On tame operators between non-archimedean power series spaces

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Abstract. Let $p \in \{1, \infty\}$. We show that any continuous linear operator T from $A_1(a)$ to $A_p(b)$ is tame i.e. there exists a positive integer c such that $\sup_x \|Tx\|_k / |x|_{ck} < \infty$ for every $k \in \mathbb{N}$. Next we prove that a similar result holds for operators from $A_\infty(a)$ to $A_p(b)$ if and only if the set $M_{b,a}$ of all finite limit points of the double sequence $(b_j/a_i)_{i,j \in \mathbb{N}}$ is bounded. Finally we show that the range of every tame operator from $A_\infty(a)$ to $A_\infty(b)$ has a Schauder basis.

1 Introduction

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [6] - [9] and [12].

Let Γ be the family of all non-decreasing unbounded sequences of positive real numbers. Let $a = (a_n), b = (b_n) \in \Gamma$. The power series spaces of finite type $A_1(a)$ and infinite type $A_\infty(b)$ are the most known and important examples of nuclear Fréchet spaces with a Schauder basis. They were studied in [1] and [13] - [15]. Let $p, q \in \{1, \infty\}$.

The problem when $A_p(a)$ has a subspace (or quotient) isomorphic to $A_q(b)$ was studied in [13]. In particular, the spaces $A_p(a)$ and $A_q(b)$ are isomorphic if and only if $p = q$ and the sequences a, b are equivalent i.e. $0 < \inf_n(a_n/b_n) \le \sup_n(a_n/b_n) < \infty$ ([13], Corollary 6).

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N. De Grande-De Kimpe has proved ([1], Proposition 4.3) that any continuous linear operator from $A_1(a)$ to $A_\infty(b)$ is compactoid (the assumption that the field $\mathbb{K}$ is spherically complete can be easy omitted). Hence $A_1(a)$ has no quotient isomorphic to $A_\infty(b)$, and $A_\infty(b)$ has no subspace isomorphic to $A_1(a)$.

In [14], we have proved that the range of every continuous linear operator from $A_1(a)$ to $A_p(b)$ has a Schauder basis ([14], Theorem 10); a similar result holds for continuous linear operators from $A_\infty(a)$ to $A_p(b)$, if the set $M_{b,a}$ of all finite limit points of the double sequence $(b_i/a_j)_{i,j\in\mathbb{N}}$ is bounded ([14], Theorem 10). In particular, any complemented subspace F of $A_1(a)$ has a Schauder basis ([14], Corollary 13); in fact, F is isomorphic to $A_1(c)$ for some subsequence c of a ([14], Proposition 14). Similar results hold for complemented subspaces of $A_\infty(a)$, if the set $M_{a,a}$ is bounded ([14], Corollary 13 and Proposition 14).

It is not known whether the range of every continuous linear operator from $A_\infty(a)$ to $A_\infty(b)$ has a Schauder basis.

Let E and F be Fréchet spaces with fixed bases of continuous seminorms $(|\cdot|_k)$ and $(\|\cdot\|_k)$, respectively. A continuous linear operator $T : E \to F$ is *tame* (or *linearly tame*) if there exists a positive integer c such that

$$\sup_x \|Tx\|_k/|x|_{ck} < \infty \text{ for all } k \in \mathbb{N};$$

clearly, any bounded linear operator from E to F is tame. The pair (E, F) is *tame* if every continuous linear operator from E to F is tame. The space E is *tame* if the pair (E, E) is tame.

In this paper we study tame operators from $A_p(a)$ to $A_q(b)$ (and from $A_p(a, r)$ to $A_q(b, s)$). First we show that the pair $(A_1(a), A_p(b))$ is tame for all $a, b \in \Gamma$ and $p \in \{1, \infty\}$ (Theorem 1); in particular, the space $A_1(a)$ is tame for every $a \in \Gamma$.

On the other hand, if $a \in \Gamma$ with $M_{a,a} \neq \{0, 1\}$ and $r = (r_k) \subset \mathbb{R}$ is a strictly increasing sequence with $\lim_k r_k = 0$ and $\lim_k(r_{2k}/r_k) = 1$ then the space $A_1(a, r)$ is not tame (Theorem 4).

Next, using the Grothendieck's factorization theorem (Theorem 7), we prove that the pair $(A_\infty(a), A_p(b))$ is tame if and only if the set $M_{b,a}$ is bounded (Theorem 9).

Finally we show that the range of every tame operator from $A_\infty(a)$ to $A_\infty(b)$ has a Schauder basis (Theorem 11).

In our paper we use and develop some ideas of [2] and [5].

2 Preliminaries

The linear span of a subset A of a linear space E is denoted by $[A]$.

By a *seminorm* on a linear space E we mean a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}$, $x \in E$ and $p(x + y) \leq max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\ker p := \{x \in E : p(x) = 0\} = \{0\}$.

Let E, F be locally convex spaces. A map $T : E \to F$ is called an isomorphism if it is linear, bijective and the maps T, T^{-1} are continuous. If there exists an isomorphism $T : E \to F$, then we say that E is isomorphic to F. The family of all continuous linear maps from E to F we denote by $L(E, F)$. An operator $T \in L(E, F)$ is *bounded* if the range of some neighbourhood of zero in E is bounded in F. The *range* of $T \in L(E, F)$ is the subspace $T(E)$ of F.

The set of all continuous seminorms on a lcs E is denoted by $\mathcal{P}(E)$. A non-decreasing sequence (p_n) of continuous seminorms on a metrizable lcs E is a *base* in $\mathcal{P}(E)$ if for every $p \in \mathcal{P}(E)$ there are $C > 0$ and $k \in \mathbb{N}$ such that $p \leq C p_k$. A metrizable complete lcs is called a *Fréchet space*.

Let (x_n) be a sequence in a Fréchet space E. The series $\sum_{n=1}^{\infty} x_n$ is convergent in E if and only if $\lim_n x_n = 0$.

A normable Fréchet space is a *Banach space*.

Put $B_{\mathbb{K}} = \{\alpha \in \mathbb{K} : |\alpha| \leq 1\}$. Let A be a subset of a lcs E. The set $\text{co}A = \{\sum_{i=1}^{n} \alpha_i a_i : n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in B_{\mathbb{K}}, a_1, \ldots, a_n \in A\}$ is the *absolutely convex hull* of A; its closure in E is denoted by $\overline{\text{co}}^E A$. A subset A of a lcs E is *absolutely convex* if $\text{co}A = A$.

A subset B of a lcs E is *compactoid* (or a *compactoid*) if for each neighbourhood U of 0 in E there exists a finite subset A of E such that $B \subset U + \text{co}A$.

An operator $T \in L(E, F)$ is *compactoid* if for some neighbourhood U of zero in E the set $T(U)$ is compactoid in F; clearly, any compactoid operator is bounded.

For any seminorm p on a lcs E the map $\overline{p} : E/\ker p \to [0, \infty)\, x + \ker p \to p(x)$ is a norm on $E_p = E/\ker p$.

A lcs E is nuclear if for every $p \in \mathcal{P}(E)$ there exists $q \in \mathcal{P}(E)$ with $q \geq p$ such that the map

$$\varphi_{q,p} : (E_q, \overline{q}) \to (E_p, \overline{p})\,, x + \ker q \to x + \ker p$$

is compactoid. Any nuclear Fréchet space E is a *Fréchet-Montel space* i.e. every bounded subset of E is compactoid.

Let U be an absolutely convex neighbourhood of zero in a lcs E. The Minkowski functional of U

$$p_U : E \to [0,\infty), p_U(x) = \inf\{|\alpha| : \alpha \in \mathbb{K} \text{ and } x \in \alpha U\}$$

is a continuous seminorm on E.

A sequence (x_n) in an lcs E is a *Schauder basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$, and the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n (n \in \mathbb{N})$ are continuous.

An infinite matrix $A = (a_{n,k})$ of real numbers is a *Köthe matrix* if $0 \leq a_{n,k} \leq a_{n,k+1}$ for all $n, k \in \mathbb{N}$, and $\sup_k a_{n,k} > 0$ for $n \in \mathbb{N}$. Let A be a Köthe matrix.

