On closed subspaces with Schauder bases in non-archimedean Fréchet spaces

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ABSTRACT

The main purpose of this paper is to prove that a non-archimedean Fréchet space of countable type is normable (respectively nuclear; reflexive; a Montel space) if and only if any its closed subspace with a Schauder basis is normable (respectively nuclear; reflexive; a Montel space). It is also shown that any Schauder basis in a non-normable non-archimedean Fréchet space has a block basic sequence whose closed linear span is nuclear. It follows that any non-normable non-archimedean Fréchet space contains an infinite-dimensional nuclear closed subspace with a Schauder basis. Moreover, it is proved that a non-archimedean Fréchet space $E$ with a Schauder basis contains an infinite-dimensional complemented nuclear closed subspace with a Schauder basis if and only if any Schauder basis in $E$ has a subsequence whose closed linear span is nuclear.

INTRODUCTION

In this paper all linear spaces are over a non-archimedean non-trivially valued field $K$ which is complete under the metric induced by the valuation $| \cdot | : K \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [5], [7] and [6]. Schauder and orthogonal bases in locally convex spaces are studied in [1], [2], [3] and [4].

Any infinite-dimensional Banach space $E$ of countable type is isomorphic to the Banach space $c_0$ of all sequences in $K$ converging to zero (with the sup-norm) (see [6], Theorem 3.16), so every closed subspace of $E$ has a Schauder basis.

There exist Fréchet spaces of countable type without a Schauder basis (see [9]). Nevertheless, any infinite-dimensional Fréchet space $F$ of finite type is

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isomorphic to the Fréchet space $\mathbb{K}^N$ of all sequences in $\mathbb{K}$ with the topology of pointwise convergence (see [3], Theorem 3.5), so every closed subspace of $F$ has a Schauder basis. Moreover, any infinite-dimensional Fréchet space contains an infinite-dimensional closed subspace with a Schauder basis (see [8]). It is also known that any closed subspace of $c_0 \times \mathbb{K}^N$ has a Schauder basis (see [12], Proposition 9). On the other hand any infinite-dimensional Fréchet space which is not isomorphic to any of the following spaces: $c_0$, $\mathbb{K}^N$, $c_0 \times \mathbb{K}^N$, contains a closed subspace without a Schauder basis (see [12], Theorem 7).

In this paper we study closed subspaces with Schauder bases in Fréchet spaces.

In Section 1 we investigate normable closed subspaces. First, we show that a Fréchet space is normable if and only if each of its closed subspaces with a Schauder basis is normable (Theorem 1.5). Then we prove that a Fréchet space with a Schauder basis $(x_n)$ contains a closed subspace isomorphic to $c_0$ if and only if $(x_n)$ has a subsequence $(x_{k_n})$ whose closed linear span is isomorphic to $c_0$ (Proposition 1.6). It is known that a Fréchet space contains a closed subspace isomorphic to $c_0$ if and only if it contains a bounded non-compactoid subset (see [4], Corollary 7.6). It follows that a Fréchet space of countable type is a Montel space (respectively a reflexive space) if and only if each of its closed subspaces with a Schauder basis is a Montel space (respectively a reflexive space) (Corollaries 1.11 and 1.12).

In Section 2 we are interested in nuclear closed subspaces. First, we prove that a Fréchet space of countable type is nuclear if and only if each of its closed subspaces with a Schauder basis is nuclear (Theorem 2.2). Next, we show that any Schauder basis in a non-normable Fréchet space has a block sequence whose closed linear span is nuclear (Theorem 2.3). It follows that any non-normable Fréchet space contains an infinite-dimensional nuclear closed subspace with a Schauder basis (Theorem 2.7). It is of interest to note that there exists a non-normable metrizable lcs $E$ such that any nuclear subspace of $E$ is finite-dimensional (Example 2.8). We also show that a Fréchet space $E$ with a Schauder basis $(x_n)$ contains an infinite-dimensional complemented nuclear closed subspace with a Schauder basis if and only if $(x_n)$ has a subsequence $(x_{k_n})$ whose closed linear span is nuclear (Proposition 2.6).

PRELIMINARIES

The linear hull of a subset $A$ in a linear space $E$ is denoted by $\text{lin}A$.

Let $(y_n)$ be a sequence in a linear space $E$. Let $(k_n) \subset \mathbb{N}$ be an increasing sequence and let $(\beta_n) \subset \mathbb{K}$. Put $z_n = \sum_{i=k_n}^{k_{n+1}-1} \beta_i y_i$ for $n \in \mathbb{N}$. The sequence $(z_n)$ is a block sequence of $(y_n)$ if $\max_{k_n \leq i < k_{n+1}} |\beta_i| > 0$ for any $n \in \mathbb{N}$.

Let $E, F$ be locally convex spaces. A map $T : E \to F$ is called a linear homeomorphism if $T$ is linear, one-to-one, surjective and the maps $T, T^{-1}$ are continuous. $E$ is isomorphic to $F$ if there exists a linear homeomorphism $T : E \to F$.

A sequence $(x_n)$ in a lcs $E$ is a Schauder basis in $E$ if each $x \in E$ can be written
uniquely as \( x = \sum_{n=1}^{\infty} \alpha_n x_n \) with \((\alpha_n) \subset K\) and the coefficient functionals \(f_n : E \to K, x \mapsto \alpha_n (n \in \mathbb{N})\) are continuous.

By a seminorm on a linear space \( E \) we mean a function \( p : E \to [0, \infty) \) such that \( p(\alpha x) = |\alpha| p(x) \) for all \( \alpha \in K, x \in E \) and \( p(x + y) \leq \max\{p(x), p(y)\} \) for all \( x, y \in E \). A seminorm \( p \) on \( E \) is a norm if \( \ker p := \{ x \in E : p(x) = 0 \} = \{0\} \).

The set of all continuous seminorms on a metrizable \( E \) is denoted by \( \mathcal{P}(E) \). A non-decreasing sequence \((p_k) \subset \mathcal{P}(E)\) is a base in \( \mathcal{P}(E) \) if for every \( p \in \mathcal{P}(E) \) there exists \( k \in \mathbb{N} \) with \( p \leq p_k \). A sequence \((p_k)\) of norms on \( E \) is a base of norms in \( \mathcal{P}(E) \) if it is a base in \( \mathcal{P}(E) \).

Any metrizable \( E \) possesses a base \((p_k)\) in \( \mathcal{P}(E) \) and every metrizable \( E \) with a continuous norm has a base of norms \((p_k)\) in \( \mathcal{P}(E) \).

