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Composition operators on the space of smooth functions

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Operatory kompozycji na przestrzeni funkcji gładkich

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Abstract

The aim of this thesis is to investigate several properties of composition operators and weighted composition operators on the space of smooth functions, i.e., operators of the form $C_\psi : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto F \circ \psi$, or of the form $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, where $\Omega \subset \mathbb{R}^d$ is open, $C^\infty(\Omega)$ is the Fréchet space of smooth functions on Ω, and the functions $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ are smooth. Those operators are very natural examples of operators acting on the space of smooth functions, which is very important in analysis.

The first part of this dissertation is devoted to the question of for which smooth symbols $\psi : \mathbb{R} \to \mathbb{R}$ the range of the composition operator C_ψ is closed in $C^\infty(\mathbb{R})$. We present several necessary and sufficient conditions for this property. In particular, we prove that if $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth semiproper function which has no flat points, then the range of C_ψ is closed.

The second part of this dissertation is devoted to the study of several dynamical properties of composition operators and weighted composition operators acting on $C^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is open. We characterize hypercyclic (in case if the weight is real valued), weakly mixing, and mixing weighted composition operators. As a by-product we obtain a characterization of hypercyclic, weakly mixing, and mixing composition operators. Then we show that those three classes of operators coincide in the one-dimensional case.

Streszczenie

Celem niniejszej rozprawy jest zbadanie kilku własności operatorów kompozycji i wagowych operatorów kompozycji działających na przestrzeni funkcji gładkich, to jest, operatorów postaci $C_\psi : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto F \circ \psi$, lub postaci $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, gdzie $\Omega \subset \mathbb{R}^d$ jest zbiorem otwartym, $C^\infty(\Omega)$ jest przestrzenią Frécheta funkcji gładkich na Ω, a funkcje $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ są gładkie. Operatory te są bardzo naturalnymi przykładami operatorów działających na przestrzeni funkcji gładkich, która jest bardzo ważnym obiektem w analizie.

Pierwsza część rozprawy poświęcona jest pytaniu dla jakich gładkich funkcji $\psi : \mathbb{R} \to \mathbb{R}$ obraz operatora kompozycji C_ψ jest domknięty w $C^\infty(\mathbb{R})$. Podajemy kilka warunków koniecznych i dostatecznych. W szczególności dowodzimy, że jeśli $\psi : \mathbb{R} \to \mathbb{R}$ jest gładką funkcją semiwłaściwą, która nie ma punktów płaskich, to obraz operatora kompozycji C_ψ jest domknięty.

Druga część rozprawy poświęcona jest badaniu kilku dynamicznych własności operatorów kompozycji i wagowych operatorów kompozycji działających na $C^\infty(\Omega)$, gdzie $\Omega \subset \mathbb{R}^d$ jest zbiorem otwartym. Charakteryzujemy hipercykliczne (w przypadku wag rzeczywistych), słabo mieszające i mieszające wagowe operatory kompozycji. Jako wniosek otrzymujemy charakteryzację hipercyklicznych, słabo mieszających i mieszających operatorów kompozycji. Następnie pokazujemy, że te trzy klasy operatorów pokrywają się w przypadku jednowymiarowym.

Contents

Introduction

Let $\Omega \subset \mathbb{R}^d$ be open and let $C^\infty(\Omega)$ be the space of real or complex valued smooth functions on Ω endowed with the topology of uniform convergence on compact sets with respect to all (partial) derivatives. This space is one of the most natural Fréchet spaces of classical analysis - a natural environment for differential equations. The goal of this dissertation is to investigate several properties of composition operators acting on $C^\infty(\Omega)$, i.e., operators of the form

$$C_\psi : C^\infty(\Omega) \to C^\infty(\Omega),\ F \mapsto F \circ \psi,$$

where $\psi : \Omega \to \Omega$ is smooth, and weighted composition operators on $C^\infty(\Omega)$, i.e., operators of the form

$$C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega),\ F \mapsto w \cdot (F \circ \psi),$$

where $\psi : \Omega \to \Omega$ and $w : \Omega \to \mathbb{C}$ are smooth.

Our first goal is to investigate for which one-dimensional smooth symbols $\psi : \mathbb{R} \to \mathbb{R}$ the range of the composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$, is closed, i.e., when the set

$$\operatorname{Im} C_\psi := \left\{F \circ \psi : F \in C^\infty(\mathbb{R})\right\}$$

is closed in $C^\infty(\mathbb{R})$. Let us observe that $\operatorname{Im} C_\psi$ is a subalgebra of $C^\infty(\mathbb{R})$. Let us also note that from the Open Mapping Theorem for Fréchet spaces it follows that the operator C_ψ has closed range if and only if it is open onto its image.

Our second goal is to investigate several dynamical properties of the weighted composition operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, i.e., we want to know for which smooth functions $\psi : \Omega \to \Omega$ and $w : \Omega \to \mathbb{C}$ this operator is hypercyclic, weakly mixing, mixing, or chaotic.

Below we present the state of the art of those two problems and we describe our contributions to those topics. Since both those problems are quite different, we divide this survey into two parts. After presenting the historical background, we will briefly describe the content of this dissertation.

Closed range composition operators

The beginning of the study of closed range composition operators acting on the spaces of smooth functions goes back to a nice result of Whitney who proved in 1943 the following theorem (see [48]).

Theorem. *(Whitney) Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth and even function. Then there exists a smooth function $F:\mathbb{R}\to\mathbb{R}$ such that*

$$f(x) = F\left(x^2\right) \text{ for every } x\in\mathbb{R}.$$

Since the subspace of even functions is closed in $C^\infty(\mathbb{R})$, this result is equivalent with the statement that the composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, where $\psi(x) = x^2$, has closed range.

Let now $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open sets and let $\psi : U \to V$ be a smooth function. Please recall that, by definition, ψ is a semiproper function if for every compact set $K \subset V$ there exists a compact set $L\subset U$ such that $K \cap \psi(U) \subset \psi(L)$. From the very beginning it was well-known that if the range of the composition operator $C_\psi : C^\infty(V) \to C^\infty(U)$, $F \mapsto F\circ\psi$, is closed, then ψ must be a semiproper function (see [13, Prop. 1.4.1] without a proof, for the proof see [33, Prop. 2.9]).

An important contribution to the problem of deciding if the range of a composition operator is closed was made by Glaeser in [25]. His result can be formulated in the following way. Let us emphasize that in this theorem the symbol of the composition operator is a real analytic function.

Theorem. *(Glaeser) Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open, where $n \le m$. Let $\psi : U \to V$ be a semiproper real analytic function. If the rank of the Jacobian matrix of ψ is equal to n on a dense open subset of U, then the composition operator $C_\psi : C^\infty(V) \to C^\infty(U)$, $F \mapsto F\circ\psi$, has closed range.*

As an immediate consequence of this theorem we can deduce the following corollary.

Corollary. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a real analytic function. The range of the composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F\circ\psi$, is closed if and only if ψ is semiproper.*

Please note that Glaeser in [25] observed that for $\psi:\mathbb{R}\to\mathbb{R}$ defined via the formula

$$\psi(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

we have that

$$\sqrt{\psi} = \overline{\operatorname{Im} C_\psi} \setminus \operatorname{Im} C_\psi.$$

Since ψ is a semiproper function, this example shows that the semiproperness of a smooth function $\psi:\mathbb{R}\to\mathbb{R}$ is *not* a sufficient condition for $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ to have closed range.

Let us note that the idea from Glaeser example, to use the square root when investigating the closedness of the range of the composition operator, plays a significant role in this dissertation (see, e.g., the proof of Theorem 2.4.1).

The problem of deciding for which real analytic functions ψ the range of the composition operator C_ψ is closed was finally solved by Bierstone and Milman [8, 10, 11] and Bierstone, Milman and Pawłucki [12]. They have related the problem of closedness of the range of the composition operator C_ψ with some geometrical properties of the image of the symbol. In fact, the property of the closed range of a composition operator corresponding to a semiproper real analytic symbol ψ depends only on the image of ψ (see [12, Cor. 1.5]), which is a closed subanalytic set (for the theory of semianalytic and subanalytic sets we refer to [9]).

Let now $X \subset \mathbb{R}^d$ be arbitrary. By $C^\infty(X)$ we denote the space of those real valued functions on X which can be extended to a smooth function on $\mathbb{R}^d$, and by $C^{(\infty)}(X)$ we denote the space of those real valued functions on X which can be extended to a C^k function on $\mathbb{R}^d$, for each $k \in \mathbb{N}$. It is obvious that we always have the inclusion

$$C^\infty(X) \subset C^{(\infty)}(X).$$

Surprisingly, the above inclusion may be strict (see [43] for an example of a set in $\mathbb{R}^2$ for which the above inclusion is strict, see [42] for an example of a subanalytic set in $\mathbb{R}^5$ which has the same property). For a detailed information for which sets the equality $C^\infty(X) = C^{(\infty)}(X)$ holds we refer to [42]. Let us just note that this is true if, for example, $X \subset \mathbb{R}$ or if X is from the class of Nash subanalytic sets.

The following deep theorem gives a characterization of closed range composition operators corresponding to real analytic symbols (see [11, Theorem 1.13], where several other equivalent characterizations are described).

Theorem. *(Bierstone, Milman, Pawłucki) Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open and let $\psi : U \to V$ be a semiproper real analytic function. The composition operator $C_\psi : C^\infty(V) \to C^\infty(U)$, $F \mapsto F \circ \psi$, has closed range if and only if*

$$C^\infty(\psi(U)) = C^{(\infty)}(\psi(U)).$$

In particular, this is so when $\psi(U)$ is semianalytic.

The above result shows that the study of closed range composition operators leads to a better understanding of the space of smooth functions and its elements.

In the general case, when the symbol of the composition operator C_ψ is only smooth, the situation seems to be far more difficult and the characterization of closed range composition operators is not known. Probably the most important step towards such a characterization is a better understanding what kind of functions can be in the closure of the image of the composition operator. A big step in this direction was made by Allan et al. (see [3], see also [2] and [4]). They showed that for *injective* $\psi : \mathbb{R} \to \mathbb{R}^d$ a function f is in the closure of the image of C_ψ if and only if at every point it has the "right kind" of Taylor series, i.e., they proved the following theorem.

Theorem. *(Allan, Kakiko, O'Farell, Watson) Let $\psi : \mathbb{R} \to \mathbb{R}^d$ be smooth and injective. Then*

$$\overline{\operatorname{Im} C_\psi} = \left\{ f \in C^\infty(\mathbb{R}) : \forall a \in \mathbb{R}\ \exists F \in C^\infty(\mathbb{R}^d)\ \forall n \geq 0\ f^{(n)}(a) = (F \circ \psi)^{(n)}(a) \right\}.$$

It is well-known that the proper non injective analogue of the set from the right hand side of the equality in the theorem above should be the set

$$\widehat{\operatorname{Im} C_\psi} := \left\{ f \in C^\infty(\mathbb{R}) : \forall b \in \psi(\mathbb{R})\ \exists F_b \in C^\infty(\mathbb{R}^d)\ \forall a \in \psi^{-1}(\{b\}) \forall n \geq 0\ f^{(n)}(a) = (F_b \circ \psi)^{(n)}(a) \right\}.$$

Unfortunately, for an arbitrary smooth $\psi : \mathbb{R} \to \mathbb{R}^d$ we only know that $\overline{\operatorname{Im} C_\psi} \subset \widehat{\operatorname{Im} C_\psi}$ (see, e.g., [2, Cor. 5]).

Please note that Tougeron in [47] showed that if $\psi : \mathbb{R}^n \to \mathbb{R}^m$ is a real analytic function, then $\overline{\operatorname{Im} C_\psi} = \widehat{\operatorname{Im} C_\psi}$ (with the analogous definition of $\widehat{\operatorname{Im} C_\psi}$ for many variables).

Let us also note that there is a similar description of the closure of an arbitrary ideal contained in the space of smooth functions (see the paper of Whitney [49], or the book of Malgrange [37]).

In 2011 Kenessey and Wengenroth [34], using the above results of Allan et al., obtained the following full characterization of composition operators with closed range corresponding to smooth *injective* symbols $\psi : \mathbb{R} \to \mathbb{R}^d$.

Theorem. *(Kenessey, Wengenroth) Let $\psi : \mathbb{R} \to \mathbb{R}^d$ be a smooth and injective function. The composition operator $C_\psi : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$, has closed range if and only if ψ is semiproper, has Whitney regular image and has no flat points.*

Please note that a point x is called a flat point of a function $\psi = (\psi_1, \ldots, \psi_d) : \mathbb{R} \to \mathbb{R}^d$ if for every $1 \leq i \leq d$ all derivatives of the function ψ_i vanish at x. Please also note that Whitney regularity of a set, which appears in the theorem above, is a geometrical property and in case if $\psi : \mathbb{R} \to \mathbb{R}$ it is trivially satisfied for $\psi(\mathbb{R})$. Thus, the following corollary is true.

Corollary. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth and injective. Then the composition operator C_ψ : $C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$, has closed range if and only if ψ is a semiproper function which has no flat points.*

The above result shows that in the smooth case the property of the composition operator C_ψ to have closed range depends not only on the properties of the image of ψ. Indeed, if $\psi_1 : \mathbb{R} \to \mathbb{R}$ is defined as $\psi_1(x) = x$ and if $\psi_2 : \mathbb{R} \to \mathbb{R}$ is defined as

$$\psi_2(x) = \begin{cases} \operatorname{sgn}(x) \cdot \exp\left(-\frac{1}{x^2}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

then $\psi_1(\mathbb{R}) = \psi_2(\mathbb{R})$, the range of C_{ψ_1} is closed, and the range of C_{ψ_2} is not closed.

Furthermore, Kenessey in [33] obtained some specific sufficient and necessary conditions for C_ψ to have closed range, where $\psi : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth and *injective* function.

Finally, let us note that question about the closed range of a composition operator was also extensively studied, when the underlying space is a suitable space of holomorphic functions, or the space of real analytic functions (see, for example, the books of Cowen, MacCluer [18], and Shapiro [46], papers of Zorboska [52], [53] for closed range composition operators on various spaces of holomorphic functions, and papers of Domański and Langenbruch [21], [22], [23], Domański, Goliński and Langenbruch [20] for closed range composition operators on the the space of real analytic functions).

In this dissertation we obtain some sufficient and necessary conditions for the closed range of the composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$, where $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth, not necessary injective, function.

For the sufficiency part we prove, using ideas of Kenessey and Wengenroth, the following result (see Theorem 2.2.3). Please note that by the fiber of ψ over b we mean the set $\psi^{-1}(\{b\})$.

Theorem. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth semiproper function which satisfies the following conditions:*

1. *Every fiber of ψ over a boundary point of $\psi(\mathbb{R})$ contains a non-flat point.*
2. *Every fiber of ψ over an interior point of $\psi(\mathbb{R})$ contains either a non-flat non extreme point or both a non-flat local minimum and a non-flat local maximum.*

Then

$$\operatorname{Im} C_\psi = \overline{\operatorname{Im} C_\psi} = \widehat{\operatorname{Im} C_\psi}.$$

As an immediate consequence of this result we obtain that if $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth semiproper function without flat points, then the range of the operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$, is closed (see Corollary 2.2.6). Thus, our theorem extends the sufficiency parts of the results of Glaeser and Kenessey and Wengenroth in case if $\psi : \mathbb{R} \to \mathbb{R}$.

Let us now note that if a not constant, smooth, and semiproper function $\psi : \mathbb{R} \to \mathbb{R}$ does not satisfy the conditions of our theorem above, then there exists $b \in \psi(\mathbb{R})$ such that the boundary of the set $\{x \in \mathbb{R} : \psi(x) > b\}$ or the boundary of the set $\{x \in \mathbb{R} : \psi(x) < b\}$ is nonempty and contains only flat points. We strongly believe that in this case the range of the composition operator C_ψ is not closed. Unfortunately, we were not able to show that this is true. We only managed to prove the following weaker result (see Theorem 2.4.1). Please note that a point x_0 is called a nice flat point of a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ if for every $n \geq 1$ and every $0 < \theta < 1$ there is a neighbourhood U of x_0 such that for every $x \in U$ we have $\left|\psi^{(n)}(x)\right| \leq \left|\psi(x) - \psi(x_0)\right|^\theta$. Let us emphasize that the most standard examples of flat points of smooth functions are in fact nice flat points (see Example 2.3.8), however there are examples of smooth functions which posses a flat point which is not a nice flat point (see Example 2.3.10).

Theorem. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be smooth. If there exists $b \in \psi(\mathbb{R})$ such that the boundary of the set $\{x \in \mathbb{R} : \psi(x) > b\}$ or the boundary of the set $\{x \in \mathbb{R} : \psi(x) < b\}$ is nonempty and contains* *only nice flat points of ψ, then the range of the composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$, is not closed.*

Connecting our both results, we obtain the following characterization of closed range composition operators in a special case (see Corollary 2.4.3).

Corollary. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that every flat point of ψ is a nice flat point of ψ. The range of the composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$, is closed if and only if ψ is a semiproper function which is ether constant or it satisfies the following conditions*

1. *Every fiber of ψ over a boundary point of $\psi(\mathbb{R})$ contains a non-flat point.*
2. *Every fiber of ψ over an interior point of $\psi(\mathbb{R})$ contains either a non-flat non extreme point or both a non-flat local minimum and a non-flat local maximum.*

At first glance, the notion of a nice flat point seems to be not natural, however we were able to prove the following result which shows that those points are very important in studying the smoothness of roots of smooth functions (please note that this result is a crucial ingredient in the proof of our necessary condition for C_ψ to have closed range).

Theorem. *Let $g : \mathbb{R} \to \mathbb{R}$ be smooth. The function $|g|^\theta$ is smooth for every $0 < \theta < 1$ if and only if every point $x_0 \in g^{-1}(\{0\})$ is a nice flat point of g.*

Let us note that in general, the square root of a smooth nonnegative function g is not smooth even if the function g is flat on the set $g^{-1}(\{0\})$. The first example of such a situation was indicated by Glaeser (see [26]). For further information about the regularity of the square root of a smooth function we refer to [1, 17].

Finally, let us note that the proof of our necessary condition (Theorem 2.2.3) was already published (see [40]). Let us also note that the author of this thesis has also published a paper (see [41]), where he claims to give a full characterization of closed range composition operators corresponding to one-dimensional smooth symbols. Unfortunately, that papers contains a major error (to be more precise, [41, Lemma 2.3 and Lemma 2.4] are not correct). At this moment it is not clear if the main result stated there is true. Although, in this thesis we use a lot of ideas contained in that paper.

Dynamical properties of composition operators and weighted composition operators

The second part of our thesis is devoted to the study of dynamical properties of composition operators and weighted composition operators on the space of smooth functions. This part of our research belongs to linear dynamics, a fascinating and beautiful branch of functional analysis. As the name indicates, linear dynamics is a theory which investigates the behaviour of iterates of linear transformations. Let us briefly recall basics concepts of this theory.

Let X be a Fréchet space and let $T : X \to X$ be an operator (i.e, linear and continuous). For every $n \geq 1$ the operator $T^n : X \to X$ is defined as the n-th iterate of T, i.e,

$$T^n = \underbrace{T \circ \cdots \circ T}_{n\ times}.$$

One of the most fundamental notions in linear dynamics are hypercyclic operators. An operator $T : X \to X$ is called hypercyclic if there exists $x \in X$ such that the set

$$\mathrm{orb}(x, T) := \left\{T^n(x) : n \geq 1\right\}$$

is dense in X. Such an x is called a hypercyclic vector of T. By the famous Birkhoff's Transitivity Theorem (see Theorem 1.2.4), an operator T on a Fréchet space X is hypercyclic if and only if it is topologically transitive, i.e., for every two nonempty open sets $U, V \subset X$ there is $n \in \mathbb{N}$ such that $T^n(U) \cap \mathrm{V} \neq \emptyset$. Other interesting classes of operators investigated by linear dynamics are weakly mixing, mixing, or chaotic operators (for the definitions of those notions we refer to the Preliminaries of this dissertation, we also refer to two recently published monographs [7, 30] for a great exposition of the subject of linear dynamics).

Let us note that the notion of a hypercyclic operator is closely related with the famous invariant subspace problem, which is still open. This problem asks if every operator T on a Hilbert space H posses a closed non-trivial invariant subspace, i.e., if there is a closed non-trivial subspace H_1 of H such that $T(H_1) \subset H_1$. Please observe that if every non-zero vector $x \in H$ is a hypercyclic vector of T, then T does not have a non-trivial invariant subspace (it even does not have a non-trivial invariant subset). It is well known that the invariant subspace problem has a negative answer in the Banach space setting (see the paper of Enflo [24] and the paper of Read [44], where he constructs an operator on l_1, for which every non-zero vector is hypercyclic). Let us also mention the paper of Goliński [28], where, for example, he constructs an operator on the Schwartz space of rapidly decreasing functions for which every non-zero vector is hypercyclic.

At first glance, hypercyclicity of an operator seems to be an extremely rare property. Let us note that it is well-known that there are no hypercyclic operators acting on finite dimensional spaces (see [30, Theorem 2.58]). However, in the infinite dimensional setting, there are plenty of natural examples of hypercyclic operators. Below we list the three classical ones. Please note that it is known that for every infinite dimensional separable Fréchet space there exists a hypercyclic operator acting on it (see [6, 16]).

A fundamental example of a hypercyclic operator is due to Rolewicz, who proved in [45] that the operator λB, where $|\lambda| > 1$ and B is the backward shift, is a hypercyclic operator on the l_p spaces. The second example is due to Birkhoff [14], who showed that the translation operator on the space of entire functions on the complex plain is hypercyclic. The third one is due to MacLane [36], generalized by Godefroy and Shapiro [27]. They proved that differential operators with constant coefficients on the space of entire functions are hypercyclic. Let us notice that all the mentioned operators are even mixing and chaotic (for details see, for example, the book of Grosse-Erdmann and Peris [30]).

Let us now observe that the translation operator investigated by Birkhoff is a particular case of a composition operator (on the space of entire functions). Simple arguments from linear dynamics allow us to transfer Birkhoff's result to the smooth case, i.e., to show that the composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ with the symbol $\psi(x) = (x + b)$, where b a non-zero real number, is hypercyclic, mixing, and even chaotic.

