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Turán and Ramsey numbers for 3-uniform hyperpaths

A dissertation for the degree
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Contents

Chapter 1

Introduction

Loosely speaking, Ramsey Theory is an area of discrete mathematics which is concerned with the existence of well-ordered subsets in large sets. Ramsey Theory plays an important role in many branches of mathematics, for instance in combinatorics, number theory, geometry, and logic. Crucial applications of Ramsey Theory can be found in many other disciplines of science, like information theory or game theory (see [31]).

In 1928 the English mathematician Frank Plumpton Ramsey proved in his seminal paper [29] a theorem which, in the special case of graphs, states that in any edge-colored large complete graph one can always find a monochromatic clique of a given size. This publication has had a considerable impact on the growth of combinatorics.

Strictly related to Ramsey's theorem is the Ramsey number $R(a, b)$, which in a classical form is defined as the least integer N such that every red-blue coloring of the edges of the complete graph K_N on N vertices contains a red complete subgraph K_a or a blue K_b. Determining Ramsey numbers is one of the most difficult problems related to Ramsey's theorem (see [10] for details). In this dissertation we will focus on hypergraph Ramsey numbers (see definition in Section 2.2).

Our main result asserts that the Ramsey number $R(P_3^3; r)$ for the 3-uniform loose path of length 3, P_3^3, and for the number of colors $r \leqslant 7$, equals $r + 6$ (Theorem 2.1 for $r = 3$ and Theorem 2.2). In order to prove these results we determine some Turán numbers and their extensions: Turán numbers of higher orders and conditional Turán numbers. Most importantly, we give a relatively simple inductive proof of an exact formula for the Turán number $\mathrm{ex}_3(n; P_3^3)$ for every n (Theorem 2.7). Moreover, we also determine the second order Turán number $\mathrm{ex}_3^{(2)}(n; P_3^3)$ (Theorem 2.9) and the third order Turán number $\mathrm{ex}_3^{(3)}(12; P_3^3)$ (Theorem 2.10). These results are published in my paper [15], as well as two joint papers with Polcyn and Ruciński: [16] and [17]. However, some proofs presented in the thesis have been modified compared with the journal versions. Moreover, my thesis contains also material which has never been published. In particular, a second proof of Theorem 2.1, as well as an alternative proof of Theorem 2.7. In order to distinguish my own results from those of other authors, I will follow the convention of not quoting my own papers next to the theorem number.

The thesis is divided into six chapters including Introduction and Conclusions. Chapter 2 is devoted to basic terminology and definitions and a review of selected earlier results on hypergraph Ramsey and Turán numbers. Moreover, in Section 2.2 we formulate our Theorem 2.1 and its generalization, Theorem 2.2, which, in my opinion, is the main result of this dissertation. Then, in Section 2.3, we state our results on Turán numbers, Theorems 2.7, 2.9, and 2.10, which are instrumental in proving Theorem 2.2.

The main goal of Chapter 3 is to prove Theorem 2.1 and Theorem 2.2. At the beginning we will present my original, self-contained, unpublished proof of Theorem 2.1. Then, we recall a general strategy of finding upper bounds on Ramsey numbers based on Turán numbers and employ it to conduct a second proof of Theorem 2.1. We finish Chapter 3 with the proof of Theorem 2.2, emphasizing the necessity of applying the Turán numbers of higher orders.

In our proofs we will need another variation of Turán numbers, called *conditional*. We define them at the beginning of Chapter 4 and state Theorem 4.1 which captures a relation between conditional and ordinary Turán numbers for P_3^3. Then we present two proofs of Theorem 2.7. The first one was published in [16], while the idea of the second proof of Theorem 2.7 was suggested by one of the reviewers of our paper (see acknowledgements in [16]).

The main purpose of Chapter 5 is to prove Theorems 2.9 and 2.10. We finish the dissertation with Chapter 6, where we summarize our impact on hypergraph Ramsey and Turán Theory, briefly present recent results by other authors related to our research, and suggest some open problems for further studies.

Chapter 2

Preliminaries and main results

In this Chapter, after formulating some basic definitions of hypergraph theory, we present the main results of the thesis.

2.1. Basic definitions

Definition 2.1. *(k-uniform hypergraph)*
For $k \geqslant 2$, a *k-uniform hypergraph* (or *k-graph*, for short) is an ordered pair $H = (V, E)$, where V is a finite, non-empty set of vertices and $E \subseteq \binom{V}{k}$ is a set of distinct k-element subsets of V, called edges. A hypergraph H is often identified with its edge-set E ; for instance, $|H|$ stands for the number of edges of H. Sometimes, a hypergraph is also called *a set system* or *a family of sets*.

For a given k-graph $H = (V, E)$ we say that a k-graph $H' = (V', E')$ is a *sub-k-graph* of H if $V' \subseteq V$ and $E' \subseteq E$. *A copy* of a k-graph F in another k-graph H is a sub-k-graph of H isomorphic to F.

Definition 2.2. *(Vertex degree)*
We say that a vertex v is *of degree i* in H when it belongs to exactly i edges of H. We denote the degree of a vertex v in H by $deg_H(v)$.

Definition 2.3. *($\mathcal{F}$-free k-graph)*
For a given family of k-graphs $\mathcal{F}$, we say that a k-graph H is *$\mathcal{F}$-free* if H contains no copy of a member of $\mathcal{F}$. If $\mathcal{F} = \{F\}$, then we write F-free instead of $\{F\}$-free.

Definition 2.4. *(Intersecting k-graph)*
If for all $e_1, e_2 \in E$, $e_1 \cap e_2 \neq \emptyset$, then H is called *intersecting*.

A trivial example of an intersecting k-graph is a star.

Definition 2.5. *(Star)*
A star is a k-uniform hypergraph with a vertex, called the center, belonging to all its edges (see Figure 2.1). Note that a star may have up to k centers. An n-vertex k-uniform star, $n > k$ is called *full* and denoted by S^k_n, if it has $\binom{n-1}{k-1}$ edges, that is the degree of its (unique) center equals $\binom{n-1}{k-1}$.

Definition 2.6. *(Complete k-graph)*
A complete k-graph K^k_n (or *the clique*) is a k-graph on a set V of n vertices in which every k-element subset of V forms an edge. Thus, $|K^k_n| = \binom{n}{k}$. When $n < k$, K^k_n is just a set of n isolated vertices, that is $E(K^k_n) = \emptyset$.

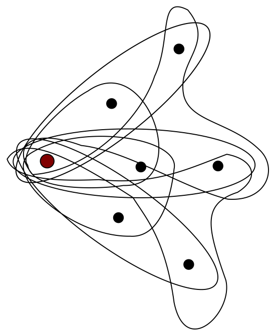

Figure 2.1. A star

There are several natural definitions of k-uniform paths and cycles. In this thesis we focus on the symmetric case when the intersections of consecutive edges have a fixed size (see for instance [32]).

Definition 2.7. *(Hyperpath and hypercycle)*
An ℓ-overlapping k-uniform path with m edges, denoted by $P_m^{k,\ell}$, is a k-graph whose vertex set has size $k+(m-1)(k-\ell)$ and can be linearly ordered in such a way that the edges are segments of that order and every two consecutive edges share exactly ℓ vertices (see Figure 2.2). *An ℓ-overlapping k-uniform cycle with m edges*, denoted by $C_m^{k,\ell}$, is defined similarly. Note that for $\ell > k/2$ also non-consecutive edges may have non-empty intersections.

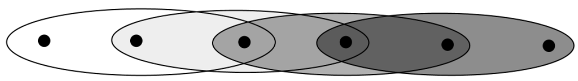

Figure 2.2. A tight path P_m^3

For $\ell = 1$ we use the terms *loose paths* and *loose cycles* (see Figure 2.3). Some authors, for instance Füredi, Jiang, and Seiver in [8] and Kostochka, Mubayi, and Verstraëte in [20], call such paths and cycles *linear*, while by *loose* they mean paths in which consecutive edges may intersect on more vertices (but non-consecutive edges are disjoint). Notice that for $k = 2$ we get the usual graph definitions of the path P_m and the cycle C_m with m edges.

In this dissertation we focus on loose paths, mainly of length 3. To simplify notation we will write P_m^k instead of $P_m^{k,1}$ and P instead of P_3^3 (see Figure 2.4).

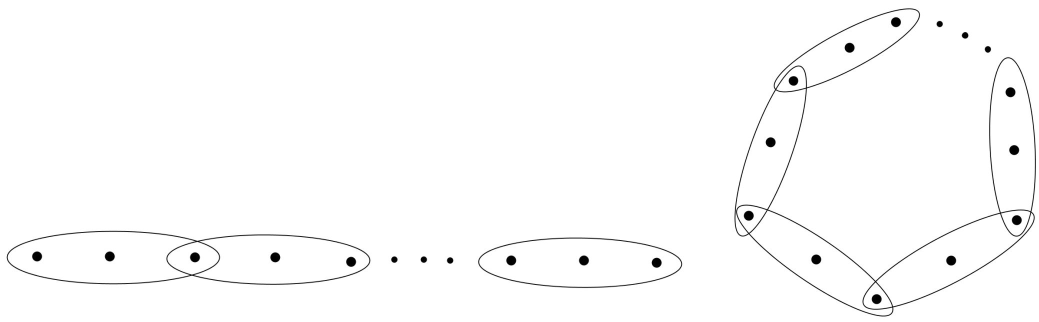

Figure 2.3. A loose path and a loose cycle

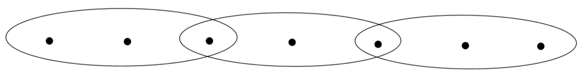

Figure 2.4. The loose path P

As a helpful tool, in our proofs we will use a 3-uniform loose cycle of length 3, C_3^3, which will be called *a triangle*. For convenience, we will write C instead of C_3^3 (see Figure 2.5).

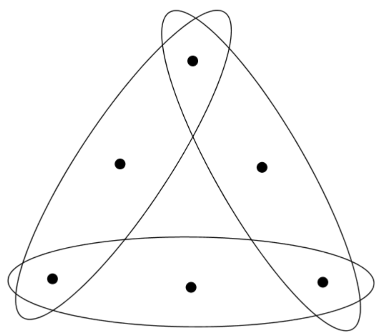

Figure 2.5. The triangle C

2.2. Results on Ramsey numbers

One of the most basic definitions in my thesis is that of a Ramsey number.

Definition 2.8. *(Ramsey number)*
For a given k-graph F and a natural number $r \geqslant 2$, the (r-colored) *Ramsey number* $R(F;r)$ is the least integer n such that every coloring of the edges of a clique K_n^k with r colors results in a monochromatic copy of F.

If F itself is a clique, we are dealing with *classical Ramsey numbers* which are hard to determine and thus very little is known. In fact, the only known value of a classical Ramsey number for $k \geqslant 3$ is $R(K_4^3;2) = 13$ and was computed by Radziszowski and McKay in 1991 ([24]) with some help of computer. Due to this hardness, sometimes instead of cliques researchers consider other structures with smaller density, for instance, hyperpaths and hypercycles.

There are many results in graph Ramsey theory related to cycles and paths (see [28]). In the classical case of two colors ($r = 2$), Gerencsér and Gyárfás [9] proved that

$$R(P_m;2) = \left\lfloor \tfrac{3m+1}{2} \right\rfloor, \text{ where } m \geqslant 1,$$

while Károlyi and Rosta (see [30], [18]) and Faudree, and Schelp (see [4]), showed that

$$R(C_m;2) = \begin{cases} 2m-1 & \text{for odd } m \text{ and } m > 3, \\ \frac{3m}{2} - 1 & \text{for even } m \text{ and } m > 4. \end{cases}$$

Let us now turn to hypergraphs. Recall that subscript m in P_m^k and C_m^k stands for the number of edges not vertices. First, for $r = 2$ and $k = 3$, it was proved by Haxell, Łuczak, Peng, Rödl, Ruciński, Simonovits, and Skokan in [13] that $R(C_m^3;2)$ and $R(P_m^3;2)$ are asymptotically equal to $\frac{5m}{2}$. Subsequently, Omidi and Shahsiah in [25] proved that

$$R(P_m^3;2) = R(C_m^3;2) + 1 = \left\lfloor \tfrac{5m+1}{2} \right\rfloor \text{ for } m \geqslant 2.$$

For higher dimensions ($k \geqslant 4$), but small m, Gyárfás and Raeisi [11] proved that

$$\begin{aligned} R(P_2^k;2) &= 2k-1, \\ R(P_3^k;2) &= R(C_3^k;2) + 1 = 3k-1, \\ R(P_4^k;2) &= R(C_4^k;2) + 1 = 4k-2, \end{aligned}$$

while in [12] Gyárfás, Sárközy, and Szemerédi established an asymptotic formula for fixed k and $m \to \infty$:

$$R(C_m^k;2) \sim \tfrac{2k-1}{2} m.$$

Taking into account, a standard construction with $\lceil \frac{m}{2} \rceil - 1$ vertices on the left, $(k-1)m$ vertices on the right, where all k-tuples with at least one vertex on the left are colored blue and the remaining k-tuples are colored red, we conclude that also $R(P_m^k;2) \sim \frac{2k-1}{2} m$.

For $r \geqslant 3$, Axenovich, Gyárfás, Liu, and Mubayi in [1] determined the value of $R(P_2^3;r)$ for an infinite subsequence of integers r (including $2 \leqslant r \leqslant 10$) and for $r \to \infty$ they proved that

$$R(P_2^3; r) \sim \sqrt{6r}.$$

For cycles and $r \geqslant 3$, we only know (see [11]) that $R(C; 3) = 8$. Proving an analog of the latter result for P was the starting point of our research. Indeed, Chapter 2 of this thesis brings two proofs of the following result from [15].

Theorem 2.1. $R(P; 3) = 9$.

The first proof, presented in Section 3.2, relies on a detailed case analysis. In the second proof, in order to establish an upper bound on Ramsey numbers for P, we apply Turán numbers which are determined for all n in Theorem 2.7, stated in Section 2.3, and proved in Section 4.2.

In the last section of Chapter 3, we prove our main contribution to the theory of Ramsey numbers which appeared in [17].

Theorem 2.2. *For all* $r \leqslant 7$, $R(P; r) = r + 6$.

2.3. Results on Turán numbers

We begin this section with the definitions of Turán numbers and extremal k-graphs.

Definition 2.9. *(Turán number)*
For a family of k-graphs $\mathcal{F}$ and a positive integer n, the *Turán number* $\mathrm{ex}_k(n; \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free k-graph on n vertices.

Definition 2.10. *(Extremal k-graph)*
An n-vertex k-graph H is called *extremal* with respect to $\mathcal{F}$ if H is $\mathcal{F}$-free and $|H| = \mathrm{ex}_k(n; \mathcal{F})$. We denote by $\mathrm{Ex}_k(n; \mathcal{F})$ the set of all pairwise non-isomorphic n-vertex k-graphs which are extremal with respect to $\mathcal{F}$.

If $\mathcal{F} = \{F\}$, then we write $\mathrm{ex}_k(n; F)$ and $\mathrm{Ex}_k(n; F)$ instead of $\mathrm{ex}_k(n; \{F\})$ and $\mathrm{Ex}_k(n; \{F\})$, respectively.

One of the most important results in extremal set theory is the celebrated theorem of Erdős, Ko, and Rado which asserts that for $n \geqslant 2k+1$ the full star S_n^k is the unique largest intersecting k-graph. This result is formulated below in terms of the Turán numbers. Let M_2^k be a k-graph consisting of two disjoint edges.

Theorem 2.3. [3] *For all* $n \geqslant 2k$, $\mathrm{ex}_k(n; M_2^k) = \binom{n-1}{k-1}$. *Moreover, for all* $n \geqslant 2k+1$, $\mathrm{Ex}_k(n; M_2^k) = \{S_n^k\}$.

Another important ingredient of our proofs is the Turán number for the triangle C. It was firstly computed by Frankl and Füredi in [7], but only for $n \geqslant 75$. Csákány and Kahn in [2] improved this result to cover all n.

Theorem 2.4. [2] *For all* $n \geqslant 6$,

$$\mathrm{ex}_3(n; C) = \binom{n-1}{2}.$$

Moreover, for all $n \geqslant 8$,

$$\mathrm{Ex}_3(n; C) = \{S_n^3\}.$$

From Theorem 2.4 we can conclude the following far reaching generalization the proof of which is deferred to Section 4.2.

Corollary 2.5. *For a family $\mathcal{F}$ such that $C \in \mathcal{F}$ and $\mathcal{F}$ does not contain any star,*

$$\textit{for } n \geqslant 6 \ \mathrm{ex}_3(n; \mathcal{F}) = \binom{n-1}{2} \quad \textit{and for } n \geqslant 8 \quad \mathrm{Ex}_3(n; \mathcal{F}) = \{S_n^3\}.$$

For $k \geqslant 4$, the Turán number for P_2^k, the loose path of length two, was determined in [5] for large n and in [19] for all n: $\mathrm{ex}_k(n; P_2^k) = \binom{n-2}{k-2}$ and the extremal k-graph is the maximal star with a 2-element center. In my thesis, we will need a much simpler analog of this result for $k = 3$, first observed in [19].