A \( E \) is of finite type if for each continuous seminorm \( p \) on \( E \) the quotient space \( (E/ \ker p) \) is finite-dimensional. A metrizable \( E \) is of countable type if it contains a linearly dense countable subset.

Norms \( p, q \) on a linear space \( E \) are equivalent if there exist positive numbers \( a, b \) such that \( ap(x) < q(x) < bp(x) \) for every \( x \in E \). Every two norms on a finite-dimensional linear space are equivalent. Every \( n \)-dimensional \( E \) is linearly homeomorphic to the Banach space \( K^n \).

Let \( p \) be a seminorm on a linear space \( E \) and \( t \in (0, 1) \). An element \( x \in E \) is \( t \)-orthogonal to a subspace \( M \) of \( E \) with respect to \( p \) if \( p(\alpha x + y) \geq t \max \{p(\alpha x), p(y)\} \) for all \( \alpha \in K, y \in M \). A sequence \((x_n) \subset E \) is \( t \)-orthogonal with respect to \( p \) if \( p(\sum_{i=1}^{n} \alpha_i x_i) \geq t \max_{1 \leq i \leq n} p(\alpha_i x_i) \) for all \( n \in \mathbb{N} \) and \( \alpha_1, \ldots, \alpha_n \in K \).

Let \( (t_k) \subset (0, 1) \). A sequence \((x_n)\) in a metrizable \( E \) is \( (t_k) \)-orthogonal with respect to \((p_k) \subset \mathcal{P}(E)\) if \( (x_n) \) is \( t_k \)-orthogonal with respect to \( p_k \) for every \( k \in \mathbb{N} \). (If \( t_k = 1 \) for \( k \in \mathbb{N} \), then we shall write \( 1 \)-orthogonal instead of \((1)\)-orthogonal.)

A sequence \((x_n)\) in a metrizable \( E \) is orthogonal if it is \( 1 \)-orthogonal with respect to some base \((p_k)\) in \( \mathcal{P}(E) \). (In [6], a sequence \((x_n)\) in a normed space \((E, \| \cdot \|)\) is called orthogonal if it is \( 1 \)-orthogonal with respect to the norm \( \| \cdot \| \).)

An orthogonal sequence \((x_n)\) of non-zero elements in a metrizable \( E \) is a basic orthogonal sequence in \( E \). A linearly dense basic orthogonal sequence in a metrizable \( E \) is an orthogonal basis in \( E \).

Any block sequence of an orthogonal basis in a metrizable \( E \) is a basic orthogonal sequence in \( E \).

A sequence \((x_n)\) in a metrizable \( E \) is orthogonal in \( E \) if and only if it is \( (t_k) \)-orthogonal with respect to some base \((p_k)\) in \( \mathcal{P}(E) \) for some \((t_k) \subset (0, 1) \) (see [3], Proposition 2.6).

Every orthogonal basis in a metrizable \( E \) is a Schauder basis in \( E \) (see [3], Proposition 1.4) and every Schauder basis in a Fréchet space \( E \) is an orthogonal basis in \( E \) (see [3], Proposition 1.7).

A subset \( A \) of a lcs \( E \) is compactoid if for each neighbourhood \( U \) of 0 in \( E \) there exists a finite subset \( B = \{b_1, \ldots, b_n\} \) of \( E \) such that \( A \subset U + \text{co } B \), where \( \text{co } B = \{ \sum_{i=1}^{n} \alpha_i b_i : \alpha_1, \ldots, \alpha_n \in K, |\alpha_1|, \ldots, |\alpha_n| \leq 1 \} \) is the absolutely convex hull of \( B \).
A bounded subset $A$ in a lcs $E$ is compactoid if and only if any orthogonal sequence $(x_n) \subseteq A$ tends to $0$ in $E$ (see [3], Theorem 2.2).

Let $E$ and $F$ be locally convex spaces. The linear map $T : E \rightarrow F$ is compact if there exists a neighbourhood $U$ of $0$ in $E$ such that $T(U)$ is compactoid in $F$.

For any seminorm $p$ on a lcs $E$ the map $\overline{p} : (E/\ker p) \rightarrow [0, \infty), x + \ker p \mapsto p(x)$ is a norm on $(E/\ker p)$.

A lcs $E$ is nuclear if for every continuous seminorm $p$ on $E$ there exists a continuous seminorm $q$ on $E$ with $q \geq p$ such that the canonical map

$$\varphi_{p,q} : ((E/\ker q), \overline{q}) \rightarrow ((E/\ker p), \overline{p}), x + \ker q \mapsto x + \ker p$$

is compact. A subspace of a nuclear lcs is nuclear (see [7], Proposition 1.2).

Let $E$ be a Fréchet space with a Schauder basis $(x_n)$ which is $1$-orthogonal with respect to a base of norms $(p_k)$ in $\mathcal{P}(E)$. Then $E$ is nuclear if and only if

$$\forall k \in \mathbb{N} \exists m > k : \lim_{n} \frac{p_k(x_n)}{p_m(x_n)} = 0$$

(see [2], Propositions 2.4 and 3.5).

Let $B = (b_{k,n})$ be an infinite matrix consisting of positive real numbers such that $b_{k,n} \leq b_{k+1,n}$ for all $k, n \in \mathbb{N}$. The Köthe space associated with the matrix $B$ is the space $K(B) = \{(\alpha_n) \in \ell^1 : \lim_{n} |\alpha_n| b_{k,n} = 0 \text{ for all } k \in \mathbb{N}\}$ with the following standard base of norms $(p_k)$: $p_k((\alpha_n)) = \max_{n} |\alpha_n| b_{k,n}, k \in \mathbb{N}$. The space $K(B)$ is a Fréchet space and the sequence $(e_n)$ of coordinate vectors forms the standard Schauder basis in $K(B)$ (see [2], Proposition 2.2). The basis $(e_n)$ is $1$-orthogonal with respect to the base $(p_k)$.

1. ON NORMABLE CLOSED SUBSPACES

Using the ideas of the proofs of Lemma 1, [8], Theorem 2, [8], and Proposition 9, [12], we show the following three lemmas.

**Lemma 1.1.** Let $n \in \mathbb{N}$ and let $p_1, \ldots, p_n$ be continuous seminorms on a metrizable lcs $E$ of countable type. Let $M$ be a finite-dimensional subspace of $E$. Then for every $t \in (0, 1)$ there exists a closed subspace $L$ of $E$ with $\dim(E/L) < \infty$ such that any $x \in L$ is $t$-orthogonal to $M$ with respect to $p_i$ for all $1 \leq i \leq n$.