There is an extensive literature devoted to various dynamical properties of (weighted) composition operators on many types of spaces of functions, we only mention some of it. In [15], Bonet and Domański obtained, among other results, a characterization of topologically transitive composition operators acting on the space of real analytic functions. Grosse-Erdmann and Mortini (see [29]) characterized hypercyclic composition operators acting on the space of holomorphic functions $H(\Omega)$, where $\Omega \subset \mathbb{C}$ is open. This was later on generalized by Zając [51], who obtained a characterization of hypercyclic composition operators acting on $H(\Omega)$, where $\Omega \subset \mathbb{C}^d$ is a domain of holomorphy, or if it is even a Stein manifold. There are results about dynamical properties of composition operators acting on L^p spaces and spaces of continuous functions due to Kalmes [31]. Moreover, Kalmes and Niess [32] obtained results about dynamical properties of composition operators on the kernels of non constant linear partial differential equations (for instance for harmonic functions). Finally, let us mention the paper [50] of Yousefi and Rezaei, where they investigate hypercyclicity of weighted composition operators on the space of holomorphic functions. To our best knowledge, there are no results about dynamical properties of (weighted) composition operators on the space of smooth functions.

To give a feeling what is going on in this area, let us state two results about the dynamical behaviour of composition operators on the spaces of holomorphic functions. The first result is due to Grosse-Erdmann and Mortini [29].

Theorem. *(Grosse-Erdmann, Mortini) Let $\Omega \subset \mathbb{C}$ be open and let $\psi : \Omega \to \Omega$ be a bijective holomorphic function.*

1. *If Ω is finitely connected but not simply connected, then the composition operator $C_\psi : H(\Omega) \to H(\Omega)$, $F \mapsto F \circ \psi$, is never hypercyclic.*
2. *If Ω is simply connected or infinitely connected, then the composition operator $C_\psi : H(\Omega) \to H(\Omega)$, $F \mapsto F \circ \psi$, is hypercyclic if and only if the function ψ has the run-away property, i.e., for every compact set $K \subset \Omega$ there is $n \in \mathbb{N}$ such that $K \cap \psi_n(K) = \emptyset$.*

We will see in a moment that in the smooth case the dynamical behaviour of the weighted composition operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$ does not depend on the topological properties of Ω. We will also see that in the smooth case the run-away property of the symbol of the operator plays a crucial role.

The following result is due to Shapiro (see, e.g, [30, Theorem 4.37]).

Theorem. *(Shapiro) Let $\Omega \subset \mathbb{C}$ be simply connected and let $\psi : \Omega \to \Omega$ be a bijective holomorphic function. The following conditions are equivalent.*

(1) The composition operator $C_\psi : H(\Omega) \to H(\Omega)$, $F \mapsto F \circ \psi$, is hypercyclic.

(2) The composition operator $C_\psi : H(\Omega) \to H(\Omega)$, $F \mapsto F \circ \psi$, is mixing.

(3) The composition operator $C_\psi : H(\Omega) \to H(\Omega)$, $F \mapsto F \circ \psi$, is chaotic.

(4) The function ψ has the run-away property.

(5) The function ψ has no fixed points.

The second goal of this dissertation is to investigate various dynamical properties of the weighted composition operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w\cdot(F\circ\psi)$, where $\Omega \subset \mathbb{R}^d$ is open and the functions $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ are smooth. As noted above, to our best knowledge, previously the results about those properties were not known. Let us also note, that in our study we use several techniques and ideas, which were developed in the study of dynamical properties of composition operators on other spaces of functions.

The main results of this part of our dissertation are Theorem 3.2.6, Corollary 3.2.7, Theorem 3.2.9, and Theorem 3.4.4. In Theorem 3.2.6 we obtain the following full characterization of weakly mixing weighted composition operators. Let us emphasize, that in this context, the run-away property of the symbol of the operator plays a crucial role (as in the mentioned above theorem due to Grosse-Erdmann and Mortini). Please not that by $[\psi'(x)]$ we denote the Jacobian matrix of ψ at x.

Theorem. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ be smooth. The weighted composition operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, is weakly mixing if and only if the following conditions are satisfied:*

1. *For every $x \in \Omega$ we have that $w(x) \neq 0$.*
2. *The function ψ is injective.*
3. *For every $x \in \Omega$ we have that* $\det[\psi'(x)] \neq 0$.
4. *The function ψ has the run-away property.*

Unfortunately, we were not able to fully characterize those weighted composition operators which are hypercyclic. We only managed to find the following full description of hypercyclic operators $C_{w,\psi}$ under an additional assumption that the function w is real valued (see Corollary 3.2.7).

Corollary. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{R}$ be smooth. The following conditions are equivalent:*

(1) The operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, is weakly mixing.

(2) The operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, is hypercyclic.

(3) The following conditions are satisfied:

- *For every $x \in \Omega$ we have $w(x) \neq 0$.*
- *The function ψ is injective.*
- *For every $x \in \Omega$ we have $\det[\psi'(x)] \neq 0$.*
- *The function ψ has the run-away property.*

In Theorem 3.2.9 we give the following characterization of mixing weighted composition operators. Please note that a smooth function $\psi : \Omega \to \Omega$ has the strong run-away property if for every compact set $K \subset \Omega$ there is $N \in \mathbb{N}$ such that for every $n \geq N$ we have $\psi_n(K) \cap K = \emptyset$.

Theorem. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ be smooth. The weighted composition operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, is mixing if and only if the following conditions are satisfied:*

1. *For every $x \in \Omega$ we have $w(x) \neq 0$.*
2. *The function ψ is injective.*
3. *For every $x \in \Omega$ we have $\det[\psi'(x)] \neq 0$.*
4. *The function ψ has the strong run-away property.*

Unfortunately, we were not able to decide if, in general, every hypercyclic (or weakly mixing) weighted composition operator on the space of smooth functions is automatically mixing (in order to prove this, one has to show that every smooth injective function which has the run-away property also has the strong run-away property). However, we managed to show the following result (see Theorem 3.4.4), which shows that all mentioned dynamical properties of weighted composition operators coincide in the one-dimensional case.

Theorem. *Let $\psi : \mathbb{R} \to \mathbb{R}$, $w : \mathbb{R} \to \mathbb{C}$ be smooth and assume that $w(x) \neq 0$ for every $x \in \mathbb{R}$. The following conditions are equivalent:*

(1) The operator $C_{w,\psi} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto w \cdot (F \circ \psi)$ is hypercyclic.

(2) The operator $C_{w,\psi} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto w \cdot (F \circ \psi)$, is weakly mixing.

(3) The operator $C_{w,\psi} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto w \cdot (F \circ \psi)$, is mixing.

(4) *The operator* $C_{w,\psi} : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$, $F \mapsto w \cdot (F \circ \psi)$, *is chaotic.*

(5) *For all* $x \in \mathbb{R}$ *we have* $\psi'(x) \neq 0$ *and* ψ *has the run-away property.*

(6) *For all* $x \in \mathbb{R}$ *we have* $\psi'(x) \neq 0$ *and* ψ *has the strong run-away property.*

(7) *For all* $x \in \mathbb{R}$ *we have* $\psi'(x) \neq 0$ *and* ψ *has no fixed points.*

Content of the dissertation

This dissertation is divided into 3 chapters. In the first chapter we establish notation and we present some, well-known by now, results which will be useful for our purposes.

The second chapter is devoted to the question of for which smooth symbols $\psi : \mathbb{R} \to \mathbb{R}$ the composition operator $C_{\psi} : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$, $F \mapsto F \circ \psi$, has closed range. In Section 2.1 we recall the notion of a semiproper function and we give a different proof of a well-known fact saying that semiproperness of ψ is a necessary condition for C_{ψ} to have closed range (see Proposition 2.1.6). In the next section we give a sufficient condition ensuring that the operator C_{ψ} has closed range (see Theorem 2.2.3). As a consequence of this result we obtain that if $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth semiproper function without flat points, then the range of C_{ψ} is closed (see Corollary 2.2.6). The goal of Section 2.3 is to introduce the notion of a nice flat point and to characterize those smooth functions $g : \mathbb{R} \to \mathbb{R}$ such that for every $0 < \theta < 1$ the function $|g|^{\theta}$ is smooth (see Theorem 2.3.4). In Section 2.4 we use this characterization to obtain a necessary condition for C_{ψ} to have closed range (see Theorem 2.4.1). As a consequence we obtain a characterization of closed range composition operators in a special case (see Corollary 2.4.3). Several examples illustrate the results obtained.

The third chapter of this thesis is devoted to the study of several dynamical properties of composition operators and weighted composition operators on the space of smooth functions. In Section 3.1 we introduce all the necessary notions from linear dynamics which are specific for (weighted) composition operators. In Section 3.2 we characterize weakly mixing weighted composition operators (see Theorem 3.2.6). As a corollary, we obtain a characterization of hypercyclic weighted composition operators corresponding to real weights (see Corollary 3.2.7). In Theorem 3.2.9 we characterize mixing weighted composition operators. Section 3.3 is a by-product of the previous one. In Theorem 3.3.1 and Theorem 3.3.2 we characterize hypercyclic, weakly mixing and mixing composition operators. In the last section we investigate what happens in the one dimensional case (i.e, when $\Omega \subset \mathbb{R}$). We show (see Theorem 3.4.4) that in this case all the mentioned dynamical notions coincide for weighted composition operators. The last section contains few examples.

CHAPTER 1

PRELIMINARIES

The aim of this chapter is to introduce necessary notions and to collect basic facts which will be used in this thesis.

By $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ we denote the set of all natural numbers $\{1,2,\ldots\}$, the field of real numbers, and the field of complex numbers, respectively. If z is a complex number, then by $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, and $\mathrm{Arg}(z)$ we denote the real part of z, the imaginary part of z, and the principal value of the argument of z, i.e., $\mathrm{Arg}(z) \in [0,2\pi)$, respectively.

By a Fréchet space we mean a locally convex, metrizable, and complete linear topological space over $\mathbb{R}$ or $\mathbb{C}$. If X, Y are Fréchet spaces then a map $T: X \to Y$ is called an operator if it is linear and continuous.

All notions from functional analysis are explained in [38]. For a great introduction to linear dynamics we refer to two recently published monographs [7, 30].

1.1 The space of smooth functions

Let $\Omega \subset \mathbb{R}^d$ be open. By a real (or complex) space $C^\infty(\Omega)$, we denote the linear space of all real (or complex) valued smooth functions on Ω, equipped with the standard topology of uniform convergence of functions and all partial derivatives on compact set, i.e., the topology generated by the family of seminorms

$$\left\{||\cdot||_{K,n} : K \subset \Omega \text{ compact}, \ n \geq 0\right\},$$

where

$$||f||_{K,n} = \max_{x\in K} \max_{0\leq|\alpha|\leq n} \left|\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)\right|,$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$, and $|\alpha| = \alpha_1 + \ldots + \alpha_d$. It is well known that $C^\infty(\Omega)$ is a Fréchet space (see, e.g. [38, Example 5.18(4)]).

The following result is due to E. Borel (see, e.g., [35, Theorem 2.2.1]). Please note that if $\Omega \subset \mathbb{R}$ and if $f \in C^\infty(\Omega)$, then by definition $f^{(0)} = f$ and $f^{(n)}$ is the n-th derivative of f.

Theorem 1.1.1. *Let $\{a_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real (or complex) numbers. There exists $f \in C^\infty((-1,1))$ such that*

$$f^{(n)}(0) = a_n \text{ for every } n \geq 0.$$

Definition 1.1.2. *Let I be a closed, connected subset of $\mathbb{R}$. We say that $f : I \to \mathbb{C}$ is a smooth function on I if all the derivatives of f exists on the interior on I and can be extended in a continuous way to the whole I.*

The following corollary is a immediate consequence of the theorem of E. Borel.

Corollary 1.1.3. *Let I be a closed, connected subset of $\mathbb{R}$ and let f be a smooth function on I. There exists $\tilde{f} \in C^\infty(\mathbb{R})$ such that*

$$\tilde{f}_{|I} = f.$$

The following formula of Faá di Bruno (see, e.g., [35, Lemma 1.3.1]) will play an important role in this thesis.

Theorem 1.1.4. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions. For every $n \geq 1$ and for every $x \in \mathbb{R}$ we have*

$$(f \circ g)^{(n)}(x) = \sum_{A_n} B_{m_1,\ldots,m_n} \cdot f^{(m_1+m_2+\ldots+m_n)}(g(x)) \cdot \prod_{j:m_j\neq 0} \left(g^{(j)}(x)\right)^{m_j},$$

where A_n is the set of all n - tuples of nonnegative integers $m_1, m_2, \ldots, m_n$ satisfying the condition

$$1 \cdot m_1 + 2 \cdot m_2 + \ldots + m_n \cdot n = n$$

and

$$B_{m_1,\ldots,m_n} = \frac{n!}{(1!)^{m_1} \cdot m_1! \cdot \ldots \cdot (n!)^{m_n} \cdot m_n!}.$$

The following propositions are easy and well-known.

Proposition 1.1.5. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$ be smooth. The mapping*

$$C_\psi : C^\infty(\Omega) \to C^\infty(\Omega),\ F \mapsto F \circ \psi,$$

is linear and continuous.

Proposition 1.1.6. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ be smooth. The mapping*

$$C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega),\ F \mapsto w \cdot (F \circ \psi),$$

is linear and continuous.

One of the most important tools available when working with smooth functions are smooth partitions of unity. In this thesis we will use the following proposition (see, e.g., [39, Cor. 1.2.6]).

Proposition 1.1.7. *Let $\Omega \subset \mathbb{R}^d$ be open, let X be a closed subset of Ω and let U be an open subset of Ω containing X. There exists $\varphi \in C^\infty(\Omega)$ such that $\varphi(x) = 1$ if $x \in X$ and $\varphi(x) = 0$ if $x \notin U$.*

We will also need the following well-known lemma (see, e.g., [3, Lemma 5]).

Lemma 1.1.8. *There exists a sequence $\{C_n\}_{n\in\mathbb{N}}$ of positive numbers such that for every natural number k there exists a smooth function $\Phi_k : \mathbb{R} \to \mathbb{R}$ with the following properties:*

1. $0 \le \Phi_k(x) \le 1$ *for all* $x \in \mathbb{R}$;
2. $\Phi_k(x) = 0$ *for* $x \in (-\infty, \frac{1}{2k}]$, $\Phi_k(x) = 1$ *for* $x \in [\frac{1}{k}, \infty)$;
3. $\left|(\Phi_k)^{(i)}(x)\right| \le C_i \cdot k^i$ *for all* $x \in \mathbb{R}$ *and all* $i \in \mathbb{N}$.

1.2 Linear dynamics

Throughout this section X will always denote a Fréchet space. Our aim is to collect definitions and facts from linear dynamics which will be used in this thesis. We start with the following definitions.

Definition 1.2.1. *Let $T : X \to X$ be an operator. For every $n \ge 1$, the operator $T^n : X \to X$ is defined as the n-th iteration of T, i.e.,*

$$T^n = \underbrace{T \circ \cdots \circ T}_{n \ times}.$$

Definition 1.2.2. *An operator $T : X \to X$ is called hypercyclic if there exists $x \in X$ such that the set*

$$\operatorname{orb}(x, T) := \left\{T^n(x) : n \ge 1\right\}$$

is dense in X. Such an x is called a hypercyclic vector of T.

Definition 1.2.3. *An operator $T : X \to X$ is called topologically transitive if for every two nonempty open sets $U, V \subset X$ there is $n \in \mathbb{N}$ such that*

$$T^n(U) \cap V \ne \emptyset.$$

The following famous theorem, due to Birkhoff (see, e.g, [30, Theorem 1.16]), shows that the above two notions are in fact equivalent (in the context of operators acting on Fréchet spaces).

Theorem 1.2.4. *Let X be a Fréchet space and let $T : X \to X$ be an operator. The following conditions are equivalent:*

(1) The operator T is hypercyclic.

(2) The operator T is topologically transitive.

Moreover, if one of the above conditions is satisfied, then the set of hypercyclic vectors of T is dense in X.

In our thesis we will also investigate the following three classes of operators.

Definition 1.2.5. *Let $T : X \to X$ be an operator. We say that T is weakly mixing if the operator $T \times T : X \times X \to X \times X$ is topologically transitive, i.e., for every four nonempty open sets $U_1, U_2, V_1, V_2 \subset X$ there is $n \in \mathbb{N}$ such that*

$$T^n(U_1) \cap V_1 \neq \emptyset \text{ and } T^n(U_2) \cap V_2 \neq \emptyset.$$

Definition 1.2.6. *An operator $T : X \to X$ is called mixing if for every two nonempty open sets $U, V \subset X$ there is $N \in \mathbb{N}$ such that*

$$T^n(U) \cap V \neq \emptyset \text{ for every } n \geq N.$$

Definition 1.2.7. *An operator $T : X \to X$ is called chaotic if it is hypercyclic and has a dense set of periodic points, i.e., points $x \in X$ for which there exists $n \in \mathbb{N}$ such that $T^n(x) = x$.*

Remark 1.2.8. 1. From the very definitions and from Theorem 1.2.4 it is clear that every mixing operator is weakly mixing and that every weakly mixing operator is hypercyclic.
2. The first example of an operator which is hypercyclic but is not weakly mixing was given by de la Rosa and Read (see [19]).
3. Examples of mixing but not weakly mixing operators can be found in [30, Example 4.9].
4. It is known that every chaotic operator is already weakly mixing (see [5]).

The following powerful result is due to Ansari (see, e.g, [30, Theorem 6.2]).

Theorem 1.2.9. *Let X be a Fréchet space and let $T : X \to X$ be a hypercyclic operator. Then for every $n \geq 1$ the operator T^n is hypercyclic.*

The following result gives a nice criterion for an operator T to be weakly mixing (see [30, Theorem 1.54]).

Theorem 1.2.10. *Let X be a Fréchet space and let $T : X \to X$ be an operator. Then T is weakly mixing if and only if for every two nonempty open sets $U, V \subset X$ and every $k \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that*

$$T^{n+i}(U) \cap V \neq \emptyset \text{ for every } 1 \leq i \leq k.$$

Let now X be a *real* Fréchet space. The complexification $\widetilde{X}$ of X is defined as

$$\widetilde{X} := \{x + iy : x, y \in X\}.$$

With addition, multiplication by complex scalars, and the topology defined in a natural way, $\widetilde{X}$ is a *complex* Fréchet space. If $T : X \to X$ is an operator, then its complexification $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ is defined via the formula

$$\widetilde{T}(x + iy) = T(x) + iT(y).$$

The following proposition is a direct consequence of [30, Theorem 3.15] and [30, Proposition 3.19].

Proposition 1.2.11. *Let $T : X \to X$ be an operator on a real Fréchet space X. Let $\widetilde{T}$ be its complexification. The following conditions are equivalent:*

(1) The operator $T : X \to X$ is weakly mixing.

(2) The operator $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ is weakly mixing.

(3) The operator $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ is hypercyclic.

CHAPTER 2

CLOSED RANGE COMPOSITION OPERATORS FOR ONE-DIMENSIONAL SMOOTH SYMBOLS

The goal of this chapter is to investigate the question of for which smooth symbols $\psi : \mathbb{R} \to \mathbb{R}$ the composition operator

$$C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}),\ F \mapsto F \circ \psi,$$

has closed range, i.e., when the set

$$\operatorname{Im} C_\psi := \{F \circ \psi : F \in C^\infty(\mathbb{R})\}$$

is closed in $C^\infty(\mathbb{R})$.

First, we recall the basic properties of semiproper functions and we present a proof of a well-known fact saying that semiproperness of ψ is a necessary condition for C_ψ to have closed range. Then, in Theorem 2.2.3, we give a sufficient condition ensuring that the range of C_ψ is closed. As an immediate consequence of this result we obtain that if $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth semiproper function which has no flat points, then the range of C_ψ is closed (see Corollary 2.2.6).

The next part of this chapter deals with the problem of when all the roots of a smooth function are smooth. In Theorem 2.3.4 we characterize those smooth functions $g : \mathbb{R} \to \mathbb{R}$ such that for every $0 < \theta < 1$ the function $|g|^\theta$ is smooth. Then we apply this result to obtain a necessary condition for C_ψ to have closed range (see Theorem 2.1.6).

Unfortunately, we were not able to obtain a full characterization of closed range composition operators corresponding to one-dimensional smooth symbols, nevertheless we have obtained such a characterization in a special case, see Corollary 2.4.3.

Let us note that the following easy proposition is true.

Proposition 2.0.1. *The composition operator C_ψ has closed range on the complex space $C^\infty(\mathbb{R})$ if and only if it has closed range on the real space $C^\infty(\mathbb{R})$.*

Therefore, in this chapter, we may assume that $C^\infty(\mathbb{R})$ is the space of real valued functions.

2.1 Semiproper functions

As noted in the Introduction of this thesis, from the very beginning it was well-known that the following notion is very important in studying closed range composition operators.

Definition 2.1.1. *We say that a function $\psi : \mathbb{R}^n \to \mathbb{R}^m$ is* semiproper *if for every compact set $K \subset \mathbb{R}^m$ there exists a compact set $L \subset \mathbb{R}^n$ such that $K \cap \psi(\mathbb{R}^n) \subset \psi(L)$.*

The following lemma is easy and well-known (see, e.g., [13, Prop. 1.4.1]).

Lemma 2.1.2. *Let $\psi : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous semiproper function. Then the set $\psi(\mathbb{R}^n)$ is closed in $\mathbb{R}^m$.*

In the one-dimensional case the converse also holds, i.e., the following lemma is true.

Lemma 2.1.3. *A continuous function $\psi : \mathbb{R} \to \mathbb{R}$ is semiproper if and only if the set $\psi(\mathbb{R})$ is closed in $\mathbb{R}$.*

Proof. If ψ is semiproper, then the set $\psi(\mathbb{R})$ is closed in $\mathbb{R}$ by Lemma 2.1.2. To prove the converse, let us assume that the set $\psi(\mathbb{R})$ is closed in $\mathbb{R}$. Let $K \subset \mathbb{R}$ be an arbitrary compact set. We have to show that there exists a compact set $L \subset \mathbb{R}$ such that $K \cap \psi(\mathbb{R}) \subset \psi(L)$. If $K \cap \psi(\mathbb{R}) = \emptyset$, then this assertion is clearly true. So let us assume that $K \cap \psi(\mathbb{R}) \neq \emptyset$. Let

$$a := \min\left\{x : x \in K \cap \psi(\mathbb{R})\right\} \quad \text{and} \quad b := \max\left\{x : x \in K \cap \psi(\mathbb{R})\right\}.$$

The numbers a and b are well defined since $K \cap \psi(\mathbb{R})$ is a nonempty compact set (as an intersection of the compact set K with the closed set $\psi(\mathbb{R})$). Let x_1 and x_2 be such that

$$\psi(x_1) = a \text{ and } \psi(x_2) = b.$$

The closed interval

$$L = \{x_1 + t(x_2 - x_1) : 0 \le t \le 1\}$$

is clearly compact. From the Intermediate Value Theorem we obtain that

$$[a, b] = [\psi(x_1), \psi(x_2)] \subset \psi(L),$$

and hence

$$K \cap \psi(\mathbb{R}) \subset [a, b] \subset \psi(L).$$

This completes the proof.

