof with the use of Lemma 4.1.7. □

An analogous result can be proved in the Roumieu case.

Theorem 5.1.4. *Let ω be a non-quasianalytic weight. If X is an interpolating variety for $A_{\{\omega\}}$ then it satisfies $\{\omega\}$-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$.*

Proof. We will only write the proof for $\mathbb{H}_+$. The proof for $\mathbb{H}_-$ is similar. For $\lambda' \in \Lambda\cap\mathbb{H}_+$ let $f_{\lambda'} = f_{\lambda',0}$ be given by Lemma 2.1.4 and H be the function from Lemma 5.1.2. Define

$$h_{\lambda'}(z) = \frac{f_{\lambda'}(z)e^{iMz}}{H^{\frac{1}{m}}(z)}$$

with constants $M, m \in \mathbb{N}$ to be chosen later. We have

$$\ln|h_{\lambda'}(z)| = \frac{1}{m}\ln\left|\frac{(f_{\lambda'}(z)e^{iMz})^m}{H(z)}\right|.$$

Since the function under the module is holomorphic in the upper half-plane, $\ln|h_{\lambda'}(\cdot)|$ is subharmonic there. Moreover,

$$\exists\, N\in\mathbb{N}\ \forall\, n\in\mathbb{N}\ \exists\, C_n>0$$

$$\begin{aligned}
|h_{\lambda'}(z)| = \left|\frac{f_{\lambda'}(z)e^{iMz}}{H^{\frac{1}{m}}(z)}\right| &\leq C_n e^{N|\operatorname{Im} z|+\frac{1}{n}\omega(z)}e^{-M|\operatorname{Im} z|}e^{-\frac{1}{m}\ln|H(z)|}\\
&\leq C_n e^{N|\operatorname{Im} z|+\frac{1}{n}\omega(z)}e^{-M|\operatorname{Im} z|}e^{-\frac{1}{4m}\omega(z)}\\
&= C_n e^{(N-M)|\operatorname{Im} z|+(\frac{1}{n}-\frac{1}{4m})\omega(z)}
\end{aligned}$$

and

$$\begin{aligned}
|h_{\lambda'}(\lambda')| &= e^{-M|\operatorname{Im}\lambda'|-\frac{1}{m}\ln|H(\lambda')|}\\
&\geq e^{-M|\operatorname{Im}\lambda'|-\frac{1}{m}(C\omega(\lambda')+C+C|\operatorname{Im}\lambda'|)}\\
&= e^{-\frac{C}{m}-(\frac{C}{m}+M)|\operatorname{Im}\lambda'|-\frac{C}{m}\omega(\lambda')}.
\end{aligned}$$

Fix M satisfying $M \geq N$. Then for every $m\in\mathbb{N}$ we can choose $n > 4m$. This yields that $h_{\lambda'}$ is a bounded holomorphic function and thus $\ln|h_{\lambda'}(\cdot)|$ admits a least harmonic majorant $k_{\lambda'}$. Applying the Poisson-Jensen formula for subharmonic functions in the upper half-plane for $\ln|h_{\lambda'}(\cdot)|$ (see [Ran95, Theorem 4.5.4]) we obtain

$$\ln|h_{\lambda'}(\lambda')| = k_{\lambda'}(\lambda') - \sum_{\substack{z\in\mathbb{H}_+\\ h_{\lambda'}(z)=0}} m_z \ln\left|\frac{\lambda'-\bar{z}}{\lambda'-z}\right|.$$

Therefore

$$\sum_{\substack{\lambda\in\Lambda\cap\mathbb{H}_+\\ \lambda\neq\lambda'}} m_\lambda \ln\left|\frac{\lambda'-\bar{\lambda}}{\lambda'-\lambda}\right| \le k_{\lambda'}(\lambda') - \ln|h_{\lambda'}(\lambda')|.$$

We have that $k_{\lambda'}(\lambda') < D_m$ for some constant $D_m > 0$ depending on m. Finally, we obtain

$$\sum_{\substack{\lambda\in\Lambda\cap\mathbb{H}_+\\ \lambda\neq\lambda'}} m_\lambda \ln\left|\frac{\lambda'-\bar{\lambda}}{\lambda'-\lambda}\right| \le D_m + C + (C+M)|\operatorname{Im}\lambda'| + \frac{C}{m}\omega(\lambda'). \qquad \square$$

There is an alternative proof of Theorem 5.1.3 without the need of constructing a non-vanishing holomorphic function with prescribed growth given in Lemma 5.1.2. Instead we will use the analytic characterisation of interpolating varieties in the Beurling case.

Alternative proof of Theorem 5.1.3. Let f be the function from Theorem 3.1.1 vanishing on X and satisfying

$$\frac{|f^{(m_\lambda)}(\lambda)|}{m_\lambda!} \ge e^{-C|\operatorname{Im}\lambda| - C\omega(\lambda)}$$

for some $C > 0$ and every $\lambda \in \Lambda$. Then

$$f_\lambda(z) := \frac{f(z)}{(z-\lambda)^{m_\lambda}}$$

is an entire function and thus $u_\lambda(z) = \ln|f_\lambda(z)|$ is subharmonic on $\mathbb{C}$. Moreover, by the maximum modulus principle $f_\lambda \in A_{(\omega)}$. Hence

$$u_\lambda(t) \le \ln|f_\lambda(t)| \le M\omega(t) + M$$

for some $M > 0$ and every $t \in \mathbb{R}$. By the non-quasianalyticity of ω the Poisson transform P_ω of ω on $\mathbb{H}_*$ exists. Therefore $MP_\omega + M$ is a harmonic majorant of u_λ on $\mathbb{H}_*$. Then by the Poisson-Jensen formula ([Ran95, Theorem 4.5.4]) applied to u_λ we obtain

$$u_{\lambda'}(\lambda') \le MP_\omega(\lambda') + M - \sum_{\substack{z\in\mathbb{H}_*\\ f_{\lambda'}(z)=0}} m_z \ln\left|\frac{\lambda'-\bar{z}}{\lambda'-z}\right|.$$

for $\lambda' \in \Lambda$. Finally, using

$$u_{\lambda'}(\lambda') = \ln\frac{|f^{(m_{\lambda'})}(\lambda')|}{m_{\lambda'}!} \ge -C|\operatorname{Im}\lambda'| - C\omega(\lambda')$$

and [BMT90, Lemma 2.2] we get

$$\sum_{\substack{\lambda\in\Lambda\cap\mathbb{H}_*\\ \lambda\neq\lambda'}} m_\lambda \ln\left|\frac{\lambda'-\bar{\lambda}}{\lambda'-\lambda}\right| \leq MP_\omega(\lambda') + M + C|\operatorname{Im}\lambda'| + C\omega(\lambda')$$

$$\leq MC(\omega(\lambda') + |\operatorname{Im}\lambda'| + 1) + M + C|\operatorname{Im}\lambda'| + C\omega(\lambda')$$

for some $C > 0$ and every $\lambda' \in \Lambda$. The use of Lemma 4.1.7 completes the proof. □

Now, we can complete this section.

Theorem 5.1.5. *Let ω be a non-quasianalytic weight and X a multiplicity variety.*

(1) If X is interpolating for $A_{(\omega)}$ then X satisfies $(B(\omega))$.

(2) If X is interpolating for $A_{\{\omega\}}$ then X satisfies $(B\{\omega\})$.

Proof. (1) follows from Theorems 5.1.1, 5.1.3 and Corollary 4.2.13.
(2) follows from Theorems 5.1.1, 5.1.4 and Proposition 4.2.14. □

5.2 Sufficient conditions in the Beurling case

In the previous section we found out that every interpolating variety for $A_{(\omega)}$ is $(\omega)^*$-sparse and, in the non-quasianalytic case, satisfies (ω)-Carleson condition and $(B(\omega))$. In this section we will prove that these conditions are also sufficient. It turns out that the sufficiency does not need the assumption of non-quasianalyticity of the weight at all. At the end of this section we will give purely geometric characterisation of interpolating varieties in the non-quasianalytic Beurling case. In the quasianalytic case such a description is yet unknown. The implications (1) ⇒ (4) ⇒ (3) ⇒ (1) for $h = 1$ in Theorem 5.2.1 were proved in the paper of Massaneda, Ortega-Cerdá, and Ounaïes [MOCO03] for non-quasianalytic subadditive weights. We use their methods, but with some changes in the proofs we are able to show that subadditivity was superfluous. We prove also that the condition

$$\sup_{x\in\mathbb{R}} \sum_{\lambda\in\Lambda:\ |\operatorname{Im}\lambda|>\omega(\lambda)} \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} < \infty$$

can be replaced by $(B(\omega))$ or (ω)-Carleson condition.

The main theorem of this section reads as follows.

Theorem 5.2.1. *Let ω be a non-quasianalytic Beurling weight and $X = \{(\lambda, m_\lambda) |\ \lambda \in \Lambda\}$ a multiplicity variety. Then the following conditions are equivalent*

(1) *X is (ω)-sparse and satisfies*

$$\sup_{x\in\mathbb{R}} \sum_{\lambda\colon |\mathrm{Im}\,\lambda|>h\omega(\lambda)} m_\lambda \frac{|\mathrm{Im}\,\lambda|}{|x-\lambda|^2} < \infty$$

for some $h > 0$,

(2) *X is $(\omega)^*$-sparse and satisfies $(B(\omega))$,*

(3) *X is $(\omega)^*$-sparse and satisfies (ω)-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$,*

(4) *X is interpolating for $A_{(\omega)}$.*

The rest of this Section will be mainly devoted for the proof of this theorem.

In this section we will encounter several particular distributions. For the sake of convenience we introduce a notation for them.

Definition 5.2.2. For $c > 0$, $\lambda \in \mathbb{C}$, $\varphi \in C_c^\infty(\mathbb{C})$ we define a distribution ${\mathrm{Avg}^1}_{\lambda,c}$ counting the average of φ over the circle $S(\lambda, c) = \{z \in \mathbb{C} |\; |z - \lambda| = c\}$

$$\langle {\mathrm{Avg}^1}_{\lambda,c}, \varphi\rangle = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\lambda + ce^{i\theta})\,\mathrm{d}\theta$$

and a distribution ${\mathrm{Avg}^2}_{\lambda,c}$ counting the average of φ over the disk $D(\lambda, c)$

$$\langle {\mathrm{Avg}^2}_{\lambda,c}, \varphi\rangle = \frac{1}{\pi c^2}\int_{D(\lambda,c)} \varphi(z)\,\mathrm{dz}.$$

In Lemma 1.5.6 we proved that values of a weight do not differ too much if points are close. In a particular case we can make this estimate more precise. This lemma will be useful later.

Lemma 5.2.3. *Let ω be a weight. Denote $p(z) = |\mathrm{Im}\,z|+\omega(z)$. For any multiplicity variety X such that $0 \notin \Lambda$ there exists $\epsilon > 0$ and $C > 0$ such that*

$$dist\Big(\bigcup_{\lambda\in\Lambda} D(\lambda, \epsilon p(\lambda)), 0\Big) > 0$$

and

$$\frac{1}{C}p(z) \le p(\lambda) \le Cp(z)$$

for all pairs (z, λ) with $z \in D(\lambda, \epsilon p(\lambda))$.

Proof. Denote by Ω_ϵ the set of all pairs (z,λ) with $z \in D(\lambda, \epsilon p(\lambda))$. By condition (β) of the weight we have constants $C, D > 0$ such that $p(z) \leq C|z| + D$ for all $z \in \mathbb{C}$. Then for $(z,\lambda) \in \Omega_\epsilon$

$$|z| \geq |\lambda| - \epsilon p(\lambda) \geq (1-\epsilon C)|\lambda| - \epsilon D.$$

If we choose $\epsilon < \frac{1}{2C}$, then for $\lambda > 2\epsilon D + 2$ we will have $|z| > 1$. Furthermore, there are only finitely many points $\lambda \in \Lambda$ with $\lambda \leq 2\epsilon D + 2$. Denote the set of these points by Λ_0. Then we can adjust ϵ so that the sets $D(\lambda, \epsilon p(\lambda))$ does not contain 0 for any $\lambda \in \Lambda_0$. In conclusion

$$\operatorname{dist}\Big(\bigcup_{\lambda\in\Lambda} D(\lambda, \epsilon p(\lambda)), 0\Big) > 0. \tag{5.2}$$

From Lemma 1.5.7 (2) for all pairs $(z,\lambda) \in \Omega_\epsilon$ (possibly with smaller ϵ) we have $p(\lambda) \leq Cp(z) + D$. By (5.2), p has a positive infimum on the set $\bigcup_{\lambda\in\Lambda} D(\lambda, \epsilon p(\lambda))$. Hence we can choose $C > 0$ such that $p(\lambda) \leq Cp(z)$ for all $(z,\lambda) \in \Omega_\epsilon$.

Further, by Lemma 1.5.7 (1) we have $p(z) \leq Cp(\lambda) + D$ for all $(z,\lambda) \in \Omega_\epsilon$. Since p has a positive infimum on Λ we can find another constant $C' > 0$ such that $Cp(\lambda) + D \leq C'p(\lambda)$ for all $\lambda \in \Lambda$. □

The usual way of solving interpolation problems for holomorphic functions is to take a C^∞ solution, which in many cases is easy to find, and then apply to it the Hörmander's theorem for solving $\bar\partial$ equation. In the weighted case one uses the Hörmander's theorem with estimates on weighted norms. Suppose we have a C^∞ solution F of some interpolation problem on a variety X. The standard method to make it holomorphic is to solve the equation

$$\bar\partial\psi = -\frac{\bar\partial F}{G}$$

where G is holomorphic and equals zero on X. Then one takes $f = F + \psi G$, which is already holomorphic and solves the interpolation problem. In this section we will use another approach. The weighted Hörmander's theorem gives a solution of the equation $\bar\partial u = v$ with the estimate

$$\int_\Omega \frac{u(z)e^{-\psi(z)}}{(1+|z|^2)^2}\,\mathrm{dz} \lesssim \int_\Omega v(z)e^{-\psi(z)}\,\mathrm{dz}$$

where ψ is some subharmonic function. It is assumed that the right-hand side integral is finite, hence the left-hand side integral is finite as well. Using this theorem we would like to solve the equation $\bar\partial u = \bar\partial F$ and then take $f = F - u$. For f to be a solution to the interpolation problem we need then to push u to be zero (maybe with derivatives up to some order) at points on which we are interpolating. This could be done by a careful choice of the function ψ. In Lemma 5.2.5 we will provide such a mapping related to a given weight. In the proof of it we will need the following formula.

Lemma 5.2.4. *The following equality holds:*

$$\frac{1}{\pi r^2}\int\limits_{D(\xi,r)} \ln|\zeta|^2 \,\mathrm{d}\zeta = \begin{cases} \ln|\xi|^2 & \text{if } |\xi| > r, \\ \frac{|\xi|^2}{r^2} + \ln r^2 - 1 & \text{if } |\xi| \le r, \end{cases}$$

for any $\xi \in \mathbb{C}$ and $r > 0$.

Proof. First, we will show that

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \ln|\xi - re^{i\theta}| \,\mathrm{d}\theta = \begin{cases} \ln|\xi| & \text{if } |\xi| > r, \\ \ln r & \text{if } |\xi| \le r, \end{cases} \tag{5.3}$$

for any $\xi \in \mathbb{C}$ and $r > 0$. Let $|\xi| > r$. Then $u(z) = \ln|\xi - z|$ is harmonic on the disc $D(0,(1+\eta)r)$ for $\eta > 0$ such that $(1+\eta)r < |\xi|$. Hence, by the mean value property,

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \ln|\xi - re^{i\theta}| \,\mathrm{d}\theta = \frac{1}{2\pi}\int\limits_0^{2\pi} u(re^{i\theta}) \,\mathrm{d}\theta = u(0) = \ln|\xi|.$$

For $|\xi| = r$ we have

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \ln|\xi - re^{i\theta}| \,\mathrm{d}\theta = \ln|\xi| + \frac{1}{2\pi}\int\limits_0^{2\pi} \ln|1 - e^{i\theta}| \,\mathrm{d}\theta = \ln r$$

by [Rud87, Lemma 15.17]. Finally, assume that $|\xi| < r$. The function $z \mapsto \frac{\xi - z}{z}$ is holomorphic on $\mathbb{C}_\infty \setminus \{0\}$ and equals zero only in ξ, hence $u(z) = \ln|\xi - z| - \ln|z|$ is harmonic on $\mathbb{C}_\infty \setminus D(0,(1-\eta)r)$ for $\eta > 0$ such that $|\xi| < (1-\eta)r$. By the mean value property

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \ln|\xi - re^{i\theta}| - \ln|re^{i\theta}| \,\mathrm{d}\theta = \frac{1}{2\pi}\int\limits_0^{2\pi} u(re^{i\theta}) \,\mathrm{d}\theta = u(\infty) = 0.$$

This yields

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \ln|\xi - re^{i\theta}| \,\mathrm{d}\theta = \frac{1}{2\pi}\int\limits_0^{2\pi} \ln|re^{i\theta}| \,\mathrm{d}\theta = \ln r.$$

Further, for any $\xi \in \mathbb{C}$ and $r > |\xi|$ we have

$$\frac{1}{\pi r^2} \int\limits_{D(\xi,r)} \ln|\zeta|^2 \, \mathrm{d}\zeta = \frac{2}{\pi r^2} \int\limits_0^r \int\limits_0^{2\pi} \ln|\xi - se^{i\theta}| s \, \mathrm{d}\theta \, \mathrm{d}s$$

$$= \frac{2}{\pi r^2} \int\limits_0^{|\xi|} s \int\limits_0^{2\pi} \ln|\xi - se^{i\theta}| \, \mathrm{d}\theta \, \mathrm{d}s + \frac{2}{\pi r^2} \int\limits_{|\xi|}^{r} s \int\limits_0^{2\pi} \ln|\xi - se^{i\theta}| \, \mathrm{d}\theta \, \mathrm{d}s.$$

Then, by (5.3),

$$\begin{aligned}
\frac{1}{\pi r^2} \int\limits_{D(\xi,r)} \ln|\zeta|^2 \, \mathrm{d}\zeta &= \frac{4}{r^2} \int\limits_0^{|\xi|} s \ln|\xi| \, \mathrm{d}s + \frac{4}{r^2} \int\limits_{|\xi|}^{r} s \ln s \, \mathrm{d}s \\
&= \frac{2|\xi|^2}{r^2} \ln|\xi| + \frac{4}{r^2} \left[\frac{1}{2} s^2 \left(\ln s - \frac{1}{2} \right) \right]_{|\xi|}^{r} \\
&= \frac{2|\xi|^2}{r^2} \ln|\xi| + 2\ln r - 1 - 2\frac{|\xi|^2}{r^2} \ln|\xi| + \frac{|\xi|^2}{r^2} \\
&= \ln r^2 - 1 + \frac{|\xi|^2}{r^2}.
\end{aligned}$$

On the other hand, for any $\xi \in \mathbb{C}$ and $r \leq |\xi|$ we obtain immediately that

$$\begin{aligned}
\frac{1}{\pi r^2} \int\limits_{D(\xi,r)} \ln|\zeta|^2 \, \mathrm{d}\xi &= \frac{2}{\pi r^2} \int\limits_0^r s \int\limits_0^{2\pi} |\xi - se^{i\theta}| \, \mathrm{d}\theta \, \mathrm{d}s \\
&= \frac{4}{r^2} \ln|\xi| \int\limits_0^r s \, \mathrm{d}s = \ln|\xi|^2.
\end{aligned}$$

□

The next lemma provides the subharmonic function ψ associated with a given weight ω, which is to be used in the Hörmander's theorem.

Lemma 5.2.5. *Assume:*

(1) ω is a weight,

(2) $X = \{(\lambda, m_\lambda) |\ \lambda \in \Lambda\}$ is an (ω)-sparse variety such that $0 \notin \Lambda$,

(3) $\tilde{p}\colon \mathbb{C} \to [0, \infty)$ is a subharmonic function such that for every $R > 0$ there exists $C_R > 0$ for which the following condition is satisfied

$$\frac{1}{C_R} \tilde{p}(z) \leq |\operatorname{Im} z| + \omega(z) \leq C_R \tilde{p}(z)$$

for every $z \in \mathbb{C} \setminus D(0, R)$, *(4) there exist $\epsilon_0 > 0$ and $C_0 > 0$ such that for all $z \in \bigcup_{\lambda\in\Lambda} D(\lambda, \epsilon_0 p(\lambda))$*

$$\frac{1}{\tilde{p}(z)} \leq C_0 \Delta \tilde{p}(z).$$

Then there exist constants $K > 0$, $\beta_0 > 0$ and a function $v\colon \mathbb{C} \to \mathbb{R}$ satisfying $v(z) \leq 0$ for every $z \in \mathbb{C}$ such that for every $\beta > \beta_0$ the function $\psi(z) = \beta\tilde{p}(z) + v(z)$ has the following properties:

(i) ψ is subharmonic,

(ii)

$$|\psi(z) - \beta\tilde{p}(z)| \leq K\tilde{p}(z)$$

for all $z \in \bigcup_{\lambda\in\Lambda} R(\lambda, \frac{1}{2}\delta_\lambda, \delta_\lambda)$, where δ_λ are the separation radii of X,

(iii) for each $\lambda \in \Lambda$ there is a constant $C_{\lambda,\beta} > 0$ such that

$$e^{\psi(z)} \leq C_{\lambda,\beta}|z - \lambda|^{2m_\lambda}$$

for $z \in D(\lambda, 1)$.

Proof. Denote $p(z) = |\operatorname{Im} z| + \omega(z)$. Since $0 \notin \Lambda$ we can choose $R_0 > 0$ such that $\Lambda \cap D(0, R_0) = \emptyset$. Therefore, by assumption (3),

$$\frac{1}{C_R}\tilde{p}(\lambda) \leq p(\lambda) \leq C_R\tilde{p}(\lambda) \tag{5.4}$$

for some $C_R > 0$ and every $\lambda \in \Lambda$.

