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ON NON-ARCHIMEDEAN FRÉCHET SPACES WITH NUCLEAR KÖTHE QUOTIENTS

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ABSTRACT. Assume that $\mathbb{K}$ is a complete non-Archimedean valued field. We prove that every infinite-dimensional Fréchet-Montel space over $\mathbb{K}$ which is not isomorphic to $\mathbb{K}^{\mathbb{N}}$ has a nuclear Köthe quotient. If the field $\mathbb{K}$ is non-spherically complete, we show that every infinite-dimensional Fréchet space of countable type over $\mathbb{K}$ which is not isomorphic to the strong dual of a strict LB-space has a nuclear Köthe quotient.

1. Introduction

In this paper all linear spaces are over a non-Archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$. For fundamentals of normed spaces and Hausdorff locally convex spaces (lcs) we refer to [6] and [8, 9].

Any infinite-dimensional Banach space of countable type is isomorphic to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero with the sup-norm, and any closed subspace of c_0 is complemented ([6], Th. 3.16). Any infinite-dimensional Fréchet space of finite type is isomorphic to the Fréchet space $\mathbb{K}^{\mathbb{N}}$ of all sequences in $\mathbb{K}$ with the product topology.

By a Köthe space we mean an infinite-dimensional Fréchet space with a Schauder basis and with a continuous norm.

We investigated quotients of Fréchet spaces in [10, 11, 12].

If the field $\mathbb{K}$ is spherically complete, then there exist non-normable Fréchet spaces (over $\mathbb{K}$) of countable type with a continuous norm and without a nuclear Köthe quotient ([12], Theorem 10).

In this paper we study when a Fréchet space of countable type has a nuclear Köthe quotient.

We show that for every Fréchet space E with a continuous norm and for every biorthogonal sequence $((x_n, f_n)) \subset E \times E'$ such that (x_n) is linearly dense in E and (f_n) is equicontinuous, there exists an infinite subset J of $\mathbb{N}$ such that the quotient $(E/\bigcap_{n\in J} \ker f_n)$ of E is a Köthe space (Corollary 3.7). It follows that an infinite-dimensional Fréchet space of countable type has a Köthe quotient if and only if it is not isomorphic to $\mathbb{K}^{\mathbb{N}}$ (Corollary 3.10).

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Next we prove that a Fréchet space E of countable type has a nuclear Köthe quotient if and only if it has a non-decreasing base $(\|\cdot\|_k)$ of continuous seminorms such that the dual norms $\|\cdot\|_k^{'}, k \in \mathbb{N}$, are pairwise non-equivalent on the subspace $E_1^{'} = \{f \in E^{'} : \|f\|_1^{'} < \infty\}$ of E' (Theorem 3.11).

Using this theorem we show that every infinite-dimensional Fréchet-Montel space E which is not isomorphic to $\mathbb{K}^{\mathbb{N}}$ has a nuclear Köthe quotient (Theorem 3.12).

If $\mathbb{K}$ is non-spherically complete, then every infinite-dimensional Fréchet space E of countable type which is not isomorphic to the strong dual of a strict LB-space has a nuclear Köthe quotient (Theorem 3.13).

In our paper we use and develop some ideas of [1].

2. Preliminaries

The field $\mathbb{K}$ is *spherically complete* if any decreasing sequence of closed balls in $\mathbb{K}$ has a non-empty intersection. Let $B_{\mathbb{K}}$ denote the set $\{\alpha \in \mathbb{K} : |\alpha| \leq 1\}$.

For $S \subset \mathbb{N}$ we put $c_{00}(S) = \{(x_n) \in c_{00} : x_n = 0$ for any $n \in (\mathbb{N} \setminus S)\}$, where $c_{00} = \{(x_n) \in \mathbb{K}^{\mathbb{N}} : x_n = 0$ for almost all $n \in \mathbb{N}\}$.

Let E be a linear space.

The linear span of a subset A of E is denoted by lin$A$.

A set $A \subset E$ is *absolutely convex* if for any $\alpha, \beta \in B_{\mathbb{K}}$ and any $x, y \in A$ we have $\alpha x + \beta y \in A$. Let A be an absolutely convex set in E. We put $A^e = A$ if the valuation of $\mathbb{K}$ is discrete and $A^e = \bigcap\{\alpha A : \alpha \in \mathbb{K}$ with $|\alpha| > 1\}$ otherwise.

If $A \subset E$, then the set co$A = \{\sum_{i=1}^n \alpha_i a_i : n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in B_{\mathbb{K}}, a_1, \ldots, a_n \in A\}$ is the smallest absolutely convex subset of E that contains A.

A set $A \subset E$ is $\mathbb{K}$-*polar* if for each $x \in (E \setminus A)$ there exists a linear functional f on E such that $|f(x)| > 1$ and $|f(a)| \leq 1$ for any $a \in A$.

A *seminorm* on E is a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x + y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$.

Let $t \in (0, 1]$ and let p be a seminorm on a linear space E. A sequence $(x_n) \subset E$ is *t-orthogonal with respect to p* if $p(\sum_{i=1}^n \alpha_i x_i) \geq t \max\{p(\alpha_i x_i) : 1 \leq i \leq n\}$ for all $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$. (In [6], a sequence (x_n) in a normed space $(E, ||\cdot||)$ is called orthogonal if it is 1-orthogonal with respect to the norm $||\cdot||$.)

A seminorm p on E is a *norm* if $\ker p = \{0\}$.

For any seminorm p on E the map $\overline{p} : E_p \to [0, \infty), x + \ker p \to p(x)$ is a norm on $E_p = (E/\ker p)$.

Norms p, q on E are *equivalent* if there exist positive numbers a, b such that $ap(x) \leq q(x) \leq bp(x)$ for any $x \in E$; then we write $p \approx q$.

Any two norms on a finite-dimensional linear space are equivalent.

In this paper by an *lcs* we mean a Hausdorff locally convex space.

Let E be an lcs.

The set of all continuous seminorms on E is denoted by $\mathcal{P}(E)$.

E is *of finite type* if for any $p \in \mathcal{P}(E)$ the space E_p is finite-dimensional.

E is *of countable type* if for any $p \in \mathcal{P}(E)$ the normed space $(E_p, \overline{p})$ contains a linearly dense countable subset.

The topological dual of E we denote by E'. If $A \subset E$ and M is a subspace of E, we set $A^\circ = \{f \in E' : |f(x)| \leq 1$ for $x \in A\}$ and $M^\perp = \{f \in E' : f(x) = 0$ for $x \in M\}$. If $B \subset E'$ and W is a subspace of E', we put ${}^\circ B = \{x \in E : |f(x)| \leq 1$ for $f \in B\}$ and ${}^\perp W = \{x \in E : f(x) = 0$ for $f \in W\}$. It is easy to see that $M^\perp = M^\circ$ and

${}^{\perp}W = {}^{\circ}W$. If A is an absolutely convex subset of E, then ${}^{\circ}(A^{\circ}) = B^e$, where B is the closure of A in $(E, \sigma(E, E'))$ ([8], Proposition 4.10).

A subset A of E is *polar* if ${}^{\circ}(A^{\circ}) = A$. E is *polar* if for any $p \in \mathcal{P}(E)$ there exists $q \in \mathcal{P}(E)$ with $q \geq p$ such that the set $\{x \in E : q(x) \leq 1\}$ is polar.

A set $A \subset E$ is *bornivorous* if it absorbs any bounded subset of E. E is *bornological* if any absolutely convex bornivorous subset of E is a neighbourhood of zero. E is *polarly barreled* if any polar barrel in E is a neighbourhood of zero. E is *polarly bornological* if any $\mathbb{K}$-polar bornivorous subset of E is a neighbourhood of zero.

A subset B of an lcs E is *compactoid* if for any neighbourhood U of 0 in E there exists a finite subset S of E such that $B \subset U + \mathrm{co}S$.

A subspace D of E has the *weak extension property* in E if for any $g \in D'$ there exists an $f \in E'$ with $f|_D = g$.

Let $\mathcal{B}(E)$ denote the family of all bounded subsets of E. The strong dual of E, that is, the topological dual E' of E with the topology $b(E', E)$ of uniform convergence on bounded subsets of E, is denoted by E'_b.

E is *reflexive* if the canonical map $j : E \to (E'_b)'_b$ is an isomorphism.

Let E and F be an lcs. The space of all linear continuous maps from E to F is denoted by $L(E, F)$. An operator $T \in L(E, F)$ is an *isomorphism* if T is injective, surjective and the inverse map T^{-1} is continuous. E is *isomorphic* to F ($E \simeq F$) if there exists an isomorphism $T : E \to F$. A linear map $T : E \to F$ is *compact* if there exists a neighbourhood U of 0 in E such that $T(U)$ is compactoid in F.

An lcs E is *nuclear* if for any $p \in \mathcal{P}(E)$ there exists $q \in \mathcal{P}(E)$ with $q \geq p$ such that the map $\varphi_{p,q} : (E_q, \overline{q}) \to (E_p, \overline{p})$, $x + \ker q \to x + \ker p$ is compact.

Let E be a metrizable lcs. E is of countable type if and only if it contains a linearly dense countable subset. A sequence $(p_k) \subset \mathcal{P}(E)$ is a *base* in $\mathcal{P}(E)$ if for any $p \in \mathcal{P}(E)$ there exists $k \in \mathbb{N}$ such that $p \leq p_k$.