**Proof.** Let $1 \leq i \leq n$ and $F_i = E/\ker p_i$. Let $\pi_i : E \rightarrow F_i$ be the quotient mapping. Denote by $(\tilde{G}_i, \tilde{p}_i)$ the completion of the normed space $(F_i, \tilde{p}_i)$ of countable type. Then there exists a linear continuous projection $Q_i$ of $\tilde{G}_i$ onto $\pi_i(M)$ of norm less than or equal to $t^{-1}$ (see [6], Theorem 3.16 and its proof). Let $H_i = F_i \cap \ker Q_i$ and $E_i = \pi^{-1}_i(H_i)$. Any $x \in E_i$ is $t$-orthogonal to $M$ with respect to $p_i$. Indeed, let $\alpha \in \mathbb{K}, m \in M, z = \pi_i(m)$ and $y = \pi_i(x)$. Since $z = Q_i(\alpha y + z)$, then $\overline{p_i}(z) \leq t^{-1} \overline{p_i}(\alpha y + z)$. Hence $\overline{p_i}(\alpha y + z) \geq t \max\{\overline{p_i}(\alpha y), \overline{p_i}(z)\}$ (see [6], Lemma 3.2). Thus $p_i(\alpha y + m) \geq t \max\{p_i(\alpha y), p_i(m)\}$.

Let $L = \bigcap_{i=1}^{n} E_i$. Any $x \in L$ is $t$-orthogonal to $M$ with respect to $p_i$ for all $1 \leq i \leq n$. Clearly, $L$ is a closed subspace of $E$ and
\[
\dim(E/L) \leq \sum_{i=1}^{n} \dim(E/E_i) = \sum_{i=1}^{n} \dim(F_i/H_i) \\
\leq \sum_{i=1}^{n} \dim(G_i/\ker Q_i) < \infty. \quad \square
\]

**Lemma 1.2.** Let \( E \) be a metrizable lcs with a base \((p_k)\) in \( \mathcal{P}(E) \). Assume that \( (s_n) \subset (0,1) \) with \( s = \prod_{n=1}^{\infty} s_n > 0 \). Then any sequence \( (y_n) \subset (E \setminus \ker p_1) \) such that \( y_{n+1} \) is \( s_{n+1} \)-orthogonal to \( \text{lin}\{y_1, \ldots, y_n\} \) with respect to \( p_i \) for all \( 1 \leq i \leq n \) and \( n \in \mathbb{N} \), is orthogonal in \( E \).

**Proof.** It is enough to show that the sequence \((y_n)\) is \((t_m)\)-orthogonal with respect to \((p_m)\) for some \((t_m) \subset (0,1]\) (see [3], Proposition 2.6).

Let \( m \in \mathbb{N} \) and \( \alpha_1, \ldots, \alpha_m \in \mathbb{K} \). Then
\[
p_1 \left( \sum_{i=1}^{m} \alpha_i y_i \right) \geq \left( \prod_{i=1}^{m} s_i \right) \max_{1 \leq i \leq m} p_1(\alpha_i y_i).
\]
Let \( E_m = \text{lin}\{y_1, \ldots, y_m\} \). Since the norms \( p_1|E_m, p_m|E_m \) are equivalent then there exists \( d_m \in (0,1) \) such that for arbitrary \( \alpha_1, \ldots, \alpha_m \in \mathbb{K} \) we have
\[
p_m \left( \sum_{i=1}^{m} \alpha_i y_i \right) \geq d_m \max_{1 \leq i \leq m} p_m(\alpha_i y_i).
\]
Let \( k > m \) and \( \alpha_1, \ldots, \alpha_k \in \mathbb{K} \). Then
\[
p_m \left( \sum_{i=1}^{k} \alpha_i y_i \right) \geq \left( \prod_{i=m+1}^{k} s_i \right) d_m \max_{1 \leq i \leq k} p_m(\alpha_i y_i) \geq s d_m \max_{1 \leq i \leq k} p_m(\alpha_i y_i).
\]
Thus the sequence \((y_n)\) is \((sd_m)\)-orthogonal with respect to \((p_m)\). \( \square \)

**Lemma 1.3.** Let \( E \) be a Fréchet space with a base of norms \((p_k)\) in \( \mathcal{P}(E) \). Assume that for any \( k \in \mathbb{N} \) the norms \( p_k \) and \( p_k + 1 \) are equivalent on some subspace \( E_k \) of finite codimension in \( E \). Then \( E \) is normable.

**Proof.** First, we show that for any \( k \in \mathbb{N} \) the norms \( p_k \) and \( p_k + 1 \) are equivalent on some dense subspace \( F_k \) of the normed space \( (E,p_k) \). Let \( k \in \mathbb{N} \). Denote by \( G_k \) the closure of \( E_k + 1 \) in \( (E,p_k) \). Put \( n = \dim(E/G_k) \). Clearly \( n < \infty \).

If \( n = 0 \), then we can take \( F_k = E_k + 1 \).

If \( n > 0 \), then by Lemma 3.14, [6], there exist \( e_1, \ldots, e_n \in E \) such that \( \text{lin}\{e_1, \ldots, e_n\} \setminus G_k = E_k \) and
\[
p_k \left( \sum_{i=1}^{n} \alpha_i e_i + x \right) \geq 2^{-n} \max \left\{ \max_{1 \leq i \leq n} p_k(\alpha_i e_i), p_k(x) \right\}
\]
for all \( \alpha_1, \ldots, \alpha_n \in \mathbb{K} \) and \( x \in G_k \). Set \( F_k = \text{lin}\{e_1, \ldots, e_n\} + E_k + 1 \). Of course, \( F_k \) is dense in \( (E,p_k) \). The norms \( p_k, p_k + 1 \) are equivalent on \( F_k \). Indeed, put
\[
C = \max \left\{ \max_{1 < i < n} \left[ p_{k+1}(e_i)/p_k(e_i) \right], \max_{x \in E_{k+1}} \left[ p_{k+1}(x)/p_k(x) \right] \right\}.
\]
Clearly $C < \infty$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ and $x \in E_{k+1}$. Then
\[
 p_{k+1}\left(\sum_{i=1}^{n} \alpha_i e_i + x\right) \leq \max_{1 \leq i \leq n} \max_{1 \leq j \leq n} p_{k+1}(\alpha_i e_i), p_{k+1}(x) \leq C \max_{1 \leq i \leq n} p_{k}(\alpha_i e_i), p_{k}(x) \leq 2^n C p_{k}\left(\sum_{i=1}^{n} \alpha_i e_i + x\right).
\]