The space $K(A) = \{x = (x_n) \in \mathbb{K}^{\mathbb{N}} : \lim_n |x_n| a_{n,k} = 0 \text{ for every } k \in \mathbb{N}\}$ with the canonical base $(|\cdot|_k)$ of seminorms, where

$$|x|_k = \max_n |x_n| a_{n,k}, k \in \mathbb{N},$$

is a Fréchet space. The sequence (e_j), where $e_j = (\delta_{j,n})$, is an unconditional Schauder basis in $K(A)$. It is orthogonal with respect to the canonical base $(|\cdot|_k)$ of seminorms i.e. for all $k, n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ we have

$$|\sum_{i=1}^{n} \alpha_i e_i|_k = \max_{1 \leq i \leq n} |\alpha_i e_i|_k.$$

Any infinite-dimensional Fréchet space E with a Schauder basis is isomorphic to $K(A)$ for some Köthe matrix (see [1], Proposition 2.4 and its proof).

By a Köthe space we mean a Fréchet space with a Schauder basis and with a continuous norm. Any Köthe space is isomorphic to $K(A)$ for some Köthe matrix with $a_{n,k} > 0$ for all $n, k \in \mathbb{N}$ (see [1], Proposition 2.4). Let $E = K(A)$ be a Köthe space. For any continuous linear functional f on E there exists a sequence $(z_j) \subset \mathbb{K}$ such that $f(x) = \sum_{n=1}^{\infty} x_n z_n$ for any $x \in E$ and $\sup_n(|z_n|/a_{n,k}) < \infty$ for some $k \in \mathbb{N}$ ([1], Proposition 2.2). Then $|f|_k^* := \sup_x(|f(x)|/|x|_k) = \sup_n(|z_n|/a_{n,k})$ for $k \in \mathbb{N}$.

Let $a = (a_n) \in \Gamma$. Then the following Köthe spaces are nuclear (see [1]):

1. $A_1(a) = K(A)$ with $A = (a_{n,k}), a_{n,k} = e^{-a_n/k}$;

2. $A_\infty(a) = K(A)$ with $A = (a_{n,k}), a_{n,k} = e^{k a_n}$.

$A_1(a)$ and $A_\infty(a)$ are the *power series spaces* (of *finite type* and *infinite type*, respectively).

Let $p \in \{1, \infty\}$. Denote by Λ_p the family of all strictly increasing sequences $r = (r_k)$ of real numbers such that $\lim_k r_k = 0$ if $p = 0$ and $\lim_k r_k = \infty$ if $p = \infty$. Let $a \in \Gamma$ and $r \in \Lambda_p$. Clearly, the Köthe space $A_p(a, r) = K(A)$ with $A = (a_{n,k}), a_{n,k} = e^{r_k a_n}$ is isomorphic to $A_p(a)$.

Let $(E, \|\cdot\|)$ be a normed space and let $t \in (0, 1]$. A sequence $(x_n) \subset E$ is *t-orthogonal* if for all $m \in \mathbb{N}, \alpha_1, \ldots, \alpha_m \in \mathbb{K}$ we have

$$\|\sum_{i=1}^{m} \alpha_i x_i\| \geq t \max_{1 \leq i \leq m} \|\alpha_i x_i\|.$$

If $(x_n) \subset (E \setminus \{0\})$ is t-orthogonal and linearly dense in E then it is *t-orthogonal basis* in E. Every t-orthogonal basis in E is a Schauder basis ([7], [8]).

3 Results

First we shall prove that the pair $(A_1(a), A_p(b))$ is tame for all $a, b \in \Gamma$ and $p = 1$; for $p = \infty$ it follows by [1], Proposition 4.3.

Theorem 1. *Let $a, b \in \Gamma$. If $r = (r_k), s = (s_k) \in \Lambda_1$ with $\inf_{c \geq 1} \limsup_k (r_{ck}/s_k) = 0$, then the pair $(A_1(a, r), A_1(b, s))$ is tame. If $r \in \Lambda_1$ and $s \in \Lambda_\infty$, then the pair $(A_1(a, r), A_\infty(b, s))$ is tame. In particular, the pair $(A_1(a), A_p(b))$ is tame for any $p \in \{1, \infty\}$.*

Proof. (1) Let $r = (r_k), s = (s_k) \in \Lambda_1$ with $\inf_{c \geq 1} \limsup_k (r_{ck}/s_k) = 0$. Denote by $(|\cdot|_k)$ and $(\|\cdot\|_k)$ the canonical bases in $\mathcal{P}(A_1(a, r))$ and $\mathcal{P}(A_1(b, s))$, respectively. Let $T \in L(A_1(a, r), A_1(b, s))$. Then there exist increasing functions $C, \varphi : \mathbb{N} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N} \forall x \in A_1(a) : \|Tx\|_k \leq C(k)|x|_{\varphi(k)}.$$

Let $(t_{n,j}) \subset \mathbb{K}$ with $Te_n = \sum_{j=1}^{\infty} t_{n,j} e_j, n \in \mathbb{N}$. For some function $p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ we have $\|Te_n\|_k = |t_{n,p(n,k)}| \exp(s_k b_{p(n,k)})$ for $n, k \in \mathbb{N}$. Then for $k, l, n \in \mathbb{N}$ we have

$$\frac{\|Te_n\|_l}{\|Te_n\|_k} \geq \frac{|t_{n,p(n,k)}| \exp(s_l b_{p(n,k)})}{|t_{n,p(n,k)}| \exp(s_k b_{p(n,k)})} = \exp[(s_l - s_k) b_{p(n,k)}].$$

Hence for all $c, l, n, k \in \mathbb{N}$ with $[(s_l - s_k) b_{p(n,k)} + (r_{ck} - r_{\varphi(l)}) a_n] \geq 0$ we have

$$\frac{\|Te_n\|_l}{\|Te_n\|_k} \frac{|e_n|_{ck}}{|e_n|_{\varphi(l)}} \geq 1, \text{ so } (*) \ \frac{\|Te_n\|_k}{|e_n|_{ck}} \leq \frac{\|Te_n\|_l}{|e_n|_{\varphi(l)}} \leq C(l).$$

Now we shall prove that there exist $A > 0, K \geq 1$ and $c \geq \varphi(K)$ such that

$$(**)\ \forall k \geq K \exists l_k > k : \frac{s_{l_k} - s_k}{r_{\varphi(l_k)} - r_{ck}} > A > \frac{s_k - s_1}{r_{ck} - r_{\varphi(1)}}.$$

Put $A = 2s_1/r_{\varphi(1)}$. Clearly $\lim_k[(s_k - s_1)/(r_{ck} - r_{\varphi(1)})] = A/2$ for $c \in \mathbb{N}$. By our assumption, for some $c_0 \geq 1$ we have $\limsup_k(r_{c_0 k}/s_k) < A^{-1}$. Since the sequences $(r_{ck}/s_k)_{c=1}^{\infty}$ and $[(s_k - s_1)/(r_{ck} - r_{\varphi(1)})]_{c=1}^{\infty}$ are decreasing for every $k > \varphi(1)$, we have

$$\exists k_0 > \varphi(1) \forall k \geq k_0 \forall c \geq c_0 : \frac{r_{ck}}{s_k} < \frac{1}{A}$$

and

$$\exists K \geq k_0 \forall k \geq K \forall c \geq c_0 : \frac{s_k - s_1}{r_{ck} - r_{\varphi(1)}} < A.$$

Let $c \geq \max\{c_0, \varphi(K)\}$. Clearly $\lim_l[(s_l - s_k)/(r_{\varphi(l)} - r_{ck})] = s_k/r_{ck} > A$ for $k \geq K$, so we get

$$\forall k \geq K \exists l_k > k : \frac{s_{l_k} - s_k}{r_{\varphi(l_k)} - r_{ck}} > A.$$

Thus we have shown $(**)$.

Clearly $\|Tx\|_k \leq C(k)|x|_{ck}$ for $x \in A_1(a, r)$ and $1 \leq k < K$.
Let $k \geq K$. Let $n \in \mathbb{N}$. Consider two cases.

Case 1: $b_{p(n,k)} \leq a_n/A$. Then

$$(s_k - s_1)b_{p(n,k)} \leq (r_{ck} - r_{\varphi(1)})Ab_{p(n,k)} \leq (r_{ck} - r_{\varphi(1)})a_n,$$

so $[(s_1 - s_k)b_{p(n,k)} + (r_{ck} - r_{\varphi(1)})a_n] \geq 0$. Using $(*)$ we get $\|Te_n\|_k \leq C(1)|e_n|_{ck}$.