By Propositions 4.1.10 and 4.1.5, X is necessarily weakly (ω)-separable. Hence, by (5.4), there exist $C_s > 0$ and δ satisfying $0 < \delta < 1$ such that for

$$\delta_\lambda = \delta e^{-C_s \frac{\tilde{p}(\lambda)}{m_\lambda}}$$

the disks $D(\lambda, 2\delta_\lambda)$ are pairwise disjoint for all $\lambda \in \Lambda$.

By Lemma 5.2.3 and condition (5.4), for some ϵ_1 satisfying $0 < \epsilon_1 \leq \epsilon_0$ it holds

$$\operatorname{dist}\Big(\bigcup_{\lambda\in\Lambda} D(\lambda, \epsilon_1\tilde{p}(\lambda)), 0\Big) > 0.$$

Hence we can choose R satisfying $0 < R \leq R_0$ and such that

$$\Big(\bigcup_{\lambda\in\Lambda} D(\lambda, \epsilon_1\tilde{p}(\lambda))\Big) \cap D(0, R) = \emptyset.$$

Therefore using Lemma 5.2.3 and assumption (3) we obtain

$$\frac{1}{H}\tilde{p}(z) \leq \tilde{p}(\lambda) \leq H\tilde{p}(z) \tag{5.5}$$

for some $H > 0$ and all pairs (z, λ) with $z \in D(\lambda, \epsilon_1 \tilde{p}(\lambda))$.

Furthermore, since $\inf_{\lambda\in\Lambda} \tilde{p}(\lambda) \geq \frac{1}{C_R} \inf_{\lambda\in\Lambda} p(\lambda) > 0$, we may adjust δ so that

$$\delta_\lambda = \delta e^{-C_s \frac{\tilde{p}(\lambda)}{m_\lambda}} \leq \delta \leq \epsilon_1 \tilde{p}(\lambda)$$

for every $\lambda \in \Lambda$.

We set

$$\psi(z) = \beta\tilde{p}(z) + v(z) \quad (z \in \mathbb{C})$$

where

$$v(z) = \sum_{\lambda\in\Lambda} m_\lambda \Bigg[\ln|z-\lambda|^2 - \frac{1}{\pi\epsilon^2\tilde{p}^2(\lambda)} \int\limits_{D(\lambda,\epsilon\tilde{p}(\lambda))} \ln|z-\xi|^2 \,\mathrm{d}\xi \Bigg].$$

Constants $0 < \epsilon \leq \epsilon_1$ and $\beta > 0$ are to be determined later on. We divide the long proof into several parts mainly according to the parts of the assertion.

Claim I. The following equality holds:

$$v(z) = \sum_{\lambda:|\lambda-z|\leq\epsilon\tilde{p}(\lambda)} m_\lambda \left[\ln \frac{|z-\lambda|^2}{\epsilon^2\tilde{p}^2(\lambda)} + 1 - \frac{|z-\lambda|^2}{\epsilon^2\tilde{p}^2(\lambda)} \right]$$

for every $z \in \mathbb{C}$. In particular, $v(z) = 0$ for $z \notin \bigcup_{\lambda\in\Lambda} D(\lambda, \epsilon\tilde{p}(\lambda))$ and $v(z) \leq 0$ for all $z \in \mathbb{C}$.

Proof. By Lemma 5.2.4, we have

$$\ln|\xi|^2 - \frac{1}{\pi r^2} \int\limits_{D(\xi,r)} \ln|\zeta|^2 \,\mathrm{d}\zeta = \begin{cases} 0 & \text{if } |\xi| > r, \\ \ln\frac{|\xi|^2}{r^2} + 1 - \frac{|\xi|^2}{r^2} & \text{if } |\xi| \leq r, \end{cases}$$

for any $\xi \in \mathbb{C}$ and $r > 0$. Let now $z \in \mathbb{C}$ and $\lambda \in \Lambda$ be given. Using the above equality for $\xi = z - \lambda$ and $r = \epsilon\tilde{p}(\lambda)$, and applying

$$\frac{1}{\pi\epsilon^2\tilde{p}^2(\lambda)} \int\limits_{D(z-\lambda,\epsilon\tilde{p}(\lambda))} \ln|\xi|^2 \,\mathrm{d}\xi = \frac{1}{\pi\epsilon^2\tilde{p}^2(\lambda)} \int\limits_{D(\lambda,\epsilon\tilde{p}(\lambda))} \ln|z-\xi|^2 \,\mathrm{d}\xi,$$

we obtain the form of v given in the assertion.

Finally, since $\ln x + 1 - x \leq 0$ for all $x > 0$, we get $v(z) \leq 0$ for $z \in \mathbb{C}$. □

Claim II. The following inequality between distributions holds:

$$\Delta v(z) \geq -4 \sum_{\lambda:|\lambda-z|\leq\epsilon\tilde{p}(\lambda)} \frac{m_\lambda}{\epsilon^2\tilde{p}^2(\lambda)}.$$

Proof. Define

$$k_\lambda(z) = \begin{cases} m_\lambda \ln \frac{|z-\lambda|^2}{\epsilon^2 \tilde{p}^2(\lambda)} & \text{for } z \in D(\lambda, \epsilon\tilde{p}(\lambda)), \\ 0 & \text{for } z \in \mathbb{C} \setminus D(\lambda, \epsilon\tilde{p}(\lambda)), \end{cases}$$

and

$$h_\lambda(z) = \begin{cases} m_\lambda \Big(1 - \frac{|z-\lambda|^2}{\epsilon^2 \tilde{p}^2(\lambda)}\Big) & \text{for } z \in D(\lambda, \epsilon\tilde{p}(\lambda)), \\ 0 & \text{for } z \in \mathbb{C} \setminus D(\lambda, \epsilon\tilde{p}(\lambda)). \end{cases}$$

Denote $k(z) = \sum_{\lambda\in\Lambda} k_\lambda(z)$, $h(z) = \sum_{\lambda\in\Lambda} h_\lambda(z)$ and $c_\lambda = \epsilon\tilde{p}(\lambda)$. Then according to Lemmas A.3 and A.4 of Appendix A we have

$$\Delta v = \Delta k + \Delta h = 4\pi \sum_{\lambda\in\Lambda} m_\lambda(\delta_\lambda - \mathrm{Avg}^2{}_{\lambda,c_\lambda}) \geq -4\pi \sum_{\lambda\in\Lambda} m_\lambda \,\mathrm{Avg}^2{}_{\lambda,c_\lambda}\,.$$

Let φ be a positive test function. Then

$$\begin{aligned} \Big\langle \sum_{\lambda\in\Lambda} m_\lambda \,\mathrm{Avg}^2{}_{\lambda,c_\lambda}, \varphi \Big\rangle &= \sum_{\lambda\in\Lambda} m_\lambda \Big\langle \mathrm{Avg}^2{}_{\lambda,c_\lambda}, \varphi \Big\rangle = \sum_{\lambda\in\Lambda} m_\lambda \frac{1}{\pi c_\lambda^2} \int\limits_{D(\lambda,c_\lambda)} \varphi(z)\,\mathrm{dz} \\ &= \frac{1}{\pi} \sum_{\lambda\in\Lambda} \int\limits_{\mathbb{C}} \frac{m_\lambda}{c_\lambda^2} \chi_{D(\lambda,c_\lambda)}(z)\varphi(z)\,\mathrm{dz} \\ &= \frac{1}{\pi} \int\limits_{\mathbb{C}} \sum_{\lambda\in\Lambda} \frac{m_\lambda}{c_\lambda^2} \chi_{D(\lambda,c_\lambda)}(z)\varphi(z)\,\mathrm{dz} \\ &= \frac{1}{\pi} \int\limits_{\mathbb{C}} \sum_{\lambda:|\lambda-z|\leq c_\lambda} \frac{m_\lambda}{c_\lambda^2} \varphi(z)\,\mathrm{dz} \\ &= \frac{1}{\pi} \Big\langle \sum_{\lambda:|\lambda-z|\leq c_\lambda} \frac{m_\lambda}{c_\lambda^2}, \varphi \Big\rangle. \end{aligned}$$

Therefore

$$\Delta v(z) \geq -4 \sum_{\lambda:|\lambda-z|\leq\epsilon\tilde{p}(\lambda)} \frac{m_\lambda}{\epsilon^2 \tilde{p}^2(\lambda)}. \qquad \square$$

Claim III. Denote

$$h(z) = \begin{cases} -\frac{1}{\tilde{p}(z)} & \textit{for } z \in \bigcup_{\lambda\in\Lambda} D(\lambda, \epsilon\tilde{p}(\lambda)), \\ 0 & \textit{otherwise.} \end{cases}$$

Then $\Delta v(z) \geq Ch(z)$ for some $C > 0$ in the sense of distributions. In particular, ψ is subharmonic if β is chosen big enough.

Proof. By Claim II and (5.5) we have

$$\Delta v(z) \geq -4H^2 \sum_{\lambda:|\lambda-z|\leq \epsilon H\tilde{p}(z)} \frac{m_\lambda}{\epsilon^2 \tilde{p}^2(z)} = -\frac{4H^2}{\epsilon^2} \frac{n(z, \epsilon H\tilde{p}(z))}{\tilde{p}^2(z)}.$$

Using Lemma 4.1.20 we obtain that for some, possibly smaller, ϵ and a constant $C > 0$

$$\Delta v \geq -\frac{4H^2 C}{\epsilon^2} \frac{1}{\tilde{p}}. \tag{5.6}$$

By assumption (4) for $z \in \bigcup_{\lambda\in\Lambda} D(\lambda, \epsilon\tilde{p}(\lambda))$ we have

$$\frac{1}{\tilde{p}(z)} \leq C_0 \Delta\tilde{p}(z).$$

Taking $\beta \geq \frac{4H^2 C C_0}{\epsilon^2}$ and using condition (5.6) we obtain

$$\Delta\psi \geq \beta\Delta\tilde{p} - \frac{4H^2 C}{\epsilon^2}\frac{1}{\tilde{p}} \geq \Big(\beta - \frac{4H^2 C C_0}{\epsilon^2}\Big)\Delta\tilde{p} \geq 0.$$

Recalling that $v(z) = 0$ for $z \notin \bigcup_{\lambda\in\Lambda} D(\lambda, \epsilon\tilde{p}(\lambda))$ we see that ψ is subharmonic on $\mathbb{C}$. □

Claim IV. It holds

$$|\psi(z) - \beta\tilde{p}(z)| \leq K\tilde{p}(z)$$

for some $K > 0$ independent of β, and all $z \in \bigcup_{\lambda\in\Lambda} R(\lambda, \frac{1}{2}\delta_\lambda, \delta_\lambda)$.

Proof. If $0 < x \leq 1$ then

$$\ln x \leq x - 1 \leq 0 = \ln x - \ln x = \ln x + \ln\frac{1}{x}$$

hence

$$0 \leq x - 1 - \ln x \leq \ln\frac{1}{x}$$

and

$$|x - 1 - \ln x| \leq \ln\frac{1}{x}.$$

Thus for all $z \in \mathbb{C}$

$$\begin{aligned} |\psi(z) - \beta\tilde{p}(z)| = |v(z)| &\leq \sum_{\lambda:|\lambda-z|\leq\epsilon\tilde{p}(\lambda)} m_\lambda \ln\frac{\epsilon^2\tilde{p}^2(\lambda)}{|z-\lambda|^2} \\ &\leq \sum_{\lambda:|\lambda-z|\leq\epsilon H\tilde{p}(z)} m_\lambda \ln\frac{\epsilon^2 H^2\tilde{p}^2(z)}{|z-\lambda|^2}, \end{aligned} \tag{5.7}$$

where the second inequality follows from (5.5).

By Proposition 1.10.2 for any $z \in \mathbb{C}$ we have

$$N(z,r) = \sum_{\lambda:\, 0<|\lambda - z|\le r} m_\lambda \ln \frac{r}{|z-\lambda|} + n(z,0)\ln r.$$

Assume that $z \notin \Lambda$. Then

$$N(z,r) = \sum_{\lambda: |\lambda - z|\le r} m_\lambda \ln \frac{r}{|z-\lambda|}$$

which with (5.7) gives

$$|\psi(z) - \beta\tilde{p}(z)| \le 2N(z, \epsilon H \tilde{p}(z)). \tag{5.8}$$

We leave the above inequality at this stage for the moment.

Now we will show that for $z \in R(\lambda, \frac{1}{2}\delta_\lambda, \delta_\lambda)$

$$N(z,r) \le N(\lambda, 2r+\delta_\lambda) + C\tilde{p}(\lambda) \tag{5.9}$$

for some constant $C > 0$ independent of z, λ and r. First, observe that

$$N(z,r) = \sum_{\substack{\lambda':\, |\lambda' - z|\le r \\ \lambda' \ne \lambda}} m_{\lambda'} \ln \frac{r}{|z-\lambda'|} + m_\lambda \ln \frac{r}{|z-\lambda|}$$

and

$$N(\lambda, 2r+\delta_\lambda) = \sum_{\lambda':\, 0<|\lambda' - \lambda|\le 2r+\delta_\lambda} m_{\lambda'} \ln \frac{2r+\delta_\lambda}{|\lambda-\lambda'|} + m_\lambda \ln(2r+\delta_\lambda).$$

For $z \in R(\lambda, \frac{1}{2}\delta_\lambda, \delta_\lambda)$ we have $\bar{D}(z,r) \subset \bar{D}(\lambda, r+\delta_\lambda) \subset \bar{D}(\lambda, 2r+\delta_\lambda)$. Thus

$$\begin{aligned}
N(z,r) - N(\lambda, 2r+\delta_\lambda) &= \sum_{\lambda' \in \bar{D}(z,r)\setminus\{\lambda\}} m_{\lambda'} \ln \frac{r}{|z-\lambda'|} + m_\lambda \ln \frac{r}{|z-\lambda|} \\
&\quad - \sum_{\lambda' \in \bar{D}(\lambda, 2r+\delta_\lambda)\setminus\{\lambda\}} m_{\lambda'} \ln \frac{2r+\delta_\lambda}{|\lambda-\lambda'|} - m_\lambda \ln(2r+\delta_\lambda) \\
&= \sum_{\lambda' \in \bar{D}(z,r)\setminus\{\lambda\}} m_{\lambda'} \ln \frac{r|\lambda-\lambda'|}{|z-\lambda'|(2r+\delta_\lambda)} \\
&\quad + m_\lambda \ln \frac{r}{|z-\lambda|(2r+\delta_\lambda)} \\
&\quad - \sum_{\lambda' \in \bar{D}(\lambda, 2r+\delta_\lambda)\setminus(\bar{D}(z,r)\cup\{\lambda\})} m_{\lambda'} \ln \frac{2r+\delta_\lambda}{|\lambda-\lambda'|} \\
&\le \sum_{\lambda' \in \bar{D}(z,r)\setminus\{\lambda\}} m_{\lambda'} \ln \frac{r|\lambda-\lambda'|}{|z-\lambda'|(2r+\delta_\lambda)} + m_\lambda \ln \frac{1}{|z-\lambda|}.
\end{aligned}$$

Moreover, $|z-\lambda| \leq \delta_\lambda$ and $|z-\lambda'| \geq \delta_\lambda$ for $\lambda' \neq \lambda$ since the discs $D(\lambda, 2\delta_\lambda)$ and $D(\lambda', 2\delta_{\lambda'})$ are disjoint. Hence

$$\frac{|z-\lambda|}{|z-\lambda'|} \leq 1 \leq \frac{r+\delta_\lambda}{r} = \frac{2r+\delta_\lambda}{r} - 1,$$

and thus

$$1 \geq \frac{r}{2r+\delta_\lambda}\left(\frac{|z-\lambda|}{|z-\lambda'|}+1\right) = \frac{r}{2r+\delta_\lambda}\left(\frac{|z-\lambda|}{|z-\lambda'|}+\frac{|z-\lambda'|}{|z-\lambda'|}\right) \geq \frac{r}{2r+\delta_\lambda}\frac{|\lambda-\lambda'|}{|z-\lambda'|}.$$

Therefore, using $|z-\lambda| \geq \frac{1}{2}\delta_\lambda$ and the formula for δ_λ, we obtain

$$N(z,r) - N(\lambda, 2r+\delta_\lambda) \leq m_\lambda \ln \frac{1}{|z-\lambda|} \leq m_\lambda \ln\left(2\delta_\lambda^{-1}\right) \leq m_\lambda \ln \frac{2}{\delta} + C_s\tilde{p}(\lambda).$$

As X is (ω)-sparse we can use Proposition 4.1.18 to get (5.9).

Therefore, from (5.8) and (5.9),

$$|\psi(z) - \beta\tilde{p}(z)| \leq 2N(z, \epsilon H\tilde{p}(z)) \leq 2N(\lambda, 2\epsilon H\tilde{p}(z)+\delta_\lambda) + C\tilde{p}(\lambda)$$

for some $C > 0$ and all pairs $z \in \mathbb{C}$, $\lambda \in \Lambda$ such that $z \in R(\lambda, \frac{1}{2}\delta_\lambda, \delta_\lambda)$.

To finish the proof we have to estimate $N(\lambda, 2\epsilon H\tilde{p}(z)+\delta_\lambda)$ by $\tilde{p}(\lambda)$ multiplied by a constant. Then by the use of (5.5) we will obtain an estimate of the form $C\tilde{p}(z)$, as required.

By (5.5) we have

$$2\epsilon H\tilde{p}(z) + \delta_\lambda \leq 2\epsilon H^2\tilde{p}(\lambda) + \delta_\lambda = 2\epsilon H^2\tilde{p}(\lambda) + \delta e^{-C_s\frac{\tilde{p}(\lambda)}{m_\lambda}}.$$

For points $\lambda \in \Lambda$ with $\tilde{p}(\lambda) \geq 2$ we have

$$2\epsilon H^2\tilde{p}(\lambda) + e^{-C_s\frac{\tilde{p}(\lambda)}{m_\lambda}} \leq 2\epsilon H^2\tilde{p}(\lambda) + \frac{1}{2}\tilde{p}(\lambda),$$

which is smaller than $\tilde{p}(\lambda)$ for ϵ small enough. Then by the sparsity of X we get

$$N(\lambda, 2\epsilon H\tilde{p}(z)+\delta_\lambda) \lesssim \tilde{p}(\lambda)$$

for every $\lambda \in \Lambda$ satisfying $\tilde{p}(\lambda) \geq 2$, and $z \in R(\lambda, \frac{1}{2}\delta_\lambda, \delta_\lambda)$.

On the other hand, there are only finitely many points $\lambda \in \Lambda$ with $\tilde{p}(\lambda) < 2$. Hence

$$\sup_{\lambda\in\Lambda:\,\tilde{p}(\lambda)<2}\ \sup_{z\in R(\lambda,\frac{1}{2}\delta_\lambda,\delta_\lambda)} N(\lambda, 2\epsilon H\tilde{p}(z)+\delta_\lambda) < \infty.$$

Then, by Lemma 4.1.7 and (5.4), for some $C > 0$, every λ satisfying $p(\lambda) < 2$, and every $z \in R(\lambda, \frac{1}{2}\delta_\lambda, \delta_\lambda)$

$$N(\lambda, 2\epsilon H\tilde{p}(z)+\delta_\lambda) \leq Cp(\lambda) \leq CC_R\tilde{p}(\lambda).$$

□

Claim V. For each $\lambda \in \Lambda$ there is a constant $C_{\lambda,\beta} > 0$ such that

$$e^{\psi(z)} \le C_{\lambda,\beta}|z-\lambda|^{2m_\lambda} \tag{5.10}$$

for $z \in D(\lambda, 1)$.

Proof. First, using subharmonicity of $\ln|z-w|^2$ with respect to w when z is fixed we get that every element of the sum in the definition of v is negative. Hence

$$\begin{aligned} v(z) &= \sum_{\lambda' \in \Lambda} m_{\lambda'} \left[\ln|z-\lambda'|^2 - \frac{1}{\pi\epsilon^2\tilde{p}^2(\lambda')} \int\limits_{D(\lambda',\epsilon\tilde{p}(\lambda'))} \ln|z-\xi|^2 \, \mathrm{d}\xi \right] \\ &\le m_\lambda \left[\ln|z-\lambda|^2 - \frac{1}{\pi\epsilon^2\tilde{p}^2(\lambda)} \int\limits_{D(\lambda,\epsilon\tilde{p}(\lambda))} \ln|z-\xi|^2 \, \mathrm{d}\xi \right]. \end{aligned}$$

Then using the formula

$$\int\limits_{D(0,r)} \ln|z|^2 \, \mathrm{dz} = 2\pi \left(r^2 \ln r - \frac{r^2}{2} \right)$$

we obtain

$$\begin{aligned} -\frac{m_\lambda}{\pi\epsilon^2\tilde{p}^2(\lambda)} \int\limits_{D(\lambda,\epsilon\tilde{p}(\lambda))} \ln|z-\xi|^2 \, \mathrm{d}\xi &\le -\frac{m_\lambda}{\pi\epsilon^2\tilde{p}^2(\lambda)} \int\limits_{D(\lambda,\epsilon\tilde{p}(\lambda))} \ln|\lambda-\xi|^2 \, \mathrm{d}\xi \\ &= -\frac{2m_\lambda}{\epsilon^2\tilde{p}^2(\lambda)} \left((\epsilon\tilde{p}(\lambda))^2 \ln(\epsilon\tilde{p}(\lambda)) - \frac{(\epsilon\tilde{p}(\lambda))^2}{2} \right) \\ &= m_\lambda(1 - \ln(\epsilon^2\tilde{p}^2(\lambda))). \end{aligned}$$

Therefore, for $z \in D(\lambda, 1)$

$$e^{\psi(z)} \le e^{\beta\tilde{p}(z)} e^{v(z)} \le e^{\beta\tilde{p}(z)} e^{m_\lambda(1-\ln(\epsilon^2\tilde{p}^2(\lambda)))} e^{m_\lambda \ln|z-\lambda|^2} \le C_{\lambda,\beta}|z-\lambda|^{2m_\lambda}$$

where

$$C_{\lambda,\beta} = e^{m_\lambda(1-\ln(\epsilon^2\tilde{p}^2(\lambda)))} \sup_{w \in D(\lambda,1)} e^{\beta\tilde{p}(w)}.$$

□

The procedure we are going to use to find a solution to the interpolation problem consists of two steps. First, we will construct a C^∞ solution F with some control of $\bar{\partial}F$ and then by the use of the Hörmander's theorem we will make it holomorphic. However, the interpolation problem was originally defined for holomorphic functions, and it involved complex derivatives of a function. To extend the restriction operator R to C^∞ functions we introduce the following differential operator

$$\partial = \frac{1}{2}(\partial_x - i\partial_y).$$

For a holomorphic function this operator just computes its complex derivative, but it is also well defined for C^∞ functions. Then we define

$$R(F) = \Big(\frac{\partial^l F(\lambda)}{l!}\Big)_{\lambda\in\Lambda, 0\le l<m_\lambda}$$

for $F \in C^\infty(\mathbb{C})$.

