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Multidimensional wavelet bases in Besov and Triebel-Lizorkin spaces

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Contents

Introduction

Methods of wavelet analysis are an important tool in investigating of properties of function spaces. Due to wavelet bases we can define isomorphisms between function spaces of Hardy-Sobolev-Triebel type and corresponding sequence spaces. These isomorphisms reduce many problems from the function spaces level to the sequence spaces level. The main advantage of that approach is that interesting issues often simplify in sequence spaces. So the question about existence of an unconditional basis in function spaces or wavelet characterization is very important to investigate their properties. That way of research can be used to investigate boundedness, compactness and spectral properties of operators acting between function spaces. Among operators special attention is paid on Sobolev embeddings, because they are simple and on the other side many other operators can be factorized by them.

Weighted function spaces are the subject of a research, because of many applications in theory of differential equations, spectral operators theory, etc. Local Muckenhoupt weights (class A_∞^{loc}), that are important to the main part of that dissertation, and spaces with that weights were introduced by V.Rychkov in 2001 in [34]. That weights are generalizations of classical Muckenhoupt weights A_∞ as well as admissible weights, which are smooth. One of the breakthroughs in the history of function spaces of Sobolev-Besov-Hardy type was wavelets characterizations of that spaces. The unweighted case was considered by Y.Meyer et al. in [31] and [3]. Wavelet bases in function spaces with admissible weights were constructed by D.Haroske and H.Triebel in [19]. In subsequent years Haroske and Skrzypczak showed wavelet characterization of spaces with Muckenhoupt weights, cf. [16]. Most recently Izuki and Sawano proved that there exists wavelet bases in function spaces with weights of $\mathcal{A}_\infty^{loc}$ class, cf. [24]. Weighted function spaces with Muckenhoupt weights are applied to consider differential operators like in [12], [18]. They have many applications in theory of differential equations, spectral operators theory etc.

Wavelet theory is widely developed. Orthonormal wavelet bases in L_2 are used in theoretical mathematics as well as in computer science. In that dissertation we deal with inhomogeneous Besov and Triebel-Lizorkin spaces, which are described in monographic series "Theory of Function Spaces" by H.Triebel, [39]-[41]. In definition of weighted function spaces we replace Lebesgue measure by $w\,dx$ measure, where w is positive, locally integrable function called weight. V. Rychkov introduce the theory of function spaces with local Muckenhoupt weights, which are generalizations of earlier results. Izuki and Sawano continue to develop that theory, by showing atomic decomposition in such a spaces, [23]. By now there has been shown wavelet characterization of Besov and Triebel-Lizorkin spaces with local Muckenhoupt weights, [24].

In Chapter 1 there are introduced wavelet bases. Basic kinds of wavelets are defined. Special attention is paid to Daubechies wavelets. These systems of wavelets are orthogonal bases in $L_2(\mathbb{R}^n)$. They are especially useful because of compact supports and smoothness to some order. That properties of Daubechies wavelets make proofs of theorems about isomorphisms between function spaces and sequence spaces easier.

In further part of Chapter 1 we introduce inhomogeneous Besov and Triebel-Lizorkin spaces. These are quasi-Banach spaces of tempered distributions. We recall development of theory of function spaces following [40].

In Chapter 2 we concern weights and weighted function spaces. Classical Muckenhoupt weights $\mathcal{A}_p$ was introduced by Muckenhoupt as a class of weights, for which Hardy-Littlewood maximal operator is bounded in proper function spaces of p-integrable functions. Bui, Taibleson and Weiss have developed theory of weighted Besov and Triebel-Lizorkin spaces as spaces of tempered distributions. In the same time there have been developed theory of function spaces with weights from another class, which consist of smooth weights with such a behavior at infinity that they do not belong to $\mathcal{A}_p$, cf. [35]. Moreover, at least for some kind of weights, there was no possibility to continue the theory inside spaces of tempered distributions. In 2001 Rychkov introduced in [34] so called local Muckenhoupt weights, i.e. $\mathcal{A}_p^{loc}$ class, which embrace every considered classes of weights. It revealed that local Muckenhoupt weights are natural family of weights for inhomogeneous Besov and Triebel-Lizorkin spaces. Rychkov proved some properties of local Muckenhoupt weights. Keep on researching we develop theory of properties of that class of weights to get analogous theory of properties of classical Muckenhoupt weights, for example the local version of the reverse Hölder inequality, the theorem about representation of weights, etc.

The main aim of Chapter 3 is to formulate the wavelet characterization of $B^{s,w}_{p,q}(\mathbb{R}^n)$ and $F^{s,w}_{p,q}(\mathbb{R}^n)$ spaces with local Muckenhoupt weight w. We follow H.Triebel's approach in [44]. He made an observation that Daubechies wavelets can serve as kernels of local means and as atoms. It can be adopted to spaces with local Muckenhoupt weights $\mathcal{A}_\infty^{loc}$. To prove local means representation as well as Daubechies wavelet representation we use atomic decomposition theorem for function spaces with local Muckenhoupt weights proved by Izuki and Sawano in [23].

In the last part of Chapter 3 we show some applications of wavelet characterization of $B^{s,w}_{p,q}(\mathbb{R}^n)$ and $F^{s,w}_{p,q}(\mathbb{R}^n)$ spaces with local Muckenhoupt weight w. First we consider continuous embeddings. Then we obtain results about dual spaces to function spaces with local Muckenhoupt weights. At last we get complex interpolation for $F^{s,w}_{p,q}(\mathbb{R}^n)$ spaces.

Sobolev embeddings of Besov and Triebel-Lizorkin spaces were widely studied. In paper by D. Haroske and L. Skrzypczak [16] there were stated characterization of embeddings between function spaces with classical Muckenhoupt weights. Using results from [27] we get analogous results for embeddings of function spaces with local Muckenhoupt weights. It turns out that in some special cases we get better results even for classical Muckenhoupt weights cases when we use local Muckenhoupt weights. In this section we also study embeddings from $B^{s,w}_{p,q}(\mathbb{R}^n)$ and $F^{s,w}_{p,q}(\mathbb{R}^n)$ spaces to function spaces outside the Besov and Triebel-Lizorkin scales, for example to $C(\mathbb{R}^n)$, $L_\infty(\mathbb{R}^n)$. In particular we study conditions on parameters s, p and q whether weighted function spaces consist of regular distributions.

In the book [43] H. Triebel proves, that Haar wavelets can be used to characterization of unweighted function spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ as far as absolute value of smoothness parameter is small enough. The aim of Chapter 4 is to show the weighted version of that result. The conditions on smoothness s and weight w, that guarantee that Haar functions form unconditional bases in $B^{s,w}_{p,q}(\mathbb{R}^n)$ and $F^{s,w}_{p,q}(\mathbb{R}^n)$ are formulated. Here p and q must be finite.

In Chapter 5 we turn to L_p spaces. Wavelet systems in weighted L_p spaces were investigated by several authors. P. G. Lemarié-Rieusset considered one dimensional case, cf. [29]. He proved that the homogeneous wavelet system of Daubechies type is an unconditional basis in $L_p(\mathbb{R}^n, d\mu)$, $1 < p < \infty$, if and only if $d\mu = wdx$, where w is a weight belonging to the Muckenhoupt class $\mathcal{A}_p$. He also found a sufficient and necessary condition for inhomogeneous systems to be unconditional bases. Other one dimensional systems were investigated by Kazarian [28] and García-Cuerva, Kazarian [14]. Multidimensional homogeneous wavelet systems were considered by Aimar, Bernardis, Martín-Reyes, cf. [1]. They proved that a homogeneous wavelet system satisfying certain regularity conditions is an unconditional basis in $L_p(d\mu)$ if and only if $d\mu = wdx$ with $w \in \mathcal{A}_p$.

The aim of Chapter 5 is to prove the counterpart of Lemarié-Rieusset's result for multidimensional inhomogeneous wavelet systems. To formulate the necessary and sufficient condition we use the class of local Muckenhoupt weights $\mathcal{A}^{loc}_\infty$. Furthermore it leads to necessary conditions for the Paley-Littlewood characterization of L_p spaces with weights.

Chapter 1

Definitions and notation

1.1 Notation

Let us fix some notation. By $\mathbb{N}$ we denote the set of natural numbers, by $\mathbb{N}_0$ the set $\mathbb{N}\cup\{0\}$, by $\mathbb{C}$ the complex plane, by $\mathbb{R}^n$ the euclidean n-space, $n \in \mathbb{N}$ and by $\mathbb{Z}^n$ the set of all lattice points in $\mathbb{R}^n$ having integer components.

The positive part of a real function f is given by $f_+(x) = \max(f(x), 0)$. For two positive real sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ we mean by $a_k \sim b_k$ that there exist constants $c_1, c_2 > 0$ such that $c_1 a_k \le b_k \le c_2 a_k$ for all $k \in \mathbb{N}$, similarly for positive functions.

Given two quasi-Banach spaces X and Y we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

All unimportant positive constants are denoted by c, occasionally with subscripts. Let dx and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure. Log is always taken with respect to a base 2.

We denote by $\mathcal{D}(\mathbb{R}^n)$ the space of C^∞ functions with compact support. $\mathcal{D}'(\mathbb{R}^n)$ is its topological dual, the space of distributions.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. By $\mathcal{S}'(\mathbb{R}^n)$ we denote its topological dual, the space of tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$ is the standard quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f|L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p}$$

with the obvious modification if $p = \infty$.

Let $C(\mathbb{R}^n)$ be the Banach space of all complex-valued uniformly continuous bounded functions in $\mathbb{R}^n$ and let for $r \in \mathbb{N}$,

$$C^r(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) : D^\gamma f \in C(\mathbb{R}^n), |\gamma| \le r\},$$

obviously normed, where we use the standard abbreviation D^γ for derivatives.

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi) = (\mathcal{F}\varphi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi x} \varphi(x)\, dx, \quad \xi \in \mathbb{R}^n, \tag{1.1.1}$$

denotes the Fourier transform of φ. Here ξx is the scalar product in $\mathbb{R}^n$. As usual, $\mathcal{F}^{-1}\varphi$ or $\varphi^\vee$, stands for the inverse Fourier transform, given by the right-hand side of (1.1.1) with i in place of $-i$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way.

By a cube Q we understand the cube in $\mathbb{R}^n$ with sides parallel to the axis and $|Q|$ denotes the Lebesgue measure of Q. For some $d > 0$ by dQ we understand a cube with the same center as Q and sides scaled by d. A dyadic cube is the cube with sides 2^{-j} and center $2^{-j}(m + \frac{1}{2})$, where $m + \frac{1}{2} = (m_1 + \frac{1}{2}, m_2 + \frac{1}{2}, \dots, m_n + \frac{1}{2})$, denoted by Q_{jm}, for $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$.

Let $B(x, r)$ denote an open ball in $\mathbb{R}^n$ centered at x with radius $r > 0$.

Given $1 \leq p \leq \infty$, its dual index is the number $1 \leq p' \leq \infty$ satisfying

$$1 = \frac{1}{p} + \frac{1}{p'}$$

and given $0 < p < 1$ its dual index is ∞, where we use the conventions that $1/0 = \infty$ and $1/\infty = 0$.

By

$$\Delta^N = \left(\sum_{i=1}^{n} \partial^2 / \partial x_i^2 \right)^N, \quad N \in \mathbb{N}_0,$$

we denote the N-th power of the Laplacian ($\Delta^0 = Id$).

1.2 Definitions and basic theorems

Let X be a quasi-normed vector space. A quasi-norm on X induces a locally bounded topological vector space structure on X. Conversely if X is a locally bounded vector space and B is a bounded neighborhood of zero then $\|x\|_B := \inf\{r > 0; r^{-1}x \in B\}$, $x \in X$, is a quasi-norm and different bounded neighborhoods of the origin define equivalent quasi-norms. Moreover if $\|\cdot\|$ is a quasi-norm on a linear space X then there exist $0 < p \leq 1$ and a p-norm $\|\cdot\|'$ on X equivalent to $\|\cdot\|$, i.e. the p-norm $\|\cdot\|'$ such that there is a positive constant c such that

$$c^{-1}\|x\| \leq \|x\|' \leq c\|x\|, \quad x \in X,$$

cf. [33] Theorem 3.2.1. We recall that $\|\cdot\| : X \to \langle 0, \infty)$ is called a p-norm if

(i) $\|x\| = 0 \Leftrightarrow x = 0$,

(ii) $\|\lambda x\| = |\lambda| \|x\|$, $\lambda \in \mathbb{C}$, $x \in X$,

(iii) $\|x + y\|^p \leq \|x\|^p + \|y\|^p$.

So $\|\cdot\|^p$ is a p-homogeneous F-norm, cf. [33] p. 4 for the definition.

We will always assume that quasi-Banach spaces are a vector spaces over $\mathbb{C}$.

Definition 1.1. A countable family $\{x_n\}_{n\in\mathbb{N}}$ of vectors in a separable quasi-Banach space X is a basis for X if every $x \in X$ can be written

$$x = \sum_{n\in\mathbb{N}} c_n x_n \quad \text{convergence in } X$$

for a unique choice of scalars $c_n \in \mathbb{C}$.

Definition 1.2. Let $\{x_n\}_{n\in\mathbb{N}}$ be a countable family of vectors from a quasi-Banach space X. The series $\sum_{n\in\mathbb{N}} x_n$ is *unconditionally convergent* if for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum_{k=0}^{\infty} x_{\sigma(k)}$ is convergent in X.

Theorem 1.1. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a countable family of vectors from a quasi-Banach space X. $\sum_{n\in\mathbb{N}} x_n$ converges unconditionally if and only if $\sum_{n\in\mathbb{N}} \varepsilon_n x_n$ converges for every choice of signs $\varepsilon_n = \pm 1$.*

The proof can be found in [20] (Theorem 3.10) or [33] (Theorem 3.8.2).

Definition 1.3. A basis $\{x_n\}_{n\in\mathbb{N}}$ in a separable quasi-Banach space X is called *unconditional* if for any $x \in X$ the series

$$x = \sum_{n\in\mathbb{N}} c_n x_n$$

is unconditionally convergent.

Remark 1.1. Obviously $\{x_n\}_{n\in\mathbb{N}}$ is an unconditional basis in X if and only if $\{x_{\sigma(n)}\}_{n\in\mathbb{N}}$ is a basis in X for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$.

The basis that is not an unconditional basis is called a conditional basis. All the bases we will work with in the thesis are unconditional.

Definition 1.4. Given a basis $\{x_n\}_{n\in\mathbb{N}}$ for a quasi-Banach space X, then the sequence of linear functionals x_n^* defined by

$$x = \sum_{n\in\mathbb{N}} x_n^*(x) x_n$$

is called sequence of *coefficient functionals* for $\{x_n\}_{n\in\mathbb{N}}$.

Definition 1.5. Let $\{x_n\}_{n\in\mathbb{N}}$ be a basis for a quasi-Banach space X and $\{x_n^*\}_{n\in\mathbb{N}}$ be the coefficient functionals. Then we say that $\{x_n\}_{n\in\mathbb{N}}$ is *a Schauder basis* for X if each x_n^* is continuous.

Theorem 1.2. *A countable family $\{x_n\}_{n\in\mathbb{N}}$ is a Schauder basis for a quasi-Banach space X if and only if $\{x_n\}_{n\in\mathbb{N}}$ is a basis for X.*

The proof of the theorem can be found in [20], Theorem 4.13 for the Banach spaces and [33] Corollary 2.6.2 for the quasi-Banach case.

Definition 1.6. Let X be a Banach space. System $\{x_n, x_n^*\}_{n\in\mathbb{N}}$ of x_n from X and functionals x_n^* from X^* we call *a biorthogonal system* if

$$x_n^*(x_m) = \begin{cases} 1 & \text{for } n = m, \\ 0 & \text{for } n \neq m. \end{cases}$$

Lemma 1.1. *Let X, Y be quasi-Banach spaces. If $\{x_n\}_{n\in\mathbb{N}}$ is a basis for X and $T : X \to Y$ is a topological isomorphism, then $\{Tx_n\}_{n\in\mathbb{N}}$ is a basis for Y.*

The above lemma can be found in [20], Lemma 4.18.

Theorem 1.3. *If $\{x_n\}_{n\in\mathbb{N}}$ is a (unconditional) basis for a reflexive Banach space X, then its biorthogonal system $\{x_n^*\}_{n\in\mathbb{N}}$ is a (unconditional) basis for X^*.*

The proof of the theorem can be found in [20], Corollary 5.22.

We can define partial sum operators

$$S_N x = \sum_{n=1}^{N} x_n^*(x)x_n, \quad x \in X.$$

We say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is complete in a quasi-normed space X if $\overline{\operatorname{span}}\{x_n\}_{n\in\mathbb{N}} = X$.

A family $\mathcal{U}$ of continuous operators from a quasi-normed space X into a quasi-normed space Y is called equicontinuous if for each positive ε there is a positive δ such that

$$\sup\{\|Ax\| : A \in \mathcal{U}, \|x\| \leq \delta\} \leq \varepsilon.$$

Theorem 1.4. *Given a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a quasi-Banach space X, the following statements are equivalent.*

(i) *$\{x_n\}$ is a basis for X.*

(ii) *$\{x_n\}$ is complete and the operators S_N are equicontinuous on the set of all linear combinations of $\{x_n\}$.*

The above theorem for quasi-Banach spaces follows from Corollary 2.6.5 in [33], for Banach spaces it is Theorem 5.12 in [20].

Theorem 1.5. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a complete sequence in a quasi-Banach space X such that $x_n \neq 0$ for every n. Then the following statements are equivalent.*

(i) *$\{x_n\}$ is an unconditional basis for X.*

(ii) *$\{x_n\}$ is a basis, and for each bounded sequence of scalars $\Lambda = \{\lambda_n\}$ there exists a continuous linear operator $T_\Lambda : X \to X$ such that $T_\Lambda(x_n) = \lambda_n x_n$ for all $n \in \mathbb{N}$.*

For Banach spaces the theorem is stated in [20] (Theorem 6.7). For quasi-Banach it follows from Corollary 3.9.5, Proposition 3.9.13 and considerations on page 94 in [33].

Lemma 1.2 (Young's inequality)**.** *If* $\{a_k\}_{k\in\mathbb{Z}} \in \ell_p$, $\{b_k\}_{k\in\mathbb{Z}} \in \ell_q$ *and* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$ *for* $1\le p,q,r\le\infty$*. Then*

$$\left\|\sum_{l\in\mathbb{Z}} a_l b_{k-l}|\ell_r\right\| \le \|a_k|\ell_p\|\,\|b_k|\ell_q\|\,.$$

1.3 Wavelet systems

Definition 1.7. We call a *scaling function (father wavelet)* a function $\psi^F(t)\in L_2(\mathbb{R})$ and a *wavelet (mother wavelet)* a function $\psi^M(t)\in L_2(\mathbb{R})$ such that the system

$$\{\psi^M_{jm}(x)=2^{j/2}\psi^M(2^jx-m)\}_{j=0,m\in\mathbb{Z}}^{\infty}\cup\{\psi^F_{0,m}(x)=\psi^F(x-m)\}_{m\in\mathbb{Z}},$$

is an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$.

The first wavelet system consist of compactly supported functions was the Haar system. We recall briefly the construction, cf. [46], Chapter 2.1.

Definition 1.8. Let

$$h^M(x)=\begin{cases}1 & \text{if } 0\le x<1/2,\\ -1 & \text{if } 1/2\le x<1,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$h^F(x)=|h^M(x)|$$

be the *Haar wavelet* and the *Haar scaling function.*

Theorem 1.6. *The system*

$$\{h^M_{jm}(x)=2^{j/2}h^M(2^jx-m)\}_{j=0,m\in\mathbb{Z}}^{\infty}\cup\{h^F_{0,m}(x)=h^F(x-m)\}_{m\in\mathbb{Z}},$$

forms the orthogonal Haar basis in $L_2(\mathbb{R})$*.*

Haar wavelets on $\mathbb{R}^n$ we obtain by the usual tensor product procedure

$$H^G_{jm}=2^{jn/2}\prod_{r=1}^{n}h^{G_r}(2^jx_r-m_r), \tag{1.3.1}$$

where $j\in\mathbb{N}_0$, $m\in\mathbb{Z}^n$, $G=(G_1,...,G_n)\in G^j$ and $G^0=\{F,M\}^n$ and for $j>0$ $G^j=\{F,M\}^{n*}$, where * indicates that at least one G_r must be an M.

$$\left\{H^G_{jm}: j\in\mathbb{N}_0, m\in\mathbb{Z}^n, G\in G^j\right\}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$, cf. [46], Proposition 6.2.

Theorem 1.7. *The Haar system* $\{H^G_{jm} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n, G \in G^j\}$ *is a Schauder basis in* $L_p(\mathbb{R}^n)$, $1 \le p < \infty$. *If* $1 < p < \infty$ *then the basis is unconditional.*

The proof can be found in [46], Theorem 9.13.

There are no wavelets belonging to the class C^∞ with compact support. However I. Daubechies constructed systems of compactly supported wavelets with any finite smoothness, cf. [10] Chapter 6 or [46] Chapter 5. Such a system of wavelets will be called the Daubechies systems. The construction is based on the method of multiresolution analysis.

Theorem 1.8. *There exists a constant* C *such that for every* $k = 1, 2, \ldots$ *there are scaling function* $\psi^F(x)$ *and wavelet* $\psi^M(x)$ *such that*

(i) $\psi^F(x)$ *and* $\psi^M(x)$ *are in* $C^k(\mathbb{R})$.

(ii) $\psi^F(x)$ *and* $\psi^M(x)$ *have compact support and* $\operatorname{supp}\psi^F$ *and* $\operatorname{supp}\psi^M$ *are subsets of* $[-Ck, Ck]$.

The proof of the theorem can be found in [46], Theorem 5.7.

Let $\psi^F \in C^k(\mathbb{R})$ be a Daubechies scaling function and $\psi^M \in C^k(\mathbb{R})$ a Daubechies wavelet with $\int_{\mathbb{R}} \psi(x) x^v \, dx = 0$, $k \in \mathbb{N}$, $v \in \mathbb{N}_0$, $v < k$. We extend these wavelets from $\mathbb{R}$ to $\mathbb{R}^n$ by the usual tensor product procedure

$$\Psi^G_{jm} = 2^{jn/2} \prod_{r=1}^{n} \psi^{G_r}(2^j x_r - m_r), \tag{1.3.2}$$

where $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $G = (G_1, \ldots, G_n) \in G^j$ and $G^0 = \{F, M\}^n$ and for $j > 0$ $G^j = \{F, M\}^{n*}$, where * indicates that at least one G_r must be an M.

$$\left\{\Psi^G_{jm} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n, G \in G^j\right\}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$, cf. [46].

Theorem 1.9. *The Daubechies system* $\{\Psi^G_{jm} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n, G \in G^j\}$ *is a Schauder basis in* $L_p(\mathbb{R}^n)$, $1 \le p < \infty$. *If* $1 < p < \infty$ *then the basis is unconditional.*

See Theorem 9.9, [46].

1.4 Besov and Triebel-Lizorkin spaces

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi_0(x) = 1$ if $|x| \le 1$ and $\varphi_0(x) = 0$ if $|x| \ge 3/2$ and let $\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{k+1}x)$, $x \in \mathbb{R}^n$, $k \in \mathbb{N}$. Since $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ for $x \in \mathbb{R}^n$ the φ_j form a dyadic resolution of unity. The functions $(\varphi_j \widehat{f})^\vee(x)$ are entire analytic functions on $\mathbb{R}^n$ for any $f \in \mathcal{S}'(\mathbb{R}^n)$, so the pointwise operations with the function have sense.

Definition 1.9. Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the above dyadic resolution of unity.

(i) Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| f | B^s_{pq}(\mathbb{R}^n) \right\|_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \widehat{f})^\vee | L_p(\mathbb{R}^n)\|^q \right)^{1/q} < \infty$$

(with the usual modification if $q = \infty$).

Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f | F^s_{pq}(\mathbb{R}^n)\|_\varphi = \| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q \right)^{1/q} | L_p(\mathbb{R}^n)\| < \infty$$

(with the usual modification if $q = \infty$).

Remark 1.2. The theory of the above spaces may be found in [39]-[41]. The definition of Besov and Triebel-Lizorkin spaces is independent of the resolution of the unity φ up to quasi-norm equivalence. The spaces are quasi-Banach spaces. They are Banach spaces if $p \ge 1$ and $q \ge 1$. If $p, q < \infty$ the spaces are separable and the space $\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of $F^s_{pq}(\mathbb{R}^n)$ or $B^s_{pq}(\mathbb{R}^n)$. We have always

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{pq}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$$

and

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^s_{pq}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

We have the following elementary but important embeddings:

- $B^{s_1}_{pq_1}(\mathbb{R}^n) \hookrightarrow B^{s_2}_{pq_2}(\mathbb{R}^n)$ and $F^{s_1}_{pq_1}(\mathbb{R}^n) \hookrightarrow F^{s_2}_{pq_2}(\mathbb{R}^n)$ if $s_1 \ge s_2$.
- $B^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow B^s_{p,q_2}(\mathbb{R}^n)$ and $F^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow F^s_{p,q_2}(\mathbb{R}^n)$ if $q_1 \le q_2$.
- $B^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow F^s_{pq}(\mathbb{R}^n) \hookrightarrow B^s_{p,q_2}(\mathbb{R}^n)$ if $q_1 \le \min(p, q) \le \max(p, q) \le q_2$.

The both scales of function spaces have so called lift property. Let $\sigma \in \mathbb{R}$. Then $I_\sigma : f \mapsto \left((1 + |\cdot|^2)^{\sigma/2} \widehat{f} \right)^\vee$ is a topological bijection of $\mathcal{S}(\mathbb{R}^n)$ onto itself and $\mathcal{S}'(\mathbb{R}^n)$ onto itself. Furthermore

$$I_\sigma B^s_{pq}(\mathbb{R}^n) = B^{s-\sigma}_{pq}(\mathbb{R}^n)$$

and

$$I_\sigma F^s_{pq}(\mathbb{R}^n) = F^{s-\sigma}_{pq}(\mathbb{R}^n)$$

(equivalence of quasi-norms).

Remark 1.3. There are some well known special cases of these function spaces. Let $1 < p < \infty$. Then

$$L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n) \quad \text{(norm equivalence)}.$$

This is the well-known Paley-Littlewood theorem. Let $1 < p < \infty$ and $k \in \mathbb{N}_0$. Then

$$W_k^k(\mathbb{R}^n) = F_{p,2}^k(\mathbb{R}^n)$$

are the classical Sobolev spaces usually equivalently normed by

$$\left\| f|W_p^k(\mathbb{R}^n) \right\| = \left(\sum_{|\alpha|\le k} \| D^\alpha f | L_p(\mathbb{R}^n) \|^p \right)^{1/p} .$$

More generally if $1 < p < \infty$, $s \in \mathbb{R}$ then

$$F_{p,2}^s(\mathbb{R}^n) = H_p^s(\mathbb{R}^n)$$

where

$$H_p^s(\mathbb{R}^n) := I_{-s} L_p(\mathbb{R}^n)$$

is a Sobolev spaces with fractional smoothness.

One can define an equivalent quasi-norm in $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$ spaces using iterated differences

$$\Delta_h^1 f(x) = f(x+h) - f(x), \quad \Delta_h^{m+1} f(x) = \Delta_h^1 \left(\Delta_h^m f \right)(x), \ m \ge 1$$

with $x, h \in \mathbb{R}^n$.

In particular if $s > 0$ then

$$B_{\infty\infty}^s(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n).$$

Where $\mathcal{C}^s(\mathbb{R}^n)$ is the Hölder-Zygmund space that is the space consisted of continuous functions for which the following norm is finite

$$\| f|\mathcal{C}^s(\mathbb{R}^n) \| = \left\| f|C^{[s]}(\mathbb{R}^n) \right\| + \sum_{|\alpha|=[s]} \sup_{0\ne h\in\mathbb{R}^n} |h^{-\{s\}}| \left\| \Delta_h^2 D^\alpha f(x) | C(\mathbb{R}^n) \right\| , \tag{1.4.1}$$

where $s = [s] + \{s\}$, $[s]$ is an integer and $0 < \{s\} \le 1$.

Chapter 2

Function spaces with $\mathcal{A}_p^{loc}$ weights

2.1 Classes of weights

Let w be a nonnegative and locally integrable function on $\mathbb{R}^n$. Such functions are called weights and for measurable set E $w(E)$ denotes $\int_E w(x)\,dx$. We consider $L_p^w(\mathbb{R}^n)$ spaces, i.e., $L_p(\mathbb{R}^n)$ spaces with Lebesgue measure replaced with measure $w\,dx$.

2.1.1 Locally regular weights

Several classes of weights were considered in the context of Sobolev and Besov type spaces. I would like to mention some of them that are called admissible weights and regular weights.

Let $w \in C^\infty(\mathbb{R}^n)$ be a weight such that for all $\gamma \in \mathbb{N}_0^n$ there exists a positive constant c_γ

$$|D^\gamma w(x)| \le c_\gamma w(x) \quad \text{for all } x \in \mathbb{R}^n \tag{2.1.1}$$

and there exist constants $c > 0$ and $\alpha \ge 0$

$$0 < w(x) \le cw(y)\left(1 + |x-y|^2\right)^{\alpha/2} \quad \text{for all } x, y \in \mathbb{R}^n. \tag{2.1.2}$$

These weights are called *admissible weights*, [19]. For example the functions w, v given by

$$w(x) = (1+|x|^2)^{\alpha/2}, \quad v(x) = \left(1 + \log(1+|x|^2)\right)^\alpha, \quad \alpha \ge 0$$

are admissible weights.

