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Closed subspaces without Schauder bases in non-archimedean Fréchet spaces

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ABSTRACT

Let E be an infinite-dimensional non-archimedean Fréchet space which is not isomorphic to any of the following spaces: $c_0, c_0 \times \mathbb{K}^{\mathbb{N}}, \mathbb{K}^{\mathbb{N}}$. It is proved that E contains a closed subspace without a Schauder basis (even without a strongly finite-dimensional Schauder decomposition). Conversely, it is shown that any closed subspace of $c_0 \times \mathbb{K}^{\mathbb{N}}$ has a Schauder basis.

1. INTRODUCTION

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot|: \mathbb{K} \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [7], [5] and [6]. Schauder bases in locally convex spaces are studied in [2], [3] and [4].

Any infinite-dimensional Banach space of countable type is isomorphic (i.e. linearly homeomorphic) to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero (with the sup-norm) ([6], Theorem 3.16), so it has a Schauder basis. It is also known that any metrizable lcs of finite type has a Schauder basis ([3], Theorem 3.5). In [10] we proved that any infinite-dimensional metrizable lcs contains an infinite-dimensional closed subspace with an orthogonal Schauder basis.

In [12] we constructed examples of nuclear Fréchet spaces without a Schauder basis (even without the bounded approximation property). Thus we solved the problem stated in [3], whether any Fréchet space of countable type has a Schauder basis.

In this paper we obtain some results concerning the existence of closed subspaces without Schauder bases in Fréchet spaces.

Let E be a Fréchet space, which is not of finite type, such that none of its subspaces is isomorphic to c_0. Developing the ideas of [1], we show that E contains infinitely many of pairwise-nonisomorphic closed subspaces with a strongly finite-dimensional Schauder decomposition but without a Schauder basis, and a closed subspace with a finite-dimensional Schauder decomposition but without a strongly finite-dimensional Schauder decomposition (Theorem 2).

Next, we prove that every infinite-dimensional Fréchet space, which is not isomorphic to any of the following spaces: $c_0, c_0 \times \mathbb{K}^{\mathbb{N}}, \mathbb{K}^{\mathbb{N}}$, contains a closed subspace without a strongly finite-dimensional Schauder decomposition (Theorem 7).

We also show that every infinite-dimensional metrizable lcs whose completion is isomorphic to none of the following spaces: $c_0, c_0 \times \mathbb{K}^{\mathbb{N}}, \mathbb{K}^{\mathbb{N}}, c_0^{\mathbb{N}}$, contains a closed subspace without an orthogonal Schauder basis (Proposition 9).

2. PRELIMINARIES

The linear span of a subset A of a linear space E is denoted by $\operatorname{lin} A$.

The linear space of all continuous linear operators from a lcs E to itself will be denoted by $\mathcal{L}(E)$.

A sequence (x_n) in a lcs E is a *Schauder basis* of E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $\alpha_n \in \mathbb{K}, n \in \mathbb{N}$, and the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n (n \in \mathbb{N})$ are continuous.

Let E be a lcs. A sequence $(A_n) \subset \mathcal{L}(E)$ is a *Schauder partition* of E if $x = \sum_{n=1}^{\infty} A_n x$ for all $x \in E$. A Schauder partition (A_n) of E is *r-finite-dimensional* ($r \in \mathbb{N}$) if $\sup_n \dim A_n(E) \le r$, *strongly finite-dimensional* if $\sup_n \dim A_n(E) < \infty$, and *finite-dimensional* if $\dim A_n(E) < \infty$ for all $n \in \mathbb{N}$.

A Schauder partition (P_n) of a lcs E is a *Schauder decomposition* of E if $P_n P_m = \delta_{nm} P_n$ for $n, m \in \mathbb{N}$. Clearly, any lcs E with a Schauder basis has a strongly finite-dimensional Schauder decomposition.

A lcs E has the *bounded approximation property* if there exists a sequence $(A_n) \subset \mathcal{L}(E)$ with $\dim A_n(E) < \infty, n \in \mathbb{N}$, such that $\lim_n A_n x = x$ for all $x \in E$. Of course any lcs E with a finite-dimensional Schauder partition has the bounded approximation property.

By a *seminorm* on a linear space E we mean a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x + y) \le \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\ker p := \{x \in E : p(x) = 0\} = \{0\}$.

The set of all continuous seminorms on a lcs E is denoted by $\mathcal{P}(E)$. A family $\mathcal{B} \subset \mathcal{P}(E)$ is a *base* in $\mathcal{P}(E)$ if for every $p \in \mathcal{P}(E)$ there exists $q \in \mathcal{B}$ with $p \le q$.

Every metrizable lcs E has a non-decreasing sequence of continuous seminorms (p_n) which forms a (non-decreasing) base in $\mathcal{P}(E)$.

A lcs E is *of finite type* if for each continuous seminorm p on E the quotient space $E/\ker p$ is finite-dimensional. A metrizable lcs E is *of countable type* if it contains a linearly dense countable set.

A *Fréchet space* is a metrizable complete lcs.

Two norms p, q on a linear space E are *equivalent* if there exist positive numbers a, b such that $ap(x) \leq q(x) \leq bp(x)$ for every $x \in E$. Every two norms on a finite-dimensional linear space are equivalent.

Every n-dimensional lcs is isomorphic to the Banach space $\mathbb{K}^n$.