In the following lemma we prove that the only assumption needed for the existence of a C^∞ solution to the interpolation problem is that a multiplicity variety be weakly separated.

Lemma 5.2.6. *Let ω be a weight and assume that a multiplicity variety X is weakly (ω)-separated. Denote by δ_λ the separation radii for X. For any $v \in S_{(\omega)}(X)$ there exists a solution F to the interpolation problem $R(F) = v$ with the following properties:*

(1) $F \in C^\infty_{(\omega)}$ and $\bar\partial F \in C^\infty_{(\omega)}$,

(2) the support of F is contained in $\bigcup_{\lambda\in\Lambda} D(\lambda, \delta_\lambda)$,

(3) the support of $\bar\partial F$ is contained in $\bigcup_{\lambda\in\Lambda} R(\lambda, \frac{1}{2}\delta_\lambda, \delta_\lambda)$, where $R(\lambda, \frac{1}{2}\delta_\lambda, \delta_\lambda) = \{z \in \mathbb{C}|\ \frac{1}{2}\delta_\lambda \le |z-\lambda| \le \delta_\lambda\}$.

Proof. Recall that, by (ω)-separation of X,

$$\delta_\lambda = \delta e^{-C\frac{p(\lambda)}{m_\lambda}}$$

for some $C > 0$, $0 < \delta < 1$, are such that the discs $D(\lambda, \delta_\lambda)$ are disjoint for all $\lambda \in \Lambda$.

Define

$$w_\lambda(z) = \sum_{l=0}^{m_\lambda-1} v_{\lambda,l}(z-\lambda)^l$$

for $\lambda \in \Lambda$. Further, let χ be a smooth function on $\mathbb{C}$ with bounded derivatives satisfying $\chi(z) = 1$ if $|z| \le \frac{1}{2}$ and $\chi(z) = 0$ if $|z| \ge 1$. Define

$$F(z) = \sum_{\lambda\in\Lambda} w_\lambda(z)\chi\Big(\frac{z-\lambda}{\delta_\lambda}\Big).$$

Since for $z \in D(\lambda, \frac{1}{2}\delta_\lambda)$ we have $F(z) = w_\lambda(z)$, therefore

$$\frac{\partial^l F(\lambda)}{l!} = v_{\lambda,l}$$

for every $\lambda \in \Lambda$ and $0 \le l < m_\lambda$. Moreover, the supports of F and $\bar\partial F$ are as we claimed in the assertion. Further, since $\delta_\lambda \le 1$ for every $\lambda \in \Lambda$, we have for every $z \in D(\lambda, \delta_\lambda)$ and some constants $C, D > 0$

$$\begin{aligned} |F(z)| \le |w_\lambda(z)| \le \sum_{l=0}^{m_\lambda - 1} |v_{\lambda,l}||z-\lambda|^l \le \sum_{l=0}^{m_\lambda-1} |v_{\lambda,l}| \\ \le e^{C(|\operatorname{Im}\lambda| + \omega(\lambda)) + C} \le e^{D(|\operatorname{Im} z| + \omega(z)) + D}, \end{aligned}$$

by Lemma 1.5.6 in the last inequality.

Finally, for $z \in R(\lambda, \frac{1}{2}\delta_\lambda, \delta_\lambda)$

$$\begin{aligned} |\bar\partial F(z)| &= \left| \bar\partial w_\lambda(z) \chi\Big(\frac{z-\lambda}{\delta_\lambda}\Big) + w_\lambda(z)(\bar\partial\chi)\Big(\frac{z-\lambda}{\delta_\lambda}\Big)\frac{1}{\delta_\lambda} \right| \\ &= |w_\lambda(z)| \left| (\bar\partial\chi)\Big(\frac{z-\lambda}{\delta_\lambda}\Big) \right| \frac{1}{\delta_\lambda} \\ &\lesssim \frac{1}{\delta_\lambda} \sum_{l=0}^{m_\lambda-1} |v_{\lambda,l}|. \end{aligned}$$

Using the formula for δ_λ and proceeding as before we get the last part of the assertion.

□

The next lemma shows how finiteness of a particular integral of a function may cause all its Taylor coefficients to be zero up to some order. This will be useful later, as the correction of a C^∞ solution to the interpolation problem obtained by Hörmander's theorem will satisfy a certain weighted integral estimate.

Lemma 5.2.7. *Let $\lambda \in \mathbb{C}$ and $\rho > 0$. Assume that $u \in C^\infty(D(\lambda, \rho))$ is such that*

$$\int_{D(\lambda,\rho)} |u(z)|^2 \frac{1}{|z-\lambda|^{2m}} \,\mathrm{dz} < \infty. \tag{5.11}$$

for some $m \in \mathbb{N}$. Then all partial derivatives of u in λ up to order $m-1$ are equal to zero.

Proof. Without loss of generality we may assume that $\lambda = 0$. Throughout the proof we will use the notation $z = (x, y)$ or $z = x + iy$ interchangeably.

We will show that existence of a non-zero derivative of order less than m in $\lambda = 0$ contradicts the assumption about finiteness of the integral (5.11). Suppose that the smallest order of a non-zero derivative is l, and it satisfies $0 \le l < m$. Then all partial derivatives of u up to order $l-1$ are 0 in $\lambda = 0$. Therefore by Taylor's theorem

$$u(x,y) = \sum_{j=l}^{m-1} \sum_{\substack{k,n\in\mathbb{N}_0\\ k+n=j}} \frac{\partial_x^k \partial_y^n u(0)}{k!n!} x^k y^n + \sum_{\substack{k,n\in\mathbb{N}_0\\ k+n=m}} R_{k,n}(x,y) x^k y^n$$

where $R_{k,n}\colon \mathbb{C} \to \mathbb{R}$ are bounded on $D(0,\rho)$. Denote

$$P_j(x,y) = \sum_{\substack{k,n\in\mathbb{N}_0\\ k+n=j}} \frac{\partial_x^k \partial_y^n u(0)}{k!n!} x^k y^n$$

for $j = l, \ldots, m-1$. Further, let $P(x,y) = \sum_{j=l}^{m} P_j(x,y)$ and

$$R(x,y) = \sum_{\substack{k,n\in\mathbb{N}_0\\ k+n=m}} R_{k,n}(x,y) x^k y^n.$$

Let

$$\|f\| = \left(\int_{D(0,\rho)} |u(z)|^2 \frac{1}{|z-\lambda|^{2m}} \,\mathrm{dz} \right)^{\frac{1}{2}}$$

for $f \in C^\infty(D(0,\rho))$. Further, consider the following space

$$E = \{ f \in C^\infty(D(0,\rho)) |\ \|f\| < \infty \}.$$

By Minkowski's inequality, $\|\cdot\|$ is a norm on E.

We have

$$\begin{aligned} |R(x,y)| &\le \sum_{\substack{k,n\in\mathbb{N}_0\\ k+n=m}} |R_{k,n}(x,y)||x|^k|y|^n \\ &\le \sum_{\substack{k,n\in\mathbb{N}_0\\ k+n=m}} |R_{k,n}(x,y)||z|^{k+n} \le C|z|^m \end{aligned}$$

for some $C > 0$ and every $z \in D(\lambda, r)$. Therefore $\|R\| < \infty$ and, by (5.11),

$$\|P\| \le \|P+R\| + \|R\| = \|u\| + \|R\| < \infty. \tag{5.12}$$

Further, every P_j is a homogenous polynomial, i.e, $P_j(rz) = r^j P_j(z)$ for $r \ge 0$ such that $rz \in D(0,\rho)$. In other words, for every j there exists a trigonometric polynomial f_j such that $P_j(re^{i\theta}) = r^j f_j(\theta)$ for $0 \le r < \rho$, $0 \le \theta \le 2\pi$. By the assumption on derivatives of u, f_l is not equal identically zero. Hence we may choose an angle $[\alpha_1, \alpha_2]$ $(0 \le \alpha_1 < \alpha_2 < 2\pi)$ such that $f_l(\theta) > \epsilon$ for some $\epsilon > 0$ and every $\theta \in [\alpha_1, \alpha_2]$. Moreover, for some $M > 0$

$$\max_{l+1 \le j < m} \sup_{\theta \in [0,2\pi]} |f_j(\theta)| \le M.$$

Then for $\theta \in [\alpha_1, \alpha_2]$

$$|P(re^{i\theta})| = \Big|\sum_{j=l}^{m} r^j f_j(\theta)\Big| \ge r^l |f_l(\theta)| - \sum_{j=l+1}^{m} r^j |f_j(\theta)| \ge r^l \Big(\epsilon - M \sum_{j=1}^{m-l} r^j\Big).$$

But then for some η satisfying $0 < \eta < \rho$, and every r satisfying $0 \le r < \eta$

$$\epsilon - M \sum_{j=1}^{m-l} r^j \ge \frac{1}{2}.$$

Finally,

$$\begin{aligned} \|P\|^2 &= \int_{D(0,\rho)} |P(z)|^2 \frac{1}{|z-\lambda|^{2m}} \,\mathrm{dz} = \int_0^{2\pi}\int_0^{\rho} |P(re^{i\theta})|^2 r^{-2m+1} \,\mathrm{dr}\,\mathrm{d}\theta \\ &\ge \int_{\alpha_1}^{\alpha_2}\int_0^{\eta} |P(re^{i\theta})|^2 r^{-2m+1} \,\mathrm{dr}\,\mathrm{d}\theta \ge \frac{1}{4}\int_{\alpha_1}^{\alpha_2}\int_0^{\eta} r^{2l} r^{-2m+1} \,\mathrm{dr}\,\mathrm{d}\theta \end{aligned}$$

and, since $2l + 1 - m \le -1$, the last integral is divergent. □

The next theorem is still much too cumbersome to be useful in practice for checking whether a given multiplicity variety is interpolating, but, as will become clear later, it is one of the main ingredients of the interpolation theory for the spaces $A_{(\omega)}$ and $A_{\{\omega\}}$.

Theorem 5.2.8. *Let ω be a Beurling weight. Denote $p(z) = |\operatorname{Im} z| + \omega(z)$. Assume that there exist a subharmonic function $\tilde{p}\colon \mathbb{C} \to [0, \infty)$ such that for every $R > 0$ there exists $C_R > 0$ for which*

$$\frac{1}{C_R}\tilde{p}(z) \le p(z) \le C_R \tilde{p}(z) \tag{5.13}$$

for every $z \in \mathbb{C} \setminus D(0, R)$, and assume that there exist $C > 0$ and $\epsilon_0 > 0$ such that

$$\frac{1}{\tilde{p}(z)} \le C\Delta\tilde{p}(z) \tag{5.14}$$

on $\bigcup_{\lambda\in\Lambda} D(\lambda, \epsilon_0 p(\lambda))$. If a multiplicity variety X is (ω)-sparse then it is interpolating for $A_{(\omega)}$.

Proof. Finite sets are always interpolating, and finite unions of interpolating varieties whose union is weakly separated are again interpolating by Theorem 2.3.6. Hence we may assume that $0 \notin \Lambda$ or even that $\inf_{\lambda\in\Lambda}|\lambda|$ is bigger than some arbitrarily given constant. By Propositions 4.1.10 and 4.1.5, X is weakly (ω)-separated. Therefore, using (5.13) additionally, we can choose a constant $C_s > 0$ such that for

$$\delta_\lambda = \delta e^{-C_s \frac{p(\lambda)}{m_\lambda}}$$

the discs $D(\lambda, 2\delta_\lambda)$ are disjoint.

Let $\{v_{\lambda,l} |\ \lambda \in \Lambda,\ 0 \le l < m_\lambda\} \in S_{(\omega)}(X)$. By Lemma 5.2.6 there exists a $C^\infty_{(\omega)}$ solution F to the interpolation problem $R(F) = v$.

Now we want to use Hörmander's theorem (see [BG95, Theorem 2.1.3]) for the equation

$$\bar\partial u = \bar\partial F$$

with a suitable subharmonic function ψ. Then we would have

$$2\int_{\mathbb{C}} |u(z)|^2 \frac{e^{-\psi(z)}}{(1+|z|^2)^2}\,\mathrm{dz} \le \int_{\mathbb{C}} |\bar\partial F(z)|^2 e^{-\psi(z)}\,\mathrm{dz}.$$

We have to choose ψ such that the integral on the right hand side will be finite, u will belong to $C^\infty_{(\omega)}$ and will have Taylor coefficients equal 0 on X. Then our solution to the interpolation problem will be of the form $f = F - u$.

From Lemma 5.2.5 applied to $\tilde p$ we obtain constants $K > 0$, $\beta_0 > 0$ and a function $v\colon \mathbb{C} \to \mathbb{R}$ satisfying $v(z) \le 0$ such that for every $\beta > \beta_0$ the function $\psi(z) = \beta\tilde p(z) + v(z)$ is subharmonic and satisfies

$$|\psi(z) - \beta p(z)| \le K p(z)$$

for every $z \in \operatorname{supp} \bar\partial F$. Then, since $\bar\partial F \in C^\infty_{(\omega)}$, by Proposition 1.7.10 we obtain that

$$\int_{\mathbb{C}} |\bar\partial F(z)|^2 e^{-\psi(z)}\,\mathrm{dz} \le \int_{\mathbb{C}} |\bar\partial F(z)|^2 e^{(K-\beta)p(z)}\,\mathrm{dz} < \infty$$

if β is chosen big enough. Therefore we can use Hörmander's theorem to obtain $u \in C^\infty(\mathbb{C})$ such that $\bar\partial u = \bar\partial F$ and

$$2\int_{\mathbb{C}} |u(z)|^2 \frac{e^{-\psi(z)}}{(1+|z|^2)^2}\,\mathrm{dz} \le \int_{\mathbb{C}} |\bar\partial F(z)|^2 e^{-\psi(z)}\,\mathrm{dz}. \tag{5.15}$$

Further, by property (γ) of ω, there are constants $C, D > 0$ such that $\ln(1+|z|^2) \le Cp(z) + D$. In other words

$$(1+|z|^2)^2 \le e^{2D} e^{2Cp(z)}.$$

Moreover, by (5.13) and the upper semi-continuity of $\tilde{p}$

$$\tilde{p}(z) \leq C_R p(z) + A$$

for some $A > 0$ and every $z \in \mathbb{C}$. Let $\alpha = \beta C_R + 2C$. Since $v(z) \leq 0$ for all $z \in \mathbb{C}$, we obtain then

$$\begin{aligned} e^{-\alpha p(z)} = e^{-\beta C_R p(z)} e^{-2Cp(z)} &\leq e^{2D+\beta A} \frac{e^{-\beta \tilde{p}(z)}}{(1+|z|^2)^2} \\ &\leq e^{2D+\beta A} \frac{e^{-\beta \tilde{p}(z) - v(z)}}{(1+|z|^2)^2} = e^{2D+\beta A} \frac{e^{-\psi(z)}}{(1+|z|^2)^2}. \end{aligned}$$

Therefore by (5.15)

$$\int_{\mathbb{C}} |u(z)|^2 e^{-\alpha p(z)} \,\mathrm{dz} \lesssim \int_{\mathbb{C}} |u(z)|^2 \frac{e^{-\psi(z)}}{(1+|z|^2)^2} \,\mathrm{dz} < \infty.$$

This shows that $u \in W_{(\omega)}(\mathbb{C})$. Further, by Proposition 1.7.10, $F \in W_{(\omega)}(\mathbb{C})$, since $F \in C^\infty_{(\omega)}$. This yields $f = F - u \in W_{(\omega)}(\mathbb{C})$. Furthermore, $f \in H(\mathbb{C})$ hence, by Proposition 1.7.10, $f \in A_{(\omega)}$.

It remains to show that $\partial^l u(\lambda) = 0$ for all $\lambda \in \Lambda$, $0 \leq l < m_\lambda$. By (5.15) and Lemma 5.2.5 (iii) it holds

$$\begin{aligned} \infty > \int_{\mathbb{C}} |u(z)|^2 \frac{e^{-\psi(z)}}{(1+|z|^2)^2} \,\mathrm{dz} &\geq \int_{D(\lambda,1)} |u(z)|^2 \frac{e^{-\psi(z)}}{(1+|z|^2)^2} \,\mathrm{dz} \\ &\geq \frac{1}{(|\lambda|^2 + 2|\lambda| + 2)^2} \int_{D(\lambda,1)} |u(z)|^2 e^{-\psi(z)} \,\mathrm{dz} \\ &\geq \frac{1}{C_{\lambda,\beta}(|\lambda|^2 + 2|\lambda| + 2)^2} \int_{D(\lambda,1)} |u(z)|^2 \frac{1}{|z-\lambda|^{2m_\lambda}} \,\mathrm{dz}. \end{aligned}$$

Therefore we may complete the proof with the use of Lemma 5.2.7. □

In Theorem 5.2.8 one could just take $\tilde{p}(z) = |\mathrm{Im}\, z| + \omega(z)$ for the first assumption to be satisfied. But one cannot hope that the second assumption $\frac{1}{\tilde{p}(z)} \lesssim \Delta\tilde{p}(z)$ will hold for arbitrary weights. To make Theorem 5.2.8 useful, we need to find a way of constructing equivalent weights with this important property. As will be shown in Lemma 5.2.10, this can be done at least on some subset of $\mathbb{C}$. For the proof we will need the following inequalities.

Lemma 5.2.9. *The following inequalities hold:*

(1) for any $a \in \mathbb{R}$ and $r > 0$

$$\int_{a-r}^{a+r} \ln|x| \,\mathrm{dx} = \int_{-r}^{r} \ln|x+a| \,\mathrm{dx} \geq \int_{-r}^{r} \ln|x| \,\mathrm{dx},$$

(2) for any $\xi \in \mathbb{C}$ and $r > 0$

$$\int\limits_{D(\xi,r)} \ln|z|\,\mathrm{dz} = \int\limits_{D(0,r)} \ln|z+\xi|\,\mathrm{dz} \geq \int\limits_{D(0,r)} \ln|z+\xi|\,\mathrm{dz}.$$

Proof. (1) Case I: If $a + r \leq -r$ then $\ln|x| \geq \ln|t|$ for every $x \in [a-r, a+r]$ and $y \in [-r, r]$.

Case II: If $a - r < -r < a + r < r$ then $\ln|x| \geq \ln|t|$ for every $x \in [a-r, -r]$ and $t \in [a+r, r]$. Therefore

$$\int\limits_{a-r}^{a+r} \ln|x|\,\mathrm{dx} = \int\limits_{a-r}^{-r} \ln|x|\,\mathrm{dx} + \int\limits_{-r}^{a+r} \ln|x|\,\mathrm{dx} \geq \int\limits_{a+r}^{r} \ln|x|\,\mathrm{dx} + \int\limits_{-r}^{a+r} \ln|x|\,\mathrm{dx} = \int\limits_{-r}^{r} \ln|x|\,\mathrm{dx}.$$

Case III: If $-r < a - r < r < a + r$ then $\ln|x| \geq \ln|t|$ for every $x \in [r, a+r]$ and $t \in [-r, a-r]$. Therefore

$$\int\limits_{a-r}^{a+r} \ln|x|\,\mathrm{dx} = \int\limits_{a-r}^{r} \ln|x|\,\mathrm{dx} + \int\limits_{r}^{a+r} \ln|x|\,\mathrm{dx} \geq \int\limits_{a-r}^{r} \ln|x|\,\mathrm{dx} + \int\limits_{-r}^{a-r} \ln|x|\,\mathrm{dx} = \int\limits_{-r}^{r} \ln|x|\,\mathrm{dx}.$$

Case IV: If $r \leq a - r$ then $\ln|x| \geq \ln|t|$ for every $x \in [a-r, a+r]$ and $y \in [-r, r]$.

(2) By Lemma 5.2.4, we have

$$\frac{1}{\pi r^2} \int\limits_{D(\xi,r)} \ln|\zeta|^2\,\mathrm{d}\zeta = \begin{cases} \ln|\xi|^2 & \text{if } |\xi| > r, \\ \frac{|\xi|^2}{r^2} + \ln r^2 - 1 & \text{if } |\xi| \leq r, \end{cases}$$

for any $\xi \in \mathbb{C}$ and $r > 0$. Therefore

$$\int\limits_{D(\xi,r)} \ln|\zeta|\,\mathrm{d}\zeta = \begin{cases} \pi r^2 \ln|\xi| & \text{if } |\xi| > r, \\ \pi(\frac{1}{2}|\xi|^2 + r^2 \ln r - \frac{r^2}{2}) & \text{if } |\xi| \leq r, \end{cases}$$

for any $\xi \in \mathbb{C}$ and $r > 0$. Further, for $|\xi| > r$ we obtain

$$\int\limits_{D(\xi,r)} \ln|z|\,\mathrm{dz} = \pi r^2 \ln|\xi| = \int\limits_{D(0,r)} \ln|\xi|\,\mathrm{dz} \geq \int\limits_{D(0,r)} \ln|z|\,\mathrm{dz}.$$

On the other hand, for $|\xi| \leq r$

$$\int\limits_{D(\xi,r)} \ln|z|\,\mathrm{dz} = \pi\Big(\frac{1}{2}|\xi|^2 + r^2 \ln r - \frac{r^2}{2}\Big) \geq \pi\Big(r^2 \ln r - \frac{r^2}{2}\Big) = \int\limits_{D(0,r)} \ln|z|\,\mathrm{dz}. \qquad \square$$

Lemma 5.2.10. *Let ω be a Beurling weight. Denote $p(z) = |\operatorname{Im} z| + \omega(z)$. Then for every $h > 0$ there exists a function $r\colon \mathbb{C} \to [0, \infty)$ such that*

(1) $\tilde{p}(z) := |\operatorname{Im} z| + \frac{1}{\pi} r(z)$ is subharmonic on $\mathbb{C}$,

(2) there exists $C > 0$ such that

$$\frac{1}{C} p(z) \leq \tilde{p}(z)$$

for every $z \in \mathbb{C}$,

(3) for every $R > 0$ there exists $C_R > 0$ such that

$$\tilde{p}(z) \leq C_R p(z)$$

for every $z \in \mathbb{C} \setminus D(0, R)$,

(4) there exists $C > 0$ such that

$$\frac{1}{C} \Delta \tilde{p}(z) \leq \frac{1}{\tilde{p}(z)} \leq C \Delta \tilde{p}(z)$$

for every z satisfying $|\operatorname{Im} z| \leq h\omega(z)$.