If a function $w \in C^\infty(\mathbb{R}^n)$ satisfies (2.1.1) and the following exponential growth condition

$$0 < w(x) \le C \exp\left(C\,|x-y|^\beta\right) w(y) \tag{2.1.3}$$

for all $x, y \in \mathbb{R}^n$ and fixed $0 < \beta \le 1$, then it is called *a general locally regular weight*. For example the weight

$$w(x) = \exp(|x|^\beta), \quad 0 < \beta \le 1$$

is locally regular but not admissible, [35]. Of course any admissible weight is locally regular.

2.1.2 Muckenhoupt weights

Let us recall the definition of the Muckenhoupt weights, [38], Chapter V.

Definition 2.1. A weight w belongs to $\mathcal{A}_p$, $w \in \mathcal{A}_p$, $1 < p < \infty$, if

$$A_p(w) := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|^p} \int_Q w(x)\,dx \left(\int_Q w^{1-p'}(x)\,dx \right)^{p-1} < \infty$$

and $w \in \mathcal{A}_1$ if

$$A_1(w) := \sup_{Q \subset \mathbb{R}^n} \frac{w(Q)}{|Q|} \left\| w^{-1} \right\|_{L_\infty(Q)} < \infty.$$

where supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

Definition 2.2. We say that $w \in \mathcal{A}_\infty$ if for any α, $0 < \alpha < 1$, there exists β, $0 < \beta < 1$, such that for all cubes Q and all subsets $F \subset B$

$$|F| \geq \alpha |Q| \Rightarrow w(F) \geq w(Q).$$

As an example we can take

$$w(x) = |x|^\alpha \in \mathcal{A}_p \quad \text{for} \begin{cases} -n < \alpha < n(p-1), & \text{if } 1 < p < \infty, \\ -n < \alpha \leq 0, & \text{if } p = 1 \end{cases}$$

or weights with logarithmic part

$$v(x) = |x|^\alpha \begin{cases} (1 - \log|x|)^{-\beta}, & |x| \leq 1, \\ (1 + \log|x|)^{-\beta}, & |x| > 1. \end{cases}$$

Then

$$v \in \mathcal{A}_1 \quad \text{if} \begin{cases} \beta \in \mathbb{R} & \text{and } -n < \alpha < 0, \\ \beta \geq 0 & \text{and } \alpha = 0, \end{cases}$$

and

$$v \in \mathcal{A}_p, 1 < p < \infty \quad \text{if} \quad -n < \alpha < n(p-1),\ \beta \in \mathbb{R}.$$

If $\beta < 0$ and $\alpha = 0$ then $v \in \mathcal{A}_p$ for any $p > 1$ but not to $\mathcal{A}_1$, cf. [12].

2.1.3 Local Muckenhoupt weights

Definition 2.3 (Rychkov, 2001)**.** We define a class of weights $\mathcal{A}_p^{loc}$ $(1 < p < \infty)$, which consist of all nonnegative locally integrable functions w defined on $\mathbb{R}^n$ for which

$$A_p^{loc}(w) := \sup_{|Q| \leq 1} \frac{1}{|Q|^p} \int_Q w(x)\,dx \left(\int_Q w^{1-p'}(x)\,dx \right)^{p-1} < \infty. \tag{2.1.4}$$

Moreover $w \in \mathcal{A}_1^{loc}$ if

$$A_1^{loc}(w) := \sup_{|Q| \leq 1} \frac{w(Q)}{|Q|} \left\| w^{-1} \right\|_{L_\infty(Q)} < \infty. \tag{2.1.5}$$

Definition 2.4. Let f be locally integrable. Operator

$$M^{loc}f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where supremum is taken over all cubes in $\mathbb{R}^n$ for which $|Q| \leq 1$, is called *a local maximal function.*

Remark 2.1. If $w \in \mathcal{A}_1^{loc}$ then there exists $c > 0$ such that for all cubes Q, $|Q| \leq 1$,

$$\frac{w(Q)}{|Q|} \leq cw(x) \quad \text{for a.e. } x \in Q. \tag{2.1.6}$$

In consequence there exists $c' > 0$ such that

$$M^{loc}w(x) \leq c'w(x) \quad \text{a.e. } x \in \mathbb{R}^n.$$

Remark 2.2. It follows directly from the definitions that $\mathcal{A}_p \subset \mathcal{A}_p^{loc}$ and $A_p^{loc}(w) \leq A_p(w)$ for any $w \in \mathcal{A}_p$, $1 \leq p < \infty$.

Definition 2.5. We say that $w \in \mathcal{A}_\infty^{loc}$ if for any $\alpha \in (0,1)$

$$\sup_{|Q| \leq 1} \left(\sup_{F \subset Q, |F| \geq \alpha |Q|} \frac{w(Q)}{w(F)} \right) < \infty,$$

where F is taken over all measurable sets in $\mathbb{R}^n$.

Remark 2.3. Any Muckenhoupt weight of the class $\mathcal{A}_p$ belongs to the class $\mathcal{A}_p^{loc}$. But local Muckenhoupt weights cover also so called admissible weights and locally regular weights, cf. [34], [19], [35].

As an example of a weight, which is in $\mathcal{A}_\infty^{loc}$, but not in $\mathcal{A}_\infty$ and is not locally regular, we can take

$$w(x) = \begin{cases} |x|^\alpha & \text{for } |x| \leq 1, \\ \exp(|x| - 1) & \text{for } |x| > 1, \end{cases} \qquad \alpha > -n.$$

If $-n < \alpha < n(p-1)$ and $1 < p < \infty$ then $w \in \mathcal{A}_p^{loc}$. If $-n < \alpha \leq 0$ then $w \in \mathcal{A}_1^{loc}$.

2.1.4 Properties of classes $\mathcal{A}_p^{loc}$

We would like to mention some important properties of classes $\mathcal{A}_p^{loc}$.

Lemma 2.1 (Rychkov, 2001)**.** *Let $1 \leq p_1 \leq p_2 \leq \infty$. Then $\mathcal{A}_{p_1}^{loc} \subset \mathcal{A}_{p_2}^{loc} \subset \mathcal{A}_\infty^{loc}$.*
Conversely, if $w \in \mathcal{A}_\infty^{loc}$, then $w \in \mathcal{A}_p^{loc}$ for some $p < \infty$.

The last lemma implies that $\mathcal{A}_\infty^{loc} = \bigcup_{p\geq 1} \mathcal{A}_p^{loc}$. In consequence we can define for $w \in \mathcal{A}_\infty^{loc}$ a positive number

$$r_w = \inf\left\{1 \leq p < \infty : w \in \mathcal{A}_p^{loc}\right\}.$$

In an analogous way we put $\widetilde{r}_w = \inf\{1 \leq p < \infty : w \in \mathcal{A}_p\}$, for $w \in \mathcal{A}_\infty$.

Next lemma shows us an important relation between $\mathcal{A}_p$ and $\mathcal{A}_p^{loc}$ weights.

Lemma 2.2 (Rychkov, 2001)**.** *Let $1 \leq p < \infty$, $w \in \mathcal{A}_p^{loc}$ and I be a unit cube, i.e., $|I| = 1$. Then there exists a $\bar{w} \in \mathcal{A}_p$, such that $\bar{w} = w$ on I and*

$$A_p(\bar{w}) \leq c A_p^{loc}(w),$$

where constant c is independent of I.

We give an example of a weight, which is in $\mathcal{A}_p^{loc} \cap \mathcal{A}_\infty$, but not in $\mathcal{A}_p$ for some $p > 0$. Let

$$w(x) = \begin{cases} |x|^\alpha & \text{for } |x| \leq 1, \\ |x|^\beta & \text{for } |x| > 1, \end{cases}$$

for $\alpha, \beta > -n$. If $\alpha < (p-1)n$ then we have $w \in \mathcal{A}_p^{loc}$ and $r_w = \frac{\max(0,\alpha)}{n} + 1$. On the other hand if $\alpha, \beta < (p_1 - 1)n$ then we have $w \in \mathcal{A}_{p_1}$ and $\widetilde{r}_w = \frac{\max(0,\alpha,\beta)}{n} + 1$. Taking β big enough we get that w is in $\mathcal{A}_p^{loc} \cap \mathcal{A}_\infty$, but not in $\mathcal{A}_p$ and $r_w < \widetilde{r}_w$.

Lemma 2.3. *Let $w \in \mathcal{A}_p^{loc}$, $1 \leq p < \infty$. Let S be a measurable set and Q a cube with $|Q| \leq 1$ such that $S \subset Q$. Then*

$$w(Q) \leq c w(S) \left(\frac{|Q|}{|S|}\right)^p.$$

Proof. Suppose $p > 1$. By Hölder's inequality and the definition of $\mathcal{A}_p^{loc}$ class we get

$$\begin{aligned}
\left(\frac{|S|}{|Q|}\right)^p &= \left(\frac{1}{|Q|}\int_S \chi_S(x)\,dx\right)^p \leq \left(\frac{1}{|Q|}\int_Q \chi_S(x)\,dx\right)^p \\
&= \left(\frac{1}{|Q|}\int_Q \chi_S(x) w^{1/p}(x) w^{-1/p}(x)\,dx\right)^p \\
&\leq \left(\frac{1}{|Q|}\int_Q \chi_S(x) w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}(x)\,dx\right)^{p-1} \\
&\leq C\left(\frac{1}{|Q|} w(S)\right)\left(\frac{|Q|}{w(Q)}\right).
\end{aligned}$$

For $p = 1$ we get the result in analogous way. □

Theorem 2.1 (Rychkov, 2001)**.** *For $1 \leq p < \infty$ the weak (p,p) inequality*

$$w(\{x \in \mathbb{R}^n : M^{loc} f(x) > \lambda\}) \leq \frac{C}{\lambda^p}\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx$$

holds if and only if $w \in \mathcal{A}_p^{loc}$.

Definition 2.6. Let $0 < p < \infty$, $0 < q \leq \infty$ and $w \in \mathcal{A}_\infty^{loc}$. Then $\bar{\ell}_q(L_p^w(\mathbb{R}^n))$ is the set of all sequences $f = \{f_j(x)\}_{j=0}^\infty$ of Lebesgue-measurable functions on $\mathbb{R}^n$ with finite quasi-norms given by

$$\left\| f | \bar{\ell}_q(L_p^w(\mathbb{R}^n)) \right\| = \left(\sum_{j=0}^{\infty} \left\| f_j | L_p^w(\mathbb{R}^n) \right\|^q \right)^{1/q}.$$

Let $0 < p < \infty$, $0 < q \leq \infty$ or $p = q = \infty$ and $w \in \mathcal{A}_\infty^{loc}$. Then $L_p^w(\bar{\ell}_q, \mathbb{R}^n)$ is the set of all sequences $f = \{f_j(x)\}_{j=0}^\infty$ of Lebesgue-measurable functions on $\mathbb{R}^n$ with finite quasi-norms given by

$$\left\| f | L_p^w(\bar{\ell}_q, \mathbb{R}^n) \right\| = \left\| \left(\sum_{j=0}^{\infty} (f_j)^q \right)^{1/q} | L_p^w(\mathbb{R}^n) \right\|.$$

Definition 2.7. We define *a special convolution operator*

$$K_B f(x) = \int_{\mathbb{R}^n} |f(y)| 2^{-B|x-y|} dy \quad (B \geq 0).$$

The Fefferman-Stein maximal inequality holds for the operator M^{loc} and K_B and local Muckenhoupt weights.

Theorem 2.2 (Rychkov, 2001)**.** *Let $1 < p < \infty$, $1 < q \leq \infty$ and $w \in \mathcal{A}_p^{loc}$. Then for any sequence of measurable functions $\{f_j\}_{j=0}^\infty$ we have*

$$\left\| \{M^{loc} f_j\} | L_p^w(\bar{\ell}_q, \mathbb{R}^n) \right\| \leq c \left\| \{f_j\} | L_p^w(\bar{\ell}_q, \mathbb{R}^n) \right\|.$$

Also, there is a $B_0 = B_0(w, n) > 0$ such that for $B \geq B_0/p$ we have

$$\left\| \{K_B f_j\} | L_p^w(\bar{\ell}_q, \mathbb{R}^n) \right\| \leq c \left\| \{f_j\} | L_p^w(\bar{\ell}_q, \mathbb{R}^n) \right\|.$$

Lemma 2.4 (Rychkov, 2001)**.** *Let $w \in \mathcal{A}_p^{loc}$ and $1 < p < \infty$. Then*

$$w(tQ) \leq \exp(c_w t) w(Q) \quad t \geq 1, \ |Q| = 1,$$

where $c_w > 0$ is a constant depending on n and $\mathcal{A}_p^{loc}(w)$.

It follows from the above lemma that classes $\mathcal{A}_p^{loc}$ are independent of the upper bound for the cube size used in their definition, i.e. for any $C > 0$ we could have replaced $|Q| \leq 1$ by $|Q| \leq C$ in Definition 2.3.

2.1.5 Further properties of classes $\mathcal{A}_p^{loc}$

Let $f_Q = \frac{1}{|Q|} \int_Q f(x)\, dx$.

We can define *dyadic* maximal operator

$$M^\Delta f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where supremum is taken over all dyadic cubes $Q \subset \mathbb{R}^n$.

Lemma 2.5. *Let f be a locally integrable function on $\mathbb{R}^n$, and let α be a positive constant so that*

$$\Omega_\alpha = \{x : M^\Delta f(x) > \alpha\}$$

has finite measure. Then Ω_α may be written as a disjoint union of dyadic cubes $\{Q_j\}$ with

(i) $\alpha < |Q_j|^{-1} \int_{Q_j} |f(x)|\,dx$
and

(ii) $|Q_j|^{-1} \int_{Q_j} |f(x)|\,dx \leq 2^n \alpha$,
for each cube Q_j. This has the immediate consequences:

(iii) $|f(x)| \leq \alpha$ *for a.e.* $x \in \mathbb{R}^n \setminus \bigcup_j Q_j$
and

(iv) $|\Omega_\alpha| \leq \alpha^{-1} \int_{\mathbb{R}^n} |f(x)|\,dx$.

The proof of the above lemma can be found in Chapter IV, §3.1 in [35].

Next proposition is analogous to the reverse Hölder inequality, which is an important property of Muckenhoupt weights.

Proposition 2.1. *If $w \in \mathcal{A}_\infty^{loc}$, then there exists $r > 1$ and $c > 0$ (both depending on w) such that*

$$\left(\frac{1}{|Q|}\int_Q w^r\,dx\right)^{1/r} \leq \frac{c}{|Q|}\int_Q w\,dx, \tag{2.1.7}$$

for all cubes Q such that $|Q| \leq 1$.

Proof. For $w \in \mathcal{A}_\infty^{loc}$ there exist constants $\gamma, \delta \in (0,1)$ such that

$$|E| \leq \gamma\,|Q| \Rightarrow w(E) \leq \delta w(Q), \tag{2.1.8}$$

for every Q, $|Q| \leq 1$ and $E \subset Q$.

Let us consider dilation given by $D_\alpha w(t) = \alpha^{-n} w(\frac{t}{\alpha})$ for $\alpha \geq 1$. If w satisfies condition (2.1.8), then $D_\alpha w$ also does. Actually

$$D_\alpha w(E) = \int_E \frac{1}{\alpha^n} w(\frac{t}{\alpha})dt = \int_{\alpha^{-1}E} w(x)\,dx = w(\alpha^{-1}E).$$

From $|\alpha^{-1}E| \leq \gamma\,|\alpha^{-1}Q|$, we have $w(\alpha^{-1}E) \leq \delta w(\alpha^{-1}Q)$, that is $D_\alpha w(E) \leq \delta D_\alpha w(Q)$.

Because the class of w that satisfies (2.1.7) is invariant under multiplication by positive scalars, translation and dilations with $\alpha \geq 1$ similar to the class $\mathcal{A}_\infty^{loc}$, we can follow the ideas of the proof of Proposition 4 in Chapter 5, §3.1 in [35].

Let Q_0 be a cube such that $w(Q_0) = |Q_0| = 1$. We must show that

$$\int_{Q_0} w^r \leq c.$$

Let $f = w\chi_{Q_0}$. Set $E_k = \{x \in Q_0 : M^{\Delta}f(x) > 2^{Nk}\}$, where N is a large integer to be chosen momentarily. From Lemma 2.5 we know that for every $x \in E_k$ there exists maximal dyadic cube Q_j such that $x \in Q_j$ and

$$\frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx \geq 2^{Nk}.$$

These maximal dyadic cubes are pairwise disjoint, their union is E^k and every of such a cube is contained in some dyadic cube contained in E^{k-1}. Summing over all maximal dyadic cubes $Q_j \subset Q$, where cube $Q \subset E^{k-1}$, we have

$$\left|E^k \cap Q\right| = \sum_j |Q_j| \leq 2^{-Nk}\int_Q f(x)\,dx.$$

For Q

$$\int_Q f(x)\,dx \leq 2^n 2^{N(k-1)}\,|Q|\,.$$

Finally we have

$$\left|E^k \cap Q\right| \leq 2^{-Nk}2^n 2^{Nk-N}\,|Q| = 2^{n-N}\,|Q|\,.$$

Now we choose N such that $2^{n-N} \leq \gamma$ and from property of class $\mathcal{A}_\infty^{loc}$ $w(E^k \cap Q) \leq \delta w(Q)$. Taking the union over all Q consisting of E^{k-1} gives

$$w(E^k) \leq \delta w(E^{k-1})$$

and therefore

$$w(E^k) \leq \delta^k w(E^0) \leq \delta^k.$$

Now

$$\int_{Q_0} w^r(x)\,dx \leq \int_{Q_0}\left(M^{\Delta}f(x)\right)^{r-1} w(x)\,dx = \int_{Q_0\cap\{x:M^{\Delta}f(x)\leq 1\}} + \sum_{k=0}^{\infty}\int_{E^k\setminus E^{k+1}}.$$

The first integral is bounded by 1 and the k-th integral in the sum is bounded by

$$2^{N(k+1)(r-1)}w(E^k) \leq 2^{N(k+1)(r-1)}\delta^k.$$

Since $\delta < 1$, the sum

$$\sum_{k=0}^{\infty} 2^{N(k+1)(r-1)}\delta^k$$

converges if r is sufficiently close to 1. □

Proposition 2.2. *Let w_1 and w_2 be $\mathcal{A}_1^{loc}$ weights. If $1 \leq p < \infty$, then $w = w_1 w_2^{1-p}$ belongs to $\mathcal{A}_p^{loc}$. Conversely, let $w \in \mathcal{A}_p^{loc}$, then there exist w_1 and w_2 in $\mathcal{A}_1^{loc}$ such that $w = w_1 w_2^{1-p}$.*

Proof. Let $w_1, w_2 \in \mathcal{A}_1^{loc}$. Then for any cube Q it follows from (2.1.6) that $\operatorname{ess\,inf}_{x\in Q} w_i(x) > 0, i = 1, 2$. Thus for $1 \le p < \infty$, $|Q| \le 1$ and $w = w_1 w_2^{1-p}$ we have

$$\int_Q w(x)\,dx \le \int_Q w_1(x)\,dx \left(\operatorname*{ess\,inf}_{x\in Q} w_2(x)\right)^{1-p},$$

$$\left(\int_Q w^{\frac{1}{1-p}}(x)\,dx\right)^{p-1} \le \left(\int_Q w_2(x)\,dx\right)^{p-1} \left(\operatorname*{ess\,inf}_{x\in Q} w_1(x)\right)^{-1}.$$

Hence

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{\frac{1}{1-p}}(x)\,dx\right)^{p-1} \le$$
$$\le \frac{1}{|Q|}\int_Q w_1(x)\,dx \left(\operatorname*{ess\,inf}_{x\in Q} w_1(x)\right)^{-1} \left[\frac{1}{|Q|}\int_Q w_2(x)\,dx \left(\operatorname*{ess\,inf}_{x\in Q} w_2(x)\right)^{-1}\right]^{p-1}$$

is bounded by (2.1.5). So $w \in \mathcal{A}_p^{loc}$ from (2.1.4).

Now we prove the converse. Let $w \in \mathcal{A}_p^{loc}$, $p \ge 2$. Consider an operator T defined by

$$Tf = \left(w^{-1/p} M^{loc}(f^{p/p'} w^{1/p})\right)^{p'/p} + w^{1/p} M^{loc}(f w^{-1/p}).$$

Because M^{loc} is bounded on $L_p^w(\mathbb{R}^n)$ and on $L_{p'}^{w^{-p'/p}}(\mathbb{R}^n)$, cf. Lemma 2.2, thus T is bounded on $L_p(\mathbb{R}^n)$, that is

$$\begin{aligned} \|Tf\|_p &\le \left\|\left(w^{-1/p} M^{loc}\left(f^{p/p'} w^{1/p}\right)\right)^{p'/p}\right\|_p + \left\|w^{1/p} M^{loc}\left(f w^{-1/p}\right)\right\|_p \\ &\le \left\|M^{loc}\left(f^{p/p'} w^{1/p}\right)\right\|_{p', w^{-p'/p}}^{p'/p} + \left\|M^{loc}\left(f w^{-1/p}\right)\right\|_{p,w} \\ &\le C\left(\left\|f^{p/p'} w^{1/p}\right\|_{p', w^{-p'/p}}^{p'/p} + \left\|f w^{-1/p}\right\|_{p,w}\right) = C\|f\|_p, \end{aligned} \tag{2.1.9}$$

for some $C > 0$. Since $p \ge 2$, $p/p' \ge 1$, Minkowski inequality gives $T(f_1 + f_2) \le Tf_1 + Tf_2$. Fix now a nonnegative f with $\|f\|_p = 1$. It follows from (2.1.9) that the series

$$\eta = \sum_{k=1}^{\infty} (2C)^{-k} T^k(f),$$

where $T^k(f) = T(T^{k-1}(f))$ is convergent in $L_p(\mathbb{R}^n)$. But f is nonnegative, so the functions $T^k f$ are also nonnegative. Thus the series converges to η a.e. Since T is sublinear we have

$$T\eta \le \sum_{k=1}^{\infty} (2C)^{-k} T^{k+1} f = \sum_{k=2}^{\infty} (2C)^{1-k} T^k f \le (2C)\eta \quad \text{a.e}$$

Now we can write $w_1 = w^{1/p}\eta^{p/p'}$, then

$$M^{loc}w_1(x) \le M^{loc}\left(w^{1/p}\eta^{p/p'}\right)(x) \le (T(\eta))^{p/p'}\, w^{1/p} \le (2C\eta)^{p/p'}\, w^{1/p} = \tilde{C}w_1(x) \quad \text{a.e..}$$

So $w_1 \in \mathcal{A}_1^{loc}$. Similarly, if $w_2 = w^{-1/p}\eta$, then $M^{loc}w_2(x) \le \tilde{C}w_2(x)$ a.e. and $w_2 \in \mathcal{A}_1^{loc}$. Now $w = w_1 w_2^{1-p}$.

The case $p \le 2$ follows immediately by factorization for $w^{-p'/p} \in \mathcal{A}_{p'}^{loc}$, $w_1, w_2 \in \mathcal{A}_1^{loc}$, we have

$$w^{-p'/p} = w_1 w_2^{1-p'},$$

so

$$w = \left(w_1 w_2^{1-p'}\right)^{-p/p'} = w_1^{-p/p'} w_2^{-p/p'+p} = w_1^{1-p} w_2.$$

□

Proposition 2.3. *Let w satisfy conditions (2.1.1) and (2.1.3), i.e. w is a locally regular weight. Then $w \in \mathcal{A}_1^{loc}$.*

Proof. We check $A_1^{loc}(w)$ condition. Let Q be some cube with $|Q| \le 1$. Let $y \in Q$. From (2.1.3) we get

$$\begin{aligned}
&\frac{w(Q)}{|Q|}\left\|w^{-1}\right\|_{L_\infty(Q)} = \frac{1}{|Q|}\int_Q w(x)\,dx \ \operatorname*{ess\,sup}_{x\in Q} w^{-1}(x) \\
&\le c\frac{1}{|Q|}w(y)\int_Q \exp(C|x-y|^\beta)\,dx \ \operatorname*{ess\,sup}_{x\in Q} w^{-1}(x) \\
&\le \frac{c}{|Q|}w(y)\int_Q \exp(Cn^{\beta/2})\,dx \ \operatorname*{ess\,sup}_{x\in Q} w^{-1}(x) \\
&\le c\,w(y)\exp(Cn^{\beta/2}) \ \operatorname*{ess\,sup}_{x\in Q} w^{-1}(x) \\
&\le c\operatorname*{ess\,sup}_{x\in Q}\exp(C|x-y|^\beta) \le c.
\end{aligned}$$

So $w \in \mathcal{A}_1^{loc}$. □

2.2 Weighted function spaces with $\mathcal{A}_\infty^{loc}$ weights

In the definition of weighted function spaces we replace in integration Lebesgue measure with $w\,dx$ measure, where w is nonnegative, locally integrable function called a weight. In natural way then we can define weighted function spaces $L_p^w(\mathbb{R}^n)$. In [34] Rychkov define weighted Besov spaces $B_{p,q}^{s,w}(\mathbb{R}^n)$ and weighted Triebel-Lizorkin spaces $F_{p,q}^{s,w}(\mathbb{R}^n)$ related to $L_p^w(\mathbb{R}^n)$. To incorporate the wide class of weights into the theory he introduced a class of distributions which is generalization of the class of tempered distributions. Then he developed Fourier approach to the above spaces. That approach is a natural generalization of prior definitions and gathers different weighted function spaces in one theory. In that subsection we describe basic parts of that theory with properties of weighted function spaces.

Following Rychkov we define Besov and Triebel-Lizorkin spaces with local Muckenhoupt weights, [34]. Because the class of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ is too narrow for this purpose we introduce a class $\mathcal{S}_e'(\mathbb{R}^n)$, which is a topological dual to a $\mathcal{S}_e(\mathbb{R}^n)$ space. The spaces $\mathcal{S}_e(\mathbb{R}^n)$ and $\mathcal{S}_e'(\mathbb{R}^n)$ were introduced by Th. Schott in [35].

Definition 2.8. By $\mathcal{S}_e(\mathbb{R}^n)$ we denote the set of all $\psi \in C^\infty(\mathbb{R}^n)$ such that

$$q_N(\psi) := \sup_{x\in\mathbb{R}^n} e^{N|x|} \sum_{|\alpha|\le N} |D^\alpha \psi(x)| < \infty \quad \text{for all } N \in \mathbb{N}_0.$$

We equip $\mathcal{S}_e(\mathbb{R}^n)$ with the locally convex topology which is defined by the system of the semi norms q_N.

Proposition 2.4. *(i) $\mathcal{S}_e(\mathbb{R}^n)$ is a complete locally convex space.*

(ii) $\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{S}_e(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$.

(iii) $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}_e(\mathbb{R}^n)$. $\mathcal{S}_e(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.

(iv) If $w \in \mathcal{A}_\infty^{loc}$ then $\mathcal{S}_e(\mathbb{R}^n) \hookrightarrow L_p^w(\mathbb{R}^n)$ for any p, $0 < p < \infty$.

Proof. Proof of parts (i), (ii) and (iii) can be found in [35].

Proof of part (iv). It follows from Lemma 2.4 that the function $\varphi(x) = e^{-N|x|}$ belongs to $L_p^w(\mathbb{R}^n)$ for sufficiently large N. We have

$$\int_{\mathbb{R}^n} e^{-N|x|p} w(x)\,dx = \sum_{M=1}^{\infty} \int_{B(0,M)\setminus B(0,M-1)} e^{-N|x|p} w(x)\,dx$$
$$\le \sum_{M=1}^{\infty} e^{-N(M-1)p} e^{c_w M} w(B(0,1)) = c \sum_{M=1}^{\infty} e^{-(Np-c_w)M} < \infty$$

if $Np > c_w$. Thus for $f \in \mathcal{S}_e(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx \le q_N(f) \int_{\mathbb{R}^n} e^{-N|x|p} w(x)\,dx \le C q_N(f).$$

□

Definition 2.9. $\mathcal{S}_e'(\mathbb{R}^n)$ is the collection of all continuous linear forms on $\mathcal{S}_e(\mathbb{R}^n)$. We equip $\mathcal{S}_e'(\mathbb{R}^n)$ with the strong topology.

We can identify the class $\mathcal{S}_e'(\mathbb{R}^n)$ with the set of those distributions $f \in \mathcal{D}'(\mathbb{R}^n)$ for which the estimate

$$|\langle f, \psi\rangle| \le A \sup\{|D^\alpha \psi(x)| \exp(N|x|) : x \in \mathbb{R}^n, |\alpha| \le N\} \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^n),$$

is valid with some constants A, N depending on f. Such a distribution f can be extended to a continuous functional on $\mathcal{S}_e(\mathbb{R}^n)$.

We take a function $\varphi_0 \in \mathcal{D}$ such that

$$\int_{\mathbb{R}^n} \varphi_0(x)\, dx \neq 0 \tag{2.2.1}$$

We put

$$\varphi(x) = \varphi_0(x) - 2^{-n}\varphi_0(\frac{x}{2}) \tag{2.2.2}$$

and $\varphi_j(x) = 2^{(j-1)n}\varphi(2^{j-1}x)$ for $j = 1, 2, ...$ One can find φ_0 such that

$$\int_{\mathbb{R}^n} x^\beta \varphi(x)\, dx = 0 \tag{2.2.3}$$

for any multindex $\beta \in \mathbb{N}_0^n$, $|\beta| \leq B$, where B is a fixed natural number. We will write $B = -1$ if condition (2.2.3) doesn't hold. Indeed in [35] the following proposition is proved.