A metrizable complete lcs is a *Fréchet space*. Let (x_n) be a sequence in a Fréchet space E. The series $\sum_{n=1}^{\infty} x_n$ is convergent in E if and only if $\lim_{n\to\infty} x_n = 0$.

A normable Fréchet space is a *Banach space*. Any n-dimensional lcs is isomorphic to the Banach space $\mathbb{K}^n$. A *strict LB-space* is an lcs (E, τ) which is the inductive limit of an inductive sequence $((E_n, \tau_n))$ of Banach spaces such that $\tau_{n+1}|E_n = \tau_n$ for any $n \in \mathbb{N}$; for fundamentals of inductive limits of locally convex spaces we refer to [3]. A Fréchet space E is a *Fréchet-Montel space* if any bounded subset of E is compactoid.

If $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are normed spaces, then the map

$$\|\cdot\| : L(X, Y) \to [0, \infty), \|T\| = \inf\{C > 0 : \|Tx\| \leq C\|x\| \, \text{for any} \, x \in X\}$$

is a norm; the normed space $(L(X, Y), \|\cdot\|)$ is complete if $(Y, \|\cdot\|)$ is complete.

Let E be an lcs. A sequence $((x_n, f_n)) \subset E \times E'$ is biorthogonal if $f_n(x_m) = \delta_{n,m}$ for all $n, m \in \mathbb{N}$, where $\delta_{n,m} = 1$ if $n = m$ and $\delta_{n,m} = 0$ otherwise.

A sequence (x_n) in an lcs E is a *basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$. If additionally the coefficient functionals $f_n : E \to \mathbb{K}$, $x \to \alpha_n$ ($n \in \mathbb{N}$) are continuous, then (x_n) is a *Schauder basis* in E.

Let $(t_k) \subset (0, 1]$. A sequence (x_n) in a metrizable lcs E is (t_k)-*orthogonal with respect to* $(p_k) \subset \mathcal{P}(E)$ if (x_n) is t_k-orthogonal with respect to p_k for any $k \in \mathbb{N}$.

A sequence (x_n) in a metrizable lcs E is *orthogonal* in E if it is (1)-orthogonal with respect to some base (p_k) in $\mathcal{P}(E)$.

A linearly dense orthogonal sequence (x_n) of non-zero elements in a metrizable lcs E is an *orthogonal basis* in E.

Any orthogonal basis in a metrizable lcs is a Schauder basis, and any Schauder basis in a Fréchet space is an orthogonal basis ([4], Propositions 1.4 and 1.7).

3. Results

We start with the following

Lemma 3.1. *Let $(X, \|\cdot\|)$ be a normed space and let Z be a finite-dimensional subspace of X'. Then for any $\epsilon > 0$ there exists a finite-dimensional subspace W_ϵ of X with $\dim W_\epsilon = \dim Z$ such that for any $\phi \in Z'$ there is an $x \in W_\epsilon$ with $\|\phi\| \le \|x\| \le (1+\epsilon)\|\phi\|$ such that $z(x) = \phi(z)$ for any $z \in Z$.*

Proof. Denote by Y the closed linear subspace ${}^\perp Z$ of X. Then $Y^\perp = Z$ since Z is closed in $(X', \sigma(X', X))$. As in the Archimedean case one can show that the linear map

$$T : Z \to (X/Y)', (Tz)(x+Y) = z(x)$$

is an isometric isomorphism. Finite-dimensional normed spaces are reflexive ([7], Corollary 5.5), so the canonical map $\pi : (X/Y) \to (X/Y)''$ is an isometric isomorphism. Thus the map $T' \circ \pi : (X/Y) \to Z'$ is an isometric isomorphism, too.

Hence for any $\phi \in Z'$ there is an $x_0 \in X$ with $|||x_0 + Y||| = \|\phi\|$ such that $\phi(z) = z(x_0)$ for any $z \in Z$. Put $\delta > 0$. For some $x = x_{\phi,\delta} \in x_0 + Y \subset X$ we have $\|\phi\| \le \|x\| \le (1+\delta)\|\phi\|$ and $z(x) = \phi(z)$ for any $z \in Z$.

Let $\epsilon > 0$, $t = 2(2+\epsilon)^{-1}$ and $\delta = \epsilon(2+\epsilon)^{-1}$. Let $\phi_1, \ldots, \phi_n$ be a t-orthogonal basis in Z' and let $x_i = x_{\phi_i,\delta}$ for $i \le n$. Put $W_\epsilon = \operatorname{lin}\{x_i : i \le n\}$. Let $\phi \in Z'$. Then $\phi = \sum_{i=1}^n \alpha_i \phi_i$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$. For $x = \sum_{i=1}^n \alpha_i x_i$ and $z \in Z$ we have

$$z(x) = \sum_{i=1}^n \alpha_i z(x_i) = \sum_{i=1}^n \alpha_i \phi_i(z) = \phi(z).$$

Clearly

$$\|\phi\| = \sup_{z \in Z \setminus \{0\}} \frac{|\phi(z)|}{\|z\|} = \sup_{z \in Z \setminus \{0\}} \frac{|z(x)|}{\|z\|} \le \|x\|.$$

Moreover

$$\|\phi\| \ge t \max_{i \le n} \|\alpha_i \phi_i\| \ge t(1+\delta)^{-1} \max_{i \le n} \|\alpha_i x_i\| \ge (1+\epsilon)^{-1}\|x\|.$$

Thus for any $\phi \in Z'$ there exists an $x \in W_\epsilon$ with $\|\phi\| \le \|x\| \le (1+\epsilon)\|\phi\|$ such that $z(x) = \phi(z)$ for any $z \in Z$. □

We will need the following four results for biorthogonal sequences in normed spaces.

Proposition 3.2. *Let $(X, \|\cdot\|)$ be a normed space with a biorthogonal sequence $((x_n, f_n)) \subset X \times X'$ with lin$\{x_n : n \in \mathbb{N}\} = X$. Let L be an infinite subset of $\mathbb{N}$ and let $t \in (0,1)$. Then there exists an increasing sequence $(n_i) \subset L$ such that $(x_{n_i} + M)$ is a t-orthogonal basis in the quotient X/M, where $M = \bigcap_{i=1}^\infty \ker f_{n_i}$.*

Proof. Put $c = \sqrt{1/t}$. Since $c > 1$, there is a sequence (ϵ_i) of positive numbers with $\prod_{i=1}^\infty (1+\epsilon_i) < c$. Using Lemma 3.1 we can inductively choose an increasing sequence $(n_i) \subset L$ such that for any $i \in \mathbb{N}$ and any $\phi \in (\operatorname{lin}\{f_{n_j} : j \le i\})'$ there exists an $x \in \operatorname{lin}\{x_j : j < n_{i+1}\}$ with $\|\phi\| \le \|x\| \le (1+\epsilon_i)\|\phi\|$ such that $f(x) = \phi(f)$ for any $f \in \operatorname{lin}\{f_{n_j} : j \le i\}$.

We shall prove that the sequence (f_{n_i}) is t-orthogonal in X'.

Let $(\alpha_j) \subset \mathbb{K}$. Put $\mu_i = (1+\epsilon_i)^{-2}$ and $h_i = \sum_{j=1}^{i} \alpha_j f_{n_j}$ for $i \in \mathbb{N}$. Clearly $\prod_{j=1}^{i} \mu_j > t$ for any $i \in \mathbb{N}$.

We shall show that $\|h_{i+1}\| \geq \mu_i \|h_i\|$ for any $i \in \mathbb{N}$. Let $i \in \mathbb{N}$. We can assume that $h_i \neq 0$. By the Hahn-Banach theorem ([8], Theorem 4.2) there exists some $\phi \in (\operatorname{lin}\{f_{n_j} : j \leq i\})'$ with $\phi(h_i) = 1$ such that $\|\phi\| \leq (1+\epsilon_i)\|h_i\|^{-1}$. Then there is an $x \in \operatorname{lin}\{x_j : j < n_{i+1}\}$ with $\|x\| \leq (1+\epsilon_i)\|\phi\| \leq \mu_i^{-1}\|h_i\|^{-1}$ such that $h_i(x) = \phi(h_i) = 1$. Since $f_{n_{i+1}}(x) = 0$ we obtain

$$\|h_{i+1}\| \geq |h_{i+1}(x)| \|x\|^{-1} = |h_i(x)| \|x\|^{-1} \geq \mu_i \|h_i\|.$$

Thus $\|h_{i+1}\| \geq \mu_i \|h_i\|$ for any $i \in \mathbb{N}$.