Remark 5.2.11. Parts (2) and (3) of the lemma imply that $A_{(\omega)} = A_{(r)}$ and $S_{(\omega)}(X) = S_{(r)}(X)$ for every multiplicity variety X, even though r is not a weight.

Proof. Without loss of generality we may assume that $h \in \mathbb{N}$. By Lemma 1.5.5, ω is weakly subadditive. Therefore there exists $C > 0$ such that

$$\forall\, x, y \in [0, \infty) : \omega(x + y) \leq C(\omega(x) + \omega(y) + 1).$$

We may assume that this C is a positive integer. Then denote $L = 3C$ and $M = 6hL + 18$. These two constants will be appearing throughout the entire proof. By Lemma 1.5.9 there exists $t_0 \in [0, \infty)$ such that for every $t \geq t_0$

$$\frac{1}{L} \leq \frac{\omega(x)}{\omega(t)} \leq L \tag{5.16}$$

for every $x \in (t - 2M\omega(t), t + 2M\omega(t))$. Moreover, by Lemma 1.5.11 there exists $t_1 \in [0, \infty)$ such that for every $t \geq t_1$

$$\frac{1}{L} \leq \frac{\omega(x)}{\omega(t)} \leq L \tag{5.17}$$

for every $x \in (\frac{2}{3}t, \frac{4}{3}t)$.

Claim I. For every $\epsilon > 0$ there exists $R_\epsilon > 0$ such that for $z \notin D(0, R_\epsilon)$, if $|\mathrm{Im}\, z| \le h\omega(z)$ then $|z| \le (1+\epsilon)|\mathrm{Re}\, z|$.

Proof. Choose $R_\epsilon > 0$ such that $\omega(t) < \frac{\epsilon}{h(1+\epsilon)} t$ for every $t \ge R_\epsilon$. This is possible since $\omega(t) = o(t)$. Then let z satisfy $|\mathrm{Im}\, z| \le h\omega(z)$ and $|z| \ge R_\epsilon$. We have

$$|z| \le |\mathrm{Re}\, z| + |\mathrm{Im}\, z| \le |\mathrm{Re}\, z| + h\omega(z) \le |\mathrm{Re}\, z| + \frac{\epsilon}{1+\epsilon}|z|.$$

This yields the assertion. □

We will define the function r as a potential of a specific measure. Now, we will give the construction of this measure. First, we will choose inductively a sequence of real numbers $(x_n)_{n\in\mathbb{Z}}$ such that the intervals

$$I_n = \Big(x_n - \frac{\omega(x_n)}{2}, x_n + \frac{\omega(x_n)}{2}\Big)$$

will be pairwise disjoint and $\bigcup_{n\in\mathbb{Z}} \bar{I_n} = \mathbb{R}$. Denote $R_0 = \max(t_0, t_1, R_{\frac{1}{3}})$, where t_0, t_1 and $R_{\frac{1}{3}}$ are from (5.16), (5.17) and Claim I. Consider then the function $f(x) = x - \frac{\omega(x)}{2}$. It is continuous, $f(0) = 0$ and $f(x) \to \infty$ when $x \to \infty$. Thus it takes every positive real value. Put then $x_0 = 0$, $I_0 = (-R_0, R_0)$ and x_1 with $x_1 - \frac{\omega(x_1)}{2} = R_0$, $I_1 = (x_1 - \frac{\omega(x_1)}{2}, x_1 + \frac{\omega(x_1)}{2})$. Symmetrically we put $x_{-1} = -x_1$. Suppose we have chosen points up to x_{n-1} and $x_{-(n-1)} = -x_{n-1}$. Then we take x_n with

$$x_n - \frac{\omega(x_n)}{2} = R_0 + \sum_{i=1}^{n-1} \omega(x_i)$$

and $x_{-n} = -x_n$. This sequence fulfils the desired properties. We will denote the length of I_n by $\omega_n := \omega(x_n)$ for $n \neq 0$ and $\omega_0 = 2R_0$.

Second, we choose two sequences of measures. For the sake of convenience we denote the one dimensional Lebesgue measure by ν. For $n \in \mathbb{Z}$, $A \subset \mathbb{C}$ define the measure

$$\mathrm{d}\nu_\mathrm{n}(\mathrm{A}) = \int\limits_{A\cap I_n} 1 \,\mathrm{dx} = \nu(A \cap I_n).$$

In fact, these measures just divide the Lebesgue measure into parts according to I_n's, $\nu(A) = \sum_{n\in\mathbb{Z}} \mathrm{d}\nu_\mathrm{n}(\mathrm{A})$. Denote $D_n = D(0, M\omega_n)$ and define

$$\mu_n(z) := \frac{1}{M^2\pi\omega_n^2} \int\limits_{I_n} \chi_{D_n}(z - x)\,\mathrm{dx},$$

where χ_{D_n} is the characteristic function of D_n. Then let $\mathrm{d}\mu_\mathrm{n}(\mathrm{z}) := \mu_n(z)\,\mathrm{dz}$, $\mathrm{d}\mu(\mathrm{z}) = \sum_{n\in\mathbb{Z}} \mathrm{d}\mu_\mathrm{n}(\mathrm{z}) = \sum_{n\in\mathbb{Z}} \mu_n(z)\,\mathrm{dz}$. Here dz stands for the Lebesgue measure on $\mathbb{R}^2$.

Claim II. The support of μ_n is $S_n = I_n + D_n$. For every $z \in \mathbb{C}$

$$0 \leq \mu_n(z) \leq \frac{1}{M^2\pi\omega_n},$$

and $\mu_n(z) = \frac{1}{M^2\pi\omega_n}$ for $z \in D(x_n, (M-1)\omega_n)$. Moreover, there exists $K \in \mathbb{N}$ such that every $z \in \mathbb{C}$ lies in at most K sets S_n. In particular, the formula $\mu(z) = \sum_{n\in\mathbb{Z}} \mu_n(z)$ defines a positive function.

Proof. Observe that

$$\begin{aligned}
\mu_n(z) &= \frac{1}{M^2\pi\omega_n^2} \int\limits_{I_n} \chi_{D_n}(z-x)\,\mathrm{dx} = \frac{1}{M^2\pi\omega_n^2} \int\limits_{I_n} \chi_{D_n+x_n}(z-x+x_n)\,\mathrm{dx} \\
&= \frac{1}{M^2\pi\omega_n^2} \int\limits_{I_n-x_n} \chi_{D_n+x_n}(z-x)\,\mathrm{dx} = \frac{1}{M^2\pi\omega_n^2} \int\limits_{z-I_n+x_n} \chi_{D_n+x_n}(x)\,\mathrm{dx} \\
&= \frac{1}{M^2\pi\omega_n^2} \int\limits_{I_n+z-x_n} \chi_{D_n+x_n}(x)\,\mathrm{dx} = \frac{\mathrm{length}((I_n - x_n + z) \cap D(x_n, M\omega_n))}{M^2\pi\omega_n^2}.
\end{aligned}$$

We used the fact that $I_n - x_n = -I_n + x_n$. This proves the estimates for the values of μ_n. Further, we have

$$\begin{aligned}
\mu_n(z) > 0 &\Leftrightarrow (I_n - x_n + z) \cap D(x_n, M\omega_n) \neq \emptyset \Leftrightarrow (x_n - I_n + z) \cap D(x_n, M\omega_n) \neq \emptyset \\
&\Leftrightarrow (z - I_n) \cap D(0, M\omega_n) \neq \emptyset \Leftrightarrow z \in I_n + D_n.
\end{aligned}$$

This proves the assertion about the support of μ_n. Moreover, if $z \in D(x_n, (M-1)\omega_n)$ then $(I_n - x_n + z) \subset D(x_n, M\omega_n)$. Hence

$$\mu_n(z) = \frac{\mathrm{length}((I_n - x_n + z) \cap D(x_n, M\omega_n))}{M^2\pi\omega_n^2} = \frac{1}{M^2\pi\omega_n}.$$

No we are going to prove that every $z \in \mathbb{C}$ lies in at most K sets S_n, where the constant $K \in \mathbb{N}$ is independent of z. Let $z \in \mathbb{C}$. Without loss of generality we may assume that $\mathrm{Re}\, z > 0$. Furthermore, $z \in S_n$ implies $\mathrm{Re}\, z \in (x_n - (M + \frac{1}{2})\omega_n, x_n + (M + \frac{1}{2})\omega_n)$, hence it is enough to check the assertion for real z. Take $N \in \mathbb{N}$ such that for $n > N$ the intervals $(x_n - 2M\omega(x_n), x_n + 2M\omega(x_n))$ are contained in $\mathbb{R}_+$. Assume that $z \in S_n$ and $z \in I_{n_z}$ with $N \leq n_z < n$. Then

$$2M\omega_n > x_n - z \geq x_n - x_{n_z} - \frac{\omega_{n_z}}{2} = \sum_{k=n_z+1}^{n-1} \omega_k + \frac{\omega_n}{2}$$

and, by (5.16), we obtain

$$2M\omega_n \geq \frac{1}{L} \sum_{k=n_z+1}^{n-1} \omega_n + \frac{\omega_n}{2} = \Big(\frac{n - n_z - 2}{L} + \frac{1}{2}\Big)\omega_n.$$

This yields $n < n_z + 2ML - \frac{1}{2}L + 2 < n_z + 2ML + 2$. Assume now that $z \in S_n$ and $z \in I_{n_z}$ with $N < n < n_z$. Then

$$\begin{aligned} 2M\omega_n > z - x_n \geq x_{n_z} - \frac{\omega_{n_z}}{2} - x_n &= \frac{\omega_n}{2} + \sum_{k=n+1}^{n_z-1} \omega_k \\ &\geq \frac{\omega_n}{2} + \sum_{k=n+1}^{n_z-1} \omega_n = \Big(\frac{1}{2} + n_z - n - 2\Big)\omega_n. \end{aligned}$$

This yields $n > n_z - 2M - \frac{3}{2}$. Finally, z belongs to at most $2N + 2ML + 2M + 3$ sets S_n. □

Define

$$r(z) := \int_{\mathbb{C}} \ln|z - w|(\,\mathrm{d}\mu(\mathrm{w}) - \,\mathrm{d}\nu(\mathrm{w})) = \sum_{n\in\mathbb{Z}} r_n(z)$$

where

$$r_n(z) := \int_{\mathbb{C}} \ln|z - w|(\,\mathrm{d}\mu_{\mathrm{n}}(\mathrm{w}) - \,\mathrm{d}\nu_{\mathrm{n}}(\mathrm{w})).$$

Since every $z \in \mathbb{C}$ belongs to finitely many supports of μ_n's (and at the same time ν_n's, since $\operatorname{supp} \nu_n \subset \operatorname{supp} \mu_n$), r is always finite. Then we let $\tilde{p}(z) = |\operatorname{Im} z| + \frac{1}{\pi} r(z)$.

Claim III. The function r is positive and equals 0 for $z \notin \bigcup_{n\in\mathbb{Z}} S_n$. Functions r_n can be written in the following form

$$r_n(z) = \frac{1}{M^2\pi\omega_n^2} \int_{I_n} \int_{D(x,M\omega_n)} \ln \frac{|z - w|}{|z - x|} \,\mathrm{dw}\,\mathrm{dx}.$$

Moreover, $\Delta\tilde{p}(z) = 2\mu(z)$ for all $z \in \mathbb{C}$, and $\tilde{p}$ is subharmonic.

Proof. We have

$$\begin{aligned} r_n(z) &= \int_{S_n} \ln|z - w| \,\mathrm{d}\mu_{\mathrm{n}}(\mathrm{w}) - \int_{I_n} \ln|z - x| \,\mathrm{dx} \\ &= \frac{1}{M^2\pi\omega_n^2} \int_{S_n} \left(\int_{I_n} \ln|z - w| \chi_{D_n}(w - x) \,\mathrm{dx} \right) \mathrm{dw} - \int_{I_n} \ln|z - x| \,\mathrm{dx}. \end{aligned}$$

As $\ln|z|$ is locally integrable and μ_n, ν_n are finite Borel measures we can use Fubini's theorem (see [Rud87, Theorem 8.8]), hence

$$\begin{aligned}
r_n(z) &= \frac{1}{M^2\pi\omega_n^2}\int\limits_{I_n}\left(\int\limits_{S_n}\ln|z-w|\chi_{D_n}(w-x)\,\mathrm{dw}\right)\mathrm{dx}-\int\limits_{I_n}\ln|z-x|\,\mathrm{dx}\\
&= \frac{1}{M^2\pi\omega_n^2}\int\limits_{I_n}\left(\int\limits_{D(x,M\omega_n)}\ln|z-w|\,\mathrm{dw}\right)\mathrm{dx}-\int\limits_{I_n}\ln|z-x|\,\mathrm{dx} \qquad (5.18)\\
&= \int\limits_{I_n}\left(\frac{1}{M^2\pi\omega_n^2}\int\limits_{D(x,M\omega_n)}\ln|z-w|\,\mathrm{dw}-\ln|z-x|\right)\mathrm{dx}.
\end{aligned}$$

Since $\ln|z-\cdot|$ is subharmonic the expression under the integral sign is positive. This proves that r is positive. Finally, if $z\notin S_n$, $x\in I_n$ then $\ln|z-\cdot|$ is harmonic in $D(x,M\omega_n)$, thus the expression under the integral sign is 0.

Further, since

$$\int\limits_{D(x,M\omega_n)} 1\,\mathrm{dw} = M^2\pi\omega_n^2,$$

from (5.18) we obtain

$$r_n(z)=\frac{1}{M^2\pi\omega_n^2}\int\limits_{I_n}\int\limits_{D(x,M\omega_n)}\ln\frac{|z-w|}{|z-x|}\,\mathrm{dw}\,\mathrm{dx}.$$

Now, we will prove the last part of the assertion. The sum in the definition of r is locally finite, hence, by [Ran95, Theorem 3.7.4],

$$\Delta r=\sum_{n\in\mathbb{Z}}\Delta r_n=\sum_{n\in\mathbb{Z}}2\pi(\mu_n-\nu_n)=2\pi(\mu-\nu).$$

Recalling that $\Delta|\operatorname{Im} z|=2\nu$ we get $\Delta\tilde{p}=2\mu$ in the sense of distributions. But μ is an absolutely continuous measure generated by the function $\mu(\cdot)$, which is positive. Therefore $\tilde{p}$ is subharmonic. □

Claim IV. It holds

$$\{z\in\mathbb{C}|\ |\operatorname{Im} z|\le h\omega(z)\}\subset\bigcup_{n\in\mathbb{Z}}D\Big(x_n,\frac{M}{6}\,\omega_n\Big).$$

More precisely,

$$\{z\in\mathbb{C}|\ |\operatorname{Im} z|\le h\omega(z)\}\setminus D(0,R_0)\subset\bigcup_{n\in\mathbb{Z}\setminus\{0\}}D\Big(x_n,\frac{M}{6}\,\omega_n\Big)$$

and

$$D(0,R_0)\subset D(x_0,\frac{M}{6}\omega_0).$$

Proof. Let z satisfy $|\operatorname{Im} z| \leq h\omega(z)$. If $z \in D(0, R_0)$ then

$$|z - x_0| = |z| \leq R_0 = \frac{\omega_0}{2}.$$

Let $z \notin D(0, R_0)$. We may assume that $\operatorname{Re} z \geq 0$, since our construction is symmetric. Let $n \in \mathbb{N}$ be the smallest number such that $\operatorname{Re} z < x_n$. Then

$$|x_n - z| \leq |x_n - \operatorname{Re} z| + |\operatorname{Im} z| \leq 2\omega_n + h\omega(z).$$

By condition (5.17) and Claim I for $\epsilon = \frac{1}{3}$, we obtain then

$$\omega(z) \leq L\omega\Big(\frac{2}{3}|z|\Big) \leq L\omega\Big(\frac{8}{9}|\operatorname{Re} z|\Big) \leq L\omega(|\operatorname{Re} z|) \leq L\omega_n.$$

Hence $z \in D(x_n, (3 + hL)\omega_n) \subset D(x_n, \frac{M}{6}\,\omega_n)$. □

Claim V. For every $R > 0$ there exists $C_R > 0$ such that $r(z) \leq C_R\omega(z)$ for every $z \in \mathbb{C} \setminus D(0, R)$.

Proof. If $z \notin \bigcup_{n\in\mathbb{Z}} S_n$ then, by Claim III, $r(z) = 0 \leq \omega(z)$. We have

$$\nu_n(\mathbb{C}) = \int_{\mathbb{C}} \mathrm{d}\nu_\mathrm{n} = \omega_n$$

and by Fubini's theorem

$$\begin{aligned}\mu_n(\mathbb{C}) &= \int_{\mathbb{C}} \mu_n(z)\,\mathrm{dz} = \frac{1}{M^2\pi\omega_n^2}\int_{\mathbb{C}}\int_{I_n} \chi_{D_n}(z - x)\,\mathrm{dx}\,\mathrm{dz} \\ &= \frac{1}{M^2\pi\omega_n^2}\int_{I_n}\int_{\mathbb{C}} \chi_{D_n}(z - x)\,\mathrm{dz}\,\mathrm{dx} = \frac{1}{M^2\pi\omega_n^2}\int_{I_n}\int_{\mathbb{C}} \chi_{D_n}(z)\,\mathrm{dz}\,\mathrm{dx} \\ &= \int_{I_n} \mathrm{dx} = \omega_n.\end{aligned}$$

Moreover, for any $z \in \mathbb{C}$ and $n \in \mathbb{Z}$

$$\int_{\mathbb{C}} \ln\frac{|z - w|}{\omega_n}\,\mathrm{d}\nu_\mathrm{n}(\mathrm{w}) = \int_{I_n} \ln\frac{|z - x|}{\omega_n}\,\mathrm{dx} \geq \int_{I_n} \ln\frac{|\operatorname{Re} z - x|}{\omega_n}\,\mathrm{dx}$$

and as x_n is the centre of the interval I_n, by Lemma 5.2.9, we obtain

$$\int_{I_n} \ln\frac{|\operatorname{Re} z - x|}{\omega_n}\,\mathrm{dx} \geq \int_{I_n} \ln\frac{|x_n - x|}{\omega_n}\,\mathrm{dx} = \omega_n\int_{-\frac{1}{2}}^{\frac{1}{2}} \ln|x|\,\mathrm{dx} = -\omega_n \ln 2e$$

Hence

$$\int_{\mathbb{C}} \ln|z-w|\,\mathrm{d}\nu_{\mathrm{n}}(\mathrm{w}) \geq \int_{\mathbb{C}} \ln\omega_n\,\mathrm{d}\nu_{\mathrm{n}}(\mathrm{w}) - \omega_n \ln 2e$$

$$= \frac{1}{2}\int_{\mathbb{C}} \ln\omega_n(\,\mathrm{d}\mu_{\mathrm{n}}(\mathrm{w}) + \,\mathrm{d}\nu_{\mathrm{n}}(\mathrm{w})) - \omega_n \ln 2e.$$

Then for $z \in S_n$

$$\begin{aligned} r_n(z) &= \int_{\mathbb{C}} \ln|z-w|(\,\mathrm{d}\mu_{\mathrm{n}}(\mathrm{w}) - \,\mathrm{d}\nu_{\mathrm{n}}(\mathrm{w})) \\ &= \int_{\mathbb{C}} \ln|z-w|(\,\mathrm{d}\mu_{\mathrm{n}}(\mathrm{w}) + \,\mathrm{d}\nu_{\mathrm{n}}(\mathrm{w})) - 2\int_{\mathbb{C}} \ln|z-w|\,\mathrm{d}\nu_{\mathrm{n}}(\mathrm{w}) \\ &\leq \int_{\mathbb{C}} \ln|z-w|(\,\mathrm{d}\mu_{\mathrm{n}}(\mathrm{w}) + \,\mathrm{d}\nu_{\mathrm{n}}(\mathrm{w})) - \int_{\mathbb{C}} \ln\omega_n(\,\mathrm{d}\mu_{\mathrm{n}}(\mathrm{w}) + \,\mathrm{d}\nu_{\mathrm{n}}(\mathrm{w})) + 2\omega_n \ln 2e \\ &= \int_{S_n} \ln\frac{|z-w|}{\omega_n}(\,\mathrm{d}\mu_{\mathrm{n}}(\mathrm{w}) + \,\mathrm{d}\nu_{\mathrm{n}}(\mathrm{w})) + 2\omega_n \ln 2e \\ &\leq \int_{S_n} \ln 3M(\,\mathrm{d}\mu_{\mathrm{n}}(\mathrm{w}) + \,\mathrm{d}\nu_{\mathrm{n}}(\mathrm{w})) + 2\omega_n \ln 2e \\ &= (2\ln 3M + 2\ln 2e)\omega_n = (2\ln 6Me)\omega_n. \end{aligned}$$

Moreover, for $z \in S_n$ we have $\operatorname{Re} z \in (x_n - 2M\omega_n, x_n + 2M\omega_n)$. For $n \neq 0$ by condition (5.16) we obtain then

$$r_n(z) \leq (2\ln 6Me)\omega(x_n) \leq L(2\ln 6Me)\omega(|\operatorname{Re} z|) \leq L(2\ln 6Me)\omega(z).$$

Since r_0 is bounded from above and ω is bounded from below by a positive constant on $S_0 \setminus D(0,R)$ we can find a constant $C > 0$ such that

$$r_0(z) \leq C\omega(z)$$

for every $z \in S_0 \setminus D(0,R)$. As every point z lies in at most K sets S_n we obtain the assertion. □

Claim VI. There exists $C > 0$ such that $\omega(z) \leq Cr(z)$ for all z satisfying $|\operatorname{Im} z| \leq h\omega(z)$. Moreover, for some $\delta > 0$, $r_0(z) \geq \delta$ for all $z \in D(0,R_0)$, and $r(z) \geq \delta$ for every z satisfying $|\operatorname{Im} z| \leq h\omega(z)$.