Proposition 2.5. *Let $L \in \mathbb{N}$. There exist functions $\Phi_L, \Psi_L \in \mathcal{D}(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} \Phi_L(x)\, dx = 1$$

and

$$\Delta^L \Psi_L(x) = \Phi_L(x) - 2^{-n}\Phi_L(\frac{x}{2}).$$

Thus taking $\varphi_0 = \Phi_L$ and $\varphi = \Delta^L\Psi_L$ we get the pair of functions satisfying (2.2.1)-(2.2.3). In particular (2.2.3) follows by integration by parts formula if L is sufficiently large.

Definition 2.10. Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty^{loc}$. Let a function $\varphi_0 \in \mathcal{D}(\mathbb{R}^n)$ satisfy

$$\int_{\mathbb{R}^n} \varphi_0(x)\, dx \neq 0$$

and

$$\int_{\mathbb{R}^n} x^\beta \varphi(x)\, dx = 0, \quad |\beta| < B,$$

where $\varphi(x) = \varphi_0(x) - 2^{-n}\varphi_0(\frac{x}{2})$ and $B \geq [s]$. We define a weighted Besov space $B_{pq}^{s,w}(\mathbb{R}^n)$ to be a set of all $f \in \mathcal{S}_e'$ for which the following quasi-norm

$$\left\| f | B_{pq}^{s,w}(\mathbb{R}^n) \right\|_{\varphi_0} = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \varphi_j * f | L_p^w(\mathbb{R}^n) \right\|^q \right)^{1/q}$$

(with the usual modification if $q = \infty$) is finite, and a weighted Triebel-Lizorkin space $F^{s,w}_{pq}(\mathbb{R}^n)$ to be a set of all $f \in \mathcal{S}'_e$ for which the following quasi-norm

$$\left\| f | F^{s,w}_{pq}(\mathbb{R}^n) \right\|_{\varphi_0} = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\varphi_j * f|^q \right)^{1/q} | L^w_p(\mathbb{R}^n) \right\|$$

(with the usual modification if $q = \infty$) is finite.

Remark 2.4. The definition of the above spaces is independent of a choice of the function φ_0, up to the equivalence of quasi-norms. The spaces are quasi-Banach and Banach spaces if $p \geq 1$ and $q \geq 1$.

Remark 2.5. To simplify the notation we write $A^{s,w}_{pq}(\mathbb{R}^n)$ instead of $B^{s,w}_{pq}(\mathbb{R}^n)$ and $F^{s,w}_{pq}(\mathbb{R}^n)$, when both scales of spaces are meant simultaneously in some context.

Remark 2.6. The definition covers the earlier definitions of Besov and Triebel-Lizorkin spaces for Muckenhoupt weights, admissible and locally regular weights, cf. [6], [19], [35] and references given there. One can also define the Besov and Triebel-Lizorkin spaces for doubling measures, cf. [5]. This approach also covers the weighted spaces with $\mathcal{A}_\infty$ weights, but not with $\mathcal{A}^{loc}_\infty$ weights (e.g. exponential weights are not doubling). On the other hand there are the doubling measures that do not belong to the class $\mathcal{A}^{loc}_\infty$, cf. [45].

Remark 2.7. The spaces $A^{s,w}_{pq}(\mathbb{R}^n)$ have a lot of properties similar to the unweighted spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. In particular

- $A^{s_1,w}_{pq_1}(\mathbb{R}^n) \hookrightarrow A^{s_2,w}_{pq_2}(\mathbb{R}^n)$ if $s_1 \geq s_2$.
- $A^{s,w}_{p,q_1}(\mathbb{R}^n) \hookrightarrow A^{s,w}_{p,q_2}(\mathbb{R}^n)$ if $q_1 \leq q_2$.
- $B^{s,w}_{p,q_1}(\mathbb{R}^n) \hookrightarrow F^{s,w}_{pq}(\mathbb{R}^n) \hookrightarrow B^{s,w}_{p,q_2}(\mathbb{R}^n)$ if $q_1 \leq \min(p,q) \leq \max(p,q) \leq q_2$.

Moreover if $1 < p < \infty$ and $w \in \mathcal{A}^{loc}_p$ then

$$F^{0,w}_{p,2}(\mathbb{R}^n) = L^w_p(\mathbb{R}^n) \quad \text{(norm equivalence)}.$$

All the above properties can be found in [34].

Proposition 2.6. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}^n$ and $w \in \mathcal{A}^{loc}_\infty$. Then*

$$\mathcal{S}_e(\mathbb{R}^n) \hookrightarrow A^{s,w}_{pq}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'_e(\mathbb{R}^n).$$

Moreover if $q < \infty$ then $\mathcal{S}_e(\mathbb{R}^n)$ is dense in $A^{s,w}_{pq}(\mathbb{R}^n)$.

Proof. Step 1. The embeddings $A^{s,w}_{pq}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'_e(\mathbb{R}^n)$ are shown in [34], Lemma 2.15. Now it is sufficient to prove that $\mathcal{S}_e(\mathbb{R}^n) \hookrightarrow B^{s,w}_{p,q}(\mathbb{R}^n)$ for any $s \in \mathbb{R}^n$. The rest is similar or follows by elementary embeddings.

Let $f \in \mathcal{S}_e(\mathbb{R}^n)$. We take $\varphi_0 = \Phi_L$ and $\varphi = \Delta^L \Psi_L$ from Proposition 2.5. Then the properties of convolution implies

$$f * \varphi_j(x) = f * \left(\Delta^L \Psi_L\right)_j(x) = 2^{j(n-2L)} \left(\Delta^L f * \Psi_L(2^j \cdot)\right)(x).$$

But $2^{jn}\Delta^L f * \Psi_L(2^j\cdot) \to \Delta^L f$ in $\mathcal{S}_e(\mathbb{R}^n)$, cf. Proposition 2.7 in [35]. So $2^{jn}\Delta^L f * \Psi_L(2^j\cdot)$ converges uniformly to $\Delta^L f$ and

$$\begin{aligned} \left|2^{jn}\Delta^L f * \Psi_L(2^j\cdot)(x)\right| &\le e^{-N|x|} q_N \left(2^{jn}\Delta^L f * \Psi_L(2^j\cdot)(x) - \Delta^L f(x)\right) + |\Delta^L f(x)| \\ &\le Ce^{-N|x|} + |\Delta^L f(x)|. \end{aligned}$$

The function $e^{-N|x|} + |\Delta^L f(x)| \in L^w_p(\mathbb{R}^n)$ if N is sufficiently large. Therefore by Lebesgue's theorem of dominated convergence $2^{jn}\Delta^L f * \Psi_L(2^j\cdot) \to \Delta^L f$ in $L^w_p(\mathbb{R}^n)$ if $j \to \infty$. Thus

$$\begin{aligned} 2^{js}\left\|f * \varphi_j|L^w_p(\mathbb{R}^n)\right\| &\le 2^{j(s-2L)}\left\|\left(2^{jn}\Delta^L f * \Psi_L(2^j\cdot)\right)(\cdot)|L^w_p(\mathbb{R}^n)\right\| \\ &\le C2^{j(s-2L)}\left\|\Delta^L f|L^w_p(\mathbb{R}^n)\right\| \end{aligned}$$

and the constant C is independent of j. From Proposition 2.4 (iv) and if $2L > s$ we get

$$\sum_{j=0}^{\infty} 2^{jsq}\left\|f * \varphi_j|L^w_p(\mathbb{R}^n)\right\|^q < \infty.$$

Step 2. Now we prove the density of $\mathcal{S}_e(\mathbb{R}^n)$ in $A^{s,w}_{pq}(\mathbb{R}^n)$. The idea follows from Theorem 3.2 in [35]. We show that $\mathcal{D}(\mathbb{R}^n)$ is dense in $F^{s,w}_{pq}(\mathbb{R}^n)$. The proof for $B^{s,w}_{pq}(\mathbb{R}^n)$ is similar. Let $f \in F^{s,w}_{pq}(\mathbb{R}^n)$. Let $\varphi_0 \in \mathcal{D}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\varphi_0(x)\,dx \ne 0$ and $\varphi = \varphi_0 - 2^{-n}\varphi_0(\frac{\cdot}{2})$. Take $\psi_0, \psi \in \mathcal{D}(\mathbb{R}^n)$ such that for any given $A \ge 0$ ψ has vanishing moments up to A and

$$f = \sum_{k=0}^{\infty} \psi_k * \varphi_k * f \quad \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

We can find such ψ by Theorem 1.6 in [34].

Substep 2.1. Put

$$f_m = \sum_{k=0}^{m} \psi_k * \varphi_k * f, \quad m \in \mathbb{N}_0.$$

We want to show that $f_m \to f$ when $m \to \infty$, in $F^{s,w}_{pq}(\mathbb{R}^n)$. Let $0 \le m < m'$. Then

$$\left|\varphi_j * (f_{m'} - f_m)\right|(x) \le \sum_{k=m+1}^{\infty} \left|\varphi_j * \psi_k * \varphi_k * f\right|(x).$$

From Lemma 2.9 in [34] for every $A, B \ge 0$ there is a constant c such that for each $x \in \mathbb{R}^n$ we have if $j \ge m+1$

$$\left|\varphi_j * (f_{m'} - f_m)\right|(x) \le c\sum_{k=j}^{\infty} 2^{(j-k)A}2^{kn}\int_{\mathbb{R}^n} \frac{|\varphi_k * f(\,x-y)|}{\left(1+2^j|y|\right)^A 2^{|y|B}}dy$$

and if $j < m+1$

$$|\varphi_j * (f_{m'} - f_m)|\,(x) \le c \sum_{k=m+1}^{\infty} 2^{(j-k)A} 2^{kn} \int_{\mathbb{R}^n} \frac{|\varphi_k * f(\,x-y)|}{(1+2^j|y|)^A\, 2^{|y|B}} dy.$$

From Lemma 2.10 in [34], Young's inequality ($q>1$) and monotonicity of ℓ_q ($q \le 1$) we get

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{jsq} \left|\varphi_j * (f_{m'} - f_m)\right|^q (x) \\
&\le c \sum_{j=0}^{m} \left| \sum_{k=m+1}^{\infty} 2^{(j-k)(A-n+s)} 2^{ks} \left(M^{loc}(|\varphi_k * f|)(x) + K_B(|\varphi_k * f|)(x)\right) \right|^q + \\
&+ c \sum_{j=m+1}^{\infty} \left| \sum_{k=j}^{\infty} 2^{(j-k)(A-n+s)} 2^{ks} \left(M^{loc}(|\varphi_k * f|)(x) + K_B(|\varphi_k * f|)(x)\right) \right|^q \\
&\le \sum_{k=m+1}^{\infty} 2^{ksq} \left| M^{loc}(|\varphi_k * f|)(x) + K_B(|\varphi_k * f|)(x) \right|^q
\end{aligned}$$

for $A-n+s>0$.

Lemma 2.2 gives

$$\left\| f_{m'} - f_m | F^{s,w}_{pq}(\mathbb{R}^n) \right\| \le \left\| \left(\sum_{j=m+1}^{\infty} 2^{jsq} |\varphi_j * f|^q \right)^{1/q} | L^w_p(\mathbb{R}^n) \right\|.$$

By Lebesgue's theorem of dominated convergence $\{f_m\}$ is a Cauchy sequence in $F^{s,w}_{pq}(\mathbb{R}^n)$. So there is some limit element $\tilde f \in F^{s,w}_{pq}(\mathbb{R}^n)$. From the first step we get $f_m \to \tilde f$ in $\mathcal{S}'_e(\mathbb{R}^n)$. But from Theorem 1.6 in [34] we have $f_m \to f$ in $\mathcal{S}'_e(\mathbb{R}^n)$. Thus $f = \tilde f$.

Substep 2.2. Let $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi(x) = 1$ for $|x| \le 1$ and $\psi(x) = 0$ for $|x| \ge 2$. Let $\psi_m(x) = \psi(2^{-m}x)$, $m \in \mathbb{N}_0$. Let $f \in F^{s,w}_{pq}(\mathbb{R}^n)$ and fix $k \in \mathbb{N}_0$. Put $g = \psi_k * \varphi_k * f$. Then $\psi_m g \in \mathcal{D}(\mathbb{R}^n)$. To prove the density of $\mathcal{D}$ we show $\psi_m g \to g$ when $m \to \infty$ in $F^{s,w}_{pq}(\mathbb{R}^n)$.

From the first step we get

$$\begin{aligned}
&\left\| \psi_{m'} g - \psi_m g | F^{s,w}_{pq}(\mathbb{R}^n) \right\| \le c \left\| \Delta^L(\psi_{m'} g - \psi_m g) | L^w_p(\mathbb{R}^n) \right\| \\
&\le \tilde c \sum_{0 \le |\gamma| \le 2L} \left(\left(\int_{|x| \ge 2^m} |D^\gamma g(x)|^p w(x)\, dx \right)^{1/p} + \sum_{0 < |\beta| \le 2L} 2^{-m|\beta|} \left\| D^\gamma g | L^w_p(\mathbb{R}^n) \right\| \right),
\end{aligned}$$

where $0 \le m < m'$ and $2L > s$. Using inequalities from Substep 2.1 and from $\psi \in \mathcal{S}_e$ we get

$$
\begin{aligned}
&\left\|D^\gamma g|L_p^w(\mathbb{R}^n)\right\| = \left\|D^\gamma \psi_k * \varphi_k * f|L_p^w(\mathbb{R}^n)\right\| \\
&\leq \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |D^\gamma\psi_k(y)|\,|\varphi_k * f(x-y)|\,dy\right)^p w(x)\,dx\right)^{1/p} \\
&\leq c\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |D^\gamma\psi_k(y)|\,(1+|y|)^L 2^{|y|L}\frac{|\varphi_k * f(x-y)|}{(1+|y|)^L 2^{|y|L}}\,dy\right)^p w(x)\,dx\right)^{1/p} \\
&\leq c\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} q_{2L}(\psi)\frac{|\varphi_k * f(x-y)|}{(1+|y|)^L 2^{|y|L}}\,dy\right)^p w(x)\,dx\right)^{1/p} \\
&\leq \tilde{c}\left(\int_{\mathbb{R}^n}\left(M^{loc}(|\varphi_k * f|) + K_L(|\varphi_k * f|)\right)^p w(x)\,dx\right)^{1/p} \\
&\leq \tilde{c}\left(\int_{\mathbb{R}^n} |\varphi_k * f|^p\, w(x)\,dx\right)^{1/p} \leq \tilde{\tilde{c}}\left\|f|F_{pq}^{s,w}(\mathbb{R}^n)\right\|.
\end{aligned}
$$

With the same argumentation as in the conclusion of Substep 2.1 we get that $\{\psi_m\}$ is a Cauchy sequence in $F_{pq}^{s,w}(\mathbb{R}^n)$. From Proposition 2.8 in [35] we get the result.

□

Chapter 3

Local means and Daubechies wavelet bases in function spaces with $\mathcal{A}_p^{loc}$ weights

3.1 Local means and wavelet bases in weighted spaces

In this section we follow the main idea of H. Triebel from [44], that Daubechies wavelets can serve both as atoms and kernels of local means. So, first we recall the atomic decomposition of function spaces with the local Muckenhoupt weights due to Izuki and Sawano, cf. [23], also [24]. Then we introduce local means and prove characterizations of function spaces. Our approach to wavelet decomposition is more direct than this one presented in [24] since we avoid some density arguments.

3.1.1 Atomic decomposition

All results of this section come from [23].

First we define atoms, which are smooth (to some order K) functions, which satisfy moment condition up to some L.

Definition 3.1. Let $s \in \mathbb{R}$, $0 < p < \infty$, $K, L \in \mathbb{N}_0$ and $d \geq 1$. Then C^K-functions $a_{jm} : \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called *(s,p)-atoms* if

$$\text{supp } a_{jm} \subset dQ_{jm}, \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n,$$

and there exist all (classical) derivatives $D^\alpha a_{jm} \in C(\mathbb{R}^n)$ with $|\alpha| \leq K$ such that

$$|D^\alpha a_{jm}(x)| \leq 2^{-j(s-\frac{n}{p})+j|\alpha|}, \quad |\alpha| \leq K,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{3.1.1}$$

and

$$\int_{\mathbb{R}^n} x^\beta a_{jm}(x)\, dx = 0, \quad |\beta| < L,\ j \in \mathbb{N},\ m \in \mathbb{Z}^n. \tag{3.1.2}$$

Remark 3.1. Please note that the last condition is omitted if $j = 0$.

In connection with atoms and function spaces we always have sequence spaces for sequences of coefficients, which will appear in atomic decompositions.

Definition 3.2. Let $0 < p < \infty$, $0 < q \le \infty$ and $w \in \mathcal{A}_\infty^{loc}$. Then b_{pq}^w is a collection of all sequences

$$\lambda = \{\lambda_{jm} \in \mathbb{C} : j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\} \tag{3.1.3}$$

such that

$$\|\lambda|b_{pq}^w\| = \left\| \left\{ \sum_{m\in\mathbb{Z}^n} \lambda_{jm}\chi_{jm}^{(p)} \right\}_{j\in\mathbb{N}_0} |\bar{\ell}_q(L_p^w(\mathbb{R}^n)) \right\| < \infty,$$

and let $0 < p < \infty$, $0 < q \le \infty$ or $p = q = \infty$ then $f_{p,q}^w$ is a collection of all sequences λ according to (3.1.3) such that

$$\|\lambda|f_{p,q}^w\| = \left\| \left\{ \sum_{m\in\mathbb{Z}^n} \lambda_{jm}\chi_{jm}^{(p)} \right\}_{j\in\mathbb{N}_0} |L_p^w(\bar{\ell}_q, \mathbb{R}^n) \right\| < \infty,$$

where $\chi_{jm}^{(p)} = 2^{\frac{jn}{p}}\chi_{Q_{jm}}$. Once more we use the notation a_{pq}^w.

Izuki and Sawano proved in [23] that distributions from $B_{pq}^{s,w}(\mathbb{R}^n)$ and $F_{pq}^{s,w}(\mathbb{R}^n)$ admit atomic decompositions, cf. also [24].

For $w \in \mathcal{A}_\infty^{loc}$ let us define

$$\sigma_p(w) = n\left(\frac{r_w}{\min(p, r_w)} - 1\right) + (r_w - 1)n,$$

$$\sigma_q = \frac{n}{\min(1, q)} - n$$

and

$$\sigma_{pq}(w) = \max(\sigma_p(w), \sigma_q).$$

Theorem 3.1 (Izuki, Sawano)**.** *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty^{loc}$. Let $K, L \in \mathbb{Z}$ satisfy*

$$K \ge (1 + [s])_+ \quad \textit{and} \quad L \ge \max(-1, [\sigma_p(w) - s])$$

when $A_{pq}^{s,w}(\mathbb{R}^n)$ denotes $B_{pq}^{s,w}(\mathbb{R}^n)$ and

$$K \ge (1 + [s])_+ \quad \textit{and} \quad L \ge \max(-1, [\sigma_{pq}(w) - s])$$

when $A_{pq}^{s,w}(\mathbb{R}^n)$ denotes $F_{pq}^{s,w}(\mathbb{R}^n)$. Let $f \in A_{pq}^{s,w}(\mathbb{R}^n)$. Then there exists a sequence of (s,p)-atoms $\{a_{jm}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ and $\lambda \in a_{pq}^w$ such that

$$f = \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{jm}a_{jm} \quad \textit{and} \quad \|\lambda|a_{pq}^w\| \le c\,\|f|A_{pq}^{s,w}(\mathbb{R}^n)\|$$

with convergence in $\mathcal{S}'_e(\mathbb{R}^n)$*. Conversely, let* $\{a_{jm}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ *be a sequence of* (s,p)*-atoms and* $\lambda \in a^w_{pq}$*. Then the series*

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm}$$

converges in $\mathcal{S}'_e(\mathbb{R}^n)$ *and belongs to* $A^{s,w}_{pq}(\mathbb{R}^n)$ *and*

$$\left\| f|A^{s,w}_{pq}(\mathbb{R}^n)\right\| \le c \left\|\lambda|a^w_{pq}\right\| .$$

3.1.2 Characterization by local means

First let us define kernels of local means.

Definition 3.3. Let $A, B \in \mathbb{N}_0$ and $C > 0$. Then C^A-functions $k_{jm} : \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called *kernels* if

$$\text{supp } k_{jm} \subset CQ_{jm}, \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n,$$

there exist all (classical) derivatives $D^\alpha k_{jm} \in C(\mathbb{R}^n)$ with $|\alpha| \le A$ such that

$$|D^\alpha k_{jm}(x)| \le 2^{jn+j|\alpha|}, \quad |\alpha| \le A,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{3.1.4}$$

and

$$\int_{\mathbb{R}^n} x^\beta k_{jm}(x)\,dx = 0, \quad |\beta| < B,\ j \in \mathbb{N},\ m \in \mathbb{Z}^n. \tag{3.1.5}$$

Since the kernels have finite smoothness we will work with distributions of finite order.

Let us consider a set $C^m_K(\mathbb{R}^n)$ of functions φ in $C^m(\mathbb{R}^n)$ such that $supp\, \varphi \subset K$, where $K \subset \mathbb{R}^n$ is compact and a set $C^m_0(\mathbb{R}^n)$ consists of functions belonging to $C^m(\mathbb{R}^n)$ with compact support.

Definition 3.4. A distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ is *of order* m, $m \in \mathbb{N}_0$, if for every compact $K \subset \mathbb{R}^n$ there exists a constant c such that

$$|f(\varphi)| \le c \sum_{|\alpha|\le m} \sup_{x\in K} |D^\alpha \varphi(x)| \qquad \text{for every } \varphi \in C^\infty_0(\mathbb{R}^n).$$

We denote the set of all distributions of order m by $\mathcal{D}'_m(\mathbb{R}^n)$.

Distributions of finite order can be identified with continuous linear functionals on $C^m_0(\mathbb{R}^n)$.

Theorem 3.2. *If* $f \in \mathcal{D}'_m(\mathbb{R}^n)$ *then* f *can be extended to a continuous linear functional on* $C^m_0(\mathbb{R}^n)$*, moreover* $(C^m_0(\mathbb{R}^n))' = \mathcal{D}'_m(\mathbb{R}^n)$*.*

The proof of the above theorem can be found in [22], Theorem 2.1.6.

Now we can define local means as dual pairing with with distributions of finite order.

Definition 3.5. Let $f \in \mathcal{D}'_A(\mathbb{R}^n) \cap \mathcal{S}'_e(\mathbb{R}^n)$. Let k_{jm} be kernels according to Definition 3.3 (with the same constant A). Then

$$k_{jm}(f) = \langle f, k_{jm} \rangle = \int_{\mathbb{R}^n} k_{jm}(y) f(y)\, dy, \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{3.1.6}$$

are called *local means*. Furthermore we put,

$$k(f) = \{k_{jm}(f) : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}. \tag{3.1.7}$$

As in section devoted to atoms (Definition 3.2) we define sequence spaces now related to local means.

Definition 3.6. Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$ and $w \in \mathcal{A}^{loc}_\infty$. Then $\bar{b}^{s,w}_{pq}$ is a collection of all sequences λ according to (3.1.3) such that

$$\left\|\lambda | \bar{b}^{s,w}_{pq}\right\| = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \left(\int_{\mathbb{R}^n} \left| \sum_{m\in\mathbb{Z}^n} \lambda_{jm} \chi^{(p)}_{jm}(x) \right|^p w(x)\, dx \right)^{q/p} \right)^{1/q}$$

and $\bar{f}^{s,w}_{p,q}$ is a collection of all sequences λ according to (3.1.3) such that

$$\left\|\lambda | \bar{f}^{s,w}_{p,q}\right\| = \left\| \left(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{jsq} \left|\lambda_{jm}\chi_{jm}\right|^q \right)^{1/q} | L^w_p(\mathbb{R}^n) \right\| < \infty$$

The following lemma and corollary give us the conditions on the function spaces to consist of distributions of finite order.

Lemma 3.1. *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $w \in \mathcal{A}^{loc}_\infty$. Then $B^{s,w}_{pp}(\mathbb{R}^n) \subset \mathcal{D}'_l(\mathbb{R}^n)$ for any $l \ge \max(0, [-s + \frac{nr_w}{p} - \frac{n}{p}] + 1)$.*

Proof. Let $f \in B^{s,w}_{pp}(\mathbb{R}^n)$. From the atomic decomposition we have

$$f = \sum_{j,m} \lambda_{jm} a_{jm}$$

and $\lambda_{jm} \in b^w_{pp}$, with convergence in $\mathcal{D}'(\mathbb{R}^n)$. It means, that we can approximate f by functions $f_k = \sum_{j\le k, |m|\le k} \lambda_{jm} a_{jm}$, i.e. $f = \lim_{k\to\infty} f_k$ in $\mathcal{D}'(\mathbb{R}^n)$, that is

$$f(\varphi) = \lim_{k\to\infty} f_k(\varphi)$$

for all $\varphi \in C^\infty_0(\mathbb{R}^n)$.