Proposition 2.6. [19] *For $n \geqslant 1$, we have $\mathrm{ex}_3(n; P_2^3) = n$ if n is divisible by 4, $\mathrm{ex}_3(n; P_2^3) = n - 1$ if $n = 1(\mod 4)$, and $\mathrm{ex}_3(n; P_2^3) = n - 2$ in all other cases.*

Füredi, Jiang, and Seiver [8] and Kostochka, Mubayi, and Verstraëte [20] determined the Turán numbers $\mathrm{ex}_k(n; P_m^k)$ for all fixed k and m, where $k \geqslant 4$ or $m \geqslant 4$, and *sufficiently large* n. Unfortunately, there were no corresponding results for $k = m = 3$. So, in [16] we filled this gap by determining $\mathrm{ex}_3(n; P)$. What is more, our formula is valid for *all* n.

Theorem 2.7.

$$\mathrm{ex}_3(n; P) = \begin{cases} \binom{n}{3} & \text{and } \mathrm{Ex}_3(n; P) = \{K_n^3\} & \text{for } n \leqslant 6, \\ 20 & \text{and } \mathrm{Ex}_3(n; P) = \{K_6^3 \cup K_1^3\} & \text{for } n = 7, \\ \binom{n-1}{2} & \text{and } \mathrm{Ex}_3(n; P) = \{S_n^3\} & \text{for } n \geqslant 8. \end{cases}$$

As $|V(P)| = 7$, Theorem 2.7 is trivial for $n \leqslant 6$. Two proofs of Theorem 2.7 for $n \geqslant 7$ are presented in Section 4.2.

Theorem 2.7 yields quickly one of the proofs of Theorem 2.1. Unfortunately, it is not strong enough to prove Theorem 2.2 and we needed more elaborated tools, namely Turán numbers of higher order. So, we now introduce a hierarchy of Turán numbers, where in each generation we consider only k-graphs which are not sub-k-graphs of extremal k-graphs from any previous generation. An ordinary Turán number is a Turán number of the 1st order. The following definition is iterative.

Definition 2.11. For a family of k-graphs $\mathcal{F}$ and integers $d, n \geqslant 1$, set $\mathrm{Ex}_k^{(1)}(n; \mathcal{F}) = \mathrm{Ex}_k(n; \mathcal{F})$ and define the Turán number of the $(d+1)$-st order as

$$\mathrm{ex}_k^{(d+1)}(n; \mathcal{F}) = \max\{|E(H)| : |V(H)| = n,\ H \text{ is } \mathcal{F}-\text{free},$$
$$\text{and } \forall H' \in \mathrm{Ex}_k^{(1)}(n; \mathcal{F}) \cup \ldots \cup \mathrm{Ex}_k^{(d)}(n; \mathcal{F}), H \nsubseteq H'\},$$

if such a k-graph H exists. An n-vertex $\mathcal{F}$-free k-graph H is called $(d+1)$-extremal for $\mathcal{F}$ if $|E(H)| = \mathrm{ex}_k^{(d+1)}(n; \mathcal{F})$ and $\forall H' \in \mathrm{Ex}_k^{(1)}(n; \mathcal{F}) \cup \ldots \cup \mathrm{Ex}_k^{(d)}(n; \mathcal{F})$, we have $H \nsubseteq H'$; we denote by $\mathrm{Ex}_k^{(d+1)}(n; \mathcal{F})$ the family of all pairwise non-isomorphic n-vertex k-graphs which are $(d+1)$-extremal for $\mathcal{F}$.

As before, we naturally simplify notation whenever $\mathcal{F} = \{F\}$.

A historically first example of a Turán number of the second order is related to the Erdős-Ko-Rado theorem (Theorem 2.3 above). Indeed, Theorem 2.3 brought about a quite natural question: what is the largest number of edges in an n-vertex intersecting k-graph which is not a star (such a k-graph is often called *a non-trivial intersecting k-graph*). In other words we are asking about the second order Turán number $\mathrm{ex}_k^{(2)}(n; M_2^k)$. Hilton and Milner [14] fully answered this question (see [6] for a short proof). Here we state their result for $k = 3$ only and suppress the family $\mathrm{Ex}_3^{(2)}(n; M_2^3)$.

Theorem 2.8. [14] *For all $n \geqslant 7$, $\mathrm{ex}_3^{(2)}(n; M_2^3) = 3n - 8$.*

To prove Theorem 2.2, we use two results on higher order Turán numbers for P. First, in Theorem 2.9, we determine $\mathrm{ex}_3^{(2)}(n; P)$ for all n, together with the corresponding 2-extremal 3-graphs. Then, in Theorem 2.10 we determine, $\mathrm{ex}_3^{(3)}(12; P)$. The proofs of Theorems 2.9 and 2.10 are presented in Section 5.1. To state these results, we start with a definition of a comet, a 3-graph which turns out to be 2-extremal for P.

Definition 2.12. *(Comet)*
For $n \geqslant 7$, *a comet* $\mathrm{Co}(n)$ is a 3-graph with n vertices consisting of a copy of K_4^3 to which a full star S_{n-3}^3 is attached and they share only one vertex - the center of the star. This unique vertex is called *the center* of the comet, while the set of the remaining three vertices of the K_4^3 is called *the head* (see Figure 2.6).

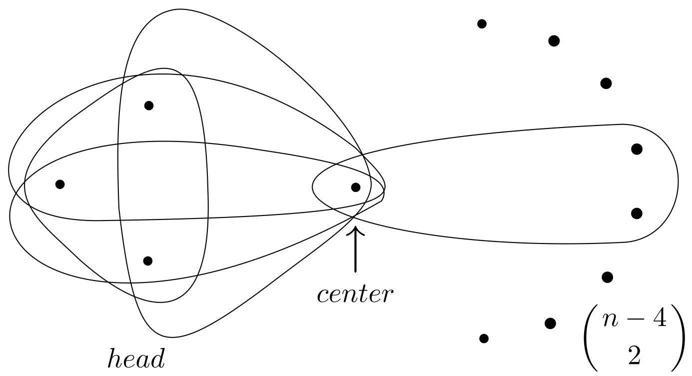

Figure 2.6. The comet $\mathrm{Co}(n)$

Let us emphasize that $\mathrm{Co}(n)$ is P-free and is not a star. The following result was proved in [17].

Theorem 2.9.

$$\mathrm{ex}_3^{(2)}(n;P) = \begin{cases} 15 & \text{and } \mathrm{Ex}_3^{(2)}(n;P) = \{S_7^3\} & \text{for} \quad n = 7, \\ 20 + \binom{n-6}{3} & \text{and } \mathrm{Ex}_3^{(2)}(n;P) = \{K_6^3 \cup K_{n-6}^3\} & \text{for} \quad 8 \leqslant n \leqslant 12, \\ 40 & \text{and } \mathrm{Ex}_3^{(2)}(n;P) = \{K_6^3 \cup K_6^3 \cup K_1^3, \mathrm{Co}(13)\} & \text{for} \quad n = 13, \\ 4 + \binom{n-4}{2} & \text{and } \mathrm{Ex}_3^{(2)}(n;P) = \{\mathrm{Co}(n)\} & \text{for} \quad n \geqslant 14. \end{cases}$$

Note that we cannot define $\mathrm{ex}_3^{(2)}(n;P)$ for $n \leqslant 6$, since then each 3-graph is a sub-3-graph of K_n^3, the unique extremal k-graph for the ordinary Turán number $\mathrm{ex}_3(n;P)$ (see Theorem 2.7 above).

At last, in [17] we also calculated the 3rd order Turán number for P, $\mathrm{ex}_3^{(3)}(12;P)$. As it was mentioned before, for our application to Ramsey numbers it is enough to determine it for $n = 12$ only.

Theorem 2.10.

$$\mathrm{ex}_3^{(3)}(12;P) = 32 \quad \textit{and} \quad \mathrm{Ex}_3^{(3)}(12;P) = \{\mathrm{Co}(12)\}.$$

Chapter 3

Ramsey numbers

In this chapter we prove our results concerning Ramsey numbers, especially Theorems 2.1 and 2.2. At the beginning we show a lemma which gives us immediately lower bounds on Ramsey numbers in question. Then we present two proofs of Theorem 2.1. As it was mentioned before, the first proof is based on a detailed analysis of all possible coloring cases. The second proof applies Theorem 2.7 and is more suitable for generalizations. We then attempt to apply Theorem 2.7 to prove Theorem 2.2 for $r = 4$ and explain why it fails. It follows that to accomplish our goal, applications of Theorems 2.9 and 2.10 are essential.

A standard approach to prove that $R(F; r) = n$ is to show, by quite different methods, that $R(F; r) \geqslant n$ and $R(F; r) \leqslant n$. To prove that $R(F; r) \geqslant n$, we have to find a coloring of the edges of K^k_{n-1} such that there is no monochromatic copy of F.

3.1. The lower bound

The derivation of the lower bound in Theorem 2.1 is based on a construction used already by Gyárfás and Raeisi in [11] to determine $R(C; 3)$. We state it here in a general form.

Lemma 3.1. *Let $r \geqslant 2$. If a k-graph F is not a star, then*

$$r + |V(F)| - 1 \leqslant R(F; r).$$

Proof. Let us consider the following r-coloring of the edges of the clique K^k_n on the vertex set $\{1, 2, \ldots n\}$, where

$$n = r + |V(F)| - 2.$$

Color an edge e by color i, for $i \in \{1, 2, ..., r - 1\}$, if the minimum vertex in e equals i, that is, $\min(e) = i$, and by color r otherwise. This way we obtain $r - 1$ monochromatic stars, in colors $1, 2, ..., r - 1$. There will be no monochromatic F in colors $1, 2, ..., r - 1$, because F is not a star. We do not obtain a copy of F in color r either, because the edges of color r form a clique K^k_{n-r+1}, while $|V(F)| = n - r + 2$.

□

As a consequence of Lemma 3.1, for $F = P$ we can formulate the following corollary.

Corollary 3.2. *For all $r \geqslant 2$, $R(P; r) \geqslant r + 6$.*

Particularly, substituting $r = 3$ we obtain $R(P; r) \geqslant 9$. To finish the proof of Theorem 2.1 we have to show that $R(P; 3) \leqslant 9$.

3.2. The first proof of Theorem 2.1

By Corollary 3.2, we have $R(P;3) \geqslant 9$, and it remains to show that $R(P;3) \leqslant 9$. We will consider an arbitrary coloring the edges of K_9^3 by three colors red, blue, and green. For convenience we label the vertices of K_9^3 by 1, ..., 9, and represent the vertex set of K_9^3 as a 3×3 grid (see Figure 3.1). In order to simplify notation introduced in Chapter 2, every edge of K_9^3 is represented by a triple ijk, where $i, j, k \in \{1, 2, \ldots, 9\}$. A copy of the path P will be written as $(ab\underline{c}d\underline{e}fg)$, where $\{a, \ldots, g\} = \{1, 2, \ldots, 9\}$, and underlined elements are vertices of degree 2 in P.

Let us suppose that there is no monochromatic copy of P in K_9^3, and consider a star with four 3-uniform edges, which intersect in exactly one vertex. From the Pigeonhole Principle we always find, in this star, two edges e and f of the same color. Without loss of generality, we may assume that the edges $e = 123$ and $f = 147$ are green (see Figure 3.1).

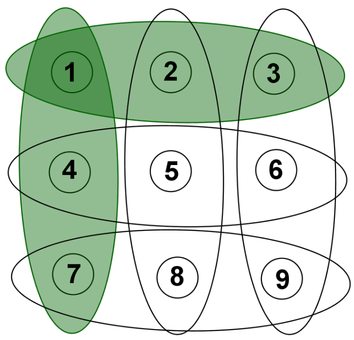

Figure 3.1. 3×3 grid of 9 labeled vertices

It can be observed that neither of the edges 258, 369, 456, and 789 is green, since otherwise we would obtain a green P. Let us consider two cases:

Case 1: The edges 258 and 369 have the same color.

As we noticed before these edges cannot be green. Without loss of generality, we assume that they are red. It implies that edges 456 and 789 are blue (red 456 would give red $(28\underline{5}4\underline{6}39)$ and red 789 would form red $(25\underline{8}7\underline{9}63)$) (see Figure 3.2).

Consequently 159 must be green, since otherwise $(82\underline{5}1\underline{9}63)$ would be red, or $(46\underline{5}1\underline{9}78)$ would be blue. Let us consider the edge 348. If it is green, then it forms green $(84\underline{3}2\underline{1}59)$. If it is red, then we get red $(96\underline{3}4\underline{8}52)$. Otherwise $(65\underline{4}3\underline{8}79)$ would be blue. Now we observe that no matter what the color of 348 is, a monochromatic P arises, a contradiction (see Figure 3.3).

Case 2: The edges 258 and 369 have different colors.

Then also the edges 456 and 789 have different colors (one of them is red and the other is blue). Indeed, by symmetry, if 456 and 789 had the same color, then we would

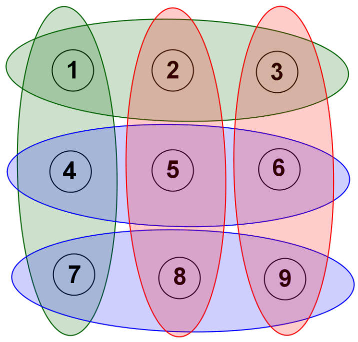

Figure 3.2. Case 1

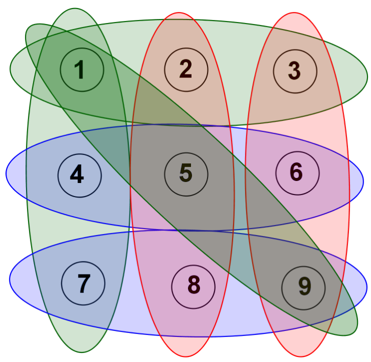

Figure 3.3. Case 1

be quick in Case 1. Without loss of generality, we may assume that 258 and 456 are red and 369, 789 blue (see Figure 3.4).

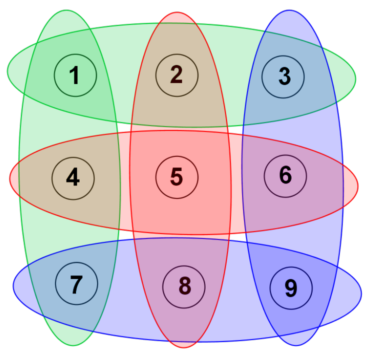

Figure 3.4. Case 2

Firstly, we consider the color of the edge 134. If 134 is red, then it forms red $(13\underline{4}6\underline{5}28)$. Similarly, if 134 is blue, then $(14\underline{3}6\underline{9}87)$ is blue. This implies that 134 is green. By the same argument 127 must be green and 458 must be red, otherwise they would form red $(17\underline{2}8\underline{5}46)$ or blue $(12\underline{7}8\underline{9}36)$, and green $(32\underline{1}7\underline{4}58)$ or blue $(45\underline{8}7\underline{9}63)$, respectively.

Let us consider the edge 126. It can be green or red, but not blue, since otherwise it would form blue $(12\underline{6}3\underline{9}87)$.

Subcase 1: 126 is green.

It implies that, in particular, 235 is red, since otherwise $(25\underline{3}6\underline{9}87)$ would be blue, or $(47\underline{1}6\underline{2}35)$ would be green. Similarly one can argue that 349 and 378 are blue.

Then no matter what the color of the edge 269 is, we have a monochromatic P. Indeed, $(96\underline{2}7\underline{1}34)$ would be green, $(96\underline{2}3\underline{5}48)$ would be red, or $(26\underline{9}4\underline{3}78)$ would be blue.

Subcase 2: 126 is red.

Analogously to Subcase 1, we pick up edges whose color is determined. The edge 358 is blue, since otherwise $(85\underline{3}4\underline{1}27)$ would be green, or $(38\underline{5}4\underline{6}21)$ would be red. The edge 345 is green, since otherwise $(54\underline{3}6\underline{9}78)$ would be blue, or $(34\underline{5}8\underline{2}16)$ would be red.

This implies that the edge 129 completes a monochromatic copy P regardless of its color. Indeed, either $(92\underline{1}7\underline{4}53)$ is green, or $(91\underline{2}8\underline{5}46)$ is red, or $(12\underline{9}7\underline{8}53)$ is blue. It contradicts our assumption and finishes the first proof of Theorem 2.1.

3.3. The second proof of Theorem 2.1

In the previous section we showed that $R(P;3) \leqslant 9$ which constitutes the harder part of the proof of Theorem 2.1. The method used there was very tedious and, therefore, not suitable for generalizations. That was our motivation for trying a different, more sophisticated approach based on Turán numbers.

It is well-known that Turán numbers provide upper bounds on Ramsey numbers (see, e.g. [11]). This relation is captured by the following result formulated in [15] and [17].

Lemma 3.3. *Let $n \geqslant r+k$, $r \geqslant 2$, and $k \geqslant 2$. If $\mathrm{ex}_k(n;F) < \frac{1}{r}\binom{n}{k}$ or $\mathrm{ex}_k(n;F) = \frac{1}{r}\binom{n}{k}$, but the only extremal k-graph is a star, then $R(F;r) \leqslant n$.*

Proof. Let us consider an r-coloring of the complete k-graph K_n^k. If we color the edges of K_n^k by r colors, then there exist at least $\frac{1}{r}\binom{n}{k}$ edges in one of the colors. If $\mathrm{ex}_k(n;F) < \frac{1}{r}\binom{n}{k}$, we obtain a copy of F in that color. The same is true if $\mathrm{ex}_k(n;F) = \frac{1}{r}\binom{n}{k}$ but there are more than $\frac{1}{r}\binom{n}{k}$ edges in one of the colors. Finally, if there are exactly $\frac{1}{r}\binom{n}{k}$ edges in each color, not all the colors may form stars. Indeed, since $n \geqslant r+k$, there would be at least k vertices which are not centers of any star. But then we would obtain at least one k-uniform edge not colored by any of the colors, a contradiction.