Let $t \in (0,1]$ and p be a seminorm on a linear space E. An element $x \in E$ is *t-orthogonal to a subspace M of E with respect to p* if $p(\alpha x + y) \geq t \max\{p(\alpha x), p(y)\}$ for all $\alpha \in \mathbb{K}, y \in M$. A sequence $(x_n) \subset E$ is *t-orthogonal with respect to p* if $p(\sum_{i=1}^{n} \alpha_i x_i) \geq t \max\{p(\alpha_i x_i) : 1 \leq i \leq n\}$ for all $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$. A sequence (x_n) in a lcs E is *orthogonal in E* if the family $\mathcal{B}$ of all continuous seminorms p on E for which (x_n) is 1-orthogonal with respect to p forms a base in $\mathcal{P}(E)$. (In [6], a sequence (x_n) in a normed space $(E, \|\cdot\|)$ is called orthogonal if it is 1-orthogonal with respect to the norm $\|\cdot\|$.)

An orthogonal sequence (x_n) of non-zero elements in a lcs E is a *basic orthogonal sequence* in E. A linearly dense basic orthogonal sequence in a lcs E is an *orthogonal basis* in E.

Let $(t_\alpha) \subset (0,1]$. A sequence (x_n) in a lcs E is *(t_α)-orthogonal with respect to* $(p_\alpha) \subset \mathcal{P}(E)$ if (x_n) is t_α-orthogonal with respect to p_α for every α.

A sequence (x_n) in a lcs E is orthogonal in E if and only if it is (t_α)-orthogonal with respect to (p_α) for some $(t_\alpha) \subset (0,1]$ and some base (p_α) in $\mathcal{P}(E)$ (cf.[3], Proposition 2.6).

Let F be a subspace of a lcs E. A sequence $(x_n) \subset F$ is orthogonal in F if and only if it is orthogonal in E ([3], Remark 1.2(i)).

Every basic orthogonal sequence in a lcs E is a Schauder basic sequence in E ([3], Proposition 1.4) and every Schauder basic sequence in a Fréchet space F is a basic orthogonal sequence in F ([3], Proposition 1.7). In particular, every orthogonal basis in a lcs E is a Schauder basis in E, and every Schauder basis in a Fréchet space F is an orthogonal basis in F.

For any sequence (L_n) of finite-dimensional subspaces of a metrizable lcs E with $\dim L_n \geq n^2, n \in \mathbb{N}$, and any non-decreasing base (q_k) in $\mathcal{P}(E)$ there exists a basic orthogonal sequence (y_n) in E with $y_n \in L_n, n \in \mathbb{N}$, that is (t_k)-orthogonal with respect to (q_{m_k}) for some $(t_k) \subset (0,1]$ and some subsequence (q_{m_k}) of (q_k) ([11], Theorem 2). In particular, any infinite-dimensional metrizable lcs contains a basic orthogonal sequence ([10], Theorem 2).

3. RESULTS

Let E be a Fréchet space, which is not of finite type, such that none of its subspaces is isomorphic to c_0. Developing the ideas of [1], we shall show that E contains infinitely many of pairwise-nonisomorphic closed subspaces with a strongly finite-dimensional Schauder decomposition but without a Schauder basis, and a closed subspace with a finite-dimensional Schauder decomposition but without a strongly finite-dimensional Schauder decomposition.

Since E is not of finite type, then it contains an infinite-dimensional subspace G with a continuous norm. Let (v_n) be a basic orthogonal sequence in G. It is easy to see that the closed linear span F of (v_n) in E has a continuous norm. Clearly, F is a non-normable Fréchet space and (v_n) is an orthogonal Schauder basis of F.

Let $(|\cdot|_k)$ be a non-decreasing sequence of norms on F which is a base in $\mathcal{P}(F)$ such that (v_n) is (1)-orthogonal with respect to $(|\cdot|_k)$, and $|\cdot|_k, |\cdot|_{k+1}$ are non-equivalent for any $k \in \mathbb{N}$. Thus there exists a sequence (N_k) of infinite subsets of $\mathbb{N}$ with $\lim_{n\in N_k} |v_n|_k |v_n|_{k+1}^{-1} = 0, k \in \mathbb{N}$. We can easily construct a partition (M_i) of $\mathbb{N}$ such that the set $M_i \cap N_k$ is infinite for all $i,k \in \mathbb{N}$. Then $\liminf_{n\in M_i} |v_n|_k |v_n|_{k+1}^{-1} = 0$ for all $i,k \in \mathbb{N}$. Put

$$\mathcal{N} = \{(p_1,\ldots,p_6) \in \mathbb{N}^6 : p_1 < p_2, p_3 < p_4, p_5 < p_6, p_1 = p_3 - p_2 = p_5 - p_4 = 1\}.$$

Let g be a one-to-one mapping from $\mathcal{N} \times \mathbb{N}^2$ onto $\mathbb{N}$ and $W_u = M_{g(u)}$ for $u \in \mathcal{N} \times \mathbb{N}^2$.

For a subspace Y of F, $f^* \in Y^*, A \in \mathcal{L}(Y)$ and $p,q \in \mathbb{N}$ we put $\|f^*\|_p = \sup\{|f^*(y)||y|_p^{-1} : y \in (Y \setminus \{0\})\}$, and $\|A\|_{p,q} = \sup\{|Ay|_p|y|_q^{-1} : y \in (Y \setminus \{0\})\}$.