For $p > 1$ from Hölder's inequality we have

$$|f_k(\varphi)| = \left|\sum_{j,|m|\le k} \lambda_{jm} a_{jm}(\varphi)\right| \le \sum_{j,|m|\le k} |\lambda_{jm}| \left|\int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx\right|$$
$$\le \left(\sum_{j,|m|\le k} 2^{jn} |\lambda_{jm}|^p w(Q_{jm})\right)^{\frac{1}{p}} \left(\sum_{j,|m|\le k} 2^{-\frac{jnp'}{p}} w^{-\frac{p'}{p}}(Q_{jm}) \left|\int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx\right|^{p'}\right)^{\frac{1}{p'}}$$

Since $\lambda \in b^w_{pp}$, we have

$$\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{jn} |\lambda_{jm}|^p w(Q_{jm}) < \infty. \tag{3.1.8}$$

From $|a_{jm}(x)| \le 2^{-j(s-\frac{n}{p})}$ we get

$$\begin{aligned}\left|\int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx\right|^{p'} &\le c\,|Q_{jm}|^{p'}\, 2^{-j(s-\frac{n}{p})p'} \sup_x |\varphi(x)|^{p'} \\ &\le c2^{-j(s-\frac{n}{p})p'-jnp'} \sup_x |\varphi(x)|^{p'}.\end{aligned}$$

Let supp $\varphi \subset K$, where K is a compact subset in $\mathbb{R}^n$. If $Q_{jm} \subset Q_{0,l}$ then from Lemma 2.3 we have

$$w^{-p'/p}(Q_{jm}) \le c w^{-p'/p}(Q_{0,l}) 2^{jnup'/p}, \tag{3.1.9}$$

since $w \in \mathcal{A}^{loc}_u$ for some $r_w < u < \infty$. So

$$\sum_{m:Q_{jm}\cap K\ne\emptyset,|m|\le k} w^{-\frac{p'}{p}}(Q_{jm}) \le 2^{jnu\frac{p'}{p}+jn} \sum_{l:Q_{0,l}\cap K\ne\emptyset} w(Q_{0,l})^{-\frac{p'}{p}}. \tag{3.1.10}$$

Now we can keep on estimating

$$\begin{aligned}&\sum_{j,|m|\le k} 2^{-jnp'/p} w^{-p'/p}(Q_{jm}) \left|\int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx\right|^{p'} \\ &\le c \sum_{j,|m|\le k} 2^{-j(s+n)p'} \sup_{x\in\mathbb{R}^n} |\varphi(x)|^{p'} w(Q_{jm})^{-\frac{p'}{p}} \\ &\le c \sum_{j\le k, l:Q_{0,l}\cap K\ne\emptyset} 2^{-j(s+n)p'} 2^{jnu\frac{p'}{p}+jn} \sup_{x\in K} |\varphi(x)|^{p'} w(Q_{0,l})^{-\frac{p'}{p}} \\ &\le C_K \sup_{x\in K} |\varphi(x)|^{p'} \sum_{j\in\mathbb{N}_0} 2^{-j(s-\frac{nu}{p}+\frac{n}{p})p'}.\end{aligned}$$

For $s > \frac{nu}{p} - \frac{n}{p}$ we have

$$|f_k(\varphi)| \le C_K \sup_{x\in K} |\varphi(x)|,$$

where C_K depends only on K. Hence

$$|f(\varphi)| \leq C_K \sup_{x\in K} |\varphi(x)| .$$

So f is a distribution of order 0 if $s > \frac{nu}{p} - \frac{n}{p}$. Now let $s \leq \frac{nu}{p} - \frac{n}{p}$ and $l > -s + \frac{nu}{p} - \frac{n}{p}$. Using the Taylor expansion of φ and the moment conditions if $j > 0$ we get

$$\begin{aligned} &\left|\int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx\right| \\ &= c\left|\int_{\mathbb{R}^n} a_{jm}(x)\sum_{|\alpha|=l} D^\alpha\varphi(x_0+\Theta(x-x_0))(x-x_0)^\alpha\,dx\right| \\ &\leq c2^{-j(l+s-\frac{n}{p}+n)}\sum_{|\alpha|=l}\sup_{x\in\mathbb{R}^n}|D^\alpha\varphi(x)| . \end{aligned} \tag{3.1.11}$$

Summing over $j, |m| \leq k$, we get from (3.1.11) and (3.1.9)

$$\begin{aligned} &\sum_{j,m} 2^{-jnp'/p} w^{-\frac{p'}{p}}(Q_{jm}) \left|\int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx\right|^{p'} \\ &\leq c_K \sum_{j\in\mathbb{N}_0} 2^{-j(l+s-\frac{nu}{p}+\frac{n}{p})p'} \left(\sum_{|\alpha|=l}\sup_{x\in K}|D^\alpha\varphi(x)|\right)^{p'} . \end{aligned}$$

Incorporating the term with $j = 0$ we get

$$|f_k(\varphi)| \leq c_K \sum_{|\alpha|\leq l}\sup_{x\in K}|D^\alpha\varphi(x)|.$$

For $0 < p \leq 1$ we have an estimate

$$
\begin{aligned}
|f_k(\varphi)| &= \left| \sum_{j,|m|\leq k} \lambda_{jm} a_{jm}(\varphi) \right| \\
&\leq \left(\sum_{\substack{j,|m|\leq k \\ Q_{jm}\cap K\neq\emptyset}} |\lambda_{jm}|^p \left| \int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx \right|^p \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{\substack{j,|m|\leq k \\ Q_{jm}\cap K\neq\emptyset}} 2^{jn} |\lambda_{jm}|^p \, w(Q_{jm}) \sup_{x\in K} |\varphi(x)|^p \, |Q_{jm}|^p \, w^{-1}(Q_{jm}) 2^{-j(s-\frac{n}{p})p} 2^{-jn} \right)^{\frac{1}{p}} \\
&\leq \sup_{x\in K} |\varphi(x)| \sup_{\substack{j,|m|\leq k \\ Q_{jm}\cap K\neq\emptyset}} 2^{-j(s+n)} w^{-\frac{1}{p}}(Q_{jm}) \left(\sum_{\substack{j,|m|\leq k \\ Q_{jm}\cap K\neq\emptyset}} 2^{jn} |\lambda_{jm}|^p \, w(Q_{jm}) \right)^{1/p}.
\end{aligned}
$$

Using the fact that $\lambda \in b^w_{pp}$ we get

$$
|f_k(\varphi)| \leq C \sup_{x\in K} |\varphi(x)| \sup_{j,|m|\leq k, Q_{jm}\cap K\neq\emptyset} 2^{-j(s+n)} w^{-\frac{1}{p}}(Q_{jm}).
$$

In the same manner as in (3.1.9) we can see that

$$
\begin{aligned}
|f_k(\varphi)| &\leq C \sup_{x\in K} |\varphi(x)| \sup_{j,l,Q_{0,l}\cap K\neq\emptyset} 2^{-j(s+n)} 2^{jnu/p} w^{-\frac{1}{p}}(Q_{0,l}) \\
&\leq C(K) \sup_{x\in K} |\varphi(x)| \sup_{j\in\mathbb{N}_0} 2^{-j(s-\frac{nu}{p}+n)}.
\end{aligned}
$$

For $s > nu/p - n$ we have

$$
|f_k(\varphi)| \leq C \sup_{x\in K} |\varphi(x)|.
$$

For $s \leq nu/p - n$ and $l > -s + nu/p - n$ using the above estimations and the same inequalities as in (3.1.11) we get

$$
\begin{aligned}
|f_k(\varphi)| &\leq \left(\sum_{j,|m|\leq k} |\lambda_{jm}|^p \left| \int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx \right|^p \right)^{1/p} \\
&\leq \sum_{|\alpha|=l} \sup_{x\in K} |D^\alpha \varphi(x)| \sup_{j,|m|\leq k, K\cap Q_{jm}\neq\emptyset} 2^{-j(l+s+n)} w^{-1/p}(Q_{jm}) \\
&\leq c_K \sum_{|\alpha|=l} \sup_{x\in K} |D^\alpha \varphi(x)| \sup_{j\in\mathbb{N}_0} 2^{-j(l+s+n-nu/p)}.
\end{aligned}
$$

So f is a distribution of order l for any $l \geq \max(0, [-s + \frac{nr_w}{p} - \frac{n}{p}] + 1)$. □

Corollary 3.1. *Let a weight w belong to the class $\mathcal{A}_\infty^{loc}$. The spaces $F_{pq}^{s,w}(\mathbb{R}^n)$ and $B_{pq}^{s,w}(\mathbb{R}^n)$ consist of distributions of finite order l for any $l \geq \max(0, [-s + \frac{nr_w}{p} - \frac{n}{p}] + 1)$.*

Proof. Let us choose $s' < s$ such that $l \geq \max(0, [-s' + \frac{nr_w}{p} - \frac{n}{p}] + 1)$. Then by the elementary embeddings and Lemma 3.1 we have

$$F_{pq}^{s,w}(\mathbb{R}^n) \subset B_{pp}^{s',w}(\mathbb{R}^n) \subset \mathcal{D}'_l(\mathbb{R}^n).$$

A similar argument works for Besov spaces. □

The next theorem gives us the characterization of Besov and Triebel-Lizorkin spaces with $\mathcal{A}_\infty^{loc}$ weights by local means.

Theorem 3.3. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. Assume that $w \in \mathcal{A}_\infty^{loc}$. Let k_{jm} be kernels according to Definition 3.3, where $A, B \in \mathbb{N}_0$ with*

$$A \geq \max\left(0, [-s + \sigma_p(w)], [\frac{nr_w}{p} - \frac{n}{p} - s] + 1\right), \quad B \geq \max(0, [s] + 1),$$

when $A_{pq}^{s,w}(\mathbb{R}^n)$ denotes $B_{pq}^{s,w}(\mathbb{R}^n)$ and

$$A \geq \max\left(0, [\sigma_{pq}(w) - s], [\frac{nr_w}{p} - \frac{n}{p} - s] + 1\right), \quad B \geq \max(0, [s] + 1),$$

when $A_{pq}^{s,w}(\mathbb{R}^n)$ denotes $F_{pq}^{s,w}(\mathbb{R}^n)$. Let $C > 0$ be fixed. Let $k(f)$ be as in (3.1.6) and (3.1.7). Then for some $c > 0$ and all $f \in A_{pq}^{s,w}(\mathbb{R}^n)$,

$$\left\| k(f) | \bar{a}_{pq}^{s,w} \right\| \leq c \left\| f | A_{pq}^{s,w}(\mathbb{R}^n) \right\|.$$

Proof. We prove the theorem for Besov spaces. The proof for $F_{pq}^{s,w}$ spaces is similar. The changes are analogous to the changes in unweighted case, cf [44].

Let

$$f(x) = \sum_{r=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \lambda_{rl} a_{rl}(x), \quad f \in B_{pq}^{s,w}(\mathbb{R}^n), \tag{3.1.12}$$

be an atomic decomposition according to Theorem 3.1 where

$$K = B \geq \max(0, [s] + 1) \quad \text{and} \quad L = A \geq \max\left(0, [-s + \sigma_p(w)], [\frac{nr_w}{p} - \frac{n}{p} - s] + 1\right)$$

For $j \in \mathbb{N}$ we split (3.1.12) into the parts

$$f = f_j + f^j = \sum_{r=0}^{j} \sum_{l \in \mathbb{Z}^n} \lambda_{rl} a_{rl} + \sum_{r=j+1}^{\infty} \sum_{l \in \mathbb{Z}^n} \lambda_{rl} a_{rl}$$

and get

$$\int_{\mathbb{R}^n} k_{jm}(y) f(y)\, dy = \int_{\mathbb{R}^n} k_{jm}(y) f_j(y)\, dy + \int_{\mathbb{R}^n} k_{jm}(y) f^j(y)\, dy.$$

Let $r \le j$ and $l \in l_r^j(m)$ where

$$l_r^j(m) = \{l : CQ_{jm} \cap DQ_{rl} \neq \emptyset\},$$

where $C, D \in \mathbb{R}$ are positive constants independent of j, r.

By the Taylor expansion of a_{rl} and properties of atoms (3.1.1) and local means (3.1.5) we have

$$\begin{aligned}
&2^{j(s-\frac{n}{p})} \left| \int_{\mathbb{R}^n} k_{jm}(y) a_{rl}(y)\, dy \right| \\
&\le c 2^{j(s-\frac{n}{p})} \sum_{|\gamma|=B} \sup_x |D^\gamma a_{rl}(x)| \int_{\mathbb{R}^n} |k_{jm}(y)| \left| y - 2^{-j}m \right|^B dy \\
&= c 2^{(j-r)(s-\frac{n}{p}-B)}.
\end{aligned}$$

Thus for any $\varepsilon > 0$ we have

$$2^{j(s-\frac{n}{p})p} |k_{jm}(f_j)|^p \le c \sum_{r=0}^{j} \sum_{l \in l_r^j(m)} |\lambda_{rl}|^p \, 2^{(j-r)(s-\frac{n}{p}-B+\varepsilon)p}.$$

Summing over $m \in \mathbb{Z}^n$ we get

$$\begin{aligned}
&2^{j(s-\frac{n}{p})p} \sum_{m \in \mathbb{Z}^n} |k_{jm}(f_j)|^p \frac{w(Q_{jm})}{|Q_{jm}|} \\
&\le c \sum_{r=0}^{j} 2^{(j-r)(s-\frac{n}{p}-B+\varepsilon)p} \sum_{m \in \mathbb{Z}^n} \sum_{l \in l_r^j(m)} |\lambda_{rl}|^p \frac{w(Q_{jm})}{|Q_{jm}|} \\
&= c \sum_{r=0}^{j} 2^{(j-r)(s-\frac{n}{p}-B+\varepsilon)p} \sum_{l \in \mathbb{Z}^n} \sum_{m : l \in l_r^j(m)} |\lambda_{rl}|^p \frac{w(Q_{jm})}{|Q_{jm}|} \\
&\le c \sum_{r=0}^{j} 2^{(j-r)(s-B+\varepsilon)p} \sum_{l \in \mathbb{Z}^n} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|}
\end{aligned} \tag{3.1.13}$$

where the last inequality is a consequence of the estimate card $l_r^j(m) \sim 1$, which follows from the assumption $r \le j$.

Now let $r > j$. Using the Taylor expansion of k_{jm} and moment conditions of atoms (3.1.2) and (3.1.4) we have

$$\begin{aligned}
&2^{j(s-\frac{n}{p})} \left| \int_{\mathbb{R}^n} k_{jm}(y) a_{rl}(y)\, dy \right| \\
&\le 2^{j(s-\frac{n}{p})} \sum_{|\gamma|=A} \sup_x |D^\gamma k_{jm}(x)| \int_{\mathbb{R}^n} |a_{rl}(y)| \left| y - 2^{-r}l \right|^A dy \\
&= c 2^{(j-r)(s-\frac{n}{p}+n+A)}.
\end{aligned}$$

Thus for any $\varepsilon > 0$ we get

$$2^{j(s-\frac{n}{p})p} \left| \int_{\mathbb{R}^n} k_{jm}(y) f^j(y)\, dy \right|^p \le c \sum_{r>j} 2^{(j-r)(s-\frac{n}{p}+n+A-\varepsilon)p} \left(\sum_{l \in l_r^j(m)} |\lambda_{rl}| \right)^p .$$

From Hölder's inequality and the estimates card $l_r^j(m) \sim 2^{n(r-j)}$

$$2^{j(s-\frac{n}{p})p} \left| \int_{\mathbb{R}^n} k_{jm}(y) f^j(y)\, dy \right|^p \le c \sum_{r>j} 2^{(j-r)(s+A-\varepsilon)p} \sum_{l \in l_r^j(m)} |\lambda_{rl}|^p .$$

Summing over $m \in \mathbb{Z}^n$

$$\begin{aligned} &2^{j(s-\frac{n}{p})p} \sum_{m \in \mathbb{Z}^n} \left| k_{jm}(f^j) \right|^p \frac{w(Q_{jm})}{|Q_{jm}|} \\ &\le c \sum_{r>j} 2^{(j-r)(s+A-\varepsilon)p} \sum_{m \in \mathbb{Z}^n} \sum_{l \in l_r^j(m)} |\lambda_{rl}|^p \frac{w(Q_{jm})}{|Q_{jm}|} \\ &\le c \sum_{r>j} 2^{(j-r)(s+A+\frac{n}{p}-\varepsilon)p} \sum_{m \in \mathbb{Z}^n} \sum_{l \in l_r^j(m)} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|} \left(\frac{|Q_{jm}|}{|Q_{rl}|} \right)^u \\ &\le c \sum_{r>j} 2^{(j-r)(s+A+\frac{n}{p}-\varepsilon-\frac{nu}{p})p} \sum_{l \in \mathbb{Z}^n} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|}, \end{aligned} \tag{3.1.14}$$

where the second inequality follows from the fact, that for $w \in \mathcal{A}_u^{loc}$ and $Q_{rl} \subset Q_{jm}$ and from Lemma 2.3 we have

$$w(Q_{jm}) \le c w(Q_{rl}) \left(\frac{|Q_{jm}|}{|Q_{rl}|} \right)^u .$$

Taking (3.1.13) and (3.1.14) together we get

$$\begin{aligned} &2^{j(s-\frac{n}{p})p} \sum_{m \in \mathbb{Z}^n} |k_{jm}(f)|^p \frac{w(Q_{jm})}{|Q_{jm}|} \le c \sum_{r=0}^{j} 2^{(j-r)(s-B+\varepsilon)p} \sum_{l \in \mathbb{Z}^n} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|} \\ &+ c \sum_{r>j} 2^{(j-r)(s+A+\frac{n}{p}-\varepsilon-\frac{nr_w}{p})p} \sum_{l \in \mathbb{Z}^n} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|} \le c \sum_{r=0}^{\infty} 2^{-|j-r|\varkappa p} \sum_{l \in \mathbb{Z}^n} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|}, \end{aligned}$$

where $\varkappa = \min(s + A + \frac{n}{p} - \frac{nr_w}{p} - \varepsilon, B - s - \varepsilon)$. Summing over j we have

$$\begin{aligned} &\left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \left(\sum_{m \in \mathbb{Z}^n} |k_{jm}(f)|^p \frac{w(Q_{jm})}{|Q_{jm}|} \right)^{q/p} \right)^{1/q} \\ &\le c \left(\sum_{j=0}^{\infty} \left(\sum_{r=0}^{\infty} 2^{-|j-r|\varkappa p} \sum_{l \in \mathbb{Z}^n} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|} \right)^{q/p} \right)^{1/q} . \end{aligned}$$

Now using the Young inequalities (Lemma 1.2) for convolution of sequences if $\frac{q}{p} \geq 1$ or monotonicity of the l_p space if $\frac{q}{p} < 1$ we proved that

$$\left\|k(f)|\bar{b}^{s,w}_{pq}\right\| \leq c\left\|\lambda|b^{w}_{pq}\right\| \leq c\left\|f|B^{s,w}_{pq}(\mathbb{R}^n)\right\|,$$

where the constant c is independent of the given atomic decomposition. □

3.1.3 Characterization by wavelets

We are going to deal with Daubechies wavelets on $\mathbb{R}^n$. We define sequence spaces related to wavelets. That spaces are defined similar to that one in Definition 3.6, but here we have additional finite sums taken on wavelet indexes G.

Definition 3.7. Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ and $w \in \mathcal{A}^{loc}_{\infty}$. Then $b^{s,w}_{pq}$ is a collection of all sequences λ according to (3.1.3) such that

$$\left\|\lambda|b^{s,w}_{pq}\right\| = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G\in G^j} \left(\int_{\mathbb{R}^n} \left|\sum_{m\in\mathbb{Z}^n} \lambda^G_{jm}\chi^{(p)}_{jm}(x)\right|^p w(x)\,dx\right)^{q/p}\right)^{1/q} < \infty.$$

and $f^{s,w}_{pq}$ is a collection of all sequences λ according to (3.1.3) such that

$$\left\|\lambda|f^{s,w}_{pq}\right\| = \left\|\left(\sum_{j,m,G} 2^{jsq}\left|\lambda^G_{jm}\chi_{jm}\right|^q\right)^{1/q} |L^w_p(\mathbb{R}^n)\right\| < \infty.$$

Theorem 3.4. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}^{loc}_{\infty}$. For wavelets defined in (1.3.2) we take*

$$k \geq \max\left(0, [s]+1, [\frac{nr_w}{p} - \frac{n}{p} - s]+1, [\sigma_p(w) - s]\right)$$

in $B^{s,w}_{pq}$ case and

$$k \geq \max\left(0, [s]+1, [\frac{nr_w}{p} - \frac{n}{p} - s]+1, [\sigma_{pq}(w) - s]\right)$$

in $F^{s,w}_{pq}$ case. Let $f \in \mathcal{S}'_e(\mathbb{R}^n)$. Then $f \in A^{s,w}_{pq}(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{j,G,m} \lambda^G_{jm} 2^{-jn/2}\Psi^G_{jm},$$

where $\lambda \in a^{s,w}_{pq}$ and the series converges in $S'_e(\mathbb{R}^n)$. This representation is unique with

$$\lambda^G_{jm} = 2^{jn/2}\langle f, \Psi^G_{jm}\rangle$$

and

$$I : f \mapsto \{2^{jn/2}\langle f, \Psi^G_{jm}\rangle\}$$

is a linear isomorphism of $A^{s,w}_{pq}(\mathbb{R}^n)$ onto $a^{s,w}_{pq}$.

If $0 < p, q < \infty$ then the system $\left\{\Psi^G_{jm}\right\}_{j,m,G}$ is an unconditional basis in $A^{s,w}_{pq}(\mathbb{R}^n)$.

Proof. Step 1.

Let $f \in \mathcal{S}'_e(\mathbb{R}^n)$ and $f = \sum_{j,G,m} \lambda^G_{jm} 2^{-jn/2}\Psi^G_{jm}$ (convergence in $\mathcal{S}'_e(\mathbb{R}^n)$) with $\lambda \in b^{s,w}_{pq}$. Then $a^G_{jm} = 2^{-j(s-\frac{n}{p})}2^{-jn/2}\Psi^G_{jm}$ is an (s,p)-atom. Indeed

$$\text{supp } a^G_{jm} \subset dQ_{jm} \qquad \text{and} \qquad \left|D^\alpha a^G_{jm}\right| \le 2^{-j(s-\frac{n}{p})+j|\alpha|}$$

for $|\alpha| \le k$ and $k = K = L$ in the definition of atoms. So $f \in B^{s,w}_{pq}(\mathbb{R}^n)$ and

$$\left\|f|B^{s,w}_{pq}(\mathbb{R}^n)\right\| \le c \left\| \left\{2^{j(s-\frac{n}{p})}\lambda^G_{jm}\right\}_{j,m,G} |b^w_{pq}\right\| = c\left\|\lambda|b^{s,w}_{pq}\right\|$$

from Theorem 3.1.

Step 2.

Now let $f \in B^{s,w}_{pq}(\mathbb{R}^n)$. We take $k^G_{jm} = 2^{jn/2}\Psi^G_{jm}$ as kernels of local means. Indeed

$$\text{supp } k^G_{jm} \subset CQ_{jm} \qquad \text{and} \qquad \left|D^\alpha k^G_{jm}(x)\right| \le 2^{jn+j|\alpha|},$$

where $|\alpha| \le k$ and $A = B = k$. So from Theorem 3.3 we have

$$\left\|k(f)|b^{s,w}_{pq}\right\| \le c\left\|f|B^{s,w}_{pq}(\mathbb{R}^n)\right\|. \tag{3.1.15}$$

From the atomic decomposition and (3.1.15) we have

$$g = \sum_{j,G,m} k^G_{jm}(f)2^{-jn/2}\Psi^G_{jm} \in B^{s,w}_{pq}(\mathbb{R}^n).$$

It follows from Lemma 3.1 that $\langle g, \Psi^{G'}_{j'm'}\rangle$ make sense. By orthogonality of wavelet basis we get

$$\langle g, \Psi^{G'}_{j'm'}\rangle = \sum_{j,G,m} k^G_{jm}(f)2^{-jn/2}\langle \Psi^G_{jm}, \Psi^{G'}_{j'm'}\rangle = \langle f, \Psi^{G'}_{j'm'}\rangle.$$

This could be extended to finite linear combinations of $\Psi^{G'}_{j'm'}$. Both distributions f and g are locally contained in the space $B^\sigma_{pp}(\mathbb{R}^n)$ for any $\sigma < s - \frac{nr_w}{p} + \frac{n}{p}$. This follows easily from the corresponding result for the spaces with Muckenhoupt weights, cf. [16], since any local Muckenhoupt weight $w \in \mathcal{A}^{loc}_p$ can be extended outside a fixed ball to a Muckenhoupt weight belonging to $\mathcal{A}_p$. Any $\varphi \in C^\infty_0(\mathbb{R}^n)$ has the unique wavelet representation. We can choose σ such that $k > \max(-\sigma + \sigma_p, \sigma)$ so this representation converges in the dual space of $B^\sigma_{pp}(\mathbb{R}^n)$, cf. [44]. This implies that $\langle g, \varphi\rangle = \langle f, \varphi\rangle$ for all $\varphi \in C^\infty_0(\mathbb{R}^n)$ and $g = f$.

Step 3.

By the above steps $f \in \mathcal{S}'_e(\mathbb{R}^n)$ belongs to $B^{s,w}_{pq}(\mathbb{R}^n)$ if and only if $f = \sum_{j,G,m} \lambda^G_{jm}2^{-jn/2}\Psi^G_{jm}$ and $\{\lambda^G_{jm}\} \in b^{s,w}_{pq}$. This representation is unique so $\lambda^G_{jm} = k^G_{jm}(f)$ and

$$\left\|f|B^{s,w}_{pq}(\mathbb{R}^n)\right\| \sim \left\|k(f)|b^{s,w}_{pq}\right\|.$$

It follows from the uniqueness of the coefficients that I is a monomorphism. We show that I is onto. Let $\left\{\lambda_{jm}^G\right\} \in b_{pq}^w$. Then by the atomic decomposition theorem

$$f = \sum_{j,G,m} \lambda_{jm}^G \Psi_{jm}^G \in B_{pq}^{s,w}(\mathbb{R}^n).$$

But the uniqueness of the coefficients implies that $\lambda_{jm}^G = \langle f, \Psi_{jm}^G \rangle$.

Step 4. Let $f \in B_{pq}^{s,w}(\mathbb{R}^n)$ and $p, q < \infty$. Then $f = \sum_{j,m,G} \lambda_{jm}^G 2^{-jn/2} \Psi_{jm}^G$ (convergence in $\mathcal{S}_e'(\mathbb{R}^n)$) with $\lambda \in b_{pq}^{s,w}$. For any $N \in \mathbb{N}$ we have

$$\begin{aligned}
0 &\le \left\| f - \sum_{j=0}^{N} \sum_{G,m} \lambda_{jm}^G 2^{-jn/2} \Psi_{jm}^G | B_{pq}^{s,w}(\mathbb{R}^n) \right\| \\
&= \left\| \sum_{j=N+1}^{\infty} \sum_{G,m} \lambda_{jm}^G 2^{-jn/2} \Psi_{jm}^G | B_{pq}^{s,w}(\mathbb{R}^n) \right\| \\
&\le C \left\| \{ 2^{j(s-\frac{n}{p})} \lambda_{jm}^G \}_{j=N+1,m,G}^{\infty} | b_{pq}^w \right\| \\
&= c \left(\sum_{j=N+1} \left(\sum_{G,m} |\lambda_{jm}^G 2^{j(s-\frac{n}{p})}|^p \right)^{q/p} \right)^{1/q} < \infty.
\end{aligned}$$

The last converges to 0 if $N \to \infty$, since $q < \infty$. This proves the convergence of the series to f in $B_{pq}^{s,w}(\mathbb{R}^n)$ using the order exhibited above.

Let $\varepsilon_{jm}^G = \pm 1$ then the sequence $\varepsilon_{jm}^G \lambda_{jm}^G$ belongs to $b_{pq}^{s,w}$. So by atomic decomposition theorem $f_\varepsilon = \sum_{j,m,G} \varepsilon_{jm}^G \lambda_{jm}^G 2^{-jn/2} \Psi_{jm}^G \in B_{pq}^{s,w}(\mathbb{R}^n)$. By the same argument as above the series converges to f_ε in $B_{pq}^{s,w}(\mathbb{R}^n)$. By the general theory of unconditional series, the series converges unconditionally.

Spaces $F_{pq}^{s,w}$ can be regarded similarly. □

3.2 Sobolev embeddings of Besov and Triebel-Lizorkin spaces

Now we prove theorems of embeddings of weighted Besov spaces into local L_p spaces and into the space of continuous functions. We will need the embeddings in Chapter 4, where we construct Haar bases in $B_{pq}^{s,w}$ and $F_{pq}^{s,w}$ spaces.

3.2.1 Embeddings for unweighted spaces

First we recall the analogous results for unweighted spaces.

Theorem 3.5. *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. Then $B^s_{pq}(\mathbb{R}^n) \subset L^{loc}_{\max(1,p)}(\mathbb{R}^n)$ if and only if*

$$s > n\left(\frac{1}{p} - 1\right)_+, \; 0 < p < \infty, \; 0 < q \le \infty \tag{3.2.1}$$

or

$$s = n\left(\frac{1}{p} - 1\right), \; 0 < p \le 1, \; 0 < q \le 1 \tag{3.2.2}$$

or

$$s = 0, \; 1 < p < \infty, \; 0 < q \le \min(p, 2). \tag{3.2.3}$$

The proof of the above theorem can be found in [37] (Theorem 3.3.2 and Corollary 3.3.1).

Theorem 3.6. *Let $s \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q \le \infty$. Then the following assertions are equivalent:*

(i) $B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$,

(ii) $B^s_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$,

(iii) either $s > \frac{n}{p}$ *or* $s = \frac{n}{p}$ *and* $0 < q \le 1$.

The above theorem can be found in [37] (Theorem 3.3.1).

3.2.2 Embeddings for weighted function spaces

The following proposition was proved in [16] for weights belonging to the $\mathcal{A}_\infty$ class, but the similar result holds also for local Muckenhoupt weights due to Daubechies wavelet characterization theorem, Theorem 3.4.

Proposition 3.1. *Let w_1 and w_2 be two $\mathcal{A}^{loc}_\infty$ weights and let $-\infty < s_2 \le s_1 < \infty$, $0 < p_1, p_2 \le \infty$, $0 < q_1, q_2 \le \infty$. We put*

$$\frac{1}{p^*} := \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+ \quad \text{and} \quad \frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+ .$$

(i) There is a continuous embedding $B^{s_1,w_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2,w_2}_{p_2,q_2}(\mathbb{R}^n)$ if, and only if,

$$\left\{2^{-\nu(s_1-s_2)} \left\| \left\{ (w_2(Q_{\nu,m}))^{1/p_2} (w_1(Q_{\nu,m}))^{-1/p_1} \right\}_m |\ell_{p^*} \right\| \right\}_\nu \in \ell_{q^*}. \tag{3.2.4}$$

(ii) The embedding $B^{s_1,w_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2,w_2}_{p_2,q_2}(\mathbb{R}^n)$ is compact if, and only if, (3.2.4) holds and, in addition,

$$\lim_{\nu\to\infty} 2^{-\nu(s_1-s_2)} \left\| \left\{ (w_2(Q_{\nu,m}))^{1/p_2} (w_1(Q_{\nu,m}))^{-1/p_1} \right\}_m |\ell_{p^*} \right\| = 0 \quad \text{if } q^* = \infty,$$

and

$$\lim_{|m|\to\infty} (w_2(Q_{\nu,m}))^{-1/p_2} (w_1(Q_{\nu,m}))^{1/p_1} = \infty \quad \text{for all } \nu \in \mathbb{N}_0 \quad \text{if } p^* = \infty.$$

The proof is rewritten version of the proof of Proposition 2.1 in [16], so it is omitted here.

Theorem 3.7. *(i) Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, $w \in \mathcal{A}_\infty^{loc}$. *There is an embedding* $B_{pq}^{s,w}(\mathbb{R}^n) \subset L_{\max(1,p)}^{loc}(\mathbb{R}^n)$ *if*

$$s > \frac{n}{p}\left(r_w - \min(1,p)\right).$$

For every $1 \le \rho < \infty$ *there exists a weight* $w \in \mathcal{A}_\infty^{loc}$ *with* $r_w = \rho$ *such that if the embedding* $B_{pq}^{s,w} \subset L_{\max(1,p)}^{loc}$ *holds, then*

$$s \ge \frac{n}{p}\left(r_w - \min(1,p)\right) \quad \text{if} \quad \begin{cases} 0 < p < \infty \text{ and } 0 < q \le 1, \\ 1 < p < \infty \text{ and } 1 < q < \infty \end{cases} \tag{3.2.5}$$

or

$$s > \frac{n}{p}\left(r_w - \min(1,p)\right) \quad \text{if} \quad \begin{cases} 0 < p \le 1 \text{ and } 1 < q < \infty, \\ 0 < p < \infty \text{ and } q = \infty. \end{cases} \tag{3.2.6}$$

(ii) Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, $w \in \mathcal{A}_1^{loc}$. *There is an embedding* $B_{pq}^{s,w}(\mathbb{R}^n) \subset L_{\max(1,p)}^{loc}(\mathbb{R}^n)$ *if*

$$s > \frac{n}{p}\left(1 - \min(1,p)\right), \ 0 < p < \infty, \ 0 < q \le \infty$$

or

$$s = \frac{n}{p}\left(1 - p\right), \ 0 < p \le 1, \ 0 < q \le 1$$

or

$$s = 0, \ 1 < p < \infty, \ 0 < q \le \min(p,2).$$

There exists an $\mathcal{A}_1^{loc}$ *weight* w *such that if the embedding* $B_{pq}^{s,w}(\mathbb{R}^n) \subset L_{\max(1,p)}^{loc}(\mathbb{R}^n)$ *holds, then the above conditions are fulfilled.*

Proof. Step 1. We begin with the first part of (i). We are interested in local embeddings. We say that a function space X is locally embedded into a function space Y, $X, Y \subset \mathcal{S}'(\mathbb{R}^n)$, if for any function $\varphi \in C_0^\infty(\mathbb{R}^n)$ an operator $X \ni f \mapsto \varphi f \in Y$ is bounded. We write $X \overset{loc}{\hookrightarrow} Y$.