Case 2: $b_{p(n,k)} > a_n/A$. Then

$$(s_{l_k} - s_k)b_{p(n,k)} \geq (r_{\varphi(l_k)} - r_{ck})Ab_{p(n,k)} > (r_{\varphi(l_k)} - r_{ck})a_n,$$

so $[(s_{l_k} - s_k)b_{p(n,k)} + (r_{ck} - r_{\varphi(l_k)})a_n] \geq 0$. Using $(*)$ we get $\|Te_n\|_k \leq C(l_k)|e_n|_{ck}$.

We have shown that $\|Te_n\|_k \leq C(l_k)|e_n|_{ck}$ for all $n \in \mathbb{N}$. It follows that $\|Tx\|_k \leq C(l_k)|x|_{ck}$ for every $x \in A_1(a, r)$ and $k \geq K$. Thus we have proved that T is tame.

(2) Let $r \in \Lambda_1$ and $s \in \Lambda_\infty$. Then every continuous linear operator T from $A_1(a, r)$ to $A_\infty(b, s)$ is bounded ([1], Proposition 4.3), so

$$\exists m \in \mathbb{N}\, \forall k \in \mathbb{N} \exists C_k > 0\, \forall x \in A_1(a) : \|Tx\|_k \leq C_k|x|_m,$$

where $(|\cdot|_k)$ and $(\|\cdot\|_k)$ are the canonical bases in $\mathcal{P}(A_1(a, r))$ and $\mathcal{P}(A_\infty(b, s))$, respectively. It follows that the pair $(A_1(a, r), A_\infty(b, s))$ is tame. □

Corollary 2. *The space $A_1(a)$ is tame for every $a \in \Gamma$.*

In connection with Corollary 2 we shall prove that for some $a \in \Gamma, r \in \Lambda_1$ the space $A_1(a,r)$ is not tame. We need the following lemma.

Lemma 3. *Let $p \in \{1,\infty\}$. For every strictly increasing sequence $(\psi_k) \subset \mathbb{N}$ there exists $r = (r_k) \in \Lambda_p$ with $\lim_k (r_{\psi_k}/r_k) = 1$.*

Proof. First we shall prove that there exists a sequence $(w_i) \subset (0,\infty)$ with $\sum_{i=1}^{\infty} w_i = \infty$ such that $\lim_k \sum_{i=k}^{\psi_k} w_i = 0$.

Let $v_1, ..., v_{\psi_1} \in (0,\infty)$. If we have v_k for some $k \in \mathbb{N}$ we choose $v_i \in (0,\infty)$ for $\psi_k < i \leq \psi_{k+1}$ such that $\sum_{i=\psi_k+1}^{\psi_{k+1}} v_i = v_k$. This way we obtain a sequence $(v_i) \subset (0,\infty)$ such that the sequence $V_k = \sum_{i=k}^{\psi_k} v_i, k \in \mathbb{N}$, is constant, since $V_{k+1} - V_k = (\sum_{i=\psi_k+1}^{\psi_{k+1}} v_i) - v_k = 0, k \in \mathbb{N}$. It follows that $\sum_{i=1}^{\infty} v_i = \infty$. Thus there exists a strictly increasing sequence $(n_l) \subset \mathbb{N}$ with $\sum_{i=\psi_{n_l}+1}^{\psi_{n_{l+1}}} v_i \geq l$ for $l \in \mathbb{N}$.
Let $w_i = v_i$ for $1 \leq i \leq \psi_{n_1}$ and $w_i = v_i/l$ for $\psi_{n_l} < i \leq \psi_{n_{l+1}}, l \in \mathbb{N}$. The series $\sum_{i=1}^{\infty} w_i$ is disconvergent, since $\sum_{i=\psi_{n_l}+1}^{\psi_{n_{l+1}}} w_i \geq 1$. The sequence $W_k = \sum_{i=k}^{\psi_k} w_i, k \in \mathbb{N}$ is convergent to 0. Indeed, for $l \in \mathbb{N}$ and $k > \psi_{n_l}$ we have $lW_k \leq \sum_{i=k}^{\psi_k} v_i = V_k = V_1$.

Put $s_k = \sum_{i=1}^{k} w_i, r_k = -\exp(-s_k)$ and $R_k = \exp s_k$ for $k \in \mathbb{N}$. Clearly $r = (r_k) \in \Lambda_1$ and $R = (R_k) \in \Lambda_\infty$. For $k \in \mathbb{N}$ we have

$$1 \leq r_k/r_{\psi_k} = R_{\psi_k}/R_k = \exp(s_{\psi_k} - s_k) < \exp W_k,$$

so $1 = \lim_k (R_{\psi_k}/R_k) = \lim_k (r_k/r_{\psi_k}) = \lim_k (r_{\psi_k}/r_k)$. $\square$

Let E and F be Fréchet spaces with fixed bases of continuous seminorms $(|\cdot|_k)$ and $(\|\cdot\|_k)$, respectively. A continuous linear operator $T : E \to F$ is *polynomially tame* if there exist positive integers c and n such that

$$\sup_x \|Tx\|_k/|x|_{ck^n} < \infty \text{ for all } k \in \mathbb{N}.$$

The pair (E,F) is *polynomially tame* if every continuous linear operator from E to F is polynomially tame. The space E is *polynomially tame* if the pair (E,E) is polynomially tame.

Theorem 4. *Let $p \in \{1,\infty\}$. Let $a \in \Gamma$ and $r \in \Lambda_p$. Assume that $M_{a,a} \neq \{0,1\}$ and $\lim_k (r_{2k}/r_k) = 1$. Then the space $A_p(a,r)$ is not tame. If $\lim_k (r_{2k^2}/r_k) = 1$, then $A_p(a,r)$ is not polynomially tame.*

Proof. Since $M_{a,a} \neq \{0,1\}$, there exist strictly increasing sequences $(i_v),(j_v) \subset \mathbb{N}$ such that (1) $A := \inf_v(a_{j_v}/a_{i_v}) > 0$ and $B := \sup_v(a_{j_v}/a_{i_v}) < 1$, if $p = 1$; (2) $A := \sup_v(a_{j_v}/a_{i_v}) < \infty$ and $B := \inf_v(a_{j_v}/a_{i_v}) > 1$, if $p = \infty$. For some $(\varphi_k) \subset \mathbb{N}$ we have (1) $\sup_k(r_{\varphi_k}/r_k) \le A$, if $p = 1$; (2) $\inf_k(r_{\varphi_k}/r_k) \ge A$, if $p = \infty$.

The operator

$$T : A_p(a,r) \to A_p(a,r), Tx = \sum_{v=1}^{\infty} x_{i_v} e_{j_v}$$

is well defined, linear and continuous. Indeed, let $x \in A_p(a,r)$. Then

$$|x_{i_v}|\|e_{j_v}\|_k = |x_{i_v}|\exp(r_k a_{j_v}) \le |x_{i_v}|\exp(A r_k a_{i_v}) \le |x_{i_v}|\exp(r_{\varphi_k} a_{i_v})$$

for all $v,k \in \mathbb{N}$. Thus $\lim_v x_{i_v} e_{j_v} = 0$ in $A_p(a,r)$ and $\|Tx\|_k \le \|x\|_{\varphi_k}$ for all $k \in \mathbb{N}$.

Now we shall prove that T is not tame. Suppose by contrary that T is tame. Then there exist $c \ge 1$ and $(C_k) \subset \mathbb{N}$ such that $\|Te_i\|_k \le C_k\|e_i\|_{ck}$ for all $k,i \in \mathbb{N}$. Hence $\exp(r_{ck}a_{i_v} - r_k a_{j_v}) \ge C_k^{-1}$ for all $v,k \in \mathbb{N}$.

By our assumptions we get $\lim_k(r_{2^t k}/r_k) = 1$ for any $t \in \mathbb{N}$, so $\lim_k(r_{ck}/r_k) = 1$.

Case 1: $p = 1$. Let $\delta \in (B,1)$. Then there exists $k_0 \in \mathbb{N}$ such that $(r_{ck}/r_k) \ge \delta > B \ge (a_{j_v}/a_{i_v})$ for all $v,k \in \mathbb{N}$ with $k \ge k_0$. Let $k \ge k_0$. Thus $r_{ck}a_{i_v} - r_k a_{j_v} \le [1-(B/\delta)]r_{ck}a_{i_v}$ for all $v \in \mathbb{N}$.

