ON METRIZABILITY OF COMPACTOID SETS IN NON-ARCHIMEDEAN LOCALLY CONVEX SPACES

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10th International Conference on P-Adic and Non-Archimedean Analysis, Michigan State University, East Lansing, Michigan, USA, June 30 to July 3, 2008

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In this paper we show a similar result for locally convex spaces with a $\mathfrak{L}$-base, i.e. with a decreasing base $(U_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ of neighbourhoods of zero. This extends the first mentioned result since every (LM)-space has a $\mathfrak{L}$-base.
We also prove that compactoid sets in (DF)-spaces are metrizable.

We start with the following useful fact ([KS, Lemma 1]).

Lemma 1

Let A be an absolutely convex compactoid set in a polar lcs E. Then for every $c > 0$ and every $f \in [A]'$ there is a $g \in E'$ with $|f(a) - g(a)| < c$ for $a \in A$.

Sketch of Proof

Put $H = [A]$. Let $c > 0$ and let $f \in H'$.
Then for some continuous polar seminorm p on E we have $|f(x)| \leq p(x)$ for every $x \in H$. Put

$$U = \{x \in E : p(x) < c/(c+1)\}.$$

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Let $a \in A$. Then a is of the form $u + d$, where $u \in U$ and $d \in D$. Hence $u \in H$ and

$$|f(a) - g(a)| = |f(u) - g(u)| \leq (c+1)p(u) < c.$$

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This completes the proof. □

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Using Lemma 1 we shall prove the following ([KS, Theorem 2])

Theorem 2

For a polar lcs E the following conditions are equivalent.
(a) *Every compactoid set in E is metrizable.*
(b) *E'_c is of countable type.*

$(a) \Rightarrow (b)$.

Let $G = E'_c$ and let $p \in \mathcal{P}(G)$. Then for some metrizable absolutely convex compactoid set A in E we have $p \leq p_A$, where $p_A : G \to [0, \infty)$, $p_A(f) = \sup_{x \in A} |f(x)|$.

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Let $\alpha \in \mathbb{K}$ with $|\alpha| > 1$. By [Sch1, Proposition 8.2] there exists a sequence $(x_n) \subset \alpha A$ with $x_n \to 0$ in E such that its closed absolutely convex hull X includes A.

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Let $H = \ker p_A$. The operator

$$T : G/H \to c_0, T(f + H) = (f(x_n)),$$

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For $f \in G$ we have

$$\max_n |f(x_n)| \leq |\alpha| p_A(f) \leq |\alpha| \sup_{x \in X} |f(x)| = |\alpha| \max_n |f(x_n)|.$$

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Let $S = \{f_n + H : n \in \mathbb{N}\}$ be a linearly dense countable subset of M.

Then the set $W = \{f_n + \ker p : n \in \mathbb{N}\}$ is linearly dense in the normed space $G_p = (G/\ker p, \overline{p})$.

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It follows that G is of countable type.

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Clearly the functional

$$\| \cdot \|_A : F' \to [0, \infty), \quad g \to \sup_{x \in A} |g(x)|$$

is a norm on F'.

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Applying Lemma 1 one can show that the set $L = \{f_n|F : n \in \mathbb{N}\}$ is linearly dense in $(F', \|\cdot\|_A)$.

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Indeed, let $f \in F'$ and $c > 0$.
By Lemma 1 there exists $g \in E'$ with $\|f - g|F\|_A \leq c$.
Moreover, there exists $h \in [\{f_n : n \in \mathbb{N}\}]$ with $p_A(g - h) \leq c$.
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It follows that all compactoid sets in E are metrizable. $\square$

Let E be a polar lcs. Since $E_\sigma = (E, \sigma(E, E'))$ is of finite type, it has no subspace isomorphic to c_0; so every bounded set in E_σ is compactoid ([GKPS3, Corollary 6.7]).

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Corollary 3

The strong dual E'_b of a polar lcs E is of countable type if and only if every bounded set in E is $\sigma(E, E')$-metrizable.