We want to use the embeddings of the type

$$B_{p,q}^{s,w}(\mathbb{R}^n) \overset{loc}{\hookrightarrow} B_{p,q}^{s_1}(\mathbb{R}^n) \tag{3.2.7}$$

and

$$B_{p,q}^{s_1}(\mathbb{R}^n) \subset L_{\max(1,p)}^{loc}(\mathbb{R}^n). \tag{3.2.8}$$

Let $s > \frac{n}{p}(r_w - \min(1,p))$. We choose $u > r_w$ such that $w \in \mathcal{A}_u^{loc}$ and $s > \frac{n}{p}(u - \min(1,p))$. We must find s_1 such that sufficient conditions for the above embeddings are fulfilled.

First we consider the embeddings (3.2.7). The condition (3.2.4) reads

$$\{2^{-j(s-s_1)}\left\|\{(w(Q_{jm}))^{-1/p}|Q_{jm}|^{1/p_1}\}_m|\ell_\infty\right\|\}_j\in\ell_\infty.$$

We get

$$\begin{aligned}&\sup_j 2^{-j(s-s_1)}\sup_m(w(Q_{jm}))^{-1/p}2^{-jn/p}\\ =&\sup_j 2^{-j(s-s_1+\frac{n}{p})}\sup_m(w(Q_{jm}))^{-1/p}<\infty.\end{aligned}\tag{3.2.9}$$

But $w\in\mathcal{A}_u^{loc}$ and by Lemma 2.3 we get

$$w(Q_{jm})^{-1}\le c2^{jnu}w(Q_{0,l})^{-1}\quad\text{ if } Q_{jm}\subset Q_{0,l}.$$

Since we get into account only finite many cubes $Q_{0,l}$ the condition (3.2.9) is equivalent to

$$\sup_j 2^{-j(s-s_1+\frac{n}{p})}2^{jnu/p}<\infty.\tag{3.2.10}$$

So the embeddings (3.2.7) hold if

$$s-s_1+\frac{n}{p}-\frac{nu}{p}\ge 0.$$

Thus

$$s_1\le s-\frac{n}{p}(u-\min(1,p))+n\left(\frac{1}{p}-1\right)_+.$$

Now we consider the embeddings (3.2.8).

It follows from (3.2.1) that we can find s_1 such that

$$n\left(\frac{1}{p}-1\right)_+<s_1\le s-\frac{n}{p}(u-\min(1,p))+n\left(\frac{1}{p}-1\right)_+.$$

Thus Theorem 3.5 implies that embeddings (3.2.7), (3.2.8) hold.

Step 2. Now we prove the second part of (i).

For $1<\rho<\infty$ we take a weight $w(x)=|x|^\alpha$ with $\alpha=n(\rho-1)$. Then $w\in\mathcal{A}_\infty^{loc}$ and $r_w=\rho$. For this weight we have

$$\begin{aligned}w(Q_{jm})&\sim 2^{-j(\alpha+n)}|m|^\alpha\quad\text{if}\quad m\neq 0,\\ w(Q_{jm})&\sim 2^{-j(\alpha+n)}\quad\text{if}\quad m=0.\end{aligned}$$

Using the atomic decomposition we construct $f\in B^{s,w}_{pq}(\mathbb{R}^n)$, that doesn't belong to $L^{loc}_{\max(1,p)}(\mathbb{R}^n)$, if the indexes do not satisfy (3.2.5) or (3.2.6).

First we construct atoms belonging to $C^K(\mathbb{R}^n)$ and with vanishing moments up to order L, where $K,L>0$.

Let $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\tilde{\varphi} \subset [-\frac{1}{2}, \frac{1}{2})^n$ and $|D^\alpha \tilde{\varphi}(x)| \le 1$, $|\alpha| \le K$, $0 \ge \tilde{\varphi}(x)$, $0 < c \le \tilde{\varphi}(x)$ on $x \in [-\frac{1}{4}, \frac{1}{4})^n$.

Let $\tilde{\varphi}_j(x) = 2^{-j(s-\frac{n}{p})}\tilde{\varphi}(2^j x)$. So $\operatorname{supp}\tilde{\varphi}_j \subset [-\frac{1}{2^{j+1}}, \frac{1}{2^{j+1}})^n$. Now let $\varphi_{j,1}$ be a translation of $\tilde{\varphi}_j$, such that $\operatorname{supp}\varphi_{j,1} \subset Q_{jm}$ and $\tilde{\varphi}_{j,1}$ be a translation of $\tilde{\varphi}_j$, such that $\operatorname{supp}\tilde{\varphi}_{j,1} \subset Q_{jm'}$, where $Q_{jm'} \subset 3Q_{jm}$, $m \ne m'$.

Taking $\psi_{j,1} = \varphi_{j,1} - \tilde{\varphi}_{j,1}$ we get an atom with

$$\begin{aligned}&\operatorname{supp}\psi_{j,1} \subset 3Q_{jm}\\ &|D^\alpha \psi_{j,1}| \le c_\alpha 2^{-j(s-\frac{n}{p})+j|\alpha|}\\ &\int \psi_{j,1}(x)\,dx = 0\end{aligned}$$

In the next step we construct $\psi_{j,2} = \psi_{j,1} - \tilde{\psi}_{j,1}$, where $\tilde{\psi}_{j,1}$ is a translation of $\psi_{j,1}$, such that $\operatorname{supp}\tilde{\psi}_{j,1} \subset 3^2 Q_{jm}$ and $\operatorname{supp}\psi_{j,1} \cap \operatorname{supp}\tilde{\psi}_{j,1} = \emptyset$. Now we get an atom with two vanishing moments:

$$\begin{aligned}\int \psi_{j,2}(x)x_k\,dx &= \int \psi_{j,1}(x)x_k\,dx - \int \psi_{j,1}(x-a)x_k\,dx\\ &= \int \psi_{j,1}(x)x_k\,dx - \int \psi_{j,1}(x)(x_k - a_k)\,dx = 0,\end{aligned}$$

where $1 \le k \le n$ and $a \in \mathbb{R}^n$.

Iterating this process we have $\psi_{j,i+1} = \psi_{j,i} - \tilde{\psi}_{j,i}$, where $\tilde{\psi}_{j,i}$ is a translation of $\psi_{j,i}$, such that $\operatorname{supp}\tilde{\psi}_{j,i} \subset 3^{i+1}Q_{jm}$, $i \ge 1$. We get atoms with vanishing moments up to order L associated with the cubes Q_{jm}, i.e. the support of the atom is contained in $3^{L+1}Q_{jm}$. We will denote that atoms by $a_{jm}^{(L)}$.

There exist sequences $\{j_k\}_{k=0}^\infty$, $j_k \in \mathbb{N}_0$, $0 = j_0 < j_1 < j_2 < \ldots$ and $\{m_k\}_{k=0}^\infty$, $m_k \in \mathbb{Z}^n$ such that the corresponding cubes Q_{j_k,m_k} satisfy $dQ_{j_k,m_k} \cap dQ_{j_l,m_l} = \emptyset$, $k \ne l$ and $dQ_{j_{k+1},m_{k+1}} \subset dQ_{j_k,0}$. Thus $\bigcup_{k=0}^\infty dQ_{j_k,m_k} \subset [-R,R)^n$ for some $R > 0$.

We take a sequence $\{\lambda_k\}_{k=0}^\infty$ of positive numbers, such that

$$\left(\sum_{k=0}^\infty |\lambda_k|^q w(Q_{j_k,m_k})^{q/p} 2^{jnq/p}\right)^{1/q} < \infty.$$

If we put

$$\lambda_{jm} = \begin{cases}\lambda_k & \text{if } j = j_k,\ m = m_k,\\ 0 & \text{otherwise,}\end{cases}$$

then $\{\lambda_{jm}\} \in b_{pq}^w$ and by the atomic decomposition theorem $f = \sum_{k=0}^\infty \lambda_k a_{j_k,m_k}^{(L)} \in B_{pq}^{s,w}(\mathbb{R}^n)$.

From our construction of atoms we have $|m_k| < c$ and $w(Q_{j_k,m_k}) \sim 2^{-j_k(\alpha+n)}$. From atomic decomposition theorem for weighted function spaces given by Izuki and Sawano in [23] we have $f \in B_{pq}^{s,w}(\mathbb{R}^n)$ if and only if

$$\left(\sum_{k=0}^\infty |\lambda_{j_k}|^q 2^{-j_k\alpha q/p}\right)^{1/q} < \infty.$$

The last condition is fulfilled for

$$\lambda_{j_k} = 2^{j_k \frac{\alpha}{p}} \quad \text{if} \quad q = \infty, \tag{3.2.11}$$

$$\lambda_{j_k} = 2^{j_k \frac{\beta}{p}}, \quad \beta < \alpha, \quad \text{if} \quad 0 < q \leq 1, \tag{3.2.12}$$

and

$$\lambda_{j_k} = 2^{j_k \frac{\alpha}{p}} k^{-1} \quad \text{if} \quad 1 < q < \infty. \tag{3.2.13}$$

Let $f_N(x) = \sum_{k=0}^N \lambda_{j_k} a^{(L)}_{j_k,m_k}(x)$ and $f(x) = \lim_{N\to\infty} f_N(x)$. By the construction of atom there is a cube $\tilde{Q}_{j_k,m_k} \subset Q_{j_k,m_k}$ of side length 4^{-j_k} such that

$$\int_{\tilde{Q}_{j_k,m_k}} a^{(L)}_{j_k,m_k}(x)\,dx \geq c2^{-j_k(s-\frac{n}{p})}2^{-j_k n} = c2^{-j_k(s+\frac{n}{p'})}.$$

In consequence

$$\begin{aligned}
\int_{[-R,R)^n} |f(x)|\,dx &= \lim_{N\to\infty} \int_{[-R,R)^n} |f_N(x)|\,dx \\
&= \lim_{N\to\infty} \sum_{k=0}^N \lambda_{j_k} \int_{[-R,R)^n} |a^{(L)}_{j_k,m_k}(x)|\,dx \\
&\geq c \lim_{N\to\infty} \sum_{k=0}^N \lambda_{j_k} 2^{-j_k(s+\frac{n}{p'})}
\end{aligned} \tag{3.2.14}$$

The sequence $\sum_{k=0}^N \lambda_{j_k} 2^{-j_k(s+\frac{n}{p'})}$ is divergent for the sequence $\{\lambda_{j_k}\}$ defined by (3.2.11) or (3.2.13) and $s \leq \frac{n}{p}(\rho - p)$. There is also a divergence if $\{\lambda_{j_k}\}$ is given by formula (3.2.12) and $s < \frac{n}{p}(\rho - p)$. Thus f is not locally integrable if $0 < p \leq 1$, $s \leq \frac{n}{p}(\rho - p)$ and $1 < q \leq \infty$ or $0 < p \leq 1$, $s < \frac{n}{p}(\rho - p)$ and $0 < q \leq 1$.

In the similar way for $p > 1$ we have

$$\begin{aligned}
\int_{[-R,R)^n} |f(x)|^p\,dx &= \lim_{N\to\infty} \int_{[-R,R)^n} |f_N(x)|^p\,dx \\
&= \lim_{N\to\infty} \sum_{k=0}^N \int_{[-R,R)^n} \lambda^p_{j_k} |a^{(L)}_{j_k,m_k}(x)|^p\,dx \\
&\geq \lim_{N\to\infty} \sum_{k=0}^N \lambda^p_{j_k} 2^{-j_k sp}.
\end{aligned}$$

Using conditions (3.2.11) to (3.2.13) we get

$$\lim_{N\to\infty} \sum_{k=0}^N 2^{j_k\alpha} 2^{-j_k sp} = \infty \quad \text{if} \quad s \leq \frac{n}{p}(\rho - 1),\ q = \infty,$$

$$\lim_{N\to\infty}\sum_{k=0}^{N} 2^{j_k\beta}2^{-j_k sp} = \infty \quad \text{if} \quad s < \frac{n}{p}(\rho-1),\ 0<q\le 1$$

and

$$\lim_{N\to\infty}\sum_{k=0}^{N} 2^{j_k\alpha}j^{-p}2^{-j_k sp} = \infty \quad \text{if} \quad s < \frac{n}{p}(\rho-1),\ 1<q<\infty.$$

Step 3. We are going to prove part (ii) of theorem. If $w\in\mathcal{A}_1^{loc}$ then $w(Q_{jm})^{-1}\le 2^{jn}w(Q_{0,l})^{-1}$ if $Q_{jm}\subset Q_{0,l}$. So instead of (3.2.10) we have

$$\sup_j 2^{-j(s-s_1)}<\infty.$$

Thus $B^{s,w}_{pq}(\mathbb{R}^n)\overset{loc}{\hookrightarrow} B^{s_1}_{pq}(\mathbb{R}^n)$ and sufficiency of the conditions follow from Theorem 3.5. Since the conditions are necessary for Lebesgue measure, they could not be weaker for $\mathcal{A}_1^{loc}$ weights.

□

Corollary 3.2. *Let $1\le p<\infty$, $0<q\le\infty$, $s\in\mathbb{R}$, $w\in\mathcal{A}_\infty^{loc}$. There is an embedding $B^{s,w}_{pq}(\mathbb{R}^n)\subset L_1^{loc}(\mathbb{R}^n)$ if*

$$s>\frac{n}{p}\left(r_w-1\right).$$

From Theorem 3.7 and elementary embeddings between weighted Besov and Triebel-Lizorkin spaces we get the following corollary

Corollary 3.3. *Let $0<p<\infty$, $0<q\le\infty$, $s\in\mathbb{R}$, $w\in\mathcal{A}_\infty^{loc}$. If $s>\frac{n}{p}\left(r_w-\min(1,p)\right)$ then $F^{s,w}_{pq}(\mathbb{R}^n)\subset L^{loc}_{\max(1,p)}(\mathbb{R}^n)$. In particular if $s>\frac{n}{p}\left(r_w-1\right)$ and $1\le p<\infty$ then $F^{s,w}_{pq}(\mathbb{R}^n)\subset L_1^{loc}(\mathbb{R}^n)$.*

Theorem 3.8. *(i) Let $0<p<\infty$, $0<q\le\infty$, $s\in\mathbb{R}$, $w\in\mathcal{A}_\infty^{loc}$. There is a continuous embedding $B^{s,w}_{pq}(\mathbb{R}^n)\hookrightarrow C(\mathbb{R}^n)$, if*

$$s>\frac{n}{p}r_w.$$

For every $1\le\rho<\infty$ there exists a weight $w\in\mathcal{A}_\infty^{loc}$ with $r_w=\rho$ such that if $s\le\frac{n}{p}r_w$ and $1<q\le\infty$ or if $s<\frac{n}{p}r_w$ and $0<q\le\infty$ then there is no embedding $B^{s,w}_{pq}(\mathbb{R}^n)\hookrightarrow L_\infty(\mathbb{R}^n)$.

(ii) Let $0<p<\infty$, $0<q\le\infty$, $s\in\mathbb{R}$, $w\in\mathcal{A}_1^{loc}$. There is a continuous embedding $B^{s,w}_{pq}(\mathbb{R}^n)\hookrightarrow C(\mathbb{R}^n)$, if

$$s>\frac{n}{p}$$

or

$$s = \frac{n}{p}, \quad 0 < q \le 1.$$

There exists an $\mathcal{A}_1^{loc}$ weight w such that if the continuous embedding $B^{s,w}_{pq}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$ holds, then the above conditions are fulfilled.

Proof. Step 1. Let us start with first part of (i). We choose $u > r_w$ such that $w \in \mathcal{A}_u^{loc}$ and $s > \frac{nu}{p}$. Analogously like in Step 1 of the proof of Theorem 3.7 we can regard embeddings (3.2.7) and (3.2.8). We take s_1 such that

$$\frac{n}{p} < s_1 \le s - \frac{nu}{p} + \frac{n}{p}.$$

Thus Theorem 3.6 implies that the continuous embeddings hold.

Step 2. Now we prove the second part of (i). For $1 < \rho < \infty$ we take a weight $w(x) = |x|^\alpha$ with $\alpha = n(\rho - 1)$. Then $w \in \mathcal{A}_\infty^{loc}$ and $r_w = \rho$.

Using the atomic decomposition we can construct $f \in B^{s,w}_{pq}(\mathbb{R}^n)$, that doesn't belong to $L_\infty(\mathbb{R}^n)$. The construction of atoms can be the same as in Step 2 of the proof of Theorem 3.7. We also take a sequence $\{\lambda_{j_k}\}_{k=0}^\infty$ with the same properties as in the proof above. We have $f = \sum_{k=0}^\infty \lambda_{j_k} a^{(L)}_{j_k m_k} \in B^{s,w}_{pq}(\mathbb{R}^n)$, cf. [23].

In the same way as in (3.2.14) we get

$$\sup_{x \in [-R,R)^n} |f(x)| \ge c \lim_{N\to\infty} \sum_{k=0}^N \lambda_{j_k} 2^{-j_k(s-\frac{n}{p})}$$

and the sequence $\sum_{k=0}^N \lambda_{j_k} 2^{-j_k(s-\frac{n}{p})}$ is divergent for the sequence $\{\lambda_{j_k}\}$ defined by (3.2.11) or (3.2.13) and $s \le \frac{n}{p}\rho$. There is also a divergence if $\{\lambda_{j_k}\}$ is given by formula (3.2.12) and $s < \frac{n}{p}\rho$.

Step 3. The rest of the proof can be rewritten from Step 3 of the proof of Theorem 3.7. □

Corollary 3.4. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, $w \in \mathcal{A}_\infty^{loc}$. If $s > \frac{n}{p} r_w$ then $F^{s,w}_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$.*

3.3 Dual spaces

In this section we calculate the dual spaces of $A^{s,w}_{p,q}(\mathbb{R}^n)$. We consider the Banach case, more precisely $1 < p < \infty$, $1 \le q < \infty$. We present two approaches. The first one covers all weights $w \in \mathcal{A}_\infty^{loc}$ and the whole range of p and q, but the duality does not coincides with the usual duality between $\mathcal{S}_e(\mathbb{R}^n)$ and $\mathcal{S}'_e(\mathbb{R}^n)$. We recall that

$$\mathcal{S}_e(\mathbb{R}^n) \hookrightarrow A^{s,w}_{pq}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'_e(\mathbb{R}^n).$$

In the second approach the representation of the functional coincides with the dual pairing $(\mathcal{S}_e(\mathbb{R}^n), \mathcal{S}'_e(\mathbb{R}^n))$ but we should assume that $w \in \mathcal{A}_p^{loc}$. The similar approach for the regular weights can be found in Th. Schott paper [35].

3.3.1 Dual spaces with general $\mathcal{A}_\infty^{loc}$ weights

By Daubechies wavelet characterization we can study the dual spaces of Besov spaces with local Muckenhoupt weights. First we determine the dual spaces of sequence spaces $b_{pq}^{s,w}(\mathbb{R}^n)$ and $f_{pq}^{s,w}(\mathbb{R}^n)$. The main idea of the proof is based on the similar assertions for spaces with doubling measures in [5].

Let $\ell_q(L_p^w(\mathbb{R}^n))$ be the set of all sequences $f = \{f_j^G(x)\}_{j=0,G\in G^j}^\infty$ of Lebesgue-measurable functions on $\mathbb{R}^n$ with finite quasi-norms given by

$$\big\|f|\ell_q(L_p^w(\mathbb{R}^n))\big\| = \left(\sum_{j=0}^\infty \sum_{G\in G^j} \big\|f_j^G|L_p^w(\mathbb{R}^n)\big\|^q\right)^{1/q}$$

and $L_p^w(\mathbb{R}^n, \ell_q)$ be the set of all sequences $f = \{f_j^G(x)\}_{j=0,G\in G^j}^\infty$ of Lebesgue-measurable functions on $\mathbb{R}^n$ with finite quasi-norms given by

$$\big\|f|L_p^w(\mathbb{R}^n, \ell_q)\big\| = \left\|\left(\sum_{j=0}^\infty \sum_{G\in G^j} |f_j^G|^q\right)^{1/q} |L_p^w(\mathbb{R}^n)\right\|$$

By dual space we understand topological dual and the norms $\|g\|$ of a continuous linear functional g is calculated in the usual way.

Proposition 3.2. *Let $1 < p < \infty$ and $0 < q < \infty$, $w \in \mathcal{A}_\infty^{loc}$. Then*

(i) $g \in \big(\ell_q(L_p^w(\mathbb{R}^n))\big)'$ *if and only if it can be represented uniquely as*

$$g(f) = \sum_{j=0}^\infty \sum_{G\in G^j} \int_{\mathbb{R}^n} g_j^G(x) f_j^G(x) w(x)\, dx$$

for every $f = \{f_j^G\}_{j,G} \in \ell_q(L_p^w(\mathbb{R}^n))$, *where* $\{g_j^G\}_{j,G} \in \ell_{q'}(L_{p'}^w(\mathbb{R}^n))$ *and* $\|g\| = \big\|\{g_j^G\}_{j,G}|\ell_{q'}(L_{p'}^w(\mathbb{R}^n))\big\|$.

(ii) $g \in \big(L_p^w(\mathbb{R}^n, \ell_q)\big)'$ *if and only if it can be represented uniquely as*

$$g(f) = \sum_{j=0}^\infty \sum_{G\in G^j} \int_{\mathbb{R}^n} g_j^G(x) f_j^G(x) w(x)\, dx$$

for every $f = \{f_j^G\}_{j,G} \in L_p^w(\mathbb{R}^n, \ell_q)$, *where* $\{g_j^G\}_{j,G} \in L_{p'}^w(\mathbb{R}^n, \ell_{q'})$ *and* $\|g\| = \big\|\{g_j^G\}_{j,G}|L_{p'}^w(\mathbb{R}^n, \ell_{q'})\big\|$.

The proof of the above theorem can be found in [39], Proposition in §2.11.1 for the case of L_p-spaces with Lebesgue measure, but it can be rewritten for measure of type $w(x)\,dx$, $w \in \mathcal{A}_\infty^{loc}$.

Proposition 3.3. *Let $1 < p < \infty$, $1 \le q < \infty$, $s \in \mathbb{R}^n$ and $w \in \mathcal{A}_\infty^{loc}$. Then $\widetilde{\lambda} \in \left(a_{pq}^{s,w}\right)'$ if and only if it can be represented uniquely as*

$$\widetilde{\lambda}(\lambda) = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^G \widetilde{\lambda}_{jm}^G w(Q_{jm}), \quad \lambda = \{\lambda_{jm}^G\}_{j,m,G} \in a_{pq}^{s,w}, \tag{3.3.1}$$

where $\{\widetilde{\lambda}_{jm}^G\}_{j,m,G} \in a_{p',q'}^{-s,w}$. Moreover $\left\|\widetilde{\lambda}\right\| = \left\|\{\widetilde{\lambda}_{jm}^G\}_{j,m,G}|a_{p',q'}^{-s,w}\right\|$.

Proof. Step 1. First we prove that the formula (3.3.1) defines a continuous functional on $b_{pq}^{s,w}$, if $\{\widetilde{\lambda}_{jm}^G\}_{j,m,G} \in b_{p',q'}^{-s,w}$. By Hölder's inequality we get

$$\begin{aligned}
\widetilde{\lambda}(\lambda) &\le \sum_j \sum_G \sum_m \left|\lambda_{jm}^G\right| \left|\widetilde{\lambda}_{jm}^G\right| \int_{Q_{jm}} w(x)\,dx \\
&= \sum_j \sum_G \sum_m \int_{\mathbb{R}^n} \left|\lambda_{jm}^G\right| \left|\widetilde{\lambda}_{jm}^G\right| \chi_{jm}(x) w(x)\,dx \\
&= \sum_j \sum_G \int_{\mathbb{R}^n} \left(\sum_m \left|\lambda_{jm}^G\right| \chi_{jm}(x)\right) \left(\sum_m \left|\widetilde{\lambda}_{jm}^G\right| \chi_{jm}(x)\right) w(x)\,dx \\
&\le \sum_{j,G} \left(\int_{\mathbb{R}^n} \left(\sum_m \left|\lambda_{jm}^G\right| \chi_{jm}(x)\right)^p w(x)\,dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \left(\sum_m \left|\widetilde{\lambda}_{jm}^G\right| \chi_{jm}(x)\right)^{p'} w(x)\,dx\right)^{\frac{1}{p'}} \\
&\le \left(\sum_j 2^{jsq} \sum_G \left(\int_{\mathbb{R}^n} \left(\sum_m \left|\lambda_{jm}^G\right| \chi_{jm}(x)\right)^p w(x)\,dx\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \times \\
&\times \left(\sum_j 2^{-jsq'} \sum_G \left(\int_{\mathbb{R}^n} \left(\sum_m \left|\widetilde{\lambda}_{jm}^G\right| \chi_{jm}(x)\right)^{p'} w(x)\,dx\right)^{\frac{q'}{p'}}\right)^{\frac{1}{q'}} \\
&\le \left(\sum_j 2^{j(s-\frac{n}{p})q} \sum_G \left(\int_{\mathbb{R}^n} \left|\sum_m \lambda_{jm}^G \chi_{jm}^{(p)}(x)\right|^p w(x)\,dx\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \times \\
&\times \left(\sum_j 2^{j(-s-\frac{n}{p'})q'} \sum_G \left(\int_{\mathbb{R}^n} \left|\sum_m \widetilde{\lambda}_{jm}^G \chi_{jm}^{(p')}(x)\right|^{p'} w(x)\,dx\right)^{\frac{q'}{p'}}\right)^{\frac{1}{q'}}
\end{aligned}$$

with usual modification if $q' = \infty$. So $b_{p',q'}^{-s,w} \subset (b_{pq}^{s,w})'$ and $\left\|\widetilde{\lambda}\right\| \le \left\|\{\widetilde{\lambda}_{jm}^G\}_{j,m,G}|b_{p',q'}^{-s,w}\right\|$.

In the same way we get that the formula (3.3.1) defines a continuous functional on $f^{s,w}_{pq}$, if $\{\widetilde{\lambda}^G_{jm}\}_{j,m,G} \in f^{-s,w}_{p',q'}$

$$\begin{aligned}
\widetilde{\lambda}(\lambda) &\leq \int_{\mathbb{R}^n} \sum_j \sum_G \sum_m \left|\lambda^G_{jm}\right| \chi_{jm} \left|\widetilde{\lambda}^G_{jm}\right| \chi_{jm} w(x)\,dx \\
&\leq \int_{\mathbb{R}^n} \left(\sum_{j,G,m} \left|\lambda^G_{jm}\right|^q \chi_{jm} \right)^{1/q} \left(\sum_{j,G,m} \left|\widetilde{\lambda}^G_{jm}\right|^{q'} \chi_{jm} \right)^{1/q'} w(x)\,dx \\
&\leq \left\| \{\lambda^G_{jm}\}_{j,m,G} | f^{s,w}_{p,q} \right\| \left\| \{\widetilde{\lambda}^G_{jm}\}_{j,m,G} | f^{-s,w}_{p',q'} \right\|
\end{aligned}$$

with usual modification if $q' = \infty$. So we get $f^{-s,w}_{p',q'} \subset (f^{s,w}_{pq})'$ and $\left\|\widetilde{\lambda}\right\| \leq \left\| \{\widetilde{\lambda}^G_{jm}\}_{j,m,G} | f^{-s,w}_{p',q'} \right\|$.

Step 2. We prove that $\widetilde{\lambda} \in (b^{s,w}_{pq})'$ can be represented according to (3.3.1). The statement that $\widetilde{\lambda} \in (f^{s,w}_{pq})'$ can be represented according to (3.3.1) can be proved similarly using appropriate norms and second part of Proposition 3.2.

Let us define $I : b^{s,w}_{pq} \to \ell_q(L^w_p(\mathbb{R}^n))$, which assigns $\{\lambda^G_{jm}\} \to \{f^G_j\}$ in the following way

$$f^G_j(x) = \sum_m \lambda^G_{jm} 2^{js} \chi_{jm}(x).$$

We have

$$\begin{aligned}
\left\| \{f^G_j\}_{j,G} | \ell_q(L^w_p(\mathbb{R}^n)) \right\| &= \left(\sum_j \sum_G \left\| f^G_j | L^w_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \\
&= \left(\sum_j \sum_G \left(\int_{\mathbb{R}^n} \left|f^G_j\right|^p w(x)\,dx \right)^{q/p} \right)^{1/q} \\
&= \left(\sum_j 2^{j(s-\frac{n}{p})q} \sum_G \left(\int_{\mathbb{R}^n} \left| \sum_m \lambda^G_{jm} \chi^{(p)}_{jm} \right|^p w(x)\,dx \right)^{q/p} \right)^{1/q},
\end{aligned}$$

so I is an isometry.