By induction we get

$$\|h_{i+1}\| \geq \prod_{j=1}^{i} \mu_j \max_{j \leq i+1} \|\alpha_j f_{n_j}\| \text{ for any } i \in \mathbb{N}. \tag{3.1}$$

Indeed, for $i = 1$ we get

$$\|h_2\| = \max\{\|\alpha_1 f_{n_1}\|, \|\alpha_2 f_{n_2}\|\} \geq \mu_1 \max\{\|\alpha_1 f_{n_1}\|, \|\alpha_2 f_{n_2}\|\}$$

if $\|\alpha_1 f_{n_1}\| \neq \|\alpha_2 f_{n_2}\|$, and

$$\|h_2\| \geq \mu_1 \|\alpha_1 f_{n_1}\| = \mu_1 \max\{\|\alpha_1 f_{n_1}\|, \|\alpha_2 f_{n_2}\|\}$$

otherwise. Assume that (3.1) is true for some $i \in \mathbb{N}$. If $\|\alpha_{i+2} f_{n_{i+2}}\| \leq \|h_{i+1}\|$ we have

$$\|h_{i+2}\| \geq \mu_{i+1} \|h_{i+1}\| \geq \prod_{j=1}^{i+1} \mu_j \max_{j \leq i+2} \|\alpha_j f_{n_j}\|;$$

otherwise we get

$$\|h_{i+2}\| = \|\alpha_{i+2} f_{n_{i+2}}\| = \max\{\|\alpha_{i+2} f_{n_{i+2}}\|, \|h_{i+1}\|\} \geq \prod_{j=1}^{i+1} \mu_j \max_{j \leq i+2} \|\alpha_j f_{n_j}\|.$$

Thus

$$\|\sum_{j=1}^{i+1} \alpha_j f_{n_j}\| \geq t \max_{j \leq i+1} \|\alpha_j f_{n_j}\|$$

for any $(\alpha_j) \subset \mathbb{K}$ and any $i \in \mathbb{N}$. This means that (f_{n_i}) is t-orthogonal in X'.

Denote by F the closure of $\operatorname{lin}\{f_{n_i} : i \in \mathbb{N}\}$ in X'. Let $(g_i) \subset F'$ be the sequence of coefficient functionals associated with the basis (f_{n_i}) in F. It is easy to check that (g_i) is a t-orthogonal sequence in F'. Denote by G the linear span of (g_i) in F'. Put $M = \bigcap_{i=1}^{\infty} \ker f_{n_i}$. Then $\operatorname{lin}\{x_{n_i} : i \in \mathbb{N}\} + M = X$, so $\operatorname{lin}\{x_{n_i} + M : i \in \mathbb{N}\} = X/M$. The map $S : X/M \to F', (S(x+M))(f) = f(x)$ is well defined, linear and injective. Moreover $S(x_{n_i} + M) = g_i$ for $i \in \mathbb{N}$; so $S(X/M) = G$.

To prove that $(x_{n_i} + M)$ is a t-orthogonal basis in X/M it is enough to show that S is an isometry. Let $x \in X$. For $y \in M$ we have

$$\|S(x+M)\| = \sup_{f \in F \setminus \{0\}} \frac{|f(x)|}{\|f\|} = \sup_{f \in F \setminus \{0\}} \frac{|f(x+y)|}{\|f\|} \leq \|x+y\|;$$

hence $\|S(x+M)\| \leq |||x+M|||$. Put $g = S(x+M)$. For some $i_0 \in \mathbb{N}$ we have $g \in \operatorname{lin}\{g_j : j \leq i_0\}$. Let $i \geq i_0$ and $F_i = \operatorname{lin}\{f_{n_j} : j \leq i\}$. Clearly, $g|_{F_i} \in F_i'$ and $\|g|_{F_i}\| \leq \|g\|$. Then there exists some $y_i \in \operatorname{lin}\{x_j : j < n_{i+1}\}$ with $\|y_i\| \leq$

$(1+\epsilon_i)\|g|_{F_i}\|$ such that $f(y_i) = g(f)$ for any $f \in F_i$. Hence $f_{n_j}(y_i) = g(f_{n_j})$ for $j \le i$; for $j > i$ we have $f_{n_j}(y_i) = 0 = g(f_{n_j})$. It follows that $f(y_i) = g(f)$ for any $f \in F$. Thus $S(y_i + M) = g$, so

$$|||S^{-1}g||| = |||y_i + M||| \le \|y_i\| \le (1+\epsilon_i)\|g|_{F_i}\| \le (1+\epsilon_i)\|g\|.$$

Since $\lim_{i\to\infty} \epsilon_i = 0$ we get $|||S^{-1}g||| \le \|g\|$; so $|||x+M||| \le \|S(x+M)\|$. We have shown that $\|S(x+M)\| = |||x+M|||$ for any $x+M \in X/M$; so S is an isometry. □

Lemma 3.3. *Let $(X, \|\cdot\|)$ be a normed space and let $((x_n, f_n)) \subset X \times X'$ be a biorthogonal sequence. Then for any finite subset A of $\mathbb{N}$ there exists $d_A \in (0,1]$ such that*

$$\|\sum_{n=1}^{\infty} \phi_n x_n\| \ge d_A \max\{\max_{n\in A} \|\phi_n x_n\|, \|\sum_{n\in\mathbb{N}\setminus A} \phi_n x_n\|\} \textit{ for all } (\phi_n) \in c_{00}.$$

Proof. Put $d_k = (\|f_k\|\|x_k\|)^{-1}$ for $k \in \mathbb{N}$; then $(d_k) \subset (0,1]$. We shall show that

$$\|\sum_{n=1}^{\infty} \phi_n x_n\| \ge d_k \max\{\|\phi_k x_k\|, \|\sum_{n\ne k} \phi_n x_n\|\} \text{ for } (\phi_n) \in c_{00} \text{ and } k \in \mathbb{N}. \tag{3.2}$$

Let $(\phi_n) \in c_{00}$ and let $k \in \mathbb{N}$. Then

$$|\phi_k| = |f_k(\sum_{n=1}^{\infty} \phi_n x_n)| \le \|f_k\|\|\sum_{n=1}^{\infty} \phi_n x_n\|.$$

Hence

$$\|\sum_{n=1}^{\infty} \phi_n x_n\| \ge |\phi_k|\|f_k\|^{-1} = d_k\|\phi_k x_k\|.$$

Using [7], Lemma 3.1, we get

$$\|\sum_{n=1}^{\infty} \phi_n x_n\| \ge d_k \max\{\|\phi_k x_k\|, \|\sum_{n\ne k} \phi_n x_n\|\}.$$

Let $m \in \mathbb{N}$. Assume that for any m-element subset B of $\mathbb{N}$ we have

$$\|\sum_{n=1}^{\infty} \phi_n x_n\| \ge \prod_{n\in B} d_n \max\{\max_{n\in B} \|\phi_n x_n\|, \|\sum_{n\in\mathbb{N}\setminus B} \phi_n x_n\|\} \text{ for } (\phi_n) \in c_{00}.$$

Let $A \subset \mathbb{N}$ be a set with $m+1$ elements and let $(\phi_n) \in c_{00}$. Take an element k of A and put $B = (A \setminus \{k\})$. Let $\psi_n = 0$ if $n \in B$ and $\psi_n = \phi_n$ if $n \in (\mathbb{N} \setminus B)$; clearly $(\psi_n) \in c_{00}$. Using (3.2) we get

$$\begin{aligned}\|\sum_{n\in\mathbb{N}\setminus B} \phi_n x_n\| = \|\sum_{n=1}^{\infty} \psi_n x_n\| &\ge d_k \max\{\|\psi_k x_k\|, \|\sum_{n\ne k} \psi_n x_n\|\} \\ &= d_k \max\{\|\phi_k x_k\|, \|\sum_{n\in\mathbb{N}\setminus A} \phi_n x_n\|\}.\end{aligned}$$

Hence, by our assumption, we have

$$\|\sum_{n=1}^{\infty}\phi_n x_n\| \geq \prod_{n\in B} d_n \max\{\max_{n\in B}\|\phi_n x_n\|, \|\sum_{n\in\mathbb{N}\setminus B}\phi_n x_n\|\}$$
$$\geq \prod_{n\in A} d_n \max\{\max_{n\in A}\|\phi_n x_n\|, \|\sum_{n\in\mathbb{N}\setminus A}\phi_n x_n\|\}.$$

Thus, by induction, we have shown our lemma. □

Lemma 3.4. *Let $(X, \|\cdot\|)$ be a normed space and let $((x_n, f_n)) \subset X \times X'$ be a biorthogonal sequence with $lin\{x_n : n \in \mathbb{N}\} = X$. Put $X_S = \bigcap_{n\in S} \ker f_n$ for $S \subset \mathbb{N}$. Assume that $C, B \subset \mathbb{N}$ and $C \setminus B$ is finite. If $(x_n + X_B)_{n\in B}$ is t-orthogonal in X/X_B for some $t \in (0, 1]$, then $(x_n + X_C)_{n\in C}$ is s-orthogonal in X/X_C for some $s \in (0, 1]$.*

Proof. Put $A = C \setminus B$, $D = C \cap B$ and $H = B \setminus C$. Let $Y_S = \text{lin}\{x_n : n \in S\}$ for $S \subset \mathbb{N}$; then we have $Y_S = \{\sum_{n=1}^{\infty}\varphi_n x_n : (\varphi_n) \in c_{00}(S)\} = X_{\mathbb{N}\setminus S}$. It is easy to see that $X_D = X_C + Y_A = X_B + Y_H$. Let d_A be as in Lemma 3.3 and put $s = td_A^2$.