Proof. Let z satisfy $|\operatorname{Im} z| \leq h\omega(z)$. By Claim IV, $z \in D(x_n, \frac{M}{6}\omega_n)$ for some $n \in \mathbb{Z}$.

Then for each $x \in I_n$ we have $|z - x| \leq (\frac{M}{6} + \frac{1}{2})\omega_n < \frac{M}{3}\,\omega_n$ and

$$\frac{1}{M^2\pi\omega_n^2} \int\limits_{D(x,M\omega_n)} \ln\frac{|z-w|}{|z-x|}\,\mathrm{dw} \geq \frac{1}{M^2\pi\omega_n^2} \int\limits_{D(x,M\omega_n)} \ln\frac{3|z-w|}{M\omega_n}\,\mathrm{dw}$$

$$= \frac{1}{9\pi} \int\limits_{D(3\frac{x-z}{M\omega_n},3)} \ln|w|\,\mathrm{dw}.$$

Then, by Lemma 5.2.9,

$$\frac{1}{M^2\pi\omega_n^2} \int\limits_{D(x,M\omega_n)} \ln\frac{|z-w|}{|z-x|}\,\mathrm{dw} \geq \frac{1}{9\pi} \int\limits_{D(0,3)} \ln|w|\,\mathrm{dw} = \ln 3 - \frac{1}{2} > 0$$

Thus, by Claim III,

$$r_n(z) = \frac{1}{M^2\pi\omega_n^2} \int\limits_{I_n} \int\limits_{D(x,M\omega_n)} \ln\frac{|z-w|}{|z-x|}\,\mathrm{dw}\,\mathrm{dx} \geq \delta\omega_n$$

for some constant $\delta > 0$. For z with $|z| \geq R_0$ by (5.17), Claim I and (5.16) we obtain then

$$\omega(z) \leq L\omega\Big(\frac{2}{3}|z|\Big) \leq L\omega(|\mathrm{Re}\, z|) \leq L\omega\Big(x_n + \frac{M}{6}\,\omega_n\Big) \leq L^2\omega_n \leq \frac{L^2}{\delta}\,r_n(z).$$

As ω is continuous there exists $C > 0$ such that

$$\omega(z) \leq C$$

for all $z \in D(0, R_0)$. Further, $D(0, R_0) \subset D(0, \frac{M}{3}R_0) = D(x_0, \frac{M}{6}\omega_0)$. Hence for z with $|z| < R_0$ we have

$$\omega(z) \leq C = \frac{C}{\delta\omega_0}\delta\omega_0 \leq \frac{C}{\delta\omega_0} r_0(z).$$

As $r_n(z) \leq r(z)$ for every $n \in \mathbb{Z}$ this completes the proof of Claim VI. □

Claim VII. There exists $C > 0$ such that $\frac{1}{C}p(z) \leq \tilde{p}(z)$ for all $z \in \mathbb{C}$. For every $R > 0$there exists $C_R > 0$ such that $\tilde{p}(z) \leq C_R p(z)$ for $z \in \mathbb{C} \setminus D(0, R)$. Therefore, assertions (2) and (3) of the lemma are satisfied.

Proof. By Claim V there exists $C > 0$ such that

$$\tilde{p}(z) = |\mathrm{Im}\, z| + \frac{1}{\pi} r(z) \leq |\mathrm{Im}\, z| + C\omega(z) \leq Cp(z)$$

for $z \in \mathbb{C} \setminus D(0, R)$. For z with $|\mathrm{Im}\, z| > h\omega(|z|)$ we have

$$p(z) \leq 2|\mathrm{Im}\, z| \leq 2\tilde{p}(z).$$

Finally, for z with $|\mathrm{Im}\, z| \leq h\omega(|z|)$ by Claim VI

$$p(z) = |\mathrm{Im}\, z| + \omega(z) \leq |\mathrm{Im}\, z| + Cr(z) \leq C\pi\tilde{p}(z).$$ □

Claim VIII. There exists $C > 0$ such that

$$\frac{1}{C}\Delta\tilde{p}(z) \le \frac{1}{\tilde{p}(z)} \le C\Delta\tilde{p}(z)$$

for all z with $|\operatorname{Im} z| \le h\omega(z)$. Therefore, condition (3) of the lemma is satisfied.

Proof. Let z satisfy $|\operatorname{Im} z| \le h\omega(z)$. Assume that $z \in D(0, R_0)$. In this case by Claim VI, Claim II, and Claim III, we obtain

$$\begin{aligned}\frac{1}{\tilde{p}(z)} &= \frac{1}{|\operatorname{Im} z| + \frac{1}{\pi}r(z)} \le \frac{\pi}{r(z)} \le \frac{\pi}{r_0(z)} \le \frac{\pi}{\delta} = \frac{\pi}{\delta\omega_0}\mu_0(z)\\ &\le \frac{M^2\pi^2\omega_0}{\delta}\mu(z) = \frac{M^2\pi^2\omega_0}{2\delta}\Delta\tilde{p}(z).\end{aligned}$$

Take now z with $|z| \ge R_0$. By Claim IV, $z \in D(x_n, \frac{M}{6}\,\omega_n)$ for some $n \in \mathbb{Z} \setminus \{0\}$. Using consecutively Claim VI, condition (5.16), Claim II, and Claim III, we obtain

$$\begin{aligned}\frac{1}{\tilde{p}(z)} &= \frac{1}{|\operatorname{Im} z| + \frac{1}{\pi}r(z)} \le \frac{\pi}{r(z)} \le \frac{C\pi}{\omega(z)} \le \frac{CL\pi}{\omega_n}\\ &= CL\pi^2M^2\mu_n(z) \le CL\pi^2M^2\mu(z) = \frac{1}{2}CL\pi^2M^2\Delta\tilde{p}(z).\end{aligned}$$

Now, we are going to prove the first inequality. Let z satisfy $|\operatorname{Im} z| \le h\omega(z)$. By Claim II there exists $n \in \mathbb{Z}$ such that $z \in S_n$ and

$$\mu(z) \le K\mu_n(z).$$

Suppose that $n = 0$. Since S_0 is compact μ and $\tilde{p}$ are bounded above there by a constant $C > 0$. Therefore,

$$\Delta\tilde{p}(z) = 2\mu(z) \le 2C \le 2C^2\frac{1}{\tilde{p}(z)}.$$

Assume now that $n \ne 0$ and $z \notin S_0$. Using consecutively Claim III, Claim II, condition (5.16), and Claim V, we obtain

$$\begin{aligned}\Delta\tilde{p}(z) &= 2\mu(z) \le 2K\mu_n(z) \le \frac{2K}{M^2\pi\omega_n} \le \frac{LK}{M^2\pi}\frac{1}{\omega(z)}\\ &\le \frac{(h+1)LK}{M^2\pi}\frac{1}{|\operatorname{Im} z| + \omega(z)} \le \frac{(h+1)LK}{M^2\pi\max(\frac{\pi}{C}, 1)}\frac{1}{|\operatorname{Im} z| + \frac{1}{\pi}r(z)}\\ &= \frac{(h+1)LK}{M^2\pi\max(\frac{\pi}{C}, 1)}\frac{1}{\tilde{p}(z)}.\end{aligned}$$

□

Now, we are able to state the first really useful theorem giving a relatively simple method of checking whether a given multiplicity variety is interpolating.

Theorem 5.2.12. *Let ω be a Beurling weight and X an (ω)-sparse multiplicity variety contained in the set $\{z \in \mathbb{C} | \ |\mathrm{Im}\, z| \leq h\omega(z)\}$ for some $h > 0$. Then X is interpolating for $A_{(\omega)}$.*

Proof. We want to use Lemma 5.2.10 and then Theorem 5.2.8. Hence the only statement to prove is

$$\bigcup_{\lambda \in \Lambda} D(\lambda, \epsilon p(\lambda)) \subset \{z \in \mathbb{C} | \ |\mathrm{Im}\, z| \leq k\omega(z)\} \tag{5.19}$$

for some $k, \epsilon > 0$. By Lemma 1.5.9 there exists $C > 0$ and $R > 0$ such that for every $t \geq R$

$$\frac{1}{C} \leq \frac{\omega(x)}{\omega(t)} \leq C \tag{5.20}$$

for every $x \in (t - (h+1)\omega(t), t + (h+1)\omega(t))$. By virtue of Theorem 2.3.6 we may assume that $|\lambda| \geq R$ for every $\lambda \in \Lambda$, since every finite variety is interpolating. Furthermore, for every $z \in D(\lambda, p(\lambda))$ we have $|z| \in (|\lambda| - p(\lambda), |\lambda| + p(\lambda))$. Then $p(\lambda) = |\mathrm{Im}\, \lambda| + \omega(\lambda) \leq (h+1)\omega(\lambda)$ yields $|z| \in (|\lambda| - (h+1)\omega(\lambda), |\lambda| + (h+1)\omega(\lambda))$. Therefore, by condition (5.20)

$$|\mathrm{Im}\, z| \leq |\mathrm{Im}\, \lambda| + p(\lambda) \leq (2h+1)\omega(\lambda) \leq C(2h+1)\omega(z).$$

This gives (5.19) with $k = C(2h+1)$ and $\epsilon = 1$. □

From Theorem 5.2.12 and Theorem 5.1.1 we can derive the following geometric characterisation of interpolating varieties, which is applicable to quasianalytic and non-quasianalytc weights.

Corollary 5.2.13. *Let ω be a Beurling weight and X a multiplicity variety contained in the set $\{z \in \mathbb{C} | \ |\mathrm{Im}\, z| \leq h\omega(z)\}$ for some $h > 0$. Then X is interpolating for $A_{(\omega)}$ if and only if X is (ω)-sparse.*

In view of this corollary, to obtain a description of all interpolating varieties we need to deal with these multiplicity varieties which are somehow far from the real line. This will be done in Theorem 5.2.16. For the proof we will need the following lemma on multipliers.

Lemma 5.2.14. *Let $\psi\colon \mathbb{C} \to \mathbb{R}$ be subharmonic and assume that for some function $m\colon \mathbb{R} \to \mathbb{R}$ satisfying $m(x) \simeq 1$ it holds $\Delta\psi = m(x)\,\mathrm{dx}$ in the sense of distributions. Then there exists an entire function f such that*

(a) $Z(f)$ is a uniformly separated sequence contained in $\mathbb{R}$, i.e.,

$$\inf_{\substack{\lambda, \lambda' \in Z(f) \\ \lambda \neq \lambda'}} |\lambda - \lambda'| > 0,$$

(b) for all $\epsilon > 0$ it holds that $|f(z)| \simeq e^{\psi(z)}$ for all points z such that $d(z, Z(f)) > \epsilon$.

Proof. By [OCS99, Lemma 3] the function

$$U(z) = \int_{\mathbb{R}} \left(\ln\left|1 - \frac{z}{x}\right| + (1 - \chi_{[-1,1]}(x)) \operatorname{Re} \frac{z}{x} \right) \frac{m(x)}{2\pi} \,\mathrm{dx}$$

is subharmonic and there exists an entire function F with a uniformly separated sequence of zeroes $Z(F)$ contained in $\mathbb{R}$ such that

(1) for $|\operatorname{Im} z| \leq 1$

$$|F(z)| \simeq e^{U(z)} \operatorname{dist}(z, Z(F)),$$

(2) for $|\operatorname{Im} z| > 1$ and some $C > 0$

$$|\log F(z) - U(z) - i\widetilde{U}(z)| \leq C$$

for a suitably defined analytic branch of the logarithm respectively in $\mathbb{H}_-$ or $\mathbb{H}_+$, where $\widetilde{U}$ is a harmonic conjugate of U in $\mathbb{C} \setminus \mathbb{R}$ such that $\widetilde{U}(z) = -\widetilde{U}(\bar{z})$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Condition (2) implies that $|\ln|F(z)| - U(z)| = |\operatorname{Re}(\log F(z) - U(z) - i\widetilde{U}(z))| \leq C$. This along with (1) yields $F(z) \simeq e^{U(z)}$ for all points z such that $d(z, Z(F)) > \epsilon$. Therefore, F satisfies all assertions of the lemma except that $F \simeq e^U$ instead of $F \simeq e^\psi$.

Let φ be any test function on $\mathbb{C}$. By [Ran95, Theorem 2.5.1] U is locally integrable, hence we can use Fubini's theorem [Rud87, Theorem 8.8] as follows

$$\begin{aligned}
\langle \Delta U, \varphi \rangle &= \int_{\mathbb{C}} U(z) \Delta\varphi(z) \,\mathrm{dz} \\
&= \int_{\mathbb{R}} \frac{m(x)}{2\pi} \int_{\mathbb{C}} \left(\ln\left|1 - \frac{z}{x}\right| + (1 - \chi_{[-1,1]}(x)) \operatorname{Re} \frac{z}{x} \right) \Delta\varphi(z) \,\mathrm{dz}\,\mathrm{dx} \\
&= \int_{\mathbb{R}} \frac{m(x)}{2\pi} \int_{\mathbb{C}} \left(\ln|x - z| - \ln|x| + (1 - \chi_{[-1,1]}(x)) \operatorname{Re} \frac{z}{x} \right) \Delta\varphi(z) \,\mathrm{dz}\,\mathrm{dx}.
\end{aligned}$$

Denote $u_x(z) = \ln|x - z|$ and $v_x(z) = -\ln|x| + (1 - \chi_{[-1,1]}(x)) \operatorname{Re} \frac{z}{x}$. For every $x \neq 0$, v_x is harmonic. Therefore, by [Ran95, Theorem 3.7.8],

$$\begin{aligned}
\langle \Delta U, \varphi \rangle &= \int_{\mathbb{R}} \frac{m(x)}{2\pi} \int_{\mathbb{C}} (u_x(z) + v_x(z)) \Delta\varphi(z) \,\mathrm{dz}\,\mathrm{dx} \\
&= \int_{\mathbb{R}} \frac{m(x)}{2\pi} \langle \Delta(u_x + v_x), \varphi \rangle \,\mathrm{dx} = \int_{\mathbb{R}} \varphi(x) m(x) \,\mathrm{dx} \\
&= \langle m(x) \,\mathrm{dx}, \varphi \rangle.
\end{aligned}$$

Hence $\Delta U = m(x)\,\mathrm{dx}$ and $\psi - U$ is harmonic. Denote $g(z) = \psi(z) - U(z)$ and let $\tilde{g}$ be a harmonic conjugate of g. Finally, take

$$f(z) = F(z)e^{g(z)+i\tilde{g}(z)}.$$

Then f is an entire function, its set of zeroes is exactly $Z(F)$ and

$$|f(z)| = |F(z)|e^{g(z)} \simeq e^{U(z)}e^{\psi(z)-U(z)} = e^{\psi(z)}$$

for all points z such that $d(z, Z(h)) > \epsilon$. □

Remark 5.2.15. The function f from Lemma 5.2.14 is called a multiplier associated with ψ.

Now we can deal with varieties far from the real line.

Theorem 5.2.16. *Let ω be a Beurling weight and X a multiplicity variety contained in the set $\{z \in \mathbb{C} |\ |\mathrm{Im}\, z| > h\omega(z)\}$ for some $h > 0$. If X is (ω)-sparse and satisfies*

$$\sup_{x\in\mathbb{R}} \sum_{\lambda:\ |\mathrm{Im}\,\lambda| > h\omega(\lambda)} m_\lambda \frac{|\mathrm{Im}\,\lambda|}{|x-\lambda|^2} < \infty \tag{5.21}$$

then it is interpolating for $A_{(\omega)}$.

Proof. Denote $p(z) = |\mathrm{Im}\, z| + \omega(z)$. By Theorem 2.3.6 we may assume without loss of generality that $i, -i \notin \Lambda$ and $p(\lambda) > 1$ for every $\lambda \in \Lambda$. Furthermore, since $X \subset \{z \in \mathbb{C} |\ |\mathrm{Im}\, z| > h\omega(z)\}$, and $\omega(t) \to \infty$ when $t \to \infty$, there are only finitely many $\lambda \in \Lambda$ such that $|\mathrm{Im}\,\lambda| \le 2$. We may therefore assume that $|\mathrm{Im}\,\lambda| > 1$ for every $\lambda \in \Lambda$.

We divide X into two parts

$$\Lambda_+ = \Lambda \cap \mathbb{H}_+, \quad X_+ = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda_+},$$
$$\Lambda_- = \Lambda \cap \mathbb{H}_-, \quad X_- = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda_-}.$$

We will show that X_+ is interpolating. A similar proof for X_- will be omitted. Then we will again use Theorem 2.3.6 to obtain that X is interpolating.

In order to prove that X_+ is interpolating we will construct a function satisfying conditions of Theorem 3.1.1. Take any entire function F such that $Z(F) = X_+$. Consider the Blaschke product for X_+ in the upper half-plane,

$$B(z) = \prod_{\lambda\in\Lambda_+} \left(\frac{|\lambda^2+1|}{\lambda^2+1} \cdot \frac{z-\lambda}{z-\bar{\lambda}} \right)^{m_\lambda}, \quad z \in \mathbb{H}_+. \tag{5.22}$$

By Proposition 4.2.9, the Blaschke condition for the upper half-plane is satisfied, hence formula (5.22) defines a holomorphic function on $\mathbb{H}_+$ with modulus less than one. Further, define

$$\phi(z) = \begin{cases} \ln\left|\frac{F(z)}{B(z)}\right| & \text{for } \operatorname{Im} z > 0, \\ \ln|F(z)| & \text{for } \operatorname{Im} z \leq 0. \end{cases}$$

Claim I. The function ϕ is subharmonic on $\mathbb{C}$, harmonic on $\mathbb{C}\setminus\mathbb{R}$, and $\Delta\phi = m(x)\mathrm{dx}$ for some bounded function $m\colon \mathbb{R} \to (0,\infty)$.

Proof. Since $\frac{F}{B}$ is holomorphic on $\mathbb{H}_+$, ϕ is harmonic on $\mathbb{C}\setminus\mathbb{R}$. Moreover, by the fact that $B(z) \leq 1$ for all $z \in \mathbb{C}$, we get that for $x \in \mathbb{R}$

$$\phi(x) = \ln|F(x)| \leq \frac{1}{2\pi}\int_0^{2\pi} \ln|F(x+re^{it})|\,\mathrm{dt} \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(x+re^{it})\,\mathrm{dt}.$$

This proves that ϕ is subharmonic on $\mathbb{C}$. Further, for $z \in \mathbb{H}_+$

$$\begin{aligned}\phi(z) = \ln\left|\frac{F(z)}{B(z)}\right| &= \ln|F(z)| - \ln|B(z)| = \ln|F(z)| - \ln \prod_{\lambda\in\Lambda_+}\left|\frac{z-\lambda}{z-\bar{\lambda}}\right|^{m_\lambda} \\ &= \ln|F(z)| + \sum_{\lambda\in\Lambda_+} m_\lambda \ln\left|\frac{z-\bar{\lambda}}{z-\lambda}\right|.\end{aligned}$$

Denote

$$u_\lambda(z) = \begin{cases} \ln|z-\lambda| & \text{for } \operatorname{Im} z > 0, \\ 0 & \text{for } \operatorname{Im} z \leq 0. \end{cases}$$

and

$$q_\lambda(z) = \begin{cases} \ln\left|\frac{z-\bar{\lambda}}{z-\lambda}\right| & \text{for } \operatorname{Im} z > 0, \\ 0 & \text{for } \operatorname{Im} z \leq 0. \end{cases}$$

Then, by [Ran95, Theorem 3.7.8],

$$\Delta\phi = \sum_{\lambda\in\Lambda_+} m_\lambda(2\pi\delta_\lambda + \Delta q_\lambda),$$

where δ_λ is Dirac's delta in λ. We may write $q_\lambda = u_{\bar{\lambda}} - u_\lambda$. Further, by Lemma A.5 of Appendix A, for any test function φ

$$\langle \Delta u_\lambda, \varphi\rangle = 2\pi\delta_\lambda - \int_{-\infty}^{\infty} \ln|x-\lambda|\frac{\partial}{\partial y}\varphi(x)\,\mathrm{dx} - \int_{-\infty}^{\infty} \frac{\operatorname{Im}\lambda}{|x-\lambda|^2}\varphi(x)\,\mathrm{dx}$$

and

$$\begin{aligned}\langle \Delta u_{\bar\lambda}, \varphi\rangle &= -\int_{-\infty}^{\infty} \ln|x-\bar\lambda| \frac{\partial}{\partial y}\varphi(x)\,\mathrm{dx} - \int_{-\infty}^{\infty} \frac{\operatorname{Im}\bar\lambda}{|x-\bar\lambda|^2}\varphi(x)\,\mathrm{dx} \\ &= -\int_{-\infty}^{\infty} \ln|x-\lambda| \frac{\partial}{\partial y}\varphi(x)\,\mathrm{dx} + \int_{-\infty}^{\infty} \frac{\operatorname{Im}\lambda}{|x-\lambda|^2}\varphi(x)\,\mathrm{dx}.\end{aligned}$$

Therefore

$$\begin{aligned}\langle \Delta\phi, \varphi\rangle &= 2\sum_{\lambda\in\Lambda_+}\int_{-\infty}^{\infty} m_\lambda \frac{\operatorname{Im}\lambda}{|x-\lambda|^2}\,\varphi(x)\,\mathrm{dx} = 2\int_{-\infty}^{\infty}\sum_{\lambda\in\Lambda_+} m_\lambda \frac{\operatorname{Im}\lambda}{|x-\lambda|^2}\,\varphi(x)\,\mathrm{dx} \\ &= \langle m(x)\,\mathrm{dx}, \varphi\rangle\end{aligned}$$

where

$$m(x) = \sum_{\lambda\in\Lambda_+} m_\lambda \frac{\operatorname{Im}\lambda}{|x-\lambda|^2}.$$

We see that $m(x) > 0$ for every $x \in \mathbb{R}$. Finally, by (5.21), m is uniformly bounded above on $\mathbb{R}$. □

Define

$$\psi(z) = M|\operatorname{Im} z| - \phi(z)$$

for $M \in \mathbb{N}$. Then by Claim I

$$\langle \Delta\psi, \varphi\rangle = \int_{-\infty}^{\infty} (2M - m(x))\varphi(x)\,\mathrm{dx}.$$

Thus $\Delta\psi \simeq \mathrm{dx}$ in the sense of distributions if $M \in \mathbb{N}$ is chosen big enough. By Lemma 5.2.14, there exists a multiplier associated with ψ, i.e. an entire function g such that

(a) $Z(g)$ is a uniformly separated sequence contained in $\mathbb{R}$,

(b) for all $\epsilon > 0$ it holds that $|g(z)| \simeq e^{\psi(z)}$ for all points z such that $d(z, Z(g)) > \epsilon$.