Case 2: $p = \infty$. Let $\delta \in (1,B)$. Then there exists $k_0 \in \mathbb{N}$ such that $r_{ck}/r_k \le B/\delta < B \le a_{j_v}/a_{i_v}$ for all $v,k \in \mathbb{N}$ with $k \ge k_0$. Let $k \ge k_0$. Thus $r_{ck}a_{i_v} - r_k a_{j_v} \le (1-\delta)r_{ck}a_{i_v}$ for all $v \in \mathbb{N}$.

It follows that $\lim_v \exp(r_{ck}a_{i_v} - r_k a_{j_v}) = 0$; a contradiction.

Similarly we show that T is not polynomially tame if $\lim_k(r_{2k^2}/r_k) = 1$. □

Neverless we have the following.

Remark. Let $a \in \Gamma$ and $r \in \Lambda_1$. Then any diagonal continuous operator T from $A_1(a,r)$ to $A_1(a,r)$ is tame. Indeed, for some $(t_i) \subset \mathbb{K}$ we have $Te_i = t_i e_i, i \in \mathbb{N}$. By the continuity of T there exist strictly increasing sequences $(C_k),(\varphi_k) \subset \mathbb{N}$ with

$$(*)\, |t_i|\exp[(r_k - r_{\varphi_k})a_i] \le C_k \text{ for all}\, i,k \in \mathbb{N}.$$

Let $k \in \mathbb{N}$. Since $\lim_l(r_l - r_{\varphi_l}) = 0$, there is an $l = l_k \in \mathbb{N}$ with $r_k - r_{k+1} \le r_l - r_{\varphi_l}$. Hence, using $(*)$ for l_k instead k, we get $|t_i|\exp[(r_k - r_{k+1})a_i] \le C_{l_k}$, so $\|Te_i\|_k \le C_{l_k}\|e_i\|_{k+1}$ for all $i \in \mathbb{N}$. It follows that $\|Tx\|_k \le C_{l_k}\|x\|_{k+1}$ for all $k \in \mathbb{N}, x \in A_1(a,r)$. □

We get also the following result.

Proposition 5. *Let $a \in \Gamma$. Then there exists a diagonal continuous operator T from $A_1(a)$ to $A_1(a)$ such that for every $r \in \Lambda_1$ we have $\sup_x \|Tx\|_k/\|x\|_k = \infty, k \in \mathbb{N}$ and $\sup_x \|Tx\|_k/\|x\|_{k+1} < \infty, k \in \mathbb{N}$, where $(\|\cdot\|_k)$is the canonical base of norms on $A_1(a, r)$.*

Proof. Put $s_k = -1/k$ for $k \in \mathbb{N}$. Put $D_{i,k} = \exp[(s_{k+1} - s_k)a_i]$ for $k, i \in \mathbb{N}$. The sequence $d_i = \max\{k \in \mathbb{N} : k \le D_{i,k}\}, i \in \mathbb{N}$, is increasing and $\lim_i d_i = \infty$. It follows that $C_k := \sup_i(d_i/D_{i,k}) < \infty$ for $k \in \mathbb{N}$, since $d_i \le D_{i,d_i} \le D_{i,k}$ if $d_i \ge k$. Clearly $b_i := \inf_k C_k D_{i,k} \ge d_i$ for $i \in \mathbb{N}$. Let $\alpha \in \mathbb{K}$ with $|\alpha| > 1$. Let $(t_i) \subset \mathbb{K}$ with $|t_i| \le b_i \le |t_i||\alpha|$ for $i \in \mathbb{N}$.

The operator $T : A_1(a) \to A_1(a), Tx = \sum_{i=1}^{\infty} t_i x_i e_i$ is well defined, linear and continuous. Indeed, let $x \in A_1(a)$. Then $|t_i x_i| \exp(s_k a_i) \le C_k \exp(s_{k+1} a_i)|x_i|$ for $k \in \mathbb{N}$, so $\lim_i t_i x_i e_i = 0$ in $A_1(a)$, $Tx \in A_1(a)$ and $|Tx|_k \le C_k |x|_{k+1}, k \in \mathbb{N}$, where $(|\cdot|_k)$ is the canonical base of norms on $A_1(a)$. Let $r \in \Lambda_1$. Clearly $\sup_i(\|Te_i\|_k/\|e_i\|_k) = \sup_i |t_i| = \infty, k \in \mathbb{N}$.

Let $k \in \mathbb{N}$. Then there exists $l = l(k) \in \mathbb{N}$ with $s_{l+1} - s_l < r_{k+1} - r_k$. Hence

$$\sup_i |t_i| \exp[(r_k - r_{k+1})a_i] \le \sup_i |t_i| \exp[(s_l - s_{l+1})a_i] \le C_l,$$

so $\|Te_i\|_k \le C_{l(k)} \|e_i\|_{k+1}$ for $i \in \mathbb{N}$. Thus $\sup_x \|Tx\|_k/\|x\|_{k+1} \le C_{l(k)}$. $\square$

To study the tameness of the power series spaces of infinite type $A_\infty(a)$ we shall need the Grothendieck's factorization theorem. To show this theorem we need the following.

Proposition 6. *Let E and F be Fréchet spaces and let $T \in L(E, F)$. Assume that the range of T is of II-category in F. Then T is open.*

Proof. Let U be an absolutely convex and open subset of E. Put $V = \overline{T(U)}^F$. Let $(\lambda_n) \subset (\mathbb{K} \setminus \{0\})$ with $\lim |\lambda_n| = \infty$. Then $T(E) = \bigcup_{n=1}^{\infty} \lambda_n T(U)$. Since $T(E)$ is II-category in F and $V = \lambda_n^{-1}\overline{\lambda_n T(U)}^F$ for $n \in \mathbb{N}$, the set V has an interior point x. We have $V - x = \overline{T(U) - x}^F \subset \overline{T(U) - T(U)}^F = V$. Thus 0 is an interior point of V. It follows that $\bigcup_{n=1}^{\infty} \lambda_n V = F$. Hence, by [7], Theorem 3.5.10 and its proof, we infer that $T(U)$ is open, so T is open. $\square$

Let E and F be locally convex spaces. If E is a linear subspace of F and the inclusion map $i : E \to F$ is continuous, we write $E \hookrightarrow F$.

Theorem 7. *(Grothendieck's Factorization Theorem; compare with [4], Theorem 24.33) Let $F_n, n \geq 0$ be Fréchet spaces and let E be a lcs. Assume that $F_0 \subset \bigcup_{n=1}^{\infty} F_n$ and $F_n \hookrightarrow E$ for $n \geq 0$. Then $F_0 \hookrightarrow F_m$ for some $m \in \mathbb{N}$*

Proof. Let $n \in \mathbb{N}$ and $H_n = \{(x, y) \in F_0 \times F_n : x = y\}$. It is easy to see that H_n is a closed subspace of the Fréchet space $F_0 \times F_n$; so H_n is a Fréchet space. The map $P_n : H_n \to F_0, P_n(x, y) = x$ is continuous. Since $F_0 \subset \bigcup_{n=1}^{\infty} F_n$, we get $F_0 = \bigcup_{n=1}^{\infty} P_n(H_n)$. By the Baire category theorem, there is an $m \in \mathbb{N}$ such that $P_m(H_m)$ is of II-category in F_0. By Proposition 6, P_m is open. Thus $F_0 = P_m(H_m)$, so $F_0 \subset F_m$. The inclusion map $i : F_0 \to F_m$ has a closed graph. By the closed graph theorem ([3], Corollary 2.2), the map i is continuous. □

We say that a pair (E, F) of Fréchet spaces is *tameable*, if there exist bases of continuous seminorms on E and F, with respect to which the pair (E, F) is tame.

We shall need the following simple result.

Proposition 8. *Let E and F be Fréchet spaces with bases of continuous seminorms $(|\cdot|_k)$ and $(\|\cdot\|_k)$, respectively. Then the following conditions are equivalent.*

(1) The pair (E, F) is tameable.