We shall prove that the normable space $(E, p_1)$ is complete. Let $(f_k^1)$ be a Cauchy sequence in $(E, p_1)$ and $k_n^0 = n, n \in \mathbb{N}$. Then there exists a subsequence $(k_n^1)$ of $(k_n^0)$ such that $p_1(f_k^1 - f_{k_n^1}^1) < n^{-1}, n \in \mathbb{N}$. Since $F_1$ is a dense subspace of $(E, p_1)$, we can choose a sequence $(f_{k_n^2}) \subset F_1$ with $p_1(f_{k_n^1}^2 - f_{k_n^2}^2) < n^{-1}, n \in \mathbb{N}$. Clearly, $(f_{k_n^2}^2)$ is a Cauchy sequence in $(E, p_1)$. Since the norms $p_1, p_2$ are equivalent on $F_1$ and $(f_{k_n^2}^2) \subset F_1$, then $(f_{k_n^2}^2)$ is a Cauchy sequence in $(E, p_2)$. In this way we can choose in turn for every $v \in \mathbb{N}$ a subsequence $(k_n^v)$ of $(k_n^{v-1})$ with $p_1(f_{k_n^v}^v - f_{k_n^{v+1}}^v) < n^{-1}, n \in \mathbb{N}$, and a sequence $(f_{k_n^v}^v) \subset F_v$ with $p_v(f_{k_n^v}^v - f_{k_n^{v+1}}^v) < n^{-1}, n \in \mathbb{N}$.

For any $n \in \mathbb{N}$ there exists $s \in \mathbb{N}$ with $s > n$ such that $k_n^{n+1} = k_n^s$. Since $p_n(f_{k_n^s}^n - f_{k_n^{n+1}}^n) < s^{-1}, n \leq s - 1$, then $p_n(f_{k_n^s}^n - f_{k_n^{n+1}}^n) < n^{-1}$. Moreover,
\[
p_n(f_{k_n^s}^n - f_{k_n^{n+1}}^n) = p_n(f_{k_n^s}^n - f_{k_n^{n+1}}^n) < s^{-1} < n^{-1}.
\]

Hence $p_n(f_{k_n^s}^n - f_{k_n^{n+1}}^n) < n^{-1}, n \in \mathbb{N}$. This follows that $p_i(f_{k_n^s}^n - f_{k_n^{n+1}}^n) \to 0$ for any $i \in \mathbb{N}$. Thus $(f_{k_n^s}^n)$ is a Cauchy sequence in $E$. Let $f$ be the limit of $(f_{k_n^s}^n)$ in $E$. Since
\[
p_1(f_{k_n^s}^n - f_{k_n^s}^n) = \max_{1 \leq i \leq n-1} p_1(f_{k_n^s}^i - f_{k_n^s}^{i+1}) < \max_{1 \leq i \leq n-1} p_1(f_{k_n^s}^i - f_{k_n^s}^{i+1}) < n^{-1}, n \in \mathbb{N},
\]

and $p_1(f_{k_n^s}^n - f) \to 0$, then $p_1(f_{k_n^s}^n - f) \to 0$. Hence $p_1(f_{k_n^s}^n - f) \to 0$, because $(f_{k_n^s}^n)$ is a Cauchy sequence in $(E, p_1)$. Thus we have proved that the normable space $(E, p_1)$ is complete. By the open mapping theorem the Fréchet space $E$ is normable. □

Immediately by Lemma 3 we obtain the following.

**Proposition 1.4.** Let $E$ be a non-normable Fréchet space with a base of norms $(p_k)$ in $\mathcal{P}(E)$. Then there exists a subsequence $(p_{n_k})$ of $(p_k)$ such that for any $k \in \mathbb{N}$ the norms $p_{n_k}$ and $p_{n_k+1}$ are non-equivalent on any subspace of finite codimension in $E$.

Now we can prove our first theorem.

**Theorem 1.5.** A Fréchet space is normable if and only if each of its closed subspaces with a Schauder basis is normable.

**Proof.** It is enough to show that any non-normable Fréchet space $E$ contains a non-normable closed subspace $G$ with a Schauder basis. Consider two cases.

Case 1. $E$ has a continuous norm. It is easy to see that $E$ contains a non-
normable closed subspace \( F \) of countable type. Let \((N_k)\) be a sequence of pairwise disjoint infinite sets with \( \bigcup_{k=1}^{\infty} N_k = \mathbb{N} \) and let \((s_n) \subseteq (0,1)\) with \( \prod_{n=1}^{\infty} s_n > 0 \). By Proposition 1.4 there exists a base of norms \((p_k)\) in \( \mathcal{P}(F) \) such that for any \( k \in \mathbb{N} \) the norms \( p_k \) and \( p_{k+1} \) are non-equivalent on any subspace of finite codimension in \( F \). Then using Lemma 1.1, we can construct inductively a sequence \((x_n) \subseteq F\) such that \( x_{n+1} \) is \( s_{n+1} \)-orthogonal to \( \text{lin}\{x_1, \ldots, x_n\} \) with respect to \( p_i \) for \( 1 \leq i \leq n, n \in \mathbb{N}, \) and \( p_k(x_n) < n^{-1} p_{k+1}(x_n) \) for all \( n \in N_k, \ k \in \mathbb{N}. \) By Lemma 1.2, \((x_n)\) is orthogonal in \( F \). Clearly, \( \inf_{n \in \mathbb{N}} \frac{p_k(x_n)}{p_{k+1}(x_n)} = 0 \) for any \( k \in \mathbb{N}. \) Hence for every \( k \in \mathbb{N} \) the norms \( p_k \) and \( p_{k+1} \) are non-equivalent on the closed linear span \( G \) of \((x_n)\). Thus \( G \) is a non-normable closed subspace with a Schauder basis in \( E \).

Case 2. \( E \) has no continuous norm. Then \( E \) contains a closed subspace \( G \) isomorphic to \( \ell^\infty \) (see [2], Proposition 2.6), so it has a non-normable closed subspace with a Schauder basis. \( \Box \)

Our next result states when a Fréchet space with a Schauder basis possesses an infinite-dimensional normable closed subspace with a Schauder basis.

**Proposition 1.6.** A Fréchet space \( E \) with a Schauder basis \((x_n)\) contains a subspace isomorphic to \( c_0 \) if and only if \((x_n)\) has a subsequence \((x_{k_n})\) whose closed linear span is isomorphic to \( c_0 \).