We will need the following

Lemma 1. *Let* $u = ((p_1,p_2,p_3,p_4,p_5,p_6),(q,r)) \in \mathcal{N} \times \mathbb{N}^2$. *The linear span of* $\{v_n : n \in W_u\} \subset F$ *contains a subspace* X_u *with* $\dim X_u = r+1$ *such that for any* r*-finite-dimensional Schauder partition* (A_k) *of* X_u *we have*

$$\max\{\|A_k\|_{p_i,p_{i+1}} : k \in \mathbb{N}, i = 1,3,5\} \geq q.$$

Proof. First we note that for all $A, B > 0$ and $m,k \in \mathbb{N}$ there exist $t \in W_u$ with $t > m$ and $a \in (\mathbb{K} \setminus \{0\})$ such that $|av_t|_k \leq A$ and $|av_t|_{k+1} \geq B$. Indeed, let $b \in \mathbb{K}$ with $0 < |b| < 1$. Since $\liminf_{n\in W_u} |v_n|_k|v_n|_{k+1}^{-1} = 0$, then there exists $t \in W_u$ with $t > m$ such that $|v_t|_k|v_t|_{k+1}^{-1} \leq |b|AB^{-1}$. Let n be an integer with $|b|^{n+1} \leq A|v_t|_k^{-1} < |b|^n$. Then $|b^{n+1}v_t|_k \leq A$ and $|b^{n+1}v_t|_{k+1} \geq B$.

Thus we can choose in turn $(x_{r+1},a_{r+1}),\ldots,(x_1,a_1),(y_1,b_1),\ldots,(y_{r+1},b_{r+1})$, $(z_{r+1},c_{r+1}),\ldots,(z_1,c_1) \in \{v_n : n \in W_u\} \times (\mathbb{K} \setminus \{0\})$ such that:

$|a_jx_j|_{p_1} \geq q\max_{i>j}|a_ix_i|_{p_2}$ for $j = r+1,\ldots,1$ (we assume that $\max\emptyset = 0$);
$|b_jy_j|_{p_2} \leq |a_jx_j|_{p_1}$, $|b_jy_j|_{p_3} \geq \max\{|a_jx_j|_{p_4}, q\max_{i<j}|b_iy_i|_{p_4}\}$ for $j = 1,\ldots,r+1$;
$|c_jz_j|_{p_4} \leq \min\{\min_i|a_ix_i|_{p_1}, \min_i|b_iy_i|_{p_3}\}$,
$|c_jz_j|_{p_5} \geq \max\{\max_i|a_ix_i|_{p_6}, \max_i|b_iy_i|_{p_6}, q\max_{i>j}|c_iz_i|_{p_6}\}$ for $j = r+1,\ldots,1$;
$x_1,\ldots x_{r+1}, y_1,\ldots,y_{r+1}, z_1,\ldots,z_{r+1}$ are distinct.

Then we have

$|b_jy_j|_{p_k}, |c_jz_j|_{p_k} \leq |a_jx_j|_{p_k}$ and $|c_jz_j|_{p_k} \leq |a_{r+1}x_{r+1}|_{p_k}$ for $1 \leq j \leq r+1, k = 1,2$;
$|a_jx_j|_{p_k}, |c_jz_j|_{p_k} \leq |b_jy_j|_{p_k}$ and $|c_jz_j|_{p_k} \leq |b_{r+1}y_{r+1}|_{p_k}$ for $1 \leq j \leq r+1, k = 3,4$;
$\max\{|a_jx_j|_{p_k}, |b_jy_j|_{p_k}\} \leq \min\{|c_jz_j|_{p_k}, |c_{r+1}z_{r+1}|_{p_k}\}$ for $1 \leq j \leq r+1, k = 5,6$;
$q|a_jx_j|_{p_2} \leq |a_ix_i|_{p_1}, q|b_iy_i|_{p_4} \leq |b_jy_j|_{p_3}, q|c_jz_j|_{p_6} \leq |c_iz_i|_{p_5}$ for $1 \leq i < j \leq r+1$.

Let $e_j = a_jx_j + b_jy_j + c_jz_j, f_j = e_j$ for $1 \le j \le r$, and
$e_{r+1} = a_{r+1}x_{r+1} + b_{r+1}y_{r+1} + c_{r+1}z_{r+1} - \sum_{j=1}^{r} c_jz_j$, $f_{r+1} = \sum_{j=1}^{r+1} e_j$.
It is easy to see that:
$|e_j|_{p_k} = |a_jx_j|_{p_k}$ for $1 \le j \le r+1, k = 1, 2$;
$|e_j|_{p_k} = |b_jy_j|_{p_k}$ for $1 \le j \le r+1, k = 3, 4$;
$|f_j|_{p_k} = |c_jz_j|_{p_k}$ for $1 \le j \le r+1, k = 5, 6$;
$\max\{|e_i|_{p_1}^{-1}|e_j|_{p_2}, |e_j|_{p_3}^{-1}|e_i|_{p_4}, |f_i|_{p_5}^{-1}|f_j|_{p_6}\} \le q^{-1}$ for $1 \le i < j \le r+1$.
Hence for all $d_1, \ldots, d_{r+1} \in \mathbb{K}$ we have:

$|\sum_{j=1}^{r+1} d_je_j|_{p_k} = |\sum_{j=1}^{r+1} d_ja_jx_j|_{p_k}$ for $k = 1, 2$;
$|\sum_{j=1}^{r+1} d_je_j|_{p_k} = |\sum_{j=1}^{r+1} d_jb_jy_j|_{p_k}$ for $k = 3, 4$;
$|\sum_{j=1}^{r+1} d_jf_j|_{p_k} = |\sum_{j=1}^{r+1} d_jc_jz_j|_{p_k}$ for $k = 5, 6$.