□

One consequence of Theorem 2.7 and Lemma 3.3 is the following upper bound on $R(P;r)$.

Corollary 3.4. *For all $r \geqslant 3$, $R(P;r) \leqslant 3r$.*

Proof. By Theorem 2.7, for $n \geqslant 8$, $\mathrm{ex}_3(n;P) = \binom{n-1}{2}$ and S_n^3 is the only extremal 3-graph. If $n = 3r$, then $\binom{n}{3} = r \cdot \mathrm{ex}_3(n;P)$ and Corollary 3.4 follows from Lemma 3.3 with $F = P$. □

Second proof of Theorem 2.1. In view of Corollary 3.2 it is enough to prove the upper bound only. On the other hand by Corollary 3.4, with $r = 3$, we have $R(P;3) \leqslant 9$. This completes the second proof of Theorem 2.1. □

3.4. The proof of Theorem 2.2

The goal of this section is to prove Theorem 2.2, stated in Chapter 2. So far, we have only proved the case $r = 3$ (Theorem 2.1) where we applied ordinary Turán numbers. Indeed, Theorem 2.7 equipped us with Turán numbers which were a useful tool in the proof of Corollary 3.4. However, in order to prove Theorem 2.2, we need Turán numbers of higher order. To realize this, let us, for instance, try to compute the Ramsey number $R(P;4)$ based on Lemma 3.3. The lower bound, by Corollary 3.2, is $R(P;4) \geqslant 10$. For the upper bound, by Theorem 2.7, we have

$$\mathrm{ex}_3(10;P) = \binom{9}{2} = 36,$$

but the average number of edges per color equals

$$\frac{1}{4}\binom{10}{3} = 30.$$

Therefore the assumptions of Lemma 3.3 are not satisfied. The only result we could obtain this way is a weaker estimate $R(P;4) \leqslant 12$. So, we indeed need some refinement of this approach.

As a source of reference for the proof of Theorem 2.2, Table 3.1 compares the values of the Turán numbers of the 1st, 2nd, and 3rd order, with the average number of edges per color in a clique K_n^3.

n	r	$\binom{n}{3}$	$\binom{n}{3}/r$	$\mathrm{ex}_3(n;P)$	$\mathrm{ex}_3^{(2)}(n;P)$	$\mathrm{ex}_3^{(3)}(n;P)$
8	2	56	28	21	-	-
9	3	84	28	28	-	-
10	4	120	30	36	24	-
11	5	165	33	45	30	-
12	6	220	36,7	55	40	32
13	7	286	40,9	66	40	-

Table 3.1. Comparison of the values of Turán numbers and the average numbers of edges per color

Now we are ready to prove Theorem 2.2. By Corollary 3.2, $R(P;r) \geqslant r+6$ holds for all $r \geqslant 2$. The case $r = 3$ was considered in Theorem 2.1, hence to prove Theorem 2.2 it is enough to show that $R(P;r) \leqslant r+6$ for each $r = 4,5,6,7$. To prove cases $r = 4,5,7$ we apply Turán numbers of the 2nd order (see Theorem 2.9), while for $r = 6$ the 3rd order Turán number (Theorem 2.10) is essential.

Proof of Theorem 2.2.

Case r = 4. Let us consider an arbitrary 4-coloring of the $\binom{10}{3} = 120$ edges of the complete 3-graph K^3_{10}. There exists a color with at least $\frac{1}{4} \cdot 120 = 30$ edges (see Table 3.1). Denote the set of these edges by H. Since, by Theorem 2.7, $\mathrm{Ex}^{(1)}_3(10; P) = \{S^3_{10}\}$, and, by Theorem 2.9, $\mathrm{ex}^{(2)}_3(10; P) = 24 < 30$, either $P \subseteq H$ or $H \subseteq S^3_{10}$. In the latter case we delete the center of the star containing H, together with the incident edges, obtaining a 3-coloring of K^3_9. Since $R(P; 3) = 9$ (see Theorem 2.1), there is a monochromatic copy of P.

Case r = 5. The proof of this case is very similar to the previous one. We consider an arbitrary 5-coloring of the $\binom{11}{3} = 165$ edges of the complete 3-graph K^3_{11}. There exists a color with at least $\frac{1}{5} \cdot 165 = 33$ edges (see Table 3.1). Denote the set of these edges by H. Again, by Theorem 2.7, $\mathrm{Ex}^{(1)}_3(11; P) = \{S^3_{11}\}$, and by Theorem 2.9, $\mathrm{ex}^{(2)}_3(11; P) = 30 < 33$, so either $P \subseteq H$ or $H \subseteq S^3_{11}$. In the latter case we delete the center of the star containing H, together with its incident edges, obtaining a 4-coloring of K^3_{10}. Since, as we have just proved in the previous case, $R(P; 4) = 10$, there is a monochromatic copy of P.

Case r = 6. This is the most difficult case in which we have to appeal to the 3rd order Turán number. We begin, as before, by considering an arbitrary 6-coloring of the complete 3-graph K^3_{12} on the set of vertices V and assuming that it does not yield a monochromatic copy of the path P. Then none of the color classes can be contained in a star S^3_{12}, since otherwise we would delete this star, obtaining a 5-coloring of K^3_{11}, which as we have just proved, contains a monochromatic P. By Theorems 2.7 and 2.9, S^3_{12} and $K^3_6 \cup K^3_6$ are, respectively, the unique 1-extremal and 2-extremal 3-graphs for P. Consequently, since the containment in a star has ben already excluded, by Theorem 2.10, every color class with more than 32 edges must be a sub-3-graph of $K^3_6 \cup K^3_6$ (see Table 3.1).

On the other hand, there exists a color class with at least $\lceil \binom{12}{3} \cdot \frac{1}{6} \rceil = 37$ edges which, as explained above, is contained in a copy K of $K^3_6 \cup K^3_6$. After deleting all edges of K from K^3_{12}, we obtain a complete bipartite 3-graph B with bipartition $V = U \cup W$, $|U| = |W| = 6$, and with

$$|E(B)| = \binom{12}{3} - 2 \cdot \binom{6}{3} = 220 - 40 = 180$$

edges, colored by 5 colors. (*Bipartite* means here that every edge of B intersects both, U and W). Thus, an average number of edges per color is $\frac{180}{5} = 36$. However, any copy K' of $K^3_6 \cup K^3_6$ may share with B at most 36 edges. It follows from the fact that for every $V' \subseteq V$, $|V'| = 6$, the quantity $|K' \cap B|$, where the cliques of K' are spanned by V' and $V \setminus V'$, is maximized (uniquely) when $|V' \cap U| = |V' \cap W| = 3$, in which case

$$|K' \cap B| = |E(B[V'])| + |E(B[V \setminus V'])| = 2 \cdot 2 \cdot 3 \cdot 3 = 36.$$

Consequently, since, as explained above, every color class with more than 32 edges (in particular with 37 edges) must be contained in $K^3_6 \cup K^3_6$, all five color classes on B have each less than 37 edges, and thus, precisely 36 edges.

Let G_i, $i = 1, 2, 3, 4, 5$, be the 5 color classes. Then, for each i, G_i is fully defined by two partitions, $U = U'_i \cup U''_i$ and $W = W'_i \cup W''_i$, where $|U'| = |U''| = |W'| = |W''| = 3$. The reason is that G_i is then a disjoint union of two copies of $K^3_6 - M^3_2$, one on the vertex set $U'_i \cup W'_i$, the other one on $U''_i \cup W''_i$, with U'_i, W'_i, U''_i, W''_i being the 4 missing edges (see Figure 3.5).

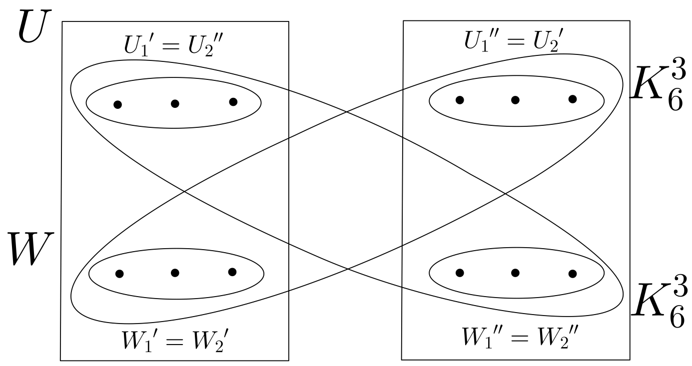

Figure 3.5. Illustration of the proof of Theorem 2.2, case $r = 6$

It can be easily seen that only 2 of these 5 color classes can be disjoint which is a contradiction (with a big cushion). Indeed, for G_1 and G_2 to be disjoint, we need that $\{U_1', U_1''\} = \{U_2', U_2''\}$ and $\{W_1', W_1''\} = \{W_2', W_2''\}$, which simply means that one of the partitions, of U or of W, must be swapped. But this implies that already G_1, G_2, and G_3 cannot be pairwise disjoint.

Case r = 7. Similarly to the cases $r = 4$ and $r = 5$ we consider an arbitrary 7-coloring of the $\binom{13}{3} = 286$ edges of the complete 3-graph K_{13}^3. There exists a color with at least $\lceil \frac{1}{7} \cdot 286 \rceil = 41$ edges (see Table 3.1). Denote the set of these edges by H. Since, by Theorem 2.7, $\mathrm{Ex}_3^{(1)}(13; P) = \{S_{13}^3\}$, and, by Theorem 2.9, $\mathrm{ex}_3^{(2)}(13; P) = 40 < 41$, either $P \subseteq H$ or $H \subseteq S_{13}^3$. In the latter case, we delete the center of the star containing H, together with the incident edges, obtaining a 6-coloring of K_{12}^3. Since $R(P; 6) = 12$, there is a monochromatic copy of P. $\square$

Chapter 4

Conditional Turán numbers and proofs of Theorem 2.7

The aim of this chapter is to prove Theorem 2.7, that is to determine the Turán number $\mathrm{ex}_3(n; P)$ and assigning a unique extremal graph for each n. The main idea of our proof is to link the presence of a copy of P with the presence of the triangle C which for a 3-graph with at least $\binom{n-1}{2}$ edges is guaranteed by Theorem 2.4.

To this end we introduce another concept of Turán numbers, namely conditional Turán numbers $\mathrm{ex}_k(n; \mathcal{F}|\mathcal{G})$, defined in the subsequent section, where an additional constraint of containing a member of a given family of k-graphs $\mathcal{G}$ is imposed. Then we state Theorem 4.1, which exhibits a relation between ordinary Turán numbers and conditional Turán numbers, and plays a crucial role in the first proof of Theorem 2.7.

In Section 4.2 we present our proof of Theorem 2.7, as well as another proof which is almost entirely based on Theorem 2.3 and Proposition 2.6. Sections 4.3 and 4.4 are totally devoted to prove, resp., Theorem 4.1 and the lemmas used therein.

Conditional Turán numbers are needed not only in the proof of Theorem 2.7, but also later in Section 5.1 to prove Theorems 2.9 and 2.10. We finish Chapter 4 with Section 4.5, where we consider some additional problems on conditional Turán numbers for non-intersecting 3-graphs and state two important results, Theorem 4.9 and Theorem 4.11, which are needed later. Their proofs are deferred to Chapter 5.

4.1. Conditional Turán numbers

Definition 4.1. *(Conditional Turán number)*
For a family of k-graphs $\mathcal{F}$, a family of $\mathcal{F}$-free k-graphs $\mathcal{G}$, and an integer $n \geqslant \min\{|V(G)| : G \in \mathcal{G}\}$, the conditional Turán number is defined as

$$\mathrm{ex}_k(n; \mathcal{F}|\mathcal{G}) = \max\{|E(H)| : |V(H)| = n,\ H \text{ is } \mathcal{F}\text{-free, and } \exists G \in \mathcal{G} :\ H \supseteq G\}.$$

Every n-vertex $\mathcal{F}$-free k-graph with $\mathrm{ex}_k(n; \mathcal{F}|\mathcal{G})$ edges and such that $H \supseteq G$ for some $G \in \mathcal{G}$ is called $\mathcal{G}$-*extremal* for $\mathcal{F}$. We denote by $\mathrm{Ex}_k(n; \mathcal{F}|\mathcal{G})$ the family of all n-vertex k-graphs which are $\mathcal{G}$-extremal for $\mathcal{F}$. (If $\mathcal{F} = \{F\}$ or $\mathcal{G} = \{G\}$, we will simply write $\mathrm{ex}_k(n; F|\mathcal{G})$, $\mathrm{ex}_k(n; \mathcal{F}|G)$, $\mathrm{ex}_k(n; F|G)$, $\mathrm{Ex}_k(n; F|\mathcal{G})$, $\mathrm{Ex}_k(n; \mathcal{F}|G)$, or $\mathrm{Ex}_k(n; F|G)$, respectively). Obviously, $\mathrm{ex}_k(n; \mathcal{F}|\mathcal{G}) \leqslant \mathrm{ex}_k(n; \mathcal{F})$.

In [16] we established a relation between the ordinary Turán number $\mathrm{ex}_3(n; P)$ and conditional Turán number $\mathrm{ex}_3(n; P|C)$ which makes the proof of Theorem 2.7 much shorter than the original one.

Theorem 4.1. *For all $n \geqslant 6$,*

$$\mathrm{ex}_3(n; P|C) = 20 + \mathrm{ex}_3(n-6; P).$$

Moreover, $\mathrm{Ex}_3(n; P|C) = \{K_6^3 \cup H_{n-6}\}$*, where* H_{n-6} *is the unique extremal* P*-free 3-graph on* $n-6$ *vertices, namely the only member of* $\mathrm{Ex}_3(n-6; P)$.

We will prove Theorem 4.1 in Section 4.4. Note that Theorem 4.1, combined with Theorem 2.7, yields immediately the explicit values of $\mathrm{ex}_3(n; P|C)$, for all n, along with the extremal sets $\mathrm{Ex}_3(n; P|C)$.

Corollary 4.2.

$$\mathrm{ex}_3(n; P|C) = \begin{cases} 20 + \binom{n-6}{3}, & \text{and} \quad \mathrm{Ex}_3(n; P|C) = \{K_6^3 \cup K_{n-6}^3\} \quad \text{for } 6 \leqslant n \leqslant 12, \\ 40, & \text{and} \quad \mathrm{Ex}_3(n; P|C) = \{K_6^3 \cup K_6^3 \cup K_1^3\} \quad \text{for } n = 13, \\ 20 + \binom{n-7}{2}, & \text{and} \quad \mathrm{Ex}_3(n; P|C) = \{K_6^3 \cup S_{n-6}^3\} \quad \text{for } n \geqslant 14. \end{cases}$$

Comparing Corollary 4.2 and Theorem 2.7 we see that for $n \geqslant 14$

$$\mathrm{ex}_3(n; P) - \mathrm{ex}_3(n; P|C) = \binom{n-1}{2} - 20 - \binom{n-7}{2} = 6n - 47.$$

Thus, in this case, the conditional Turán number is not much smaller than its unconditional counterpart. We showed in [16] that this behaviour is typical whenever the members of family $\mathcal{F}$ are connected. Let us first define a connected k-graph.

Definition 4.2. *(Connected k-graph)*
A k*-graph* $H = (V(H), E(H))$ *is connected if for every bipartition of the set of vertices* $V(H) = V_1 \cup V_2$, $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, *there exists an edge* $h \in E(H)$ *such that* $h \cap V_1 \neq \emptyset$ *and* $h \cap V_2 \neq \emptyset$.

Theorem 4.3. *If* $\mathcal{F}$ *consists of connected* k*-graphs only,* $\mathcal{F} \neq \{K_k^k\}$*, and neither* $\mathcal{F}$ *nor* $\mathcal{G}$ *depends on* n*, then, as* $n \to \infty$,

$$\mathrm{ex}_k(n; \mathcal{F}|\mathcal{G}) \sim \mathrm{ex}_k(n; \mathcal{F}).$$

Proof. Consider a disjoint union H of any $G \in \mathcal{G}$ and any extremal $\mathcal{F}$-free k-graph on $n - |V(G)|$ vertices. By the connectivity of the k-graphs $F \in \mathcal{F}$, H is $\mathcal{F}$-free. Since in addition $H \supseteq G$, setting $g = |V(G)|$, we thus have

$$|H| = \mathrm{ex}_k(n-g; \mathcal{F}) + |G| \leqslant \mathrm{ex}_k(n; \mathcal{F}|\mathcal{G}) \leqslant \mathrm{ex}_k(n; \mathcal{F}). \tag{4.1}$$

Now, let us take an extremal $\mathcal{F}$-free k-graph on n vertices H_n, and remove iteratively g vertices, each time taking away a vertex of minimum degree in the current sub-k-graph. Note that with the first vertex v_1 we removed at most

$$\frac{\sum_{v \in V(H_n)} deg_{H_n}(v)}{n} = \frac{k \cdot |H_n|}{n}$$

edges.

Let $H_{n-i} = H_n - \{v_1, v_2, \dots, v_i\}$ denote the sub-k-graph obtained from H_n after deleting the vertices $v_1, v_2, \dots, v_i$. The deletion of the next vertex v_{i+1} results in a loss of no more than

$$\frac{\sum_{v\in V(H_{n-i})} deg_{H_{n-i}}(v)}{n-i} = \frac{k\cdot |H_{n-i}|}{n-i} \leqslant \frac{k\cdot |H_n|}{n-i}$$

edges.