□

Example 2.1.4. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined as $\psi(x) = x^2$. Since $\psi(\mathbb{R}) = [0,\infty)$, we obtain by Lemma 2.1.3 that ψ is a semiproper function. In fact, it is easy to see that every polynomial of one variable with real coefficients considered as a function acting from $\mathbb{R}$ to $\mathbb{R}$ is semiproper.*

Example 2.1.5. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined as $\psi(x) = \frac{1}{x^2+1}$. Because the set $\psi(\mathbb{R}) = (0,1]$ is not closed in $\mathbb{R}$, we get from Lemma 2.1.3 that ψ is not semiproper.*

The following proposition shows that the semiproperness of ψ is a necessary condition for C_ψ to have closed range. This is true for an arbitrary smooth $\psi : \mathbb{R}^n \to \mathbb{R}^m$ (see [13, Prop. 1.4.1] without a proof, for the proof see [33, Prop. 2.9]). Below we present a different proof of this fact in the one-dimensional case.

Proposition 2.1.6. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be smooth. If the composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$, has closed range, then ψ is a semiproper function.*

Proof. Let us assume to the contrary that C_ψ has closed range and that ψ is not a semiproper function. Since every constant function is semiproper, we may assume that ψ is not constant. From Lemma 2.1.3 it follows that the set $\psi(\mathbb{R})$ is not closed in $\mathbb{R}$. Thus we can find

$$b \in \overline{\psi(\mathbb{R})} \setminus \psi(\mathbb{R}). \tag{2.1.1}$$

Since $\psi(\mathbb{R})$ is connected, either $\psi(\mathbb{R}) \subset (b,\infty)$ or $\psi(\mathbb{R}) \subset (-\infty, b)$. Let us assume that

$$\psi(\mathbb{R}) \subset (b,\infty) \tag{2.1.2}$$

(the proof in the second case goes along the same lines). Consider now the function $f : \mathbb{R} \to \mathbb{R}$ defined via formula

$$f(x) = \sqrt{\psi(x) - b}.$$

From (2.1.2) it is clear that $f \in C^\infty(\mathbb{R})$. We will show that

$$f \in \overline{\operatorname{Im} C_\psi} \setminus \operatorname{Im} C_\psi$$

and this will give a contradiction with the assumption that C_ψ has closed range.

First we will show that $f \notin \operatorname{Im} C_\psi$. To do this, let us observe that from (2.1.1) and from (2.1.2) it follows that there exists $\delta > 0$ such that

$$(b, b+\delta) \subset \psi(\mathbb{R}).$$

If $F : \mathbb{R} \to \mathbb{R}$ is such that $F \circ \psi = f$, then for every $y \in (b, b+\delta)$ the equality $F(y) = \sqrt{y-b}$ must be satisfied. But this means that the first derivative of F cannot exist at b. This shows that there does not exist a smooth function $F : \mathbb{R} \to \mathbb{R}$ such that $F \circ \psi = f$. Therefore, $f \notin \operatorname{Im} C_\psi$.

Our goal now is to show that $f \in \overline{\operatorname{Im} C_\psi}$. In order to do this we have to approximate f, in the topology of the space $C^\infty(\mathbb{R})$, by functions which are in $\operatorname{Im} C_\psi$. So let $K \subset \mathbb{R}$ be an arbitrary compact set and let $n \in \mathbb{N}$. Let

$$\delta = \min\left\{\psi(x) - b : x \in K\right\}.$$

Using (2.1.2), the smoothness of ψ, and the compactness of the set K we obtain that $\delta > 0$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function which is equal to 1 on the set $[b+\delta/2, \infty)$ and equal to 0 on the set $(-\infty, b+\delta/4]$. Such a function exists in view of Proposition 1.1.7. The function $F : \mathbb{R} \to \mathbb{R}$ defined via the formula

$$F(x) = \begin{cases} \varphi(x) \cdot \sqrt{x-b}, & x \geq b, \\ 0, & x < b, \end{cases}$$

is clearly smooth. Therefore, $F \circ \psi \in \operatorname{Im} C_\psi$. Moreover, for every $x \in \psi^{-1}((b+\delta/2, \infty))$ we have that

$$\left(F \circ \psi\right)(x) = \sqrt{\psi(x) - b} = f(x).$$

Thus the smooth functions $F \circ \psi$ and f coincide on the open set $\psi^{-1}((b+\delta/2, \infty))$, which contains the compact set K. Therefore

$$||F \circ \psi - f||_{K,n} = 0.$$

Since K and n were arbitrary, this gives that $f \in \overline{\operatorname{Im} C_\psi}$. This completes the proof.

□

Example 2.1.7. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined as $\psi(x) = \frac{1}{x^2+1}$. We know that ψ is not a semiproper function (see Example 2.1.5). Thus, by Proposition 2.1.6, we get that C_ψ does not have closed range. In fact, the proof of Proposition 2.1.6 shows that*

$$\sqrt{\psi} \in \overline{\operatorname{Im} C_\psi} \setminus \operatorname{Im} C_\psi.$$

2.2 A sufficient condition

The aim of this section is to prove Theorem 2.2.3, which gives a sufficient condition ensuring that the range of the composition operator C_ψ is closed. Before we state the result, let us introduce the necessary notation.

Definition 2.2.1. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be smooth.*

- *For every $b \in \psi(\mathbb{R})$ the set $\psi^{-1}(\{b\})$ is called the fiber of ψ over b.*
- *If for every $n \geq 1$ we have $\psi^{(n)}(x) = 0$, then x is called a flat point of ψ. Otherwise we say that x is a non-flat point of ψ.*

Example 2.2.2. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined via formula*

$$\psi(x) = \begin{cases} \exp\left(-\frac{1}{x}\right), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Then ψ is a smooth function and 0 is a flat point of ψ.

Theorem 2.2.3. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth semiproper function which satisfies the following conditions:*

1. *Every fiber of ψ over a boundary point of $\psi(\mathbb{R})$ contains a non-flat point.*

2. *Every fiber of ψ over an interior point of $\psi(\mathbb{R})$ contains either a non-flat non extreme point or both a non-flat local minimum and a non-flat local maximum.*

Then

$$\operatorname{Im} C_\psi = \overline{\operatorname{Im} C_\psi} = \widehat{\operatorname{Im} C_\psi}.$$

In the proof of this theorem, we will need the following simple observation, which can be easily proved using the Mean Value Theorem.

Lemma 2.2.4. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be smooth. If x_0 is a non-flat point of ψ, then there exists $\varepsilon > 0$ such that the only possible point in $[x_0 - \varepsilon, x_0 + \varepsilon]$ at which ψ' vanishes is x_0.*

We will also need the following, slightly reformulated, nice result due to Kenessey and Wengenroth (see [34, Lemma 2]).

Lemma 2.2.5. *Let $\psi : [a, b] \to [c, d]$ be a smooth bijection such that the only possible point at which ψ' vanishes is a (or b), which is a non-flat point of ψ. Let $f : [a, b] \to \mathbb{R}$ be a smooth function such that for every $n \geq 0$ we have that $f^{(n)}(a) = 0$ (or $f^{(n)}(b) = 0$, respectively). Then the function $f \circ \psi^{-1} : [c, d] \to \mathbb{R}$ is smooth and, moreover, for every $n \geq 0$ we have that*

$$\left(f \circ \psi^{-1}\right)^{(n)}(\psi(a)) = 0 \quad (\textit{or } \left(f \circ \psi^{-1}\right)^{(n)}(\psi(b)) = 0\,,\ \textit{respectively}).$$

Having those results, we are ready to prove the theorem.

Proof of Theorem 2.2.3. From the result of Allan et al. (see [2, Cor. 5]) we know that

$$\operatorname{Im} C_\psi \subset \overline{\operatorname{Im} C_\psi} \subset \widehat{\operatorname{Im} C_\psi}.$$

Hence, to prove the theorem, it is enough to show that

$$\widehat{\operatorname{Im} C_\psi} \subset \operatorname{Im} C_\psi.$$

So let $f \in \widehat{\operatorname{Im} C_\psi}$ be arbitrary. Please recall that

$$\widehat{\operatorname{Im} C_\psi} = \left\{ f \in C^\infty(\mathbb{R}) : \forall b \in \psi(\mathbb{R})\ \exists F_b \in C^\infty(\mathbb{R})\ \forall a \in \psi^{-1}(\{b\}) \forall n \geq 0\ f^{(n)}(a) = (F_b \circ \psi)^{(n)}(a) \right\}.$$

Our goal is to find a smooth function $\widetilde{F} : \mathbb{R} \to \mathbb{R}$ such that $\widetilde{F} \circ \psi = f$. To do this, let us first consider the function

$$F : \psi(\mathbb{R}) \to \mathbb{R},\ y = \psi(x) \mapsto f(x).$$

Claim 1. *The function F is well defined.*

Proof of the claim. Assume that $y \in \psi(\mathbb{R})$ is equal to $\psi(x_1)$ and $\psi(x_2)$, where x_1 and x_2 are two different points. By the definition of $\widehat{\operatorname{Im} C_\psi}$, we have that $f(x_1) = f(x_2)$. Therefore, the value of F at y does not depend on the choice of $x \in \psi^{-1}(\{y\})$. This shows that F is well defined.

By our assumptions ψ is semiproper and hence, by Lemma 2.1.3, the set $\psi(\mathbb{R})$ is a closed, possibly unbounded, interval. We will show now that the following is true.

Claim 2. *The function F is smooth on $\psi(\mathbb{R})$.*

Proof of the claim. Let $b \in \psi(\mathbb{R})$ be arbitrary. According to our assumptions, there are three possible cases.

Case 1. The point b is a boundary point of $\psi(\mathbb{R})$ and the fiber of ψ over b contains a non-flat extreme point a_0.

Let us assume that b is the smallest value of ψ. Since ψ is not flat at a_0, by Lemma 2.2.4 we can find $\varepsilon > 0$ such that the only point in $[a_0 - \varepsilon, a_0 + \varepsilon]$ at which ψ' vanishes is a_0. Let us consider the function

$$\widetilde{\psi} : [a_0, a_0 + \varepsilon] \to [\psi(a_0), \psi(a_0 + \varepsilon)], \ x \mapsto \psi(x).$$

Since ψ' does not vanish in $(a_0, a_0 + \varepsilon)$, we obtain that $\widetilde{\psi}$ is a smooth bijection. From the definition of $\widehat{\operatorname{Im} C_\psi}$ we can find $F_b \in C^\infty(\mathbb{R})$ such that

$$(F_b \circ \psi)^{(n)}(a_0) - f^{(n)}(a_0) = 0 \text{ for every } n \geq 0.$$

Using Lemma 2.2.5, we obtain that the function

$$(F_b \circ \psi - f) \circ \widetilde{\psi}^{-1} : [\psi(a_0), \psi(a_0 + \varepsilon)] \to \mathbb{R}$$

is smooth. It is clear that

$$(F_b \circ \psi - f) \circ \widetilde{\psi}^{-1} = F_b - F \text{ on } [\psi(a_0), \psi(a_0 + \varepsilon)],$$

and hence $F_b - F$ is smooth on $[\psi(a_0), \psi(a_0 + \varepsilon)]$. But $F_b \in C^\infty(\mathbb{R})$, so F is a smooth function on $[\psi(a_0), \psi(a_0 + \varepsilon)]$.

If b is the biggest value of ψ, then in a similar way we obtain that F is smooth on $[b - \varepsilon, b]$ for some $\varepsilon > 0$.

Case 2. The point b is an interior point of $\psi(\mathbb{R})$ and the fiber of ψ contains a non-flat non extreme point a_0.

Because a_0 is a non-flat point, by Lemma 2.2.4, there exists $\varepsilon > 0$ such that the only possible point in $[a_0-\varepsilon, a_0+\varepsilon]$ at which ψ' vanishes is a_0. Since a_0 is a non extreme point of ψ and a_0 is the only possible point at which ψ' vanishes in $[a_0-\varepsilon, a_0+\varepsilon]$, we get that ψ is either an increasing or a decreasing function on $[a_0-\varepsilon, a_0+\varepsilon]$. Let us assume that ψ is an increasing function on $[a_0-\varepsilon, a_0+\varepsilon]$ (the proof in the other case goes along the same lines).

It is clear that the functions

$$\widetilde{\psi_1} : [a_0, a_0+\varepsilon] \to [\psi(a_0), \psi(a_0+\varepsilon)],\ x \mapsto \psi(x),$$

and

$$\widetilde{\psi_2} : [a_0-\varepsilon, a_0] \to [\psi(a_0-\varepsilon), \psi(a_0)],\ x \mapsto \psi(x),$$

are smooth bijections. Using the definition of $\widehat{\operatorname{Im} C_\psi}$, we can find $F_b \in C^\infty(\mathbb{R})$ such that

$$(F_b \circ \psi)^{(n)}(a_0) - f^{(n)}(a_0) = 0 \text{ for every } n \geq 0.$$

Using twice Lemma 2.2.5, we get that the functions

$$(F_b \circ \psi - f) \circ \widetilde{\psi_1}^{-1} : [\psi(a_0), \psi(a_0+\varepsilon)] \to \mathbb{R}$$

and

$$(F_b \circ \psi - f) \circ \widetilde{\psi_2}^{-1} : [\psi(a_0-\varepsilon), \psi(a_0)] \to \mathbb{R}$$

are smooth and, moreover, we get that

$$\left((F_b \circ \psi - f) \circ \widetilde{\psi_1}^{-1}\right)^{(n)}(\psi(a_0)) = \left((F_b \circ \psi - f) \circ \widetilde{\psi_2}^{-1}\right)^{(n)}(\psi(a_0)) = 0 \tag{2.2.1}$$

for every $n \geq 0$. Observe now that

$$(F_b \circ \psi - f) \circ \widetilde{\psi_1}^{-1} = F_b - F \text{ on } [\psi(a_0), \psi(a_0+\varepsilon)]$$

and

$$(F_b \circ \psi - f) \circ \widetilde{\psi_2}^{-1} = F_b - F \text{ on } [\psi(a_0-\varepsilon), \psi(a_0)].$$

Together with (2.2.1), this gives that the function $F_b - F$ is smooth on $[\psi(a_0-\varepsilon), \psi(a_0+\varepsilon)]$. Thus F is smooth on $[\psi(a_0-\varepsilon), \psi(a_0+\varepsilon)]$.

Case 3. The point b is an interior point of $\psi(\mathbb{R})$ and the fiber of ψ contains a non-flat local minimum a_1 of ψ and a non-flat local maximum a_2 of ψ.

Since ψ is not flat at a_1 and it is not flat at a_2, by Lemma 2.2.4 there exists $\varepsilon > 0$ such that the only point in $[a_1-\varepsilon, a_1+\varepsilon]$ at which ψ' vanishes is a_1 and the only point in $[a_2-\varepsilon, a_2+\varepsilon]$ at which ψ' vanishes is a_2. It is easy to see that the functions

$$\widetilde{\psi_1} : [a_1, a_1+\varepsilon] \to [\psi(a_1), \psi(a_1+\varepsilon)],\ x \mapsto \psi(x),$$

and

$$\widetilde{\psi_2} : [a_2 - \varepsilon, a_2] \to [\psi(a_2 - \varepsilon), \psi(a_2)],\ x \mapsto \psi(x),$$

are smooth bijections. From the definition of $\widehat{\operatorname{Im} C_\psi}$ there exists $F_b \in C^\infty(\mathbb{R})$ such that

$$(F_b \circ \psi)^{(n)}(a_1) - f^{(n)}(a_1) = 0 = (F_b \circ \psi)^{(n)}(a_2) - f^{(n)}(a_2)$$

for every $n \geq 0$. Using Lemma 2.2.5, we obtain that the functions

$$(F_b \circ \psi - f) \circ \widetilde{\psi_1}^{-1} : [\psi(a_1), \psi(a_1 + \varepsilon)] \to \mathbb{R}$$

and

$$(F_b \circ \psi - f) \circ \widetilde{\psi_2}^{-1} : [\psi(a_2 - \varepsilon), \psi(a_2)] \to \mathbb{R}$$

are smooth and, moreover,

$$\left((F_b \circ \psi - f) \circ \widetilde{\psi_1}^{-1}\right)^{(n)} (\psi(a_1)) = \left((F_b \circ \psi - f) \circ \widetilde{\psi_2}^{-1}\right)^{(n)} (\psi(a_2)) = 0$$

for every $n \geq 0$. Since

$$(F_b \circ \psi - f) \circ \widetilde{\psi_1}^{-1} = F_b - F \text{ on } [\psi(a_1), \psi(a_1 + \varepsilon)]$$

and

$$(F_b \circ \psi - f) \circ \widetilde{\psi_2}^{-1} = F_b - F \text{ on } [\psi(a_2 - \varepsilon), \psi(a_2)],$$

this gives that $F_b - F$ is smooth on $[\psi(a_2 - \varepsilon), \psi(a_1 + \varepsilon)]$. Therefore, F is a smooth function on $[\psi(a_2 - \varepsilon), \psi(a_1 + \varepsilon)]$.

Altogether, the above shows that F is a smooth function on $\psi(\mathbb{R})$.

Now, we are able to finish the proof of the theorem. By Claim 2 the function F is smooth on $\psi(\mathbb{R})$, which as noted before, is a closed, possibly unbounded interval. Using Corollary 1.1.3 we can extend F to a smooth function on $\mathbb{R}$, denote this extension by $\widetilde{F}$. Of course, for every $x \in \mathbb{R}$ we have that

$$\left(\widetilde{F} \circ \psi\right)(x) = \widetilde{F}(\psi(x)) = F(\psi(x)) = f(x).$$

This shows that $f \in \operatorname{Im} C_\psi$ and this proves the theorem.

□

As an immediate consequence of Theorem 2.2.3, we obtain the following corollary.

Corollary 2.2.6. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth semiproper function which has no flat points. Then the composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$, has closed range.*

Please note, that every semiproper real analytic function satisfies the conditions of the statement above. In particular, C_ψ has closed range if ψ is a polynomial.

Let now φ be an arbitrary smooth function with values in $[0,1]$ which is 0 on $(-\infty, 0]$ and 1 on $[1, \infty)$. In the following examples we will try to illustrate the conditions of Theorem 2.2.3.

Example 2.2.7. *Let*

$$\psi(x) = \begin{cases} \exp(1/x), & x < 0 \\ \varphi(x)(x-1)^2, & x \geq 0. \end{cases}$$

Since $\psi(\mathbb{R}) = [0,\infty)$, by Lemma 2.1.3 we get that ψ is semiproper. Observe that every fiber of ψ over an interior point of $\psi(\mathbb{R})$ contains a non-flat point which is not a extreme point and the fiber over 0, which is the only boundary point of $\psi(\mathbb{R})$, contains 1 which is a non-flat point. Thus, by Theorem 2.2.3 the composition operator C_ψ has closed range.

Example 2.2.8. *Let*

$$\psi(x) = \begin{cases} \varphi(-x)(x+2)^3, & x < 0 \\ \varphi(x)(x-2)^3, & x \geq 0. \end{cases}$$

Since $\psi(\mathbb{R}) = \mathbb{R}$, we get that ψ is a semiproper function. For every $y \neq 0$ the fiber of ψ over y contains a non-flat non extreme point. Moreover, the fiber over 0 contains 2 which is a non-flat and non-extreme point. By our theorem C_ψ has closed range.

Example 2.2.9. *Let*

$$\psi(x) = \begin{cases} -\varphi(-x)(x+1)^2, & x < 0 \\ \varphi(x)(x-1)^2, & x \geq 0. \end{cases}$$

Then ψ is semiproper and for every $y \neq 0$ the fiber of ψ over y contains a non-flat non extreme point. Moreover, the fiber over 0 contains -1 which is a non-flat local maximum, and contains 1 which is a non-flat local minimum. Hence the range of C_ψ is closed.

Remark 2.2.10. Theorem 2.2.3 gives a description of a wide class of smooth symbols $\psi : \mathbb{R} \to \mathbb{R}$ for which the composition operator C_ψ has closed range. We do not know if the range of C_ψ is closed in case if ψ is a not constant semiproper function and does not satisfy the conditions of this result, i.e., when either there is a boundary point b of $\psi(\mathbb{R})$ such that the fiber of ψ over b contains only flat points of ψ or if there is an interior point b of $\psi(\mathbb{R})$ such that the fiber of ψ over b does not contain a non-flat non extreme point and does not contain either a non-flat local minimum or a non-flat local maximum. It is easy to see that the above conditions precisely hold if there is $b \in \psi(\mathbb{R})$ such that either the boundary of the set $\{x \in \mathbb{R} : \psi(x) > b\}$ is nonempty and contains only flat points of ψ or the boundary of the set $\{x \in \mathbb{R} : \psi(x) < b\}$ is nonempty and contains only flat points of ψ.

Unfortunately, we were not able to solve this problem in full generality. We only managed to show (see Theorem 2.4.1 below) that if there exists $b \in \psi(\mathbb{R})$ such that the boundary of the set $\{x \in \mathbb{R} : \psi(x) > b\}$ or the boundary of the set $\{x \in \mathbb{R} : \psi(x) < b\}$ is nonempty and contains only *nice* flat points of ψ, then the range of the composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$, is not closed (the definition of a nice flat point of ψ is given in the next section).

Let us note that at this moment, we strongly believe that the following conjectures are true.

Conjecture 2.2.11. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be smooth. The composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$, has closed range if and only if ψ is a semiproper function which is either constant or it satisfies the following conditions:*

1. *Every fiber of ψ over a boundary point of $\psi(\mathbb{R})$ contains a non-flat point.*
2. *Every fiber of ψ over an interior point of $\psi(\mathbb{R})$ contains either a non-flat non extreme point or both a non-flat local minimum and a non-flat local maximum.*

Conjecture 2.2.12. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be smooth. Then*

$$\overline{\operatorname{Im} C_\psi} = \widehat{\operatorname{Im} C_\psi}.$$

2.3 Roots of smooth functions

Now, our aim is to characterize those smooth functions $g : \mathbb{R} \to \mathbb{R}$ such that for every $0 < \theta < 1$ the function $|g|^\theta$ is smooth (for the sake of completeness, let us note that the function $|g|^\theta$ is defined via formula $|g|^\theta(x) = |g(x)|^\theta$ for every $x \in \mathbb{R}$). In the next section we will use this characterization to obtain a necessary condition for C_ψ to have closed range.