Let $X_u = \text{lin}\{e_j : 1 \le j \le r+1\}$. Then $X_u = \text{lin}\{f_j : 1 \le j \le r+1\}$, and $\dim X_u = r+1$. Thus there exist $e_j^*, f_j^* \in X_u^*$ for $1 \le j \le r+1$ such that $e_j^*(e_i) = f_j^*(f_i) = \delta_{ji}$ for $1 \le j, i \le r+1$. Clearly, $f_j^* = e_j^* - e_{r+1}^*$ for $1 \le j \le r$, and $f_{r+1}^* = e_{r+1}^*$.

Let $1 \le j \le r+1, k = 1, 2$ and $x \in X_u$. Then

$$|x|_{p_k} = |\sum_{i=1}^{r+1} e_i^*(x)e_i|_{p_k} = |\sum_{i=1}^{r+1} e_i^*(x)a_ix_i|_{p_k} \ge |e_j^*(x)a_jx_j|_{p_k} = |e_j^*(x)||e_j|_{p_k}.$$

Hence $\|e_j^*\|_{p_k} \le |e_j|_{p_k}^{-1}$. Since $|e_j^*(e_j)| = 1$, then $\|e_j^*\|_{p_k} \ge |e_j|_{p_k}^{-1}$. Thus
$\|e_j^*\|_{p_k} = |e_j|_{p_k}^{-1}$, for $1 \le j \le r+1, k = 1, 2$. Similarly we obtain that
$\|e_j^*\|_{p_k} = |e_j|_{p_k}^{-1}$, for $1 \le j \le r+1, k = 3, 4$, and
$\|f_j^*\|_{p_k} = |f_j|_{p_k}^{-1}$, for $1 \le j \le r+1, k = 5, 6$.

Let (A_k) be an r-finite-dimensional Schauder partition of X_u. Let $k \in \mathbb{N}$. Put $s_k = \max(\{|e_i^*A_ke_j| : 1 \le i < j \le r+1\} \cup \{|e_j^*A_ke_i| : 1 \le i < j \le r+1\} \cup \{|f_j^*A_kf_{r+1}| : 1 \le j \le r\})$.

We prove that $|e_1^*A_ke_1| \le s_k$. Since $\dim A_k(X_u) \le r$ then there exist $d_1, \ldots, d_{r+1} \in \mathbb{K}$ and $j_0 \in \{1, \ldots, r+1\}$ such that $\sum_{j=1}^{r+1} d_jA_ke_j = 0$ and $\max_j |d_j| = |d_{j_0}| = 1$. Hence $|e_{j_0}^*A_ke_{j_0}| = |-\sum_{j \ne j_0} d_{j_0}^{-1}d_je_{j_0}^*A_ke_j| \le s_k$. For $1 \le j \le r$ we have

$$|e_j^*A_ke_j - e_{r+1}^*A_ke_{r+1}| = |f_j^*A_kf_{r+1} - \sum_{i \ne j} e_j^*A_ke_i + \sum_{i=1}^{r} e_{r+1}^*A_ke_i| \le s_k,$$

since $f_j^* = e_j^* - e_{r+1}^*, f_{r+1} = \sum_{i=1}^{r+1} e_i$. Thus $|e_i^*A_ke_i - e_j^*A_ke_j| \le s_k$ for $1 \le i, j \le r+1$. Hence $|e_1^*A_ke_1| \le \max\{|e_1^*A_ke_1 - e_{j_0}^*A_ke_{j_0}|, |e_{j_0}^*A_ke_{j_0}|\} \le s_k$.

Let $1 \le i < j \le r+1$. Then we have

$$|e_i^*A_ke_j| \le \|e_i^*\|_{p_1}|A_ke_j|_{p_1} \le \|e_i^*\|_{p_1}\|A_k\|_{p_1,p_2}|e_j|_{p_2} =$$
$$|e_i|_{p_1}^{-1}|e_j|_{p_2}\|A_k\|_{p_1,p_2} \le q^{-1}\|A_k\|_{p_1,p_2},$$
$$|e_j^*A_ke_i| \le \|e_j^*\|_{p_3}|A_ke_i|_{p_3} \le \|e_j^*\|_{p_3}\|A_k\|_{p_3,p_4}|e_i|_{p_4} =$$
$$|e_j|_{p_3}^{-1}|e_i|_{p_4}\|A_k\|_{p_3,p_4} \le q^{-1}\|A_k\|_{p_3,p_4}.$$

Let $1 \le j \le r$. Then we obtain

$$|f_j^* A_k f_{r+1}| \leq \|f_j^*\|_{p_5} |A_k f_{r+1}|_{p_5} \leq \|f_j^*\|_{p_5} \|A_k\|_{p_5,p_6} |f_{r+1}|_{p_6} =$$
$$|f_j|_{p_5}^{-1} |f_{r+1}|_{p_6} \|A_k\|_{p_5,p_6} \leq q^{-1} \|A_k\|_{p_5,p_6}.$$

Thus $|e_1^* A_k e_1| \leq s_k \leq q^{-1} \max\{\|A_k\|_{p_i,p_{i+1}} : i = 1,3,5\}, k \in \mathbb{N}$. Hence we have $1 = |\sum_{k=1}^{\infty} e_1^* A_k e_1| \leq \max_k |e_1^* A_k e_1| \leq q^{-1} \max\{\|A_k\|_{p_i,p_{i+1}} : k \in \mathbb{N}, i = 1,3,5\}$. It follows that $\max\{\|A_k\|_{p_i,p_{i+1}} : k \in \mathbb{N}, i = 1,3,5\} \geq q$. □

Let P_u be the natural linear projection from F onto the closed linear span F_u of $\{v_n : n \in W_u\} \subset F$ for $u \in \mathcal{N} \times \mathbb{N}^2$. Clearly, $\{P_u : u \in \mathcal{N} \times \mathbb{N}^2\}$ forms a Schauder decomposition of F.