**Proof.** Assume that \((x_n)\) is \( 1 \)-orthogonal with respect to a base \((p_i)\) in \( \mathcal{P}(E) \) and \( F \) is a subspace of \( E \) isomorphic to \( c_0 \). Then there is \( k \in \mathbb{N} \) such that \( p_k|F \) is a norm on \( F \) and

\[ (*) \quad \forall j \geq k \exists s_j > 0 \quad \forall y \in F : p_k(y) \geq s_j p_j(y). \]

Put \( N_j = \{ n \in \mathbb{N} : p_k(x_n) \geq s_j p_j(x_n) \} \) for \( j \geq k \). It is easy to check that there exists a sequence \((y_n) \subseteq (F \setminus \{0\})\) such that \( y_n = \sum_{i=n}^{\infty} \alpha_{n,i} x_i, \ n \in \mathbb{N}, \) for some \((\alpha_{n,i})_{n \in \mathbb{N}} \subseteq \mathbb{K}. \)

Let \( n, j \in \mathbb{N} \) with \( j \geq k \). Then \( p_k(y_n) = \max_{i \geq n} p_k(\alpha_{n,i} x_i) = p_k(\alpha_{n,i} x_i) \) for some \( i_n \geq n, \) and \( p_j(y_n) = \max_{i \geq n} p_j(\alpha_{n,i} x_i) \geq p_j(\alpha_{n,i} x_i). \) By \( (*) \) we get \( p_k(\alpha_{n,i} x_i) = p_k(y_n) \geq s_j p_j(y_n) \geq s_j p_j(\alpha_{n,i} x_i). \) Hence \( p_k(x_{i_n}) \geq s_j p_j(x_{i_n}), \) so \( i_n \in N_j. \) Thus \( \{ i_n : n \in \mathbb{N} \} \subseteq \bigcap_{j=k}^{\infty} N_j. \) Since \( i_n \geq n \) for any \( n \in \mathbb{N}, \) the set \( N_0 = \bigcap_{j=k}^{\infty} N_j \) is infinite. Denote by \( G \) the closed linear span of \( \{x_n : n \in N_0\}. \) Clearly,

\[ (**) \quad \forall n \in N_0 \forall j \geq k : p_k(x_n) \geq s_j p_j(x_n) \geq s_j p_k(x_n). \]

Since \( \forall n \in N_0 \exists j \geq k : p_j(x_n) > 0, \) then \( \forall n \in N_0 : p_k(x_n) > 0. \) Hence \( p_k|G \) is a norm on \( G. \) Moreover, \( \forall j \geq k \forall x \in G : p_k(x) \geq s_j p_j(x). \) Indeed, let \( j \geq k \) and \( x \in G. \) Then \( x = \sum_{n \in N_0} \alpha_n x_n \) for some \((\alpha_n)_{n \in N_0} \subseteq \mathbb{K} \) and by \( (**) \) we have

\[ p_k(x) = \max_{n \in N_0} p_k(\alpha_n x_n) \geq s_j \max_{n \in N_0} p_j(\alpha_n x_n) = s_j p_j(x). \]

This follows that for any \( j \geq k \) the norms \( p_j|G \) and \( p_k|G \) are equivalent. Thus \( G \) is normable, so it is isomorphic to \( c_0. \) \( \Box \)
Corollary 1.7. A metrizable lcs $E$ with an orthogonal basis $(x_n)$ contains an infinite-dimensional normable subspace if and only if $(x_n)$ has a subsequence $(x_{k_n})$ whose closed linear span is normable.

By the proof of Proposition 1.6 we obtain

Remark 1.8. Let $(x_n)$ be a Schauder basis in a Fréchet space $E$. Assume that $(x_n)$ is 1-orthogonal with respect to a base $(p_k)$ in $P(E)$. Then $(x_n)$ has a subsequence $(x_{k_n})$ whose closed linear span is isomorphic to $c_0$ if and only if there exist an infinite subset $M$ of $\mathbb{N}$, a sequence $(d_k) \subset (0, 1)$ and $k_0 \in \mathbb{N}$ such that $p_k(x_n) \geq d_{k+1}p_{k+1}(x_n) > 0$ for all $k \geq k_0$ and $n \in M$.

Clearly, any Fréchet space which contains a closed subspace isomorphic to $c_0$ is non-nuclear. The following example shows that the converse is not true.

Example 1.9. Let $(N_i)$ be a sequence of pairwise disjoint infinite sets with $\bigcup_{i=1}^{\infty} N_i = \mathbb{N}$. For $i \in \mathbb{N}$ and $n \in N_i$ we put $b_{k,n} = k^i$ if $k \leq i$, and $b_{k,n} = k^{i+1}$ if $k > i$. Clearly, $0 < b_{k,n} \leq b_{k+1,n}$ for all $k, n \in \mathbb{N}$. Let $B = (b_{k,n})$ and $E = K(B)$.

The Köthe space $E$ is non-nuclear and has no subspace isomorphic to $c_0$.

Indeed, let $(e_n)$ be the standard basis in $E$ and let $(p_k)$ be the standard base in $P(E)$. Since $[p_1(e_n)/p_i(e_n)] = i$ for $i \in \mathbb{N}$ and $n \in N_i$, then $\lim_n [p_1(e_n)/p_i(e_n)] = 0$ for none of $i \in \mathbb{N}$. Thus $E$ is non-nuclear.

Let $N_0$ be an infinite subset of $\mathbb{N}$. If the set $M_i = N_0 \cap N_i$ is infinite for some $i \in \mathbb{N}$, then $\lim_n \in M_i [p_k(e_n)/p_{k+1}(e_n)] = \lim_n \in M_i [k/(k+1)]^{m_i} = 0$ for any $k > i$, so the closed linear span $X_0$ of $\{e_n : n \in N_0\}$ is non-normable. If the set $M_i$ is finite for any $i \in \mathbb{N}$, then there exist two increasing sequences $(n_i), (m_i) \subset \mathbb{N}$ such that $n_i \in M_{m_i}$ for any $i \in \mathbb{N}$. Thus $\lim_n [p_k(e_n)/p_{k+1}(e_n)] = \lim_n [k/(k+1)]^{m_i} = 0$ for any $k \in \mathbb{N}$; so $X_0$ is non-normable, too. By Proposition 1.6 we infer that $E$ has no subspace isomorphic to $c_0$.