We start with the following definition.

Definition 2.3.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be smooth. We say that $x \in \mathbb{R}$ is a nice flat point of g if for every $n \geq 1$ and every $0 < \theta < 1$ there is a neighbourhood U of x_0 such that for every $x \in U$ we have*

$$\left|g^{(n)}(x)\right| \leq \left|g(x) - g(x_0)\right|^\theta.$$

Remark 2.3.2. Let $g : \mathbb{R} \to \mathbb{R}$ be smooth. From the very definition it follows that every nice flat point of g is a flat point of g.

The following proposition shows that the definition of a nice flat point can be formulated in a formally much stronger or a much weaker way.

Proposition 2.3.3. *Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function and let $x_0 \in \mathbb{R}$. The following conditions are equivalent:*

(1) For every $n \geq 1$, for every $0 < \theta < 1$, and every $C > 0$ there is a neighbourhood U of x_0 such that for every $x \in U$ we have

$$\left|g^{(n)}(x)\right| \leq C\left|g(x) - g(x_0)\right|^\theta.$$

(2) For every $n \geq 1$ and for every $0 < \theta < 1$ there is $D > 0$ and there is a neighbourhood U of x_0 such that for every $x \in U$ we have

$$\left|g^{(n)}(x)\right| \leq D\left|g(x) - g(x_0)\right|^\theta.$$

Proof. It is obvious that (1) implies (2). So let us now assume that (2) is satisfied and let us fix $0 < \theta_0 < 1$, $n \in \mathbb{N}$, and $C > 0$. We choose θ_1 such that $0 < \theta_0 < \theta_1 < 1$. From our assumptions there exists a neighbourhood U of x_0 such that

$$\left|g^{(n)}(x)\right| \leq D\left|g(x) - g(x_0)\right|^{\theta_1} \text{ for every } x \in U. \tag{2.3.1}$$

There exists a neighbourhood W of x_0 such that

$$D\left|g(x) - g(x_0)\right|^{\theta_1 - \theta_0} \leq C \text{ for every } x \in W. \tag{2.3.2}$$

The set $U \cap W$ is a neighbourhood of x_0 and, by (2.3.1) and (2.3.2), we get that for all $x \in U \cap W$ we have

$$\left|g^{(n)}(x)\right| \leq D\left|g(x) - g(x_0)\right|^{\theta_1} \leq D\left|g(x) - g(x_0)\right|^{\theta_1 - \theta_0} \cdot \left|g(x) - g(x_0)\right|^{\theta_0} \leq C\left|g(x) - g(x_0)\right|^{\theta_0}.$$

This completes the proof.

□

Now, we are ready to formulate and to prove the main result of this section.

Theorem 2.3.4. *Let $g : \mathbb{R} \to \mathbb{R}$ be smooth. The function $|g|^\theta$ is smooth for every $0 < \theta < 1$ if and only if every point $x_0 \in g^{-1}(\{0\})$ is a nice flat point of g. Moreover, if one of the above conditions holds, then for every $0 < \theta < 1$ the function $|g|^\theta$ is flat on $g^{-1}(\{0\})$.*

Please note that if $|g|^\theta$ is smooth for every $0 < \theta < 1$, then $|g|^s$ is smooth for every $s > 0$ (just observe that $|g|^s = |g|^{\theta \cdot k}$ for a suitable natural number k).

In the proof of this theorem we will need the following almost obvious lemma. We omit its simple proof.

Lemma 2.3.5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then the function $|g|$ is smooth on the open set $\{x \in \mathbb{R} : g(x) \neq 0\}$ and, moreover, for every every $n \geq 1$ and every $x \in \mathbb{R}$ such that $g(x) \neq 0$ we have*

$$\left||g|^{(n)}(x)\right| = \left|g^{(n)}(x)\right|.$$

Proof of Theorem 2.3.4. Assume first that the function $|g|^\theta$ is smooth for every $0 < \theta < 1$.

Claim 1. *The function g is flat on the set $g^{-1}(\{0\})$.*

Proof of the claim. Assume to the contrary that this is not the case, i.e., assume that there exists $x_0 \in g^{-1}(\{0\})$ at which g is not flat. Let $n_0 \in \mathbb{N}$ be the smallest natural number such that $g^{(n_0)}(x_0) \neq 0$. Using n_0 times l'Hôpital's Rule we obtain that

$$\lim_{x \to x_0} \frac{g(x)}{(x - x_0)^{n_0}} = \frac{g^{(n_0)}(x_0)}{n_0!} \neq 0.$$

This shows that the one-sided limit

$$\lim_{x\to x_0^+} \frac{|g(x)|}{(x-x_0)^{n_0}}$$

exists and is not equal to 0. From this we obtain that

$$\lim_{x\to x_0^+} \frac{\sqrt[n_0+1]{|g(x)|} - \sqrt[n_0+1]{|g(x_0)|}}{x-x_0} = \lim_{x\to x_0^+} \frac{\sqrt[n_0+1]{|g(x)|}}{x-x_0} = \lim_{x\to x_0^+} \sqrt[n_0+1]{\frac{|g(x)|}{(x-x_0)^{n_0}} \cdot \frac{1}{x-x_0}} = \infty.$$

This gives a contradiction, because this limit should be finite since the function $|g|^{\frac{1}{n_0+1}}$ is smooth by our assumptions. This shows that the claim is true.

Claim 2. *For every* $0 < \theta < 1$ *the function* $|g|^\theta$ *is flat on the set* $g^{-1}(\{0\})$.

Proof of the claim. Assume to the contrary that there exists $0 < \theta < 1$ and $x_0 \in g^{-1}(\{0\})$ such that the function $f = |g|^\theta$ is not flat at x_0. Let n_0 be the smallest natural number such that $f^{(n_0)}(x_0) \neq 0$. As in the proof of the previous claim we obtain that the one-sided limit

$$\lim_{x\to x_0^+} \frac{|f(x)|}{(x-x_0)^{n_0}}$$

exists and is not equal to 0. This implies that

$$\lim_{x\to x_0^+} \frac{\sqrt[n_0+1]{|f(x)|} - \sqrt[n_0+1]{|f(x_0)|}}{x-x_0} = \lim_{x\to x_0^+} \frac{\sqrt[n_0+1]{|f(x)|}}{x-x_0} = \lim_{x\to x_0^+} \sqrt[n_0+1]{\frac{|f(x)|}{(x-x_0)^{n_0}} \cdot \frac{1}{x-x_0}} = \infty.$$

But this gives that

$$\infty = \lim_{x\to x_0^+} \frac{\sqrt[n_0+1]{|f(x)|} - \sqrt[n_0+1]{|f(x_0)|}}{x-x_0} = \lim_{x\to x_0^+} \frac{|g|^{\frac{\theta}{n_0+1}}(x) - |g|^{\frac{\theta}{n_0+1}}(x_0)}{x-x_0},$$

which is impossible since the function $|g|^{\frac{\theta}{n_0+1}}$ is smooth by our assumption. This proves the claim.

Now, we will show the following claim, and this will complete the proof of this part of the theorem.

Claim 3. For every $n \geq 1$, for every $x_0 \in g^{-1}(\{0\})$, for every $\varepsilon > 0$, and for every $0 < \theta < 1$ there exists a neighbourhood U of x_0 such that

$$\left|g^{(n)}(x)\right| \leq \varepsilon \left|g(x)\right|^\theta \text{ for every } x \in U.$$

Proof of the claim. We proceed by induction on n.
Step 1.

Let $n = 1$. We fix $\varepsilon > 0$, $0 < \theta < 1$, and $x_0 \in g^{-1}(\{0\})$. By our assumptions, the function $|g|^{1-\theta}$ is smooth and, moreover, by Claim 2 it is flat on the set $g^{-1}(\{0\})$. Since $\left(|g|^{1-\theta}\right)'(x_0) = 0$, the set

$$W = \left\{x : \left|\left(|g|^{1-\theta}\right)'(x)\right| < (1-\theta)\varepsilon\right\}$$

is a neighbourhood of x_0. If $x \in W$ and $g(x) = 0$, then from Claim 1 we get $g'(x) = 0$ and the inequality

$$\left|g'(x)\right| \leq \varepsilon \left|g(x)\right|^{\theta}$$

holds. If $x \in W$ and $g(x) \neq 0$, then calculating the derivative of $|g|^{1-\theta}$ at x we obtain that

$$\left|(1-\theta)\frac{|g|'(x)}{|g|^{\theta}(x)}\right| < (1-\theta)\varepsilon. \tag{2.3.3}$$

By Lemma 2.3.5 we know that $\left||g|'(x)\right| = |g'(x)|$ for every $x \in \mathbb{R}$ such that $g(x) \neq 0$. Thus the inequality in (2.3.3) gives that

$$\left|g'(x)\right| \leq \varepsilon \left|g(x)\right|^{\theta}.$$

This proves the claim for $n = 1$.
Step 2.

Assume that Claim 3 is true for $j = 1, \ldots, n-1$, fix $\varepsilon > 0$, $0 < \theta < 1$, and $x_0 \in g^{-1}(\{0\})$. Let A_n^* be the set of $(n-1)$ - tuples of nonnegative integers $m_1, \ldots, m_{n-1}$ satisfying

$$1 \cdot m_1 + \ldots + (n-1) \cdot m_{n-1} = n$$

and let

$$C_{m_1,\ldots,m_n} := B_{m_1,\ldots,m_n} \cdot (1-\theta) \cdot (1-\theta-1) \cdot \ldots \cdot (1-\theta-m_1-\ldots-m_n+1),$$

where $B_{m_1,\ldots,m_n}$ are as in Theorem 1.1.4. We choose $\delta > 0$ such that

$$\sum_{A_n^*} \left(\left|C_{m_1,\ldots,m_{n-1},0}\right| \cdot \delta^{m_1+\ldots+m_{n-1}}\right) \leq \frac{(1-\theta)\varepsilon}{2}. \tag{2.3.4}$$

For $(m_1, \ldots, m_{n-1}) \in A_n^*$ let $\#(m_1, \ldots, m_{n-1})$ be the number of those $1 \leq j \leq n-1$ for which $m_j \neq 0$. From the inductive assumption there exists a family

$$\left\{U_{m_1,\ldots,m_{n-1}} : (m_1, \ldots, m_{n-1}) \in A_n^*\right\}$$

of neighbourhoods of x_0 such that for every $(m_1, \ldots, m_{n-1}) \in A_n^*$

$$\left|g^{(j)}(x)\right| \leq \delta \cdot \left|g(x)\right|^{1-\frac{1-\theta}{m_j \cdot \#(m_1,\ldots,m_{n-1})}} \tag{2.3.5}$$

if $x \in U_{m_1,\ldots,m_{n-1}}$ and $1 \leq j \leq n-1$ is such that $m_j \neq 0$.

By our assumptions, the function $|g|^{1-\theta}$ is smooth and, moreover, by Claim 2, it is flat on the set $g^{-1}(\{0\})$. Since $\left(|g|^{1-\theta}\right)^{(n)}(x_0) = 0$, the set

$$U_n = \left\{x : \left|\left(|g|^{1-\theta}\right)^{(n)}(x)\right| < \frac{(1-\theta)\varepsilon}{2}\right\} \tag{2.3.6}$$

is a neighbourhood of x_0 and hence the set

$$W = U_n \cap \bigcap_{(m_1,\ldots,m_{n-1})\in A_n^*} U_{m_1,\ldots,m_{n-1}}$$

is a neighbourhood of x_0. If $x \in W$ and $g(x) = 0$, then by Claim 1 we have $g^{(n)}(x) = 0$ and the inequality

$$\left|g^{(n)}(x)\right| \le \varepsilon \left|g(x)\right|^{\theta}$$

holds. If $x \in W$ and $g(x) \neq 0$, then using the formula (Theorem 1.1.4) for the the n - th derivative of $|g|^{1-\theta}$ at x and the inequality in (2.3.6) we obtain that

$$\left|\sum_{A_n}\left(C_{m_1,\ldots,m_n}|g|^{1-\theta-m_1-\ldots-m_n}(x)\prod_{j:m_j\neq 0}\left(|g|^{(j)}(x)\right)^{m_j}\right)\right| \le \frac{(1-\theta)\varepsilon}{2}.$$

This gives that

$$\left|(1-\theta)\frac{|g|^{(n)}(x)}{|g|^{\theta}(x)} + \sum_{A_n^*}\left(C_{m_1,\ldots,m_{n-1,0}}|g|^{1-\theta-m_1-\ldots-m_{n-1}}(x)\prod_{j:m_j\neq 0}\left(|g|^{(j)}(x)\right)^{m_j}\right)\right| \le \frac{(1-\theta)\varepsilon}{2}. \tag{2.3.7}$$

Using Lemma 2.3.5 and the triangle inequality we obtain that

$$\begin{aligned}\left|\sum_{A_n^*}\left(C_{m_1,\ldots,m_{n-1,0}}|g|^{1-\theta-m_1-\ldots-m_{n-1}}(x)\prod_{j:m_j\neq 0}\left(|g|^{(j)}(x)\right)^{m_j}\right)\right| &\le \\ \le \sum_{A_n^*}\left(\left|C_{m_1,\ldots,m_{n-1,0}}\right||g|^{1-\theta-m_1-\ldots-m_{n-1}}(x)\prod_{j:m_j\neq 0}\left(\left|g^{(j)}(x)\right|\right)^{m_j}\right) & \\ = \sum_{A_n^*}\left(\left|C_{m_1,\ldots,m_{n-1,0}}\right|\prod_{j:m_j\neq 0}\left(\frac{\left|g^{(j)}(x)\right|}{|g|^{1-\frac{1-\theta}{m_j\cdot\#(m_1,\ldots,m_n)}}(x)}\right)^{m_j}\right), &\end{aligned} \tag{2.3.8}$$

Using now the inequalities in (2.3.4) and (2.3.5) we obtain that

$$\begin{aligned}\left|\sum_{A_n^*}\left(C_{m_1,\ldots,m_{n-1,0}}|g|^{1-\theta-m_1-\ldots-m_{n-1}}(x)\prod_{j=1}^{n-1}\left(|g|^{(j)}(x)\right)^{m_j}\right)\right| & \\ \le \sum_{A_n^*}\left(\left|C_{m_1,\ldots,m_{n-1,0}}\right|\prod_{m_j\neq 0}\delta^{m_j}\right) & \\ \le \frac{(1-\theta)\varepsilon}{2}. &\end{aligned} \tag{2.3.9}$$

Thus from (2.3.7) and (2.3.9) we obtain that

$$\left|(1-\theta)\frac{|g|^{(n)}(x)}{|g|^{\theta}(x)}\right| \leq (1-\theta)\varepsilon.$$

Together with Lemma 2.3.5, this gives that

$$\left|g^{(n)}(x)\right| \leq \varepsilon \left|g(x)\right|^{\theta}.$$

By induction it follows that Claim 3 is true. This ends the proof of this part of the theorem.

Assume now that every $x_0 \in g^{-1}(\{0\})$ is a nice flat point of g, i.e., assume that for every $x_0 \in g^{-1}(\{0\})$, for every $n \geq 1$, and every $0 < \theta < 1$ there is a neighbourhood U of x_0 such that for every $x \in U$ we have $|g^{(n)}(x)| \leq |g(x)|^{\theta}$. Fix $0 < \theta_0 < 1$. Our aim now is to show that the function $|g|^{\theta_0}$ is smooth.

Claim 4. *The function g is flat on the set $g^{-1}(\{0\})$. In fact, for every $x_0 \in g^{-1}(\{0\})$ and for every $k \geq 1$ there exists a neighbourhood U of x_0 such that*

$$|g(x)| \leq |x - x_0|^k \text{ for every } x \in U.$$

Proof of the claim. The first assertion of the claim follows immediately from our assumptions since for every $x_0 \in g^{-1}(\{0\})$ and every $n \geq 1$ we have that

$$|g^{(n)}(x_0)| \leq |g(x_0)|^{\theta_0} = 0.$$

Let now $x_0 \in g^{-1}(\{0\})$ be arbitrary. By Taylor's Formula there exists a neighbourhood U of x_0 such that

$$|g(x)| \leq |x - x_0|^k \text{ for every } x \in U.$$

This proves the claim.

We will show now the following is true and this will complete the proof of the theorem.

Claim 5. *Let $n \geq 1$ be arbitrary. The n-th derivative of $|g|^{\theta_0}$ exists on $\mathbb{R}$ and this derivative vanishes on the set $g^{-1}(\{0\})$. Moreover, for every $x_0 \in g^{-1}(\{0\})$ there exists $0 < \tau < 1$, $C > 0$, and a neighbourhood U of x_0, such that*

$$\left|\left(|g|^{\theta_0}\right)^{(n)}(x)\right| \leq C|g(x)|^{\tau} \text{ for every } x \in U.$$

Proof of the claim. We proceed by induction on n.

Step 1.

Let $n = 1$. It is clear that the first derivative of $|g|^{\theta_0}$ exists on the open set

$$\left\{x \in \mathbb{R} : g(x) \neq 0\right\}.$$

Let $x_0 \in g^{-1}(\{0\}$ be arbitrary and let $k \geq 1$ be such that $k \cdot \theta_0 > 1$. Using Claim 4 we can find a neighbourhood U of x_0 such that

$$|g(x)| \leq |x - x_0|^k \text{ for every } x \in U.$$

For every $x \in U$ we have that

$$0 \leq \left| \frac{|g|^{\theta_0}(x)}{x - x_0} \right| \leq \frac{|x - x_0|^{k \cdot \theta_0}}{|x - x_0|},$$

and this implies that

$$\lim_{x \to x_0} \frac{|g|^{\theta_0}(x) - |g|^{\theta_0}(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{|g|^{\theta_0}(x)}{x - x_0} = 0.$$

Thus the first derivative of $|g|^{\theta_0}$ exists on $\mathbb{R}$ and, moreover, it is equal to 0 on the set $g^{-1}(\{0\}$.

Let us now fix $x_0 \in g^{-1}(\{0\})$ and $0 < \tau < 1$ such that $0 < 1 - \theta_0 + \tau < 1$. From our assumption there exists a neighbourhood W of x_0 such that

$$|g'(x)| \leq |g|^{1-\theta_0+\tau}(x) \text{ for every } x \in W. \tag{2.3.10}$$

Let $C = \theta_0$. We will show that

$$\left| \left(|g|^{\theta_0}\right)'(x) \right| \leq C|g(x)|^{\tau} \text{ for every } x \in W.$$

and this will end Step 1 of the proof.

So let $x \in W$ be arbitrary. If $g(x) = 0$, then we have that $\left(|g|^{\theta_0}\right)'(x) = 0$ as we have proved above, and hence the desired inequality is satisfied. If $g(x) \neq 0$, then from Lemma 2.3.5 we get that

$$\left(|g|\right)'(x) = |g'(x)|.$$

Calculating the derivative of $|g|^{\theta_0}$ at x and using the inequality in (2.3.10) we get that

$$\begin{aligned}
\left| \left(|g|^{\theta_0}\right)'(x) \right| &= \left| \theta_0 \cdot \left|g\right|^{\theta_0 - 1}(x) \cdot |g|'(x) \right| \\
&= \theta_0 \cdot \left|g\right|^{\theta_0 - 1}(x) \cdot |g'(x)| \\
&\leq \theta_0 \cdot \left|g\right|^{\theta_0 - 1}(x) \cdot |g|^{1-\theta_0+\tau}(x) \\
&= C \cdot |g(x)|^{\tau}.
\end{aligned}$$

This completes the proof in case $n = 1$.

Step 2.

Assume now that Claim 5 is true for $j = n-1$. It is clear that the n-th derivative of $|g|^{\theta_0}$ exists on the open set

$$\left\{x \in \mathbb{R} : g(x) \neq 0\right\}.$$

Let $x_0 \in g^{-1}(\{0\}$ be arbitrary. By the inductive hypothesis there exists $0 < \tau_{n-1} < 1$, $C_{n-1} > 0$, and a neighbourhood W of x_0 such that

$$\left(|g|^{\theta_0}\right)^{(n-1)}(x) \leq C_{n-1}|g(x)|^{\tau_{n-1}} \text{ for every } x \in W.$$

Let $k \geq 1$ be such that $k \cdot \tau_{n-1} > 1$. From Claim 4 there exits a neighbourhood V of x_0 such that

$$|g(x)| \leq |x - x_0|^k \text{ for every } x \in V.$$

For every $x \in V \cap W$ we have that

$$\left|\left(|g|^{\theta_0}\right)^{(n-1)}(x)\right| \leq C_{n-1}|g(x)|^{\tau_{n-1}} \leq C_{n-1}|x - x_0|^{k\cdot\tau_{n-1}}.$$

This easily implies that

$$\lim_{x\to x_0} \frac{\left(|g|^{\theta_0}\right)^{(n-1)}(x) - \left(|g|^{\theta_0}\right)^{(n-1)}(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{\left(|g|^{\theta_0}\right)^{(n-1)}(x)}{x - x_0} = 0.$$

Thus the n-th derivative of $|g|^{\theta_0}$ exists on $\mathbb{R}$ and this derivative is equal to 0 on $g^{-1}(\{0\})$.

Let us now fix $x_0 \in g^{-1}(\{0\})$ and let

$$C_{m_1,\ldots,m_n} = B_{m_1,\ldots,m_n} \cdot \theta_0 \cdot (\theta_0 - 1) \cdot \ldots \cdot (\theta - m_1 - \ldots - m_n + 1),$$

where $B_{m_1,\ldots,m_n}$ are as in Theorem 1.1.4. Moreover, let A_n be as in Theorem 1.1.4 and let

$$C = \sum_{A_n} |C_{m_1,\ldots,m_n}|.$$

Choose now $0 < \tau < 1$ and $0 < \alpha < 1$ such that the following holds: if an n-tuple of nonnegative integers $m_1, \ldots, m_n$ satisfies

$$1 \cdot m_1 + \ldots + n \cdot m_n = n,$$

then

$$\theta_0 + (\alpha - 1)(m_1 + \ldots + m_n) > \tau.$$

It is clear that we can find such α and τ. By our assumptions we can find a neighbourhood W of x_0 such that

$$\left|g^{(j)}(x)\right| \leq |g|^{\alpha}(x) \text{ for every } x \in W \text{ and } j = 1,\ldots,n.$$

Let

$$U = W \cap \{x : |g(x)| < 1\}.$$

Clearly U is a neighbourhood of x_0. We will show that

$$\left(|g|^{\theta_0}\right)^{(n)}(x) \leq C|g|^{\tau}(x) \text{ for every } x \in U$$

and this will complete the proof of the inductive step.