From the Hahn-Banach Theorem we get that there exists $\widetilde{\widetilde{\lambda}} \in (\ell_q(L^w_p(\mathbb{R}^n)))'$, such that $\widetilde{\widetilde{\lambda}} \circ I = \widetilde{\lambda}$ and $\left\|\widetilde{\widetilde{\lambda}}\right\| = \left\|\widetilde{\lambda}\right\|$. By Proposition 3.2 we have $\widetilde{\widetilde{\lambda}}(f) = \langle f, g \rangle$, where $g \in \ell_{q'}(L^w_{p'}(\mathbb{R}^n))$ and $f \in \ell_q(L^w_p(\mathbb{R}^n))$, with dual pairing

$$\langle f, g \rangle = \sum_{j=0}^{\infty} \sum_{G \in G^j} \int_{\mathbb{R}^n} g^G_j(x) f^G_j(x) w(x)\,dx.$$

We define projection $P : \ell_{q'}(L^w_{p'}(\mathbb{R}^n)) \to b^{-s,w}_{p',q'}$ by

$$P\left(\{h_j^G\}_{j,G}\right) = \{\widetilde{\lambda}^G_{jm}\}_{j,m,G} = \left\{ \frac{2^{js}}{w(Q_{jm})} \int_{Q_{jm}} h_j^G(x) w(x)\, dx \right\}_{j,m,G}$$

for $\{h_j^G\}_{j,G} \in \ell_{q'}(L^w_{p'}(\mathbb{R}^n))$.

From Hölder's inequality we have

$$\begin{aligned}
&\left\| P\left(\{h_j^G\}_{j,G}\right) | b^{-s,w}_{p',q'} \right\| \\
&= \left(\sum_j 2^{j(-s-\frac{n}{p'})q'} \sum_G \left(\int_{\mathbb{R}^n} \left| \sum_m \frac{2^{js}}{w(Q_{jm})} \int_{Q_{jm}} h_j^G(x) w(x)\, dx \chi^{(p)}_{jm}(y) \right|^{p'} w(y) dy \right)^{\frac{q'}{p'}} \right)^{\frac{1}{q'}} \\
&\approx \left(\sum_j 2^{-jsq'} \sum_G \left(\sum_m \left| \frac{2^{js}}{w(Q_{jm})} \int_{Q_{jm}} h_j^G(x) w(x)\, dx \right|^{p'} w(Q_{jm}) \right)^{\frac{q'}{p'}} \right)^{\frac{1}{q'}} \\
&= \left(\sum_j 2^{-jsq'} \sum_G \left(\sum_m 2^{jsp'} \left| \int_{Q_{jm}} h_j^G(x) w(x)\, dx \right|^{p'} w(Q_{jm})^{-p'} w(Q_{jm}) \right)^{\frac{q'}{p'}} \right)^{\frac{1}{q'}} \\
&\leq \left(\sum_j \sum_G \left(\sum_m \int_{Q_{jm}} \left| h_j^G(x) \right|^{p'} w(x)\, dx w(Q_{jm})^{\frac{p'}{p}} w(Q_{jm})^{1-p'} \right)^{\frac{q'}{p'}} \right)^{\frac{1}{q'}} \\
&= \left(\sum_j \sum_G \left(\sum_m \int_{Q_{jm}} \left| h_j^G(x) \right|^{p'} w(x)\, dx \right)^{\frac{q'}{p'}} \right)^{\frac{1}{q'}} \\
&= \left(\sum_j \sum_G \left(\int_{\mathbb{R}^n} \left| h_j^G(x) \right|^{p'} w(x)\, dx \right)^{\frac{q'}{p'}} \right)^{\frac{1}{q'}} = \left\| \{h_j^G\}_{j,G} | \ell_{q'}(L^w_{p'}(\mathbb{R}^n)) \right\|.
\end{aligned}$$

Thus the projection is continuous.

Using Proposition 3.2 we get

$$
\begin{aligned}
\widetilde{\lambda}(\lambda) &= \widetilde{\widetilde{\lambda}}(I(\lambda)) \\
&= \sum_{j=0}^{\infty} \sum_{G \in G^j} \int_{\mathbb{R}^n} g_j^G(x) f_j^G(x) w(x)\, dx \\
&= \sum_{j=0}^{\infty} \sum_{G \in G^j} \int_{\mathbb{R}^n} g_j^G(x) \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^G 2^{js} \chi_{jm}(x) w(x)\, dx \\
&= \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^G 2^{js} \int_{\mathbb{R}^n} g_j^G(x) \chi_{jm}(x) w(x)\, dx \\
&= \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^G 2^{js} \int_{Q_{jm}} g_j^G(x) w(x)\, dx \\
&= \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^G 2^{js} w(Q_{jm})^{-1} \int_{Q_{jm}} g_j^G(x) w(x)\, dx w(Q_{jm}) \\
&= \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^G P(g_j^G) w(Q_{jm}) = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^G \widetilde{\lambda}_{jm}^G w(Q_{jm}).
\end{aligned}
$$

Step 3. Norm equivalence we get by Step 1 and from Step 2 with inequalities

$$
\begin{aligned}
\left\| \{\widetilde{\lambda}_{jm}^G\}_{j,m,G} | b_{p',q'}^{-s,w} \right\| &= \left\| P\left(\{g_j^G\}_{j,G}\right) | b_{p',q'}^{-s,w} \right\| \\
&\leq \left\| \{g_j^G\}_{j,G} | \ell_{q'}(L_{p'}^w(\mathbb{R}^n)) \right\| = \left\| \widetilde{\widetilde{\lambda}} \right\| = \left\| \widetilde{\lambda} \right\|
\end{aligned}
$$

and similarly for $f_{pq}^{s,w}$ case. □

Now it follows from Proposition 3.3 and Theorem 3.4:

Theorem 3.9. *Let* $1 < p < \infty$, $1 \leq q < \infty$, $s \in \mathbb{R}^n$ *and* $w \in \mathcal{A}_\infty^{loc}$. *Then*

$$
\left(A_{pq}^{s,w}(\mathbb{R}^n)\right)' = A_{p',q'}^{-s,w}(\mathbb{R}^n).
$$

3.3.2 Dual spaces with weights with $r_w > p$

Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 < q < \infty$ and $w \in \mathcal{A}_\infty^{loc}$. If $r_w < p$ then $w \in \mathcal{A}_p^{loc}$. It follows easily from the definition of $\mathcal{A}_\infty^{loc}$ class that $w^{1-p'} \in \mathcal{A}_{p'}^{loc}$. Thus the assumption $r_w < p$ implies $w, w^{1-p'} \in \mathcal{A}_\infty^{loc}$. This allows to prove another representation of continuous functionals on $A_{pq}^{s,w}(\mathbb{R}^n)$.

The space $\mathcal{S}_e(\mathbb{R}^n)$ is dense in $A_{pq}^{s,w}(\mathbb{R}^n)$ and $\mathcal{S}_e(\mathbb{R}^n) \hookrightarrow A_{pq}^{s,w}(\mathbb{R}^n)$, cf. Theorem 2.6. Therefore any continuous functional on $A_{pq}^{s,w}(\mathbb{R}^n)$ can be incorporated as a distribution belonging to $\mathcal{S}_e'(\mathbb{R}^n)$. In that sense we have

$$
\left(A_{pq}^{s,w}(\mathbb{R}^n)\right)' = \{f \in \mathcal{S}_e'(\mathbb{R}^n) : \exists c > 0, |f(\varphi)| \leq c\|\varphi | A_{pq}^{s,w}(\mathbb{R}^n)\| \text{ for all } \varphi \in \mathcal{S}_e(\mathbb{R}^n)\}
$$

If $r_w < p$ then $\left(A^{s,w}_{pq}(\mathbb{R}^n)\right)'$ can be identify with $A^{-s,w^{1-p'}}_{p',q'}(\mathbb{R}^n)$.

The operator $T_w : f \mapsto w^{1/p} f$ is an isometry from $\ell_q(L^w_p(\mathbb{R}^n))$ onto $\ell_q(L_p(\mathbb{R}^n))$ and from $L^w_p(\mathbb{R}^n, \ell_q)$ onto $L_p(\mathbb{R}^n, \ell_q)$. Therefore we have the following counterpart of Proposition 3.2.

Proposition 3.4. *Let $1 < p < \infty$ and $0 < q < \infty$, $w \in \mathcal{A}^{loc}_\infty$ and $r_w < p$. Then*

(i) $g \in \left(\ell_q(L^w_p(\mathbb{R}^n))\right)'$ if and only if it can be represented uniquely as

$$g(f) = \sum_{j=0}^{\infty} \sum_{G \in G^j} \int_{\mathbb{R}^n} g^G_j(x) f^G_j(x)\, dx$$

for every $f = \{f^G_j\}_{j,G} \in \ell_q(L^w_p(\mathbb{R}^n))$, where $\{g^G_j\}_{j,G} \in \ell_{q'}(L^{w^{1-p'}}_{p'}(\mathbb{R}^n))$ and $\|g\| = \left\|\{g^G_j\}_{j,G} | \ell_{q'}(L^{w^{1-p'}}_{p'}(\mathbb{R}^n))\right\|$.

(ii) $g \in \left(L^w_p(\mathbb{R}^n, \ell_q)\right)'$ if and only if it can be represented uniquely as

$$g(f) = \sum_{j=0}^{\infty} \sum_{G \in G^j} \int_{\mathbb{R}^n} g^G_j(x) f^G_j(x)\, dx$$

for every $f = \{f^G_j\}_{j,G} \in L^w_p(\mathbb{R}^n, \ell_q)$, where $\{g^G_j\}_{j,G} \in L^{w^{1-p'}}_{p'}(\mathbb{R}^n, \ell_{q'})$ and $\|g\| = \left\|\{g^G_j\}_{j,G} | L^{w^{1-p'}}_{p'}(\mathbb{R}^n, \ell_{q'})\right\|$.

In consequence we get also

Proposition 3.5. *Let $1 < p < \infty$, $1 < q < \infty$, $s \in \mathbb{R}^n$, $w \in \mathcal{A}^{loc}_\infty$ and $r_w < p$. Then $\widetilde{\lambda} \in \left(a^{s,w}_{pq}\right)'$ if and only if it can be represented uniquely as*

$$\widetilde{\lambda}(\lambda) = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^G_{jm} \widetilde{\lambda}^G_{jm} |Q_{jm}|, \quad \lambda = \{\lambda^G_{jm}\}_{j,m,G} \in a^{s,w}_{pq},$$

where $\{\widetilde{\lambda}^G_{jm}\}_{j,m,G} \in a^{-s,w^{1-p'}}_{p',q'}$. Moreover $\left\|\widetilde{\lambda}\right\| = \left\|\{\widetilde{\lambda}^G_{jm}\}_{j,m,G} | a^{-s,w^{1-p'}}_{p',q'}\right\|$.

Corollary 3.5. *Let $1 < p < \infty$, $1 < q < \infty$, $s \in \mathbb{R}^n$ and $w \in \mathcal{A}^{loc}_\infty$ with $r_w < p$. If $f \in A^{-s,w^{1-p'}}_{p',q'}(\mathbb{R}^n)$ then the mapping*

$$\mathcal{S}_e(\mathbb{R}^n) \ni \varphi \mapsto f(\varphi) \in \mathbb{C}$$

can be extended to a continuous linear functional on $A^{s,w}_{pq}(\mathbb{R}^n)$ and

$$|f(\varphi)| \le \left\|f | A^{-s,w^{1-p'}}_{p',q'}(\mathbb{R}^n)\right\| \left\|\varphi | A^{s,w}_{pq}(\mathbb{R}^n)\right\|. \tag{3.3.2}$$

On the other hand any $g \in \left(A^{s,w}_{pq}(\mathbb{R}^n)\right)'$ can be represented uniquely by $f \in A^{s,w^{1-p'}}_{p',q'}(\mathbb{R}^n)$ with (3.3.2).

3.4 The Calderón product and the complex interpolation

The last consequence of Theorem 3.4 we would like to prove is a characterization of the complex interpolation spaces for the pairs of $F^{s,w}_{pq}(\mathbb{R}^n)$ spaces. The general introduction to the complex method of interpolation can be found in [2]. We follow the approach developed by Frazier and Jawerth in [13] and Mendez and Mitrea in [30]. The approach used the concept of Calderón product of quasi-Banach lattices. We recall the needed definitions.

Definition 3.8. Suppose (M, μ) is a measure space and X is a quasi-Banach space of μ-measurable functions. Let g be a measurable function on M. Then X is said to be a *quasi-Banach lattice* on M if the conditions $f \in X$ and $|g(x)| \leq |f(x)|$ μ-a.e. imply that $g \in X$ and $\|g\|_X \leq \|f\|_X$.

Definition 3.9. A quasi-Banach lattice of functions $(X, \|\cdot\|_X)$ is called lattice r-convex if

$$\left\| \left(\sum_{j=1}^{m} |f_j|^r \right)^{1/r} \right\|_X \leq \left(\sum_{j=1}^{m} \|f_j\|_X^r \right)^{1/r}$$

for any finite family $\{f_j\}_{j=1}^m$ of functions belonging to X.

Remark 3.2. Spaces $f^{s,w}_{p,q}$ are quasi-Banach lattices and lattice r-convex for any $0 < r \leq \min(p, q, 1)$. We have from the Minkowski inequality

$$\left\| \left(\sum_{k=1}^{l} |\lambda^k|^r \right)^{1/r} |f^{s,w}_{pq} \right\| = \left(\int_{\mathbb{R}^n} \left(\sum_{j,m,G} 2^{jsq} \left(\sum_{k=1}^{l} |\lambda^{G,k}_{jm}|^r \right)^{q/r} \chi_{jm}(x) \right)^{p/q} w(x)dx \right)^{1/p}$$

$$= \left(\int_{\mathbb{R}^n} \left(\sum_{j,m,G} 2^{jsq} \left(\sum_{k=1}^{l} |\lambda^{G,k}_{jm}|^r \right)^{q/r} \chi_{jm}(x) \right)^{pr/qr} w(x)dx \right)^{(r/p)(1/r)}$$

$$\leq \left(\int_{\mathbb{R}^n} \sum_{k=1}^{l} \left(\sum_{j,m,G} 2^{jsq} |\lambda^{G,k}_{jm}|^q \chi_{jm}(x) \right)^{pr/qr} w(x)dx \right)^{(r/p)(1/r)}$$

$$\leq \left(\sum_{k=1}^{l} \left(\int_{\mathbb{R}^n} \left(\sum_{j,m,G} 2^{jsq} |\lambda^{G,k}_{jm}|^q \chi_{jm}(x) \right)^{p/q} w(x)dx \right)^{(r/p)} \right)^{(1/r)} \leq \left(\sum_{k=1}^{l} \|\lambda^k | f^{s,w}_{pq}\|^r \right)^{1/r}$$

for any finite family $\{\lambda^k\}_{k=1}^l$ of $\lambda^k = \{\lambda^{G,k}_{jm}\}_{j,m,G} \in f^{s,w}_{pq}$.

Definition 3.10. Let X_0, X_1 be quasi-Banach lattices on (M, μ). *The Calderón product* $X_0^{1-\theta} X_1^{\theta}$, where $0 < \theta < 1$, is a space of μ-measurable functions f on M such that there exist $f_0 \in X_0$ and $f_1 \in X_1$ such that

$$|f(x)| \le |f_0(x)|^{1-\theta} |f_1(x)|^{\theta} \quad \text{for } \mu - a.e. x \in M.$$

We put

$$\|f\|_{X_0^{1-\theta} X_1^{\theta}} := \inf\{\|f_0\|_{X_0}^{1-\theta} \|f_1\|_{X_1}^{\theta} : |f| \le |f_0|^{1-\theta}|f_1|^{\theta}, f_0 \in X_0, f_1 \in X_1\}.$$

The next proposition describes the relation between complex interpolation spaces $[X_0, X_1]_\theta$ and Calderón product of quasi-Banach function spaces.

Proposition 3.6. *Let M be a separable complete metric space, μ a σ-finite Borel measure on M and X_0, X_1 be a pair of quasi-Banach function spaces on (M, μ). Then if both X_0 and X_1 are lattice r-convex for some $r > 0$ and separable it follows that $X_0 + X_1$ is lattice r-convex and $[X_0, X_1]_\theta = X_0^{1-\theta} X_1^{\theta}$.*

The above proposition is due to Kalton and Mitrea, [25], [26], see also [30].
The next theorem is a weighted version of Theorem 8.2 proved in [13].

Theorem 3.10. *Let $s_0, s_1 \in \mathbb{R}$, $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 < \infty$, $w \in \mathcal{A}_\infty^{loc}$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$ and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$f_{pq}^{s,w} = \left(f_{p_0,q_0}^{s_0,w}\right)^{(1-\theta)} \left(f_{p_1,q_1}^{s_1,w}\right)^{\theta}.$$

Proof. The proof of the above theorem can be rewritten similarly as the proof of Theorem 8.2 in [13], because a weight doesn't play an important role in calculations. The only change is that instead of Proposition 2.7 in [13] we use the following proposition:

Proposition 3.7. *Let $\varepsilon > 0$, $0 < p, q < \infty$, $s \in \mathbb{R}$, $w \in \mathcal{A}_\infty^{loc}$. Let for every cube Q_{jm}, $j \ge 0, m \in \mathbb{Z}^n$, $E_{jm} \subseteq Q_{jm}$ be a measurable set with $|E_{jm}|/|Q_{jm}| \ge \varepsilon$. Then*

$$\|\lambda | f_{pq}^{s,w}\| \sim \left\| \left(\sum_{j,m,G} 2^{jsq} |\lambda_{jm}^G|^q \chi_{E_{jm}} \right)^{1/q} | L_p^w(\mathbb{R}^n) \right\|.$$

Proof. The proof is similar to the one of Theorem 2.7 in [13] and is based on Fefferman-Stein inequality for maximal function. So in our case it is an easy consequence of Theorem

2.2. For $0 < v < \min\left(1, \frac{p}{r_w}, q\right)$ we have

$$\begin{aligned}
\|\lambda|f^{s,w}_{pq}\| &= \left\| \left(\sum_{j,m,G} 2^{jsq} |\lambda^G_{jm}|^q \chi_{Q_{jm}} \right)^{1/q} |L^w_p(\mathbb{R}^n) \right\| \\
&= \left\| \left(\sum_{j,m,G} 2^{jsq} \left(\frac{1}{|E_{jm}|} \int_{\mathbb{R}^n} |\lambda^G_{jm}| \chi_{E_{jm}}(y)\, dy \chi_{Q_{jm}} \right)^q \right)^{1/q} |L^w_p(\mathbb{R}^n) \right\| \\
&\leq \left\| \left(\sum_{j,m,G} 2^{jsq} \left(\frac{|Q_{jm}|}{|E_{jm}|} M^{loc}\left(|\lambda^G_{jm}| \chi_{E_{jm}} \right) \right)^q \right)^{1/q} |L^w_p(\mathbb{R}^n) \right\| \\
&\leq \frac{1}{\varepsilon} \left\| \left(\sum_{j,m,G} 2^{jsq} \left(M^{loc}\left(|\lambda^G_{jm}|^v \chi_{E_{jm}} \right) \right)^{q/v} \right)^{v/q} |L^w_{p/v}(\mathbb{R}^n) \right\|^{1/v} \\
&\leq \frac{c}{\varepsilon} \left\| \left(\sum_{j,m,G} 2^{jsq} |\lambda^G_{jm}|^q \chi_{E_{jm}} \right)^{1/q} |L^w_p(\mathbb{R}^n) \right\|.
\end{aligned}$$

The other direction is trivial since $\chi_{E_{jm}} \leq \chi_{Q_{jm}}$. □

□

Next theorem follows immediately from Theorem 3.4, Proposition 3.6 and Theorem 3.10.

Theorem 3.11. *Let $s_0, s_1 \in \mathbb{R}$, $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 < \infty$, $w \in \mathcal{A}^{loc}_\infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$ and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$f^{s,w}_{pq} = \left[f^{s_0,w}_{p_0,q_0}, f^{s_1,w}_{p_1,q_1} \right]_\theta$$

and

$$F^{s,w}_{pq}(\mathbb{R}^n) = \left[F^{s_0,w}_{p_0,q_0}(\mathbb{R}^n), F^{s_1,w}_{p_1,q_1}(\mathbb{R}^n) \right]_\theta .$$

Remark 3.3. Rychkov in [34] proved real interpolation for Besov spaces with local Muckenhoupt weights.

Theorem 3.12. *Let $0 < p < \infty$, $0 < q, q_0, q_1 \leq \infty$, $-\infty < s_0 < s_1 < \infty$, $0 < \theta < 1, s = (1-\theta)s_0 + \theta s_1$ and $w \in \mathcal{A}^{loc}_\infty$. Then*

$$\left(F^{s_0,w}_{p,q_0}, F^{s_1,w}_{p,q_1} \right)_{\theta,q} = \left(B^{s_0,w}_{p,q_0}, B^{s_1,w}_{p,q_1} \right)_{\theta,q} = B^{s,w}_{pq}.$$

Chapter 4

Haar bases in weighted function spaces

Haar functions are the simplest example of compactly supported wavelets. Their plain construction based on characteristic function of a unit cube allows many applications. In this chapter we consider the Haar system in the spaces $B^{s,w}_{pq}(\mathbb{R}^n)$ and $F^{s,w}_{pq}(\mathbb{R}^n)$. We follow the approach in the book [43], where H. Triebel proves, that Haar wavelets can be used to characterization of unweighted function spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ as far as absolute value of smoothness parameter is small enough.

4.1 Haar wavelets as local means

We want to use Theorem 3.3 to our purpose, i.e. to treat Haar wavelets as local means.

To do this we must restrict the definition of the local means to regular distributions (locally integrable functions). If $A = 0$ and $B = 1$ then Theorem 3.3 is valid for spaces $A^{s,w}_{pq}(\mathbb{R}^n)$ consisting of regular distributions.

Taking constants $A = 0$ and $B = 1$ we get the following inequalities

$$0 \geq \max\left(0, [-s + \sigma_p(w)], [\frac{nr_w}{p} - \frac{n}{p} - s] + 1\right) \tag{4.1.1}$$

and

$$1 \geq \max(0, [s] + 1) \tag{4.1.2}$$

in B-case and

$$0 \geq \max\left(0, [-s + \sigma_{pq}(w)], [\frac{nr_w}{p} - \frac{n}{p} - s] + 1\right) \tag{4.1.3}$$

and

$$1 \geq \max(0, [s] + 1)$$

in F-case. From (4.1.1) we get

$$\begin{aligned} &0 \geq [-s + \sigma_p(w)] \\ &1 > -s + \sigma_p(w) \\ &s > n\left(\frac{r_w}{\min(p, r_w)} - 1\right) + n\,(r_w - 1) - 1 \end{aligned}$$

and

$$
\begin{aligned}
0 &\geq [\frac{nr_w}{p} - \frac{n}{p} - s] + 1 \\
-1 &\geq [\frac{nr_w}{p} - \frac{n}{p} - s] \\
s &> \frac{n}{p}(r_w - 1).
\end{aligned}
$$

From (4.1.2) we get

$$
\begin{aligned}
1 &\geq \max(0, [s] + 1) \\
[s] &\leq 0 \\
s &< 1.
\end{aligned}
$$

Thus we get

$$\max\left(\sigma_p(w) - 1, \frac{n}{p}(r_w - 1)\right) < s < 1$$

for $B^{s,w}_{pq}(\mathbb{R}^n)$ spaces.

From (4.1.3) we get

$$
\begin{aligned}
&0 \geq [-s + \sigma_{pq}(w)] \\
&1 > -s + \sigma_{pq}(w) \\
&s > \max\left(n\left(\frac{r_w}{\min(p, r_w)} - 1\right) + n\left(r_w - 1\right), \frac{n}{\min(1, q)} - n\right) - 1
\end{aligned}
$$

and

$$s > \frac{n}{p}(r_w - 1).$$

Thus we get

$$\max\left(\sigma_{pq}(w) - 1, \frac{n}{p}(r_w - 1)\right) < s < 1$$

for $F^{s,w}_{pq}(\mathbb{R}^n)$ spaces.

We are looking for the weight $w \in \mathcal{A}^{loc}_\infty$ such that the corresponding Besov and Triebel-Lizorkin spaces satisfy the above inequalities and Corollary 3.2 and Corollary 3.3 respectively.

4.2 Smoothness of series of characteristic functions

In this section we find the sufficient condition for the function $\sum_{j=0, m\in\mathbb{R}^n}^{\infty} \mu_{jm}\chi_{jm}$ to belong to the spaces $B^{s,w}_{pq}$ and $F^{s,w}_{pq}$.

It appears that we need some regularity conditions on weights to prove further results.

Definition 4.1. Let $w \in \mathcal{A}_\infty^{loc}$. We say that the weight w satisfies *a regularity condition* if there exist constants $D > 1$ and $d > 0$ such that for every $k, j \in \mathbb{N}_0$, $k > j$ the following inequalities

$$\frac{w(Q_{kl})}{|Q_{kl}|} \leq d \frac{w(Q_{jm})}{|Q_{jm}|} \tag{4.2.1}$$

hold if $DQ_{kl} \cap Q_{jm} \neq \emptyset$ and $DQ_{kl} \nsubseteq Q_{jm}$.

Remark 4.1. The regularity condition is independent of the choice of a constant D in the sense that if it holds for some $D > 1$ with $d > 0$ then it holds for any $\tilde{D} > 1$ with some $\tilde{d}$. It should be clear that if the condition (4.2.1) holds for $D > 1$ and $d > 0$ then it holds for $1 < \tilde{D} < D$ and $\tilde{d} = d$. We show that if the condition (4.2.1) holds for $D > 1$ then it holds also for $\tilde{D} = 2D$.

Let Q_{kl} and Q_{jm} be cubes such that $k > j$, $2DQ_{kl} \cap Q_{jm} \neq \emptyset$ and $2DQ_{kl} \nsubseteq Q_{jm}$. Let us assume that $DQ_{kl} \cap Q_{jm} = \emptyset$ or $DQ_{kl} \subseteq Q_{jm}$.

If $DQ_{kl} \subseteq Q_{jm}$ and $2DQ_{kl} \nsubseteq Q_{jm}$ then there exist two cubes $Q_{kl'}$ and $Q_{kl''}$ such that $Q_{kl'} \cap DQ_{kl} \neq \emptyset$, $Q_{kl''} \cap DQ_{kl'} \neq \emptyset$ and $DQ_{kl''} \nsubseteq Q_{jm}$. Then

$$\frac{w(Q_{kl})}{|Q_{kl}|} \leq d \frac{w(Q_{kl'})}{|Q_{kl'}|} \leq d^2 \frac{w(Q_{kl''})}{|Q_{kl''}|} \leq d^3 \frac{w(Q_{jm})}{|Q_{jm}|}. \tag{4.2.2}$$

If $DQ_{kl} \cap Q_{jm} = \emptyset$ and $2DQ_{kl} \cap Q_{jm} \neq \emptyset$ then there exist two cubes $Q_{kl'} \cap DQ_{kl} \neq \emptyset$, $Q_{kl''} \cap DQ_{kl'} \neq \emptyset$ and $DQ_{kl''} \cap Q_{jm} \neq \emptyset$, $DQ_{kl''} \nsubseteq Q_{jm}$ such that (4.2.2) holds.

The following weights satisfy the regularity condition:

1. Polynomial weights

$$w(x) = |x|^\alpha, \quad -n < \alpha < n(p-1), \ p > 1,$$

then $w \in \mathcal{A}_p^{loc}$.

$$w(Q_{jm}) \sim \begin{cases} |m|^\alpha 2^{-j(\alpha+n)}, & m \neq 0 \\ 2^{-j(\alpha+n)}, & m = 0. \end{cases}$$

We can check the regularity condition explicitly:

$$\frac{w(Q_{kl})}{w(Q_{jm})} = \frac{\max(1,|l|^\alpha) 2^{-k(\alpha+n)}}{\max(1,|m|^\alpha) 2^{-j(\alpha+n)}} = 2^{(j-k)n} 2^{(j-k)\alpha} \left(\frac{\max(1,|l|^\alpha)}{\max(1,|m|^\alpha)} \right). \tag{4.2.3}$$

We take $D = 2$. Then $DQ_{kl} \cap Q_{jm} \neq \emptyset$ and $DQ_{kl} \nsubseteq Q_{jm}$ implies

$$|2^{-k} l - 2^{-j} m| \leq 2^{-j} \sqrt{n}$$

and

$$2^{-k} \max(1,|l|) \sim 2^{-j} \max(1,|m|).$$

So

$$\frac{w(Q_{kl})}{w(Q_{jm})} \leq c 2^{(j-k)n}.$$

2. Weights with logarithmic factor

$$v(x) = |x|^\alpha \begin{cases} (1 - \log|x|)^{-\beta}, & |x| \le 1, \\ (1 + \log|x|)^{-\beta}, & |x| > 1, \end{cases}$$

with $-n < \alpha < n(p-1)$, $p > 1$, $\beta \in \mathbb{R}$, then $w \in \mathcal{A}_p^{loc}$.

$$w(Q_{jm}) \sim \begin{cases} |m|^\alpha 2^{-j(\alpha+n)} \max((1 - \log(|m|2^{-j}))^{-\beta}, (1 + \log(|m|2^{-j}))^{-\beta}), & m \neq 0, \\ 2^{-j(\alpha+n)}(1 - \log(2^{-j}))^{-\beta}, & m = 0. \end{cases}$$

Using above estimations we check the condition similarly as in (4.2.3).

3. Polynomial weight with different powers near zero and at infinity

$$w(x) = \begin{cases} |x|^\alpha, & |x| \le 1 \\ |x|^\beta, & |x| > 1. \end{cases}$$

and $\alpha, \beta > -n$, $\alpha < n(p-1)$, $p > 1$, then $w \in \mathcal{A}_p^{loc}$.