The obtained k-graph H_{n-g} is still $\mathcal{F}$-free, has $n-g$ vertices, and at least

$$|H_n| - \sum_{i=0}^{g-1} \frac{k\cdot|H_n|}{n-i} \geqslant |H_n| - g\cdot \frac{k\cdot |H_n|}{n-g+1}$$

edges.

Hence, recalling that $|H_n| = \mathrm{ex}_k(n;\mathcal{F})$ and $|H_{n-g}| \leqslant \mathrm{ex}_k(n-g;\mathcal{F})$ we infer that

$$\mathrm{ex}_k(n-g;\mathcal{F}) \geqslant \mathrm{ex}_k(n;\mathcal{F})\left(1 - \frac{kg}{n-g+1}\right). \tag{4.2}$$

Since $\mathcal{F} \neq \{K_k^k\}$, by considering a matching $M_{\lfloor \frac{n}{k} \rfloor}$ we have

$$\mathrm{ex}_k(n;\mathcal{F}) \geqslant |M_{\lfloor \frac{n}{k} \rfloor}| = \left\lfloor \frac{n}{k} \right\rfloor \to \infty$$

on $n \to \infty$. Hence by the (4.1) and (4.2) it follows that

$$\mathrm{ex}_k(n;\mathcal{F})\left(1 - \frac{kg}{n-g+1}\right) \leqslant \mathrm{ex}_k(n-g;\mathcal{F}) + |G| \leqslant \mathrm{ex}_k(n;\mathcal{F}|\mathcal{G}) \leqslant \mathrm{ex}_k(n;\mathcal{F}).$$

From the Squeeze Theorem of Calculus we conclude that

$$\lim_{n\to\infty} \frac{\mathrm{ex}_k(n;\mathcal{F}|\mathcal{G})}{\mathrm{ex}_k(n;\mathcal{F})} = 1.$$

□

In striking contrast, for a family $\mathcal{F}$ containing a disconnected k-graph, conditioning on the presence of specified sub-k-graphs may cause a Turán number drop significantly. A prime example of this phenomenon is furnished by Theorems 2.3 and 2.8 stated in Chapter 2.

Indeed, one can check that an intersecting 3-graph is not a star if, and only if, it contains either the triangle C, or K_4^3, or a 3-graph F_5 where $V(F_5) = \{a,b,c,d,e\}$, $E(F_5) = \{\{a,b,c\},\{c,d,e\},\{e,a,b\}\}$. Hence, setting $M := M_2^3$,

$$\mathrm{ex}_3^{(2)}(n;M) = \mathrm{ex}_3(n;M|\{C, K_4^3, F_5\}).$$

We can thus reformulate Theorem 2.8 in terms of a conditional Turán number.

Corollary 4.4. [14] *For 3-graph H, where $|V(H)| = n$ and $n \geqslant 6$, we have*
$$\mathrm{ex}_3(n;M|\{C, K_4^3, F_5\}) = 3n-8.$$

Hence, for $\mathcal{F} = \{M\}$, a conditional Turán number can be much smaller than the unconditional one (linear vs. quadratic function of n).

Returning to the main theme of this chapter, in [17] we stated a lemma which is crucial in the proof of Theorem 4.1. Let $\mathrm{ex}_3^{con}(n; P|C)$ be defined as $\mathrm{ex}_3(n; P|C)$, but where the maximum number of edges is taken over all connected 3-graphs only.

Lemma 4.5. *For all $n \geqslant 7$,*

$$\mathrm{ex}_3^{con}(n; P|C) = 3n - 8.$$

Comparing the results of Lemma 4.5 and Corollary 4.2 we see that this time the assumption of connectivity may lead to a significant drop in the order of magnitude of a Turán number. Furthermore, it is not a coincidence that in Lemma 4.5 and Theorem 2.8 we have the same extremal number $3n-8$. Indeed, in that proof we consider the intersecting and non-intersecting case. Due to the presence of C the intersecting case is necessarily non-trivial and the bound of Theorem 2.8 applies. In the non-intersecting case, the size of H drops even further (cf. Corollary 4.7 below).

The proof of Lemma 4.5 requires a few preparations and for this purpose it is deferred to Section 4.4. As a main tool in that proof we apply another lemma, Lemma 4.6, where we additionally assume the existence of M, which means that 3-graph is non-intersecting. To formulate this result we need one more construction.

Definition 4.3. *(Satellite)*
A satellite $\mathrm{Sa}(n)$ is a 3-graph with n vertices consisting of a copy of K_5^3 to which $n-5$ 3-uniform edges are attached, and all these edges intersect in exactly two vertices of K_5^3. The K_5^3 is the core of the satellite, while the remaining $n-5$ edges form the tail of the satellite (see Figure 4.1).

Note that $\mathrm{Sa}(n)$ is a P-free 3-graph, $M, C \subset \mathrm{Sa}(n)$, and $|\mathrm{Sa}(n)| = n+5$.

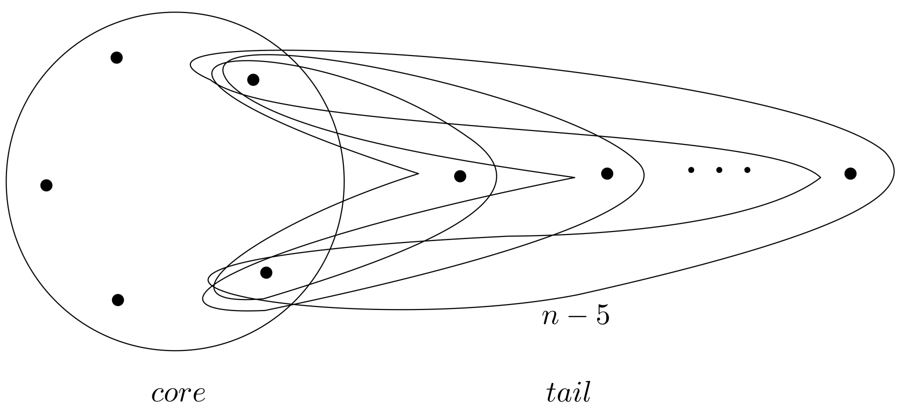

Figure 4.1. A satellite $\mathrm{Sa}(n)$

Lemma 4.6. *Let H be a connected, P-free 3-graph with $n \geqslant 7$ vertices and such that $H \supset M, C$. Then $H \subset \mathrm{Sa}(n)$.*

Lemma 4.6 is very strong, because it characterizes the structure of all connected P-free graphs containing C and M, paving way for potential applications (see Chapter 6). In addition, it immediately yields the corresponding Turán numbers.

Corollary 4.7. *For all $n \geqslant 7$*

$$\mathrm{ex}_3^{con}(n; P|\{C, M\}) = n + 5, \qquad \mathrm{Ex}_3^{con}(n; P|\{C, M\}) = \mathrm{Sa}(n).$$

Comparing Lemma 4.5 and Corollary 4.7, we notice that imposing the additional constraint that a 3-graph H is non-intersecting results in a further drop of the conditional Turán number in question.

Both, Lemma 4.6 and Corollary 4.7, first appeared in [26], where [17] was pointed to as a source of an implicit proof. In Section 4.4 we give these proofs explicitly.

4.2. Proofs of Theorem 2.7

In this section we conduct two proofs of Theorem 2.7. The first of the presented proofs was published in [16]. That result relies heavily on Theorem 2.4, that is, on the presence of the triangle C in every 3-graph with more than $\binom{n-1}{2}$ edges. Here we present a streamline version of that proof where a crucial role is played by Theorem 4.1.

Since the first proof of Theorem 2.7 uses Corollary 2.5, we need to prove that result first.

Proof of Corollary 2.5. Let $C \in \mathcal{F}$ and $\mathcal{F}$ not contain any star. By Theorem 2.4 and monotonicity, for $n \geqslant 6$ we get

$$\mathrm{ex}_3(n; \mathcal{F}) \leqslant \mathrm{ex}_3(n; C) = \binom{n-1}{2}.$$

On the other hand, because there are no stars in $\mathcal{F}$, the full star S_n^3 is $\mathcal{F}$-free. Consequently

$$\mathrm{ex}_3(n; \mathcal{F}) \geqslant |S_n^3| \geqslant \binom{n-1}{2},$$

which proves that

$$\mathrm{ex}_3(n; \mathcal{F}) = \binom{n-1}{2}.$$

Clearly, $S_n^3 \in \mathrm{Ex}_3(n; \mathcal{F})$. If for $n \geqslant 8$ there was another extremal 3-graph for $\mathcal{F}$, it would also be an extremal 3-graph for C, a contradiction with Theorem 2.4. $\square$

First proof of Theorem 2.7. Recall that we want to determine the Turán number $\mathrm{ex}_3(n; P)$ for all n. As Theorem 2.7 is trivial for $n \leqslant 6$, we assume that $n \geqslant 7$. By considering whether or not a 3-graph contains the triangle C, we infer that

$$\mathrm{ex}_3(n;P) = \max\{\mathrm{ex}_3(n;P|C), \mathrm{ex}_3(n;\{P,C\})\}. \tag{4.3}$$

By Theorem 2.4, for $n \geqslant 6$,

$$\mathrm{ex}_3(n;\{P,C\}) = \binom{n-1}{2}, \tag{4.4}$$

and, for $n \geqslant 8$,

$$\mathrm{Ex}_3(n;\{P,C\}) = \{S_n^3\}. \tag{4.5}$$

$n = 7$ (initial step).

By Theorem 4.1, $\mathrm{ex}_3(7;P|C) = 20$, while, by (4.4), $\mathrm{ex}_3(n;\{P,C\}) = 15$. So, by (4.3),

$$\mathrm{ex}_3(7;P) = \mathrm{ex}_3(7;P|C) = 20.$$

Moreover,

$$\mathrm{Ex}_3(7;P) = \mathrm{Ex}_3(7;P|C) = \{K_6^3 \cup K_1^3\}.$$

Indeed, the second equation above follows by the second part of Theorem 4.1, while the first one follows from the observation that there cannot be a C-free extremal 3-graph for P, because $\mathrm{ex}_3(n;\{P,C\}) = 15$.

$n \geqslant 8$ (inductive step).

Again by Theorem 4.1,

$$\mathrm{ex}_3(n;P|C) = 20 + \mathrm{ex}_3(n-6;P) = 20 + \begin{cases} \binom{n-6}{3}, & 6 \leqslant n \leqslant 12, \\ 20, & n = 13, \\ \binom{n-7}{2}, & n \geqslant 14, \end{cases}$$

where the second equality comes from the induction assumption.

It is easy to check that in each case the above quantity is strictly smaller than

$$\binom{n-1}{2} = \mathrm{ex}_3(n;\{P,C\}).$$

Hence, by (4.3),

$$\mathrm{ex}_3(n;P) = \binom{n-1}{2}$$

and, by (4.5), the only extremal 3-graph is the same as that for $\mathrm{ex}_3(n;C)$, i.e. S_n^3.

□

The second proof of Theorem 2.7, suggested by an anonymous referee of paper [16], avoids any reference to Theorem 2.4 and instead is based on the Erdős-Ko-Rado Theorem, Theorem 2.3, as well as Proposition 2.6. The proof is similar to, but simpler than the proof of Theorem 4.11 in Chapter 5.

Second proof of Theorem 2.7. For $n = 7$, suppose that H is P-free and $|H| \geqslant 20$. By Theorem 2.3, H contains two disjoint edges, e_1 and e_2. Let v be the vertex of H not in $e_1 \cup e_2$. If $deg_H(v) = 0$, we obtain the extremal 3-graph $K_6^3 \cup K_1^3$. If $\{v, x, y\} \in H$, then, since H is P-free, there is $i = 1, 2$ such that $\{x, y\} \subset e_i$. However, the presence of such an edge excludes from $H[e_1 \cup e_2]$ six edges of the form $\{a, b, c\}$, where $a \in e_i$, $a \neq x, y$, $b \in \{x, y\}$, and $c \in e_{3-i}$, because each of them, together with e_{3-i} and $\{v, x, y\}$, would form a copy of P. Hence, in this case, $|H| \leqslant \max\{3 + 20 - 6, 6 + 20 - 12\} = 17 < 20$, a contradiction.

The rest of the proof is by induction on n. For $n \geqslant 8$, suppose that H is P-free and $|H| \geqslant \binom{n-1}{2}$. By Theorem 2.3, either H is a full star and we are done or H contains two disjoint edges e_1 and e_2. Let $S = V \setminus (e_1 \cup e_2)$. We will consider two cases. Let $P_2^3 \cup K_3^3$ consist of a copy of P_2^3 and an edge K_3^3 vertex-disjoint from it.

Case 1. H is $(P_2^3 \cup K_3^3)$-free

For $n \geqslant 9$, we employ Proposition 2.6 to, say, $V \setminus e_1$, obtaining an upper bound of $n-3$ for the number edges induced in H by this set, provided $4|(n-3)$, and $n-4$ otherwise. Observe further that there are at most

$$\max\{3, n-6\} = n - 6$$

edges between e_1 and S (always with two vertices in e_1). Also, there are at most 19 edges spanned by $e_1 \cup e_2$, excluding e_2 which we have already counted in. Altogether, we have at most

$$2n + 10 < \binom{n-1}{2}$$

edges, for $n \geqslant 10$, a contradiction.

For $n = 9$, we have at most

$$2 \cdot 9 + 9 < \binom{9-1}{2} = 28$$

edges, a contradiction again.

For $n = 8$, we argue similarly but more subtly, using also the argument from the case $n = 7$. By Proposition 2.6 there are at most 4 edges within, say, $V \setminus e_1$. But any such edge, except for e_2 itself, due to the P-freeness of H, forbids the presence of 6 edges of $H[e_1 \cup e_2]$. Observe further that there are at most $\max\{3, 2\} = 3$ edges between e_1 and S, but any one of them forbids 6 edges of $H[e_1 \cup e_2]$, which, however, are different from the 6 edges excluded earlier. Altogether,

$$|H| \leqslant \max\{4 + 3 + 19 - 12, 1 + 3 + 19 - 6, 1 + 0 + 19\} = 20 < 21,$$

a contradiction.

Case 2. H contains a copy of $P_2^3 \cup K_3^3$

For a pair of vertices $\{x, y\}$ we call a vertex z a *co-neighbor* of $\{x, y\}$ in H if the triple $\{x, y, z\}$ is an edge of H.

Let $e_1 = \{a_1, a_2, a_3\}, e_2 = \{b_1, b_2, b_3\}$, and $f = \{c_1, c_2, a_3\}$ form a copy of $P_2^3 \cup K_3^3$ in H. For a pair of vertices $\{x, y\}$ contained in $e_1 \cup f$, let $N(x, y)$ denote the set of co-neighbors of $\{x, y\}$ outside $e_1 \cup f$.

Observe that if either $|N(a_1, a_2)| \geqslant 2$ or $|N(c_1, c_2)| \geqslant 2$, then $N(a_i, c_j) = \emptyset$ for all $i, j \in \{1, 2\}$; otherwise, we could find a loose path P in H. Similarly, if $|N(a_i, c_j)| \geqslant 2$ for some i, j, then $|N(a_1, a_2)| = |N(c_1, c_2)| = \emptyset$. Further, we have

$$|N(a_1, a_2)| + |N(c_1, c_2)| \leqslant 2n - 13, \tag{4.6}$$

since otherwise we could find $i, j \in \{1, 2, 3\}$, $i \neq j$ such that $\{a_1, a_2, b_i\}, \{c_1, c_2, b_j\} \in H$, which together with $\{b_1, b_2, b_3\}$ form P in H.

Let us now make more observations about $N(a_i, c_j)$, where $i, j \in \{1, 2\}$. If a_i, c_j have a co-neighbor b_k in $\{b_1, b_2, b_3\}$, then f, $\{a_i, c_j, b_k\}$, and e_2 form P in H. So, $N(a_i, c_j) \subseteq V \setminus (e_1 \cup f \cup e_2)$. From this we can see that

$$|N(a_1, c_1)| + |N(a_2, c_2)| \leqslant n - 7. \tag{4.7}$$

Indeed, otherwise $|N(a_1, c_1) \cap N(a_2, c_2)| \geqslant 2$, and so we could find two different vertices d_1, d_2 outside $e_1 \cup e_2 \cup f$ such that $\{a_1, c_1, d_1\}, \{a_2, c_2, d_2\}$, and $\{c_1, c_2, a_3\}$ form P in H. For a similar reason, we have

$$|N(a_1, c_2)| + |N(a_2, c_1)| \leqslant n - 7. \tag{4.8}$$

Let $W = \{a_1, a_2, c_1, c_2\}$ and $\sigma_W = \sum_{\{x,y\} \subseteq W} |N(x, y)|$. By our previous observations, if $|N(a_1, a_2)| \geqslant 2$ or $|N(c_1, c_2)| \geqslant 2$, then, by (4.6) $\sigma_W \leqslant 2n - 13$. If $|N(a_i, c_j)| \geqslant 2$ for some i, j, then, by (4.7) and (4.8) $\sigma_W \leqslant 2n - 14$. In the remaining case, when for all $\{x, y\} \subseteq W$, $|N(x, y)| \leqslant 1$, we have trivially $\sigma_W \leqslant 6$. In summary,

$$\sum_{\{x,y\} \subseteq W} |N(x, y)| \leqslant \max\{2n - 13, 6\}. \tag{4.9}$$

Since H does not contain P, no edge of H intersects W in exactly one vertex. The number of edges of H that intersect W and are contained in $\{a_1, a_2, a_3, c_1, c_2\}$ is at most $\binom{5}{3} = 10$. If $n - 4 \geqslant 7$, then by (4.9) the number of edges incident to W is at most

$$10 + \sigma_W \leqslant 2n - 3.$$

Thus, we can delete W and apply induction to $H - W$ to get

$$|H| \leqslant \binom{n-5}{2} + 2n - 3 < \binom{n-1}{2}.$$

If $n \in \{8, 9\}$ then we have

$$|H| \leqslant \binom{n-4}{3} + 10 + \sigma_W \leqslant \binom{n-4}{3} + 16 < \binom{n-1}{2}.$$

It remains to consider the case where $n = 10$. Our previous estimates yield only

$$|H| \leqslant \binom{6}{3} + 17 = 37.$$

However, if $|H - W| \geqslant 19$, then the edge e_1 together with two edges of $H - W$ (found greedily) would form a copy of P in H. So, $|H - W| \leqslant 18$ and, consequently, $|H| \leqslant 18 + 17 = 35 < \binom{9}{2}$. □

Comparing these two proofs, the first proof of Theorem 2.7 seems considerably shorter than the second one. However, it relies on Theorems 2.4 and 4.1, both having quite involved and technical proofs (see, respectively, [2] and the next section).