If $g(x) = 0$, then we already know (see the previous page) that $\left(|g|^{\theta_0}\right)^{(n)}(x) = 0$ and thus the desired inequality is trivially satisfied. If $x \in U$ and $g(x) \neq 0$, then, by Theorem 1.1.4, we have

$$\begin{aligned}
\left|\left(|g|^{\theta_0}\right)^{(n)}(x)\right| &= \left|\sum_{A_n}\left(C_{m_1,\ldots,m_n}\cdot|g|^{\theta_0-m_1-\ldots-m_n}(x)\cdot\prod_{j:m_j\neq 0}\left(|g|^{(j)}(x)\right)^{m_j}\right)\right| \\
&\leq \sum_{A_n}\left(\left|C_{m_1,\ldots,m_n}\right|\cdot|g|^{\theta_0-m_1-\ldots-m_n}(x)\cdot\prod_{j:m_j\neq 0}\left||g|^{(j)}(x)\right|^{m_j}\right) \\
&\leq \sum_{A_n}\left(\left|C_{m_1,\ldots,m_n}\right|\cdot|g|^{\theta_0-m_1-\ldots-m_n}(x)\cdot\prod_{j:m_j\neq 0}\left(|g|^{\alpha}(x)\right)^{m_j}\right) \\
&= \sum_{A_n}\left(\left|C_{m_1,\ldots,m_n}\right|\cdot|g|^{\theta_0+(\alpha-1)(m_1+\ldots+m_n)}(x)\right) \\
&\leq \sum_{A_n}\left(\left|C_{m_1,\ldots,m_n}\right|\cdot|g|^{\tau}(x)\right) \\
&= C|g|^{\tau}(x).
\end{aligned}$$

By induction it follows that Claim 5 is true. This completes the proof of the theorem.

□

Corollary 2.3.6. *Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth nonnegative function such that every point in $g^{-1}(\{0\})$ is a nice flat point of g. Then the function $h = \sqrt{g}$ is smooth. Moreover, every point in $h^{-1}(\{0\})$ is a nice flat point of h.*

Proof. The first assertion follows immediately from Theorem 2.3.4. The same result implies that for every $0 < \theta < 1$ the function $h^{\theta} = g^{\frac{1}{2}\theta}$ is smooth. Using again Theorem 2.3.4 we obtain that every point in $h^{-1}(\{0\})$ is a nice flat point of h.

□

The following examples show that the most standard flat points of smooth functions are in fact nice flat points.

Example 2.3.7. *Let $g:\mathbb{R}\to\mathbb{R}$ be defined via the formula*

$$g(x)=\begin{cases}\exp\left(-\frac{1}{x}\right), & x>0,\\ 0, & x\le 0.\end{cases}$$

It is clear that for every $0<\theta<1$ the function g^θ is smooth. Theorem 2.3.4 implies that 0 is a nice flat point of g. Moreover, every $x\in(-\infty,0)$ is a nice flat point of g.

Example 2.3.8. *Let $g:\mathbb{R}\to\mathbb{R}$ be defined via the formula*

$$g(x)=\begin{cases}\exp\left(-\frac{1}{p(x)}\right), & p(x)>0,\\ 0, & p(x)\le 0,\end{cases}$$

where $p:\mathbb{R}\to\mathbb{R}$ is a real analytic function. One can check that g is a smooth function and that for every $0<\theta<1$ the function g^θ is smooth. Theorem 2.3.4 implies that the set $g^{-1}(\{0\})$ consists only of nice flat points of g.

Remark 2.3.9. In 1963, in [26], Glaeser gave an example of a nonnegative smooth function $g:\mathbb{R}\to\mathbb{R}$ which is 0 only at 0 and flat there, such that the function $\sqrt{g}$ is not even twice differentiable. In view of Theorem 2.3.4 this means that in this case 0 is a flat point of g but it is not a nice flat point of g.

For more information about the properties of the square root of a nonnegative smooth function we refer to [17].

The following example is taken from [1, page 5].

Example 2.3.10. *Let $g:\mathbb{R}\to\mathbb{R}$ be defined via the following formula*

$$g(x)=\begin{cases}\exp\left(-\frac{1}{x}\right)\sin^2\left(\frac{1}{x}\right)+\exp\left(-\frac{2}{x}\right), & x>0,\\ 0, & x\le 0.\end{cases}$$

It is easy to verify that g is a smooth function which is flat on the set $g^{-1}(\{0\})$. For every $x>0$ we have that

$$\begin{aligned}\left(\sqrt{g}\right)''(x)=&\frac{4\exp\left(-\frac{2}{x}\right)-\exp\left(-\frac{1}{x}\right)\sin^2\left(\frac{1}{x}\right)+2\exp\left(-\frac{1}{x}\right)\cos^2\left(\frac{1}{x}\right)-4\exp\left(-\frac{1}{x}\right)\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)}{2x^4\cdot\sqrt{\exp\left(-\frac{1}{x}\right)\sin^2\left(\frac{1}{x}\right)+\exp\left(-\frac{2}{x}\right)}}\\ &+\frac{2\exp\left(-\frac{2}{x}\right)-\exp\left(-\frac{1}{x}\right)\sin^2\left(\frac{1}{x}\right)+2\exp\left(-\frac{1}{x}\right)\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)}{x^3\cdot\sqrt{\exp\left(-\frac{1}{x}\right)\sin^2\left(\frac{1}{x}\right)+\exp\left(-\frac{2}{x}\right)}}\\ &+\frac{2\exp\left(-\frac{2}{x}\right)-\exp\left(-\frac{1}{x}\right)\sin^2\left(\frac{1}{x}\right)-2\exp\left(-\frac{1}{x}\right)\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)}{4x^2\cdot\left(\exp\left(-\frac{1}{x}\right)\sin^2\left(\frac{1}{x}\right)+\exp\left(-\frac{2}{x}\right)\right)^{\frac{3}{2}}}.\end{aligned}$$

One can easily calculate that

$$\lim_{n\to\infty}\left(\sqrt{g}\right)''\left(\frac{1}{2\pi n}\right)=\infty.$$

Therefore, the function $\sqrt{g}$ is not smooth. Thus, from Theorem 2.3.4 it follows that the point 0 *is not a nice flat point of g.*

2.4 A necessary condition

The goal of this section is to prove the following theorem which gives a necessary condition for C_ψ to have closed range. Let us emphasize that together with Theorem 2.2.3, this result does not give a full description of closed range composition operators corresponding to smooth symbols $\psi:\mathbb{R}\to\mathbb{R}$ (see Remark 2.2.10). Nevertheless, together with Theorem 2.2.3 it gives such a characterization in a special case (see Corollary 2.4.3).

Theorem 2.4.1. *Let $\psi:\mathbb{R}\to\mathbb{R}$ be smooth. If there exists $b\in\psi(\mathbb{R})$ such that the boundary of the set $\{x\in\mathbb{R}:\psi(x)>b\}$ or the boundary of the set $\{x\in\mathbb{R}:\psi(x)<b\}$ is nonempty and contains only nice flat points, then the range of the composition operator $C_\psi: C^\infty(\mathbb{R})\to C^\infty(\mathbb{R})$, $F\mapsto F\circ\psi$, is not closed.*

Please note that the assumptions of the theorem above imply that ψ is not constant.

Proof. Let us assume that there exists $b\in\psi(\mathbb{R})$ such that the boundary of the set

$$\{x\in\mathbb{R}:\psi(x)>b\}$$

is nonempty and contains only nice flat points of ψ (the proof in the other case goes along the same lines). Our aim is to show that the range of C_ψ is not closed. In order to do this, we will construct a smooth function $g\in\overline{\operatorname{Im}C_\psi}\setminus\operatorname{Im}C_\psi$.

Let us first consider the function $h:\mathbb{R}\to\mathbb{R}$ defined via formula

$$h(x)=\begin{cases}\psi(x)-b, & \text{if } \psi(x)\geq b,\\ 0, & \text{if } \psi(x)<b.\end{cases}$$

It is clear that this function is continuous. Even more, outside the boundary B of the set $\{x\in\mathbb{R}:\psi(x)>b\}$ it is smooth and for every $x_0\in B$ and every $n\in\mathbb{N}$

$$\lim_{\substack{x\to x_0\\ x\notin B}}\frac{h^{(n)}(x)}{x-x_0}=0.$$

Thus inductively, by l'Hôpital's Rule, $h^{(n)}$ exists and vanishes on B for all $n\in\mathbb{N}$. For reader's convenience we provide some details.

Claim 1. *For every $n \geq 1$ the n-th derivative of the function h exists and, moreover,*

$$h^{(n)}(x) = \begin{cases} \psi^{(n)}(x), & \text{if } \psi(x) > b, \\ 0, & \text{if } \psi(x) \leq b. \end{cases}$$

Proof of the claim. We proceed by induction on n.

Step 1.

Let $n = 1$. It is clear that the first derivative of h exists on the open set $\{x \in \mathbb{R} : \psi(x) \neq b\}$. It is also clear that $h'(x) = \psi'(x)$ for every $x \in \mathbb{R}$ such that $\psi(x) > b$ and that $h'(x) = 0$ for every $x \in \mathbb{R}$ such that $\psi(x) < b$.

Let now x_0 be such that $\psi(x_0) = b$. If there exists a neighbourhood U of x_0 such that $\psi(x) \leq b$ for every $x \in U$, then $h(x) = 0$ on U and hence the first derivative of h exists at x_0 and $h'(x_0) = 0$. If such a neighbourhood does not exists, then x_0 must be in the boundary of the set $\{x \in \mathbb{R} : \psi(x) > b\}$, and hence, by our assumptions, x_0 must be a nice flat point of ψ. In particular, x_0 must be a flat point of ψ. Thus

$$\lim_{x \to x_0} \left| \frac{\psi(x) - \psi(x_0)}{x - x_0} \right| = 0. \tag{2.4.1}$$

Observe now that

$$0 \leq |h(x)| \leq \left|\psi(x) - b\right| \text{ for every } x \in \mathbb{R}.$$

Together with (2.4.1) this gives that

$$\lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{h(x)}{x - x_0} = 0.$$

This completes the proof in case $n = 1$.

Step 2.

Assume that Claim 1 is true for $j = n - 1$. It is obvious that the n-th derivative of h exists on the open set $\{x \in \mathbb{R} : \psi(x) \neq b\}$. It is also clear that $h^{(n)}(x) = \psi^{(n)}(x)$ for every $x \in \mathbb{R}$ such that $\psi(x) > b$ and that $h^{(n)}(x) = 0$ for every $x \in \mathbb{R}$ such that $\psi(x) < b$.

Let us fix now x_0 such that $\psi(x_0) = b$. If there exists a neighbourhood U of x_0 such that $h^{(n-1)}(x) = 0$ for every $x \in U$, then the n-th derivative of h exists at x_0 and $h^{(n)}(x_0) = 0$. If such a neighbourhood does not exists, then it is easy to see that b must be in the boundary of the set $\{x \in \mathbb{R} : \psi(x) > b\}$, and hence b must be a nice flat point of ψ. In particular, x_0 must be a flat point of ψ. Thus

$$\lim_{x \to x_0} \frac{\psi^{(n-1)}(x)}{x - x_0} = \lim_{x \to x_0} \frac{\psi^{(n-1)}(x) - \psi^{(n-1)}(x_0)}{x - x_0} = 0,$$

and therefore

$$\lim_{x \to x_0} \left| \frac{\psi^{(n-1)}(x)}{x - x_0} \right| = 0. \tag{2.4.2}$$

Let us observe now that

$$0 \le \left|h^{(n-1)}(x)\right| \le \left|\psi^{(n-1)}(x)\right| \text{ for every } x \in \mathbb{R}.$$

Together with (2.4.2) this gives that

$$\lim_{x\to x_0} \frac{h^{(n-1)}(x) - h^{(n-1)}(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{h^{(n-1)}(x)}{x - x_0} = 0.$$

This completes the proof of the inductive step. By induction, Claim 1 is true.

Claim 2. *Every point in $h^{-1}(\{0\})$ is a nice flat point of h.*

Proof of the claim. Let us fix $x_0 \in h^{-1}(\{0\})$, $n \ge 1$, and $0 < \theta < 1$. If there exists a neighbourhood U of x_0 such that $\psi(x) \le b$ for every $x \in U$, then $h(x) = 0$ for every $x \in U$. This easily implies that

$$\left|h^{(n)}(x)\right| \le |h(x)|^\theta \text{ for every } x \in U.$$

If such a neighbourhood does not exists, then it is easy to see that x_0 must be in the boundary of the set $\{x \in \mathbb{R} : \psi(x) > b\}$ (in particular, $\psi(x_0) = b$). Thus, by our assumptions, x_0 must be a nice flat point of ψ. Therefore, there exists a neighbourhood W of x_0 such that

$$\left|\psi^{(n)}(x)\right| \le \left|\psi(x) - b\right|^\theta \text{ for every } x \in W. \tag{2.4.3}$$

If $x \in W$ and $h^{(n)}(x) = 0$, then the inequality

$$\left|h^{(n)}(x)\right| \le |h(x)|^\theta \tag{2.4.4}$$

is trivially satisfied. If $x \in W$ and $h^{(n)}(x) \ne 0$, then from Claim 1 we get that $\psi(x) > b$ and $h^{(n)}(x) = \psi^{(n)}(x)$. From (2.4.3) we thus get that

$$\left|h^{(n)}(x)\right| = \left|\psi^{(n)}(x)\right| \le \left|\psi(x) - b\right|^\theta = |h(x)|^\theta \tag{2.4.5}$$

Connecting (2.4.4) and (2.4.5) we obtain that

$$\left|h^{(n)}(x)\right| \le |h(x)|^\theta \text{ for every } x \in W.$$

This proves the claim.

Let us now consider the function

$$g = \sqrt{h}.$$

From Claim 1, Claim 2, and Corollary 2.3.6 we obtain that g is a smooth function and, moreover, that every point in $g^{-1}(\{0\})$ is a nice flat point of g. We will show that

$$g \in \overline{\operatorname{Im} C_\psi} \setminus \operatorname{Im} C_\psi$$

and this will complete the proof of the theorem.

Claim 3. *We have that* $g \notin \operatorname{Im} C_\psi$.

Proof of the claim. The assumption that there exists $b \in \psi(\mathbb{R})$ such that the boundary of the set $\{x \in \mathbb{R} : \psi(x) > b\}$ is nonempty implies that there exists $\delta > 0$ such that

$$[b, b+\delta) \subset \psi(\mathbb{R}). \tag{2.4.6}$$

If $G : \mathbb{R} \to \mathbb{R}$ is such that $G \circ \psi = g$, then for every $y \in [b, b+\delta)$ the equality $G(y) = \sqrt{y-b}$ must hold. But this implies that the first derivative of G cannot exist at b. This shows that there does not exist a smooth function $G : \mathbb{R} \to \mathbb{R}$ such that $G \circ \psi = g$. This proves the claim.

Claim 4. *Let K be a compact subset of* $\mathbb{R}$, $0 < \theta < 1$, *and* $N \in \mathbb{N}$. *There exists a natural number* k_0 *such that for every* $k \geq k_0$, *for every* $x \in K$, *and every* $1 \leq i \leq N$ *the condition* $g(x) \leq 1/k$ *implies that*

$$\left|g^{(i)}(x)\right| \leq (1/k)^\theta.$$

Proof of the claim. Assume that this is not the case. Then there exists $1 \leq i_0 \leq N$, an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers and a sequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of points from K such that for every $k \in \mathbb{N}$ we have

$$g(x_{n_k}) \leq 1/n_k \text{ and } \left|g^{(i_0)}(x_{n_k})\right| > (1/n_k)^\theta. \tag{2.4.7}$$

Since the set K is compact we may assume that the sequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges to a point $x_0 \in K$. From (2.4.7) we obtain that $g(x_0) = 0$ (the function g is nonnegative) and hence $\psi(x_0) \leq b$. Since the derivatives of g are nonzero only at points from the set $\{x \in \mathbb{R} : \psi(x) > b\}$, we get from (2.4.7) that for every $k \in \mathbb{N}$ we have

$$x_{n_k} \in \{x \in \mathbb{R} : \psi(x) > b\}.$$

Therefore, x_0 is in the boundary of the set $\{x \in \mathbb{R} : \psi(x) > b\}$. Since $x_0 \in g^{-1}(\{0\})$, we have that x_0 is a nice flat point of g. Thus we can find a neighbourhood U of x_0 such that

$$\left|g^{(i_0)}(x)\right| \leq \left|g(x)\right|^\theta \text{ for every } x \in U. \tag{2.4.8}$$

Because the sequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges to x_0, we can find an index k_0 such that $x_{n_{k_0}}$ in U. Combining (2.4.7) and (2.4.8) we obtain a contradiction

$$\left(1/n_{k_0}\right)^\theta < \left|g^{(i_0)}(x_{n_{k_0}})\right| \leq \left|g(x_{n_{k_0}})\right|^\theta \leq \left(1/n_{k_0}\right)^\theta.$$

Claim 5. *We have that* $g \in \overline{\operatorname{Im} C_\psi}$.

Proof of the claim. In order to prove the claim, we have to approximate the function g by functions from $\operatorname{Im} C_\psi$ (in the topology of the space $C^\infty(\mathbb{R})$).

By Lemma 1.1.8 there exists a sequence $\{C_n\}_{n\in\mathbb{N}}$ of positive numbers such that for every natural number k there exists a smooth function $\Phi_k : \mathbb{R} \to \mathbb{R}$ with the following properties:

1. $0 \le \Phi_k(x) \le 1$ for all $x \in \mathbb{R}$;
2. $\Phi_k(x) = 0$ for $x \in (-\infty, \frac{1}{2k}]$, $\Phi_k(x) = 1$ for $x \in [\frac{1}{k}, \infty)$;
3. $\left|(\Phi_k)^{(i)}(x)\right| \le C_i \cdot k^i$ for all $x \in \mathbb{R}$ and every $i \in \mathbb{N}$.

For every $k \ge 1$ let

$$g_k := g \cdot (\Phi_k \circ g).$$

Let us observe that for every$k \ge 1$ the function

$$G_k(x) = \begin{cases} \sqrt{x-b} \cdot \Phi_k\left(\sqrt{x-b}\right), & \text{if } x \ge b, \\ 0, & \text{otherwise.} \end{cases}$$

is smooth and $G_k \circ \psi = g_k$. Therefore, $g_k \in \operatorname{Im} C_\psi$ for every $k \ge 1$.

We will show now that $g_k \xrightarrow{k\to\infty} g$ in the topology of the space $C^\infty(\mathbb{R})$ and this will complete the proof of the claim. So let K be an arbitrary compact set and let n be an arbitrary natural number. We have to show that

$$\sup_{x\in K}\left|g^{(n)}(x) - g_k^{(n)}(x)\right| \xrightarrow{k\to\infty} 0.$$

Using the product rule and Faá di Bruno's formula (Theorem 1.1.4) we obtain that

$$\begin{aligned} g^{(n)}(x) - g_k^{(n)}(x) &= g^{(n)}(x) - \left(g \cdot (\Phi_k \circ g)\right)^{(n)}(x) \\ &= g^{(n)}(x) - g^{(n)}(x) \cdot \Phi_k(g(x)) - \sum_{i=1}^{n} \binom{n}{i} g^{(n-i)}(x) (\Phi_k \circ g)^{(i)}(x) \\ &= g^{(n)}(x) - g^{(n)}(x) \cdot \Phi_k(g(x)) \\ &- \sum_{i=1}^{n} \left(\binom{n}{i} g^{(n-i)}(x) \cdot \left(\sum_{A_i} B_{m_1,\dots,m_i} \cdot \Phi_k^{(m_1+m_2+\ldots+m_i)}(g(x)) \cdot \prod_{j:m_j\neq 0} \left(g^{(j)}(x)\right)^{m_j} \right)\right) \end{aligned}$$

We choose now $0 < \theta < 1$ such that for every $1 \le i \le n$ the following condition holds: if an i - tuple of nonnegative integers $m_1, m_2, \dots, m_i$ satisfies

$$1 \cdot m_1 + 2 \cdot m_2 + \cdots + m_i \cdot i = i,$$

then

$$\theta + (\theta - 1)(m_1 + m_2 + \cdots + m_i) \ge 1/2.$$

It is easy to see that such a θ exists. Using now Claim 4 we can find $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, for every $x \in K$, and every $0 \leq i \leq n$ the condition $g(x) \leq 1/k$ implies that

$$\left|g^{(i)}(x)\right| < (1/k)^{\theta}.$$

Case 1. If $k \geq k_0$, $x \in K$ and $g(x) > 1/k$, then from the properties of the function Φ_k we have that

$$(\Phi_k \circ g)(x) = 1 \text{ and } (\Phi_k \circ g)^{(s)}(x) = 0 \text{ for } s \in \mathbb{N}.$$

This gives that for all $x \in K$ such that $g(x) > 1/k$ we have

$$g^{(n)}(x) - g_k^{(n)}(x) = 0.$$

Case 2. If $k \geq k_0$, $x \in K$ and $g(x) \leq 1/k$, then

$$\begin{aligned}
&\left|g^{(n)}(x) - g_k^{(n)}(x)\right| \leq \left|g^{(n)}(x)\right| + \left|g^{(n)}(x)\cdot\Phi_k(g(x))\right| \\
&+ \sum_{i=1}^{n}\left(\binom{n}{i}\left|g^{(n-i)}(x)\right|\cdot\left(\sum_{A_i}\left|B_{m_1,\dots,m_i}\right|\cdot\left|\Phi_k^{(m_1+m_2+\dots+m_i)}(g(x))\right|\cdot\prod_{j:m_j\neq 0}\left(\left|g^{(j)}(x)\right|\right)^{m_j}\right)\right) \\
&\leq \frac{1}{k^{\theta}} + \frac{1}{k^{\theta}} + C\cdot\sum_{i=1}^{n}\left(\frac{1}{k^{\theta}}\cdot\sum_{A_i}k^{m_1+m_2+\dots+m_i}\cdot\prod_{j:m_j\neq 0}\left(\frac{1}{k}\right)^{\theta\cdot m_j}\right) \\
&= \frac{2}{k^{\theta}} + C\cdot\sum_{i=1}^{n}\left(\sum_{A_i}\left(\frac{1}{k}\right)^{\theta+(\theta-1)\cdot(m_1+m_2+\dots+m_i)}\right) \\
&\leq \frac{2}{k^{\theta}} + D\cdot\left(\frac{1}{k}\right)^{\frac{1}{2}},
\end{aligned}$$

where the constants C and D depend only on n.