Let X^r be the closed linear span of $\bigcup\{X_u : u \in \mathcal{N} \times \mathbb{N} \times \{r\}\} \subset F$ for $r \in \mathbb{N}$ and let X be the closed linear span of $\bigcup\{X^r : r \in \mathbb{N}\} \subset F$.

Using Lemma 1 we obtain the following

Theorem 2

(a) Let $r \in \mathbb{N}$. The closed subspace X^r of E has an $(r+1)$-finite-dimensional Schauder decomposition and admits no r-finite-dimensional Schauder partition.

(b) The spaces X^r, X^s are non-isomorphic for distinct $r, s \in \mathbb{N}$.

(c) The closed subspace X of E has a finite-dimensional Schauder decomposition and admits no strongly finite-dimensional Schauder partition.

Proof. (a) Since $P_v(X^r) = X_v$ for $v \in \mathcal{N} \times \mathbb{N} \times \{r\}$ and $P_v(X^r) = \{0\}$ for $v \in \mathcal{N} \times \mathbb{N} \times (\mathbb{N} \setminus \{r\})$, then $\{P_v|X^r : v \in \mathcal{N} \times \mathbb{N} \times \{r\}\}$ forms an $(r+1)-$ finite- dimensional Schauder decomposition of X^r.

Suppose, by contradiction, that X^r has an r-finite-dimensional Schauder partition (B_k). By the Banach-Steinhaus theorem ([5], Theorem 3.37), the operators $B_k, k \in \mathbb{N}$, are equicontinuous. Hence there exists $(p_1,p_2,p_3,p_4,p_5,p_6) \in \mathcal{N}$ and a constant $C > 0$ such that

$$\max_k |B_k x|_{p_i} \leq C|x|_{p_{i+1}} \quad \text{for } x \in X^r, i = 1,3,5.$$

Let $q \in \mathbb{N}$ with $q > C$ and $u = ((p_1,p_2,p_3,p_4,p_5,p_6), q, r)$. Then $u \in \mathcal{N} \times \mathbb{N}^2$ and $P_u|X^r$ is a linear projection of X^r onto X_u and $|P_u x|_k \leq |x|_k$ for all $x \in F, k \in \mathbb{N}$. Let $A_k = (P_u B_k)|X_u, k \in \mathbb{N}$. Then (A_k) is an r-finite-dimensional Schauder partition of X_u. Let $k \in \mathbb{N}, x \in (X_u \setminus \{0\}), i = 1,3,5$. Then

$$|A_k x|_{p_i} |x|_{p_{i+1}}^{-1} = |P_u B_k x|_{p_i} |x|_{p_{i+1}}^{-1} \leq |B_k x|_{p_i} |x|_{p_{i+1}}^{-1} \leq C < q.$$

Hence $\max\{\|A_k\|_{p_i,p_{i+1}} : k \in \mathbb{N}, i = 1,3,5\} < q$, contrary to Lemma 1.

(b) It follows by (a).

(c) Since $P_v(X) = X_v$ for $v \in \mathcal{N} \times \mathbb{N}^2$, then $\{P_v|X : v \in \mathcal{N} \times \mathbb{N}^2\}$ forms a finite-dimensional Schauder decomposition of X. Suppose, on the contrary, that X has an r-finite-dimensional Schauder partition (A_k) for some $r \in \mathbb{N}$. Let P^r be the natural projection of F onto the closed linear span F^r of $\bigcup\{F_v : v \in \mathcal{N} \times \mathbb{N} \times \{r\}\} \subset F$ and $B_k = (P^r A_k)|X^r, k \in \mathbb{N}$. Since $P^r|X$ is a continuous linear projection of X onto X^r, then (B_k) is an r-finite-dimensional Schauder partition of X^r, contrary to (a). □

By the proof of Theorem 2 we obtain the following

Corollary 3. *Any non-normable Fréchet space F with a continuous norm and with a Schauder basis contains an infinite-dimensional closed subspace without a strongly finite-dimensional Schauder decomposition.*

In order to get our next theorem we need three lemmas. The first one is a simple modification of Proposition 2.2, [9].