Define $f = gF$. We will show that f is the function we are searching for.

Claim II. It holds $f \in A_{(\omega)}$ and $X_+ \subset Z(f)$.

Proof. Let $z \in \mathbb{C} \setminus \bigcup_{x\in Z(g)} D(x,\epsilon)$. Then

$$|f(z)| \lesssim e^{\psi(z)+\ln|F(z)|} = e^{M|\operatorname{Im} z| - \phi(z) + \ln|F(z)|} \le e^{M|\operatorname{Im} z|} \le e^{Mp(z)}.$$

Hence by the maximum modulus principle we get that $f \in A_{(\omega)}$. Moreover $X_+ \subset Z(f)$ since $X_+ = Z(F)$. □

The last statement to prove is that there exist $\epsilon, C > 0$ such that

$$\left|\frac{f^{(m_\lambda)}(\lambda)}{m_\lambda!}\right| \geq \epsilon\, e^{-Cp(\lambda)}.$$

for all $\lambda \in \Lambda_+$.

By Propositions 4.1.10 and 4.1.5, X is weakly (ω)-separated. Therefore there exist $C_s > 0$, $0 < \delta < 1$ such that for

$$\delta_\lambda = \delta e^{-C_s \frac{p(\lambda)}{m_\lambda}}$$

the discs $D(\lambda, 2\delta_\lambda)$ are disjoint for $\lambda \in \Lambda$. Since, by our assumption, $|\operatorname{Im}\lambda| > 2$ for every $\lambda \in \Lambda$, the distance from every $D(\lambda, \delta_\lambda)$ to $\mathbb{H}_-$ is at least 1.

We are going to prove now that there exists $C > 0$ such that

$$|B(z)| \geq \epsilon e^{-Cp(z)}$$

for all $\lambda \in \Lambda_+$ and all $z \in \partial D_\lambda$.

Claim III. For some $C > 0$

$$m_\lambda \ln\left|\frac{z-\bar\lambda}{z-\lambda}\right| \leq Cp(\lambda).$$

Proof. We have

$$m_\lambda \ln\left|\frac{z-\bar\lambda}{z-\lambda}\right| = m_\lambda \ln|z-\bar\lambda| + m_\lambda \ln\frac{1}{|z-\lambda|}.$$

We estimate the second summand

$$m_\lambda \ln\frac{1}{|z-\lambda|} = m_\lambda \ln\frac{1}{\delta_\lambda} = m_\lambda \ln\delta^{-1} + m_\lambda \ln e^{C_s\frac{p(\lambda)}{m_\lambda}} = m_\lambda \ln\delta^{-1} + C_s p(\lambda).$$

Using Proposition 4.1.18 we get the desired estimate. Further,

$$\begin{aligned} m_\lambda \ln|z-\bar\lambda| &\leq m_\lambda \ln(|z-\lambda| + 2\operatorname{Im}\lambda) = m_\lambda \ln(\delta_\lambda + 2\operatorname{Im}\lambda) \\ &\leq m_\lambda \ln(3\operatorname{Im}\lambda) \leq m_\lambda \ln 3 + m_\lambda \ln p(\lambda) \end{aligned}$$

and one more use of Proposition 4.1.18 completes the proof. □

Claim IV. For some $k \in \mathbb{N}$

$$\left|\frac{z-\bar\lambda'}{z-\lambda'}\right| \leq \left|\frac{\lambda-\bar\lambda'}{\lambda-\lambda'}\right|^k.$$

Proof. Since $|z-\bar{w}|^2 = |z-w|^2 + 4\operatorname{Im} z \operatorname{Im} w$ for all $z, w \in \mathbb{C}$, we can rewrite the inequality in the assertion as

$$1 + 4\frac{\operatorname{Im} z \operatorname{Im} \lambda'}{|z-\lambda'|^2} \leq \left(1 + 4\frac{\operatorname{Im} \lambda \operatorname{Im} \lambda'}{|\lambda-\lambda'|^2}\right)^k = \sum_{j=0}^{k} \binom{k}{j} \left(4\frac{\operatorname{Im} \lambda \operatorname{Im} \lambda'}{|\lambda-\lambda'|^2}\right)^j.$$

We have

$$\operatorname{Im} z \leq \operatorname{Im} \lambda + \delta_\lambda \leq 2\operatorname{Im} \lambda$$

and

$$|\lambda - \lambda'| \leq |z-\lambda| + |z-\lambda'| \leq 2|z-\lambda'|$$

hence

$$\frac{\operatorname{Im} z \operatorname{Im} \lambda'}{|z-\lambda'|^2} \leq 8\frac{\operatorname{Im} \lambda \operatorname{Im} \lambda'}{|\lambda-\lambda'|^2}.$$

This yields

$$1 + 4\frac{\operatorname{Im} z \operatorname{Im} \lambda'}{|z-\lambda'|^2} \leq \sum_{j=0}^{1} \binom{8}{j} \left(4\frac{\operatorname{Im} \lambda \operatorname{Im} \lambda'}{|\lambda-\lambda'|^2}\right)^j \leq \sum_{j=0}^{8} \binom{8}{j} \left(4\frac{\operatorname{Im} \lambda \operatorname{Im} \lambda'}{|\lambda-\lambda'|^2}\right)^j. \qquad \square$$

Claim V. For some $\epsilon, C > 0$

$$|B(z)| \geq \epsilon e^{-Cp(z)}$$

for all $\lambda \in \Lambda_+$ and all $z \in \partial D_\lambda$.

Proof. By virtue of Proposition 4.1.18 we may use Proposition 4.2.16 to obtain that X_+ satisfies (ω)-Blaschke condition for $\mathbb{H}_+$. Therefore

$$\sum_{\lambda' \in \Lambda_+ \setminus \{\lambda\}} m_{\lambda'} \ln\left|\frac{\lambda - \bar{\lambda}'}{\lambda - \lambda'}\right| \leq Cp(\lambda)$$

for some $C > 0$ and every $\lambda \in \Lambda_+$. Let $z \in \partial D_\lambda$ for some $\lambda \in \Lambda_+$. Using Claims III and IV and then Lemma 1.5.6, we obtain for some $C, k > 0$

$$\begin{aligned}
\sum_{\lambda' \in \Lambda_+} m_{\lambda'} \ln\left|\frac{z - \bar{\lambda}'}{z - \lambda'}\right| &= m_\lambda \ln\left|\frac{z-\bar{\lambda}}{z-\lambda}\right| + \sum_{\lambda' \in \Lambda_+ \setminus \{\lambda\}} m_{\lambda'} \ln\left|\frac{z - \bar{\lambda}'}{z - \lambda'}\right| \\
&\leq Cp(\lambda) + k \sum_{\lambda' \in \Lambda_+ \setminus \{\lambda\}} m_{\lambda'} \ln\left|\frac{\lambda - \bar{\lambda}'}{\lambda - \lambda'}\right| \\
&\leq C(k+1)p(\lambda) \leq C(k+1)(Cp(z) + C).
\end{aligned}$$

This estimate is equivalent to the assertion of the claim. $\square$

Claim VI. For some $\epsilon, C > 0$

$$\left|\frac{f^{(m_\lambda)}(\lambda)}{m_\lambda!}\right| \geq \epsilon e^{-Cp(z)}.$$

Proof. For $z \in \bigcup_{\lambda\in\Lambda_+} \partial D(\lambda, \delta_\lambda)$ we have

$$|f(z)| = |g(z)|\frac{|F(z)|}{|B(z)|}|B(z)| = |g(z)|e^{\phi(z)}|B(z)| \simeq e^{\psi(z)+\phi(z)}|B(z)| = e^{M|\operatorname{Im} z|}|B(z)|.$$

Since $|\operatorname{Im} z| > 1$ for every $z \in \bigcup_{\lambda\in\Lambda_+} \partial D(\lambda, \delta_\lambda)$, using Claim V we obtain that $|f(z)| \geq \epsilon e^{-Cp(z)}$ for some constants $\epsilon, C > 0$ and for every $\lambda \in \Lambda_+$, $z \in \partial D_\lambda$. Define

$$h(z) = \frac{f(z)}{(z-\lambda)^{m_\lambda}}.$$

Then h is holomorphic, non-vanishing in D_λ and for $z \in \partial D_\lambda$

$$|h(z)| = |f(z)|\delta_\lambda^{-m_\lambda} \geq |f(z)|,$$

since $\delta_\lambda \leq 1$. Thus h satisfies the same inequality as f on ∂D_λ. But as h is non-vanishing, by the minimum principle it satisfies this inequality on entire D_λ. Therefore

$$\left|\frac{f^{(m_\lambda)}(\lambda)}{m_\lambda!}\right| = |h(\lambda)| \geq \epsilon e^{-Cp(\lambda)},$$

which completes the proof of the claim and of the theorem. □

Theorem 5.2.17 is the final result about sufficiency of geometric conditions in the Beurling case.

Theorem 5.2.17. *Let ω be a Beurling weight and X a multiplicity variety. Assume that X satisfies one of the following conditions*

(1) *X is (ω)-sparse and satisfies*

$$\sup_{x\in\mathbb{R}} \sum_{\lambda:\ |\operatorname{Im}\lambda|>h\omega(\lambda)} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} < \infty$$

for some $h > 0$,

(2) *X is $(\omega)^*$-sparse and satisfies $(B(\omega))$,*

(3) *X is (ω)-sparse and satisfies (ω)-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$.*

Then

(4) *X is interpolating for $A_{(\omega)}$.*

More precisely, the following implications hold: (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (1), (1) $\Rightarrow$ (4).

Proof. (2) $\Rightarrow$ (3) The assertion follows from Corollary 4.2.13.

(3) $\Rightarrow$ (1) Denote $U = \{z \in \mathbb{C} |\ |\mathrm{Im}\, z| > \omega(z)\}$. Since (ω)-sparsity and (ω)-Carleson condition are carried over to subvarieties, $X' = X \cap U$ is (ω)-sparse and satisfies (ω)-Carleson condition. Therefore, in view of Proposition 4.1.18, we may obtain (1) for $h = 1$ by the use of Proposition 4.2.16.

(1) $\Rightarrow$ (4) Assume that X is (ω)-sparse and satisfies

$$\sup_{x \in \mathbb{R}} \sum_{\lambda\colon |\mathrm{Im}\,\lambda| > h\omega(\lambda)} m_\lambda \frac{|\mathrm{Im}\,\lambda|}{|x - \lambda|^2} < \infty$$

for some $h > 0$. Denote

$$\begin{aligned}
\Omega_0 &= \{z \in \mathbb{C} |\ |\mathrm{Im}\, z| \le h\omega(z)\},\\
\Omega_+ &= \{z \in \mathbb{C} |\ \mathrm{Im}\, z > h\omega(z)\},\\
\Omega_- &= \{z \in \mathbb{C} |\ \mathrm{Im}\, z < -h\omega(z)\},
\end{aligned}$$

and divide X into three parts

$$\begin{aligned}
\Lambda_0 &= \Lambda \cap \Omega_0, \quad X_0 = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda_0},\\
\Lambda_+ &= \Lambda \cap \Omega_+, \quad X_+ = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda_+},\\
\Lambda_- &= \Lambda \cap \Omega_-, \quad X_- = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda_-}.
\end{aligned}$$

Using Theorem 5.2.12 we obtain that X_0 is interpolating. Theorem 5.2.16 asserts that X_- and X_+ are interpolating. Finally, X is interpolating by Theorem 2.3.6. □

Assuming additionally that the weight is non-quasianalytic we can obtain a complete geometric characterisation of interpolating varieties in the Beurling case, which was stated in Theorem 5.2.1. We recall this theorem here.

Theorem 5.2.18. *Let ω be a non-quasianalytic Beurling weight and X a multiplicity variety. Then the following conditions are equivalent*

(1) *X is (ω)-sparse and satisfies*

$$\sup_{x \in \mathbb{R}} \sum_{\lambda\colon |\mathrm{Im}\,\lambda| > h\omega(\lambda)} m_\lambda \frac{|\mathrm{Im}\,\lambda|}{|x - \lambda|^2} < \infty$$

for some $h > 0$,

(2) *X is $(\omega)^*$-sparse and satisfies $(B(\omega))$,*

(3) *X is (ω)-sparse and satisfies (ω)-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$,*

(4) *X is interpolating for $A_{(\omega)}$.*

Remark. Condition (1) is somehow the smallest one among (1), (2) and (3). It requires the estimate for the function N only for $r = p(\lambda)$, and the series

$$\sum_{\lambda\colon |\operatorname{Im}\lambda| > h\omega(\lambda)} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2}$$

is taken only over a subset of Λ. However, calculating the supremum over $x \in \mathbb{R}$ is rather hard in most cases. The situation becomes simple whenever $X \subset \{z \in \mathbb{C} | \, |\operatorname{Im} z| \leq h\omega(z)\}$ for some $h > 0$. Then this condition is trivially fulfilled. Conditions (2) and (3) give explicitly more information about an interpolating variety. Additionally, conditions like (ω)-Carleson condition involving reduced Blaschke products are well established in the theory of interpolation problem for many spaces of holomorphic functions.

Proof. In view of Theorem 5.2.17 we have (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (1), (1) $\Rightarrow$ (4). Therefore it is enough to prove (4) $\Rightarrow$ (2). And this implication is given by Theorems 5.1.1 and 5.1.5. □

The reader might have already realised that Theorem 5.2.1 yields some consequences about the relation of sparsity, (ω)-Carleson condition and $(B(\omega))$. We will state this and other consequences precisely in Section 5.4.

5.3 Sufficient conditions in the Roumieu case

In the Roumieu case results are quite similar to those in the Beurling case. We start with a theorem applicable to all weights.

Theorem 5.3.1. *Let ω be a Roumieu weight and X a multiplicity variety. Assume that X satisfies one of the following conditions*

(1) *X is $\{\omega\}$-sparse and for some Beurling weight $\sigma = o(\omega)$ it satisfies*

$$\sup_{x\in\mathbb{R}} \sum_{\lambda\colon |\operatorname{Im}\lambda| > h\sigma(\lambda)} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} < \infty$$

for some $h > 0$,

(2) *X is $\{\omega\}$-sparse and satisfies $(B\{\omega\})$,*

(3) *X is $\{\omega\}$-sparse and satisfies $\{\omega\}$-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$.*

Then

(4) *X is interpolating for $A_{\{\omega\}}$.*

More precisely, the following implications hold: (2) $\Rightarrow$ (3)*,* (3) $\Rightarrow$ (1)*,* (1) $\Rightarrow$ (4)*.*

Proof. (2) $\Rightarrow$ (3) The assertion follows from Proposition 4.2.14.

(3) $\Rightarrow$ (1) By Proposition 4.2.3 there exists a Beurling weight σ satisfying $\sigma = o(\omega)$ and such that X satisfies (σ)-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$. On the other hand, by Proposition 4.1.19

$$\exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, C_m > 0\ \forall\, \lambda \in \Lambda\colon \quad m_\lambda \le C_m + M|\mathrm{Im}\,\lambda| + \frac{1}{m}\omega(\lambda).$$

Denote $q(z) = \inf_{m\in\mathbb{N}} C_m + \frac{1}{m}\omega(z)$. By Lemma 1.5.16 there exists a Beurling weight ν satisfying $q = o(\nu)$ and $\nu = o(\omega)$. Then

$$m_\lambda \le M|\mathrm{Im}\,\lambda| + q(\lambda) \le C(|\mathrm{Im}\,\lambda| + \nu(\lambda) + 1)$$

for some $C > 0$ and every $\lambda \in \Lambda$.

Consider $\mu = \max(\sigma, \nu)$ and $X' = X \cap \{z \in \mathbb{C}|\ |\mathrm{Im}\,z| > \mu(z)\}$. Then X' satisfies (μ)-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$. Moreover, with the use of Lemma 4.1.7 we obtain

$$m_\lambda \le C(|\mathrm{Im}\,\lambda| + \mu(\lambda))$$

for some $C > 0$. Now, we may use Proposition 4.2.16 to obtain

$$\sup_{x\in\mathbb{R}} \sum_{\lambda\colon |\mathrm{Im}\,\lambda| > \mu(\lambda)} m_\lambda \frac{|\mathrm{Im}\,\lambda|}{|x-\lambda|^2} < \infty.$$

(1) $\Rightarrow$ (4) By Proposition 4.1.3, X is (ν)-sparse for some Beurling weight ν satisfying $\mu = o(\omega)$. We have that $\mu = \max(\sigma, \nu)$ is a Beurling weight satisfying $\mu = o(\omega)$. Then X is (μ)-sparse and

$$\sup_{x\in\mathbb{R}} \sum_{\lambda\colon |\mathrm{Im}\,\lambda| > h\mu(\lambda)} m_\lambda \frac{|\mathrm{Im}\,\lambda|}{|x-\lambda|^2} \le \sup_{x\in\mathbb{R}} \sum_{\lambda\colon |\mathrm{Im}\,\lambda| > h\sigma(\lambda)} m_\lambda \frac{|\mathrm{Im}\,\lambda|}{|x-\lambda|^2} < \infty.$$

Denote

$$\begin{aligned}
\Omega_0 &= \{z \in \mathbb{C}|\ |\mathrm{Im}\,z| \le h\mu(|z|)\},\\
\Omega_+ &= \{z \in \mathbb{C}|\ \mathrm{Im}\,z > h\mu(|z|)\},\\
\Omega_- &= \{z \in \mathbb{C}|\ \mathrm{Im}\,z < -h\mu(|z|)\},
\end{aligned}$$

and divide X into three parts

$$\begin{aligned}\Lambda_0 = \Lambda \cap \Omega_0, \quad X_0 = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda_0},\\ \Lambda_+ = \Lambda \cap \Omega_+, \quad X_+ = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda_+},\\ \Lambda_- = \Lambda \cap \Omega_-, \quad X_- = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda_-}.\end{aligned}$$

Using Theorem 5.2.12 we obtain that X_0 is interpolating for $A_{(\mu)}$. Theorem 5.2.16 asserts that X_- and X_+ are interpolating for $A_{(\mu)}$. Then X is $A_{(\mu)}$-interpolating by Theorem 2.3.6. Finally, X is $A_{\{\omega\}}$ interpolating by Corollary 3.3.1. □

Using the results we have already obtained, we are able to give a geometric characterisation of interpolating varieties in the non-quasianalytic Roumieu case.

Theorem 5.3.2. *Let ω be a non-quasianalytic Roumieu weight and X a multiplicity variety. Then the following conditions are equivalent:*

(1) *X is $\{\omega\}$-sparse and for some Beurling weight $\sigma = o(\omega)$ it satisfies*

$$\sup_{x\in\mathbb{R}} \sum_{\lambda\colon |\operatorname{Im}\lambda| > h\sigma(\lambda)} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} < \infty$$

for some $h > 0$,

(2) *X is $\{\omega\}$-sparse and satisfies $(B\{\omega\})$,*

(3) *X is $\{\omega\}$-sparse and satisfies $\{\omega\}$-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$,*

(4) *for some Beurling weight $\sigma = o(\omega)$, X is $(\sigma)^*$-sparse and satisfies $(B(\sigma))$,*

(5) *for some Beurling weight $\sigma = o(\omega)$, X is (σ)-sparse and satisfies (σ)-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$,*

(6) *for some Beurling weight $\sigma = o(\omega)$, X is interpolating for $A_{(\sigma)}$,*

(7) *X is interpolating for $A_{\{\omega\}}$.*

Proof. (2) ⇒ (3), (3) ⇒ (1) (1) ⇒ (7) Follow from Theorem 5.3.1.

(7) ⇒ (2) Follows from Theorems 5.1.1 and 5.1.5.

(4) ⇔ (5) ⇔ (6) Follow from Theorem 3.1.1.

(2) ⇒ (4) By Proposition 4.1.3 we have that X is $(\sigma)^*$-sparse for some Beurling weight σ satisfying $\sigma = o(\omega)$. By Proposition 4.2.4 we have that X satisfies $(B(\nu))$ for some Beurling weight ν satisfying $\nu = o(\omega)$. Then $\mu = \max(\sigma, \nu)$ is a Beurling weight satisfying $\mu = o(\omega)$. Moreover, X is (μ)-sparse and satisfies $(B(\mu))$.