For $m \in D$ we have $|||x_m + X_D||| \geq d_A|||x_m + X_C|||$. Indeed, let $m \in D$ and let $x \in X_D$. Then $x = y + \sum_{n=1}^{\infty}\psi_n x_n$ for some $y \in X_C$ and some $(\psi_n) \in c_{00}(A)$. Using Lemma 3.3 we get

$$\|x_m+x\| = \|x_m+y+\sum_{n=1}^{\infty}\psi_n x_n\| \geq d_A \max\{\max_{n\in\mathbb{N}}\|\psi_n x_n\|, \|x_m+y\|\} \geq d_A|||x_m+X_C|||.$$

Hence we obtain

$$|||\sum_{n=1}^{\infty}\alpha_n x_n + X_D||| \geq td_A \max_{n\in\mathbb{N}}|||\alpha_n x_n + X_C||| \text{ for } (\alpha_n) \in c_{00}(D).$$

Indeed, let $(\alpha_n) \in c_{00}(D)$ and let $x \in X_D$. For some $y \in X_B$ and some $(\beta_n) \in c_{00}(H)$ we have $x = y + \sum_{n=1}^{\infty}\beta_n x_n$. Then

$$\|\sum_{n=1}^{\infty}\alpha_n x_n + x\| \geq |||\sum_{n=1}^{\infty}\alpha_n x_n + \sum_{n=1}^{\infty}\beta_n x_n + X_B||| \geq t\max_{n\in\mathbb{N}}|||\alpha_n x_n + X_B|||$$
$$\geq t\max_{n\in\mathbb{N}}|||\alpha_n x_n + X_D||| \geq td_A \max_{n\in\mathbb{N}}|||\alpha_n x_n + X_C|||.$$

We shall prove that the sequence $(x_n + X_C)_{n\in C}$ is s-orthogonal in X/X_C. To show this it is enough to prove that

$$|||\sum_{n=1}^{\infty}\phi_n x_n + X_C||| \geq s\max_{n\in\mathbb{N}}|||\phi_n x_n + X_C||| \text{ for } (\phi_n) \in c_{00}(C).$$

Let $(\phi_n) \in c_{00}(C)$. Then $(\phi_n) = (\gamma_n) + (\alpha_n)$ for some $(\gamma_n) \in c_{00}(A)$ and some $(\alpha_n) \in c_{00}(D)$. Let $x \in X_C$. Using Lemma 3.3 we get

$$\begin{aligned} \|\sum_{n=1}^{\infty} \phi_n x_n + x\| &= \|\sum_{n=1}^{\infty} \gamma_n x_n + \sum_{n=1}^{\infty} \alpha_n x_n + x\| \\ &\geq d_A \max\{\max_{n\in\mathbb{N}} \|\gamma_n x_n\|, \|\sum_{n=1}^{\infty} \alpha_n x_n + x\|\} \\ &\geq d_A \max\{\max_{n\in\mathbb{N}} |||\gamma_n x_n + X_C|||, |||\sum_{n=1}^{\infty} \alpha_n x_n + X_D|||\} \\ &\geq t d_A^2 \max\{\max_{n\in\mathbb{N}} |||\gamma_n x_n + X_C|||, \max_{n\in\mathbb{N}} |||\alpha_n x_n + X_C|||\} = s \max_{n\in\mathbb{N}} |||\phi_n x_n + X_C|||. \end{aligned}$$

□

Lemma 3.5. *Let $(X, \|\cdot\|)$ be a normed space and let $((x_n, f_n)) \subset X \times X'$ be a biorthogonal sequence such that the subspace $X_0 = lin\{x_n : n \in \mathbb{N}\}$ is dense in X. Let $L \subset \mathbb{N}$. Put $W = \bigcap_{n\in L} \ker f_n$ and $W_0 = X_0 \cap W$. If the sequence $(x_n + W_0)_{n\in L}$ is t-orthogonal in X_0/W_0 for some $t \in (0, 1]$, then $(x_n + W)_{n\in L}$ is a t-orthogonal basis in X/W and*

$$\|f_n\|^{-1} \leq |||x_n + W||| \leq (t\|f_n\|)^{-1} \text{ for any } n \in L.$$

Proof. For any $x \in X$ we have $\inf\{\|x + y\| : y \in W_0\} = \inf\{\|x + y\| : y \in \overline{W}_0\}$, where $\overline{W}_0$ is the closure of W_0 in X. Thus $(x_n + \overline{W}_0)_{n\in L}$ is t-orthogonal in $(X/\overline{W}_0)$.

Denote by π the quotient map $X \to X/\overline{W}_0$. We have

$$\pi(X_0) = \text{lin}\{x_n + \overline{W}_0 : n \in L\},$$

so $(x_n + \overline{W}_0)$ is linearly dense in $X/\overline{W}_0$. It follows that $(x_n + \overline{W}_0)_{n\in L}$ is a t-orthogonal basis in $X/\overline{W}_0$.

We shall prove that $\overline{W}_0 = W$. For any $n \in L$ the functional

$$\tilde{f}_n : X/\overline{W}_0 \to \mathbb{K},\ \tilde{f}_n(x + \overline{W}_0) = f_n(x)$$

is well defined, linear and continuous. Indeed, for any neighbourhood V of zero in $\mathbb{K}$ we have $f_n^{-1}(V) = \pi^{-1}(\tilde{f}_n^{-1}(V))$, so the set $\tilde{f}_n^{-1}(V) = \pi(f_n^{-1}(V))$ is open in $X/\overline{W}_0$. Clearly $\overline{W}_0 \subset W$. Let $w \in W$. Then for some $(\alpha_m) \subset \mathbb{K}$ we have $w + \overline{W}_0 = \sum_{m\in L} \alpha_m(x_m + \overline{W}_0)$. Hence for any $n \in L$ we get

$$0 = f_n(w) = \tilde{f}_n(w + \overline{W}_0) = \sum_{m\in L} \alpha_m \tilde{f}_n(x_m + \overline{W}_0) = \alpha_n.$$

Thus $w + \overline{W}_0 = 0$, so $w \in \overline{W}_0$. It follows that $\overline{W}_0 = W$.

We have shown that $(x_n + W)_{n\in L}$ is a t-orthogonal basis in X/W.

Let $n \in L$. It is easy to see that

$$x + W = \sum_{m\in L} f_m(x)(x_m + W) \text{ for } x \in X.$$

For $x \in X$ we get

$$\|x\| \geq |||x + W||| \geq t \max_{m\in L} |f_m(x)|\, |||x_m + W||| \geq t|f_n(x)|\, |||x_n + W|||.$$

Thus

$$\|f_n\| = \sup_{x\in X\setminus\{0\}} |f_n(x)|\|x\|^{-1} \leq (t|||x_n + W|||)^{-1}.$$

Hence $|||x_n + W||| \leq (t\|f_n\|)^{-1}$. On the other hand we have

$$1 = |f_n(x_n + w)| \leq \|f_n\|\|x_n + w\| \text{ for } w \in W.$$

Thus $\|f_n\|^{-1} \leq |||x_n + W|||$. This completes the proof. □

Let E be a metrizable lcs with a non-decreasing base $(\|\cdot\|_k)$ in $\mathcal{P}(E)$. For $k \in \mathbb{N}$ we denote by $E_k^{'}$ (or $E_{\|\cdot\|_k}^{'}$) the linear subspace of $E^{'}$ consisting of all $f \in E^{'}$ such that for some $C > 0$ we have $|f(x)| \leq C\|x\|_k$ for any $x \in E$. Clearly, the map

$$\|\cdot\|_k^{'} : E_k^{'} \to [0, +\infty), \|f\|_k^{'} = \inf\{C > 0 : |f(x)| \leq C\|x\|_k \text{ for any } x \in E\}$$

is a norm on $E_k^{'}$, and the normed space $(E_k', \|\cdot\|_k')$ is complete. Moreover we have $\bigcup_{l=1}^{\infty} E_l^{'} = E^{'}$, $E_k^{'} \subset E_{k+1}^{'}$ and $\|f\|_{k+1}^{'} \leq \|f\|_k^{'}$ for $f \in E_k^{'}$ and $k \in \mathbb{N}$.

Clearly, a metrizable lcs has a non-decreasing base of continuous norms if and only if it has a continuous norm.

For biorthogonal sequences in a metrizable lcs with a continuous norm we get

Proposition 3.6. *Let E be a metrizable lcs with a non-decreasing base $(\|\cdot\|_k)$ of continuous norms and let $((x_n, f_n)) \subset E \times E_1^{'}$, where $E_1 = (E, \|\cdot\|_1)$, is a biorthogonal sequence such that (x_n) is linearly dense in E. Let L be an infinite subset of $\mathbb{N}$. Then there exists an infinite subset J of L such that the sequence $(x_n + W)_{n\in J}$, where $W = \bigcap_{n\in J} \ker f_n$, is an orthogonal basis in the quotient space E/W with a continuous norm. The space E/W is nuclear if*

$$\lim_{n\in J} \frac{\|f_n\|_k'}{\|f_n\|_{k+1}'} = \infty \text{ for any } k \in \mathbb{N}.$$

Proof. Let $t \in (0, 1)$. Put $E_0 = \text{lin}\{x_n : n \in \mathbb{N}\}$, $E_k = (E, \|\cdot\|_k)$ and $E_{(k)} = (E_0, \|\cdot\|_k | E_0)$ for $k \in \mathbb{N}$. Clearly, $E_{(1)}' \subset E_{(k)}'$ and E_0 is dense in E_k for any $k \in \mathbb{N}$. By Proposition 3.2 we can inductively choose a decreasing sequence (J_k) of infinite subsets of L such that the sequence $(x_n + W_k)_{n\in J_k}$ is t-orthogonal in $E_{(k)}/W_k$ for $k \in \mathbb{N}$, where $W_k = \bigcap_{n\in J_k} \ker f_n \cap E_0$.