4.3. Proof of Theorem 4.1

In this section, first of all, we focus on the proof of Theorem 4.1. Then we make some preparations toward the proof of Lemma 4.5, and finally we prove it. To prove Theorem 4.1 we need the following fact.

Fact 4.1. *If $\mathcal{F}$ is a family of connected k-graphs then $ex_k(n; \mathcal{F})$, as a function of n, is superadditive, that is, for any pair of natural numbers n_1, n_2*

$$\mathrm{ex}_3(n_1; \mathcal{F}) + \mathrm{ex}_3(n_2; \mathcal{F}) \leqslant \mathrm{ex}_3(n_1 + n_2; \mathcal{F}).$$

Proof. Let $\mathcal{F}$ be a family of connected k-graphs, and n_1, n_2 be natural numbers. Choose $H_1 \in \mathrm{Ex}_k(n_1; \mathcal{F})$ and $H_2 \in \mathrm{Ex}_k(n_2; \mathcal{F})$. By the connectivity of all k-graphs in $\mathcal{F}$, the vertex disjoint union $H_1 \cup H_2$ is also $\mathcal{F}$-free. Consequently

$$\mathrm{ex}_3(n_1; \mathcal{F}) + \mathrm{ex}_3(n_2; \mathcal{F}) = |H_1| + |H_2| = |H_1 \cup H_2| \leqslant \mathrm{ex}_3(n_1 + n_2; \mathcal{F}).$$

□

Proof of Theorem 4.1. Let H be a P-free 3-graph with $V(H) = V$, $|V| = n \geqslant 7$, and let $C \subset H$. Further, denote by H_1 a connected component of H containing a copy of C and set $H_2 = H - V(H_1)$. Set also $n_i = |V(H_i)|$, $i = 1, 2$, and note that $n_1 + n_2 = n$. Clearly, $n_1 \geqslant 6$. Note that $|H_2| \leqslant \mathrm{ex}_3(n_2; P)$.

If $n_1 = 6$, then

$$|H| \leqslant |K_6^3| + \mathrm{ex}_3(n - 6; P) = 20 + \mathrm{ex}_3(n - 6; P),$$

and every 3-graph H which reaches this bound is a disjoint union of $H_1 = K_6^3$ and a 3-graph $H_2 \in \mathrm{Ex}_3(n - 6; P)$. To finish the proof, it is enough to show that if $n_1 \geqslant 7$, then

$$|H| < 20 + \mathrm{ex}_3(n - 6; P).$$

We first estimate $|H_1|$. For $7 \leqslant n_1 \leqslant 12$, by Lemma 4.5 and Theorem 2.7 (for $n \leqslant 6$),

$$|H_1| \leqslant 3n_1 - 8 < 20 + \binom{n_1 - 6}{3} = 20 + \mathrm{ex}_3(n_1 - 6; P).$$

For $n_1 \geqslant 13$, again by Lemma 4.5 (applied twice),

$$|H_1| \leqslant 3n_1 - 8 < 20 + 3(n_1 - 6) - 8 = 20 + \mathrm{ex}_3^{con}(n_1 - 6; P|C) \leqslant 20 + \mathrm{ex}_3(n_1 - 6; P).$$

Finally, by the above estimates of $|H_1|$ and by Fact 4.1,

$$|H| = |H_1| + |H_2| \leqslant 20 + \mathrm{ex}_3(n_1 - 6; P) + \mathrm{ex}_3(n_2; P) \leqslant 20 + \mathrm{ex}_3(n - 6; P).$$

□

4.4. Proofs of Lemma 4.5 and Lemma 4.6

Before we prove Lemma 4.6, we have to prepare some background. Let H be a connected, P-free 3-graph with $V(H) = V$ and $|V| = n \geqslant 7$ vertices, containing a copy of the triangle. With some abuse of notation, let us denote by C a fixed copy of triangle in H. Set

$$U = V(C) = \{x_1, x_2, x_3, y_1, y_2, y_3\},\ U = U_1 \cup U_2,$$

where $U_1 = \{y_1, y_2, y_3\}$ is the set of vertices of degree one in C, while $U_2 = \{x_1, x_2, x_3\}$ is the set of vertices of degree two.

Let us also recall that we identify the edge set of a 3-graph with the 3-graph itself, hence (see Figure 4.2)

$$C = \{\{x_1, y_2, x_3\}, \{x_3, y_1, x_2\}, \{x_2, y_3, x_1\}\}.$$

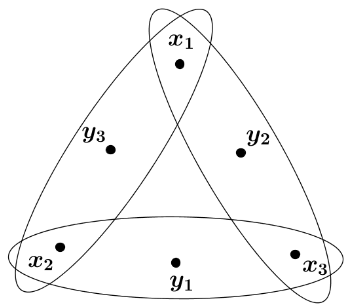

Figure 4.2. A fixed copy of triangle in H

Then, for $s \geqslant 1$, let $W = V \backslash U = \{w_1, ..., w_s\}$, where $|W| = s = n - 6$. Further, let us divide the set of edges of H into three sets (see Figure 4.3):

.

$$H[U] = H \cap \binom{U}{3},\ H[W] = H \cap \binom{W}{3} \text{ and } H(U, W) = H \backslash (H[U] \cup H[W]).$$

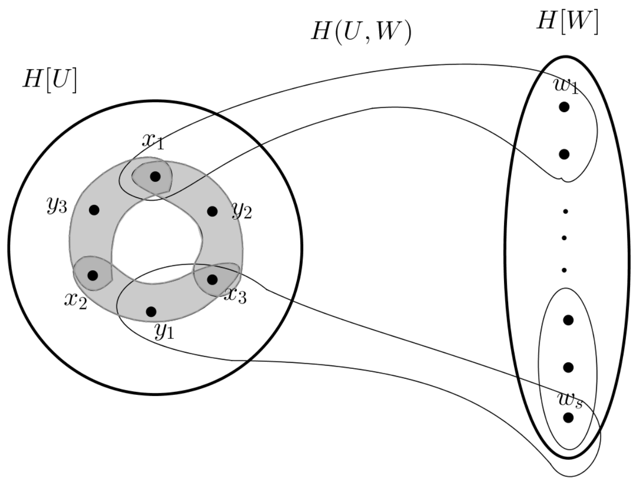

Figure 4.3. The division of the set of edges of H

Notice that $H(U,W)$ is the set of all edges of H which intersect both U and W. Finally, we define two sets of triples (not necessarily in H):

$$E_1 = \{\{x_i, y_i, w_l\} : 1 \leqslant i \leqslant 3, 1 \leqslant l \leqslant s\},$$

$$E_2 = \{\{x_i, x_j, w_l\} : 1 \leqslant i < j \leqslant 3, 1 \leqslant l \leqslant s\},$$

and set

$$E' = E_1 \cup E_2. \tag{4.10}$$

The edges in E' are formed by taking one vertex from the set W and two vertices from C. For an edge in E_1, one vertex is of degree 1 in C, another of degree 2, but they do not belong to the same edge of C. Similarly, the edges in E_2 are formed by one vertex from the set W and two vertices of degree 2 in C (see Figure 4.4).

Now, let us formulate several simple observations. Fact 4.2 below has been first observed in [15] (Facts 1-3,6) and then used in [16] (Facts 1-2). It implies that although, in principle, $H(U,W)$ may consist of edges having one vertex in U and two in W, the assumption that H is P-free makes it impossible. For the same reason, out of all potential edges with two vertices in U and one in W, only those belonging E' can actually occur in H.

Fact 4.2.

a) $H = H[U] \cup H(U,W)$*, that is,* $H[W] = \emptyset$*,*

b) $H(U,W) \subseteq E' = E_1 \cup E_2$*.*

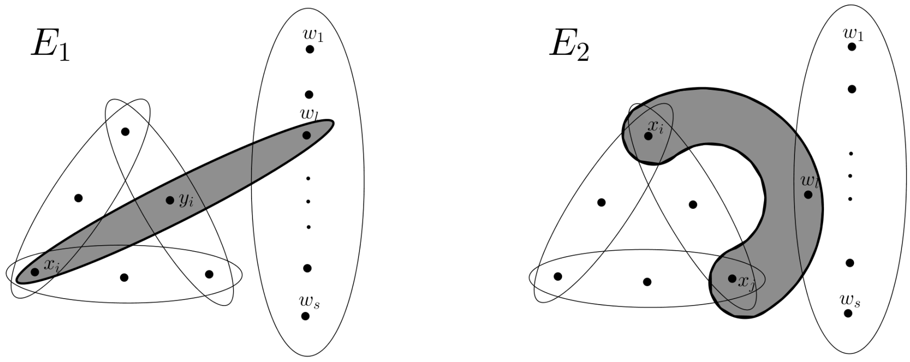

Figure 4.4. The edges from the family $E' = E_1 \cup E_2$ are shaded

Proof. By the P-freeness of H, no edge of H intersects U in just one vertex. If $|f \cap U| = 2$, then there is $e_f \in C : |e_f \cap f| = 1$. Suppose $HW \neq \emptyset$. Then, by the connectivity of H, there exist $g \in H[W]$ and $f \in H(U, W)$ such that $|g \cap f'| = 1$. Consequently, $\{e_f, g, f\}$ is a copy of P (see Figure 4.5), a contradiction. Hence, $H[W] = \emptyset$ and $a)$ holds.

If there were an edge $e \in C$ and $f \in H(U, W)$ such that $e \cap f = \emptyset$, then $\{e, e_f, f\}$ would be a copy of P, a contradiction. Thus, every edge of $H(U, W)$ must intersect all three edges of C, and thus belong to E'. Hence, $H(U, W) \subseteq E'$ and $b)$ holds. $\square$

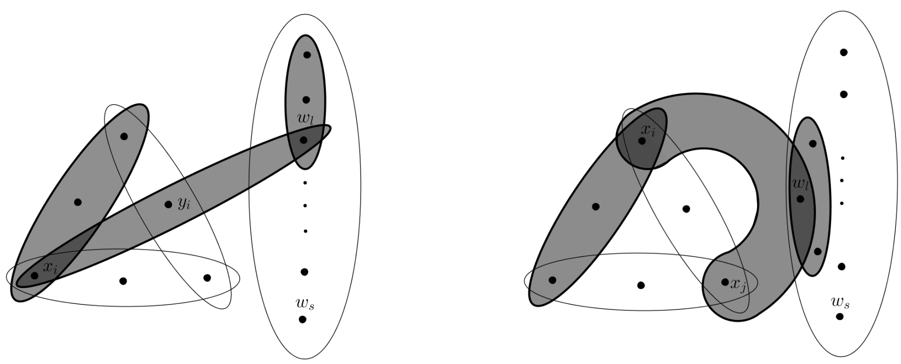

Figure 4.5. Illustrations to the proof of Fact 4.2

Fact 4.3. *$H(U, W)$ is intersecting.*

Proof. Recall that $H(U, W) \subseteq E'$ and note that E_2 is intersecting by definition. Suppose there exist $e \in E_1$ and $f \in E'$ such that $e \cap f = \emptyset$. Then $C \cup \{e\} \cup \{f\} \supset P$, a contradiction. Thus, either e or f cannot be in H, and $H(U, W)$ is, indeed, intersecting. $\square$

Now, we are ready to prove Lemma 4.6.

Proof of Lemma 4.6. Let H be a connected, P-free 3-graph on $n \geqslant 7$ vertices, containing M and C. We use notation introduced at the beginning of this section. Let $f, h \in H$

satisfy $f \cap h = \emptyset$. By Facts 4.2 and 4.3, at least one of f and h belongs to $H[U]$. Without loss of generality we assume that $f \in H[U]$. If also $h \in H[U]$, then clearly, $f \cup h = U$. Since $n \geqslant 7$ and H is connected, $H(U,W) \neq \emptyset$. By the P-freeness of H, each $e \in H(U,W)$ needs to be disjoint from either f or h. Either way, there exist two disjoint edges $e, f \in H$ such that $e \in H(U,W)$ and $f \in H[U]$.

Notice that if $e \in E_1$, then $C \cup \{e\} \cup \{f\} \supset P$, a contradiction. Thus, $e \in E_2$, say, $e \cap U = \{x_1, x_2\}$. Then, the only edge in $H[U]$ disjoint from e, which does not create a copy of the path P with e and an edge of C is $f = \{x_3, y_1, y_2\}$ (see Figure 4.6).

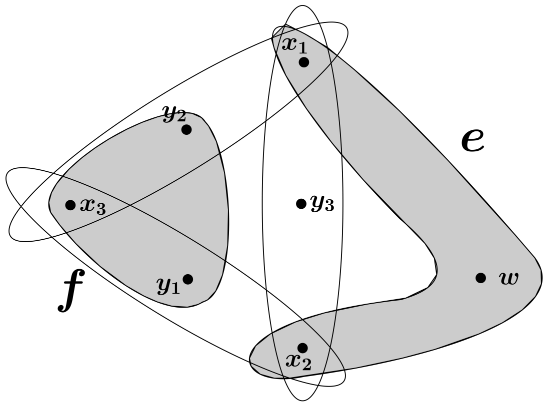

Figure 4.6. Illustration to the proof of Lemma 4.6

Further, observe that all triples in E', except those of the type $\{x_1, x_2, w\}$, $w \in W$, form a copy of P with f and some edge of C (see Figure 4.7). Hence,

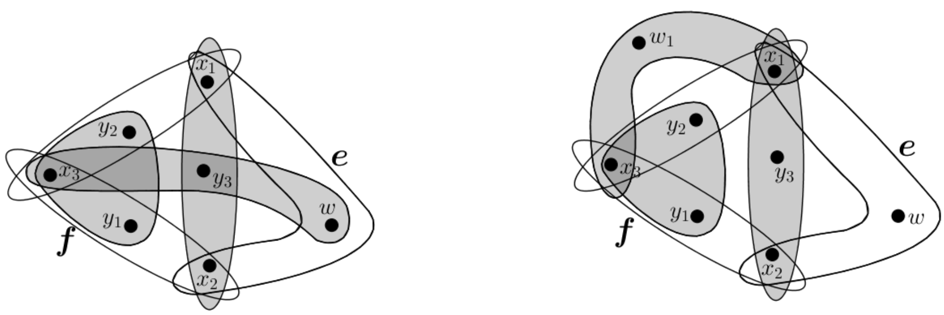

Figure 4.7. Examples of triples in E' which form a copy of P

$$H(U,W) \subseteq \{\{x_1, x_2, w\} : w \in W\}. \tag{4.11}$$

Let

$$X = \left\{ h \in \binom{U}{3} \setminus C : y_3 \in h \right\}.$$

For each $h \in X$, we have $C \cup \{e, f, h\} \supset P$, and thus

$$H[U] \subseteq \binom{U}{3} \setminus X. \tag{4.12}$$

It means, however, that the set $H[U]$ consists of the edges formed by vertices $\{x_1, x_2, x_3, y_1, y_2\}$, which form the core of a satellite, and the edge $\{x_1, y_3, x_2\}$, which, along with the edges of $H(U, W)$, forms the tail of the satellite (cf. (4.11)). We conclude that $H \subseteq \mathrm{Sa}(n)$ (see Figure 4.8).

□

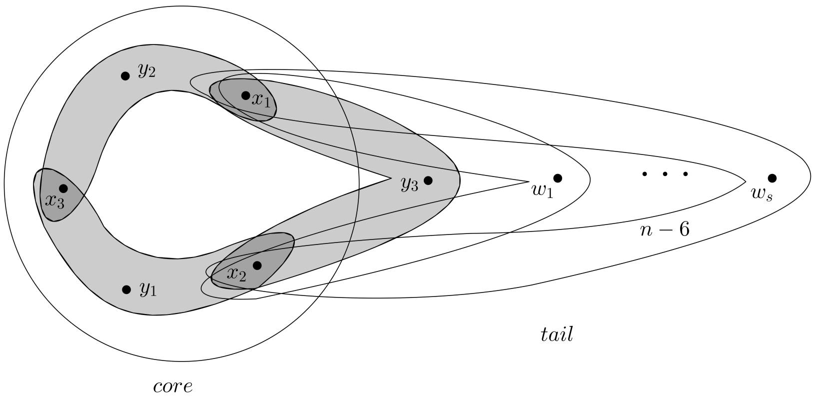

Figure 4.8. Satellite as an extremal graph for H in a proof of Lemma 4.6

Last, but not least, we prove Lemma 4.5.