(2) There exists a function $S : \mathbb{N} \to \mathbb{N}$ such that

$$\forall T \in L(E, F)\ \exists d \in \mathbb{N}\ \forall k \geq d : \sup_x \|Tx\|_k / |x|_{S(k)} < \infty.$$

(3) There exists a function $S : \mathbb{N} \to \mathbb{N}$ such that

$$\forall T \in L(E, F)\ \exists c \in \mathbb{N}\ \forall k \in \mathbb{N} : \sup_x \|Tx\|_k / |x|_{cS(k)} < \infty.$$

Proof. (1) ⇒ (2). Let $(|\cdot|'_k)$ and $(\|\cdot\|'_k)$ be bases of continuous seminorms on E and F, respectively, with respect to which the pair (E, F) is tame. Then for every $T \in L(E, F)$ there is a $c = c(T) \in \mathbb{N}$ such that

$$C_{T,k} := \sup_{x \in E} \|Tx\|'_k / |x|'_{ck} < \infty, k \in \mathbb{N}.$$

For some increasing functions $C, D, \varphi, \psi : \mathbb{N} \to \mathbb{N}$ we have

$$|x|'_k \leq D(k)|x|_{\psi(k)} \text{ and } \|y\|_k \leq C(k)\|y\|'_{\varphi(k)} \text{ for all } x \in E, y \in F \text{ and } k \in \mathbb{N}.$$

Put $S(k) = \psi(k\varphi(k)), k \in \mathbb{N}$. For $T \in L(E, F), x \in E$ and $k \geq c = c(T)$ we have

$$\|Tx\|_k \leq C(k)\|Tx\|'_{\varphi(k)} \leq C_{T,\varphi(k)}C(k)|x|'_{c\varphi(k)} \leq W_{T,k}|x|_{\psi(c\varphi(k))} \leq W_{T,k}|x|_{S(k)},$$

where $W_{T,k} := D(c\varphi(k))C_{T,\varphi(k)}C(k)$.

(2) $\Rightarrow$ (3). Let $T \in L(E,F)$. Clearly there is $c \in \mathbb{N}$ with $\sup_x \|Tx\|_k/|x|_c < \infty$ for $1 \le k \le d$. Then $\sup_x \|Tx\|_k/|x|_{cS(k)} < \infty$ for all $k \in \mathbb{N}$.

(3) $\Rightarrow$ (1). Without loss of generality we can assume that the function $S : \mathbb{N} \to \mathbb{N}$ is increasing and $S(k) \ge 2k$ for $k \in \mathbb{N}$. Put $|\cdot|'_k = |\cdot|_{S^k(k)}$ and $\|\cdot\|'_k = \|\cdot\|_{S^k(k)}$ for all $k \in \mathbb{N}$. Clearly $(|\cdot|'_k)$ and $(\|\cdot\|'_k)$ are bases of continuous seminorms on E and F, respectively, with respect to which the pair (E,F) is tame. Indeed, let $T \in L(E,F)$ and $c \in \mathbb{N}$ with $\sup_x \|Tx\|_k/|x|_{cS(k)} < \infty$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then

$$\sup_x \frac{\|Tx\|'_k}{|x|'_{(c+1)k}} = \sup_x \frac{\|Tx\|_{S^k(k)}}{|x|_{S^{ck+k}(ck+k)}} \le \sup_x \frac{\|Tx\|_{S^k(k)}}{|x|_{cS^{k+1}(k)}} < \infty,$$

since $S^{ck+k}(ck+k) \ge S^{ck-1}(S^{k+1}(k)) \ge 2^{ck-1}S^{k+1}(k) \ge cS^{k+1}(k)$.

Thus (E,F) is tameable. □

Now we shall prove that the pair $(A_\infty(a), A_p(b))$ is tame if and only if the set $M_{b,a}$ is bounded.

Remark. Nyberg proved that for $a, b \in \Gamma$ the set $M_{b,a}$ is bounded if and only if there exist strictly increasing sequences $(m_i), (n_i) \subset \mathbb{N}$ such that $\sup_i (b_{m_{i+1}}/a_{n_i+1}) < \infty$ and $\lim_i (b_{m_i+1}/a_{n_i}) = \infty$ ([5], Lemma 5.1).

Theorem 9. *Let $p \in \{1, \infty\}$. Let $a, b \in \Gamma$. Then the following conditions are equivalent.*

(1) The pair $(A_\infty(a), A_p(b))$ is tame.

(2) The pair $(A_\infty(a), A_p(b))$ is tameable.

(3) The set $M_{b,a}$ of all finite limit points of the double sequence $(b_i/a_j)_{i,j\in\mathbb{N}}$ is bounded.

Proof. Denote by $(|\cdot|_k)$ and $(\|\cdot\|_k)$ the canonical bases of continuous norms on $A_\infty(a)$ and $A_p(b)$, respectively. Put $H = L(A_\infty(a), A_p(b))$. For $T \in H$ and $(k,n) \in \mathbb{N}\times\mathbb{N}$ we put $\|T\|_{k,n} = \sup_x \|Tx\|_k/|x|_n$. For $k \in \mathbb{N}$ we set $r_k = -1/k$ if $p = 1$ and $r_k = k$ if $p = \infty$.

The implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Denote by $\mathcal{B}$ the family of all bounded subsets of $A_\infty(a)$. For any $(n, B) \in \mathbb{N}\times\mathcal{B}$ the functional $q_{n,B} : H \to [0,\infty), T \to \sup_{x\in B} \|Tx\|_n$, is a seminorm on H. Denote by τ the locally convex topology on H generated by these seminorms. Then $H = (H,\tau)$ is a locally convex space. Let $s : \mathbb{N} \to \mathbb{N}$. Denote by H_s the family

of all $T \in H$ such that $\|T\|_{k,s(k)} < \infty$ for any $k \in \mathbb{N}$. Clearly H_s is a linear subspace of H and functionals $\|\cdot\|_{k,s(k)} |_{H_s}, k \in \mathbb{N}$ are norms on H_s.

It is not hard to check that H_s with the metrizable locally convex topology τ_s generated by these norms is complete. Thus $H_s = (H_s, \tau_s)$ is a Fréchet space. It is easy to see that $H_s \hookrightarrow H$.

By Proposition 8 there is a function $S : \mathbb{N} \to \mathbb{N}$ such that for every $T \in H$ there exists a positive integer c such that $\|T\|_{k,cS(k)} < \infty, k \in \mathbb{N}$. Let $c \in \mathbb{N}$. Denote by F_c the Fréchet space H_{s_c}, where $s_c : \mathbb{N} \to \mathbb{N}, k \to cS(k)$. Then $\bigcup_{c=1}^{\infty} F_c = H$.

Let g be a strictly increasing continuous mapping of $[0, \infty)$ onto itself with $g(k) \geq S(k+2), k \in \mathbb{N}$. Put $G(x) = \int_0^x g(t)dt$ and $f(x) = xG(x)$ for $x > 0$. Let $u(x) = x^2 f'(x)$ for $x > 0$. Then f', u and their inverse functions $h = (f')^{-1}, w = u^{-1}$ are strictly increasing mappings of $(0, \infty)$ onto itself. Clearly $S(k) \leq g(k-2) \leq G(k-1)$ for $k \geq 3$.

Denote by F_0 the Fréchet space H_{s_0}, where $s_0 : \mathbb{N} \to \mathbb{N}$ with $f(k) < s_0(k) \leq f(k) + 1, k \in \mathbb{N}$. By the Grothendieck's factorization theorem there is an $m \in \mathbb{N}$ such that $F_0 \hookrightarrow F_m$. Then we have

$$(*)\, \forall k \in \mathbb{N}\, \exists n_k \in \mathbb{N}\, \exists C_k > 1\, \forall T \in F_0 : \|T\|_{k,s_m(k)} \leq C_k \max_{1 \leq n \leq n_k} \|T\|_{n,s_0(n)}.$$

Let $T_{i,j} : A_\infty(a) \to A_p(b), x \to x_i e_j$ for $i, j \in \mathbb{N}$. Clearly $T_{i,j} \in H$ and

$$\|T_{i,j}\|_{k,n} = \sup_x |x_i| \|e_j\|_k / |x|_n = \exp(r_k b_j - n a_i)$$

for all $i, j, n, k \in \mathbb{N}$. Using $(*)$ we get

$$\forall k \in \mathbb{N} \exists n_k \in \mathbb{N} \exists C_k > 0 \forall i, j \in \mathbb{N} : \exp(r_k b_j - s_m(k) a_i) \leq C_k \max_{1 \leq n \leq n_k} \exp(r_n b_j - s_0(n) a_i).$$

Consider two cases.

Case 1: $p = \infty$. Then we have $\forall k \in \mathbb{N} \exists n_k \in \mathbb{N} \exists D_k > 0 \forall i, j \in \mathbb{N} :$

$$k(b_j/a_i) - mS(k) \leq D_k/a_i + \max_{1 \leq n \leq n_k} [n(b_j/a_i) - f(n)],$$

so

$$\forall k \in \mathbb{N} \forall A \in M_{b,a} : kA - mS(k) \leq \sup_n [An - f(n)].$$

It is easy to see that $\sup_{t>0}(At - f(t)) = Ah(A) - f(h(A))$ for $A > 0$.