Altogether, the above gives that

$$\sup_{x\in K}\left|g^{(n)}(x) - g_k^{(n)}(x)\right| \xrightarrow{k\to\infty} 0.$$

This proves the claim and completes the proof of Theorem 2.4.1.

□

Example 2.4.2. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined via the formula*

$$\psi(x) = \begin{cases} \exp\left(-\frac{1}{x}\right), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Observe that 0 *is the only point in the boundary of the set* $\{x \in \mathbb{R} : \psi(x) > 0\}$. *From Example 2.3.7 we know that* 0 *is a nice flat point of* ψ. *By Theorem 2.4.1, the range of* C_ψ *is not closed. In fact, the proof of that theorem shows that* $\sqrt{\psi} \in \overline{\operatorname{Im} C_\psi} \setminus \operatorname{Im} C_\psi$.

We were not able to obtain a full characterization of closed range composition operators corresponding to smooth symbols $\psi : \mathbb{R} \to \mathbb{R}$. Nevertheless, we have the following special result.

Corollary 2.4.3. *Let* $\psi : \mathbb{R} \to \mathbb{R}$ *be a smooth function such that every flat point of* ψ *is a nice flat point of* ψ. *The range of the composition operator* $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$, *is closed if and only if* ψ *is a semiproper function which is ether constant or it satisfies the following conditions*

1. *Every fiber of* ψ *over a boundary point of* $\psi(\mathbb{R})$ *contains a non-flat point.*
2. *Every fiber of* ψ *over an interior point of* $\psi(\mathbb{R})$ *contains either a non-flat non extreme point or both a non-flat local minimum and a non-flat local maximum.*

Proof. Combine Lemma 2.1.6, Theorem 2.2.3, and Theorem 2.4.1.

□

CHAPTER 3

DYNAMICAL PROPERTIES OF WEIGHTED COMPOSITION OPERATORS

Throughout this chapter $C^\infty(\Omega)$ is the space of real or complex valued smooth functions on Ω. All the results below are valid in both cases (of course, if we consider the weighted composition on the real space $C^\infty(\Omega)$, then the weight has to be real valued).

The goal of this part of the thesis is to investigate several dynamical properties of composition operators and weighted composition operators on the space of smooth functions. In Theorem 3.2.6 we characterize weakly mixing weighted composition operators. Corollary 3.2.7 gives a characterization of hypercyclic weighted composition operators under the assumption that the weight is real valued. In Theorem 3.2.9 we give a description of mixing weighted composition operators. As an immediate consequence of those results, we obtain characterizations of hypercyclic, weakly mixing, and mixing composition operators (see Theorem 3.3.1 and Theorem 3.3.2). In the last section we show that the classes of hypercyclic, weakly mixing, mixing, and chaotic weighted composition operators coincide in the one-dimensional case (see Theorem 3.4.4). From what follows it will be clear that the dynamical properties of the weighted composition operator $C_{w,\psi}$ heavily rely on the dynamical properties of the function ψ and that the weight function w plays a secondary role in this context.

3.1 Notation

For all the basic notions and facts from linear dynamics we refer to the Preliminaries of this dissertation. In this section we will fix up some notation and we will introduce some specific concepts which will be important in studying dynamical properties of (weighted) composition operators.

Since the main point of our interests is the behaviour of the iterates of (weighted) composition operators acting on the space of smooth functions, we start with the following definitions.

Definition 3.1.1. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ be smooth. For every $n \geq 1$, the operator $C^n_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$ is defined as the n-th iteration of $C_{w,\psi}$, i.e.,*

$$C^n_{w,\psi} = \underbrace{C_{w,\psi} \circ \cdots \circ C_{w,\psi}}_{n\ times}.$$

Let us emphasize that in the following definition we do *not* assume that the function ψ is surjective.

Definition 3.1.2. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$.*

- *We define $\psi_0 : \Omega \to \Omega$ as $\psi_0(x) = x$ for every $x \in \Omega$.*
- *For every $n \geq 1$ we define $\psi_n : \Omega \to \Omega$ inductively via the formula $\psi(x) = \psi(\psi_{n-1}(x))$ for every $x \in \Omega$.*
- *If ψ is injective, then for every $n < 0$ we define $\psi_n : \psi_{-n}(\Omega) \to \Omega$ via the rule: $\psi_n(x) = y$ if and only if $\psi_{-n}(y) = x$.*

Remark 3.1.3. Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ be smooth. Then for every $F \in C^\infty(\Omega)$ and every $n \geq 1$ we have that

$$\left(C^n_{w,\psi}(F)\right)(x) = \left(\prod_{i=0}^{n-1} w(\psi_i(x))\right) \cdot F(\psi_n(x)).$$

The following notions are crucial in studying dynamical properties of (weighted) composition operators.

Definition 3.1.4. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$.*

- *We say that ψ has the* run-away *property if for every compact set $K \subset \Omega$ there exists $n \in \mathbb{N}$ such that $\psi_n(K) \cap K = \emptyset$.*
- *We say that ψ has the* strong run-away *property if for every compact set $K \subset \Omega$ there exists $N \in \mathbb{N}$ such that $\psi_n(K) \cap K = \emptyset$ for every $n \geq N$.*

Remark 3.1.5. Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$. It is clear that if ψ has the strong run-away property, then ψ has also the run-away property. In Lemma 3.4.2 we will show that the converse is true if we assume that $\psi : \mathbb{R} \to \mathbb{R}$ is smooth and injective. We doubt if this is also true in higher dimensions.

Example 3.1.6. *Let $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be defined via the formula*

$$\psi(x_1,\dots,x_d) = (x_1+1,\dots,x_d+1).$$

It is easy to verify that ψ has the run-away property and the strong run-away property.

Example 3.1.7. *Let $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be defined via the formula*

$$\psi(x_1,\dots,x_d) = (e^{x_1},\dots,e^{x_d}).$$

One can easily check that ψ has the run-away property and the strong run-away property.

3.2 Hypercyclic, weakly mixing, and mixing weighted composition operators

In this section we characterize hypercyclic (for real weights), weakly mixing, and mixing weighted composition operators. First, in the following four lemmas, we show that if the operator $C_{w,\psi}$ is hypercyclic, weakly mixing, or mixing, then the functions w and ψ have to satisfy some quite natural conditions. Please recall that if $\Omega \subset \mathbb{R}^d$ is open and $\psi : \Omega \to \Omega$ is smooth, then for every $x \in \Omega$ the derivative of ψ at x, which we denote by $\psi'(x)$, is a linear map from $\mathbb{R}^d$ to $\mathbb{R}^d$. By $[\psi'(x)]$ we denote the Jacobian matrix of ψ at x. Similarly, if $f : \Omega \to \mathbb{C}$ is a smooth function, then the derivative of f at x, which we denote by $f'(x)$, is a linear map from $\mathbb{R}^d$ to $\mathbb{R}^2$.

Lemma 3.2.1. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ be smooth. If the weighted composition operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w\cdot(F\circ\psi)$, is hypercyclic, then the following conditions are satisfied:*

1. *For every $x \in \Omega$ we have that $w(x) \neq 0$.*
2. *The function ψ is injective.*
3. *For every $x \in \Omega$ we have that* $\det[\psi'(x)] \neq 0$.

In the proof of this lemma we will several times use the following simple observation: if the the set $\operatorname{Im} C_{w,\psi} = \{w\cdot(F\circ\psi) : F \in C^\infty(\Omega)\}$ is not dense in $C^\infty(\Omega)$, then the operator $C_{w,\psi}$ cannot be hypercyclic.

Proof.
1. Assume to the contrary that $C_{w,\psi}$ is a hypercyclic operator and that there exists $x_0 \in \Omega$ with $w(x_0) = 0$. Then for every $F \in C^\infty(\Omega)$ we have

$$\big(C_{w,\psi}(F)\big)(x_0) = w(x_0)\cdot(F\circ\psi)(x_0) = 0.$$

This gives that

$$\operatorname{Im} C_{w,\psi} \subset \left\{ f \in C^\infty(\Omega) : f(x_0) = 0 \right\}.$$

But the set $\left\{ f \in C^\infty(\Omega) : f(x_0) = 0 \right\}$ is a proper and closed linear subspace of $C^\infty(\Omega)$, so the operator $C_{w,\psi}$ cannot be hypercyclic. This gives a contradiction.

2. Assume to the contrary that the operator $C_{w,\psi}$ is hypercyclic and that we can find two distinct points $x_1, x_2 \in \Omega$ with $\psi(x_1) = \psi(x_2)$. By the first part of this lemma we know that $w(x) \neq 0$ for every $x \in \Omega$. For an arbitrary $F \in C^\infty(\Omega)$ we have

$$\left(C_{w,\psi}(F)\right)(x_1) = w(x_1) \cdot F(\psi(x_1)) = \frac{w(x_1)}{w(x_2)} \cdot w(x_2) \cdot F(\psi(x_2)) = \frac{w(x_1)}{w(x_2)} \cdot \left(C_{w,\psi}(F)\right)(x_2).$$

This gives that

$$\operatorname{Im} C_{w,\psi} \subset \left\{ f \in C^\infty(\Omega) : f(x_1) = \frac{w(x_1)}{w(x_2)} \cdot f(x_2) \right\}.$$

But the latter is a proper and closed linear subspace of $C^\infty(\Omega)$, and hence the operator $C_{w,\psi}$ cannot be hypercyclic. This gives a contradiction.

3. Assume that $C_{w,\psi}$ is hypercyclic. By the first part of this lemma we have that $w(x) \neq 0$ for every $x \in \Omega$. Let us suppose that there exists $x_0 \in \Omega$ with

$$\det[\psi'(x_0)] = 0.$$

Simple facts from linear algebra allow us to find $0 \neq h \in \mathbb{R}^d$ such that

$$\psi'(x_0)h = (0, \ldots, 0).$$

For every $F \in C^\infty(\Omega)$ we have

$$\begin{aligned}
\left(C_{w,\psi}(F)\right)'(x_0)h &= \left(w \cdot (F \circ \psi)\right)'(x_0)h \\
&= F(\psi(x_0))\, w'(x_0)h + w(x_0) \cdot F'(\psi(x_0))\psi'(x_0)h \\
&= F(\psi(x_0))\, w'(x_0)h \\
&= w(x_0) \cdot (F(\psi(x_0))) \cdot \frac{w'(x_0)h}{w(x_0)} \\
&= \left(C_{w,\psi}(F)\right)(x_0) \cdot \frac{w'(x_0)h}{w(x_0)}.
\end{aligned}$$

Therefore

$$\operatorname{Im} C_{w,\psi} \subset \left\{ f \in C^\infty(\Omega) : f'(x_0)h = f(x_0) \cdot \frac{w'(x_0)h}{w(x_0)} \right\}.$$

But the latter is a proper, closed linear subspace of $C^\infty(\Omega)$ and therefore the operator $C_{w,\psi}$ cannot be hypercyclic. This gives a contradiction.

□

Unfortunately, we can only prove the following lemma in case if the weight is a real valued function. We do not know if it is true for complex valued weights.

Please recall that for a complex number z, $\mathrm{Re}(z)$ denotes its real part.

Lemma 3.2.2. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{R}$ be smooth. If the weighted composition operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, is hypercyclic, then ψ has the run-away property.*

Proof. Assume to the contrary that $C_{w,\psi}$ is a hypercyclic operator and that the function ψ does not have the run-away property. Let $\mathcal{M}$ be the family of those natural numbers m which satisfy the following condition: there exists a compact set $K \subset \Omega$ whose intersection with precisely m connected components of Ω is nonempty and, moreover, there exists a natural number N such that $\psi_n(K) \cap K \neq \emptyset$ for every $n \geq N$. The assumption that ψ does not have the run-away property, together with the fact that every compact subset of Ω is contained in finitely many connected components of Ω, gives that $\mathcal{M}$ is nonempty. Let m_0 be the smallest element of $\mathcal{M}$. There are two possible cases: either $m_0 = 1$ or $m_0 > 1$. We will show that each of this cases leads to a contradiction and this will end the proof.

Case 1. Assume that $m_0 = 1$. From the definition of m_0 and the definition of the family $\mathcal{M}$, there exists a compact set $K \subset \Omega$ which is contained in a connected component of Ω, we denote this component by Ω_0, and there exists $N \in \mathbb{N}$ such that

$$\psi_n(K) \cap K \neq \emptyset \text{ for every } n \geq N. \tag{3.2.1}$$

Claim 1.1. *For every $n \geq 1$ we have that $\psi_n(\Omega_0) \subset \Omega_0$.*

Proof of the claim. We know that

$$\psi_N(K) \cap K \neq \emptyset \text{ and } \psi_{N+1}(K) \cap K \neq \emptyset.$$

Therefore

$$\psi_N(\Omega_0) \subset \Omega_0 \text{ and } \psi_{N+1}(\Omega_0) \subset \Omega_0.$$

This gives that ψ maps the set $\psi_N(\Omega_0)$, which is a subset of Ω_0, into Ω_0. Therefore $\psi(\Omega_0) \subset \Omega_0$, by connectivity of Ω_0, and from this it follows that $\psi_n(\Omega_0) \subset \Omega_0$ for every $n \geq 1$. This proves the claim.

Claim 1.2. *For every $n \geq 1$ and every $x \in \Omega_0$ we have that*

$$\prod_{i=0}^{2n-1} w(\psi_i(x)) > 0.$$

Proof of the claim. Since the operator $C_{w,\psi}$ is hypercyclic, we get from Lemma 3.2.1 that $w(x) \neq 0$ for every $x \in \Omega$. Because Ω_0 is connected, this gives that w has constant sign on Ω_0. From Claim 1.1 we obtain that for every $x \in \Omega_0$ and every $n \geq 1$ the signs of $w(x)$ and $w(\psi_n(x))$ are equal. This implies that for every $x \in \Omega_0$ and every $n \geq 1$ the desired inequality holds, since the product of an even number of nonzero numbers which have the same sign is always positive.

We are now ready to finish the proof in this case. By our assumptions the operator $C_{w,\psi}$ is hypercyclic. Using Ansari's result (see Theorem 1.2.9) we obtain that the operator $C^2_{w,\psi}$ is also hypercyclic. Because the set of hypercyclic vectors of $C^2_{w,\psi}$ is dense in $C^\infty(\Omega)$ (see Birkhoff's Theorem 1.2.4), we can find $h \in C^\infty(\Omega)$ which is a hypercyclic vector of $C^2_{w,\psi}$ and which satisfies the inequality

$$\operatorname{Re}(h(x)) > 1 \text{ for every } x \in K. \tag{3.2.2}$$

Since h is a hypercyclic vector of $C^2_{w,\psi}$, we can find an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that the sequence $\left\{C^{2n_k}_{w,\psi}(h)\right\}_{k\in\mathbb{N}}$ converges (in the topology of $C^\infty(\Omega)$) to the function which is equal to -1 on Ω. In particular, there is $k_0 \in \mathbb{N}$ such that

$$\operatorname{Re}\left(\left(C^{2n_k}_{w,\psi}(h)\right)(x)\right) < 0 \text{ for every } x \in K \text{ and every } k \geq k_0. \tag{3.2.3}$$

Let $k_1 > k_0$ be such that $2n_{k_1} \geq N$. In view of (3.2.1) we can find $x_0 \in K$ such that $\psi_{2n_{k_1}}(x_0) \in K$. From (3.2.3) (see also Remark 3.1.3) we obtain that

$$\left(\prod_{i=0}^{2n_{k_1}-1} w(\psi_i(x_0))\right) \cdot \operatorname{Re}\left(h(\psi_{2n_{k_1}}(x_0))\right) < 0.$$

Using (3.2.2) we thus get that

$$\prod_{i=0}^{2n_{k_1}-1} w(\psi_i(x_0)) < 0$$

and this gives a contradiction with the statement of Claim 1.2. This completes the proof in this case.

Case 2. Assume that $m_0 > 1$. From the definition of m_0 and the definition of the family $\mathscr{M}$, there exists a compact set $K \subset \Omega$ whose intersection with precisely m_0 connected components of Ω is nonempty and, moreover, there is a natural number N such that

$$\psi_n(K) \cap K \neq \emptyset \text{ for every } n \geq N. \tag{3.2.4}$$

Let $\Omega_1, \ldots, \Omega_{m_0}$ be those connected components of Ω whose intersection with K is nonempty. Moreover, for $1 \leq i \leq m_0$ let $K_i = K \cap \Omega_i$. It is clear that K_i is a compact set for every $1 \leq i \leq m_0$.

Claim 2.1. *For every* $1 \le i \le m_0$ *there exists* $1 \le j \le m_0$ *and there is* $n \in \mathbb{N}$ *such that*

$$\psi_n(\Omega_i) \subset \Omega_j.$$

Proof of the claim. Assume that this is not the case, i.e., assume that there exists $1 \le i_0 \le m_0$ such that for every $1 \le j \le m_0$ and every $n \ge 1$ we have

$$\psi_n(\Omega_{i_0}) \not\subset \Omega_j.$$

Because for every $n \in \mathbb{N}$ the set $\psi_n(\Omega_{i_0})$ is connected, this means that

$$\psi_n(\Omega_{i_0}) \cap \left(\bigcup_{i=1}^{m_0} \Omega_i\right) = \emptyset \text{ for every } n \ge 1. \tag{3.2.5}$$

In particular,

$$\psi_n(K_{i_0}) \cap K = \emptyset \text{ for every } n \ge 1. \tag{3.2.6}$$

Let us now consider the set

$$M := K \setminus K_{i_0}.$$

It is clear that M is compact. Moreover, the intersection of M with precisely $m_0 - 1$ connected components of Ω is nonempty. We will show that there exists a natural number N_M such that $\psi_n(M) \cap M \neq \emptyset$ for every $n \ge N_M$. This will give a contradiction with the assumption that m_0 is the smallest element of $\mathscr{M}$.

In order to show that there exists a natural number N_M such that for every $n \ge N_M$ we have $\psi_n(M) \cap M \neq \emptyset$, let us first observe that there exists $N_1 \in \mathbb{N}$ such that

$$\psi_n(M) \cap K_{i_0} = \emptyset \text{ for every } n \ge N_1. \tag{3.2.7}$$

Indeed, if this would not be the case, then it would be possible to find $1 \le j \neq i_0 \le m$ and $n_1 < n_2$ such that

$$\psi_{n_1}(K_j) \cap K_{i_0} \neq \emptyset \text{ and } \psi_{n_2}(K_j) \cap K_{i_0} \neq \emptyset.$$

Hence $\psi_{n_1}(K_j) \subset \Omega_{i_0}$ since the set K_j is contained in the connected component Ω_j of Ω. But this impossible in view of (3.2.5) since

$$\psi_{n_2}(K_j) \cap K_{i_0} = \psi_{n_2-n_1}(\psi_{n_1}(K_j)) \cap K_{i_0} \subset \psi_{n_2-n_1}(\Omega_{i_0}) \cap K_{i_0} \subset \psi_{n_2-n_1}(\Omega_{i_0}) \cap \left(\bigcup_{i=0}^{m_0} \Omega_i\right) = \emptyset.$$

Let now $N_M = \max\{N, N_1\}$. From (3.2.4), (3.2.6), and (3.2.7), for every $n \ge N_M$ we obtain that

$$\begin{aligned}
\emptyset \neq &\psi_n(K) \cap K = \psi_n(M \cup K_{i_0}) \cap (M \cup K_{i_0}) \\
&= \big(\psi_n(M) \cap (M \cup K_{i_0})\big) \cup \big(\psi_n(K_{i_0}) \cap (M \cup K_{i_0})\big) \\
&= \psi_n(M) \cap (M \cup K_{i_0}) \\
&= \big(\psi_n(M) \cap M\big) \cup \big(\psi_n(M) \cap K_{i_0}\big). \\
&= \psi_n(M) \cap M.
\end{aligned}$$

Hence

$$\psi_n(M) \cap M \neq \emptyset \text{ for every } n \geq N_M.$$

As explained above, this gives a contradiction with our assumptions. This completes the proof of the claim.

Claim 2.2. *For every $1 \leq i \leq m_0$ there exists $1 \leq j_i \leq m_0$ and there are two natural numbers $N_{i,1} < N_{i,2}$ such that*

$$\psi_{N_{i,1}}(\Omega_i) \subset \Omega_{j_i} \text{ and } \psi_{N_{i,2}}(\Omega_i) \subset \Omega_{j_i}.$$

Proof of the claim. Let us fix $1 \leq i_0 \leq m_0$. By Claim 2.1 there exists $n_1 \in \mathbb{N}$ and $1 \leq i_1 \leq m_0$ such that

$$\psi_{n_1}(\Omega_{i_0}) \subset \Omega_{i_1}.$$

Using Claim 2.1 once again we can find $n_2 \in \mathbb{N}$ and $1 \leq i_2 \leq m_0$ such that

$$\psi_{n_1+n_2}(\Omega_{i_0}) \subset \psi_{n_2}(\Omega_{i_1}) \subset \Omega_{i_2}.$$

Hence inductively we can find a sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers and a sequence $\{i_k\}_{k\in\mathbb{N}}$ of natural numbers between 1 and m_0 such that

$$\psi_{n_1+\ldots+n_k}(\Omega_{i_0}) \subset \Omega_{i_k} \text{ for every } k \in \mathbb{N}.$$

By the Pigeonhole Principle we can find $k_1 < k_2$ such that $i_{k_1} = i_{k_2}$. This proves the claim.

Let $N_0 = \max\{N_{i,1} : 1 \leq i \leq m_0\}$, where the numbers $N_{i,1}$ are chosen according to the statement in Claim 2.2. We will investigate now the properties of the compact set

$$L := \psi_{N_0}(K).$$

Claim 2.3. *For every $n \geq N$ we have that $\psi_n(L) \cap L \neq \emptyset$.*

Proof of the claim. Since the operator $C_{w,\psi}$ is hypercyclic, by Lemma 3.2.1 we obtain that ψ is an injective function. By (3.2.4) we thus obtain that for every $n \geq N$ we have

$$\emptyset \neq \psi_{N_0}(\psi_n(K) \cap K) = \psi_{N_0}(\psi_n(K)) \cap \psi_{N_0}(K) = \psi_n(\psi_{N_0}(K)) \cap \psi_{N_0}(K) = \psi_n(L) \cap L$$

and this proves the claim.