Proof of Lemma 4.5. Let H be a connected P-free 3-graph with $n \geqslant 7$ vertices, containing a copy of the triangle C. Clearly H cannot be a subset of the star S^3_n. If $M \nsubseteq H$, then by Theorem 2.8,

$$|H| \leqslant \mathrm{ex}^{(2)}_3(n; \{P, M\}) \leqslant \mathrm{ex}^{(2)}_3(n; M) = 3n - 8$$

and the assertion follows. On the other hand, if $M \subseteq H$, then, by Lemma 4.6, $H \subseteq \mathrm{Sa}(n)$ and therefore

$$|H| \leqslant |\mathrm{Sa}(n)| = n + 5 < 3n - 8.$$

□

4.5. Turán numbers for non-intersecting 3-graphs

In this section we focus on conditional Turán numbers for non-intersecting H, that is, when $H \supset M := M^3_2$. We start with the following result which was first proved in [16].

Proposition 4.8. *We have*

$$\mathrm{ex}_3^{(2)}(n; P) = \mathrm{ex}_3(n; P|M) \quad \text{for} \quad n \geqslant 11$$

and

$$\mathrm{ex}_3^{(2)}(n; C) = \mathrm{ex}_3(n; C|M) \quad \text{for} \quad n \geqslant 8.$$

Proof. Observe that, for each $F \in \{P, C\}$

$$\mathrm{ex}_3^{(2)}(n; F) = \max\{\mathrm{ex}_3(n; F|M), \mathrm{ex}_3^{(2)}(n; \{F, M\})\} \tag{4.13}$$

and by Theorem 2.8

$$\mathrm{ex}_3^{(2)}(n; \{F, M\}) \leqslant \mathrm{ex}_3^{(2)}(n; M) = 3n - 8.$$

Let us consider a comet $\mathrm{Co}(n)$ for $n \geqslant 6$ (see Figure 2.6). Note that $\mathrm{Co}(n) \supseteq M$ and $\mathrm{Co}(n) \not\supseteq P$, and so

$$\mathrm{ex}_3(n; P|M) \geqslant \mathrm{Co}(n) = \binom{n-4}{2} + 4 \geqslant 3n - 8$$

for $n \geqslant 11$, which, in turn, by (4.13), implies that

$$\mathrm{ex}_3^{(2)}(n; P) = \mathrm{ex}_3(n; P|M).$$

To prove the second equality, we use the following construction. For $n \geqslant 7$, consider a 3-graph $H(n; C|P)$ consisting of an edge $\{x, y, z\}$ and all edges of the form $\{x, y, w\}$, $w \neq z$, and $\{z, w', w''\}$, where $\{w', w''\} \cap \{x, y\} = \emptyset$ (see Figure 4.9).

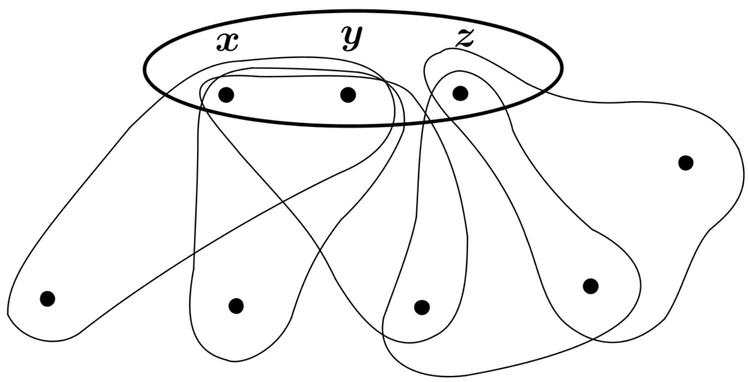

Figure 4.9. Part of the 3-graph $H(n; C|P)$

Note that $H(n; C|P) \supseteq P$ and $H(n; C|P) \not\supseteq C$ and thus

$$\mathrm{ex}_3(n; C|P) \geqslant |H(n; C|P)| = 1 + (n-3) + \binom{n-3}{2} = \binom{n-2}{2} + 1. \tag{4.14}$$

Since $M \subset P$,

$$\mathrm{ex}_3(n;C|M) \geqslant \mathrm{ex}_3(n;C|P) \geqslant |H(n;C|P)| \geqslant \binom{n-2}{2} + 1 \geqslant 3n-8$$

for $n \geqslant 8$, and thus, we also have

$$\mathrm{ex}_3^{(2)}(n;C) = \mathrm{ex}_3(n;C|M).$$

□

Inspired by the above result, we state now the following theorem from [17] which is going to be instrumental in the proofs of Theorems 2.9 and 2.10. It will be proved in Section 5.2.

Theorem 4.9.

$$\mathrm{ex}_3(n;P|M) = \begin{cases} 20 + \binom{n-6}{3} & \text{and} \quad \mathrm{Ex}_3(n;P|M) = \{K_6^3 \cup K_{n-6}^3\} & \text{for} \quad 6 \leqslant n \leqslant 12, \\ 40 & \text{and} \quad \mathrm{Ex}_3(n;P|M) = \{K_6^3 \cup K_6^3 \cup K_1^3, \mathrm{Co}(13)\} & \text{for} \quad n = 13, \\ 4 + \binom{n-4}{2} & \text{and} \quad \mathrm{Ex}_3(n;P|M) = \{\mathrm{Co}(n)\} & \text{for} \quad n \geqslant 14. \end{cases}$$

We can draw the following corollary of Theorem 4.9.

Corollary 4.10. *For $n \geqslant 14$,*

$$\mathrm{ex}_3(n;C|M) = \mathrm{ex}_3(n;C|P).$$

Proof. By (4.14) and Theorem 4.9, for $n \geqslant 14$

$$\mathrm{ex}_3(n;C|P) \geqslant |H(n;C|P)| = \binom{n-2}{2} + 1 \geqslant 4 + \binom{n-4}{2} = \mathrm{ex}_3(n;P|M).$$

Thus,

$$\mathrm{ex}_3(n;C|M) = \max\{\mathrm{ex}_3(n;C|\{M,P\}), \mathrm{ex}_3(n;\{C,P\}|M)\} \leqslant$$

$$\max\{\mathrm{ex}_3(n;C|P), \mathrm{ex}_3(n;P|M)\} = \mathrm{ex}_3(n;C|P).$$

The inverse inequality is trivial (as explained in the proof of Proposition 4.8). □

In [17] we also considered a more restricted version of Theorem 4.9, where we forbad the pair $\{P,C\}$, conditioning on a 3-graph H being non-intersecting. This result, interesting in its own right, serves as a main tool in the proof of Theorem 4.9.

Theorem 4.11.

$$\mathrm{ex}_3(n;\{P,C\}|M) = \begin{cases} 2n-4 & \text{for} \quad 6 \leqslant n \leqslant 9, \\ 20 & \text{for} \quad n = 10, \\ 4 + \binom{n-4}{2} \quad \text{and } \mathrm{Ex}_3(n;\{P,C\}|M) = \{\mathrm{Co}(n)\} & \text{for} \quad n \geqslant 11. \end{cases}$$

Note that the Turán numbers $\mathrm{ex}_3(n;P|M)$, and $\mathrm{ex}_3^{(2)}(n;P)$ coincide for $n \geqslant 8$, and moreover they coincide with the Turán number $\mathrm{ex}_3(n;\{P,C\}|M)$ for $n \geqslant 13$.

Chapter 5

Determining Turán numbers of higher order

In this chapter we focus on the proofs of Theorem 2.9, determining the second order Turán number for all n, and Theorem 2.10, determining the third order Turán number for $n = 12$. In their proofs we will use some results stated earlier, most notably Theorem 4.9.

To prove Theorem 4.9 we will need Theorem 4.11, while in the proof of Theorem 4.11 we will make use of a lemma, formulated in [17], which states that if, additionally to $\{P, C\}$, we forbid also $P_2^3 \cup K_3^3$, then the formula for $\mathrm{ex}_3(n; \{P, C\}|M)$ valid only for $6 \leqslant n \leqslant 9$ (cf. Theorem 4.11), takes over for all values of n.

Lemma 5.1. *For $n \geqslant 6$*

$$\mathrm{ex}_3(n; \{P, C, P_2^3 \cup K_3^3\}|M) = 2n - 4.$$

In Section 5.1 we prove Theorems 2.9 and 2.10. Then, in Section 5.2 we give a proof of Theorem 4.9 in which we use Theorem 4.11. The proof of this latter theorem is very technical and relies, in turn, on Lemma 5.1. Both these results are proved in the last section, Section 5.3.

5.1. Proofs of Theorems 2.9 and 2.10

Proof of Theorem 2.9. We begin the proof with considering the case $n = 7$ separately.

Case (n=7). By Theorem 2.7,

$$\mathrm{ex}_3(7; P) = 20 \quad \text{and} \quad \mathrm{Ex}_3(7; P) = \{K_6^3 \cup K_1^3\}.$$

By Definition 2.11 to determine $\mathrm{ex}_3^{(2)}(7; P)$ we need to find the largest number of edges in a 7-vertex P-free 3-graph H which is not a sub-3-graph of $K_6^3 \cup K_1^3$. Note that

$$P \nsubseteq S_7^3 \nsubseteq K_6^3 \cup K_1^3,$$

and thus,

$$\mathrm{ex}_3^{(2)}(7; P) \geqslant |S_7^3| = \binom{7-1}{2} = 15.$$

If H is a 7-vertex P-free 3-graph with $|H| > 15$, then, by Theorem 2.4, $C \subset H$. But then, since $H \nsubseteq K_6^3 \cup K_1^3$, H must be connected. Consequently, by Lemma 4.5, $|H| \leqslant 3 \cdot 7 - 8 = 13$, a contradiction. Thus, $\mathrm{ex}_3^{(2)}(7; P) = 15$.

To finish the proof in the case $n = 7$, we have to show that S^3_7 is the unique 2-extremal 3-graph for P. Let H be a 7-vertex P-free 3-graph with 15 edges and such that $H \not\subseteq K^3_6 \cup K^3_1$ and $H \neq S^3_7$. Since $H \neq S^3_7$ and $|H| > 13$, by Theorem 2.8, $M \subset H$. Let $e_1 = \{a, b, c\}$ and $e_2 = \{x, y, z\}$ be two disjoint edges of H, and let v be the seventh vertex. Since $H \not\subseteq K^3_6 \cup K^3_1$, $deg_H(v) > 0$. However, since H is P-free, there are no edges containing v and one vertex from each e_1 and e_2. W.l.o.g., let $\{v, a, b\} \in H$. Then to avoid P, none of the following six edges can be present in H (we skip brackets for convenience): $cax, cay, caz, cbx, cby, cbz$. If there is also an edge containing v and intersecting e_2, then another six edges must be missing from H. Thus, in this case $|H| \leqslant 6 + 20 - 12 = 14$, a contradiction. Otherwise we have to refine our count and observe that, due to the presence of $\{v, a, b\}$ in H, at least one edge out of each of the following pairs must be absent from H: axy or cyz, axz or cxy, azy or cxz. This means that $|H| \leqslant 3 + 20 - 6 - 3 = 14$, again a contradiction. Hence $\mathrm{Ex}^{(2)}_3(7; P) = \{S^3_7\}$.

Case ($\mathbf{n \geqslant 8}$). By Theorem 2.7,

$$\mathrm{ex}_3(n; P) = \binom{n-1}{2} \quad \text{and} \quad \mathrm{Ex}_3(n; P) = \{S^3_n\}.$$

Therefore, to determine $\mathrm{ex}^{(2)}_3(n; P)$ for $n \geqslant 8$ we need to find the largest number of edges in an n-vertex P-free 3-graph H which is not a star. We have,

$$\mathrm{ex}^{(2)}_3(n; P) = \max\{\mathrm{ex}^{(2)}_3(n; \{P, M\}), \mathrm{ex}_3(n; P|M)\}.$$

By monotonicity and Theorem 2.8,

$$\mathrm{ex}^{(2)}_3(n; \{P, M\}) \leqslant \mathrm{ex}^{(2)}_3(n; M) = 3n - 8,$$

while $\mathrm{ex}_3(n; P|M)$ is given by Theorem 4.9. One can easily verify that for $n \geqslant 8$ we have $\mathrm{ex}_3(n; P|M) > 3n - 8$. Consequently,

$$\mathrm{ex}^{(2)}_3(n; P) = \mathrm{ex}_3(n; P|M)$$

and Theorem 2.9 for $n \geqslant 8$ follows from Theorem 4.9. □

Proof of Theorem 2.10. Let us recall that we are to show that

$$\mathrm{ex}^{(3)}_3(12; P) = 32 \quad \text{and} \quad \mathrm{Ex}^{(3)}_3(12; P) = \{\mathrm{Co}(12)\}.$$

By Theorem 2.9,

$$\mathrm{ex}^{(2)}_3(12; P) = 40 \quad \text{and} \quad \mathrm{Ex}^{(2)}_3(12; P) = \{K^3_6 \cup K^3_6\},$$

while by Theorem 2.7,

$$\mathrm{Ex}^{(1)}_3(12; P) = \{S^3_{12}\}.$$

Therefore, to determine $\mathrm{ex}_3^{(3)}(12;P)$ we have to find the largest number of edges in a 12-vertex P-free 3-graph H such that

$$H \nsubseteq S_{12}^3 \quad \text{and} \quad H \nsubseteq K_6^3 \cup K_6^3. \tag{5.1}$$

The comet Co(12) satisfies all the above conditions and has 32 edges. Let H be a 12-vertex P-free 3-graph satisfying conditions (5.1) but $H \neq$Co(12). In order to prove Theorem 2.10, we have to show that $|H| < 32$.

If H is non-trivially intersecting then, by Theorem 2.8,

$$|H| \leqslant 3 \cdot 12 - 8 = 28 < 32.$$

If $H \supset M$ and H is disconnected, then, since $H \nsubseteq K_6^3 \cup K_6^3$, by Theorems 2.7 and 4.9,

$$|H| \leqslant \max\{\mathrm{ex}_3(7;P) + \mathrm{ex}_3(5;P), \mathrm{ex}_3(8;P) + \mathrm{ex}_3(4;P),$$

$$\mathrm{ex}_3(9;P) + \mathrm{ex}_3(3;P), \mathrm{ex}_3(10;P|M), \mathrm{ex}_3(11;P|M)\} =$$

$$\max\{20 + 10, 21 + 4, 28 + 1, 24, 30\} = 30 < 32.$$

Finally, assume that H is connected and $H \supset M$. If, in addition, $C \subseteq H$, then, by Lemma 4.5, we have

$$|H| \leqslant \mathrm{ex}_3^{con}(n;P|C) = 3 \cdot 12 - 8 = 28 < 32.$$

Otherwise, H is a $\{P, C\}$-free 3-graph containing M and, by Theorem 4.11,

$$|H| \leqslant \mathrm{ex}_3(12;\{P,C\}|M) = 4 + \binom{12-4}{2} = 32.$$

However, again by Theorem 4.11, the comet Co(12) is the only M-extremal 3-graph for $\{P, C\}$ and, consequently, $|H| < 32$. $\square$

5.2. Proof of Theorem 4.9

In this section we present the proof of Theorem 4.9.

Proof of Theorem 4.9. Our goal is to determine $\mathrm{ex}_3(n;P|M)$ and $\mathrm{Ex}_3(n;P|M)$ for all $n \geqslant 6$. But first we need another piece of notation. Let $\widehat{\mathrm{ex}_3}(n;P|\{C,M\})$ be the largest number of edges in an n vertex P-free 3-graph H which contains both, a copy of M *and* a copy of C. Note that the definition of $\mathrm{ex}_3(n;P|\{C,M\})$ given in Section 4.1 requires only that there is a copy of M in H *or* there is a copy of C in H. (In [17] we erroneously used notation $\mathrm{ex}_3(n;P|\{C,M\})$ for $\widehat{\mathrm{ex}_3}(n;P|\{C,M\})$).

By considering whether or not a 3-graph H contains a copy of the triangle C, we infer that

$$\mathrm{ex}_3(n;P|M) = \max\{\mathrm{ex}_3(n;\{P,C\}|M), \widehat{\mathrm{ex}_3}(n;P|\{C,M\})\}.$$

The number $\mathrm{ex}_3(n;\{P,C\}|M)$ is given by Theorem 4.11, whereas

$$\widehat{\mathrm{ex}_3}(n;P|\{C,M\}) = \mathrm{ex}_3(n;P|C),$$

since the unique extremal graph from Corollary 4.2 contains M. Now, we are going to show that for $6 \leqslant n \leqslant 12$,

$$\mathrm{ex}_3(n;P|C) > \mathrm{ex}_3(n;\{P,C\}|M),$$

for $n = 13$,

$$\mathrm{ex}_3(n;P|C) = \mathrm{ex}_3(n;\{P,C\}|M),$$

while, for $n \geqslant 14$,

$$\mathrm{ex}_3(n;P|C) < \mathrm{ex}_3(n;\{P,C\}|M).$$

Indeed,

for $6 \leqslant n \leqslant 9$,

$$\mathrm{ex}_3(n;P|C) = 20 + \binom{n-6}{3} > 2n-4 = \mathrm{ex}_3(n;\{P,C\}|M),$$

for $n = 10$,

$$\mathrm{ex}_3(n;P|C) = 24 > 20 = \mathrm{ex}_3(10;\{P,C\}|M),$$

for $11 \leqslant n \leqslant 12$,

$$\mathrm{ex}_3(n;P|C) = 20 + \binom{n-6}{3} > 4 + \binom{n-4}{2} = \mathrm{ex}_3(n;\{P,C\}|M),$$

for $n = 13$,

$$\mathrm{ex}_3(13;P|C) = 40 = 4 + \binom{13-4}{2} = \mathrm{ex}_3(13;\{P,C\}|M),$$

while for $n \geqslant 14$,

$$\mathrm{ex}_3(n;P|C) = 20 + \binom{n-7}{2} < 4 + \binom{n-4}{2} = \mathrm{ex}_3(n;\{P,C\}|M).$$

Hence, Theorem 4.9 follows immediately from the respective parts of Corollary 4.2 and Theorem 4.11. $\square$

5.3. Proof of Theorem 4.11

This section of my dissertation is entirely devoted to the proof of Theorem 4.11. Recall, that Theorem 4.11 determines the maximal number of edges in an n-vertex 3-graph H which is P-free and C-free but is not intersecting, that is, H contains M.