Let $(m_k) \subset \mathbb{N}$ be an increasing sequence such that $m_k \in J_k$ for $k \in \mathbb{N}$. Put $J = \{m_k : k \in \mathbb{N}\}$, $W = \bigcap_{n\in J} \ker f_n$ and $W_0 = E_0 \cap W$. Clearly, the set $(J \setminus J_k)$ is finite for any $k \in \mathbb{N}$. By Lemma 3.4 there exists $(s_k) \subset (0, 1]$ such that the sequence $(x_n + W_0)_{n\in J}$ is s_k-orthogonal in $(E_{(k)}/W_0)$ for any $k \in \mathbb{N}$.

Using Lemma 3.5 we infer that the sequence $(x_n + W)_{n\in J}$ is an s_k-orthogonal basis in the quotient space E_k/W for any $k \in \mathbb{N}$. It follows that $(x_n + W)_{n\in J}$ is an orthogonal basis in E/W (see [4], Proposition 2.6). W is closed in E_1, so the space E/W has a continuous norm.

If $\lim_{n\in J}[\|f_n\|_k'/\|f_n\|_{k+1}'] = \infty$ for any $k \in \mathbb{N}$, then using Lemma 3.5 we get

$$\lim_{n\in J} \frac{|||x_n + W|||_k}{|||x_n + W|||_{k+1}} = 0 \text{ for any } k \in \mathbb{N}.$$

It follows that E/W is nuclear (see [2], Proposition 3.5, and its proof). □

Corollary 3.7. *Let E be a Fréchet space with a continuous norm. For any biorthogonal sequence $((x_n, f_n)) \subset E \times E^{'}$ such that (x_n) is linearly dense in E and (f_n) is equicontinuous, there exists an infinite subset J of $\mathbb{N}$ such that $(E/\bigcap_{n\in J} \ker f_n)$ is a Köthe space.*

Corollary 3.8. *Any infinite-dimensional metrizable lcs E of countable type with a continuous norm has an infinite-dimensional quotient with a continuous norm and with an orthogonal basis.*

Proof. Let $(\|\cdot\|_k)$ be a non-decreasing base of continuous norms on E and let (y_n) be a linearly independent sequence in E such that its linear span E_0 is dense in E. Put $E_1 = (E, \|\cdot\|_1)$.

Using the p-adic Hahn-Banach theorem we can inductively choose a sequence $((x_n, f_n)) \subset E_1 \times E_1^{'}$ such that $\mathrm{lin}\{x_i : i \leq n\} = \mathrm{lin}\{y_i : i \leq n\}$, $x_n \in \bigcap_{k<n} \ker f_k$, $f_n \in \ (\mathrm{lin}\{x_i : i < n\})^{\perp}$ and $f_n(x_n) = 1$ for any $n \in \mathbb{N}$. Clearly, the sequence $((x_n, f_n))$ is biorthogonal and $\mathrm{lin}\{x_n : n \in \mathbb{N}\} = E_0$. Using Proposition 3.6 completes the proof. □

Corollary 3.9. *Any infinite-dimensional Fréchet space E of countable type with a continuous norm has a Köthe quotient.*

Clearly, any Fréchet space which is not of finite type has an infinite-dimensional quotient with a continuous norm. Hence using the previous corollary we get the following one (see [10], Corollary 13).

Corollary 3.10. *An infinite-dimensional Fréchet space E of countable type has a Köthe quotient if and only if it is not isomorphic to $\mathbb{K}^{\mathbb{N}}$.*

Let E be a Fréchet space with a non-decreasing base $(\|\cdot\|_k)$ in $\mathcal{P}(E)$. Let M be a closed subspace of E. Let $k \in \mathbb{N}$ and $f \in E_k^{'} \cap M^{\perp}$. Then the functional $\phi_M(f) : E/M \to \mathbb{K}$, $x + M \to f(x)$ is well defined and linear. Moreover we have

$$|(\phi_M(f))(x+M)| = |f(x)| = |f(x+y)| \leq \|f\|_k^{'} \|x+y\|_k \text{ for all } x \in E \text{ and } y \in M.$$

Hence $|(\phi_M(f))(x+M)| \leq \|f\|_k^{'} |||x+M|||_k$ for $x \in E$, so $\phi_M(f) \in (E/M)_k^{'}$ and $|||\phi_M(f)|||_k^{'} \leq \|f\|_k^{'}$. On the other hand we get $\|f\|_k^{'} \leq |||\phi_M(f)|||_k^{'}$ since

$$|f(x)| = |(\phi_M(f))(x+M)| \leq |||\phi_M(f)|||_k^{'} |||x+M|||_k \leq |||\phi_M(f)|||_k^{'} \|x\|_k \text{ for } x \in E.$$

We have shown that $|||\phi_M(f)|||_k^{'} = \|f\|_k^{'}$ for all $f \in E_k' \cap M^{\perp}$ and $k \in \mathbb{N}$.

If $k \in \mathbb{N}$, $g \in (E/M)_k^{'}$ and $\pi : E \to E/M$ is the quotient map, then $g \circ \pi \in E_k^{'} \cap M^{\perp}$. It follows that the map

$$\phi_M : M^{\perp} \to (E/M)^{'},\ f \to \phi_M(f)$$

is well defined and surjective.

Clearly, ϕ_M is linear and injective. Thus ϕ_M is an isomorphism. Moreover we have

$$|||\phi_M(f)|||_k^{'} = \|f\|_k^{'} \text{ for all } f \in E_1' \cap M^{\perp} \text{ and } k \in \mathbb{N}.$$

Now we can prove our main result.

Theorem 3.11. *A Fréchet space E of countable type has a nuclear Köthe quotient if and only if for some non-decreasing base $(\|\cdot\|_k)$ in $\mathcal{P}(E)$ the norms $\|\cdot\|_k'|_{E_1'}$ and $\|\cdot\|_{k+1}'|_{E_1'}$ are not equivalent for any $k \in \mathbb{N}$.*

Proof. (A) Assume that E has a nuclear Köthe quotient $Z = E/M$. Let (z_n) be a Schauder basis in Z and let $(g_n) \subset Z^{'}$ be a sequence of coefficient functionals associated with the basis (z_n). Clearly, (z_n) is (1)-orthogonal with respect to some non-decreasing base of norms (p_k) in $\mathcal{P}(Z)$. By the nuclearity of Z we can assume that $\lim_{n\to\infty}[p_{k+1}(z_n)/p_k(z_n)] = \infty$ for $k \in \mathbb{N}$ (see [2], Proposition 3.5).

We shall prove that $\lim_{n\to\infty}[p_k^{'}(g_n)/p_{k+1}^{'}(g_n)] = \infty$ for $k \in \mathbb{N}$. We have $p_k(z) = \max_{m\in\mathbb{N}} |g_m(z)|p_k(z_m) \geq |g_n(z)|p_k(z_n)$ for $z \in Z$ and $k, n \in \mathbb{N}$, so $(g_n) \subset Z_{p_1}^{'}$ and $p_k^{'}(g_n) \leq [p_k(z_n)]^{-1}$ for $k, n \in \mathbb{N}$. On the other hand we have $1 = |g_n(z_n)| \leq p_k^{'}(g_n)p_k(z_n)$ for $k, n \in \mathbb{N}$. Thus $p_k^{'}(g_n) = [p_k(z_n)]^{-1}$ for all $k, n \in \mathbb{N}$. Hence we get $\lim_{n\to\infty}[p_k^{'}(g_n)/p_{k+1}^{'}(g_n)] = \infty$ for $k \in \mathbb{N}$.

Let $(|\cdot|_k)$ be a non-decreasing base in $\mathcal{P}(E)$ and let $(|||\cdot|||_k)$ be the base in $\mathcal{P}(Z)$ induced by $(|\cdot|_k)$. Passing to subsequences we can assume that $p_k \leq |||\cdot|||_k \leq p_{k+1}$ for $k \in \mathbb{N}$. Then $p_{k+1}^{'}|_{Z_{p_1}^{'}} \leq |||\cdot|||_k^{'}|_{Z_{p_1}^{'}} \leq p_k^{'}|_{Z_{p_1}^{'}}$ for $k \in \mathbb{N}$.

For $g \in Z_{p_1}^{'} = (E/M)_{p_1}^{'}$ and $h = \phi_M^{-1}(g)$ we have

$$|h(x)| = |(\phi_M(h))(x+M)| = |g(x+M)|$$
$$\leq p_1^{'}(g)p_1(x+M) \leq p_1^{'}(g)|||x+M|||_1 \leq p_1'(g)|x|_1$$

for $x \in E$, so $\phi_M^{-1}(g) \in E_{|\cdot|_1}^{'}$ for $g \in Z_{p_1}^{'}$.