Suppose that there exists $A \in M_{b,a}$ such that $h(A) > m + 1$. Then for $k \in \mathbb{N}$ with $h(A) < k \leq h(A) + 1$ we have

$$kA - mS(k) \leq Ah(A) - f(h(A)) \leq Ak - f(k-1),$$

so $(k-1)G(k-1) = f(k-1) \leq mS(k) \leq mG(k-1)$. Thus $h(A) < k \leq m+1$; a contradiction.

It follows that $A \leq h^{-1}(m+1)$ for every $A \in M_{b,a}$, so $M_{b,a}$ is bounded.

Case 2: $p = 1$. Then we have $\forall k \in \mathbb{N} \exists n_k \in \mathbb{N} \exists D_k > 0 \forall i, j \in \mathbb{N}$:

$$\frac{-1}{k}\frac{b_j}{a_i} - mS(k) \leq \frac{D_k}{a_i} + \max_{1 \leq n \leq n_k}\left(\frac{-1}{n}\frac{b_j}{a_i} - f(n)\right),$$

so

$$\forall k \in \mathbb{N} \forall A \in M_{b,a} : -\frac{A}{k} - mS(k) \leq \sup_n \left(-\frac{A}{n} - f(n)\right).$$

It is easy to see that $\sup_{t>0}(-A/t - f(t)) = -A/w(A) - f(w(A))$ for $A > 0$.

Suppose that there exists $A \in M_{b,a}$ with $w(A) > m+1$. Then for $k \in \mathbb{N}$ with $w(A) < k \leq w(A) + 1$ we have

$$-A/k - mS(k) \leq -A/w(A) - f(w(A)) \leq -A/k - f(k-1),$$

so $(k-1)G(k-1) = f(k-1) \leq mS(k) \leq mG(k-1)$. Thus $w(A) < k \leq m+1$; a contradiction.

It follows that $A \leq w^{-1}(m+1)$ for every $A \in M_{b,a}$, so $M_{b,a}$ is bounded.

(3) $\Rightarrow$ (1). Let $B > \sup M_{b,a}$. Let $T \in H$. Then there exists $\varphi : \mathbb{N} \to \mathbb{N}$ such that $C_k := \|T\|_{k,\varphi(k)} < \infty, k \in \mathbb{N}$. Let $(t_{n,j}) \subset \mathbb{K}$ with $Te_n = \sum_{j=1}^{\infty} t_{n,j} e_j, n \in \mathbb{N}$. For all $n, k \in \mathbb{N}$ there exists $v(n,k) \in \mathbb{N}$ with

$$\|Te_n\|_k = |t_{n,v(n,k)}| \exp(r_k b_{v(n,k)}).$$

Then for all $l, n, k \in \mathbb{N}$ we have

$$\frac{\|Te_n\|_k}{\|Te_n\|_l} \geq \frac{|t_{n,v(n,l)}| \exp(r_k b_{v(n,l)})}{|t_{n,v(n,l)}| \exp(r_l b_{v(n,l)})} = \exp[(r_k - r_l) b_{v(n,l)}].$$

Hence for all $c, l, n, k \in \mathbb{N}$ with $[(r_k - r_l) b_{v(n,l)} + (cl - \varphi(k)) a_n] \geq 0$ we have

$$\frac{\|Te_n\|_k}{\|Te_n\|_l} \frac{|e_n|_{cl}}{|e_n|_{\varphi(k)}} \geq 1, \text{ so } (*) \ \frac{\|Te_n\|_l}{|e_n|_{cl}} \leq \frac{\|Te_n\|_k}{|e_n|_{\varphi(k)}} \leq C_k.$$

Let c be an integer greater than $B + \varphi(1)$.

Let $l \in \mathbb{N}$. Any positive integer n satisfies one of the following conditions.

$(*_1)$ $b_{v(n,l)}/a_n \leq B$. Then

$$(r_l - r_1) b_{v(n,l)}/a_n \leq (r_l - r_1)B \leq (cl - \varphi(1)).$$

Hence $(r_1 - r_l) b_{v(n,l)} + (cl - \varphi(1)) a_n \geq 0$. Using $(*)$ we get $\|Te_n\|_l \leq C_1 |e_n|_{cl}$.

$(*_2)$ $b_{v(n,l)}/a_n \geq \varphi(2l)2l$. Then

$$(r_{2l} - r_l)b_{v(n,l)} + (cl - \varphi(2l))a_n \geq (\varphi(2l) + cl - \varphi(2l))a_n > 0.$$

Using $(*)$ we obtain $\|Te_n\|_l \leq C_{2l}|e_n|_{cl}$.

$(*_3)$ $B < b_{v(n,l)}/a_n < \varphi(2l)2l$. By the definition of the set $M_{b,a}$ the set of all positive integers n satisfying $(*_3)$ is finite.

It follows that $D_l := \sup_n \|Te_n\|_l/|e_n|_{cl} < \infty$ for every $l \in \mathbb{N}$. Hence $\|Tx\|_l \leq D_l|x|_{cl}$ for every $x \in A_\infty(a)$, so T is tame. Thus the pair $(A_\infty(a), A_p(b))$ is tame. □

Corollary 10. *The space $A_\infty(a)$ is tame if and only if the set $M_{a,a}$ is bounded.*

In [14] we have shown that the range of any continuous linear operator from $A_\infty(a)$ to $A_\infty(b)$ has a Schauder basis, if the set $M_{b,a}$ is bounded ([14], Theorem 10). It is not known whether the assumption on $M_{b,a}$ is necessary. We shall prove the following.

Theorem 11. *Let $a, b \in \Gamma$. Then the range of every tame operator S from $A_\infty(a)$ to $A_\infty(b)$ has a Schauder basis.*

Proof. By $(|\cdot|_k)$ we denote the canonical base in $\mathcal{P}(A_\infty(c))$ for every $c \in \Gamma$. It is easy to see that there exist two strictly increasing sequences $(s_n), (t_n) \subset \mathbb{N}$ and $d = (d_n) \in \Gamma$ with $\sup_n(d_{n+1} - d_n) < \infty$ such that $d_{s_n} = a_n$ and $d_{t_n} = b_n$ for all $n \in \mathbb{N}$. The operator $R : A_\infty(d) \to A_\infty(a), (x_n) \to (x_{s_n})$, is well defined, linear and $|Rx|_k \leq |x|_k$ for all $x \in A_\infty(d), k \in \mathbb{N}$. Moreover $R(A_\infty(d)) = A_\infty(a)$.

For $y = (y_n) \in A_\infty(b)$ we put $z_y = (z_{y,n})$, where $z_{y,n} = y_k$ if $n = t_k$ for some $k \in \mathbb{N}$, and $z_{y,n} = 0$ otherwise. Then the operator $Q : A_\infty(b) \to A_\infty(d), Qy = z_y$ is well defined, linear and $|Qy|_k = |y|_k$ for all $y \in A_\infty(b), k \in \mathbb{N}$. It is easy to see that the linear operator $T : A_\infty(d) \to A_\infty(d), T = QSR$, is tame and the range of T is isomorphic to the range of S, so it is enough to show that the range of T has a Schauder basis. Put $E = A_\infty(d)$. By tameness of T we have

$$\exists c \in \mathbb{N} \forall k \in \mathbb{N} \exists C_k \in \mathbb{N} \forall x \in E : |Tx|_k \leq C_k|x|_{ck};$$

clearly we can assume that the sequence (C_k) is strictly increasing.

Let $(t_{n,j}) \subset \mathbb{K}$ with $Te_n = \sum_{j=1}^\infty t_{n,j}e_j, n \in \mathbb{N}$. Then $Tx = \sum_{j=1}^\infty(\sum_{n=1}^\infty t_{n,j}x_n)e_j$ for every $x = (x_n) \in E$. Put $\alpha_n = \exp d_n, n \in \mathbb{N}$. Then $D := \sup_n(\alpha_{n+1}/\alpha_n) < \infty$. For all $k, n \in \mathbb{N}$ we have

$$(*_1) \quad \max_j |t_{n,j}|\alpha_j^k = |Te_n|_k \leq C_k|e_n|_{ck} = C_k\alpha_n^{ck}.$$

Put $\mathbb{N}_0 = (\mathbb{N} \cup \{0\})$, $C_0 = 1$ and $M_k = \prod_{i=0}^{k} C_i$ for $k \geq 0$.