Before going through the proof of Theorem 4.11, we first prove Lemma 5.1.

Proof of Lemma 5.1. Let V be a set with $|V| = n \geqslant 6$. Fix four different vertices $v_1, v_2, v_3, v_4 \in V$ and define a 3-graph $H_n^{(0)}$ on V as

$$H_n^{(0)} = \left\{ h \in \binom{V}{3} : \{v_i, v_{i+1}\} \subset h, i \in \{1,3\} \right\}.$$

Note that $M \subset H_n^{(0)}$ and $|H_n^{(0)}| = 2n - 4$. Moreover, since every edge contains one of the pairs $\{v_1, v_2\}$ or $\{v_3, v_4\}$, among any three edges at least two share two vertices. Therefore, $H_n^{(0)}$ is $\{P, C, P_2^3 \cup K_3^3\}$-free and, thus,

$$\mathrm{ex}_3(n; \{P, C, P_2^3 \cup K_3^3\}|M) \geqslant 2n - 4.$$

To show the opposite inequality, consider a $\{P, C, P_2^3 \cup K_3^3\}$-free 3-graph H containing a matching $M = \{e, f\}$, with $V(H) = V$, $|V| = n \geqslant 6$. Since H is $P_2^3 \cup K_3^3$-free, $H[V \backslash e]$ and $H[V \backslash f]$ are P_2^3-free, and by Proposition 2.6

$$|H[V \backslash e]| \leqslant n - 3 \quad \text{and} \quad |H[V \backslash f]| \leqslant n - 3.$$

Since H is P-free, there is no edge $h \in H$ with $|h \cap e| = |h \cap f| = 1$. Hence, if $|H[e \cup f]| = 2$, then $|H| \leqslant 2(n-3) = 2n - 6$.

On the other hand, if there exists an edge $h \in H[e \cup f] \backslash \{e, f\}$, then, since H is $P_2^3 \cup K_3^3$-free, all edges of H intersect one of e or f on at least two vertices. Let

$$F_e = \{h \in H : |h \cap e| = 2\}, \; F_f = \{h \in H : |h \cap f| = 2\}.$$

If there existed $h_1, h_2 \in F_e$ with $|h_1 \cap h_2| = 1$, then, depending on whether $|(h_1 \cup h_2) \cap f| = 0, 1$ or 2, the edges $\{h_1, h_2, f\}$ would form, respectively, a copy of C, P or $P_2^3 \cup K_3^3$ (see Figure 5.1).

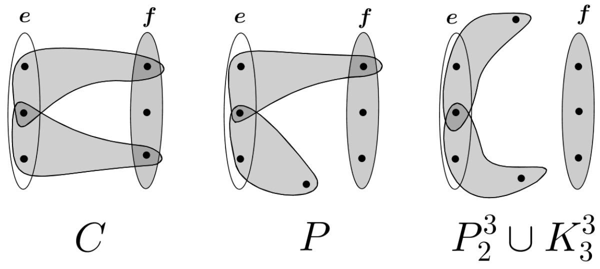

Figure 5.1. Illustration to the proof of Lemma 5.1

Thus,

$$\forall h_1, h_2 \in F_e, \quad |h_1 \cap h_2| = 2,$$

so, either all pairs $h_1, h_2 \in F_e$ share two vertices of e or all pairs $h_1, h_2 \in F_e$ share one vertex of $V \backslash e$ (and another in e).

This implies that

$$|F_e| \leqslant \max\{n-3, 3\} = n-3.$$

Similarly, $|F_f| \leqslant n-3$ and, consequently,

$$|H| = |\{e, f\}| + |F_e| + |F_f| \leqslant 2 + (n-3) + (n-3) = 2n-4,$$

which completes the proof.

□

We still have to make some more preparations before proceeding with the proof of Theorem 4.11. Note that since $|V(P_2^3 \cup K_3^3)| = 8$, no n-vertex 3-graph, $n = 6, 7$, contains a copy of $P_2^3 \cup K_3^3$ and therefore, by Lemma 5.1,

$$\mathrm{ex}_3(n; \{P, C\}|M) = \mathrm{ex}_3(n; \{P, C, P_2^3 \cup K_3^3\}|M) = 2n-4.$$

Thus, from now on we will be assuming that $n \geqslant 8$. Define a sequence of 3-graphs

$$H_n = \begin{cases} H_n^{(0)} & \text{for} \quad 8 \leqslant n \leqslant 9, \\ K_5^3 \cup K_5^3 & \text{for} \quad n = 10, \\ \mathrm{Co}(n) & \text{for} \quad n \geqslant 11, \end{cases}$$

where $H_n^{(0)}$ is the 3-graph introduced in the proof of Lemma 5.1. By an inspection, we can see that H_n is $\{P, C\}$-free and contains M. Hence

$$\mathrm{ex}_3(n; \{P, C\}|M) \geqslant |H_n| = \begin{cases} 2n-4 & \text{for} \quad 8 \leqslant n \leqslant 9, \\ 20 & \text{for} \quad n = 10, \\ 4 + \binom{n-4}{2} & \text{for} \quad n \geqslant 11. \end{cases}$$

The main difficulty lies in showing the reverse inequality, namely, that any $\{P, C\}$-free 3-graph H on $n \geqslant 8$ vertices, containing M, satisfies $|H| \leqslant |H_n|$. Moreover, for $n \geqslant 11$, we want to show that the equality is reached by the extremal 3-graph H_n =Co(n) only. We may assume that H contains a copy $P_2^3 \cup K_3^3$. On the other hand, by Lemma 5.1,

$$|H| \leqslant 2n - 4 \leqslant |H_n|,$$

where, in fact, $2n - 4 < |H_n|$ for $n \geqslant 10$. Before turning to the actual proof of Theorem 4.11, we need to introduce some notation and prove facts related to the structure of H.

Let us assume that H is $\{P, C\}$-free and contains a copy of $P_2^3 \cup K_3^3$. Let $e_1, e_2 \in H$ and $x \in V = V(H)$ be such that $e_1 \cap e_2 = \{x\}$ and there is an edge in H disjoint from $e_1 \cup e_2$. We know that such a choice of e_1, e_2, x exists, because $P_2^3 \cup K_3^3 \subseteq H$. We split $V = U \cup W$, where

$$U = e_1 \cup e_2, \quad \text{and} \quad W = V \backslash U.$$

Note that $|U| = 5$ and $|W| = n - 5$. Further set

$$H(U, W) = H \backslash (H[U] \cup H[W])$$

for the sub-3-graph of H consisting of all edges intersecting both, U and W. Notice that $H[W] \neq \emptyset$, and thus the set W_0 of vertices of degree 0 in $H[W]$ has size

$$|W_0| \leqslant n - 8. \tag{5.2}$$

Set also $W_1 = W \backslash W_0$ (see Figure 5.2).

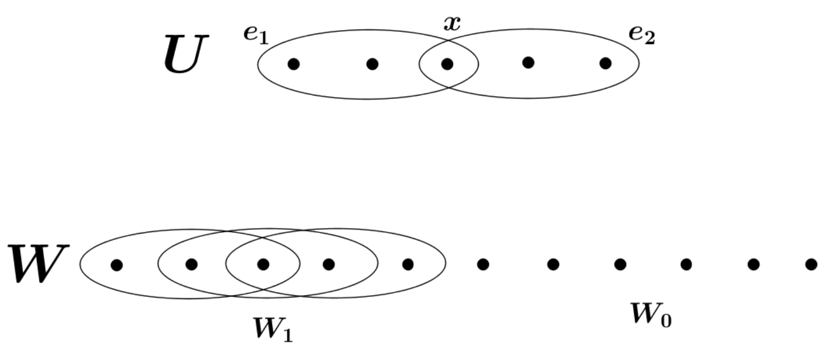

Figure 5.2. The structure of H in the proof of Theorem 4.11

Let us split

$$H[U] = \{e_1, e_2\} \cup E(x) \cup E(\overline{x}),$$

where $E(x)$ contains all edges of $H[U]$ which contain vertex x, except for e_1 and e_2, while $E(\overline{x})$ contains all other edges of $H[U]$. Note that

$$\max\{|E(x)|, |E(\overline{x})|\} \leqslant 4. \tag{5.3}$$

We also split the set of edges of $H(U, W)$. First, notice that if for some $h \in H(U, W)$ we have $|h \cap U| = 1$, then $h \cap U = \{x\}$, since otherwise h together with e_1 and e_2 would form a copy of P in H. Let

$$F^0 = \{h \in H(U, W) : h \cap U = \{x\}\}.$$

The edges $h \in H(U, W)$ with $|h \cap U| = 2$ must satisfy $h \cap U \subset e_1$ or $h \cap U \subset e_2$, since otherwise h together with edges e_1 and e_2 would form a copy of C in H. For $k = 1, 2$ we define

$$F^k = \{h \in H(U, W) : |h \cap (U \backslash \{x\})| = k\}.$$

We have $H(U, W) = F^0 \cup F^1 \cup F^2$. (Note that in each case $k = 0, 1, 2$, the superscript k stands for the common size of the set $h \cap (U \backslash \{x\})-$ see Figure 5.3).

For a sub-3-graph $F \subseteq H(U, W)$ and $i = \{0, 1\}$, set

$$F_i = \{h \in F : h \cap W \subset W_i\},$$

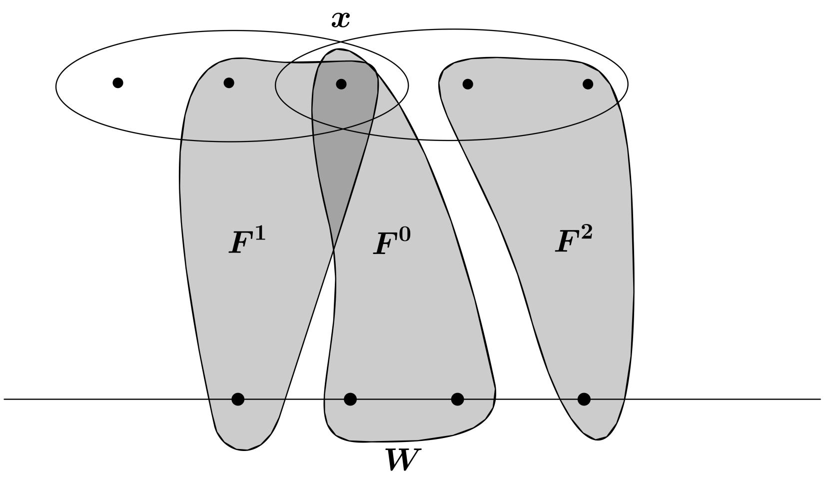

Figure 5.3. Illustration of the three types of edges in $H(U, W)$

which in the particular case when $F = H(U, W)$ will be abbreviated to H_i. Especially, for $i = \{0, 1\}$, $H_i = F_i^0 \cup F_i^1 \cup F_i^2$, where F_i^0 is the subset of edges $h \in F^0$ with $|h \cap W_i| = 2$, while F_i^k, $k = \{1, 2\}$, is the subset of edges of F^k whose unique vertex in W lies in W_i.

A crucial observation is that, since H is P-free, for every two disjoint edges in H, no edge may intersect each of them in exactly one vertex. Thus, there is no edge in H with one vertex in each of the sets, U, W_0 and W_1. Therefore,

$$H(U, W) = H_0 \cup H_1, \tag{5.4}$$

and consequently,

$$H = H[U] \cup H(U, W) \cup H[W] = H[U] \cup H_0 \cup H_1 \cup H[W] = H[U \cup W_0] \cup H_1 \cup H[W]. \tag{5.5}$$

Furthermore, by the same principle, if $e \in F_1^0$, then the pair $e \cap W_1$ must be *non-separable* in $H[W_1]$, that is, every edge of $H[W_1]$ must contain both these vertices or none. Since the non-separable pairs in $H[W_1]$ form a graph of maximum degree at most 2, there are at most $|W_1|$ of them. Therefore,

$$|F_1^0| \leqslant |W_1|. \tag{5.6}$$

The above crucial observation brings about another consequence, namely $F_1^1 = \emptyset$. Thus,

$$H_1 = F_1^0 \cup F_1^2. \tag{5.7}$$

If we bring equations (5.6), and (5.7) together, we need to bound $|F_1^2|$ which, however, requires a detailed analysis of the degrees of vertices $v \in W$ in the 3-graphs F^k, $k = \{0,1,2\}$. For $v \in W$ and $F \subseteq H$, let $F(v)$ be a *a neighborhood* of v in F, that is, the set of pairs of vertices which together with v form edges of F, thus $|F(v)| = deg_F(v)$. Notice that, since H is P-free, for every $v \in W$ either

$$F^0(v) = \emptyset \quad \text{or} \quad F^2(v) = \emptyset. \tag{5.8}$$

Moreover, by the definition of F^1 and F^2 (and the remarks preceding them),

$$|F^1(v)| \leqslant 4 \quad \text{and} \quad |F^2(v)| \leqslant 2. \tag{5.9}$$

For $v \in W_0$, by the remark preceding equation (5.4), $|F^0(v)| \leqslant |W_0| - 1$, and thus, by (5.8), (5.9) and (5.2),

$$|H(v)| = |F^0(v)| + |F^1(v)| + |F^2(v)| \leqslant 4 + \max\{2, n-9\}.$$

In particular, for $n = 10$,

$$\forall v \in W_0, \ |H(v)| \leqslant 6, \tag{5.10}$$

while for $n \geqslant 11$,

$$\forall v \in W_0, \ |H(v)| \leqslant n-5, \tag{5.11}$$

where the equality for $n \geqslant 12$ is achieved only when $|F^0(v)| = n-9$, $|F^1(v)| = 4$, and $F^2(v) = \emptyset$.

Consider now $v \in W_1$. For each $e \in F^0$, the pair $e \cap W_1$ must be non-separable in W_1 and, v belongs to at most two non-separable pairs. Thus, $|F^0(v)| \leqslant 2$ and, consequently, by (5.7), (5.8) and (5.9),

$$\forall v \in W_1, \ |H_1(v)| = |F^0(v)| + |F^2(v)| \leqslant 2. \tag{5.12}$$

We can also show that

$$|F_1^2| \leqslant \max\{|W_1|, 2|W_1| - 4\}. \tag{5.13}$$

Indeed, if for all $v \in W_1$ we have $|F^2(v)| = |F_1^2(v)| = 1$, then $|F_1^2| \leqslant |W_1|$. Otherwise, let $v \in W_1$ have, by (5.9), $|F^2(v)| = 2$ and let $\{v, v', v''\} \in H[W]$. Since H is P-free, $F^2(v') = F^2(v'') = \emptyset$, and therefore, again by (5.9),

$$|F_1^2| \leqslant 2(|W_1| - 2) = 2|W_1| - 4.$$

Now we are ready to set bounds on the number of edges in H_1, as well as in $H[U] \cup H_1$, which will be repeatedly used in the proof of Theorem 4.11. Recall that $|W_1| \geqslant 3$.

Fact 5.1. *We have*

$$|H_1| \leqslant 2|W_1| - 3 \tag{5.14}$$

and for $|W_1| \geqslant 4$,

$$|H[U]| + |H_1| \leqslant 2|W_1| + 2. \tag{5.15}$$

Proof. Let $h \in H[W]$. Notice that $\sum_{v \in h} |H_1(v)| \leqslant 3$, while for $v \in W_1 \backslash h$, by (5.12), $|H_1(v)| \leqslant 2$. This yields $|H_1| \leqslant 3 + 2(|W_1| - 3)$ and takes care of (5.14).

If $H_1 = \emptyset$ then (5.15) holds, as $|H[U]| \leqslant 10$. To prove (5.15) also when $H_1 \neq \emptyset$, we need a better bound on $|H[U]|$. To this end, note that if $F^0 \neq \emptyset$ then $E(\overline{x}) = \emptyset$, while if $F_1^2 \neq \emptyset$ then $E(x) = \emptyset$. Hence, by (5.7) and (5.3),

$$H_1 \neq \emptyset \quad \Rightarrow \quad |H[U]| \leqslant 6. \tag{5.16}$$

So, if one of the sets, F_1^0 or F_1^2, is empty, then we get (5.15) by (5.16), (5.6), and (5.13). If both these sets are non-empty, then $E(\overline{x}) = E(x) = \emptyset$, and thus $|H[U]| = 2$. In this case (5.15) follows by (5.14) with a margin. $\square$

Since H is C-free, on several occasions our proof relies on two instances of Theorem 2.4. Namely, if $|W_0| \geqslant 1$ then

$$|H[U \cup W_0]| \leqslant \binom{|W_0| + 4}{2}, \tag{5.17}$$

while if $|W| = n - 5 \geqslant 6$ then

$$|H[W]| \leqslant \binom{n-6}{2}. \tag{5.18}$$

Finally, there cannot be too many edges between U and the vertex set of a copy of P_2^3 in $H[W]$ if there happens to be one. For a subset $W' \subset W$, we denote by $H(U, W')$ the bipartite, induced sub-3-graph of H with bipartition (U, W').