Put $\|\cdot\|_k = |\cdot|_{2k-1}$ for $k \in \mathbb{N}$. Then $(\|\cdot\|_k)$ is a non-decreasing base in $\mathcal{P}(E)$ and $(\phi_M^{-1}(g_n)) \subset E_{|\cdot|_1}^{'} = E_{\|\cdot\|_1}^{'}$. Moreover we have

$$\frac{\|\phi_M^{-1}(g_n)\|_k^{'}}{\|\phi_M^{-1}(g_n)\|_{k+1}^{'}} = \frac{|\phi_M^{-1}(g_n)|_{2k-1}^{'}}{|\phi_M^{-1}(g_n)|_{2k+1}^{'}} = \frac{|||g_n|||_{2k-1}^{'}}{|||g_n|||_{2k+1}^{'}} \geq \frac{p_{2k}^{'}(g_n)}{p_{2k+1}^{'}(g_n)} \text{ for all } k, n \in \mathbb{N}.$$

It follows that the norms $\|\cdot\|_k^{'}|_{E_{\|\cdot\|_1}^{'}}$ and $\|\cdot\|_{k+1}^{'}|_{E_{\|\cdot\|_1}^{'}}$ are not equivalent for any $k \in \mathbb{N}$.

(B) Now assume that E has a non-decreasing base $(\|\cdot\|_k)$ in $\mathcal{P}(E)$ such that the norms $\|\cdot\|_k^{'}|_{E_1^{'}}$ and $\|\cdot\|_{k+1}^{'}|_{E_1^{'}}$ are not equivalent for any $k \in \mathbb{N}$. Without loss of generality we can assume that $\|\cdot\|_k$, $k \in \mathbb{N}$, are norms. Indeed, put $M = ^{\perp} E_1'$ and $Z = E/M$. Let $x \in E$ with $|||x+M|||_1 = 0$. Then there exists $(y_n) \subset M$ such that $\lim_{n\to\infty} \|x-y_n\|_1 = 0$. Let $f \in E_1'$. Then $|f(x)| = \lim_{n\to\infty} |f(x-y_n)| = 0$, so $f(x) = 0$. It follows that $x \in M$, so $(|||\cdot|||_k)$ is a non-decreasing base of norms in $\mathcal{P}(Z)$. Clearly $E_1^{'} \subset M^{\perp}$. Thus $|||\phi_M(f)|||_k^{'} = \|f\|_k^{'}$ for all $f \in E_1'$ and $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then there exists a sequence $(h_n) \subset E_1^{'}$ such that $\lim_{n\to\infty}[\|h_n\|_k^{'}/\|h_n\|_{k+1}^{'}] = \infty$. Hence $(\phi_M(h_n)) \subset Z_{|||\cdot|||_1}^{'}$ and $\lim_{n\to\infty}[|||\phi_M(h_n)|||_k^{'}/|||\phi_M(h_n)|||_{k+1}^{'}] = \infty$, so the norms $|||\cdot|||_k^{'}|_{Z_{|||\cdot|||_1}^{'}}$ and $|||\cdot|||_{k+1}^{'}|_{Z_{|||\cdot|||_1}^{'}}$ are not equivalent for any $k \in \mathbb{N}$. If Z has a nuclear Köthe quotient, then E has one.

Thus from now on we shall assume that $\|\cdot\|_k$, $k \in \mathbb{N}$, are norms on E. We show that for any linear subspace G of $E_1^{'}$ with $\dim(E_1^{'}/G) < \infty$ we have

$$\sup\{\|g\|_k^{'}/\|g\|_{k+1}^{'} : g \in (G \setminus \{0\})\} = \infty \text{ for } k \in \mathbb{N}. \tag{3.3}$$

Indeed, suppose by contradiction that for some subspace G of finite codimension in E_1' there exist some $k \in \mathbb{N}$ and $C > 0$ such that $\|g\|_k^{'} \leq C\|g\|_{k+1}^{'}$ for any $g \in G$. Denote by G_k the closure of G in the Banach space $(E_k^{'}, \|\cdot\|_k^{'})$. For any $h \in G_k$ there exists $(g_n) \subset G$ with $\lim_{n\to\infty} \|h-g_n\|_k^{'} = 0$. Since $\|h-g_n\|_{k+1}^{'} \leq \|h-g_n\|_k^{'}$, then $\lim_{n\to\infty} \|g_n\|_{k+1}^{'} = \|h\|_{k+1}^{'}$ and $\lim_{n\to\infty} \|g_n\|_k^{'} = \|h\|_k^{'}$. By our assumption we get $\|g_n\|_k^{'} \leq C\|g_n\|_{k+1}^{'}$ for $n \in \mathbb{N}$. Thus $\|h\|_{k+1}^{'} \leq \|h\|_k^{'} \leq C\|h\|_{k+1}^{'}$ for any $h \in G_k$. It follows that the norms $\|\cdot\|_{k+1}^{'}|_{G_k}$ and $\|\cdot\|_k^{'}|_{G_k}$ are equivalent, so the normed space $(G_k, \|\cdot\|_{k+1}^{'}|_{G_k})$ is complete. Hence G_k is closed in $(E_{k+1}^{'}, \|\cdot\|_{k+1}^{'})$. For some finite-dimensional subspace S of E_1' we have $E_1^{'} = G + S$. Put $H_k = G_k + S$.

Then $E_1' \subset H_k$; moreover H_k is closed in $E_k^{'}$ and $E_{k+1}^{'}$ ([6], Lemma 3.14). Thus the norms $\|\cdot\|_k^{'}|_{H_k}$ and $\|\cdot\|_{k+1}^{'}|_{H_k}$ are equivalent, a contradiction.

Using (3.3) we shall construct a biorthogonal sequence $((x_n, f_n)) \subset E \times E_1^{'}$ such that the sequence (x_n) is linearly dense in E and $\lim_{n\to\infty}[\|f_n\|_{k+1}^{'}/\|f_n\|_k^{'}] = 0$ for $k \in \mathbb{N}$. Let E_0 be an $\aleph_0$-dimensional dense subspace of E with a Hamel basis (y_n). Put $x_1 = y_1$. Let $f_1 \in E_1^{'}$ with $f_1(x_1) = 1$. Suppose that for some $n \geq 2$ we have $\{(x_k, f_k) : k < n\} \subset E \times E_1'$ with $\operatorname{lin}\{x_k : k < n\} = \operatorname{lin}\{y_k : k < n\}$ such that $f_k(x_i) = \delta_{k,i}$ for $k < n$, $i < n$, and $[\|f_m\|_{k+1}^{'}/\|f_m\|_k^{'}] < m^{-1}$ for $m < n$ and $k < m$. Let $g_n \in E_1^{'} \cap (\operatorname{lin}\{y_k : k < n\})^{\perp}$ such that $g_n(y_n) = 1$ and $|g_n(y_{n+1})| > \|y_{n+1}\|_n$; then $\|g_n\|_n' > 1$. Using (3.3) we can inductively choose

$$g_{n-1}, \ldots, g_1 \in E_1^{'} \cap (\operatorname{lin}\{y_s : s \leq n\})^{\perp}$$

such that

$$\|g_k\|_{k+1}^{'} < 1 \text{ and } \|g_k\|_k^{'} > n \max_{k<i\leq n} \|g_i\|_k^{'} \text{ for } k = n-1, \ldots, 1.$$

Put $x_n = y_n - \sum_{k=1}^{n-1} f_k(y_n)x_k$ and $f_n = \sum_{k=1}^{n} g_k$. Then $(x_n, f_n) \in E \times E_1'$ and $\operatorname{lin}(\{x_k : k \leq n\}) = \operatorname{lin}(\{y_k : k \leq n\})$. Moreover $f_n(x_n) = 1$ and $f_i(x_n) = 0 = f_n(x_i)$ for $i < n$. We shall show that $[\|f_n\|_{k+1}^{'}/\|f_n\|_k^{'}] < n^{-1}$ for $k < n$. Let $k < n$. Then

$$\max_{i\leq k} \|g_i\|_{k+1}' \leq \max_{i\leq k} \|g_i\|_{i+1}' < 1 < \|g_n\|_n' \leq \|g_n\|_{k+1}' \leq \max_{k<i\leq n} \|g_i\|_{k+1}'$$

and

$$\max_{k<i\leq n} \|g_i\|_{k+1}' \leq \max_{k<i\leq n} \|g_i\|_k' < n^{-1}\|g_k\|_k'.$$

Thus $\|f_n\|_{k+1}^{'} < n^{-1}\|g_k\|_k'$. On the other hand we have

$$\|g_i\|_k^{'} \leq \|g_i\|_{i+1}^{'} < 1 < \|g_n\|_n' \leq \|g_n\|_k' < \|g_k\|_k^{'} \text{ for } i < k$$

and $\|g_i\|_k^{'} < \|g_k\|_k^{'}$ for $k < i \leq n$. Thus $\|g_i\|_k^{'} < \|g_k\|_k^{'}$ for $i \leq n$ with $i \neq k$. Hence $\|f_n\|_k^{'} = \|g_k\|_k^{'}$. It follows that $[\|f_n\|_{k+1}^{'}/\|f_n\|_k^{'}] < n^{-1}$.