By Theorem 2 and its proof we get as well ([KS, Corollary 3])

Corollary 4

For a lcs E the following conditions are equivalent.
(a) *Every compactoid set in E is metrizable.*
(b) *For every subspace F of countable type in E the space F'_c is of countable type.*

Definition of a resolution

For $\alpha = (\alpha_n), \beta = (\beta_n) \in \mathbb{N}^{\mathbb{N}}$ we write $\alpha \leq \beta$ if $\alpha_n \leq \beta_n$ for all $n \in \mathbb{N}$.

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A family $(A_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ of sets is *increasing* if $A_\alpha \subset A_\beta$ for all $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ with $\alpha \leq \beta$.

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A resolution $(A_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ of a lcs E is *compactoid* [*bounded*] if the sets $A_\alpha, \alpha \in \mathbb{N}^{\mathbb{N}}$, are compactoid [bounded] in E.

We will use the following known fact ([MS, Lemma 2.1]).

Lemma A

Let $(A_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ be a resolution of an uncountable set A. Then for some $\beta \in \mathbb{N}^\mathbb{N}$ the set A_β is infinite.

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Using the concept of a *t-frame* in a normed space we show the following ([KS, Proposition 5])

Proposition 5

Any lcs E with a compactoid resolution $(A_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ is of countable type.

Sketch of Proof

Let $p \in \mathcal{P}(E)$, $E_p = E/\ker p$ and let $\pi_p : E \to E_p$ be the quotient map.

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Let $C_\alpha = \pi_p(A_\alpha)$ for $\alpha \in \mathbb{N}^\mathbb{N}$. Then $(C_\alpha)_{\alpha \in \mathbb{N}^\mathbb{N}}$ is a compactoid resolution of the normed space $E_p = (E_p, \overline{p})$.

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By Lemma A, for some $\beta \in \mathbb{N}^{\mathbb{N}}$ the set $B_\beta = C_\beta \cap X$ is infinite. The infinite compactoid t-frame B_β in the normed space E_p is distant from 0.
It is in contradiction with [GPS, Proposition 2.2].

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Thus every t-frame in the normed space E_p is countable. Applying [GPS, Theorem 2.5] we deduce that the space E_p is of countable type. It follows that E is of countable type. $\square$

Using Proposition 5 we get ([KS, Corollary 6])

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Let $\beta \in \mathbb{K}$ with $|\beta| > 1$. For $\alpha = (\alpha_n) \in \mathbb{N}^{\mathbb{N}}$ we put

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By Proposition 5 the space E is of countable type. □

Last result yields the following known fact ([GPS, Theorem 3.1]).

Corollary 7

Every Fréchet-Montel space is of countable type.

A resolution $(A_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ of a lcs E is *countably compactoid* if every countable subset of $A_\alpha, \alpha \in \mathbb{N}^\mathbb{N}$, is compactoid.

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Following the proof of Proposition 5 we get the following.

Remark 8

Any lcs E with a countably compactoid resolution is of countable type.

We will need the following fact which follows from [Sch1, Lemma 10.6] and [Sch2, Proposition 4.5].

Lemma 9

For a lcs E every equicontinuous set A in E' is compactoid in E'_c.

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PROOF. There is a neighbourhood U of zero in E such that $A \subset U^\circ$. The set U° is compactoid in E'_σ ([Per2, Theorem 4.2]).
Applying [Sch1, Lemma 10.6] one gets that the topologies $\sigma(E', E)$ and $c(E', E)$ coincide on U°.
Finally using [Sch2, Proposition 4.5] we deduce that the set U° is compactoid in E'_c. $\square$

Making use of Lemma 9, Proposition 5 and Corollary 4 we get ([KS, Theorem 10])

Theorem 10

Let E be a lcs with a $\mathfrak{L}$-base i.e. with a decreasing base $(U_\alpha)_{\alpha \in \mathbb{N}^\mathbb{N}}$ of neighbourhoods of zero.
Then every compactoid set in E is metrizable.

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PROOF. Let F be a subspace of countable type in E. Let $V_\alpha = U_\alpha \cap F$ for $\alpha \in \mathbb{N}^\mathbb{N}$. Clearly, $(V_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ is a $\mathfrak{L}$-base in F.

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Then $(V_\alpha^\circ)_{\alpha\in\mathbb{N}^\mathbb{N}}$ is a compactoid resolution of F'_c.
By Proposition 5, the space F'_c is of countable type.