(4) ⇒ (2) Follows from Propositions 4.1.3 and 4.2.4. □

A particularly important case covered by Theorem 5.3.2 are the Gevrey classes. We exhibit it separately as a corollary. Recall that for $s > 1$

$$G_s(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) |\ \forall\, M \in \mathbb{N}\ \exists\, h > 0 : \sup_{x \in [-M,M]} \sup_{n \in \mathbb{N}_0} \frac{|f^{(n)}(x)|}{h^n (n!)^s} < \infty \right\}.$$

Corollary 5.3.3. *A variety X is interpolating for the Fourier-Laplace transform image of the space of Gevrey ultradistributions with compact support $\widehat{G}'_s(\mathbb{R})$ $(s > 1)$ if and only if for $\omega(t) = t^{1/s}$, X is $\{\omega\}$-sparse and satisfies $\{\omega\}$-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$.*

5.4 Consequences of characterisations of interpolating varieties and examples

Interpolation theorems have many consequences. In Section 4.1 we introduced (ω)-sparsity and $(\omega)^*$-sparsity. Using the characterisation of interpolating varieties in the Beurling case we are able to prove that for some multiplicity varieties these conditions are equivalent.

Proposition 5.4.1. *Let ω be a non-quasianalytic Beurling weight and X a multiplicity variety contained in the set $\{z \in \mathbb{C} |\ |\mathrm{Im}\, z| \leq h\omega(z)\}$ for some $h > 0$. Then X is (ω)-sparse if and only if it is $(\omega)^*$-sparse.*

Proof. The condition

$$\sup_{x \in \mathbb{R}} \sum_{\lambda :\ |\mathrm{Im}\, \lambda| > h\omega(\lambda)} m_\lambda \frac{|\mathrm{Im}\, \lambda|}{|x - \lambda|^2} < \infty$$

is satisfied automatically, hence implication (1) ⇒ (2) or (1) ⇒ (3) of Theorem 5.2.1 yields the assertion. □

In Section 4.2 we started to study the relation between sparsity, conditions (B) and the weighted Carleson conditions. Now, using characterisation theorems we can deduce much more.

Proposition 5.4.2. *Let ω be a non-quasianalytic Beurling weight and X a multiplicity variety. Denote $U_{h_1,h_2} = \{z \in \mathbb{C} |\ h_1\omega(z) \leq |\mathrm{Im}\, z| \leq h_2\omega(z)\}$ for $h_2 > h_1 > 0$. If X is (ω)-sparse then*

$$\sup_{x \in \mathbb{R}} \sum_{\lambda \in \Lambda \cap U_{h_1,h_2}} m_\lambda \frac{|\mathrm{Im}\, \lambda|}{|x - \lambda|^2} < \infty$$

for all $h_2 > h_1 > 0$.

Proof. The variety $Y = X \cap \{z \in \mathbb{C} |\ |\operatorname{Im} z| \leq h_2\omega(z)\}$ satisfies condition (1) of Theorem 5.2.1. Hence, by the implication (1) $\Rightarrow$ (2) of this theorem, it satisfies $(B(\omega))$. Using $(B(\omega))$ with U_{h_1,h_2} we obtain the assertion. $\square$

Proposition 5.4.3. *Let ω be a non-quasianalytic Beurling weight. Let X be an (ω)-sparse multiplicity variety contained in the set $\{z \in \mathbb{C} |\ |\operatorname{Im} z| \leq h\omega(z)\}$ for some $h > 0$. Then X satisfies $(B(\omega))$ and (ω)-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$.*

Proof. The condition

$$\sup_{x\in\mathbb{R}} \sum_{\lambda\colon |\operatorname{Im}\lambda|>h\omega(\lambda)} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} < \infty$$

is satisfied automatically, hence implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) of Theorem 5.2.1 yield the assertion. $\square$

Condition $(B(\omega))$ reads that for an arbitrary set $U \subset \mathbb{H}_+, \mathbb{H}_-$

$$\sum_{\lambda\colon \Lambda\cap U} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} \leq C \frac{|\operatorname{Im}\lambda_x| + \omega(\lambda_x)}{|\operatorname{Im}\lambda_x|}$$

for some $C > 0$, every $x \in \mathbb{R}$ and $\lambda_x \in \Lambda \cap U$ such that $d(x, \lambda_x) = d(x, \Lambda \cap U)$. That sparsity implies this condition is not obvious but also not very surprising, since for $X \subset \{z \in \mathbb{C} |\ |\operatorname{Im} z| \leq h\omega(z)\}$ we have

$$\sum_{\lambda\colon \Lambda\cap U} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2} \leq h \sum_{\lambda\colon \Lambda\cap U} m_\lambda \frac{\omega(\lambda)}{|x-\lambda|^2}.$$

Further, $(\omega)^*$-sparsity gives that points of the variety are placed somehow "linearly" (comp. Proposition 4.1.10), hence

$$\sum_{\lambda\colon \Lambda\cap U} m_\lambda \frac{\omega(\lambda)}{|x-\lambda|^2}$$

is "close" to the integral

$$\int_{\mathbb{R}} \frac{\omega(t)}{|x-t|^2}\, \mathrm{dt}$$

which is convergent as ω is non-quasianalytic. This reasoning breaks down in the quasianalytic case and one may expect that Propositions 5.4.2 and 5.4.3 are not true for quasianalytic weights. In fact, we will even show that condition $(B(\omega))$ and (ω)-Carleson condition do not have to be satisfied by an interpolating variety in the quasianalytic case. So they are not necessary for interpolation in this case.

Example 5.4.4. Let ω be a quasi-analytic Beurling weight. Consider the variety

$$X = \{(m + i\omega(m), 1) \colon m \in \mathbb{N}\}$$

and denote λ_m for $m \in \mathbb{N}$. We have $n(\lambda_m, r) \leq 2r + 1$ for all $m \in \mathbb{N}$ and $r > 0$. Moreover, X is (ω)-separated, since $d(\lambda_m, \lambda_k) \geq 1$ for $m \neq k$. Hence by Proposition 4.1.12, X is $(\omega)^*$-sparse. The second condition of Theorem 5.2.17 (1) is trivially satisfied, hence X is interpolating for $A_{(\omega)}$.

On the other hand, for $x \in [m, m+1]$

$$\omega(x) \leq \omega(2m) \leq C\omega(m)$$

for some $C > 0$ independent of m, by property (α) of the weight. This gives

$$\frac{\omega(m)}{m^2} \geq \frac{1}{C}\frac{\omega(x)}{x^2}$$

for $x \in [m, m+1]$. Since $\omega(t) = o(t)$ there is $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$ it holds $\omega(m) \leq m$. Then

$$\sum_{\lambda \colon \Lambda \cap \mathbb{H}_+} m_\lambda \frac{|\mathrm{Im}\,\lambda|}{|\lambda|^2} = \sum_{m \in \mathbb{N}} \frac{\omega(m)}{m^2 + \omega^2(m)} \geq \frac{1}{2} \sum_{m \geq m_0} \frac{\omega(m)}{m^2}$$
$$\geq \frac{1}{2C} \sum_{m \geq m_0} \int_m^{m+1} \frac{\omega(x)}{x^2}\,\mathrm{dx} = \frac{1}{2C} \int_{m_0}^{\infty} \frac{\omega(x)}{x^2}\,\mathrm{dx} = \infty.$$

Hence X does not satisfy $(B(\omega))$. Due to Corollary 4.2.13 it cannot satisfy (ω)-Carleson condition as well.

It is natural to ask if the condition

$$\sup_{x \in \mathbb{R}} \sum_{\lambda \colon |\mathrm{Im}\,\lambda| > h\omega(\lambda)} m_\lambda \frac{|\mathrm{Im}\,\lambda|}{|x - \lambda|^2} < \infty$$

is necessary in the quasianalytic case for a multiplicity variety to be interpolating. The answer is yet unknown.

Theorem 2.3.6 stated that a finite union of interpolating varieties for $A_{(\omega)}$ is interpolating whenever the union of these varieties is weakly separated. Now we can prove an analogue of this result for the non-quasianalytic Roumieu case.

Theorem 5.4.5. *Let ω be a non-quasianalytic Roumieu weight and assume that $X = \{(\lambda, m_\lambda) |\ \lambda \in \Lambda\}$ and $Y = \{(\eta, m_\eta) |\ \eta \in \mathcal{K}\}$ are interpolating for $A_{\{\omega\}}$. Suppose that*

$$\exists\, \delta > 0, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, C_m > 0\ \forall\, \lambda \in \Lambda, \eta \in \mathcal{K}$$
$$|\lambda - \eta| \geq \max(e^{-C_m - M|\mathrm{Im}\,\lambda| - \frac{1}{m}\omega(\lambda)}, e^{-C_m - M|\mathrm{Im}\,\eta| - \frac{1}{m}\omega(\eta)})$$

Then $X \cup Y$ is interpolating for $A_{\{\omega\}}$.

Proof. Theorem 5.3.2 and Corollary 3.1.2 imply that for some Beurling weight $\sigma = o(\omega)$, X and Y are interpolating for $A_{(\sigma)}$. Further, we can introduce $q(z) = \inf_m C_m + \frac{1}{m}\omega(z)$ and by the use of Lemma 1.5.16 obtain another weight ν satisfying $q = o(\nu)$ and $\nu = o(\omega)$. Denote $\mu = \max(\sigma, \nu)$. Then μ is a weight satisfying $q = o(\mu)$ and $\mu = o(\omega)$. Moreover, X and Y are interpolating for $A_{(\mu)}$ hence (μ)-sparse by Theorem 5.2.1, and weakly (μ)-separated by Propositions 4.1.10 and 4.1.5. Furthermore, by the assumption

$$\begin{aligned}|\lambda - \eta| &\geq \delta \max(e^{-M|\operatorname{Im}\lambda| - q(\lambda)}, e^{-M|\operatorname{Im}\eta| - q(\eta)}) \\ &\geq \delta e^{-C} \max(e^{-M|\operatorname{Im}\lambda| - \mu(\lambda)}, e^{-M|\operatorname{Im}\eta| - \mu(\eta)})\end{aligned}$$

for some $C > 0$ and all $\lambda \in \Lambda$, $\eta \in \mathcal{K}$. Thus $X \cup Y$ is weakly (μ)-separated. Theorem 2.3.6 yields then that $X \cup Y$ is interpolating for $A_{(\mu)}$. We can complete the proof using Corollary 3.3.1. □

In Chapter 3 we proved for the Roumieu case that a multiplicity variety is interpolating if there exists a function which equals zero on this variety and has particular Taylor coefficients not decreasing too fast (see Theorem 3.2.1). For the Beurling case this was a characterisation of interpolating varieties (see Theorem 3.1.1). Now we are able to give an analogous characterisation in the non-quasianalytic Roumieu case. The quasianalytic case is yet unsolved.

Theorem 5.4.6. *Let ω be a non-quasianalytic Roumieu weight and X a multiplicity variety. Then X is an interpolating variety for $A_{\{\omega\}}$ if and only if there exists $f \in A_{\{\omega\}}$ with $X \subset Z(f)$ satisfying that for some $M \in \mathbb{N}$ and every $m \in \mathbb{N}$ there exists $C_m \in \mathbb{R}$ such that for every $\lambda \in \Lambda$*

$$\frac{|f^{(m_\lambda)}(\lambda)|}{m_\lambda!} \geq e^{-C_m - M|\operatorname{Im}\lambda| - \frac{1}{m}\omega(\lambda)}.$$

Proof. Assume that X is interpolating for $A_{\{\omega\}}$. By Theorem 5.3.2, there exists a Beurling weight $\sigma = o(\omega)$ such that X is $A_{(\sigma)}$ interpolating. Then by Theorem 3.1.1 there exists a function $f \in A_{(\sigma)}$ with $X \subset Z(f)$ satisfying

$$\frac{|(f^{m_\lambda})(\lambda)|}{m_\lambda!} \geq e^{-M|\operatorname{Im}\lambda| - M\sigma(\lambda)}$$

for some $M \in \mathbb{N}$ and every $\lambda \in \Lambda$. As $\sigma = o(\omega)$, this implies the assertion of the theorem. Finally, Proposition 1.7.12 implies $f \in A_{\{\omega\}}$.

The converse is given by Theorem 3.2.1. □

This theorem allows us to state a corollary similar to 3.1.2. However, this would only work for non-quasianalytic weights. Due to the geometric descriptions of interpolating varieties we will be able to show a slightly stronger result.

Corollary 5.4.7. *Let ν be a non-quasianalytic weight and X be an interpolating variety for $A_{\{\nu\}}$. Then X is interpolating for $A_{\{\omega\}}$ for every weight ω satisfying $\nu = O(\omega)$.*

Remark 5.4.8. The outcome of the geometric descriptions is that we could omit the assumption about the non-quasianalyticity of ω.

Proof. By Theorem 5.3.2, X is $\{\nu\}$-sparse and satisfies $\{\nu\}$-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$. But it is a trivial observation that then X is $\{\omega\}$-sparse and satisfies $\{\omega\}$-Carleson condition for $\mathbb{H}_+$ and $\mathbb{H}_-$. Hence by Theorem 5.3.1, X is interpolating for $A_{\{\omega\}}$. □

Another corollary of the analytic characterisation is an analogue of Theorem 2.2.5 about multiplying all multiplicities by a given constant.

Corollary 5.4.9. *Let ω be a non-quasianalytic Roumieu weight. If $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ is interpolating for $A_{\{\omega\}}$ then for any $n \in \mathbb{N}$ the variety $X = \{(\lambda, n \cdot m_\lambda)\}_{\lambda\in\Lambda}$ (we multiply all multiplicities by n) is also interpolating for $A_{\{\omega\}}$.*

Proof. By Theorem 5.4.6 there exists $f \in A_{\{\omega\}}$ with $X \subset Z(f)$ satisfying that for some $M \in \mathbb{N}$, for all $m \in \mathbb{N}$ there exists $C_m \in \mathbb{R}$ such that for all $\lambda \in \Lambda$

$$\frac{|f^{(m_\lambda)}(\lambda)|}{m_\lambda!} \geq e^{-C_m - M|\operatorname{Im}\lambda| - \frac{1}{m}\,\omega(\lambda)}.$$

Then by the Leibniz formula for $\lambda \in \Lambda$

$$\begin{aligned}\left|\frac{1}{(2m_\lambda)!}\big(f(z)f(z)\big)^{(2m_\lambda)}|_{z=\lambda}\right| &= \left|\frac{1}{(2m_\lambda)!}\sum_{k=0}^{2m_\lambda}\binom{2m_\lambda}{k} f^{(k)}(\lambda) f^{(2m_\lambda-k)}(\lambda)\right| \\ &= \left|\frac{1}{(2m_\lambda)!}\binom{2m_\lambda}{m_\lambda} f^{(m_\lambda)}(\lambda) f^{(m_\lambda)}(\lambda)\right| \\ &\geq e^{-2C_m - 2M|\operatorname{Im}\lambda| - \frac{2}{m}\,\omega(\lambda)}.\end{aligned}$$

Moreover, $f^2 \in A_{\{\omega\}}$. One more use of Theorem 5.4.6 completes the proof. □

Now, we are going to present some examples of interpolating and non-interpolating varieties and test their behaviour against bounded perturbation, diffusion and rotation (see Section 2.2 for definitions of these operations). We already know that the interpolation property is stable under shifts (see Proposition 2.2.3). We have seen in Examples 4.1.14 and 4.1.15 that

$$X = \{(m, 1) |\ m \in \mathbb{N}\}$$

is an (ω)-sparse variety for any weight ω, for which bounded perturbation or diffusion can break sparsity down. Moreover, X satisfies $(B(\omega))$, $(B\{\omega\})$ and the weighted

Carleson conditions trivially. Hence it is interpolating for any $A_{(\omega)}$ and $A_{\{\omega\}}$ by Theorems 5.2.17 and 5.3.1. Furthermore, $i \cdot X = \{(mi, 1)|\ m \in \mathbb{N}\}$ is not a Blaschke sequence, thus $i \cdot X$ cannot be interpolating for any $A_{(\omega)}$ or $A_{\{\omega\}}$ where ω is a non-quasianalytic weight. This could be seen with the use of Theorem 5.1.5 and Proposition 4.2.9. Therefore the only admissible operation for interpolating varieties is translation.

The following example shows that lattices are never interpolating.

Example 5.4.10. Let $z, \xi \in \mathbb{C}$ be linearly independent. Consider

$$X = z \cdot \mathbb{Z} + \xi \cdot \mathbb{Z},$$

a lattice generated by z and ξ. It is known that the number of points of such a lattice in a disc $D(\lambda, r)$ is proportional to r^2 for big r. Denoting $p(z) = |\operatorname{Im} z| + \omega(z)$ for a weight ω, we obtain $n(\lambda, p(\lambda)) \geq \epsilon p(\lambda)^2$ for some $\epsilon > 0$ and all $\lambda \in \Lambda$. Hence X is not (ω)-sparse by Proposition 4.1.10. Therefore it is not interpolating for $A_{(\omega)}$. As every $\{\omega\}$-sparse variety is (σ)-sparse for some $\sigma = o(\omega)$ by Proposition 4.1.3, X cannot be interpolating for any $A_{\{\omega\}}$.

The following example distinguishes the sets of interpolating varieties for different weights.

Example 5.4.11. Let ν and ω be two Beurling weights such that $\nu = o(\omega)$. Consider

$$X = \{(m + i\omega(m), 1), (m + e^{-\omega(m)} + i\omega(m), 1)|\ m \in \mathbb{N}\}.$$

Denote $\lambda_m = m + i\omega(m)$ and $\eta_m = m + e^{-\omega(m)} + i\omega(m)$. Then

$$|\operatorname{Im} \lambda_m| = \omega(m) \quad \text{and} \quad |\operatorname{Im} \eta_m| = \omega(m).$$

Hence $X \subset \{z \in \mathbb{C}|\ |\operatorname{Im} z| \leq \omega(z)\}$. Further,

$$|\lambda_m - \eta_m| = e^{-\omega(m)},$$

which shows that X is (ω)-separated. Moreover, $n(\lambda, r) \leq 4r + 2$ for every $\lambda \in X$ and $r > 0$. Hence X is (ω)-sparse by Proposition 4.1.12 and interpolating for $A_{(\omega)}$ by Theorem 5.2.17. On the other hand, X is not (ν)-separated hence not interpolating for $A_{(\nu)}$.

Now let ν and ω be two Roumieu weights satisfying $\nu = o(\omega)$. Using Lemma 1.5.16 we can find another weight σ such that $\nu = o(\sigma)$ and $\sigma = o(\omega)$. Then consider

$$X = \{(m + i\sigma(m), 1), (m + e^{-\sigma(m)} + i\sigma(m), 1)|\ m \in \mathbb{N}\}.$$

We have seen earlier that X is interpolating for $A_{(\sigma)}$. By Corollary 3.3.1, X is interpolating for $A_{\{\omega\}}$. But it is not (ν)-sparse and thus cannot be $\{\nu\}$-sparse. Finally, X is not interpolating for $A_{\{\nu\}}$ by Theorem 5.1.1.

Appendix A: Calculation of derivatives of some distributions

In this appendix we calculate the laplacian of several particular distributions. All computations are standard, and probably can be found in some books as exercises. We introduce the following notation.

Definition A.1. For $c > 0$, $\lambda \in \mathbb{C}$, $\varphi \in C_c^\infty(\mathbb{C})$ we define a distribution ${\mathrm{Avg}^1}_{\lambda,c}$ counting the average of φ over the circle $S(\lambda, c) = \{z \in \mathbb{C} |\ |z - \lambda| = c\}$

$$\langle {\mathrm{Avg}^1}_{\lambda,c}, \varphi \rangle = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\lambda + ce^{i\theta})\, \mathrm{d}\theta$$

and a distribution ${\mathrm{Avg}^2}_{\lambda,c}$ counting the average of φ over the disk $D(\lambda, c)$

$$\langle {\mathrm{Avg}^2}_{\lambda,c}, \varphi \rangle = \frac{1}{\pi c^2} \int_{D(\lambda,c)} \varphi(z)\, \mathrm{dz}.$$

Further, to simplify the notation we introduce the shift operator

$$\tau_\lambda \colon C_c^\infty(\mathbb{C}) \to C_c^\infty(\mathbb{C})$$

for $\lambda \in \mathbb{C}$ by the formula $(\tau_\lambda \varphi)(z) = \varphi(z + \lambda)$ for $z \in \mathbb{C}$, $\varphi \in C_c^\infty(\mathbb{C})$.

The following lemma will be helpful later.