The function $q : \mathbb{N} \to \mathbb{N}_0, q(t) = \max\{k \in \mathbb{N}_0 : C_k \leq \alpha_t\}$, is non-decreasing and $\lim_t q(t) = \infty$.

Let $f : \mathbb{N} \to (0, \infty), f(t) = \alpha_t^{q(t)}/M_{q(t)}$. Then $f(t) = \max_{k \geq 0} \alpha_t^k/M_k$ for $t \in \mathbb{N}$, since $\alpha_t^{k-1}/M_{k-1} \leq \alpha_t^k/M_k$ if and only if $k \leq q(t)$ for all $k, t \in \mathbb{N}$. Thus f is non-decreasing, $f(1) \geq 1$ and $\lim_t f(t) = \infty$.

Let $(n_k) \subset \mathbb{N}$ be a strictly increasing sequence with $q(n_k) > k$ for every $k \in \mathbb{N}$. For $n \in \mathbb{N}$ we have

$$M_{q(n+1)} \leq C_{q(n+1)}^{q(n+1)-q(n)} M_{q(n)} \leq \alpha_{n+1}^{q(n+1)-q(n)} M_{q(n)}.$$

Let $k \in \mathbb{N}$. For $n \geq n_k$ we get

$$(*_2) \quad \frac{\alpha_{n+1}^k}{f(n+1)} = \frac{M_{q(n+1)}}{\alpha_{n+1}^{q(n+1)-k}} \leq \frac{M_{q(n)}}{\alpha_{n+1}^{q(n)-k}} \leq \frac{M_{q(n)}}{\alpha_n^{q(n)-k}} = \frac{\alpha_n^k}{f(n)}.$$

The function $r : \mathbb{N} \to \mathbb{N}_0, r(t) = \max\{k \in \mathbb{N}_0 : C_k \leq \alpha_t^{2c}\}$ is non-decreasing and $\lim_t r(t) = \infty$.

Let $g : \mathbb{N} \to (0, \infty), g(t) = \alpha_t^{2cr(t)}/M_{r(t)}$. Then $g(t) = \max_{k \geq 0} \alpha_t^{2ck}/M_k$ for $t \in \mathbb{N}$, since $\alpha_t^{2c(k-1)}/M_{k-1} \leq \alpha_t^{2ck}/M_k$ if and only if $k \leq r(t)$ for all $t, k \in \mathbb{N}$. Thus g is non-decreasing and $g(t) \geq f(t)$ for $t \in \mathbb{N}$.

For $n \in \mathbb{N}$ we have

$$M_{r(n+1)} \leq C_{r(n+1)}^{r(n+1)-r(n)} M_{r(n)} \leq \alpha_{n+1}^{2c(r(n+1)-r(n))} M_{r(n)}.$$

Let $k \in \mathbb{N}$. For $n \geq n_k$ we get $r(n) \geq q(n) > k$ and

$$(*_3) \quad \frac{\alpha_{n+1}^{2ck}}{g(n+1)} = \frac{M_{r(n+1)}}{\alpha_{n+1}^{2c(r(n+1)-k)}} \leq \frac{M_{r(n)}}{\alpha_{n+1}^{2c(r(n)-k)}} \leq \frac{M_{r(n)}}{\alpha_n^{2c(r(n)-k)}} = \frac{\alpha_n^{2ck}}{g(n)}.$$

Put $\|x\|_1 = \sup_j f(j)|x_j|$ and $\|x\|_2 = \sup_j g(j)|x_j|$ for $x = (x_j) \in E$. Clearly, $\|x\|_1 \leq \|x\|_2$ for $x \in E$. Moreover, $|x|_k \leq M_k \|x\|_1$ for $x \in E, k \in \mathbb{N}$, since $\alpha_n^k \leq M_k f(n)$ for $k, n \in \mathbb{N}$.

We shall prove that there exists $C > 0$ such that $\|Tx\|_1 \leq C\|x\|_2$ for every $x \in E$. Let $x \in E$ with $\|x\|_2 < \infty$. Then $\|Tx\|_1 = \sup_j f(j)|\sum_{n=1}^{\infty} t_{n,j} x_n|$. Let $j \geq n_1$. Then $q(j) \geq 2$, so using $(*_1)$ we get

$$f(j)|\sum_{n=1}^{\infty} t_{n,j} x_n| \leq \max_n \frac{\alpha_j^{q(j)} |t_{n,j}||x_n|}{M_{q(j)}} \leq \max_n \frac{C_{q(j)} \alpha_n^{cq(j)} |x_n|}{M_{q(j)}} =$$

$$\max_n \frac{\alpha_n^{cq(j)}|x_n|}{M_{q(j)-1}} \le \max_n \frac{\alpha_n^{2c(q(j)-1)}}{M_{q(j)-1}}|x_n| \le \max_n g(n)|x_n| = \|x\|_2.$$

Put $P : E \to E, (x_1, x_2, ...) \to (x_1, x_2, ..., x_{n_1}, 0, 0, ...)$. Since $\dim P(E) < \infty$ there exists $C_1' > 1$ such that $\|x\|_1 \le C_1'|x|_1$ for every $x \in P(E)$. Hence for $C = C_1'C_1M_c$ we have

$$\max_{1\le j\le n_1} f(j)|\sum_{n=1}^{\infty} t_{n,j}x_n| = \|PTx\|_1 \le C_1'|PTx|_1 \le C_1'|Tx|_1 \le$$

$$C_1'C_1|x|_c \le C\|x\|_1 \le C\|x\|_2.$$

Thus $\|Tx\|_1 = \sup_j f(j)|\sum_{n=1}^{\infty} t_{n,j}x_n| \le C\|x\|_2$ for every $x \in E$.

The set $B = \{x \in E : \lim_n g(n)|x_n| = 0 \text{ and } \|x\|_2 \le 1\}$ is an absolutely convex compactoid in E. Indeed, let $\varphi \in \mathbb{K}$ with $|\varphi| > 1$. Let $(\gamma_j) \subset \mathbb{K}$ with $1 \le |\gamma_j|g(j) < |\varphi|$ for $j \in \mathbb{N}$; clearly $(\gamma_j) \in c_0$. If $x = (x_j) \in B$, then $\sup_j |x_j/\gamma_j| \le 1$; so $B \subset \overline{co}\{\gamma_j e_j : j \in \mathbb{N}\}$. For $j, k \in \mathbb{N}$ we have

$$|\gamma_j e_j|_k < |\varphi|\frac{\alpha_j^k}{g(j)} \le |\varphi|\alpha_j^{-ck} \sup_n \frac{\alpha_n^{2ck}}{g(n)} \le |\varphi|\frac{M_k}{\alpha_j^{ck}},$$

so $\lim_j \gamma_j e_j = 0$ in E. Thus B is compactoid in E.

Therefore $V = T(B)$ is an absolutely convex compactoid in $G = T(E)$.

Denote by F the completion of the normed space $(G, |\cdot|_1)$. Clearly, V is an absolutely convex compactoid in F. Let $t \in (0, 1)$. By [8], Lemma 4.36 and Theorem 4.37, there exists a t-orthogonal sequence (g_n) in F with $(g_n) \subset (\varphi V) \setminus \{0\}$ such that $V \subset \overline{co}^F\{g_n : n \in \mathbb{N}\}$ and $\lim_n |g_n|_1 = 0$; without loss of generality we can assume that the sequence $(|g_n|_1)$ is non-increasing. Clearly $(\gamma_j e_j) \subset \varphi B$, so B is linearly dense in E. Hence V is linearly dense in G, so (g_n) is linearly dense in F. Thus (g_n) is a t-orthogonal basis in F. Let $(g_n^*) \subset F^*$ be the sequence of coefficient functionals associated with the Schauder basis (g_n) in F.