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Then $(V_\alpha^\circ)_{\alpha\in\mathbb{N}^\mathbb{N}}$ is a compactoid resolution of F_c'.
By Proposition 5, the space F_c' is of countable type.

Corollary 4 completes the proof. □

Applying Theorem 10 we get the following well-known result ([GKP, Theorem 3.1]).

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Now it is enough to use Theorem 10 to complete the proof. □

We also get the following ([KS, Corollary 12]).

Corollary 12

Let E be a metrizable lcs. Then every compactoid set in the strong dual E'_b of E is metrizable.

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$$A_\alpha = \bigcap_{n=1}^{\infty} \beta^{\alpha_n} U_n$$

for $\alpha = (\alpha_n) \in \mathbb{N}^\mathbb{N}$, where (U_n) is a base of absolutely convex neighbourhoods of zero in E and $\beta \in \mathbb{K}$ with $|\beta| > 1$.

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for $\alpha = (\alpha_n) \in \mathbb{N}^\mathbb{N}$, where (U_n) is a base of absolutely convex neighbourhoods of zero in E and $\beta \in \mathbb{K}$ with $|\beta| > 1$.

Clearly, for every bounded set A in E there exists $\alpha \in \mathbb{N}^\mathbb{N}$ such that $A \subset A_\alpha$. Thus $(A_\alpha^\circ)_{\alpha\in\mathbb{N}^\mathbb{N}}$ is a $\mathfrak{L}$-base in the space E'_b. $\square$

Denote by $\mathcal{L}$ the family of all locally convex spaces E with a $\mathfrak{L}$-base $(U_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$. Clearly all metrizable lcs belong to $\mathcal{L}$.

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It is easy to check the following ([KS, Proposition 13])

Proposition 13

Let $E \in \mathcal{L}$. Then every subspace F of E belongs to $\mathcal{L}$. If F is a closed subspace of E, then $E/F \in \mathcal{L}$. The completion of E belongs to $\mathcal{L}$.

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We show that the class $\mathcal{L}$ is stable by countable products, locally convex countable inductive and projective limits and locally convex countable direct sums ([KS, Propositions 14 and 15 and Corollaries 16 and 17]).

Proposition 14

If $(E_n) \subset \mathcal{L}$, *then* $E = \prod_{n=1}^{\infty} E_n$ *belongs to* $\mathcal{L}$.

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Corollary 17

If $(E_n) \subset \mathcal{L}$ *is a projective sequence of lcs, then the projective limit* $\varprojlim E_n$ *belongs to* $\mathcal{L}$.

Next we shall prove that compactoid sets in (DF)-spaces are metrizable.

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Recall that a lcs E is a (DF)-*space* if it has a fundamental sequence (B_n) of bounded sets and for every sequence (V_n) of absolutely convex neighbourhoods of zero in E such that the set $V = \bigcap_{n=1}^{\infty} V_n$ is bornivorous, V is a neighbourhood of zero in E.

One can show the following lemma ([KS, Lemma 18])

Lemma 18

If $\mathbb{K}$ is not spherically complete, $\mathbb{L}$ is the spherical completion of $\mathbb{K}$ and E is a (DF)-space over $\mathbb{K}$, then the space $H = \mathbb{L} \otimes E$ is a (DF)-space over $\mathbb{L}$.

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Using this lemma we prove the following ([KS, Theorem 19])

Theorem 19

Every compactoid set in a (DF)-space E is metrizable.

First we show that every countable bounded subset $F = \{f_n : n \in \mathbb{N}\}$ of E'_b is equicontinuous. Let $F_n = \{f_i : 1 \leq i \leq n\}$ for $n \in \mathbb{N}$. Then ${}^\circ F = \bigcap_{n=1}^{\infty} {}^\circ F_n$. Clearly

$$ {}^\circ F_n = \bigcap_{k=1}^{n} f_k^{-1}(B_{\mathbb{K}}) $$

is an absolutely convex neighbourhood of zero in E for $n \in \mathbb{N}$. Moreover, ${}^\circ F$ is a bornivorous set in E. Indeed, let B be a bounded set in E. Then for some $\alpha \in \mathbb{K}$ we have $F \subset \alpha B^\circ$, so

$$ {}^\circ F \supset \alpha^{-1}({}^\circ(B^\circ)) \supset \alpha^{-1} B. $$

Thus ${}^\circ F$ is a neighbourhood of zero in E, so F is equicontinuous.