Thus we have inductively constructed a biorthogonal sequence $((x_n, f_n)) \subset E \times E_1^{'}$ with $\operatorname{lin}\{x_n : n \in \mathbb{N}\} = E_0$ such that $[\|f_n\|_{k+1}^{'}/\|f_n\|_k^{'}] < n^{-1}$ for all $k, n \in \mathbb{N}$ with $k < n$, so $\lim_{n\to\infty}[\|f_n\|_{k+1}^{'}/\|f_n\|_k^{'}] = 0$ for any $k \in \mathbb{N}$.

Using Proposition 3.6 we infer that E has a nuclear Köthe quotient; in fact for any infinite subset L of $\mathbb{N}$ there exists an infinite subset J of L such that E/W is a nuclear Köthe quotient of E, where $W = \bigcap_{n\in J} \ker f_n$. □

In [10], Theorem 11, we have shown that any nuclear Fréchet space which is not of finite type has a nuclear Köthe quotient. Now we can generalize that result by proving the following.

Theorem 3.12. *Any Fréchet-Montel space E which is not of finite type has a nuclear Köthe quotient.*

Proof. By Theorem 3.11 it is enough to show that for some non-decreasing base $(\|\cdot\|_k)$ in $\mathcal{P}(E)$ the norms $\|\cdot\|_k'|_{E_1'}$ and $\|\cdot\|_{k+1}'|_{E_1'}$ are not equivalent for any $k \in \mathbb{N}$. Suppose, by contradiction, that it is not true. Let $(|\cdot|_k)$ be a non-decreasing base in $\mathcal{P}(E)$. Then for any $l \in \mathbb{N}$ there is some $k \geq l$ such that $|\cdot|_k^{'}|_{E_l'} \approx |\cdot|_j^{'}|_{E_l'}$ for any $j > k$. Passing to a subsequence we can assume that $|\cdot|_{l+1}^{'}|_{E_l'} \approx |\cdot|_j^{'}|_{E_l'}$ for all $l, j \in \mathbb{N}$ with $l+1 < j$. Let F_l denote the closure of $E_l^{'}$ in $(E_{l+1}^{'}, |\cdot|_{l+1}^{'})$ for $l \in \mathbb{N}$.

We have $|\cdot|_{l+1}^{'}|_{F_l} \approx |\cdot|_{l+2}^{'}|_{F_l}$ for $l \in \mathbb{N}$. Indeed, let $l \in \mathbb{N}$. For some $C > 0$ we have $|f|_{l+1}^{'} \leq C|f|_{l+2}^{'}$ for any $f \in E_l'$. Let $g \in F_l$ and let $(f_n) \subset E_l^{'}$ with $\lim_{n\to\infty} |f_n - g|_{l+1}^{'} = 0$. Thus $\lim_{n\to\infty} |f_n - g|_{l+2}^{'} = 0$, so $\lim_{n\to\infty} |f_n|_{l+1}^{'} = |g|_{l+1}^{'}$ and $\lim_{n\to\infty} |f_n|_{l+2}^{'} = |g|_{l+2}^{'}$. Hence $|g|_{l+2}' \leq |g|_{l+1}^{'} \leq C|g|_{l+2}^{'}$, so $|\cdot|_{l+1}^{'}|_{F_l} \approx |\cdot|_{l+2}^{'}|_{F_l}$.

It follows that the normed space $(F_l, |\cdot|_{l+2}^{'}|_{F_l})$ is complete, so F_l is a closed subspace of $F_{l+1} = (F_{l+1}, |\cdot|_{l+2}^{'}|_{F_{l+1}})$. Thus (F_l) is a strict inductive sequence with $\bigcup_{l=1}^{\infty} F_l = E^{'}$. Let $F = \varinjlim F_l$.

It is easy to see that the identity map $I : F \to E_b'$ is continuous. E is a Fréchet-Montel space, so its strong dual E_b' is an LB-space ([3], Corollary 2.5.9). Hence, by the open mapping theorem for LF-spaces ([5], Theorem 3.1), the map I is open. Thus $E_b' = \varinjlim F_l$. Since E_b' is a Montel space ([8], Theorem 10.7) the Banach spaces $F_n, n \in \mathbb{N}$, are finite-dimensional. Hence the spaces $E_n', n \in \mathbb{N}$, are finite-dimensional, too. It follows that E is of finite type, a contradiction. □

In the case when $\mathbb{K}$ is not spherically complete we get

Theorem 3.13. *Assume that $\mathbb{K}$ is not spherically complete. Let E be an infinite-dimensional Fréchet space of countable type which is not isomorphic to the strong dual of a strict LB-space. Then E has a nuclear Köthe quotient.*

Proof. Suppose, by contradiction, that E has no nuclear Köthe quotient. Then, as in the proof of Theorem 3.12, we get a non-decreasing base $(|\cdot|_k)$ in $\mathcal{P}(E)$ such that the sequence (F_l), where F_l is the closure of E_l' in $(E_{l+1}', |\cdot|_{l+1}')$, is a strict inductive sequence with $\bigcup_l F_l = E'$. Let $F = \varinjlim F_l$. We shall prove that $F_b^{'} = (E_b^{'})_b^{'}$.

Let $V_n = \{x \in E : |x|_n \leq 1\}$ and $B_n = \{f \in E_n' : |f|_n^{'} \leq 1\}$ for $n \in \mathbb{N}$. Clearly $B_n \subset \{f \in F_n : |f|_{n+1}' \leq 1\} \subset B_{n+1}$ for $n \in \mathbb{N}$.

We shall prove that (B_n) is a fundamental sequence of bounded subsets of $E_b^{'}$. Let A be a closed absolutely convex bounded subset of E and let $n \in \mathbb{N}$. Then $^\circ(A^\circ) = A^e$ ([8], p. 199), so $^\circ(A^\circ) \subset \alpha V_n \subset \alpha(^\circ B_n)$ for some $\alpha \in \mathbb{K}$. Hence $B_n \subset \alpha[^\circ(A^\circ)]^\circ \subset \alpha A^\circ$. Thus $(B_n) \subset \mathcal{B}(E_b^{'})$. Let $B \in \mathcal{B}(E_b^{'})$ and let $\beta \in \mathbb{K}$ with $|\beta| > 1$. Then $^\circ B$ is a barrel in E, so it is a neighbourhood of zero in E. Therefore $\beta V_k \subset {}^\circ B$ for some $k \in \mathbb{N}$. Hence $B \subset (^\circ B)^\circ \subset \beta^{-1} V_k^\circ \subset B_k$.

It follows that $\mathcal{B}(F) = \mathcal{B}(E_b^{'})$.

Let f be a linear functional on F which is bounded on bounded sets. Then $f|F_n$ is bounded, so continuous, for any $n \in \mathbb{N}$. Hence f is continuous on F ([3], Proposition 1.1.6). By Proposition 3.14 (see below) any linear functional on $E_b^{'}$ which is bounded on bounded sets is continuous. It follows that $F_b^{'} = (E_b^{'})_b^{'}$.

The Fréchet space E is reflexive since $\mathbb{K}$ is not spherically complete ([8], Theorem 10.3). Thus E is isomorphic to the strong dual of a strict LB-space, a contradiction. □

In the proof of our previous theorem we used the following

Proposition 3.14. *Assume that $\mathbb{K}$ is not spherically complete. Let E be a Fréchet space of countable type. Then any linear functional f on $E_b^{'}$ which is bounded on bounded subsets of $E_b^{'}$ is continuous.*

Proof. Let (p_k) be a non-decreasing base in $\mathcal{P}(E)$. Let $k \in \mathbb{N}$. Denote by (G_k, q_k) the Banach space (E'_k, p'_k). By our assumption the functional $f|_{G_k}$ is continuous on (G_k, q_k). Let $\pi_k : E \to E_{p_k}$ be the quotient map. For any $h \in G_k$ the functional $h_k : (E_{p_k}, \overline{p_k}) \to \mathbb{K}$, $\pi_k(x) \to h(x)$ is well defined, linear and continuous and the linear map $(G_k, q_k) \to (E_{p_k}, \overline{p_k})' : h \to h_k$ is an isomorphism. The space $(E_{p_k}, \overline{p_k})$ is of countable type, so its completion $(\tilde{E_{p_k}}, \tilde{p_k})$ is a reflexive Banach space ([8], Corollary 9.9).

Thus $(G_k, q_k)'$ is isomorphic to $(\tilde{E}_{p_k}, \tilde{p}_k)$, so there exists $y_k \in \tilde{E}_{p_k}$ such that $f(h) = \tilde{h_k}(y_k)$ for any $h \in G_k$, where $\tilde{h_k} \in (\tilde{E_{p_k}}, \tilde{p_k})'$ with $\tilde{h_k}|_{E_{p_k}} = h_k$.

It follows that $\tilde{h_k}(y_k) = \tilde{h_{k+1}}(y_{k+1})$ for all $h \in G_k$, $k \in \mathbb{N}$.