Fact 5.2. *If $H[W]$ contains a copy Q of P_2^3 with $V(Q) \subseteq W$, then*

$$|H(U, V(Q))| \leqslant 4. \tag{5.19}$$

Proof. Recall that, the edges of $H(U, V(Q))$ with one vertex in U are those belonging to F^0 that is those containing the central vertex x. Note that due to the fact that H is P-free, there are at most two such edges in $H(U, V(Q))$. By symmetry, there are also at most two edges in $H(U, V(Q))$ with one vertex in W, which yields (5.19). □

Only now we are fully prepared for the proof of Theorem 4.11.

Proof of Theorem 4.11. At the beginning of the proof we will consider the three smallest cases for $n = 8, 9, 10$ separately. Afterwards, we turn to the main case of $n \geqslant 11$. Here, after quickly taking care of the subcase $W_0 = \emptyset$, we will assume that $W_0 \neq \emptyset$ and proceed by induction on n with $n = 11$ being the initial step.

Let H be a $\{P, C\}$-free n-vertex 3-graph which contains a copy of $P_2^3 \cup K_3^3$. We use the same notation and terminology as it was introduced at the beginning of this section. In addition, for $v \in V$, we will write $H - v$ for $H[V \backslash \{v\}]$.

Case n = 8.

We have $|W_1| = 3$, $|H[W]| = 1$, and $W_0 = \emptyset$. If $H(U, W) = H_1 = \emptyset$, then

$$|H| = |H[U]| + |H[W]| \leqslant 10 + 1 < 12 = |H_8|.$$

Otherwise, by (5.16), $|H[U]| \leqslant 6$ and, therefore by (5.14),

$$|H| = |H[U]| + |H_1| + |H[W]| \leqslant 6 + 3 + 1 < 12.$$

Case n = 9.

We have $|W| = 4$ and $|H[W]| \leqslant \binom{4}{3} = 4$.

If $W_0 = \emptyset$ then, by (5.15),

$$|H| = |H[U]| + |H_1| + |H[W]| \leqslant 2|W_1| + 2 + 4 = 14 = |H_9|.$$

If $W_0 \neq \emptyset$, then $|W_0| = 1$, $|W_1| = 3$ and $|H[W]| = 1$. By (5.14), $|H_1| \leqslant 3$, and consequently, by (5.5) and (5.17),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant 10 + 3 + 1 = 14.$$

Case n = 10.

We have $|W| = 5$, $|W_0| \leqslant 2$ and $|H[W]| \leqslant \binom{5}{3} = 10$. If $W_0 = \emptyset$ then, by (5.15),

$$|H[U]| + |H_1| \leqslant 2|W| + 2 = 12.$$

If, additionally, $|H[W]| \leqslant 5$, then

$$|H| = |H[U]| + |H_1| + |H[W]| \leqslant 12 + 5 < 20 = |H_{10}|.$$

Otherwise, by Proposition 2.6, $H[W]$ contains a copy Q of P_2^3 (note that $V(Q) = W_1$), and, by (5.19), $|H_1| \leqslant 4$. Hence, using (5.16) along the way,

$$|H| = |H[U]| + |H_1| + |H[W]| \leqslant \max\{10 + 0, 6 + 4\} + 10 = 20.$$

Now, let $W_0 \neq \emptyset$. Fix $v \in W_0$ and notice that $H - v$ is $\{P, C\}$-free and contains M. Since we have already proved that $\mathrm{ex}_3(9; \{P, C\}|M) = 14$,

$$|H - v| \leqslant 14.$$

Moreover, by (5.10), $|H(v)| \leqslant 6$, and consequently,

$$|H| = |H - v| + |H(v)| \leqslant 14 + 6 = 20.$$

Case n $\geqslant$ 11.

The proof is by induction on n with $n = 11$ being the base case. First, however, we take care of a subcase when $W_0 = \emptyset$, for which, by (5.15) and (5.18),

$$|H| = |H[U]| + |H_1| + |H[W]| \leqslant 2(n-5) + 2 + \binom{n-6}{2} = 3 + \binom{n-4}{2} < |H_n|.$$

Hence, in what follows we will be assuming that $W_0 \neq \emptyset$.

Case n = 11 (initial step).

Suppose first that $H[W]$ contains a copy Q of P_2^3. Then $|W_0| = 1$, $|W_1| = 5$, $V(Q) = W_1$, and by (5.19), $|H_1| \leqslant 4$. Consequently, by (5.5), (5.17), and (5.18),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant 10 + 4 + 10 < 25 = |H_{11}|.$$

In the remainder of this part of the proof, besides the assumption that $W_0 \neq \emptyset$, we will be also assuming that $H[W]$ is P_2^3-free and thus, by Proposition 2.6, $|H[W]| \leqslant 6$. We consider three cases with respect to the size of $|W_0|$.

$|\mathbf{W_0}| = \mathbf{1}$.

We have $|W_1| = 5$ and, by (5.14), $|H_1| \leqslant 7$. Consequently, by (5.5) and (5.17),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant 10 + 7 + 6 < 25.$$

$|\mathbf{W_0}| = \mathbf{2}$.

We have $|W_1| = 4$ and therefore $|H[W]| \leqslant \binom{4}{3} = 4$. Moreover, by (5.14), $|H_1| \leqslant 5$ and finally, by (5.5) and (5.17),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant 15 + 5 + 4 < 25.$$

$|\mathbf{W_0}| = \mathbf{3}$.

We have $|W_1| = 3$ and therefore $|H[W]| = 1$. Moreover, by (5.14), $|H_1| \leqslant 3$ and thus, by (5.5) and (5.17),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leqslant 21 + 3 + 1 = 25,$$

with equality only when $|H_1| = 3$ and $|H[U \cup W_0]| = 21$. The latter, by the second part of Theorem 2.4, is possible only when $H[U \cup W_0]$ is a star (with the center at x). This, in turn, implies that $F^2 = \emptyset$ (otherwise H would not be P-free) and, further, by (5.7), that $H_1 = F_1^0$. Hence, $H =$Co(11) with x at the center and W_1 as the head.

Case $\mathbf{n \geqslant 12}$ (inductive step).

Fix $v \in W_0$. By the induction hypothesis

$$|H - v| \leqslant 4 + \binom{n-5}{2}$$

with the equality only when $H - v =$Co$(n-1)$. Looking at the structure of $H - v$, if it is a comet, then it must have the center at x and the head must be the unique edge of $H[W]$. Moreover, by (5.11), $|H(v)| \leqslant n - 5$, with the equality only when $|F^0(v)| = n - 9$, $|F^1(v)| = 4$, and $F^2(v) = \emptyset$. Consequently,

$$|H| = |H - v| + |H(v)| \leqslant 4 + \binom{n-5}{2} + n - 5 = |H_n|.$$

and this bound is achieved only when both $H - v = \mathrm{Co}(n-1)$ and $|H(v)| = n - 5$. This, however, implies that $H = \mathrm{Co}(n)$ (with the same center and head as in $H - v$). This way we have proved Theorem 4.11.

$\square$

Chapter 6

Conclusions

The main subject of this dissertation were Ramsey and Turán numbers for the 3-uniform path P of length 3. We began by introducing the basic definitions and terminology used in this thesis, together with the most important results related to Ramsey and Turán numbers. In Section 2.2 we stated Theorem 2.1 and, the most crucial in our research, Theorem 2.2. Then, we moved to results connected to Turán numbers. We quoted the well-known Erdős-Ko-Rado Theorem (Theorem 2.3), Theorem 2.4 concerning the Turán number for the triangle C and its generalization. Nonetheless, the most important result in this thesis, related to Turán numbers is Theorem 2.7 which gives values of $\mathrm{ex}_3(n;P)$ for all n. Moreover, in Chapter 2 we stated our two other results, Theorem 2.9 and Theorem 2.10, which were useful in the proof of Theorem 2.2.

Then in Chapter 3, we focused exclusively on the proofs of Theorems 2.1 and 2.2. In Section 3.1 we introduced a standard approach towards determining lower bounds on Ramsey numbers and concentrated on two proofs of Theorem 2.1. The first proof was based on detailed analysis of all possible 3-colorings in a 3-graph which contains 9 vertices, very strenuous and, thus, not suitable for generalizations. The second proof was based on Turán numbers. The relation between Turán and Ramsey numbers is laid down in Lemma 3.3. We finished Chapter 3 with the proof of Theorem 2.2, where we employed another variant of Turán numbers, so called Turán number of higher order, which had first appeared in [14] (see Theorem 2.8).

In Chapter 4 we defined a new type of Turán numbers, the conditional Turán numbers and used them to prove Theorem 2.7 which determines Turán numbers $\mathrm{ex}_3(n;P)$ for all n. We also formulated a few results such that Theorems 4.9 and 4.11, and Corollary 4.10, about conditional Turán numbers. Chapter 5 was mainly devoted to the proofs of Theorem 2.9, Theorem 2.10, Theorem 4.9, and Theorem 4.11.

The most important, in my opinion, result of this dissertation, Theorem 2.2 gives the exact value $R(P;r) = r+6$ for $r \leqslant 7$. In [27] Polcyn and Ruciński used the third and fourth order Turán numbers for P to show that $R(P;r) = r+6$ also for $r \in \{8,9\}$ and soon after Polcyn [26] proved that $R(P;10) = 16$ but she had to use the fifth order Turán number for P, which she determined for all n. Quite recently, the authors of [22] improved the upper bound $R(P;r) \leqslant 3n$ down to $R(P;r) \leqslant 2n + \sqrt{18n+1} + 2$ and in [23] Łuczak, Polcyn, and Ruciński gave bounds on $R(P^k;r)$, for k-uniform paths of length 3, which, interestingly, turned out to be independent of k (for k large). It would be interesting to decide if $R(P;r) = r+6$ also for all $r \geqslant 11$. To achieve this task it seem to be essential to compute Turán numbers of order s, $\mathrm{ex}^{(s)}(c;P)$, for $s \geqslant 6$ or, otherwise, come up with a totally new idea.

There are some related problems too. For instance the only facts we know about Ramsey number $R(C;r)$ are that $R(C;r) = r+5$ for $r = 2,3$ and $R(C;r) \geqslant r+5$ for all r [11]. Gyárfás and Raeisi conjectured in [11] that $R(C;r) = r+5$ for all r. To solve this problem one may need to compute $\mathrm{ex}_3^{(s)}(n; C)$ for $s \geqslant 2$ and at least for some small values of n.

It would also be interesting to compute the conditional Turán numbers $\mathrm{ex}_3(n; C|M) = \mathrm{ex}_3(n; C|P)$ (c.f. Corollary 4.10). By (4.14) we know that lower bound on $\mathrm{ex}_3(n; C|P)$ is $\binom{n-2}{2} + 1$. We dare to formulate the following conjecture.

Conjecture 6.1. *With a possible exception of some small values of n,*

$$\mathrm{ex}_3(n; C|P) = \binom{n-2}{2} + 1.$$

Finally, Łuczak and Polcyn in [21], Sect. 4, formulated a question of characterizing the structure of all connected P-free 3-graphs H. One important case of this problem, when H contains the triangle C and is not intersecting (i.e., H contains also M), is covered by Lemma 4.6, proved in my thesis, which asserts that such an H must be a sub-3-graph of a satellite. It would be very desirable to complete this characterization.

Bibliography

[1] M. Axenovich, A. Gyárfás, H. Liu, and D. Mubayi, *Multicolor ramsey numbers for triple systems*, Discrete Mathematics **322** (2014), 69–77.

[2] R. Csákány and J. Kahn, *A homological Approach to Two Problems on Finite Sets*, Journal of Algebraic Combinatorics **9** (1999), 141–149.

[3] P. Erdős, C. Ko, and R. Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. **(2)** (1961), no. 12, 313–320.

[4] R. J. Faudree and R. H. Schelp, *All Ramsey numbers for cycles in graphs*, Discrete Mathematics **(8)** (1974), no. 4, 313–329.

[5] P. Frankl, *On families of finite sets no two of which intersect in a singleton*, Bulletin of the Australian Mathematical Society **(1)** (1977), no. 17, 125–134.

[6] P. Frankl and Z. Füredi, *Non-trivial Intersecting Families*, Journal of Combinatorial Theory, Series A **(1)** (1986), no. 41, 150–153.

[7] P. Frankl and Z. Füredi, *Exact solution of some Turán-type problems*, Journal of Combinatorial Theory, Series A **(2)** (1987), no. 45, 226–262.

[8] Z. Füredi, T. Jiang, and R. Seiver, *Exact solution of the hypergraph Turán problem for k-uniform linear paths*, Combinatorica **(3)** (2014), no. 34, 299–322.

[9] L. Gerencsér and A. Gyárfás, *On Ramsey-Type Problems*, Annales Universitatis Scientiarum Budapestinensis, Eötvös Sect. Math. **10** (1967), 167–170.

[10] R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey theory, 2nd edition*, John Wiley & Sons (1980).

[11] A. Gyárfás and G. Raeisi, *The Ramsey number of loose triangles and quadrangles in hypergraphs*, Electronic Journal of Combinatorics **(2)** (2012), no. 19, # P30.

[12] A. Gyárfás, G. N. Sárközy, and E. Szemerédi, *The Ramsey Number of Diamond-Matchings and Loose Cycles in Hypergraphs*, Electronic Journal of Combinatorics **(15)** (2008), #R126.

[13] P. Haxell, T. Łuczak, Y. Peng, V. Rödl, A. Ruciński, M. Simonovits, and J. Skokan, *The Ramsey number for hypergraph cycles I*, Journal of Combinatorial Theory, Series A **(1)** (2006), no. 113, 67–83.

[14] A. J. W. Hilton and E. C. Milner, *Some intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. **(1)** (1967), no. 18, 369–384.

[15] E. Jackowska, *The 3-color ramsey number for a 3-uniform loose path of length 3*, The Australasian Journal of Combinatorics **(2)** (2015), no. 63, 314–320.

[16] E. Jackowska, J. Polcyn, and A. Ruciński, *Turán numbers for linear 3-uniform paths of length 3*, Electronic Journal of Combinatorics **(2)** (2016), no. 23, # P2.30.

[17] E. Jackowska, J. Polcyn, and A. Ruciński, *Multicolor ramsey numbers and restricted turán numbers for the loose 3-uniform path of length three*, Electronic Journal of Combinatorics **24** (2017), no. 3, #P3.5.

[18] G. Károlyi and V. Rosta, *Generalized and Geometric Ramsey Numbers for Cycles*, Theoretical Computer Science **(1-2)** (2001), no. 263, 87–98.

[19] P. Keevash, D. Mubayi, and R. M. Wilson, *Set systems with no singleton intersection*, SIAM J. Discrete Math. **(4)** (2006), no. 20, 1031–1041.

[20] A. Kostochka, D. Mubayi, and J. Verstraëte, *Turán Problems and Shadows I: Paths and Cycles*, Journal of Combinatorial Theory, Series A **129** (2015), 57–79.

[21] T. Łuczak and J. Polcyn, *Paths in hypergraphs: a rescaling phenomenon*, arXiv:1706.08465v1 **submitted**.

[22] T. Łuczak and J. Polcyn, *On Multicolor Ramsey number for 3-paths of length three*, Electronic Journal of Combinatorics **(1)** (2017), no. 24, # P1.27.

[23] T. Łuczak, J. Polcyn, and A. Ruciński, *On Multicolor Ramsey number for k-paths of length three*, arxiv (2017).

[24] B. D. McKay and S. P. Radziszowski, *The First Classical Ramsey Number for Hypergraphs is Computed*, Proceedings of the Second Annual ACM-SIAM Symposium on Discrete Algorithms SODA'91 San Francisco (1991), 304–308.

[25] G. R. Omidi and M. Shahsiah, *Ramsey Numbers of 3-Uniform Loose Paths and Loose Cycles*, Journal of Combinatorial Theory, Series A **121** (2014), 64–73.

[26] J. Polcyn, *One more Turán number and Ramsey number for the Loose 3-uniform path of length three*, Discussiones Mathematicae Graph Theory **37** (2017), 443–464.

[27] J. Polcyn and A. Ruciński, *Refined Turán numbers and Ramsey numbers for the loose 3-uniform path of length three*, Discrete Mathematics **(2)** (2017), no. 340, 107–118.

[28] S. P. Radziszowski, *Small Ramsey numbers*, Electronic Journal of Combinatorics **1** (1994, Dynamic Surveys, revision #15: March 3, 2017).

[29] F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. **30** (1928), 264–286.

[30] V. Rosta, *On a ramsey-type problem of J.A. Bondy and P. Erdős, I, II*, Journal of Combinatorial Theory, Series B **(1)** (1973), no. 15, 94–120.

[31] V. Rosta, *Ramsey theory applications*, Electronic Journal of Combinatorics **3** (2004), #DS13.

[32] A. Ruciński and V. Rödl, *Dirac-type questions for hypergraphs - a survey (or more problems for Endre to solve)*, An Irregular Mind (Szemerédi is 70) Bolyai Soc. Math. Studies **21** (2010), 561–590.