Now we give an example of a weight, that doesn't satisfy the regularity condition Let $-n < \alpha < 0$. We put

$$w(x) = \begin{cases} 1, & |x| \ge 1 \text{ or } |x| < 1 \text{ and } x_1 \le 0, \\ |x|^\alpha, & |x| < 1 \text{ and } x_1 > 0 \end{cases}$$

for $x \in \mathbb{R}^n$.

We will show that $w \in \mathcal{A}_1^{loc}$. Outside the ball $B(0,1)$ the weight is equal to 1, so it is sufficient to check $A_1^{loc}(w)$ condition near zero. Let $Q \subset [-2,2]^n$ be such that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$, where $Q_1 = Q \cap \{x \in \mathbb{R}^n : x_1 \le 0\}$ and $Q_2 = Q \cap \{x \in \mathbb{R}^n : x_1 > 0\}$.

$$\begin{aligned} \frac{w(Q)}{|Q|} ||w^{-1}|L_\infty(Q)|| &\le \frac{c}{|Q|} \int_Q |x|^\alpha \, dx \frac{1}{\inf_{x \in Q} w(x)} \\ &\le \frac{c}{|Q|} \int_Q |x|^\alpha \, dx \frac{1}{\min(\inf_{x \in Q} |x|^\alpha, 1)} \le C \end{aligned}$$

since $|x|^\alpha \ge c > 0$ for any $x \in [-2,2]^n$.

This weight doesn't satisfy the regularity condition. Let us choose cubes Q_{jm} and Q_{kl} with $j = 0, m = (-1, 0, ..., 0)$ and $k > 0, l = (0, ..., 0)$. With that choice we have $w(Q_{jm}) = 2^{-jn}$ and $w(Q_{kl}) \sim 2^{-k(\alpha+n)}$. Therefore

$$\frac{w(Q_{kl})}{w(Q_{jm})} 2^{(k-j)n} \ge c \frac{2^{-k(\alpha+n)}}{2^{-jn}} 2^{(k-j)n} = c 2^{-k\alpha}.$$

Taking k arbitrarily large condition (4.2.1) is not satisfied since $\alpha < 0$.

Let us recall that $\bar{b}_{pq}^{s,w}$, $\bar{f}_{pq}^{s,w}$ spaces are defined in Definition 3.6. Now we can state the representation of weighted Besov and Triebel-Lizorkin spaces by characteristic functions of dyadic cubes.

Proposition 4.1. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Let $w \in \mathcal{A}^{loc}_{\infty}$ satisfy the regularity condition. If*

$$\frac{n}{p}\left(r_w - \min(1,p)\right) < s < \frac{1}{p},$$

then $f = \sum_{jm} \mu_{jm}\chi_{jm}$ (convergence in $\mathcal{S}'_e(\mathbb{R}^n)$) with $\mu \in \bar{b}^{s,w}_{pq}$ belongs to $B^{s,w}_{pq}(\mathbb{R}^n)$ and

$$\left\|f|B^{s,w}_{pq}(\mathbb{R}^n)\right\| \le c\left\|\mu|\bar{b}^{s,w}_{pq}\right\|$$

for some $c > 0$ and all $\mu \in \bar{b}^{s,w}_{pq}$.

If $q < \infty$ and

$$n\left(\max\left(1, \frac{r_w}{p}, \frac{1}{q}\right) - 1\right) < s < \min\left(\frac{1}{p}, \frac{1}{q}\right),$$

then $f = \sum_{jm} \mu_{jm}\chi_{jm}$ (convergence in $\mathcal{S}'_e(\mathbb{R}^n)$) with $\mu \in \bar{f}^{s,w}_{pq}$ belongs to $F^{s,w}_{pq}(\mathbb{R}^n)$ and

$$\left\|f|F^{s,w}_{pq}(\mathbb{R}^n)\right\| \le c\left\|\mu|\bar{f}^{s,w}_{pq}\right\|$$

for some $c > 0$ and all $\mu \in \bar{f}^{s,w}_{pq}$.

Proof. Let

$$f = \sum_{jm} \mu_{jm}\chi_{jm} \tag{4.2.4}$$

with $\mu \in \bar{a}^{s,w}_{pq}$. Let $w \in \mathcal{A}^{loc}_u$ for some $u \ge r_w$. We take Daubechies wavelet expansion of χ_{jm} in $L_2(\mathbb{R}^n)$ by wavelets belonging to $C^r(\mathbb{R}^n)$, where $r \ge \max(1, [s]+1, [\frac{nr_w-1}{p} - s] + 1, [\sigma_p(w) - s])$ in the case of Besov spaces and $r \ge \max(1, [s]+1, [\frac{nr_w-1}{p} - s] + 1, [\sigma_{pq}(w) - s])$ in the case of Triebel-Lizorkin spaces. Then $\operatorname{supp}\psi^G_{kl} \subset DQ_{kl}$ for some $D = D(r)$ and $\psi^G_{k,l}$ satisfy moment conditions of order r for every $j \in \mathbb{N}_0$, $k \in \mathbb{Z}^n$ and $G \in \{F, M\}^{n*}$. We have

$$\chi_{jm}(x) = \sum_{k,l,G} \lambda^G_{kl} 2^{-kn/2}\psi^G_{kl}, \tag{4.2.5}$$

where $\lambda^G_{kl} = 2^{kn/2}\langle\chi_{jm}, \psi^G_{kl}\rangle$.

First we estimate $\lambda^G_{kl}(\chi_{jm})$. We have

$$\begin{aligned}
\left|\lambda^G_{kl}(\chi_{jm})\right| &= \left|\int_{\mathbb{R}^n} 2^{kn/2}\chi_{jm}(y)\psi^G_{kl}(y)dy\right| \\
&\le 2^{kn/2}\int_{\mathbb{R}^n} \left|\chi_{jm}(y)\psi^G_{kl}(y)\right| dy \\
&= 2^{kn/2}\int_{Q_{jm}} \left|\psi^G_{kl}(y)\right| dy \\
&= 2^{kn}\int_{Q_{jm}} \left|\psi^G(2^k y - l)\right| dy \le c\min(1, 2^{n(k-j)}).
\end{aligned} \tag{4.2.6}$$

We also know that $\lambda_{kl}^G(\chi_{jm}) = 0$ if $\operatorname{supp}\psi_{kl}^G \cap Q_{jm} = \emptyset$ or $\operatorname{supp}\psi_{kl}^G \subset Q_{jm}$ if $G \in \{F, M\}^{n*}$ because of the moment conditions.

We are going to show that $f \in A_{pq}^{s,w}(\mathbb{R}^n)$. According to (4.2.4) and (4.2.5) we get

$$\begin{aligned} f &= \sum_{jm} \mu_{jm} \sum_{k,l,G} \lambda_{kl}^G 2^{-kn/2} \psi_{kl}^G \\ &= \sum_{k,l,G} \sum_{j,m} 2^{-kn/2} \mu_{jm} \lambda_{kl}^G \psi_{kl}^G \\ &= \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \sum_{G} 2^{-kn/2} \sum_{j,m} \mu_{jm} \lambda_{kl}^G \psi_{kl}^G \\ &= \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \sum_{G} \nu_{kl}^G 2^{-kn/2} \psi_{kl}^G \\ &= \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \sum_{G} \left(\nu_{kl}^{G,1} + \nu_{kl}^{G,2} \right) 2^{-kn/2} \psi_{kl}^G, \end{aligned}$$

where

$$\nu_{kl}^G = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \mu_{jm} \lambda_{kl}^G,$$

and

$$\nu_{kl}^{G,1} = \sum_{j \geq k} \sum_{m \in \mathbb{Z}^n} \mu_{jm} \lambda_{kl}^G, \tag{4.2.7}$$

$$\nu_{kl}^{G,2} = \sum_{j=0}^{k-1} \sum_{m \in \mathbb{Z}^n} \mu_{jm} \lambda_{kl}^G. \tag{4.2.8}$$

Step 1. We start with the sum over $j \geq k$. We estimate

$$\begin{aligned} \left| \nu_{kl}^{G,1} \right| &= \left| \sum_{j \geq k} \sum_{m \in \mathbb{Z}^n} \mu_{jm} \lambda_{kl}^G \right| \leq \sum_{j \geq k} \sum_{m \in \mathbb{Z}^n} |\mu_{jm}| \left| \lambda_{kl}^G \right| \\ &\leq c \sum_{j \geq k} \sum_{m : \lambda_{kl}^G(\chi_{jm}) \neq 0} 2^{n(k-j)} |\mu_{jm}|, \end{aligned} \tag{4.2.9}$$

cf. (4.2.6).

From $\mu \in \bar{b}_{pq}^{s,w}$ we have

$$\sum_{j=0}^{\infty} 2^{jsq} \left(\sum_{m \in \mathbb{Z}^n} |\mu_{jm}|^p \, w(Q_{jm}) \right)^{q/p} < \infty.$$

Substep 1.1.
First we consider the case $p \le 1$. We have

$$\sum_{l\in\mathbb{Z}^n} \left|\nu_{kl}^{G,1}\right|^p w(Q_{kl}) \le c \sum_{l\in\mathbb{Z}^n} \left(\sum_{j\ge k} \sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0} 2^{n(k-j)} |\mu_{jm}| \right)^p w(Q_{kl})$$
$$\le c \sum_{l\in\mathbb{Z}^n} \sum_{j\ge k} \sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0} 2^{np(k-j)} |\mu_{jm}|^p w(Q_{kl})$$
$$\le c \sum_{j\ge k} 2^{np(k-j)} \sum_{l\in\mathbb{Z}^n} \sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0} |\mu_{jm}|^p w(Q_{kl}).$$

But $\lambda_{k,l}^G(\chi_{j,m}) \neq 0$ only if $\operatorname{supp} \psi_{kl}^G \cap Q_{jm} \neq \emptyset$. Moreover $j \ge k$ so the relation $\lambda_{kl}^G(\chi_{jm}) \neq 0$ implies that there exists $C \ge 1$ independent of j and k such that $Q_{jm} \subset CQ_{kl}$. Now the properties of the weights belonging to $\mathcal{A}_u^{loc}$ (Lemma 2.3) give us

$$w(Q_{kl}) \le w(CQ_{kl}) \le cw(Q_{jm})2^{(j-k)nu}. \tag{4.2.10}$$

Then

$$\begin{aligned}\sum_{l\in\mathbb{Z}^n} \left|\nu_{kl}^{G,1}\right|^p w(Q_{kl}) &\le c \sum_{j\ge k} 2^{(j-k)nu} 2^{np(k-j)} \sum_{l\in\mathbb{Z}^n} \sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0} |\mu_{jm}|^p w(Q_{jm}) \\ &\le c \sum_{j\ge k} 2^{-n(j-k)(p-u)} \sum_{m\in\mathbb{Z}^n} |\mu_{jm}|^p w(Q_{jm}),\end{aligned} \tag{4.2.11}$$

since $\left|\{l : \lambda_{kl}^G(\chi_{jm}) \neq 0\}\right| \le c$.
Substep 1.2.
Now we consider the case $p > 1$. The inequality $j \ge k$ implies $\left|\{m : \lambda_{kl}^G(\chi_{jm}) \neq 0\}\right| \le c2^{n(j-k)}$. Using Hölder's inequality and (4.2.9) we get

$$\left|\nu_{kl}^{G,1}\right| \le c \sum_{j\ge k} \sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0} 2^{n(k-j)} |\mu_{jm}|$$
$$\le c \sum_{j\ge k} 2^{n(k-j)} 2^{n(j-k)/p'} \left(\sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0} |\mu_{jm}|^p \right)^{1/p}.$$

For any $\varepsilon > 0$ we have

$$\sum_{l\in\mathbb{Z}^n} \left|\nu_{kl}^{G,1}\right|^p w(Q_{kl}) \le c \sum_{l\in\mathbb{Z}^n} \sum_{j\ge k} 2^{-np(j-k)(\frac{1}{p}-\varepsilon)} \sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0} |\mu_{jm}|^p w(Q_{kl})$$
$$\le c \sum_{j\ge k} 2^{-np(j-k)(\frac{1}{p}-\varepsilon)} 2^{nu(j-k)} \sum_{m\in\mathbb{Z}^n} |\mu_{jm}|^p w(Q_{jm})$$
$$= c \sum_{j\ge k} 2^{-np(j-k)(\frac{1-u}{p}-\varepsilon)} \sum_{m\in\mathbb{Z}^n} |\mu_{jm}|^p w(Q_{jm}),$$

where we used inequality (4.2.10) once more.

Substep 1.3. In case of F-spaces we use the maximal inequality of Fefferman-Stein type

$$\left\| \left(\sum_{k,l} (M^{loc}|g_{kl}|^v)(\cdot)^{q/v} \right)^{1/q} |L_p^w(\mathbb{R}^n) \right\| \leq \left\| \left(\sum_{k,l} |g_{kl}(\cdot)|^q \right)^{1/q} |L_p^w(\mathbb{R}^n) \right\|$$

for $0 < v < \min(1, \frac{p}{u}, q)$, cf. Theorem 2.2. We have for $x \in Q_{kl}$

$$\begin{aligned}
\left(\chi_{kl}(x) \sum_{m:\lambda^G_{kl}(\chi_{jm})\neq 0} |\mu_{jm}| \right)^v &\leq \chi_{kl}(x) \sum_{m:\lambda^G_{kl}(\chi_{jm})\neq 0} |\mu_{jm}|^v \\
&= \chi_{kl}(x) 2^{jn} \int_{\mathbb{R}^n} \sum_{m:\lambda^G_{kl}(\chi_{jm})\neq 0} |\mu_{jm}|^v \chi_{jm}(y) dy \\
&\leq c 2^{n(j-k)} M^{loc} \left(\sum_{m:\lambda^G_{kl}(\chi_{jm})\neq 0} |\mu_{jm}|^v \chi_{jm}(\cdot) \right)(x).
\end{aligned}$$

It follows from (4.2.9) and the above inequality that for $\varepsilon > 0$ we have

$$\begin{aligned}
&\left| v^{G,1}_{kl} \chi_{kl}(x) \right|^q \\
&\leq c \left| \sum_{j\leq k} 2^{n(k-j)} \sum_{m:\lambda^G_{kl}(\chi_{jm})\neq 0} |\mu_{jm}| \chi_{kl}(x) \right|^q \\
&\leq c \sum_{j\leq k} 2^{n(j-k)q(\varepsilon-1)} \left(\sum_{m:\lambda^G_{kl}(\chi_{jm})\neq 0} |\mu_{jm}| \chi_{kl}(x) \right)^q \\
&\leq c \sum_{j\leq k} 2^{n(j-k)q(\varepsilon-1)} 2^{n(j-k)\frac{q}{v}} M^{loc} \left(\sum_{m:\lambda^G_{kl}(\chi_{jm})\neq 0} |\mu_{jm}|^v \chi_{jm}(\cdot) \right)(x)^{q/v}
\end{aligned}$$

In consequence

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}^n} \left| \nu^{G,1}_{kl} \right|^q \chi_{kl}(x) \\
&\leq c \sum_{l\in\mathbb{Z}^n} \sum_{j\geq k} 2^{-nq(j-k)(-\frac{1}{v}+1-\varepsilon)} M^{loc} \left(\sum_{m:\lambda^G_{kl}(\chi_{jm})\neq 0} |\mu_{jm}|^v \chi_{jm}(\cdot) \right)(x)^{q/v}.
\end{aligned} \tag{4.2.12}$$

Since $s > n\left(\max\left(1, \frac{r_w}{p}, \frac{1}{q}\right) - 1\right)$ we can choose $\varepsilon > 0$, $u > r_w$ and $v < \min\left(1, \frac{p}{u}, q\right)$ such that $s > n\left(\frac{1}{v} + \varepsilon - 1\right)$. Then by (4.2.12)

$$\begin{aligned}
&\left\| \left(\sum_{k,l,G} 2^{ksq} \left|\nu_{kl}^{G,1}\right|^q \chi_{kl}(\cdot) \right)^{1/q} |L_p^w(\mathbb{R}^n) \right\| \\
&\le c \left\| \left(\sum_{k,G} 2^{ksq} \sum_{l\in\mathbb{Z}^n} \sum_{j\ge k} 2^{-nq(j-k)(-\frac{1}{v}+1-\varepsilon)} M^{loc} \left(\sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0} |\mu_{jm}|^v \chi_{jm}(\cdot) \right)^{q/v} \right)^{v/q} |L_p^w(\mathbb{R}^n) \right\| \\
&\le c \left\| \left(\sum_{k,G} 2^{ksq} \sum_{l\in\mathbb{Z}^n} \sum_{j\ge k} 2^{-nq(j-k)(-\frac{1}{v}+1-\varepsilon)} \sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0} |\mu_{jm}|^q \chi_{jm}(\cdot) \right)^{1/q} |L_p^w(\mathbb{R}^n) \right\| \\
&\le c \left\| \left(\sum_{j} 2^{jsq} \sum_{k\le j} 2^{-q(j-k)(-\frac{n}{v}+n-n\varepsilon+s)} \sum_{m\in\mathbb{Z}^n} |\mu_{jm}|^q \chi_{jm}(\cdot) \right)^{1/q} |L_p^w(\mathbb{R}^n) \right\| \\
&\le c \left\| \left(\sum_{j,m} 2^{jsq} |\mu_{jm}|^q \chi_{jm}(\cdot) \right)^{1/q} |L_p^w(\mathbb{R}^n) \right\|
\end{aligned} \tag{4.2.13}$$

Step 2.

We are going to estimate the second term in the decomposition of f. We have

$$\begin{aligned}
\left|\nu_{kl}^{G,2}\right| &= \left| \sum_{j<k} \sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0} \mu_{jm} \lambda_{kl}^G \right| \\
&\le c \sum_{j<k} \sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0} |\mu_{jm}|
\end{aligned}$$

from (4.2.6).

Substep 2.1.

Let $p \le 1$. Now $\left|\{l : \lambda_{kl}^G(\chi_{jm}) \neq 0\}\right| \le c2^{(k-j)(n-1)}$. Because w satisfies the regularity condition we get

$$\sum_{l\in\mathbb{Z}^n}\left|\nu_{kl}^{G,2}\right|^p w(Q_{kl}) \le c\sum_{j<k}\sum_{l\in\mathbb{Z}^n}\sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0}|\mu_{jm}|^p w(Q_{kl})$$
$$\le c\sum_{j<k}\sum_{l\in\mathbb{Z}^n}\sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0}|\mu_{jm}|^p w(Q_{kl})w(Q_{jm})w(Q_{jm})^{-1}$$
$$\le c\sum_{j<k}\sum_{m\in\mathbb{Z}^n}\sum_{l:\lambda_{kl}^G(\chi_{jm})\neq 0}|\mu_{jm}|^p w(Q_{jm})2^{(j-k)n} \tag{4.2.14}$$
$$\le c\sum_{j<k}\sum_{m\in\mathbb{Z}^n}2^{(k-j)(n-1)}|\mu_{jm}|^p w(Q_{jm})2^{(j-k)n}$$
$$\le c\sum_{j<k}\sum_{m\in\mathbb{Z}^n}2^{j-k}|\mu_{jm}|^p w(Q_{jm}).$$

Substep 2.2.

Now let $p>1$ and $\varepsilon>0$. The inequality $j<k$ implies $\left|\{m:\lambda_{kl}^G(\chi_{jm})\neq 0\}\right|\le c$ for some constant c independent of j,k and l. Using Hölder's inequality, (4.2.6), the regularity condition on w and the estimate $\left|\{l:\lambda_{kl}^G(\chi_{jm})\neq 0\}\right|\le c2^{(k-j)(n-1)}$ we get

$$\sum_{l\in\mathbb{Z}^n}\left|\nu_{kl}^{G,2}\right|^p w(Q_{kl}) \le \sum_{l\in\mathbb{Z}^n}\left(\sum_{j<k}\sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0}|\mu_{jm}|\right)^p w(Q_{kl})$$
$$\le c\sum_{l\in\mathbb{Z}^n}\left(\sum_{j<k}\left(\sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0}|\mu_{jm}|^p\right)^{1/p}\right)^p w(Q_{kl})$$
$$\le c\sum_{l\in\mathbb{Z}^n}\sum_{j<k}2^{(k-j)\varepsilon p}\sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0}|\mu_{jm}|^p w(Q_{kl})$$
$$\le c\sum_{j<k}2^{(k-j)\varepsilon p}\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|^p\sum_{l:\lambda_{kl}^G(\chi_{jm})\neq 0}w(Q_{kl})$$
$$\le c\sum_{j<k}2^{(k-j)\varepsilon p}\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|^p\,2^{(j-k)n}2^{(k-j)(n-1)}w(Q_{jm})$$
$$\le c\sum_{j<k}2^{(k-j)(\varepsilon p-1)}\sum_{m\in\mathbb{Z}^n}|\mu_{jm}|^p w(Q_{jm}).$$

Substep 2.3.

In case of F-spaces we get

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}^n} \left|\nu_{kl}^{G,2}\right|^q \chi_{kl}(x) \\
&\leq c \sum_{l\in\mathbb{Z}^n} \sum_{j<k} 2^{-q(j-k)\varepsilon} \sum_{m:\lambda_{kl}^G(\chi_{jm})\neq 0} |\mu_{jm}|^q \chi_{kl}(x) \\
&\leq c \sum_{j<k} 2^{-q(j-k)\varepsilon} \sum_{m\in\mathbb{Z}^n} |\mu_{jm}|^q \sum_{l:\lambda_{kl}^G(\chi_{jm})\neq 0} \chi_{kl}(x) \\
&\leq c \sum_{j<k} 2^{-q(j-k)\varepsilon} \sum_{m\in\mathbb{Z}^n} |\mu_{jm}|^q \chi_{dQ_{jm}}(x),
\end{aligned} \tag{4.2.15}$$

where $d>1$ is such that $\bigcup_{l:\lambda_{kl}^G(\chi_{jm})\neq 0} DQ_{kl} \subset dQ_{jm}$ and is finite. Moreover, since $j<k$ the constant d can be chosen independently of k.

Let us assume that $s<0$.

$$\begin{aligned}
&\left\| \left(\sum_{k,l,G} 2^{ksq} \left|\nu_{kl}^{G,2}\right|^q \chi_{kl}(\cdot) \right)^{1/q} |L_p^w(\mathbb{R}^n) \right\| \\
&\leq c \left\| \left(\sum_{k,G} 2^{ksq} \sum_{j<k} 2^{-q(j-k)\varepsilon} \sum_{m\in\mathbb{Z}^n} |\mu_{jm}|^q \chi_{dQ_{jm}}(\cdot) \right)^{1/q} |L_p^w(\mathbb{R}^n) \right\| \\
&\leq c \left\| \left(\sum_{j} 2^{jsq} \sum_{k>j} 2^{-q(j-k)(\varepsilon+s)} \sum_{m\in\mathbb{Z}^n} |\mu_{jm}|^q \chi_{dQ_{jm}}(\cdot) \right)^{1/q} |L_p^w(\mathbb{R}^n) \right\| \\
&\leq c \left\| \left(\sum_{j,m} 2^{jsq} |\mu_{jm}|^q \chi_{jm}(\cdot) \right)^{1/q} |L_p^w(\mathbb{R}^n) \right\|
\end{aligned}$$

where the last inequality follows from Proposition 3.7.

Step 3.

For $p \le 1$ we get by (4.2.7), (4.2.8), (4.2.11) and (4.2.14)

$$\sum_{k=0}^{\infty} 2^{ksq} \sum_{G} \left(\sum_{l \in \mathbb{Z}^n} \left| \nu_{kl}^{G} \right|^p w(Q_{kl}) \right)^{q/p}$$

$$\le c \sum_{k=0}^{\infty} 2^{ksq} \sum_{G} \left(\sum_{j \ge k} 2^{-n(j-k)(p-u)} \sum_{m \in \mathbb{Z}^n} |\mu_{jm}|^p w(Q_{jm}) \right.$$

$$\left. + \sum_{j<k} 2^{(j-k)} \sum_{m \in \mathbb{Z}^n} |\mu_{jm}|^p w(Q_{jm}) \right)^{q/p}$$

$$\le \tilde{c} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} 2^{jsp} \sum_{m} |\mu_{jm}|^p w(Q_{jm}) 2^{-|k-j|\varkappa p} \right)^{q/p},$$

where $\varkappa = \min\left(n - \frac{nu}{p} + s, \frac{1}{p} - s\right)$.

From monotonicity of ℓ_p spaces if $\frac{q}{p} \le 1$ or Young's inequalities if $\frac{q}{p} > 1$ we have for $\varkappa > 0$

$$\left(\sum_{k=0}^{\infty} 2^{ksq} \sum_{G} \left(\sum_{l \in \mathbb{Z}^n} \left| \nu_{kl}^{G} \right|^p w(Q_{kl}) \right)^{q/p} \right)^{1/q}$$

$$\le \tilde{c} \left(\sum_{j=0}^{\infty} 2^{jsq} \left(\sum_{m} |\mu_{jm}|^p w(Q_{jm}) \right)^{q/p} \right)^{1/q}.$$

In the same way we get the result for $p > 1$ with $\varkappa = \min\left(\frac{n}{p} - \frac{nu}{p} - n\varepsilon + s, \frac{1}{p} - \varepsilon - s\right)$.

Thus

$$\left\| \nu | b_{pq}^{s,w} \right\| \le c \left\| \mu | \bar{b}_{pq}^{s,w} \right\|.$$

Step 4. To prove the result for $f_{pq}^{s,w}$ spaces we use the complex interpolation.

The mapping

$$\nu^2 : \{\mu_{jm}\}_{j,m} \mapsto \{\nu_{kl}^{G,2} = \sum_{j<k} \sum_{m \in \mathbb{Z}^n} \mu_{jm} \lambda_{jm}^{G}\}_{k,l,G}$$

is a linear operator. It follows from (4.2.13) that it is a bounded operator from $\bar{f}_{pq}^{s,w}$ to $f_{pq}^{s,w}$ if $s < 0$ and from $\bar{b}_{pq}^{s,w}$ to $b_{pq}^{s,w}$ if $\frac{n}{p}(r_w - \min(1,p)) < s < \frac{1}{p}$. In consequence it is a bounded operator from $\bar{f}_{pp}^{s,w}$ to $f_{pp}^{s,w}$ if $s < \frac{1}{p}$.

Because of Theorem 3.11 we have

$$[f_{p_0,q_0}^{s_0,w}, f_{p_1,q_1}^{s_1,w}]_\theta = f_{pq}^{s,w}$$

for $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$ and $s = (1-\theta)s_0 + \theta s_1$. We show that if $s_0 < 0$ and $s_1 < \frac{1}{p_1}$ we can reach any $s < \min\left(\frac{1}{p}, \frac{1}{q}\right)$ for $0 < p, q < \infty$.

We choose a sequence $\{p_1^{(j)}\}_{j\in\mathbb{N}}$ in such a way that $s < \frac{1}{p_1^{(j)}} \to s$ if $j \to \infty$ and afterwards sequences $\{s_1^{(j)}\}_j$, $\{\theta^{(j)}\}_j$ and $\{p_0^{(j)}\}_j$ such that

$$s < s_1^{(j)} < \frac{1}{p_1^{(j)}},$$

$$s = (1-\theta^{(j)})s_0 + \theta^{(j)} s_1^{(j)}, \quad \text{for some } s_0 > 0,$$
$$\frac{1}{p} = \frac{1-\theta^{(j)}}{p_0^{(j)}} + \frac{\theta^{(j)}}{p_1^{(j)}}.$$

Then $\theta^{(j)} \to 1$ since $s_1^{(j)} \to s$ if $j \to \infty$. Moreover

$$\frac{1}{q^{(j)}} = \frac{1-\theta^{(j)}}{q_0} + \frac{\theta^{(j)}}{p_1^{(j)}} = s + \varepsilon^{(j)} + \frac{1-\theta^{(j)}}{q_0}.$$

with $\varepsilon^{(j)} \to 0$ if $j \to \infty$ since $\frac{s-s_0}{p_1^{(j)}-(s_1^{(j)}-s_0)} \to s$ if $j \to \infty$. Thus if $\theta^{(j)} \to 1$ then $\frac{1}{q^{(j)}} \to s$. So any $\frac{1}{q} > s$ can be reached. In consequence

$$\left\|\nu^2|f_{pq}^{s,w}\right\| \le c \left\|\mu|\bar{f}_{pq}^{s,w}\right\|.$$

for $s < \min\left(\frac{1}{p}, \frac{1}{q}\right)$.

Now the inequality

$$\left\|\nu|f_{pq}^{s,w}\right\| \le c \left\|\mu|\bar{f}_{pq}^{s,w}\right\|$$

follows from the above inequality and (4.2.13).

Step 5. From the Daubechies wavelet characterization (Theorem 3.4) we get $f \in A_{pq}^{s,w}(\mathbb{R}^n)$ and

$$\left\|f|A_{pq}^{s,w}(\mathbb{R}^n)\right\| \le c \left\|\mu|\bar{a}_{pq}^{s,w}\right\|.$$

□

4.3 Characterization of function spaces by Haar wavelets

We recall that sequence spaces $b_{pq}^{s,w}$ and $f_{pq}^{s,w}$ are defined in Definition 3.7.

Theorem 4.1. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty^{loc}$ satisfies the regularity condition.*

(i) Let $r_w < \min(1,p)\left(1+\frac{1}{n}\right)$. Let one of the following conditions be satisfied

- $0 < p < \infty,\ 0 < q \le \infty$, $\max\left(\frac{n}{p}\left(r_w - \min(1,p)\right), \sigma_p(w) - 1\right) < s < \min\left(1, \frac{1}{p}\right)$.