Since a Fréchet space of countable type is a Montel space if and only if it has no subspace isomorphic to $c_0$ (see [4], Corollary 7.6), then we get

Corollary 1.10. A Fréchet space $F$ with a Schauder basis $(x_n)$ is a Montel space if and only if $(x_n)$ has no subsequence $(x_{k_n})$ whose closed linear span is isomorphic to $c_0$.

Corollary 1.11. A Fréchet space $E$ of countable type is a Montel space if and only if each of its closed subspaces with a Schauder basis is a Montel space.

Using [7], Corollary 9.9, Theorem 10.3 and Theorem 10.4 we obtain

Corollary 1.12. A Fréchet space of countable type is reflexive if and only if each of its closed subspaces with a Schauder basis is reflexive.
2. ON NUCLEAR CLOSED SUBSPACES

First, we show the following lemma.

**Lemma 2.1.** Let $E$ be a metrizable lcs with a base $(p_k)$ in $\mathcal{P}(E)$. Assume that
\[
\forall k \in \mathbb{N} \exists m(k) > k \forall \varepsilon > 0 \exists F < E : \dim(E/F) < \infty \forall x \in F : p_k(x) \leq \varepsilon p_{m(k)}(x).
\]
Then $E$ is nuclear.

**Proof.** Let $k \in \mathbb{N}, m = m(k)$ and $E_i = (E/\ker p_i)$ for $i \in \mathbb{N}$. We shall prove that the canonical map
\[
\varphi : (E_m, \overline{p_m}) \to (E_k, \overline{p_k}), x + \ker p_m \to x + \ker p_k
\]
is compact. Let $\varepsilon > 0$. Then there exists a subspace $F$ of $E$ with $\dim(E/F) < \infty$ such that $p_k(x) \leq 2^{-1} \varepsilon p_m(x)$ for any $x \in F$. Without loss of generality we can assume that $F \supset \ker p_m$ and $G_m = (F/\ker p_m)$ is a closed subspace of the normed space $(E_{m}, \overline{p_{m}})$. Put $B_m = \{z \in E_m : \overline{p_m}(z) \leq 1\}, \ B_k = \{z \in E_k : \overline{p_k}(z) \leq \varepsilon\},$ and $n = \dim(E/F)$. Clearly, $\overline{p_k}(\varphi(y)) \leq 2^{-1} \overline{p_m}(y)$ for $y \in G_m$.

If $n = 0$, then $\varphi(B_m) \subset B_k$.

If $n \geq 1$, then by Lemma 3.14, [6], there exist $z_1, \ldots, z_n \in E_m$ such that
\[
\text{lin}\{z_i : 1 \leq i \leq n\} + G_m = E_m
\]
and
\[
\overline{p_m} \left( \sum_{i=1}^{n} \alpha_i z_i + y \right) \geq \left[ 2^{-1/n} \right] \max \{ \max_{1 \leq i \leq n} \overline{p_m}(\alpha_i z_i), \overline{p_m}(y) \}
\]
for all $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ and $y \in G_m$. Clearly, we can assume that $\overline{p_m}(z_i) \geq 2$ for all $1 \leq i \leq n$. Then
\[
\varphi(B_m) \subset \text{co}\{\varphi(z_i) : 1 \leq i \leq n\} + B_k
\]
Indeed, let $z \in B_m$. Then there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ and $y \in G_m$ such that $z = \sum_{i=1}^{n} \alpha_i z_i + y$. By (*) we get $1 \geq \overline{p_m}(z) \geq \max \{ \max_{1 \leq i \leq n} |\alpha_i|, 2^{-1} \overline{p_m}(y) \}$. Hence $\max_{1 \leq i \leq n} |\alpha_i| \leq 1$ and $\overline{p_k}(\varphi(y)) \leq 2^{-1} \overline{p_m}(y) \leq \varepsilon$. Since $\varphi(z) = \sum_{i=1}^{n} \alpha_i \varphi(z_i) + \varphi(y)$, then $\varphi(z) \in \text{co}\{\varphi(z_i) : 1 \leq i \leq n\} + B_k$. This follows that $\varphi(B_m)$ is compactoid in $(E_k, \overline{p_k})$. Thus $\varphi$ is compact. Hence $E$ is nuclear. □

**Theorem 2.2.** A metrizable lcs $E$ of countable type is nuclear if and only if each of its closed subspaces with an orthogonal basis is nuclear.

In particular, a Fréchet space of countable type is nuclear if and only if each of its closed subspaces with a Schauder basis is nuclear.

**Proof.** It is enough to show that any non-nuclear metrizable lcs $E$ of countable type contains a non-nuclear closed subspace with an orthogonal basis. Let $(p_k)$ be a base in $\mathcal{P}(E)$. By Lemma 2.1 we get
\[
\exists k_0 \in \mathbb{N} \forall m \geq k_0 \forall \varepsilon_m > 0 \forall F < E : \dim(E/F) < \infty \exists x \in F : p_{k_0}(x) > \varepsilon_m p_m(x);
\]
clearly, we can assume that \( k_0 = 1. \) Let \( (N_m) \) be a sequence of pairwise disjoint infinite sets with \( \bigcup_{m=1}^{\infty} N_m = \mathbb{N} \) and let \( (s_n) \subset (0,1) \) with \( \prod_{n=1}^{\infty} s_n > 0. \) By (\( + \)) and Lemma 1.1 we can construct inductively a sequence \((y_n) \subset E\) such that \( y_{n+1} \) is \( s_{n+1} \)-orthogonal to \( \text{lin}\{y_1, \ldots, y_n\} \) with respect to \( p_i \) for \( 1 \leq i \leq n, n \in \mathbb{N}, \) and \( p_1(y_n) > \epsilon_m p_m(y_n) \) for all \( n \in N_m, m \in \mathbb{N}. \) Hence \( p_1(y_n) > 0 \) for any \( n \in \mathbb{N}. \) By Lemma 1.2, \((y_n)\) is orthogonal in \( E, \) so it is \( 1 \)-orthogonal with respect to some base \((q_k)\) in \( P(E). \) Of course, we can assume that \( q_1 \geq p_1. \) Let \( k \in \mathbb{N} \) and \( m \in \mathbb{N} \) with \( p_m \geq q_k. \) Then \( q_k(y_n) > \epsilon_m p_m(y_n) \) for all \( n \in N_m. \) Thus \( \lim_n q_k(y_n)/q_m(y_n) = 0 \) for none of \( k \in \mathbb{N}. \) Therefore the closed linear span of \((y_n)\) is non-nuclear.