Claim 2.4. *Let $s = \prod_{i=1}^{m_0}(N_{i,2} - N_{i,1})$, where the numbers $N_{i,1}$ and $N_{i,2}$ are as in Claim 2.2. Let U be a connected component of Ω such that $U \cap L \neq \emptyset$. Then $\psi_s(U) \subset U$. In particular, for every $x \in L$ the points x and $\psi_s(x)$ belong to the same connected component of Ω.*

Proof of the claim. Let $x \in L \cap U$ be arbitrary and let $1 \le i \le m_0$ be such that

$$x \in \psi_{N_0}(K_i).$$

We have that

$$x \in \psi_{N_0}(K_i) = \psi_{N_0 - N_{i,1}}(\psi_{N_{i,1}}(K_i)) \subset \psi_{N_0 - N_{i,1}}(\Omega_{j_i}).$$

Moreover, it is easy to see that

$$\psi_s(\Omega_{j_i}) \subset \Omega_{j_i}.$$

Thus

$$\psi_s(x) \in \psi_s(\psi_{N_0 - N_{i,1}}(\Omega_{j_i})) = \psi_{N_0 - N_{i,1}}(\psi_s(\Omega_{j_i})) \subset \psi_{N_0 - N_{i,1}}(\Omega_{j_i}).$$

This shows that x and $\psi_s(x)$ belong to the same connected component of Ω. Since ψ_s maps connected components of Ω into connected components of Ω, this implies that

$$\psi_s(U) \subset U.$$

Claim 2.5. *Let* $s = \prod_{i=1}^{m_0}(N_{i,2} - N_{i,1})$, *where the numbers* $N_{i,1}$ *and* $N_{i,2}$ *are as in Claim 2.2. For every* $n \ge 1$ *and every* $x \in L$ *we have that*

$$\prod_{i=0}^{2ns-1} w(\psi_i(x)) > 0.$$

Proof of the claim. Since $C_{w,\psi}$ is a hypercyclic operator, by Lemma 3.2.1 we get that $w(x) \neq 0$ for every $x \in \Omega$. This implies that w has constant sign on every connected component of Ω. Let $x \in L$ be arbitrary. From Claim 2.4 we easily get that for every $x \in L$ the points x and $\psi_{ks}(x)$ belong to the same connected component of Ω for every $k \ge 0$. Therefore, for every $x \in L$, for every $k \ge 0$ and every $i \ge 0$ the points $\psi_i(x)$ and $\psi_{ks+i}(x)$ belong to the same connected component of Ω. Thus for every $x \in L$, for every $k \ge 0$ and every $i \ge 0$ we have that

$$\prod_{k=0}^{2n-1} w(\psi_{ks+i}(x)) > 0 \text{ for every } n \ge 1.$$

Hence for every $x \in L$ and every $n \ge 1$ we get that

$$\prod_{i=0}^{2ns-1} w(\psi_i(x)) = \prod_{i=0}^{s-1}\left(\prod_{k=0}^{2n-1} w(\psi_{ks+i}(x))\right) > 0.$$

This ends the proof of the claim.

We will now proceed as in the proof in case 1. By our assumptions $C_{w,\psi}$ is a hypercyclic operator. Let

$$s = \prod_{i=1}^{m_0}(N_{i,2} - N_{i,1}),$$

where the numbers $N_{i,1}$ and $N_{i,2}$ are as in Claim 2.2. Using Ansari's result (see Theorem 1.2.9) we obtain that the operator $C^{2s}_{w,\psi}$ is hypercyclic. Since the set of hypercyclic vectors of $C^{2s}_{w,\psi}$ is dense in $C^{\infty}(\Omega)$ (see Theorem 1.2.4), we can find $h \in C^{\infty}(\Omega)$ which is a hypercyclic vector of $C^{2s}_{w,\psi}$ and which satisfies the inequality

$$\operatorname{Re}(h(x)) > 1 \text{ for every } x \in L. \tag{3.2.8}$$

Because h is a hypercyclic vector, we can find an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that the sequence $\left\{C^{2sn_k}_{w,\psi}(h)\right\}_{k\in\mathbb{N}}$ converges (in the topology of $C^{\infty}(\Omega)$) to the function which is equal to -1 on Ω. In particular, there is k_0 such that

$$\operatorname{Re}\left(\left(C^{2sn_k}_{w\psi}(h)\right)(x)\right) < 0 \text{ for every } x \in L \text{ and every } k \geq k_0. \tag{3.2.9}$$

Let $k_1 > k_0$ be such that $2sn_{k_1} \geq N$. In view of Claim 2.3 we can find $x_0 \in L$ such that $\psi_{2sn_{k_1}}(x_0) \in L$. From (3.2.9) (see also Remark 3.1.3) we obtain that

$$\left(\prod_{i=0}^{2n_{k_1}-1} w(\psi_i(x_0))\right) \cdot \operatorname{Re}\left(h(\psi_{n_{k_1}}(x_0))\right) < 0.$$

Using (3.2.8) we thus get that

$$\prod_{i=0}^{2sn_{k_1}-1} w(\psi_i(x_0)) < 0.$$

But this gives a contradiction with the statement of Claim 2.5. This completes the proof in this case.

□

Please recall that for a complex number z, by $\operatorname{Arg}(z)$ we denote the principal value of the argument of z, i.e., $\operatorname{Arg}(z) \in [0, 2\pi)$. For complex valued weights we must assume more in order to obtain the conclusion of Lemma 3.2.2.

Lemma 3.2.3. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ be smooth. If the weighted composition operator $C_{w,\psi} : C^{\infty}(\Omega) \to C^{\infty}(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, is weakly mixing, then ψ has the run-away property.*

Proof. Assume to the contrary that $C_{w,\psi}$ is a weakly mixing operator and that the function ψ does not have the run-away property. From Lemma 3.2.1 we obtain that w never vanishes. Since ψ does not have the run-away property, there exists a compact set $K \subset \Omega$ such that

$$\psi_n(K) \cap K \neq \emptyset \text{ for every } n \geq 1. \tag{3.2.10}$$

Let

$$U_1 = \left\{f \in C^{\infty}(\Omega) : \left|f(x)\right| > 1 \text{ and } \operatorname{Arg}\left(f(x)\right) \in \left(0, \frac{\pi}{4}\right) \text{ for every } x \in K\right\}$$

and

$$U_2 = \left\{ f \in C^\infty(\Omega) : \left|f(x)\right| > 1 \text{ and } \operatorname{Arg}\left(f(x)\right) \in \left(\pi, \frac{5\pi}{4}\right) \text{ for every } x \in K \right\}.$$

Clearly U_1 and U_2 are both nonempty and open in $C^\infty(\Omega)$. By the definition of a weakly mixing operator we can find $n \in \mathbb{N}$ such that

$$C^n_{w,\psi}(U_1) \cap U_1 \neq \emptyset \text{ and } C^n_{w,\psi}(U_1) \cap U_2 \neq \emptyset.$$

From (3.2.10) we can find $x_0 \in K$ such that $\psi_n(x_0) \in K$. Let $f_1 \in U_1$ be such that

$$C^n_{w,\psi}(f) \in U_1.$$

Since $\operatorname{Arg}(f(x)) \in (0, \pi/4)$ for every $x \in K$ and $\psi_n(x_0) \in K$, we have that

$$\operatorname{Arg}(f(\psi_n(x_0))) \in \left(0, \frac{\pi}{4}\right).$$

Since $x_0 \in K$, $\psi_n(x_0) \in K$, and $C^n_{w,\psi}(f) \in U_1$ we have that $f(\psi_n(x_0)) \neq 0$ and

$$\operatorname{Arg}\left(\left(C^n_{w,\psi}(f)\right)(x_0)\right) = \operatorname{Arg}\left(\left(\prod_{i=0}^{n-1} w(\psi_i(x_0))\right) \cdot f(\psi_n(x_0))\right) \in \left(0, \frac{\pi}{4}\right).$$

The above facts imply that

$$\operatorname{Arg}\left(\prod_{i=0}^{n-1} w(\psi_i(x_0))\right) \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right). \tag{3.2.11}$$

Let now $g \in U_1$ be such that

$$C^n_{w,\psi}(g) \in U_2.$$

Since $\operatorname{Arg}(g(x)) \in (0, \pi/4)$ for every $x \in K$ and $\psi_n(x_0) \in K$, we have that

$$\operatorname{Arg}(g(\psi_n(x_0))) \in (0, \pi/4).$$

Since $x_0 \in K$, $\psi_n(x_0) \in K$, and $C^n_{w,\psi}(g) \in U_2$ we have that $g(\psi_n(x_0)) \neq 0$ and

$$\operatorname{Arg}\left(\left(C^n_{w,\psi}(g)\right)(x_0)\right) = \operatorname{Arg}\left(\left(\prod_{i=0}^{n-1} w(\psi_i(x_0))\right) \cdot g(\psi_n(x_0))\right) \in \left(\pi, \frac{5\pi}{4}\right).$$

Connecting this two facts together we get that

$$\operatorname{Arg}\left(\prod_{i=0}^{n-1} w(\psi_i(x_0))\right) \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right). \tag{3.2.12}$$

This gives a contradiction since (3.2.11) and (3.2.12) cannot hold at the same time. This proves the lemma.

□

The proof of the next lemma is very similar to the previous one. We include it for the sake of completeness.

Lemma 3.2.4. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ be smooth. If the weighted composition operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, is mixing, then ψ has the strong run-away property.*

Proof. Assume to the contrary that $C_{w,\psi}$ is a mixing operator and that the function ψ does not have the strong run-away property. From Lemma 3.2.1 we obtain that w never vanishes. Since ψ does not have the strong run-away property, there exists a compact set $K \subset \Omega$ and an increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ of naturals numbers such that

$$\psi_{n_k}(K) \cap K \neq \emptyset \text{ for every } k \geq 1. \tag{3.2.13}$$

Let

$$U_1 = \left\{ f \in C^\infty(\Omega) : \left|f(x)\right| > 1 \text{ and } \operatorname{Arg}\big(f(x)\big) \in \left(0, \frac{\pi}{4}\right) \text{ for every } x \in K \right\}$$

and

$$U_2 = \left\{ f \in C^\infty(\Omega) : \left|f(x)\right| > 1 \text{ and } \operatorname{Arg}\big(f(x)\big) \in \left(\pi, \frac{5\pi}{4}\right) \text{ for every } x \in K \right\}.$$

Clearly U_1 and U_2 are both nonempty and open in $C^\infty(\Omega)$. From the definition of a mixing operator we can find N large enough such that for every $n \geq N$ we have

$$C^n_{w,\psi}(U_1) \cap U_1 \neq \emptyset \text{ and } C^n_{w,\psi}(U_1) \cap U_2 \neq \emptyset.$$

Choose $k \in \mathbb{N}$ such that $n_k \geq N$. We can find $x_0 \in K$ with $\psi_{n_k}(x_0) \in K$. Pick $f \in U_1$ such that

$$C^{n_k}_{w,\psi}(f) \in U_1.$$

Since $\operatorname{Arg}(f(x)) \in (0, \pi/4)$ for every $x \in K$ and $\psi_{n_k}(x_0) \in K$, we have that

$$\operatorname{Arg}(f(\psi_{n_k}(x_0))) \in \left(0, \frac{\pi}{4}\right).$$

Since $x_0 \in K$, $\psi_{n_k}(x_0) \in K$, and $C^{n_k}_{w,\psi}(f) \in U_1$ we have that $f(\psi_{n_k}(x_0)) \neq 0$ and

$$\operatorname{Arg}\left(\left(C^{n_k}_{w,\psi}(f)\right)(x_0)\right) = \operatorname{Arg}\left(\left(\prod_{i=0}^{n_k-1} w(\psi_i(x_0))\right) \cdot f(\psi_{n_k}(x_0))\right) \in \left(0, \frac{\pi}{4}\right).$$

Connecting this two facts together we obtain that

$$\operatorname{Arg}\left(\prod_{i=0}^{n_k-1} w(\psi_i(x_0))\right) \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right). \tag{3.2.14}$$

Pick now $g \in U_1$ such that

$$C^{n_k}_{w,\psi}(g) \in U_2.$$

Since $\mathrm{Arg}(g(x)) \in (0, \pi/4)$ for every $x \in K$ and $\psi_{n_k}(x_0) \in K$, we have that

$$\mathrm{Arg}(g(\psi_{n_k}(x_0))) \in \left(0, \frac{\pi}{4}\right).$$

Since $x_0 \in K$, $\psi_{n_k}(x_0) \in K$, and $C_{w,\psi}^{n_k}(g) \in U_2$ we have that $g(\psi_{n_k}(x_0) \neq 0$ and

$$\mathrm{Arg}\left(\left(C_{w,\psi}^{n_k}(g)\right)(x_0)\right) = \mathrm{Arg}\left(\left(\prod_{i=0}^{n_k-1} w(\psi_i(x_0))\right) \cdot g(\psi_{n_k}(x_0))\right) \in \left(\pi, \frac{5\pi}{4}\right).$$

Those two facts give that

$$\mathrm{Arg}\left(\prod_{i=0}^{n_k-1} w(\psi_i(x_0))\right) \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right). \tag{3.2.15}$$

This gives a contradiction since (3.2.14) and (3.2.15) cannot hold at the same time. This completes the proof.

□

The following technical lemma will be crucial in proving that under certain conditions on ψ and w the operator $C_{w,\psi}$ is hypercyclic, weakly mixing, or mixing.

Lemma 3.2.5. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ be smooth functions satisfying the following conditions:*

1. *For every $x \in \Omega$ we have that $w(x) \neq 0$.*
2. *The function ψ is injective.*
3. *For every $x \in \Omega$ we have that $\det[\psi'(x)] \neq 0$.*

Assume that K is a compact subset of Ω and $n_0 \in \mathbb{N}$ is such that $\psi_{n_0}(K) \cap K = \emptyset$. Let $l \in \mathbb{N}$, $\varepsilon > 0$ and $f, g \in C^\infty(\Omega)$ be arbitrary. Then

$$C_{w,\psi}^{n_0}\left(\left\{h \in C^\infty(\Omega) : ||h - f||_{K,l} < \varepsilon\right\}\right) \cap \left\{h \in C^\infty(\Omega) : ||h - g||_{K,l} < \varepsilon\right\} \neq \emptyset.$$

Proof. Our goal is to find a function $h \in C^\infty(\Omega)$ such that

$$||h - f||_{K,l} < \varepsilon \text{ and } ||C_{w,\psi}^{n_0}(h) - g||_{K,l} < \varepsilon.$$

In order to do this let us first observe that the set $\psi_{n_0}(K)$ is contained in the image of the function ψ_i for every $i = 1, \dots, n_0$. Using our assumptions and the Inverse Function Theorem we obtain that there exists an open subset W of Ω, containing the set $\psi_{n_0}(K)$, and such that the functions ψ_{-i} are well defined and smooth on W, for $i = 1, \dots, n_0$.

We know that the compact sets K and $\psi_{n_0}(K)$ are disjoint, and thus we can find disjoint open subsets U_1 and W' of Ω such that $K \subset U_1$ and $\psi_{n_0}(K) \subset W'$. Let $V_1 = W \cap W'$. We choose open subsets U_2 and V_2 of Ω such that

$$K \subset U_2 \subset \overline{U_2} \subset U_1 \text{ and } \psi_{n_0}(K) \subset V_2 \subset \overline{V_2} \subset V_1.$$

Let $\varphi_1 : \Omega \to \mathbb{R}$ be a smooth function such that $\varphi_1 \equiv 1$ on U_2 and $\varphi_1 \equiv 0$ on $\Omega \setminus U_1$. Moreover, let $\varphi_2 : \Omega \to \mathbb{R}$ be a smooth function such that $\varphi_2 \equiv 1$ on V_2 and $\varphi_2 \equiv 0$ on $\Omega \setminus V_1$. Such functions exists by Proposition 1.1.7.

Let us now consider the function h defined via formula

$$h(x) = \begin{cases} \varphi_1(x) f(x), & \text{for } x \in U_1, \\ \varphi_2(x) \cdot \frac{g(\psi_{-n_0}(x))}{\prod_{i=1}^{n_0} w(\psi_{-i}(x))}, & \text{for } x \in V_1, \\ 0, & \text{for } x \in \Omega \setminus (U_1 \cup V_1). \end{cases}$$

It is easy to verify that $h \in C^\infty(\Omega)$. Since $\varphi_1 \equiv 1$ on U_2, we obtain that the functions h and f agree on the open set U_2, which contains K. This implies that

$$||h - f||_{K,l} = 0. \tag{3.2.16}$$

For every $x \in \psi_{-n_0}(V_2)$, the fact that $\varphi_2 \equiv 1$ on V_2 implies the following equality

$$\begin{aligned} \left(C_{w,\psi}^{n_0}(h)\right)(x) &= \left(\prod_{i=0}^{n_0-1} w(\psi_i(x))\right) \cdot h(\psi_{n_0}(x)) \\ &= \left(\prod_{i=0}^{n_0-1} w(\psi_i(x))\right) \cdot \frac{g(\psi_{-n_0}(\psi_{n_0}(x)))}{\prod_{i=1}^{n_0} w(\psi_{-i}(\psi_{n_0}(x)))} \\ &= \left(\prod_{i=0}^{n_0-1} w(\psi_i(x))\right) \cdot \frac{g(x)}{\prod_{i=0}^{n_0-1} w(\psi_i(x))} \\ &= g(x). \end{aligned}$$

Thus the functions g and $C_{w,\psi}^{n_0}(h)$ agree on the open set $\psi_{-n_0}(V_2)$, which contains K. This implies that

$$||C_{w,\psi}^{n_0}(h) - g||_{K,l} = 0. \tag{3.2.17}$$

This completes the proof.

□

We are now ready to prove the main theorems of this section.

Theorem 3.2.6. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ be smooth. The weighted composition operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, is weakly mixing if and only if the following conditions are satisfied:*

1. *For every $x \in \Omega$ we have that $w(x) \neq 0$.*
2. *The function ψ is injective.*
3. *For every $x \in \Omega$ we have that* $\det[\psi'(x)] \neq 0$.

4. *The function ψ has the run-away property.*

Proof. Let us assume that the operator $C_{w,\psi}$ is weakly mixing. By Lemma 3.2.3 we have that ψ has the run-away property. Since every weakly mixing operator is hypercyclic, we obtain by Lemma 3.2.1 that the first three conditions in the theorem are satisfied. This proves the necessity part of the theorem.

Let us now assume that w and ψ satisfy the conditions $1-4$ of the theorem. In view of Theorem 1.2.10 we have to prove the following: for every two nonempty open sets U and V in $C^\infty(\Omega)$ and every $k \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that

$$C_{w,\psi}^{n+i}(U) \cap V \neq \emptyset \text{ for } i = 1, \dots, k.$$

So let us take U and V to be two nonempty open sets in $C^\infty(\Omega)$ and let $k \in \mathbb{N}$. From the description of the topology of $C^\infty(\Omega)$ we can find a compact set $K \subset \Omega$, $l \in \mathbb{N}$, $\varepsilon > 0$ and $f, g \in C^\infty(\Omega)$ such that

$$\left\{h \in C^\infty(\Omega) : ||h - f||_{K,l} < \varepsilon\right\} \subset U \tag{3.2.18}$$

and

$$\left\{h \in C^\infty(\Omega) : ||h - g||_{K,l} < \varepsilon\right\} \subset V. \tag{3.2.19}$$

Let

$$L := \bigcup_{i=0}^{k} \psi_i(K).$$

It is clear that L is a compact set. Since ψ has the run-away property, we can find $n_0 \in \mathbb{N}$ such that

$$\psi_{n_0}(L) \cap L = \emptyset.$$

This gives that for every $1 \leq i \leq k$ we have

$$\psi_{n_0+i}(K) \cap K = \psi_{n_0}(\psi_i(K)) \cap K \subset \psi_{n_0}(L) \cap L = \emptyset.$$

Thus, for every $1 \leq i \leq k$, from Lemma 3.2.5 we get that

$$C_{w,\psi}^{n_0+i}\left(\left\{h \in C^\infty(\Omega) : ||h - f||_{K,l} < \varepsilon\right\}\right) \cap \left\{h \in C^\infty(\Omega) : ||h - g||_{K,l} < \varepsilon\right\} \neq \emptyset.$$

This proves the theorem.

□

In the real weight case we obtain the following corollary.

Corollary 3.2.7. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{R}$ be smooth. The following conditions are equivalent:*

(1) The operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, is weakly mixing.

(2) The operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, is hypercyclic.

(3) The following conditions are satisfied:

- *For every $x \in \Omega$ we have $w(x) \neq 0$.*
- *The function ψ is injective.*
- *For every $x \in \Omega$ we have $\det[\psi'(x)] \neq 0$.*
- *The function ψ has the run-away property.*

Proof. It is obvious that (1) implies (2). From Lemma 3.2.1 and Lemma 3.2.2 we obtain that (2) implies (3). Finally, Theorem 3.2.6 shows that (3) implies (1).

□

Remark 3.2.8. 1. The equivalence of hypercyclicity and the weak mixing property of the weighted composition operator $C_{w,\psi}$, where w is a real valued weight, is not surprising. Indeed, if w is a real valued weight, then the operator $C_{w,\psi}$ acting on the complex space $C^\infty(\Omega)$ is the complexification of the corresponding operator $C_{w,\psi}$ acting on the real space $C^\infty(\Omega)$. By Proposition 1.2.11, such a complexification is hypercyclic if and only if it is weakly mixing.
2. We do not know if Corollary 3.2.7 is true when the weights are allowed to be complex. To obtain an analogous result for complex weights, one would have to prove Lemma 3.2.2 for complex weights. Unfortunately, this is out of our range now.
3. Please note that the condition (3) in Corollary 3.2.7 does not depend on the field (real or complex).

Theorem 3.2.9. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$, $w : \Omega \to \mathbb{C}$ be smooth. The weighted composition operator $C_{w,\psi} : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$, is mixing if and only if the following conditions are satisfied:*

1. *For every $x \in \Omega$ we have $w(x) \neq 0$.*
2. *The function ψ is injective.*
3. *For every $x \in \Omega$ we have $\det[\psi'(x)] \neq 0$.*
4. *The function ψ has the strong run-away property.*

Proof. Assume first that $C_{w,\psi}$ is mixing. Since every mixing operator is also hypercyclic, from Lemma 3.2.1, we obtain that the first three conditions in the theorem are satisfied. Lemma 3.2.4 tells us that the fourth conditions is also true.