Lemma 4. *Let $n \in \mathbb{N}$. Let q_1, q_2 be norms on an n-dimensional linear space E. Then for any $t \in (0,1)$ there exists a basis $(u_1, \ldots, u_n)$ of E that is t-orthogonal with respect to q_1 and q_2.*

Proof. We prove this lemma by induction. It is clear for $n = 1$. Assume that it is true for $n = k$. We show that it is true for $n = k+1$. Let $t \in (0,1)$. Put $s = t^{1/(k+2)}$. Let $(f_1, \ldots, f_{k+1})$ be an s-orthogonal basis of (E, q_1) ([6], Theorem 3.15(iii)). Set $a := \max_i(q_2(f_i)/q_1(f_i))$. Then $q_1(x) \geq a^{-1}sq_2(x)$ for all $x \in E$. Let $1 \leq m \leq k+1$ with $(q_2(f_m)/q_1(f_m)) = a$ and let $(e_1, \ldots, e_{k+1})$ be a basis of E such that $e_1 = f_m$ and e_{i+1} is t-orthogonal to $\text{lin}\{e_1, \ldots, e_i\}$ with respect to q_2 for $i = 1, \ldots, k$ ([6], Lemma 3.14). Put $V = \text{lin}\{e_2, \ldots, e_{k+1}\}$. For $\alpha_1 \in \mathbb{K}, e \in V$ we have $q_2(\alpha_1 e_1 + e) \geq s^k q_2(\alpha_1 e_1)$ and $q_1(\alpha_1 e_1 + e) \geq a^{-1}sq_2(\alpha_1 e_1 + e) \geq a^{-1}s^{k+1}q_2(\alpha_1 e_1) = s^{k+1}q_1(\alpha_1 e_1)$; hence $q_2(\alpha_1 e_1 + e) \geq s^k \max\{q_2(\alpha_1 e_1), q_2(e)\}$ and $q_1(\alpha_1 e_1 + e) \geq s^{k+1}\max\{q_1(\alpha_1 e_1), q_1(e)\}$ ([6], Lemma 3.2). By the assumption there exists a basis $(v_1, \ldots, v_k)$ of V that is s-orthogonal with respect to q_1 and q_2. It is clear that $(e_1, v_1, \ldots, v_k)$ is a basis of E which is t-orthogonal with respect to q_1 and q_2. □

Lemma 5. *Let E be an infinite-dimensional Fréchet space of countable type, that is not isomorphic to any of the following spaces: $c_0, c_0 \times \mathbb{K}^{\mathbb{N}}, \mathbb{K}^{\mathbb{N}}$. Then E contains a non-normable closed subspace with a continuous norm.*

Proof. Let (p_k) be a non-decreasing base in $\mathcal{P}(E)$. Consider two cases.

Case 1: There exists $m \in \mathbb{N}$ such that $\dim(\ker p_k / \ker p_{k+1}) < \infty$ for any $k \geq m$. Then $F = \ker p_m$ is a Fréchet space of finite type. Thus F is isomorphic to $\mathbb{K}^{\mathbb{N}}$ or to $\mathbb{K}^n$ for some $n = 0, 1, 2, \ldots$, and it is complemented in E ([8], Corollary 9.1(iv)). Any complement of F in E is a non-normable closed subspace with a continuous norm.

Case 2: There exists an increasing sequence (n_k) in $\mathbb{N}$ with $\dim(\ker p_{n_k} / \ker p_{n_{k+1}}) = \infty$ for any $k \in \mathbb{N}$. Let F_k be a subspace of $\ker p_{n_k}$ such that $\dim F_k = \infty$ and $(F_k \cap \ker p_{n_{k+1}}) = \{0\}$. By Theorem 2, [11](see Preliminaries) there exists a basic orthogonal sequence (y_n) in E such that the set $\{n \in \mathbb{N} : y_n \in F_k\}$ is infinite for any $k \in \mathbb{N}$. Let X be the closed linear span of $(y_n) \subset E$. Then there exists a non-decreasing base (q_k) in $\mathcal{P}(X)$ with $q_1 = 0$ such that (y_n) is (1)-orthogonal with respect to (q_k) and the set $D_k = \{n \in \mathbb{N} : y_n \in (\ker q_{k-1} \setminus \ker q_k)\}$ is infinite for any $k \in \mathbb{N}$. Denote by X_k the closed linear span of $\{y_n : n \in D_k\} \subset X, k \in \mathbb{N}$. For any $n \in \mathbb{N}$, X_n is an infinite-dimensional Fréchet space and $q_n|X_n$ is a continuous norm on X_n. If X_n is non-normable for some $n \in \mathbb{N}$, then the proof is complete. Otherwise, X_n is isomorphic to c_0 for any $n \in \mathbb{N}$.

For any $(x_n) \in \prod_{n=1}^{\infty} X_n$ the series $\sum_{n=1}^{\infty} x_n$ is convergent in X, since $q_k(x_n) = 0$ for all $n, k \in \mathbb{N}$ with $n > k$. Let P_n be the natural projection from X onto $X_n, n \in \mathbb{N}$. Clearly, (P_n) is a Schauder decomposition of X. By the open mapping theorem ([5], Corollary 2.74), the map $P : X \to \prod_{n=1}^{\infty} X_n, x \to (P_n x)$, is an isomorphism.

Thus X is isomorphic to $c_0^{\mathbb{N}}$. For any Fréchet space G of countable type, $c_0^{\mathbb{N}}$ contains a closed subspace isomorphic to G ([3], Remark 3.6). Hence X contains a non-normable closed subspace with a continuous norm (see e.g. [12]). □

Lemma 6. *Let E be a non-normable Fréchet space with a continuous norm and with a finite-dimensional Schauder decomposition (P_n). Then E contains a non-normable closed subspace F with a Schauder basis.*