- $1 < p < \infty,\ 0 < q \leq \infty,\ \frac{1}{p} - 1 < s < \frac{1}{p},\ r_w < \min\left(p, 1 + \frac{1}{n}\right)$ *and* $w^{1-p'}$ *satisfies the regularity condition and* $r_{w^{1-p'}} < 1 + \frac{1}{n}$.

Let $f \in \mathcal{S}'_e(\mathbb{R}^n)$. *Then* $f \in B^{s,w}_{pq}(\mathbb{R}^n)$ *if and only if it can be represented as*

$$f = \sum_{j,G,m} \lambda^G_{jm} 2^{-jn/2} H^G_{jm},$$

where $\lambda \in b^{s,w}_{pq}$ *and the series converges in* $S'_e(\mathbb{R}^n)$. *This representation is unique with*

$$\lambda^G_{jm} = 2^{jn/2} \langle f, H^G_{jm} \rangle$$

and

$$I : f \mapsto \{2^{jn/2} \langle f, H^G_{jm} \rangle\}$$

is a linear isomorphism of $B^{s,w}_{pq}(\mathbb{R}^n)$ *onto* $b^{s,w}_{pq}$.

If $0 < p, q < \infty$ *then the system* $\left\{H^G_{jm}\right\}_{j,m,G}$ *is an unconditional basis in* $B^{s,w}_{pq}(\mathbb{R}^n)$.

(ii) Let $r_w < p\left(\frac{\min\left(1,\frac{1}{p},\frac{1}{q}\right)}{n} + \min\left(1, \frac{1}{p}\right)\right)$. *Let one of the following conditions be satisfied*

- $0 < p < \infty,\ 0 < q < \infty,$
  $\max\left(\frac{n}{p}(r_w - \min(1,p)), \frac{n}{q}(1 - \min(1,q)), \sigma_p(w) - 1\right) < s < \min\left(1, \frac{1}{p}, \frac{1}{q}\right)$.
- $1 < p < \infty,\ 1 < q < \infty,\ \max\left(\frac{1}{p}, \frac{1}{q}\right) - 1 < s < \min\left(\frac{1}{p}, \frac{1}{q}\right)$,
  $r_w < p \min\left(1, \frac{1}{p} + \frac{1}{n}\min\left(\frac{1}{q}, \frac{1}{p}\right)\right)$ *and* $w^{1-p'}$ *satisfies the regularity condition and* $r_{w^{1-p'}} < 1 + \frac{1}{n}\min\left(1, \frac{p'}{q'}\right)$.

Let $f \in \mathcal{S}'_e(\mathbb{R}^n)$. *Then* $f \in F^{s,w}_{pq}(\mathbb{R}^n)$ *if and only if it can be represented as*

$$f = \sum_{j,G,m} \lambda^G_{jm} 2^{-jn/2} H^G_{jm},$$

where $\lambda \in f^{s,w}_{pq}$ *and the series converges in* $S'_e(\mathbb{R}^n)$. *This representation is unique with*

$$\lambda^G_{jm} = 2^{jn/2} \langle f, H^G_{jm} \rangle$$

and

$$I : f \mapsto \{2^{jn/2} \langle f, H^G_{jm} \rangle\}$$

is a linear isomorphism of $F^{s,w}_{pq}(\mathbb{R}^n)$ *onto* $f^{s,w}_{pq}$.

If $0 < p, q < \infty$ *then the system* $\left\{H^G_{jm}\right\}_{j,m,G}$ *is an unconditional basis in* $F^{s,w}_{pq}(\mathbb{R}^n)$.

Proof. Step 1. First we consider the case when

$$\max\left(\frac{n}{p}\left(r_w - \min(1,p)\right), \sigma_p(w) - 1\right) < s < \min\left(1, \frac{1}{p}\right)$$

for $A = B$ and

$$\max\left(\frac{n}{p}\left(r_w - \min(1,p)\right), \sigma_p(w) - 1, \frac{n}{q}\left(1 - \min(1,q)\right)\right) < s < \min\left(1, \frac{1}{p}, \frac{1}{q}\right)$$

for $A = F$.

Substep 1.1

Let $f \in \mathcal{S}'_e(\mathbb{R}^n)$ and $f = \sum_{j,G,m} \lambda^G_{jm} 2^{-jn/2} H^G_{jm}$. From Proposition 4.1 we obtain that $f \in A^{s,w}_{pq}(\mathbb{R}^n)$ and

$$\left\| f | A^{s,w}_{pq}(\mathbb{R}^n) \right\| \leq c \left\| \lambda | a^{s,w}_{pq} \right\|.$$

Substep 1.2.

Now let $f \in A^{s,w}_{pq}(\mathbb{R}^n)$. We take $k^G_{jm} = 2^{jn/2} H^G_{jm}$ as kernels of local means. From Theorem 3.3 we have

$$\left\| k(f) | a^{s,w}_{pq} \right\| \leq c \left\| f | A^{s,w}_{pq}(\mathbb{R}^n) \right\|.$$

The expansion

$$f = \sum_{j,G,m} k^G_{jm}(f) 2^{-jn/2} H^G_{jm}$$

follows from analogical considerations as in proof of Theorem 3.4.

Substep 1.3.

Uniqueness of the representation and unconditional convergence are also consequences of proof of Theorem 3.4.

Step 2. Now let $1 < p < \infty$, $1 < q < \infty$ and $r_w < p$. By duality we have $A^{s,w}_{pq}(\mathbb{R}^n)' = A^{-s,w^{1-p'}}_{p',q'}(\mathbb{R}^n)$. If

$$-\frac{1}{p'} = \frac{1}{p} - 1 < s < -\max\left(\frac{n}{p'}(r_{w^{1-p'}} - 1), \sigma_{p'}(w^{1-p'}) - 1\right) \tag{4.3.1}$$

then by Step 1 the Haar system is an unconditional basis in $B^{-s,w^{1-p'}}_{p',q'}(\mathbb{R}^n)$ space. Since the Haar basis is a biorthonormal system and by Theorem 1.3 it follows that the Haar system is an unconditional basis in $B^{s,w}_{pq}(\mathbb{R}^n)$ if (4.3.1) holds.

In the similar way the Haar system is a basis in $F^{s,w}_{pq}(\mathbb{R}^n)$ if

$$-\min\left(1, \frac{1}{p'}, \frac{1}{q'}\right) < s < -\max\left(\frac{n}{p'}(r_{w^{1-p'}} - \min(1,p'), \sigma_{p'}(w^{1-p'}) - 1, \frac{n}{q'}(1 - \min(1,q'))\right)$$

that is

$$\max\left(\frac{1}{p} - 1, \frac{1}{q} - 1\right) < s < -\max\left(\frac{n}{p'}(r_{w^{1-p'}} - 1), \sigma_{p'}(w^{1-p'}) - 1\right).$$

Step 3. The rest for the case of B-spaces can be proved by real interpolation

$$\left(B^{s_0,w}_{p,q_0}, B^{s_1,w}_{p,q_1}\right)_{\theta,q} = B^{s,w}_{pq},$$

where $0 < p < \infty$, $0 < q, q_0, q_1 \leq \infty$, $-\infty < s_0 < s_1 < \infty$, $0 < \theta < 1, s = (1-\theta)s_0 + \theta s_1$, proved by Rychkov in [34]. Let $1 < p, q_0 < \infty$, $0 < q, q_1 \leq \infty$ and

$$\frac{1}{p} - 1 < s_0 < -\max\left(\frac{n}{p'}(r_{w^{1-p'}} - 1), \sigma_{p'}(w^{1-p'}) - 1\right) < \max\left(\frac{n}{p}(r_w - 1), \sigma_p(w) - 1\right) < s_1 < \frac{1}{p}.$$

From Step 1 and Step 2 we know that there is the Haar bases in $B^{s_0,w}_{p,q_0}(\mathbb{R}^n)$ and $B^{s_1,w}_{p,q_1}(\mathbb{R}^n)$. Thus $\left\{H^G_{jm}\right\}_{j,m,G}$ is complete in $B^{s_1,w}_{p,q_1}(\mathbb{R}^n)$. Because $B^{s_1,w}_{p,q_1}(\mathbb{R}^n) \hookrightarrow B^{s,w}_{pq}(\mathbb{R}^n)$ and $B^{s_1,w}_{p,q_1}(\mathbb{R}^n)$ is dense in $B^{s,w}_{pq}(\mathbb{R}^n)$ we get that $\overline{\text{span}}\left\{H^G_{jm}\right\}_{j,m,G} = B^{s,w}_{pq}(\mathbb{R}^n)$. From Theorem 1.4 we know that $\sup_N \|S_N : B^{s_0,w}_{p,q_0}(\mathbb{R}^n) \to B^{s_0,w}_{p,q_0}(\mathbb{R}^n)\| < \infty$ and $\sup_N \|S_N : B^{s_1,w}_{p,q_1}(\mathbb{R}^n) \to B^{s_1,w}_{p,q_1}(\mathbb{R}^n)\| < \infty$. Interpolation gives us

$$\|S_N : B^{s,w}_{pq}(\mathbb{R}^n) \to B^{s,w}_{pq}(\mathbb{R}^n)\| \leq \|S_N : B^{s_0,w}_{p,q_0}(\mathbb{R}^n) \to B^{s_0,w}_{p,q_0}(\mathbb{R}^n)\|^{1-\theta} \|S_N : B^{s_1,w}_{p,q_1}(\mathbb{R}^n) \to B^{s_1,w}_{p,q_1}(\mathbb{R}^n)\|^{\theta} < \infty.$$

Now to prove that the Haar basis is unconditional in $B^{s,w}_{pq}(\mathbb{R}^n)$ we use Theorem 1.5 and get

$$\|T_\Lambda : B^{s,w}_{pq}(\mathbb{R}^n) \to B^{s,w}_{pq}(\mathbb{R}^n)\| \leq \|T_\Lambda : B^{s_0,w}_{p,q_0}(\mathbb{R}^n) \to B^{s_0,w}_{p,q_0}(\mathbb{R}^n)\|^{1-\theta} \|T_\Lambda : B^{s_1,w}_{p,q_1}(\mathbb{R}^n) \to B^{s_1,w}_{p,q_1}(\mathbb{R}^n)\|^{\theta} < \infty.$$

Step 4. The rest for the case of F-spaces can be proved by complex interpolation

$$\left[F^{s_0,w}_{p,q}, F^{s_1,w}_{p,q}\right]_{\theta} = F^{s,w}_{pq},$$

where $1 < p < \infty$, $1 < q < \infty$, $-\infty < s_0 < s_1 < \infty$, $0 < \theta < 1, s = (1-\theta)s_0 + \theta s_1$, from Theorem 3.11. Let

$$\max\left(\frac{1}{p}, \frac{1}{q}\right) - 1 < s_0 < -\max\left(\frac{n}{p'}(r_{w^{1-p'}} - 1), \sigma_{p'}(w^{1-p'}) - 1\right) < \max\left(\frac{n}{p}(r_w - 1), \sigma_p(w) - 1\right) < s_1 < \min\left(\frac{1}{p}, \frac{1}{q}\right).$$

Now using Step 1 and Step 2 and similar argumentation as for Besov spaces (Step 3) we get the result.

□

Corollary 4.1. *Let $1 < p < \infty$, $w \in \mathcal{A}_\infty^{loc}$ and $r_w < \min\left(p, 1 + \frac{1}{n}\min\left(1, \frac{p}{2}\right)\right)$, $r_{w^{1-p'}} < 1 + \frac{1}{n}\min\left(1, \frac{p'}{2}\right)$ and w, $w^{1-p'}$ satisfy the regularity condition. Then the Haar system is an unconditional basis in $L_p^w(\mathbb{R}^n)$.*

Proof. Let $w \in \mathcal{A}_u^{loc}$ for some $r_w < u \le p$. Then from [34] we have $L_p^w(\mathbb{R}^n) = F_{p,2}^{0,w}(\mathbb{R}^n)$. So by Theorem 4.1 we know that the Haar system is an unconditional basis in $L_p^w(\mathbb{R}^n)$. □

Remark 4.2. We give examples of weights that satisfy the assumptions of Theorem 4.1.

1. If w is an admissible weight or a general locally regular weight then w satisfies the regularity condition as well as $w^{1-p'}$. Moreover $r_w = r_{w^{1-p'}} = 1$. So for example the Haar system is an unconditional basis in $F_{pq}^{s,w}(\mathbb{R}^n)$ if $1 < p < \infty$, $1 < q < \infty$ and $\max\left(\frac{1}{p}, \frac{1}{q}\right) - 1 < s < \min\left(\frac{1}{p}, \frac{1}{q}\right)$. In particular it is a basis in $L_p^w(\mathbb{R}^n)$ for any $1 < p < \infty$.

2. If $v_\beta(x) = \begin{cases} (1 - \log|x|)^{-\beta}, & |x| \le 1, \\ (1 + \log|x|)^{-\beta}, & |x| > 1 \end{cases}$, $\beta \in \mathbb{R}$. Then $r_{v_\beta} = 1$ for any $\beta \in \mathbb{R}$ and v_β satisfies the regularity condition for any β. So the Haar system is and unconditional basis in $F_{pq}^{s,v_\beta}(\mathbb{R}^n)$ and $L_p^{v_\beta}(\mathbb{R}^n)$ with the same conditions on s and p, q as above.

3. If $w_\alpha(x) = |x|^\alpha$, $\alpha > -n$, then $r_{w_\alpha} = 1 + \frac{\max(0,\alpha)}{n}$ and w_α satisfies the regularity condition. In consequence the Haar system is an unconditional basis in $L_p^{w_\alpha}(\mathbb{R}^n)$ if $1 < p < \infty$ and $-\min\left(n, p-1, \frac{p}{2}\right) < \alpha < \min\left(1, \frac{p}{2}, n(p-1)\right)$. Moreover it is an unconditional basis in $F_{p,2}^{s,w_\alpha}(\mathbb{R}^n)$ with the same conditions on α and suitable assumptions on s and p. The situation doesn't change if we perturb w_α by a logarithmic factor v_β and take $w = w_\alpha v_\beta$.

Chapter 5

Wavelet bases in $L_p^w(\mathbb{R}^n)$ spaces

By an inhomogeneous wavelet system we understand the system consisting of integer translations of the scaling function and translations of dilations of wavelet by dyadic factor bigger or equal to 1, i.e. the system

$$\{\psi_{jm}^M\}_{j\geq 0, m\in\mathbb{Z}^n} \cup \{\psi_{0,m}^F\}_{m\in\mathbb{Z}^n}.$$

In contrast to an inhomogeneous wavelet system a homogeneous wavelet system does not contain the scaling function and its translations, but contains dilations of wavelet also by factors smaller then 1. So it is a system of the form

$$\{\psi_{jm}^M\}_{j\in\mathbb{Z}, m\in\mathbb{Z}^n}.$$

Both the homogeneous and inhomogeneous wavelet systems of Daubechies type are the orthonormal basis in the space $L_2(\mathbb{R}^n)$ and unconditional basis in the space $L_p(\mathbb{R}^n)$ $(1 < p < \infty)$. In weighted L_p spaces the behavior of the both types of wavelet systems is not equivalent. We proved that the inhomogeneous wavelet system is an unconditional basis for more general type of weights than the homogeneous one.

The main theorem of this chapter asserts that the inhomogeneous wavelet system of Daubechies type is an unconditional basis in $L_p(d\mu)$ if and only if $d\mu = w\,dx$ with $w \in \mathcal{A}_p^{loc}$. In consequence we can prove that $L_p^w(\mathbb{R}^n)$ spaces can be characterized by a square function if and only if $w \in \mathcal{A}_p^{loc}$. Thus the counterpart of the Paley-Wiener theorem holds if the power of an integration is the same as the index of weight class w belongs to. The similar results for the homogeneous wavelet system was proved by Lemarié-Rieusset [29] and Aimar, Bernardis, Martín-Reyes [1]. It was proved that the homogeneous wavelet system of Daubechies type is an unconditional basis in $L_p(d\mu, \mathbb{R}^n)$ if and only if $d\mu = wdx$ with $w \in \mathcal{A}_p$. Thus if $w \in \mathcal{A}_p^{loc} \backslash \mathcal{A}_p$ then there is a wavelet such that the corresponding inhomogeneous wavelet system is an unconditional basis in $L_p^w(\mathbb{R}^n)$ but the corresponding homogeneous system is not.

5.1 L_p spaces with local Muckenhoupt weights

In this section we prove our main result. We use the fact that wavelet projection operators satisfy condition (5.1.1) below. We follow the main idea of Aimar, Bernardis, Martín-Reyes in [1]. On the other side we have the wavelet characterization theorem stated in Theorem 3.4.

Lemma 5.1. *Let φ be a continuous function absolutely bounded by an $L_1(\mathbb{R}^n)$ radial decreasing function such that $\sum_{k\in\mathbb{Z}^n}\varphi(x-k)\neq 0$ for all $x\in\mathbb{R}^n$. Then $F(x,y)=\sum_{k\in\mathbb{Z}^n}\varphi(x-k)\overline{\varphi(y-k)}$ satisfies*

$$\{(x,y)\in\mathbb{R}^{2n}:|x-y|<\ell\}\subset\{(x,y)\in\mathbb{R}^{2n}:F(x,y)>\delta\},$$

for some positive real numbers ℓ and δ.

A proof of the above lemma can be found in [1]. Following [1] we can find that lemma applies to $P_0(x,y)=\sum_{k\in\mathbb{Z}^n}\Psi^{G_F}(x-k)\Psi^{G_F}(y-k)$ with $G_F=(F,...,F)$, where Ψ^{G_F} is a Daubechies scaling function. By the properties of the multiresolution analysis $\{\Psi^{G_F}(x-k)\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis in $V_0=\overline{\text{span}}\{\Psi^{G_F}(x-k)\}_{k\in\mathbb{Z}^n}$ and the sum $\sum_{k\in\mathbb{Z}^n}\Psi^{G_F}(x-k)$ equals to the constant with the module 1 a.e. On the other hand the function Ψ^{G_F} is a continuous function with compact support therefore it is bounded by a radial decreasing function belonging to $L_1(\mathbb{R}^n)$ and series $\sum_{k\in\mathbb{Z}^n}\Psi^{G_F}(x-k)$ is convergent to a continuous function. In consequence $\sum_{k\in\mathbb{Z}^n}\Psi^{G_F}(x-k)\neq 0$ for any $x\in\mathbb{R}^n$.

Now for a family $\{P_j(x,y)\}_{j\geq 0}=\{2^{jn}P_0(2^jx,2^jy)\}_{j\geq 0}$ we obtain that it satisfies conditions

$$\left\{(x,y)\in\mathbb{R}^{2n}:|x-y|<\ell_j\right\}\subset\left\{(x,y)\in\mathbb{R}^{2n}:P_j(x,y)>C\ell_{j+1}^{-n}\right\} \tag{5.1.1}$$

for every $j\geq 0$ and a positive constant $C>0$, where $\{\ell_j\}_{j\geq 0}$ is a decreasing sequence of positive real numbers and $\ell_j\to 0$ as $j\to\infty$.

Theorem 5.1. *Let $1<p<\infty$ and μ be a positive Borel measure on $\mathbb{R}^n$ finite on compact sets. Let $k\geq\max([n-\frac{n}{p}]+1,n(p-1))$. There exists an unconditional Daubechies wavelet basis in $L_p(\mathbb{R}^n,d\mu)$ with smoothness k if and only if $d\mu=w(x)\,dx$ with $w\in\mathcal{A}_p^{loc}$.*

Proof. Let $w\in\mathcal{A}_p^{loc}$. From Theorem 3.4 we have an unconditional basis in $F^{s,w}_{pq}(\mathbb{R}^n)$. In [34] Rychkov shows the Littlewood-Paley characterization of spaces with local Muckenhoupt weights, it means $F^{0,w}_{p,2}(\mathbb{R}^n)=L^w_p(\mathbb{R}^n)$. Hence we have an unconditional basis in $L^w_p(\mathbb{R}^n)$, cf Theorem 3.4.

On the other side. Let $\{\Psi^G_{jm}:j\in\mathbb{N}_0,m\in\mathbb{Z}^n,G\in G^j\}$ be a Daubechies wavelet system, which is an unconditional basis in $L_p(\mathbb{R}^n,d\mu)$. So every $f\in L_p(\mathbb{R}^n,d\mu)$ has the representation

$$f(x)=\sum_{j,k,G}\langle f,\Psi^G_{jk}\rangle\Psi^G_{jk}(x).$$

Operators

$$\widetilde{P}_0 f = \sum_k \langle f, \Psi_{0,k}^{G_F} \rangle \Psi_{0,k}^{G_F},$$
$$\widetilde{P}_m f = \sum_{0\le j<m,k,G} \langle f, \Psi_{jk}^{G} \rangle \Psi_{jk}^{G}, \quad m>0$$

are uniformly bounded on $L_p(\mathbb{R}^n, d\mu)$. We can write $\widetilde{P}_m f(x) = \int_{\mathbb{R}^n} \widetilde{P}_m(x,y)f(y)dy$, where $\widetilde{P}_0(x,y) = \sum_k \Psi_{0,k}^{G_F}(x)\Psi_{0,k}^{G_F}(y)$ and $\widetilde{P}_m(x,y) = \sum_{0\le j<m,k,G} \Psi_{jk}^{G}(x)\Psi_{jk}^{G}(y)$ if $m>0$, because wavelets have compact supports and we deal with locally finite sums. Hence kernels $\widetilde{P}_m(x,y)$ are bounded.

On the other hand by the properties of the multiresolution analysis the kernel $\widetilde{P}_m(x,y)$ coincides with $P_m(x,y) = 2^{mn}P_0(2^m x, 2^m y)$, if it is regarded as a kernel of projection in L_2. Again, because wavelets and the scaling function are compactly supported we deal with locally finite sums and get $P_m(x,y) = \widetilde{P}_m(x,y)$ almost everywhere. So the last equality is valid also for L_p spaces and the kernels $P_m(x,y)$ are bounded.

We are going to show that μ is absolutely continuous. Let E be a set such that $|E|=0$. For every $\varepsilon>0$ there exists an open set F such that $E\subset F$ and $\mu(F\backslash E)<\varepsilon$. Set F can be decomposed into a countable union of disjoint and dyadic cubes Q_i. From Lemma 5.1 we get that $\{P_j(x,y)\}_{j\ge 0}$ satisfies the same conditions as weakly positive family. Let $\{\ell_j\}_{j\ge0}$ be a sequence connected with that family. Without lost of generality we assume that $\ell_0 > \max(d(Q_i))$, where $d(Q)$ denote a diameter of Q and maximum is taken over all cubes Q_i from decomposition of F. For fixed i let $j_0\ge 0$ be the integer such that $\ell_{j_0+1}\le d(Q_i)<\ell_{j_0}$. If $x,y\in Q_i$ we get $|x-y|<\ell_{j_0}$ and $P_{j_0}(x,y) > C\ell_{j_0+1}^{-n}$. So for every $x\in Q_i$ we have

$$\left|P_{j_0}(\chi_{Q_i\backslash E})(x)\right| = \left|\int_{Q_i\backslash E} P_{j_0}(x,y)dy\right| > C\ell_{j_0+1}^{-n}|Q_i\backslash E|.$$

Therefore $\left|P_{j_0}(\chi_{Q_i\backslash E})(x)\right| > c_n$, for some constant depending only on C and n. From weak type inequality for operators P_j we get

$$\mu(Q_i) \le \mu\left(\{x : |P_{j_0}(\chi_{Q_i\backslash E})(x)| > c_n\}\right) \le Cc_n^{-p}\mu(Q_i\backslash E).$$

Summing over i we have

$$\mu(F) = \sum_i \mu(Q_i) \le Cc_n^{-p}\sum_i \mu(Q_i\backslash E) = Cc_n^{-p}\mu(F\backslash E) < Cc_n^{-p}\varepsilon$$

for every $\varepsilon>0$. Hence $\mu(E)=0$. From Radon-Nikodym Theorem we get that there exists locally integrable function w such that $d\mu = w(x)\,dx$.

Now we can show that $w\in\mathcal{A}_p^{loc}$. We pick a sequence $\{\ell_j\}_{j\ge0}$. Let $Q\subset\mathbb{R}^n$ be a cube with $|Q|\le\ell_0$. We can find $m_0\ge0$ with $\ell_{m_0+1}\le d(Q)<\ell_{m_0}$. Inequalities

$$|P_{m_0}(\sigma_\varepsilon\chi_Q)(x)| = \left|\int_Q P_{m_0}(x,y)\sigma_\varepsilon(y)dy\right| > C\ell_{m_0+1}^{-n}\int_Q\sigma_\varepsilon \ge c_n\,|Q|^{-1}\int_Q \sigma_\varepsilon \equiv \lambda,$$

where $\sigma_\varepsilon = (w+\varepsilon)^{-\frac{1}{p-1}}, \varepsilon > 0$, holds for every $x \in Q$. Since operators P_m are of weak type (p,p) we get

$$w(Q) \le w\left(\{x : |P_{m_0}(\sigma_\varepsilon \chi_Q)(x)| > \lambda\}\right) \le C c_n^{-p} |Q|^p \left(\int_Q \sigma_\varepsilon\right)^{-p} \int_Q \sigma_\varepsilon^p w.$$

Multiplying both sides by $\left(\int_Q \sigma_\varepsilon\right)^p \left(\int_Q \sigma_\varepsilon^p w\right)^{-1}$ and choosing ε close to zero we get

$$w(Q) \left(\int_Q w^{-\frac{1}{p-1}}\right)^p \left(\int_Q w^{-\frac{1}{p-1}}\right)^{-1} \le C\,|Q|^p$$

for every Q, $|Q| < \ell_0$. From Lemma 1.4 in [34] we know that classes $\mathcal{A}_p^{loc}$ are independent of the upper bound for the cube size in their definition. So we get a condition for $\mathcal{A}_p^{loc}$.

□

Following [34] we can state square-function characterization. Let us define

$$S(f)(x) = \left(\sum_j |\varphi_j * f(x)|^2\right)^{1/2},$$

where $\varphi_0 \in \mathcal{D}$ have nonzero integral and $\varphi = \varphi_0 - 2^{-n}\varphi_0(\frac{\cdot}{2})$ and $\varphi_j(x) = 2^{jn}\varphi(2^j x)$, $j > 0$.

Corollary 5.1. *Let $1 < p < \infty$ and $w \in \mathcal{A}_\infty^{loc}$. The following equivalence holds*

$$\left\|S(f)|L_p^w(\mathbb{R}^n)\right\| \sim \left\|f|L_p^w(\mathbb{R}^n)\right\|$$

if and only if $w \in \mathcal{A}_p^{loc}$.

Proof. Let $w \in \mathcal{A}_p^{loc}$. From [34] we have that $F_{p,2}^{0,w}(\mathbb{R}^n) = L_p^w(\mathbb{R}^n)$ with norm equivalence

$$\int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} |\varphi_j * f(x)|^2\right)^{p/2} w(x)\,dx \sim \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx.$$

Conversely, if we assume $w \in \mathcal{A}_\infty^{loc}$ then from Theorem 3.4 we get that in $F_{p,2}^{0,w}(\mathbb{R}^n)$ there exists an unconditional basis consisting of Daubechies wavelets. But $\|S(f)|L_p^w(\mathbb{R}^n)\|$ is a norm in $F_{p,2}^{0,w}(\mathbb{R}^n)$. So the norms equivalence implies that $F_{p,2}^{0,w}(\mathbb{R}^n)$ and $L_p^w(\mathbb{R}^n)$ are isomorphic. In consequence there is an unconditional basis by Daubechies wavelet in $L_p(\mathbb{R}^n)$. Now from Theorem 5.1 we obtain that $w \in \mathcal{A}_p^{loc}$.

□

Remark 5.1. It is known that above statements are not true for general Muckenhoupt weights. Taking for example weight

$$w(x) = \begin{cases} |x|^\alpha & \text{for } |x| \le 1, \\ |x|^\beta & \text{for } |x| > 1, \end{cases}$$

for $\alpha, \beta > -n$. For $\alpha < (p_1-1)n$ we have $w \in \mathcal{A}_{p_1}^{loc}$ and $r_w = \frac{\max(0,\alpha)}{n}+1$, for $\alpha, \beta < (p_2-1)n$ we have $w \in \mathcal{A}_{p_2}$ and $\widetilde{r}_w = \frac{\max(0,\alpha,\beta)}{n}+1$. Taking β big enough we get that w is in $\mathcal{A}_p^{loc} \cap \mathcal{A}_\infty$, but not in $\mathcal{A}_p$.

For $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ with $\int \varphi_0(x)\,dx \neq 0$ and $f \in \mathcal{S}'_e$ we introduce the "vertical" maximal function

$$\varphi_0^+ f(x) = \sup_{j\in\mathbb{N}} |(\varphi_0)_j * f(x)|.$$

The following corollary follows from Theorem 2.25 in [34] and Corollary 5.1.

Corollary 5.2. *Let $1 < p < \infty$ and $w \in \mathcal{A}_\infty^{loc}$. The following equivalence holds*

$$||\varphi_0^+ f|L_p^w|| \sim ||f|L_p^w||$$

if and only if $w \in \mathcal{A}_p^{loc}$.

Please note that it follows from the last corollary that if $1 < p < \infty$ and $w \in \mathcal{A}_\infty^{loc}$ then the weighted local Hardy space

$$h_p^w = \{f \in \mathcal{S}'_e : ||\varphi_0^+ f|| < \infty\}$$

coincides with L_p^w if and only if $w \in \mathcal{A}_p^{loc}$.

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