Now we show that any non-normable Fréchet space with a Schauder basis contains an infinite-dimensional nuclear closed subspace with a Schauder basis.

**Theorem 2.3.** Let \( E \) be a non-normable Fréchet space with a Schauder basis \((x_n)\). Then \((x_n)\) has a block sequence \((y_n)\) whose closed linear span is nuclear.

**Proof.** Consider two cases.

Case 1. \( E \) has a continuous norm. Assume that \((x_n)\) is \( 1 \)-orthogonal with respect to a base of norms \((p_k)\) in \( P(E). \) Without loss of generality we can assume that for any \( k \in \mathbb{N} \) the norms \( p_k \) and \( p_{k+1} \) are non-equivalent. Then \( \inf_k p_k(x_n)/p_{k+1}(x_n) = 0 \) for \( k \in \mathbb{N}. \) Let \( m, n \in \mathbb{N}. \) We can construct a finite sequence \( (\alpha_{i_1}, \ldots, \alpha_{i_{n+1}}) \subset (\mathbb{N} \setminus \{0\}) \) with \( m = i_1 < \cdots < i_{n+1} \) such that \( p_1(\alpha_{i_1} x_{i_1}) \geq 1, p_k(\alpha_{i_k} x_{i_{k+1}}) \leq 1 \) and \( p_{k+1}(\alpha_{i_k} x_{i_{k+1}}) \geq n \max_{1 \leq j \leq k} p_k(\alpha_{i_j} x_{i_j}) \) for any \( 1 \leq k \leq n. \) Let \( y_n = \sum_{i=1}^{n+1} \alpha_{i_j} x_{i_j} \) and \( 1 \leq k \leq n. \) Then \( p_1(y_n) = \max_{1 \leq j \leq n+1} p_k(\alpha_{i_j} x_{i_j}) \geq 1 \) and

\[
\frac{p_k(y_n)}{p_{k+1}(y_n)} = \frac{\max_{1 \leq j \leq n+1} p_k(\alpha_{i_j} x_{i_j})}{\max_{1 \leq j \leq n+1} p_{k+1}(\alpha_{i_j} x_{i_j})} \leq \frac{\max_{1 \leq j \leq n} p_k(\alpha_{i_j} x_{i_j})}{p_{k+1}(\alpha_{i_{n+1}} x_{i_{n+1}})} \leq n^{-1}.
\]

Thus we can construct inductively a block sequence \((y_n)\) of \((x_n)\) such that we have \( p_k(y_n)/p_{k+1}(y_n) < n^{-1} \) for all \( k, n \in \mathbb{N} \) with \( k \leq n. \) Clearly, \((y_n)\) is \( 1 \)-orthogonal with respect to \((p_k)\) and \( \lim_n p_k(y_n)/p_{k+1}(y_n) = 0, k \in \mathbb{N}. \) Hence the closed linear span of \((y_n)\) is nuclear.

Case 2. \( E \) has no continuous norm. Assume that \((x_n)\) is \( 1 \)-orthogonal with respect to a base \((p_k)\) in \( P(E). \) It is easy to see that there exist two increasing sequences \((k_n), (m_n) \subset \mathbb{N} \) such that \( x_{k_n} \in \ker p_{m_n} \setminus \ker p_{m_{n-1}}, n \in \mathbb{N}. \) Then the closed linear span \( F \) of \((x_{k_n})\) is isomorphic to \( \mathbb{K}^\mathbb{N}. \) Indeed, for any \((\alpha_n) \subset \mathbb{K}\) the sequence \((\alpha_n x_{k_n})\) is convergent to 0 in \( E. \) Hence, by the closed graph theorem
the linear map \( T : k^\mathbb{N} \to F, (\alpha_n) \to \sum_{n=1}^{\infty} \alpha_n x_{\kappa_n} \), is an isomorphism. Clearly, \( (x_{\kappa_n}) \) is a block sequence of \( (x_n) \) and \( F \) is nuclear. \( \square \)

**Corollary 2.4.** Let \( E \) be a non-normable metrizable lcs \( E \) with an orthogonal basis \( (x_n) \). Then \( (x_n) \) has a block sequence \( (y_n) \) whose closed linear span is nuclear.

The following example shows that there exists a non-normable Fréchet space \( H \) with a Schauder basis \( (x_n) \) such that for any subsequence \( (x_{\kappa_n}) \) of \( (x_n) \) the closed linear span of \( (x_{\kappa_n}) \) is non-nuclear.

**Example 2.5.** Let \( (N_i) \) be a sequence of pairwise disjoint infinite sets with \( \bigcup_{i=1}^{\infty} N_i = \mathbb{N} \). For \( i \in \mathbb{N} \) and \( n \in N_i \) we put \( b_{k,n} = 1 \) if \( k < i \) and \( b_{k,n} = n \) if \( k \geq i \). Clearly, \( 0 < b_{k,n} \leq b_{k+1,n} \) for all \( k, n \in \mathbb{N} \). Let \( B = (b_{k,n}) \) and \( H = K(B) \). Let \( (e_n) \) be the standard basis in \( H \) and let \( (p_k) \) be the standard base in \( \mathcal{P}(H) \). Since \( |p_k(e_n)/p_{k+1}(e_n)| = n^{-1} \) for any \( k \in \mathbb{N} \) and \( n \in N_{k+1} \), then \( H \) is non-normable.

Let \( N_0 \) be an infinite subset of \( \mathbb{N} \). If the set \( M_i = N_0 \cap N_i \) is infinite for some \( i \in \mathbb{N} \), then the closed linear span of \( \{e_n : n \in M_i\} \) is isomorphic to \( c_0 \), since \( p_k(e_n) = p_{j}(e_n) \) for any \( k \geq i \) and \( n \in M_i \). If the set \( M_i \) is finite for any \( i \in \mathbb{N} \), then there exist two increasing sequences \( (n_i), (m_i) \subset \mathbb{N} \) such that \( n_i \in M_{m_i} \), for any \( i \in \mathbb{N} \). Thus \( p_k(e_{n_i}) = p_{k+1}(e_{n_i}) \) for all \( i, k \in \mathbb{N} \) with \( i > k + 1 \); so the closed linear span of \( \{e_n : i \in \mathbb{N}\} \) is isomorphic to \( c_0 \).

This shows that for any infinite subset \( N_0 \) of \( \mathbb{N} \) the closed linear span \( X_0 \) of \( \{e_n : n \in N_0\} \) contains a subspace isomorphic to \( c_0 \); so \( X_0 \) is non-nuclear.