Proof. Let (r_k) be a non-decreasing sequence of norms on E that forms a base in $\mathcal{P}(E)$. By the Banach-Stainhaus theorem the operators $P_n, n \in \mathbb{N}$, are equicontinuous. Thus the norms $p_k(x) = \max_n r_k(P_n x), x \in E, k \in \mathbb{N}$, are continuous. Since $r_k \le p_k, k \in \mathbb{N}$, then (p_k) is a base in $\mathcal{P}(E)$. For any $x \in E$ the sequence $(P_n x)$ is (1)-orthogonal with respect to (p_k). Indeed, let $x \in E$, $k \in \mathbb{N}$ and $(a_n) \subset \mathbb{K}$ with $a_m = a_{m+1} = \cdots = 0$ for some $m \in \mathbb{N}$. Then $p_k(a_i P_i x) = \max_n r_k(P_n(a_i P_i x)) = r_k(a_i P_i x)$ for any $i \in \mathbb{N}$, and $p_k(\sum_{i=1}^{\infty} a_i P_i x) = \max_n r_k(P_n(\sum_{i=1}^{\infty} a_i P_i x)) = \max_n r_k(a_n P_n x) = \max_i p_k(a_i P_i x)$.

Since E is non-normable, then there is an increasing sequence $(m_k) \subset \mathbb{N}$ such that p_{m_k} and $p_{m_{k+1}}$ are non-equivalent for any $k \in \mathbb{N}$. Put $q_k = p_{m_k}$ for $k \in \mathbb{N}$.

Let $t \in (0, 1)$. Let $k \in \mathbb{N}$. Put $E_n = P_n(E), s_n = \dim E_n, n \in \mathbb{N}$. By Lemma 4, there exists a basis $(e_1^n, \ldots, e_{s_n}^n)$ of $E_n, n \in \mathbb{N}$, that is t-orthogonal with respect to $q_k|E_n$ and $q_{k+1}|E_n$. The sequence $(x_n^k)_{n=1}^{\infty} = (e_1^1, \ldots, e_{s_1}^1, e_1^2, \ldots, e_{s_2}^2, \ldots)$ is linearly dense in E, and it is t-orthogonal with respect to q_k and q_{k+1}. Indeed, let $m \in \mathbb{N}, a_1^1, \ldots, a_{s_1}^1, \ldots, a_1^m, \ldots, a_{s_m}^m \in \mathbb{K}$ and $x = \sum_{n=1}^{m} (\sum_{i=1}^{s_n} a_i^n e_i^n)$. Then

$q_j(x) = q_j(\sum_{n=1}^{m} P_n x) = \max_{1 \le n \le m} q_j(P_n x) = \max_{1 \le n \le m} q_j(\sum_{i=1}^{s_n} a_i^n e_i^n) \ge t \max_{1 \le n \le m}(\max_{1 \le i \le s_n} q_j(a_i^n e_i^n))$ for $j = k, k+1$.

For any sequence $(x_n) \subset E$, which is t-orthogonal with respect to q_k and q_{k+1}, we have:

$$(*_1) \qquad \inf\{(q_k(x)/q_{k+1}(x)) : x \in \mathrm{lin}\{x_n : n \in \mathbb{N}\}\} \ge t \inf_n (q_k(x_n)/q_{k+1}(x_n));$$

$$(*_2) \qquad \sup\{(q_k(x)/q_{k+1}(x)) : x \in \mathrm{lin}\{x_n : n \in \mathbb{N}\}\} \le t^{-1} \sup_n (q_k(x_n)/q_{k+1}(x_n)).$$

Indeed, let $m \in \mathbb{N}, a_1, \ldots a_m \in \mathbb{K}, x = \sum_{i=1}^{m} a_i x_i$. For some $i_0, j_0 \in \{1, \ldots, m\}$ we have $\max_i q_k(a_i x_i) = q_k(a_{i_0} x_{i_0})$ and $\max_i q_{k+1}(a_i x_i) = q_{k+1}(a_{j_0} x_{j_0})$. Then

$(q_k(x)/q_{k+1}(x)) = (q_k(\sum_i a_i x_i)/q_{k+1}(\sum_i a_i x_i)) \ge$

$(t \max_i q_k(a_i x_i)/\max_i q_{k+1}(a_i x_i)) \geq$
$(t q_k(a_{j_0} x_{j_0})/q_{k+1}(a_{j_0} x_{j_0})) \geq t \inf_n(q_k(x_n)/q_{k+1}(x_n))$, and
$(q_k(x)/q_{k+1}(x)) = (q_k(\sum_i a_i x_i)/q_{k+1}(\sum_i a_i x_i)) \leq$
$(\max_i q_k(a_i x_i)/t \max_i q_{k+1}(a_i x_i)) \leq (q_k(a_{i_0} x_{i_0})/t q_{k+1}(a_{i_0} x_{i_0})) \leq$
$t^{-1} \sup_n(q_k(x_n)/q_{k+1}(x_n))$. It follows $(*_1)$ and $(*_2)$.

Since q_k, q_{k+1} are non-equivalent norms on E and the sequence $(x_n^k)_{n=1}^\infty$ is linearly dense in E, we have $\inf\{(q_k(x)/q_{k+1}(x)) : x \in \operatorname{lin}\{x_n^k : n \in \mathbb{N}\}\} = 0$. By $(*_1)$, we obtain $\inf_n(q_k(x_n^k)/q_{k+1}(x_n^k)) = 0$. Thus for some an increasing sequence $(n_i) \subset \mathbb{N}$ we have $(q_k(x_{n_i}^k)/q_{k+1}(x_{n_i}^k)) \leq i^{-1}, i \in \mathbb{N}$. Put $E_m^k = \operatorname{lin}\{x_{n_i}^k : i \geq m\}$ for $m \in \mathbb{N}$. Clearly, $\dim E_m^k = \infty, m \in \mathbb{N}$. By $(*_2)$, we get