For any $k \in \mathbb{N}$ there exists $(y_{k,n}) \subset E_{p_k}$ with $\lim_{n\to\infty} \tilde{p_k}(y_k - y_{k,n}) = 0$; clearly $\lim_{n,m\to\infty} \overline{p_k}(y_{k,n} - y_{k,m}) = 0$. Moreover $h_k \circ \phi_k = h_{k+1}$, where

$$\phi_k : E_{p_{k+1}} \to E_{p_k}, \pi_{k+1}(x) \to \pi_k(x) \text{ for } k \in \mathbb{N}.$$

Thus $\lim_{n\to\infty} h_k(y_{k,n}) = \lim_{n\to\infty} h_k(\phi_k(y_{k+1,n}))$ for all $h \in G_k$, $k \in \mathbb{N}$, so we obtain $\lim_{n\to\infty} \phi(y_{k,n} - \phi_k(y_{k+1,n})) = 0$ for all $\phi \in (E_{p_k}, \overline{p_k})'$, $k \in \mathbb{N}$. We have shown that $(y_{k,n} - \phi_k(y_{k+1,n}))_{n=1}^{\infty}$ converges weakly to 0 in $(E_{p_k}, \overline{p_k})$ for any $k \in \mathbb{N}$. By [8], Proposition 4.11, we infer that $\lim_{n\to\infty} \overline{p_k}(y_{k,n} - \phi_k(y_{k+1,n})) = 0$, $k \in \mathbb{N}$.

For some $(x_{k,n}) \subset E$ we have $\pi_k(x_{k,n}) = y_{k,n}$ for all $k, n \in \mathbb{N}$. Then we obtain $\lim_{n,m\to\infty} p_k(x_{k,n} - x_{k,m}) = 0$ and $\lim_{n\to\infty} p_k(x_{k,n} - x_{k+1,n}) = 0$ for any $k \in \mathbb{N}$.

Let (ϵ_k) be a decreasing sequence of positive numbers with $\lim_{k\to\infty} \epsilon_k = 0$. Then for any $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that

$$p_k(x_{k,n} - x_{k,m}) < \epsilon_k \text{ and } p_k(x_{k,n} - x_{k+1,n}) < \epsilon_k \text{ for all } n, m \in \mathbb{N} \text{ with } n, m \geq n_k.$$

Clearly, we can assume that the sequence (n_k) is increasing.

We shall prove that $\lim_{m\to\infty} p_k(x_{k,m} - x_{m,n_m}) = 0$ for $k \in \mathbb{N}$.

Let $k \in \mathbb{N}$. For $m \in \mathbb{N}$ with $m > k$ we have

$$p_k(x_{k,m} - x_{m,n_m}) \leq \max\{p_k(x_{k,m} - x_{k,n_m}), \max_{0\leq i<m-k} p_{k+i}(x_{k+i,n_m} - x_{k+i+1,n_m})\}.$$

Let $\epsilon > 0$. Then there exists $m_0 > k$ such that $\epsilon_m \leq \epsilon$ for $m \geq m_0$.

We have $\lim_{m\to\infty} p_k(x_{k,m} - x_{k,n_m}) = 0$ and

$$\lim_{m\to\infty} p_{k+i}(x_{k+i,n_m} - x_{k+i+1,n_m}) = 0 \text{ for } 0 \leq i < m_0 - k,$$

and

$$p_{k+i}(x_{k+i,n_m} - x_{k+i+1,n_m}) \leq \epsilon_{k+i} \leq \epsilon \text{ for } m_0 - k \leq i < m - k.$$

It follows that $\lim_{m\to\infty} p_k(x_{k,m} - x_{m,n_m}) = 0$.

Put $z_m = x_{m,n_m}$ for $m \in \mathbb{N}$. Then $\lim_{m\to\infty} \overline{p_k}(y_{k,m} - \pi_k(z_m)) = 0$. Since

$$p_k(z_m - z_l) \leq \max\{p_k(x_{m,n_m} - x_{k,m}), p_k(x_{k,m} - x_{k,l}), p_k(x_{k,l} - x_{l,n_l})\}$$

we infer that (z_m) is a Cauchy sequence in E, so it converges to some z_0 in E.

Let $k \in \mathbb{N}$. For any $h \in G_k$ we have

$$f(h) = \tilde{h_k}(y_k) = \lim_{m\to\infty} h_k(y_{k,m}) = \lim_{m\to\infty} h_k(\pi_k(z_m)) = \lim_{m\to\infty} h(z_m) = h(z_0).$$

Thus $f(h) = h(z_0)$ for any $h \in E'$, so f is continuous on E'_b. □

It is known that the strong dual E'_b of a reflexive Fréchet space E over a spherically complete field is bornological ([9], Proposition 15.6). For Fréchet spaces over a non-spherically complete field we get the following.

Corollary 3.15. *Assume that $\mathbb{K}$ is not spherically complete. If E is a Fréchet space of countable type, then $E_b^{'}$ is polarly bornological.*

Proof. By [8], Lemma 9.5 and Corollary 9.9, the strong dual E_b' of E is polarly barreled. Let A be a bornivorous $\mathbb{K}$-polar subset of $E_b^{'}$. Then $A = \bigcap_{f\in P} f^{-1}(B_{\mathbb{K}})$ for some set P of linear functionals on E_b'. Since A absorbs bounded subsets of $E_b^{'}$, each $f \in P$ is bounded on bounded subsets of $E_b^{'}$. Using Proposition 3.14 we get $P \subset (E_b^{'})^{'}$. Hence A is a polar barrel in $E_b^{'}$, so it is a neighbourhood of zero in $E_b^{'}$. Thus E is polarly bornological. □

In connection with Theorem 3.13 we show the following

Proposition 3.16. *If a Fréchet space E of countable type is isomorphic to the strong dual of a strict LB-space $F = \varinjlim F_n$ such that F_n has the weak extension property in F_{n+1}, $n \in \mathbb{N}$, then E is isomorphic to a countable product of Banach spaces.*

Proof. Let $i_n : F_n \to F_{n+1}$ be the inclusion map for $n \in \mathbb{N}$. By the weak extension property the adjoint map $i_n' : F_{n+1}^{'} \to F_n^{'}$, $f \to f|_{F_n}$ is surjective for $n \in \mathbb{N}$. Put $H_1 = F_1'$ and $H_n = \ker i_{n-1}'$ for $n > 1$. The closed subspace

$$H = \{(f_n) \in \prod_{n=1}^{\infty} F_n^{'} : i_n'(f_{n+1}) = f_n \text{ for } n \in \mathbb{N}\}$$

of the Fréchet space $\prod_{n=1}^{\infty} F_n^{'}$ is the projective limit of the projective sequence $(F_n^{'})$ ([3], 1.3.2).

By [3], Theorem 1.3.5, H is isomorphic to $F_b^{'}$, so H is of countable type. Let $n \in \mathbb{N}$. It is easy to check that the linear continuous map $\pi_n : H \to F_n^{'}$, $(f_m) \to f_n$ is surjective. Thus H has a quotient isomorphic to $F_n^{'}$, so $F_n^{'}$ is of countable type. Thus H_n is complemented in F_n' ([6], Theorem 3.12), so there exists $T_n \in L(F_n', F_{n+1}')$ such that $i_n' \circ T_n$ is the identity map on F_n'.

Hence the map

$$T : \prod_{n=1}^{\infty} H_n \to H, (f_n) \to (f_1, T_1 f_1 + f_2, \ldots, T_n \circ \cdots \circ T_1 f_1 + T_n \circ \cdots \circ T_2 f_2 + \cdots + T_n f_n + f_{n+1}, \ldots)$$

is well defined. Clearly T is linear and injective.

We show that T is surjective. Let $(g_n) \in H$. Put $f_1 = g_1$. Then $g_2 - T_1 f_1 \in H_2$, so there exists an $f_2 \in H_2$ such that $g_2 = T_1 f_1 + f_2$. Assume that for some $n \geq 1$ we have a sequence $(f_1, \ldots, f_{n+1}) \in \prod_{i=1}^{n+1} H_i$ with $g_{n+1} = T_n \circ \cdots \circ T_1 f_1 + \cdots + T_n f_n + f_{n+1}$. Then $g_{n+2} - T_{n+1} g_{n+1} \in H_{n+2}$. Hence there exists an f_{n+2} in H_{n+2} such that

$$g_{n+2} = T_{n+1} g_{n+1} + f_{n+2} = T_{n+1} \circ \cdots \circ T_1 f_1 + \cdots + T_{n+1} f_{n+1} + f_{n+2}.$$

Thus we can inductively construct a sequence $(f_n) \in \prod_{n=1}^{\infty} H_n$ with $T((f_n)) = (g_n)$.

The linear maps $T_n, n \in \mathbb{N}$, are continuous, and the spaces $\prod_{n=1}^{\infty} H_n$ and H have the product topologies. It follows that the maps T and T^{-1} are continuous. Thus T is an isomorphism, so E is isomorphic to a countable product of Banach spaces. □

Corollary 3.17. *If a Fréchet-Montel space E is isomorphic to the strong dual of a polar strict LB-space $F = \varinjlim F_n$, then it is of finite type.*

Proof. F is polar and polarly barreled ([3], Proposition 1.1.10), so it is isomorphic to a subspace of $(F'_b)'_b$ ([8], Lemmas 9.2 and 9.3). Clearly $(F'_b)'_b \simeq E'_b$; by [8], Theorem 8.5, E'_b is of countable type. Thus F and $F_n, n \in \mathbb{N}$, are of countable type ([8], Proposition 4.12 and [3], Theorem 1.4.7). It follows that F_n has the weak extension property in $F_{n+1}, n \in \mathbb{N}$. Using Proposition 3.16 we infer that E is isomorphic to a countable product of Banach spaces. Thus E is of finite type since it is a Fréchet-Montel space. □

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