Let $y_n = g_n^* \circ T, n \in \mathbb{N}$; then $(y_n) \subset E^*$ and $Tx = \sum_{n=1}^{\infty} y_n(x)g_n$ in F for every $x \in E$. The set $V_0 = \varphi\overline{V}^E$ is an absolutely convex metrizable complete compactoid in E, so $\tau |_{V_0} = \tau_1 |_{V_0}$, where τ is the topology of E and τ_1 is the one generated by $|\cdot|_1$ on E ([10], Theorem 3.2). It follows that $\lim_n g_n = 0$ in E. It is not hard to check that

$$\overline{\text{co}}^F\{g_n : n \in \mathbb{N}\} = \left\{\sum_{n=1}^{\infty} \psi_n g_n : (\psi_n) \subset B_{\mathbb{K}}\right\}.$$

Thus $|y_n(x)| \le 1$ for all $x \in B, n \in \mathbb{N}$. Denote by H the linear span of B. We have $\alpha_n^k \le M_k g(n)$ for all $k, n \in \mathbb{N}$, so $H = \{x = (x_n) \in \mathbb{K}^{\mathbb{N}} : \lim_n g(n)|x_n| = 0\}$. It follows that $(H, \|\cdot\|_2)$ is a Banach space. Thus using the Banach-Steinhaus theorem we get $K = \sup_n \|y_n\|_2^* < \infty$, where $\|y_n\|_2^* = \sup_{x\in H} |y_n(x)|/\|x\|_2$.

We shall prove that the series $\sum_{n=1}^{\infty} y_n(x)g_n$ is convergent in E for every $x \in E$. In this order it is enough to show that $\lim_n y_n(x)g_n = 0$ in E for every $x \in E$. For every $n \in \mathbb{N}$ there exists $h_n \in B$ such that $g_n = \varphi T h_n$. Hence

$$(*_4)\ \ \|g_n\|_1 = |\varphi|\|Th_n\|_1 \leq |\varphi| C\|h_n\|_2 \leq C|\varphi| \ \text{ for } n \in \mathbb{N}.$$

The sequence (g_n) is t-orthogonal in $(E, |\cdot|_1)$, thus $|Tx|_1 \geq t \max_n |y_n(x)||g_n|_1$ for $x \in E$. Hence $|y_n(x)| \leq (|Tx|_1/t|g_n|_1) \leq (C_1|x|_c/t|g_n|_1)$ for all $x \in E, n \in \mathbb{N}$, so $|y_n|_c^* \leq C_1/t|g_n|_1$ for $n \in \mathbb{N}$.

Let $k \in \mathbb{N}$. Put $l = 2c(k+1)$. Let $n_0 > n_k$ with $|g_{n_0}|_1 \leq \alpha_{n_l}^c/g(n_l)$. Since $(g(n)/\alpha_n^c) \geq (\alpha_n^c/M_1)$ for $n \in \mathbb{N}$, we get $\lim_n (g(n)/\alpha_n^c) = \infty$. Thus for every $n \geq n_0$ there exists $w_n \geq n_l$ such that

$$(*_5)\ \ \frac{g(w_n)}{\alpha_{w_n}^c} \leq \frac{1}{|g_n|_1} < \frac{g(w_n+1)}{\alpha_{w_n+1}^c};$$

clearly $\lim_n w_n = \infty$. Let $n \geq n_0, w = w_n + 1$ and $s = \min\{i \in \mathbb{N} : \alpha_i \geq \alpha_w^{2c}\}$. Then $r(w) \geq r(n_k) \geq q(n_k) > k$ and $s > n_k$. We have $\alpha_{s-1} < \alpha_w^{2c} \leq \alpha_s$ and

$$\frac{g(w)}{\alpha_w^c}\frac{\alpha_s^k}{f(s)} \leq \frac{\alpha_w^{2cr(w)}}{\alpha_w^c M_{r(w)}}\frac{\alpha_s^k M_{r(w)}}{\alpha_s^{r(w)}} = \frac{\alpha_w^{2cr(w)-c}}{\alpha_s^{r(w)-k}} \leq \frac{\alpha_w^{2cr(w)-c}}{\alpha_w^{2c(r(w)-k)}} = \alpha_w^{(2k-1)c}.$$

Hence we get

$$\max\left\{\alpha_{s-1}^k, \frac{g(w)}{\alpha_w^c}\frac{\alpha_s^k}{f(s)}\right\} \leq \alpha_w^{2ck} \leq D^{2ck}\alpha_{w_n}^{2ck}.$$

Using $(*_2)$ we have for $x \in E$

$$|x|_k = \max\left\{\max_{1\leq j<s} \alpha_j^k|x_j|, \max_{j\geq s} \alpha_j^k|x_j|\right\} \leq$$

$$\max\left\{\alpha_{s-1}^k|x|_1, \frac{\alpha_s^k}{f(s)}\sup_{j\geq s} f(j)|x_j|\right\} \leq \max\left\{\alpha_{s-1}^k|x|_1, \frac{\alpha_s^k\|x\|_1}{f(s)}\right\}.$$

Hence, using $(*_4)$ and $(*_5)$, we get for $x = g_n$

$$|g_n|_k \leq \max\left\{\alpha_{s-1}^k|g_n|_1, \frac{\alpha_s^k\|g_n\|_1}{f(s)}\right\} \leq |g_n|_1 \max\left\{\alpha_{s-1}^k, \frac{\alpha_s^k}{f(s)}\frac{C|\varphi|}{|g_n|_1}\right\} \leq$$

$$C|\varphi||g_n|_1 \max\left\{\alpha_{s-1}^k, \frac{g(w)}{\alpha_w^c}\frac{\alpha_s^k}{f(s)}\right\} \leq C|\varphi||g_n|_1 D^{2ck}\alpha_{w_n}^{2ck}.$$

We have $|z(e_j)|/g(j) = |z(e_j)|/\|e_j\|_2 \le \|z\|_2^*$ for all $j \in \mathbb{N}$.

Using $(*_3)$ we get for $z = (z_j) \in E^*$

$$|z|_l^* = \sup_j \frac{|z_j|}{\alpha_j^l} \le \max\left\{\max_{j\le n_l} g(n_l)\frac{|z_j|}{g(j)}, \max_{n_l<j\le w_n} \frac{g(w_n)}{\alpha_{w_n}^l}\frac{|z_j|}{g(j)}, \sup_{j>w_n} \frac{1}{\alpha_{w_n}^{l-c}}\frac{|z_j|}{\alpha_j^c}\right\}$$

$$\le \max\left\{g(n_l)\|z\|_2^*, \frac{g(w_n)}{\alpha_{w_n}^l}\|z\|_2^*, \frac{|z|_c^*}{\alpha_{w_n}^{l-c}}\right\} \le \max\left\{\frac{\alpha_{n_l}^l}{\alpha_{w_n}^l} g(w_n)\|z\|_2^*, \frac{|z|_c^*}{\alpha_{w_n}^{l-c}}\right\}.$$

Hence, using $(*_4)$, we get for $z = y_n$ and for some constant K_l

$$|y_n|_l^* \le \max\left\{\alpha_{n_l}^l \frac{g(w_n)}{\alpha_{w_n}^l}\|y_n\|_2^*, \frac{|y_n|_c^*}{\alpha_{w_n}^{l-c}}\right\} \le \max\left\{\frac{\alpha_{n_l}^l K}{|g_n|_1\alpha_{w_n}^{l-c}}, \frac{1}{\alpha_{w_n}^{l-c}}\frac{C_1}{t|g_n|_1}\right\} \le \frac{K_l}{|g_n|_1\alpha_{w_n}^{l-c}}.$$

Thus $|g_n|_k|y_n|_l^* \le K'\alpha_{w_n}^{2ck+c-l} = K'/\alpha_{w_n}^c$ for $K' = C|\varphi|D^{2ck}K_l$ and $n > n_0$.

We have shown that for every $k \in \mathbb{N}$ there is an $l \in \mathbb{N}$ such that $\lim_n |g_n|_k|y_n|_l^* = 0$. For every $x \in E$ we have $|y_n(x)g_n|_k \le |g_n|_k|y_n|_l^*|x|_l$ for $n > n_0$, so $\lim_n y_n(x)g_n = 0$ in E for every $x \in E$. Thus the series $\sum_{n=1}^\infty y_n(x)g_n$ is convergent in E for every $x \in E$.

Since $\sum_{n=1}^\infty y_n(x)g_n = Tx$ in $(E, |\cdot|_1)$, we infer that $\sum_{n=1}^\infty y_n(x)g_n = Tx$ in E for every $x \in E$. Thus $\sum_{n=1}^\infty g_n^*(y)g_n = y$ in $G = T(E)$ for every $y \in G$. Clearly, $g_n' := g_n^*|_G \in G^*$ and $g_n'(g_m) = \delta_{n,m}$ for $n, m \in \mathbb{N}$.

It follows that (g_n) is a Schauder basis in G. □

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