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Clearly $(A_\alpha)_{\alpha\in\mathbb{N}^{\mathbb{N}}}$ is a bounded resolution of E_b'. By Lemma 9, every countable subset of $A_\alpha, \alpha \in \mathbb{N}^{\mathbb{N}}$, is compactoid in E_c'.

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Thus E'_c has a countably compactoid resolution. Using Remark 8 we deduce that E'_c is of countable type.

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Thus E'_c has a countably compactoid resolution. Using Remark 8 we deduce that E'_c is of countable type.

It follows that compactoid sets in E are metrizable.

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Let A be a compactoid set in E. Then $C = 1 \otimes A$ is a compactoid set in H.

By (1), the set C is a metrizable subset of H. But C is homeomorphic to A, so A is metrizable. $\square$

We say that a compactoid resolution $(S_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ of a lcs E is *strong* if every compactoid set in E is contained in some S_α.

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Finally, in connection to Proposition 5 we shall prove ([KS, Theorem 21])

Theorem 20

Every metrizable lcs E of countable type has a strong compactoid resolution $(S_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$.

Sketch of Proof

By [Gra2, Corollary 3.7], a subset A of c_0 is compactoid if and only if there exists $y = (y_n) \in c_0$ such that for every $x = (x_n) \in A$ we have $|x_n| \le |y_n|$ for all $n \in \mathbb{N}$.

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Clearly, $(A_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a compactoid resolution of c_0.

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Let $\alpha \in \mathbb{N}^{\mathbb{N}}$. Then there exists $y_\alpha = (y_{\alpha,n}) \in c_0$ such that $|x_n| \leq |y_{\alpha,n}|$ for all $x = (x_j) \in A_\alpha$, $n \in \mathbb{N}$.

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One can show that $(B_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a strong compactoid resolution of c_0.

Sketch of Proof

(B). Let (N_k) be a partition of $\mathbb{N}$ into infinite subsets and let $\phi_k : \mathbb{N} \to \mathbb{N}$ be strictly increasing with $\phi_k(\mathbb{N}) = N_k$ for $k \in \mathbb{N}$.

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(C). Let F be a subspace of $c_0^{\mathbb{N}}$. Then the sets

$$K_\alpha = D_\alpha \cap F, \ \alpha \in \mathbb{N}^{\mathbb{N}},$$

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$$K_\alpha = D_\alpha \cap F, \;\; \alpha \in \mathbb{N}^{\mathbb{N}},$$

form a strong compactoid resolution of F.
(D). E is a metrizable lcs of countable type, so it is isomorphic to a subspace of $c_0^{\mathbb{N}}$ ([GKPS2, Remark 3.6]).

(B). Let (N_k) be a partition of $\mathbb{N}$ into infinite subsets and let $\phi_k : \mathbb{N} \to \mathbb{N}$ be strictly increasing with $\phi_k(\mathbb{N}) = N_k$ for $k \in \mathbb{N}$.
If $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ with $\alpha \leq \beta$, then $\alpha \circ \phi_k \leq \beta \circ \phi_k$ for all $k \in \mathbb{N}$.
The map $\phi : \mathbb{N}^{\mathbb{N}} \to (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ defined by $\phi(\alpha) = (\alpha \circ \phi_n)$ is a bijection.
For $\alpha \in \mathbb{N}^{\mathbb{N}}$ we put $D_\alpha = \prod_{n=1}^{\infty} B_{\alpha \circ \phi_n}$.
Using [Gra1, Propositon 1.7] one can prove that $(D_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a strong compactoid resolution of $c_0^{\mathbb{N}}$.

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Thus E has a strong compactoid resolution. □

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For a dense sequence (x_n) in F and $\alpha = (\alpha_k) \in \mathbb{N}^{\mathbb{N}}$ we put

$$K_\alpha := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{\alpha_k} B(x_j, k^{-1}),$$

where $B(x, r)$ denotes the closed ball in F with center x and radius r.

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Consequently $(K_\alpha \cap E)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a strong compactoid resolution of E. $\square$

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