Let us now assume that the conditions in the theorem are satisfied. We have to show that the operator $C_{w,\psi}$ is mixing. So let U, V be two nonempty open subsets of $C^\infty(\Omega)$. Our goal is to find $N \in \mathbb{N}$ such that

$$C^n_{w,\psi}(U) \cap V \neq \emptyset \text{ for every } n \geq N. \tag{3.2.20}$$

Let $K \subset \Omega$ be compact, $l \in \mathbb{N}$, $\varepsilon > 0$, and $f, g \in C^\infty(\Omega)$ be such that

$$\left\{h \in C^\infty(\Omega) : ||h - f||_{K,l} < \varepsilon\right\} \subset U \tag{3.2.21}$$

and

$$\left\{h \in C^{\infty}(\Omega) : ||h - g||_{K,l} < \varepsilon\right\} \subset V. \tag{3.2.22}$$

Since the function ψ has the strong run-away property, we can find $N \in \mathbb{N}$ such that

$$\psi_n(K) \cap K = \emptyset \text{ for every } n \geq N.$$

Using Lemma 3.2.5, we obtain that for every $n \geq N$ we have

$$C^n_{w,\psi}\left(\left\{h \in C^{\infty}(\Omega) : ||h - f||_{K,l} < \varepsilon\right\}\right) \cap \left\{h \in C^{\infty}(\Omega) : ||h - g||_{K,l} < \varepsilon\right\} \neq \emptyset.$$

Connecting this with (3.2.21) and (3.2.22), we get that (3.2.20) is true. This completes the proof.

□

Remark 3.2.10. Please note that every mixing operator is weakly mixing and hypercyclic. We do not know if every every hypercyclic (weakly mixing) weighted composition operator is already mixing. In order to find a counterexample to this statement, in view of the above results, one would have to find a smooth and injective function $\psi : \Omega \to \Omega$, such that $\det[\psi'(x)] \neq 0$ for every $x \in \Omega$, which has the run-away property but does not have the strong run-away property (comp. Remark 3.1.5).

3.3 Hypercyclic, weakly mixing, and mixing composition operators

In the previous section we obtained characterizations of hypercyclic, weakly mixing and mixing weighted composition operators $C_{w,\psi} : C^{\infty}(\Omega) \to C^{\infty}(\Omega)$, $F \mapsto w \cdot (F \circ \psi)$ (see Theorem 3.2.6, Corollary 3.2.7, and Theorem 3.2.9). As an immediate consequence, taking $w : \Omega \to \mathbb{R}$ to be equal 1 on the whole set Ω, we obtain the following characterizations of hypercyclic, weakly mixing, and mixing composition operators $C_{\psi} : C^{\infty}(\Omega) \to C^{\infty}(\Omega)$, $F \mapsto F \circ \psi$.

Theorem 3.3.1. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$ be smooth. The following conditions are equivalent:*

(1) The operator $C_{\psi} : C^{\infty}(\Omega) \to C^{\infty}(\Omega)$, $F \mapsto F \circ \psi$, is hypercyclic.

(2) The operator $C_{\psi} : C^{\infty}(\Omega) \to C^{\infty}(\Omega)$, $F \mapsto F \circ \psi$, is weakly mixing.

(3) The following conditions hold:

- *The function ψ is injective.*
- *For every $x \in \Omega$ we have $\det[\psi'(x)] \neq 0$.*
- *The function ψ has the run-away property.*

Theorem 3.3.2. *Let $\Omega \subset \mathbb{R}^d$ be open and let $\psi : \Omega \to \Omega$ be smooth. The composition operator $C_\psi : C^\infty(\Omega) \to C^\infty(\Omega)$, $F \mapsto F \circ \psi$, is mixing if and only if the following conditions are satisfied:*

1. *The function ψ is injective.*
2. *For every $x \in \Omega$ we have $\det[\psi'(x)] \neq 0$.*
3. *The function ψ has the strong run-away property.*

3.4 Dynamics of weighted composition operators in the one-dimensional case

The goal of this section is to prove Theorem 3.4.4 which shows that various dynamical properties of the weighted composition operator $C_{w,\psi} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto w \cdot (F \circ \psi)$, are equivalent. We start with the following simple and well-known lemma. Please recall, that if $\psi : \mathbb{R} \to \mathbb{R}$, then every $x \in \mathbb{R}$ such that $\psi(x) = x$ is called a *fixed point* of ψ.

Lemma 3.4.1. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be smooth and injective. If ψ has no fixed points, then ψ is increasing.*

Proof. The condition that ψ has no fixed points implies that either $\psi(x) > x$ for every $x \in \mathbb{R}$ or $\psi(x) < x$ for every $x \in \mathbb{R}$. Assume first that

$$\psi(x) > x \text{ for every } x \in \mathbb{R}. \tag{3.4.1}$$

If ψ is not increasing, then the injectivity of ψ implies that ψ is decreasing. Since $\psi(0) > 0$, this gives that

$$\psi(\psi(0)) < \psi(0).$$

But his is impossible in view of (3.4.1).

Assume now that

$$\psi(x) < x \text{ for every } x \in \mathbb{R}. \tag{3.4.2}$$

As in the previous case, assume that ψ is not increasing. Since ψ is injective, this gives that ψ is decreasing. Because $\psi(0) < 0$, this implies that

$$\psi(\psi(0)) > \psi(0)$$

and this gives a contradiction with (3.4.2).

□

In the proof of the main theorem of this section we will need the following simple observation.

Lemma 3.4.2. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be smooth and injective. The following conditions are equivalent:*

(1) The function ψ has no fixed points.

(2) The function ψ has the run-away property.

(3) The function ψ has the strong run-away property.

Proof.
(1) $\Rightarrow$ (3) From the assumption that ψ has no fixed points we obtain that either $\psi(x) > x$ for every $x \in \mathbb{R}$ or $\psi(x) < x$ for every $x \in \mathbb{R}$. Moreover, from Lemma 3.4.1 we obtain that ψ is an increasing function. Thus

$$\psi([c,d]) = [\psi(c), \psi(d)] \text{ for every } c < d.$$

Let $K \subset \mathbb{R}$ be an arbitrary compact set. Our aim is to find $N \in \mathbb{N}$ such that $\psi_n(K) \cap K = \emptyset$ for every $n \geq N$. Enlarging K if necessary, we may assume that $K = [a,b]$.

Case 1. Assume that $\psi(x) > x$ for every $x \in \mathbb{R}$. For every $n \geq 0$ we have that

$$\psi_n(a) < \psi_{n+1}(a).$$

This shows that the sequence $\{\psi_n(a)\}_{n\in\mathbb{N}}$ is increasing. We claim that this sequence is unbounded. If not, then there exists $a_0 \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \psi_n(a) = a_0.$$

Using the continuity of ψ we get

$$a_0 = \lim_{n\to\infty} \psi_n(a) = \lim_{n\to\infty} \psi_{n+1}(a) = \lim_{n\to\infty} \psi(\psi_n(a)) = \psi(\lim_{n\to\infty} \psi_n(a)) = \psi(a_0),$$

which is impossible because from our assumptions ψ has no fixed points. Thus we have shown that the sequence $\{\psi_n(a)\}_{n\in\mathbb{N}}$ is increasing and unbounded. Therefore, there is $N \in \mathbb{N}$ such that $\psi_n(a) > b$ for every $n \geq N$. Thus for every $n \geq N$ we have

$$\psi_n(K) \cap K = \psi_n([a,b]) \cap [a,b] = [\psi_n(a), \psi_n(b)] \cap [a,b] = \emptyset.$$

This finishes the proof in this case.

Case 2. Assume now that $\psi(x) < x$ for every $x \in \mathbb{R}$. It is easy to see that this implies that the sequence $\{\psi_n(b)\}_{n\in\mathbb{N}}$ is decreasing. Assume for a moment that this sequence is bounded. Then there exists $b_0 \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \psi_n(b) = b_0.$$

The continuity of ψ implies that

$$b_0 = \lim_{n\to\infty} \psi_n(b) = \lim_{n\to\infty} \psi_{n+1}(b) = \lim_{n\to\infty} \psi(\psi_n(b)) = \psi(\lim_{n\to\infty} \psi_n(b)) = \psi(b_0).$$

But this is impossible because ψ has no fixed points. Therefore the decreasing sequence $\{\psi_n(b)\}_{n\in\mathbb{N}}$ is unbounded. Thus we can find $N \in \mathbb{N}$ such that $\psi_n(b) < a$ for every $n \geq N$. For every $n \geq N$ we have

$$\psi_n(K) \cap K = \psi_n([a,b]) \cap [a,b] = [\psi_n(a), \psi_n(b)] \cap [a,b] = \emptyset.$$

This finishes the proof of this part of the lemma.
(3) ⇒ (2) This implication is obvious.
(2) ⇒ (1) Assume to the contrary that ψ has a fixed point x_0, i.e., $\psi(x_0) = x_0$. For every $n \in \mathbb{N}$ we have

$$\psi_n(\{x_0\}) \cap \{x_0\} = \{x_0\},$$

and hence ψ cannot have the run-away property.

□

Lemma 3.4.3. *Let $\psi : \mathbb{R} \to \mathbb{R}$, $w : \mathbb{R} \to \mathbb{C}$ be smooth. If the weighted composition operator $C_{w,\psi} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto w \cdot (F \circ \psi)$ is hypercyclic, then ψ has no fixed points.*

Proof. Assume to the contrary that $C_{w,\psi}$ is hypercyclic and that there exists $x_0 \in \mathbb{R}$ such that $\psi(x_0) = x_0$. Let h be a hypercyclic vector of $C_{w,\psi}$ such that $h(x_0) \neq 0$. For every $n \geq 1$ we have that

$$\left(C^n_{w,\psi}(h)\right)(x_0) = (w(x_0))^n\, h(x_0).$$

Since h is a hypercyclic vector of $C_{w,\psi}$, the set

$$\left\{\left(C^n_{w,\psi}(h)\right)(x_0) : n \in \mathbb{N}\right\}$$

has to be dense in the real or complex field. But this is impossible because if $|w(x_0)| \leq 1$, then the set $\left\{\left(C^n_{w,\psi}(h)\right)(x_0) : n \in \mathbb{N}\right\}$ is bounded, and if $|w(x_0))| > 1$, then

$$\lim_{n\to\infty} \left|\left(C^n_{w,\psi}(h)\right)(x_0) : n \in \mathbb{N}\right| = \infty$$

This completes the proof of the lemma.

□

Theorem 3.4.4. *Let $\psi : \mathbb{R} \to \mathbb{R}$, $w : \mathbb{R} \to \mathbb{C}$ be smooth and assume that $w(x) \neq 0$ for every $x \in \mathbb{R}$. The following conditions are equivalent:*

(1) The operator $C_{w,\psi} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto w \cdot (F \circ \psi)$ is hypercyclic.

(2) The operator $C_{w,\psi} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto w \cdot (F \circ \psi)$ is weakly mixing.

(3) The operator $C_{w,\psi} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto w \cdot (F \circ \psi)$ is mixing.

(4) The operator $C_{w,\psi} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto w \cdot (F \circ \psi)$ is chaotic.

(5) For all $x \in \mathbb{R}$ we have $\psi'(x) \neq 0$ and ψ has the run-away property

(6) For all $x \in \mathbb{R}$ we have $\psi'(x) \neq 0$ and ψ has the strong run-away property

(7) For all $x \in \mathbb{R}$ we have $\psi'(x) \neq 0$ and ψ has no fixed points.

Proof. It is clear that (3) implies (2) and that (2) implies (1). It is also clear that (4) implies (1). From Lemma 3.2.1 and Lemma 3.4.3 we obtain that (1) implies (7). From Lemma 3.4.2 we get that (7), (6), and (5) are equivalent. By Theorem 3.2.6, (6) implies (3). To complete the proof of the theorem it is enough to show that (5) implies (4).

So let us assume that (5) holds. From Theorem 3.2.6 we obtain that the operator $C_{w,\psi}$ is weakly mixing, in particular that it is hypercyclic. In order to complete the proof we have to show that the set of periodic points of $C_{w,\psi}$ is dense in $C^\infty(\mathbb{R})$. Thus for every $f \in C^\infty(\mathbb{R})$, for every compact set $K \subset \mathbb{R}$, for every $n \in \mathbb{N}$ and every $\varepsilon > 0$ we have to find $p \in C^\infty(\mathbb{R})$ which is a periodic point of $C_{w,\psi}$ and satisfies $||f - p||_{K,n} < \varepsilon$.

Let us fix $f \in C^\infty(\mathbb{R})$, a compact set $K \subset \mathbb{R}^d$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Enlarging K if necessary, we may assume that $K = [a, b]$ for some $a < b$. Let

$$L := [a-3, b+3].$$

Moreover, let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function which is equal to 1 on $[a-1, b+1]$ and equal to 0 on $\mathbb{R}\setminus[a-2, b+2]$. Such a function exists in view of Proposition 1.1.7. Consider now the function $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ defined via formula

$$\widetilde{f}(x) = \varphi(x) \cdot f(x).$$

Clearly $\widetilde{f}$ is smooth and, moreover,

$$||f - \widetilde{f}||_{K,n} = 0. \tag{3.4.3}$$

By our assumptions ψ is an injective function and has the run-away property, hence by Lemma 3.4.2 it has the strong run-away property. Thus we can find $N \in \mathbb{N}$, such that

$$\psi_n(L) \cap L = \emptyset \text{ for every } n \geq N. \tag{3.4.4}$$

For every integer $n \geq 0$ let

$$L_n := \psi_n(L),$$

and for every integer $n < 0$ let

$$L_n := \psi_n(L \cap \psi_{-n}(\mathbb{R})).$$

Claim 1. *Let $k < l$ be two integers. Then*

$$L_{kN} \cap L_{lN} = \emptyset.$$

Proof of the claim. Assume that this is not the case, i.e., assume that there exists

$$x \in L_{kN} \cap L_{lN} \neq \emptyset.$$

This means that we can find $l_1, l_2 \in L$ such that the function ψ_{kN} is well-defined at l_1, the function ψ_{lN} is well-defined at l_2 and

$$\psi_{kN}(l_1) = \psi_{lN}(l_2).$$

Since by our assumptions ψ is an injective function, this gives that

$$l_1 = \psi_{(l-k)N}(l_2).$$

Therefore

$$\psi_{(l-k)N}(L) \cap L \neq \emptyset.$$

But this is impossible in view of (3.4.4). This finishes the proof of the claim.

Consider now the function $p : \mathbb{R} \to \mathbb{R}$ defined via formula

$$p(x) = \begin{cases} \widetilde{f}(\psi_{-nN}(x)) \cdot \left(\prod_{k=0}^{n-1}\left(\prod_{i=1}^{N} w(\psi_{-kN-i}(x))\right)\right)^{-1}, & x \in L_{nN}, \text{ where } n \geq 0, \\ \left(C_{w,\psi}^{-nN}(\widetilde{f})\right)(x), & x \in L_{nN}, \text{ where } n < 0, \\ 0, & x \in \mathbb{R} \setminus \left(\bigcup_{n\in\mathbb{Z}} L_{nN}\right). \end{cases} \tag{3.4.5}$$

Please note that $\prod_{k=0}^{-1} = 1$ by definition.

From Claim 1 and from the definition of the sets L_n, we easily observe that the function p is well-defined. Please note that ψ_{nN} is a diffeomorphism from L onto L_{nN} for every $n \geq 0$. Thus for every $n \geq 0$ the function $\widetilde{f} \circ \psi_{-nN}$ has a compact support contained in the interior of L_{nN}. In case $n < 0$, the set L_{nN} is either a compact interval, or a closed halfline, or an empty set. In the first case, the function $C_{w,\psi}^{-nN}(\widetilde{f})$ has a compact support contained in the interior of L_{nN}. In the second case, the function ψ_{-nN} maps the halfline onto the interval $(c, b+3]$ for $c > a-3$, or onto $[a-3, c)$ for $c < b+3$. In both cases, the function $C_{w,\psi}^{-nN}(\widetilde{f})$ has support containing in the interior of the halfline. Summarizing, the sets L_{nN} are closed and disjoint (possibly unbounded) intervals and the function p as defined has support contained in the interiors of those intervals. Since ψ has the run-away property (and hence also the strong run-away property), those intervals cannot accumulate at a point. This shows that the function p is smooth.

Let us now observe that the functions p and $\widetilde{f}$ are equal on the compact set L. Since the set K is contained in the interior of L, this gives that

$$||\widetilde{f} - p||_{K,n} = 0.$$

Thus from (3.4.3) we have that

$$||f - p||_{K,n} = ||\widetilde{f} - p||_{K,n} = 0.$$

We will show now that p is a periodic point of the operator $C_{w,\psi}$ and this will finish the proof of the theorem.

Claim 3. *We have that $C_{w,\psi}^N(p) = p$, i.e., p is a periodic point of $C_{w,\psi}$.*

Proof of the claim. We have to show that

$$\left(C_{w,\psi}^N(p)\right)(x) = p(x) \text{ for every } x \in \mathbb{R}.$$

There are several possible cases.
Case 1. Assume that

$$x \in \mathbb{R} \setminus \left(\bigcup_{n\in\mathbb{Z}} L_{nN} \right).$$

Then it is easy to verify that

$$\psi_N(x) \in \mathbb{R} \setminus \left(\bigcup_{n\in\mathbb{Z}} L_{nN} \right).$$

Therefore

$$\left(C_{w,\psi}^N(p)\right)(x) = \left(\prod_{i=0}^{N-1} w(\psi_i(x)) \right) \cdot p(\psi_N(x)) = 0 = p(x).$$

Case 2. Assume that $x \in L_{nN}$, where $n < -1$. Then it is clear that $\psi_N(x) \in L_{(n+1)N}$ and, therefore,

$$\begin{aligned}
\left(C_{w,\psi}^N(p)\right)(x) &= \left(\prod_{i=0}^{N-1} w(\psi_i(x)) \right) \cdot p(\psi_N(x)) \\
&= \left(\prod_{i=0}^{N-1} w(\psi_i(x)) \right) \cdot \left(C_{w,\psi}^{(-n-1)N}(\widetilde{f}) \right)(\psi_N(x)) \\
&= \left(C_{w,\psi}^N \left(C_{w,\psi}^{(-n-1)N}(\widetilde{f}) \right) \right)(x) \\
&= \left(C_{w,\psi}^{-nN}(\widetilde{f}) \right)(x) \\
&= p(x).
\end{aligned}$$

Case 3. Assume that $x \in L_{-N}$. Then $\psi_N(x) \in L$ and this gives that

$$\left(C_{w,\psi}^N(p)\right)(x) = \left(\prod_{i=0}^{N-1} w(\psi_i(x)) \right) \cdot p(\psi_N(x)) = \left(\prod_{i=0}^{N-1} w(\psi_i(x)) \right) \cdot \widetilde{f}(\psi_N(x)) = \left(C_{w,\psi}^N(\widetilde{f}) \right)(x) = p(x).$$

Case 4. Assume that $x \in L_{nN}$, where $n \geq 0$. Then $\psi_N(x) \in L_{(n+1)N}$. Therefore

$$\begin{aligned}
\left(C^N_{w,\psi}(p)\right)(x) &= \left(\prod_{i=0}^{N-1} w(\psi_i(x))\right) \cdot p(\psi_N(x)) \\
&= \left(\prod_{i=0}^{N-1} w(\psi_i(x))\right) \cdot \widetilde{f}(\psi_{(-n-1)N}(\psi_N(x))) \cdot \left(\prod_{k=0}^{n}\left(\prod_{i=1}^{N} w(\psi_{-kN-i}(\psi_N(x)))\right)\right)^{-1} \\
&= \widetilde{f}(\psi_{-nN}(x)) \cdot \frac{\prod_{i=0}^{N-1} w(\psi_i(x))}{\prod_{i=1}^{N} w(\psi_{-i}(\psi_N(x)))} \cdot \left(\prod_{k=1}^{n}\left(\prod_{i=1}^{N} w(\psi_{-kN-i}(\psi_N(x)))\right)\right)^{-1} \\
&= \widetilde{f}(\psi_{-nN}(x)) \cdot \frac{\prod_{i=0}^{N-1} w(\psi_i(x))}{\prod_{i=0}^{N-1} w(\psi_i(x))} \cdot \left(\prod_{k=1}^{n}\left(\prod_{i=1}^{N} w(\psi_{(-k+1)N-i}(x))\right)\right)^{-1} \\
&= \widetilde{f}(\psi_{-nN}(x)) \cdot \left(\prod_{k=0}^{n-1}\left(\prod_{i=1}^{N} w(\psi_{-kN-i}(x))\right)\right)^{-1} \\
&= p(x).
\end{aligned}$$

The above are the only possible cases. This ends the proof of the claim and completes the proof of the theorem.

□

3.5 Examples

In the following examples we will try to illustrate the results of this chapter.

Example 3.5.1. *Let $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be defined via formula*

$$\psi(x_1, \ldots, x_d) = (x_1 + v_1, \ldots, x_d + v_d),$$

where $(v_1, \ldots, v_d)$ in a non-zero vector in $\mathbb{R}^d$. Clearly ψ is a smooth and injective function. Moreover, for every $x \in \mathbb{R}^d$ we easily calculate that $\det[\psi'(x)] = 1$. *It is also clear that the function ψ has the strong run-away property. Thus for an arbitrary weight $w : \mathbb{R}^d \to \mathbb{C}$ which does not vanish on $\mathbb{R}^d$, we obtain from Theorem 3.2.9 that the operator $C_{w,\psi}$ is mixing. In particular, the operator C_ψ is mixing.*

Example 3.5.2. *Let $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be defined via formula*

$$\psi(x_1, \ldots, x_d) = (\psi_1(x_1), \ldots, \psi_d(x_d)),$$

where for every $1 \leq i \leq d$ the function $\psi_i : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies $\psi_i'(x) \neq 0$ for all $x \in \mathbb{R}$. Of course ψ is a smooth and injective function and for every $x \in \mathbb{R}^d$ we have that $\det[\psi'(x)] \neq 0$. *One can easily check that the function ψ has the strong run-away property if and only if it has* *the run-away property, which happens if and only if there is* $1 \le i \le d$ *such that the function* ψ_i *has no fixed points.*

Thus, if there is $1 \le i \le d$ *such that the function* ψ_i *has no fixed points, then for an arbitrary weight* $w : \mathbb{R}^d \to \mathbb{C}$ *which does not vanish on* $\mathbb{R}^d$*, we obtain from Theorem 3.2.9 that the operator* $C_{w,\psi}$ *is mixing.*

Example 3.5.3. *Let* $\psi : \mathbb{R} \to \mathbb{R}$ *be defined as*

$$\psi(x) = x + 1.$$

Since $\psi'(x) = 1$ *for every* $x \in \mathbb{R}$ *and since* ψ *has no fixed points, we get from Theorem 3.4.4 that for an arbitrary weight* $w : \mathbb{R} \to \mathbb{C}$ *which does not vanish on* $\mathbb{R}$*, the operator* $C_{w,\psi}$ *is mixing and chaotic. This also holds if* $\psi : \mathbb{R} \to \mathbb{R}$ *is defined via the formula* $\psi(x) = e^x$*.*

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