In fact, the space \( H \) has not any infinite-dimensional complemented nuclear closed subspace with a Schauder basis. This follows from our next result.

**Proposition 2.6.** Let \( E \) be a Fréchet space with a Schauder basis \( (x_n) \) and \( F \) its infinite-dimensional complemented closed subspace with a Schauder basis \( (y_n) \). If \( F \) is nuclear (respectively a Montel space), then \( (x_n) \) has a subsequence \( (x_{\kappa_n}) \) whose closed linear span is nuclear (respectively a Montel space).

**Proof.** Consider two cases.

Case 1. \( E \) has a continuous norm. Denote by \( P \) a linear continuous projection from \( E \) onto \( F \). Let \( (f_n) \) and \( (h_n) \) be the sequences of coefficient functionals associated with the bases \( (x_n) \) and \( (y_n) \), respectively. Put \( g_n(x) = h_n(Px) \) for \( n \in \mathbb{N} \) and \( x \in E \). Since

\[
1 = |g_n(y_n)| = |g_n(\sum_{k=1}^{\infty} f_k(y_n)x_k)| = |\sum_{k=1}^{\infty} f_k(y_n)g_n(x_k)| \leq \max_k |f_k(y_n)g_n(x_k)|, n \in \mathbb{N},
\]

then for any \( n \in \mathbb{N} \) there exists \( t_n \in \mathbb{N} \) with \( |f_{t_n}(y_n)g_n(x_{t_n})| \geq 1 \).

Assume that \( (x_n) \) is \( 1 \)-orthogonal with respect to a base of norms \( (p_k) \) in \( \mathcal{P}(E) \). For any \( k \in \mathbb{N} \) there exist \( q_k \in \mathcal{P}(E), s_k \in \mathbb{N} \) with \( p_k \leq q_k \leq p_{s_k} \) and
$q_k \circ P \leq p_{sk}$ such that $(y_n)$ is $l$-orthogonal with respect to $q_k$. For all $n, k \in \mathbb{N}$ we obtain

$$p_k(f_n(y_n)x_n) \leq \max_m p_k(f_m(y_n)x_m) = p_k(y_n)$$

$$\leq |g_n(x_n)|^{-1} \max_m q_k(g_m(x_n)y_m) = |g_n(x_n)|^{-1} q_k(px_n)$$

$$\leq p_{sk}(f_n(y_n)x_n).$$

Hence

$$(*) \quad p_k(f_n(y_n)x_n) \leq p_k(y_n) \leq p_{sk}(f_n(y_n)x_n) \quad \text{for all } k, n \in \mathbb{N}.$$ 

Put $r_k(y) = \max_n |h_n(y)|p_k(f_n(y_n)x_n), k \in \mathbb{N}, y \in F$.

By $(*)$, we get $r_k(y) \leq \max_n |h_n(y)|q_k(y_n) = q_k(y) \leq p_{sk}(y)$, and

$$p_k(y) \leq \max_n |h_n(y)|p_k(y_n) \leq \max_n |h_n(y)|p_{sk}(f_n(y_n)x_n) = r_{sk}(y).$$

Thus $(r_k)$ is a base of norms in $\mathcal{P}(F)$. Clearly, $(y_n)$ is $l$-orthogonal with respect to $(r_k)$ and

$$(***) \quad \frac{r_k(y_n)}{r_{k+1}(y_n)} = \frac{p_k(f_n(y_n)x_n)}{p_{k+1}(f_n(y_n)x_n)} = \frac{p_k(x_n)}{p_{k+1}(x_n)} \quad \text{for all } k, n \in \mathbb{N}.$$ 

If $F$ is nuclear, then for any $k \in \mathbb{N}$ there is $m_k \in \mathbb{N}$ with $\lim_n [r_k(y_n)/r_{mk}(y_n)] = 0$. Hence $\lim_n [p_k(x_n)/p_{mk}(x_n)] = 0$ for any $k \in \mathbb{N}$. Thus the set $\{t_n : n \in \mathbb{N}\}$ is infinite and the closed linear span of $(x_n)$ is nuclear.

If $F$ is a Montel space, then by Corollary 1.10, Remark 1.8 and $(***)$, the set $\{t_n : n \in \mathbb{N}\}$ is infinite and the closed linear span of $(x_n)$ is a Montel space.

Case 2. $E$ has no continuous norm. As in the proof of Theorem 2.3 one can prove that $(x_n)$ has a subsequence $(x_{k_n})$ whose closed linear span is isomorphic to $\mathbb{K}^N$. Clearly, $\mathbb{K}^N$ is nuclear, so it is a Montel space, too.

Lemma 5, [11], states that any non-normable Fréchet space $E$ of countable type which is not isomorphic to $c_0 \times \mathbb{K}^N$ or $\mathbb{K}^N$ contains a non-normable closed subspace with a continuous norm. It is obvious by its proof that any Fréchet space which is not isomorphic to the product of a Banach space and $\mathbb{K}^N$ contains a non-normable closed subspace with a continuous norm. Hence, using Theorems 1.5 and 2.3, we get the following.

**Theorem 2.7.** Any non-normable Fréchet space $E$ contains an infinite-dimensional nuclear closed subspace $F$ with a Schauder basis. If $E$ is not isomorphic to the product of a Banach space and $\mathbb{K}^N$, we can claim additionally that $F$ has a continuous norm.

Next example shows that there is a non-normable metrizable lcs $E$ such that:

(i) any subspace of $E$ with an orthogonal basis is normable (compare with Theorem 1.5);

(ii) any nuclear subspace of $E$ is finite-dimensional (compare with Theorem 2.7).
Example 2.8. Let $E$ be a dense subspace of $c_0 \times \mathbb{K}^\mathbb{N}$ with a continuous norm (see [10], Proposition 8 and its proof). Clearly, $E$ is non-normable.

Let $G$ be a subspace with an orthogonal basis in $E$. It is easy to check that the closure $F$ of $G$ in $c_0 \times \mathbb{K}^\mathbb{N}$ has a continuous norm (see [10], Proposition 8). But any closed subspace of $c_0 \times \mathbb{K}^\mathbb{N}$ with a continuous norm is normable (see [12], Proposition 9), so $G$ is normable.

Let $X$ be an infinite-dimensional subspace of $E$. Then $X$ contains a subspace with an orthogonal basis $(\chi_n)$ (see Lemmas 1.1 and 1.2 or [8], Theorem 2). Thus $X$ contains an infinite-dimensional normable subspace. Hence $X$ is non-nuclear.

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