Lemma A.2. *For $\lambda \in \mathbb{C}$, $\varphi \in C_c^\infty(\mathbb{C})$, a locally integrable function u and any $c > 0$ it holds*

$$\begin{aligned}
\langle \Delta(\tau_\lambda u), \varphi \rangle &= \langle \Delta u, \tau_{-\lambda} \varphi \rangle, \\
\langle \delta_0, \tau_\lambda \varphi \rangle &= \langle \delta_\lambda, \varphi \rangle, \\
\langle {\mathrm{Avg}^1}_{0,c}, \tau_\lambda \varphi \rangle &= \langle {\mathrm{Avg}^1}_{\lambda,c}, \varphi \rangle, \\
\langle {\mathrm{Avg}^2}_{0,c}, \tau_\lambda \varphi \rangle &= \langle {\mathrm{Avg}^2}_{\lambda,c}, \varphi \rangle.
\end{aligned}$$

Proof. We have

$$\begin{aligned}\langle \Delta(\tau_\lambda u), \varphi\rangle &= \int\limits_{\mathbb{C}} (\tau_\lambda u)(z)\Delta\varphi(z)\,\mathrm{dz} = \int\limits_{\mathbb{C}} u(z+\lambda)\Delta\varphi(z)\,\mathrm{dz} \\ &= \int\limits_{\mathbb{C}} u(z)\Delta\varphi(z-\lambda)\,\mathrm{dz} = \int\limits_{\mathbb{C}} u(z)\Delta(\tau_{-\lambda}\varphi)(z)\,\mathrm{dz} = \langle \Delta u, \tau_{-\lambda}\varphi\rangle.\end{aligned}$$

For the second equation we have

$$\langle \delta_0, \tau_\lambda\varphi\rangle = \delta_0(\varphi(\cdot+\lambda)) = \varphi(\lambda) = \langle \delta_\lambda, \varphi\rangle.$$

Further,

$$\langle {\mathrm{Avg}^1}_{0,c}, \tau_\lambda\varphi\rangle = \frac{1}{2\pi}\int\limits_0^{2\pi} \varphi(\lambda + ce^{i\theta})\,\mathrm{d}\theta = \langle {\mathrm{Avg}^1}_{\lambda,c}, \varphi\rangle.$$

Finally,

$$\langle {\mathrm{Avg}^2}_{0,c}, \tau_\lambda\varphi\rangle = \frac{1}{\pi c^2}\int\limits_{D(0,c)} \varphi(z+\lambda)\,\mathrm{dz} = \frac{1}{\pi c^2}\int\limits_{D(\lambda,c)} \varphi(z)\,\mathrm{dz} = \langle {\mathrm{Avg}^2}_{\lambda,c}, \varphi\rangle. \qquad \square$$

Lemma A.3. *Let $c > 0$, $\lambda \in \mathbb{C}$ and*

$$u(z) = \begin{cases} \ln \frac{|z-\lambda|^2}{c^2} & \text{for } z \in D(\lambda, c), \\ 0 & \text{for } z \in \mathbb{C}\setminus D(\lambda, c). \end{cases}$$

Then u is continuous on $\mathbb{C}\setminus\{0\}$ and

$$\Delta u = 4\pi(\delta_\lambda - {\mathrm{Avg}^1}_{\lambda,c}).$$

Proof. Assume for the moment that we have proved the lemma for $\lambda = 0$. Denote by u_λ the function u defined for $\lambda \in \mathbb{C}$. Then by Lemma A.2 we would have for any $\varphi \in C_c^\infty(\mathbb{C})$

$$\begin{aligned}\langle \Delta u_\lambda, \varphi\rangle &= \langle \Delta(\tau_{-\lambda}u_0), \varphi\rangle = \langle \Delta u_0, \tau_\lambda\varphi\rangle = \langle 4\pi(\delta_0 - {\mathrm{Avg}^1}_{0,c}), \tau_\lambda\varphi\rangle \\ &= \langle 4\pi(\delta_\lambda - {\mathrm{Avg}^1}_{\lambda,c}), \varphi\rangle.\end{aligned}$$

Therefore, without loss of generality we may assume that $\lambda = 0$.

Let $\varphi \in C_c^\infty(\mathbb{C})$. We have to calculate

$$\left\langle \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u, \varphi \right\rangle = \left\langle u, \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi \right\rangle.$$

We change Cartesian coordinates into polar coordinates. Let $\psi \in C_c^\infty(\mathbb{R} \times [0, 2\pi])$ be given by $\psi(r,\theta) = \varphi(x,y)$. Then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi(x,y) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\psi(r,\theta)$$

and $\mathrm{dx}\,\mathrm{dy} = r\,\mathrm{dr}\,\mathrm{d}\theta$. Moreover $\ln\frac{x^2+y^2}{c^2} = \ln\frac{r^2}{c^2}$. Therefore our task is to calculate

$$\int_0^{2\pi}\int_0^c r\ln\left(\frac{r^2}{c^2}\right)\left(\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\psi(r,\theta)\right)\,\mathrm{dr}\,\mathrm{d}\theta.$$

We calculate the third integral

$$\int_0^{2\pi}\frac{1}{r}\ln\left(\frac{r^2}{c^2}\right)\frac{\partial^2}{\partial\theta^2}\psi(r,\theta)\,\mathrm{d}\theta = \left[\frac{1}{r}\ln\left(\frac{r^2}{c^2}\right)\frac{\partial}{\partial\theta}\psi(r,\theta)\right]_0^{2\pi} = 0.$$

In further calculations we will use the following equalities

$$\frac{\partial}{\partial r}\ln\left(\frac{r^2}{c^2}\right) = \frac{2}{r},\quad \frac{\partial}{\partial r}\left(r\ln\left(\frac{r^2}{c^2}\right)\right) = \ln\left(\frac{r^2}{c^2}\right) + 2,\quad \frac{\partial^2}{\partial r^2}\left(r\ln\left(\frac{r^2}{c^2}\right)\right) = \frac{2}{r}.$$

We compute the second integral

$$\begin{aligned}\int_\epsilon^c \ln\left(\frac{r^2}{c^2}\right)\frac{\partial}{\partial r}\psi(r,\theta)\,\mathrm{dr} &= \left[\ln\left(\frac{r^2}{c^2}\right)\psi(r,\theta)\right]_\epsilon^c - \int_\epsilon^c\left(\frac{\partial}{\partial r}\ln\left(\frac{r^2}{c^2}\right)\right)\psi(r,\theta)\,\mathrm{dr}\\ &= -\ln\left(\frac{\epsilon^2}{c^2}\right)\psi(\epsilon,\theta) - \int_\epsilon^c\frac{2}{r}\psi(r,\theta)\,\mathrm{dr}.\end{aligned}$$

Finally, we calculate the first integral

$$\begin{aligned}\int_\epsilon^c r\ln\left(\frac{r^2}{c^2}\right)\frac{\partial^2}{\partial r^2}\psi(r,\theta)\,\mathrm{dr} &= \left[r\ln\left(\frac{r^2}{c^2}\right)\frac{\partial}{\partial r}\psi(r,\theta)\right]_\epsilon^c - \int_\epsilon^c\frac{\partial}{\partial r}\left(r\ln\left(\frac{r^2}{c^2}\right)\right)\frac{\partial}{\partial r}\psi(r,\theta)\,\mathrm{dr}\\ &= -\epsilon\ln\left(\frac{\epsilon^2}{c^2}\right)\frac{\partial}{\partial r}\psi(\epsilon,\theta) - \int_\epsilon^c\frac{\partial}{\partial r}\left(r\ln\left(\frac{r^2}{c^2}\right)\right)\frac{\partial}{\partial r}\psi(r,\theta)\,\mathrm{dr}\\ &= -\epsilon\ln\left(\frac{\epsilon^2}{c^2}\right)\frac{\partial}{\partial r}\psi(\epsilon,\theta) - \left[\left(\ln\left(\frac{r^2}{c^2}\right)+2\right)\psi(r,\theta)\right]_\epsilon^c\\ &\quad + \int_\epsilon^c\frac{\partial^2}{\partial r^2}\left(r\ln\left(\frac{r^2}{c^2}\right)\right)\psi(r,\theta)\,\mathrm{dr}\\ &= -\epsilon\ln\left(\frac{\epsilon^2}{c^2}\right)\frac{\partial}{\partial r}\psi(\epsilon,\theta) - 2\psi(c,\theta) + \left(\ln\left(\frac{\epsilon^2}{c^2}\right)+2\right)\psi(\epsilon,\theta)\\ &\quad + \int_\epsilon^c\frac{2}{r}\psi(r,\theta)\,\mathrm{dr}.\end{aligned}$$

Summarising,

$$\int_0^{2\pi}\int_\epsilon^c r\ln(r^2)\left(\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi(r,\theta)\right)\,\mathrm{d}r\,\mathrm{d}\theta =$$

$$= \int_0^{2\pi}\left(-\ln\left(\frac{\epsilon^2}{c^2}\right)\psi(\epsilon,\theta)-\epsilon\ln\left(\frac{\epsilon^2}{c^2}\right)\frac{\partial}{\partial r}\psi(\epsilon,\theta)-2\psi(c,\theta)+\left(\ln\left(\frac{\epsilon^2}{c^2}\right)+2\right)\psi(\epsilon,\theta)\right)\mathrm{d}\theta$$

$$= 2\int_0^{2\pi}\psi(\epsilon,\theta)\,\mathrm{d}\theta-2\int_0^{2\pi}\psi(c,\theta)\,\mathrm{d}\theta-\int_0^{2\pi}\epsilon\ln\left(\frac{\epsilon^2}{c^2}\right)\frac{\partial}{\partial r}\psi(\epsilon,\theta)\,\mathrm{d}\theta.$$

The last summand tends to zero with $\epsilon\to 0$, since the derivative is bounded and $\epsilon\ln(\frac{\epsilon^2}{c^2})\to\ 0$ when $\epsilon\to 0$. The first summand tends to $4\pi\psi(0,0)=4\pi\varphi(0,0)$. The middle one is an average of φ over the circle of radius c and center at 0 multiplied by 4π. Finally we can write

$$\Delta u = 4\pi(\delta_\lambda - {\mathrm{Avg}^1}_{\lambda,c}).$$ □

Lemma A.4. *Let $c>0$, $\lambda\in\mathbb{C}$ and*

$$u(z)=\begin{cases}1-\frac{|z-\lambda|^2}{c^2} & \text{for } z\in D(\lambda,c),\\ 0 & \text{for } z\in\mathbb{C}\setminus D(\lambda,c).\end{cases}$$

Then u is continuous on $\mathbb{C}\setminus\{0\}$ and

$$\Delta u = 4\pi({\mathrm{Avg}^1}_{\lambda,c}-{\mathrm{Avg}^2}_{\lambda,c}).$$

Proof. Assume for the moment that we have proved the lemma for $\lambda=0$. Denote by u_λ the function u defined for $\lambda\in\mathbb{C}$. Then by Lemma A.2 we would have for any $\varphi\in C_c^\infty(\mathbb{C})$

$$\begin{aligned}\langle\Delta u_\lambda,\varphi\rangle &= \langle\Delta(\tau_{-\lambda}u_0),\varphi\rangle=\langle\Delta u_0,\tau_\lambda\varphi\rangle=\langle 4\pi({\mathrm{Avg}^1}_{0,c}-{\mathrm{Avg}^2}_{0,c}),\tau_\lambda\varphi\rangle\\ &=\langle 4\pi({\mathrm{Avg}^1}_{\lambda,c}-{\mathrm{Avg}^2}_{\lambda,c}),\varphi\rangle.\end{aligned}$$

Therefore, without loss of generality we may assume that $\lambda=0$.

Let $\varphi\in C_c^\infty(\mathbb{C})$. We have to calculate

$$\left\langle\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u,\varphi\right\rangle=\left\langle u,\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi\right\rangle.$$

We change Cartesian coordinates into polar coordinates. Let $\psi\in C_c^\infty(\mathbb{R}\times[0,2\pi])$ be given by $\psi(r,\theta)=\varphi(x,y)$. Then

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi(x,y)=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi(r,\theta)$$

and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$. Moreover $\frac{x^2+y^2}{c^2} = \frac{r^2}{c^2}$. Therefore our task is to calculate

$$\int_0^{2\pi}\int_0^{c} r\Big(1-\frac{r^2}{c^2}\Big)\left(\Big(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\Big)\psi(r,\theta)\right)\,\mathrm{d}r\,\mathrm{d}\theta.$$

We calculate the third integral

$$\int_0^{2\pi}\frac{1}{r}\Big(1-\frac{r^2}{c^2}\Big)\frac{\partial^2}{\partial\theta^2}\psi(r,\theta)\,\mathrm{d}\theta = \left[\frac{1}{r}\Big(1-\frac{r^2}{c^2}\Big)\frac{\partial}{\partial\theta}\psi(r,\theta)\right]_0^{2\pi} = 0.$$

In further calculations we will use the following equalities

$$\frac{\partial}{\partial r}\Big(1-\frac{r^2}{c^2}\Big) = -\frac{2r}{c^2},\quad \frac{\partial}{\partial r}\left(r\Big(1-\frac{r^2}{c^2}\Big)\right) = 1-\frac{3r^2}{c^2},\quad \frac{\partial^2}{\partial r^2}\left(r\Big(1-\frac{r^2}{c^2}\Big)\right) = -\frac{6r}{c^2}.$$

The second integral can be calculated as follows

$$\begin{aligned}\int_0^{c}\Big(1-\frac{r^2}{c^2}\Big)\frac{\partial}{\partial r}\psi(r,\theta)\,\mathrm{d}r &= \left[\Big(1-\frac{r^2}{c^2}\Big)\psi(r,\theta)\right]_0^{c} - \int_0^{c}\left(\frac{\partial}{\partial r}\Big(1-\frac{r^2}{c^2}\Big)\right)\psi(r,\theta)\,\mathrm{d}r\\ &= -\psi(0,\theta) + \int_0^{c}\frac{2r}{c^2}\psi(r,\theta)\,\mathrm{d}r.\end{aligned}$$

Now, we compute the first integral

$$\begin{aligned}\int_0^{c} r\Big(1-\frac{r^2}{c^2}\Big)\frac{\partial^2}{\partial r^2}\psi(r,\theta)\,\mathrm{d}r &= \left[r\Big(1-\frac{r^2}{c^2}\Big)\frac{\partial}{\partial r}\psi(r,\theta)\right]_0^{c}\\ &\quad - \int_0^{c}\frac{\partial}{\partial r}\left(r\Big(1-\frac{r^2}{c^2}\Big)\right)\frac{\partial}{\partial r}\psi(r,\theta)\,\mathrm{d}r\\ &= -\int_0^{c}\frac{\partial}{\partial r}\left(r\Big(1-\frac{r^2}{c^2}\Big)\right)\frac{\partial}{\partial r}\psi(r,\theta)\,\mathrm{d}r\\ &= -\left[\Big(1-\frac{3r^2}{c^2}\Big)\psi(r,\theta)\right]_0^{c}\\ &\quad + \int_0^{c}\frac{\partial^2}{\partial r^2}\left(r\Big(1-\frac{r^2}{c^2}\Big)\right)\psi(r,\theta)\,\mathrm{d}r\\ &= 2\psi(c,\theta) + \psi(0,\theta) - \int_0^{c}\frac{6r}{c^2}\psi(r,\theta)\,\mathrm{d}r.\end{aligned}$$

Summarising,

$$\int_0^{2\pi}\int_0^c r\Big(1-\frac{r^2}{c^2}\Big)\left(\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi(r,\theta)\right)\,\mathrm{dr}\,\mathrm{d}\theta =$$

$$= \int_0^{2\pi}\left(\int_0^c \frac{2r}{c^2}\psi(r,\theta)\,\mathrm{dr} + 2\psi(c,\theta) - \int_0^c \frac{6r}{c^2}\psi(r,\theta)\,\mathrm{dr}\right)\mathrm{d}\theta$$

$$= \int_0^{2\pi}\left(2\psi(c,\theta) - \frac{4}{c^2}\int_0^c r\psi(r,\theta)\,\mathrm{dr}\right)\mathrm{d}\theta$$

$$= 2\int_0^{2\pi}\psi(c,\theta)\,\mathrm{d}\theta - \frac{4}{c^2}\int_0^{2\pi}\int_0^c r\psi(r,\theta)\,\mathrm{dr}\,\mathrm{d}\theta$$

$$= 2\int_0^{2\pi}\psi(c,\theta)\,\mathrm{d}\theta - \frac{4}{c^2}\int_{D(0,c)}\varphi(x,y)\,\mathrm{dx}\,\mathrm{dy}.$$

Finally, we can write

$$\Delta u = 4\pi(\mathrm{Avg^1}_{\lambda,c} - \mathrm{Avg^2}_{\lambda,c}). \qquad \square$$

Lemma A.5. *Let*

$$u_\lambda(z) = \begin{cases}\ln|z-\lambda| & \textit{for}:\ \operatorname{Im} z > 0,\\ 0 & \textit{for}\ \operatorname{Im} z \le 0,\end{cases}$$

and

$$v_\lambda(z) = \begin{cases}0 & \textit{for}\ \operatorname{Im} z \ge 0,\\ \ln|z-\lambda| & \textit{for}\ \operatorname{Im} z < 0.\end{cases}$$

Then for $\lambda \in \mathbb{H}_-$

$$\langle\Delta u_\lambda,\varphi\rangle = -\int_{-\infty}^{\infty}\ln|x-\lambda|\frac{\partial}{\partial y}\varphi(x)\,\mathrm{dx} - \int_{-\infty}^{\infty}\frac{\operatorname{Im}\lambda}{|x-\lambda|^2}\varphi(x)\,\mathrm{dx}$$

$$\langle\Delta v_\lambda,\varphi\rangle = 2\pi\delta_\lambda + \int_{-\infty}^{\infty}\ln|x-\lambda|\frac{\partial}{\partial y}\varphi(x)\,\mathrm{dx} + \int_{-\infty}^{\infty}\frac{\operatorname{Im}\lambda}{|x-\lambda|^2}\varphi(x)\,\mathrm{dx}$$

and for $\lambda \in \mathbb{H}_+$

$$\langle\Delta u_\lambda,\varphi\rangle = 2\pi\delta_\lambda - \int_{-\infty}^{\infty}\ln|x-\lambda|\frac{\partial}{\partial y}\varphi(x)\,\mathrm{dx} - \int_{-\infty}^{\infty}\frac{\operatorname{Im}\lambda}{|x-\lambda|^2}\varphi(x)\,\mathrm{dx}$$

$$\langle\Delta v_\lambda,\varphi\rangle = \int_{-\infty}^{\infty}\ln|x-\lambda|\frac{\partial}{\partial y}\varphi(x)\,\mathrm{dx} + \int_{-\infty}^{\infty}\frac{\operatorname{Im}\lambda}{|x-\lambda|^2}.$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{C})$ and $\lambda \in \mathbb{H}_-$. We assume that $z = x + iy$. Then

$$\Big\langle \frac{\partial^2}{\partial x^2} u_\lambda, \varphi \Big\rangle = \Big\langle u_\lambda, \frac{\partial^2}{\partial x^2}\varphi \Big\rangle = \frac{1}{2}\int_0^\infty \int_{-\infty}^\infty \ln|z-\lambda|^2 \frac{\partial^2}{\partial x^2}\varphi(z)\,\mathrm{dx}\,\mathrm{dy}.$$

For $z \neq \lambda$ we have

$$\frac{\partial}{\partial x}\ln|z-\lambda|^2 = \frac{2\,\mathrm{Re}(z-\lambda)}{|z-\lambda|^2},$$
$$\frac{\partial}{\partial y}\ln|z-\lambda|^2 = \frac{2\,\mathrm{Im}(z-\lambda)}{|z-\lambda|^2}$$

and

$$\frac{\partial^2}{\partial x^2}\ln|z-\lambda|^2 = \frac{(\mathrm{Im}(z-\lambda))^2 - (\mathrm{Re}(z-\lambda))^2}{|z-\lambda|^2},$$
$$\frac{\partial^2}{\partial y^2}\ln|z-\lambda|^2 = \frac{(\mathrm{Re}(z-\lambda))^2 - (\mathrm{Im}(z-\lambda))^2}{|z-\lambda|^2}.$$

Integrating by parts twice gives then

$$\begin{aligned}\Big\langle \frac{\partial^2}{\partial x^2} u_\lambda, \varphi \Big\rangle &= \frac{1}{2}\int_0^\infty \int_{-\infty}^\infty \frac{\partial^2}{\partial x^2}\ln|z-\lambda|^2 \varphi(z)\,\mathrm{dx}\,\mathrm{dy} \\ &= \frac{1}{2}\int_0^\infty \int_{-\infty}^\infty \frac{(\mathrm{Im}(z-\lambda))^2 - (\mathrm{Re}(z-\lambda))^2}{|z-\lambda|^2}\varphi(z)\,\mathrm{dx}\,\mathrm{dy}.\end{aligned}$$

Further,

$$\Big\langle \frac{\partial^2}{\partial y^2} u_\lambda, \varphi \Big\rangle = \Big\langle u_\lambda, \frac{\partial^2}{\partial y^2}\varphi \Big\rangle = \frac{1}{2}\int_{-\infty}^\infty \int_0^\infty \ln|z-\lambda|^2 \frac{\partial^2}{\partial y^2}\varphi(z)\,\mathrm{dy}\,\mathrm{dx}$$

and

$$\begin{aligned}\frac{1}{2}\int_0^\infty \ln|z-\lambda|^2 \frac{\partial^2}{\partial y^2}\varphi(z)\,\mathrm{dy} &= \frac{1}{2}\Big[\ln|z-\lambda|^2 \frac{\partial}{\partial y}\varphi(z)\Big]_0^\infty - \frac{1}{2}\int_0^\infty \frac{\partial}{\partial y}\ln|z-\lambda|^2 \frac{\partial}{\partial y}\varphi(z)\,\mathrm{dy} \\ &= -\frac{1}{2}\ln|x-\lambda|^2 \frac{\partial}{\partial y}\varphi(x) - \frac{1}{2}\Big[\frac{2\,\mathrm{Im}(z-\lambda)}{|z-\lambda|^2}\varphi(z)\Big]_0^\infty \\ &\quad + \frac{1}{2}\int_0^\infty \frac{(\mathrm{Re}(z-\lambda))^2 - (\mathrm{Im}(z-\lambda))^2}{|z-\lambda|^2}\varphi(z)\,\mathrm{dy} \\ &= -\frac{1}{2}\ln|x-\lambda|^2 \frac{\partial}{\partial y}\varphi(x) - \frac{\mathrm{Im}\,\lambda}{|x-\lambda|^2}\varphi(x) \\ &\quad + \frac{1}{2}\int_0^\infty \frac{(\mathrm{Re}(z-\lambda))^2 - (\mathrm{Im}(z-\lambda))^2}{|z-\lambda|^2}\varphi(z)\,\mathrm{dy}.\end{aligned}$$

This gives the first formula in the lemma. Proceeding similarly for v_λ when $\lambda \in \mathbb{H}_+$ we get the fourth formula. Furthermore, we have

$$\ln|z - \lambda| - u_\lambda = v_\lambda$$

in the sense of distributions. Since $\Delta \ln|z - \lambda| = 2\pi\delta_\lambda$ by virtue of [Ran95, Theorem 3.7.8], this yields the remaining formulae. □

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