$$(*_3) \qquad \sup_{x \in E_m^k} (q_k(x)/q_{k+1}(x)) \leq (tm)^{-1}.$$

By Theorem 2, [11], there exists an orthogonal sequence (y_n) in E such that for any $k, m \in \mathbb{N}$ there is $n \in \mathbb{N}$ with $y_n \in E_m^k$. Let F be the closed linear span of (y_n). Then (y_n) is a Schauder basis of F. From $(*_3)$ we have $\inf_n(q_k(y_n)/q_{k+1}(y_n)) = 0, k \in \mathbb{N}$. This follows that norms $q_k|F, q_{k+1}|F$ on F are not equivalent for any $k \in \mathbb{N}$. Since $(q_k|F)_{k=1}^\infty$ is a base in $\mathcal{P}(F)$, then F is non-normable. □

By Lemmas 5,6 and Corollary 3 we obtain the following

Theorem 7. *Let E be an infinite-dimensional Fréchet space, which is not isomorphic to any of the following spaces: $c_0, c_0 \times \mathbb{K}^{\mathbb{N}}, \mathbb{K}^{\mathbb{N}}$. Then E contains an infinite-dimensional closed subspace without a strongly finite-dimensional Schauder decomposition.*

Corollary 8. *Any non-normable Fréchet space with a continuous norm contains a closed subspace without a Schauder basis.*

Clearly any closed subspace of c_0 has a Schauder basis; it is also known that any closed subspace of $\mathbb{K}^{\mathbb{N}}$ has a Schauder basis (see Introduction). Now we prove that any closed subspace of $c_0 \times \mathbb{K}^{\mathbb{N}}$ has a Schauder basis too.

Proposition 9. *Any infinite-dimensional closed subspace of $c_0 \times \mathbb{K}^{\mathbb{N}}$ is isomorphic to one of the following spaces: $c_0, c_0 \times \mathbb{K}^{\mathbb{N}}, \mathbb{K}^{\mathbb{N}}$. In particular any closed subspace of $c_0 \times \mathbb{K}^{\mathbb{N}}$ has a Schauder basis.*

Proof. By Lemma 5 it is enough to show that any infinite-dimensional closed subspace F of $c_0 \times \mathbb{K}^{\mathbb{N}}$ with a continuous norm is normable. Put

$$p_k(\alpha, \beta) = k \max\{\max_n |\alpha_n|, \max_{1 \leq n \leq k} |\beta_n|\}$$

for $k \in \mathbb{N}, \alpha = (\alpha_n) \in c_0$ and $\beta = (\beta_n) \in \mathbb{K}^{\mathbb{N}}$. Clearly, (p_k) is a non-decreasing base in $\mathcal{P}(c_0 \times \mathbb{K}^{\mathbb{N}})$ and there exists $t \in \mathbb{N}$ such that $p_t|F$ is a norm on F. Let

$q_k = p_{t+k}|F, M_k = F \cap \{(\alpha, \beta) \in c_0 \times \mathbb{K}^{\mathbb{N}} : \beta_{t+k+1} = 0\}, k \in \mathbb{N}$. Then $M_k < F$, $\dim(F/M_k) \leq 1$ and $q_k|M_k = q_{k+1}|M_k, k \in \mathbb{N}$.

Let $k \in \mathbb{N}$. If M_k is a closed subspace of the normable space (F, q_k), then the norms q_k, q_{k+1} are equivalent. Indeed, let $x_k \in (F \setminus M_k), d_k = \inf_{m \in M_k} q_k(x_k - m)$ and $a_k = d_k(q_k(x_k))^{-1}$. Then $F = \text{lin}(\{x_k\} + M_k), a_k > 0$ and $q_k(\alpha x_k + m) \geq a_k q_k(\alpha x_k)$ for all $\alpha \in \mathbb{K}, m \in M_k$. By Lemma 3.2, [5], we obtain that

$$q_k(\alpha x_k + m) \geq a_k \max\{q_k(\alpha x_k), q_k(m)\}, \alpha \in \mathbb{K}, m \in M_k.$$

Hence for all $\alpha \in \mathbb{K}, m \in M_k$ and $b_k = (q_{k+1}(x_k)/q_k(x_k))$ we have

$$q_{k+1}(\alpha x_k + m) \leq b_k \max\{q_k(\alpha x_k), q_k(m)\} \leq a_k^{-1} b_k q_k(\alpha x_k + m).$$

Thus the norms q_k, q_{k+1} are equivalent. This follows that for any $k \in \mathbb{N}$ the norms q_k, q_{k+1} are equivalent on a dense subspace F_k of (F, q_k).

We shall prove that the normable space (F, q_1) is complete. Let (f_k^1) be a Cauchy sequence in (F, q_1) and $k_n^0 = n, n \in \mathbb{N}$. Then there exists a subsequence (k_n^1) of (k_n^0) such that $q_1(f_{k_n^1}^1 - f_{k_{n+1}^1}^1) < n^{-1}, n \in \mathbb{N}$. Since F_1 is a dense subspace of (F, q_1), we can choose a sequence $(f_{k_n^1}^2) \subset F_1$ with $q_1(f_{k_n^1}^1 - f_{k_n^1}^2) < n^{-1}, n \in \mathbb{N}$. Clearly $(f_{k_n^1}^2)$ is a Cauchy sequence in (F, q_1). Since the norms q_1, q_2 are equivalent on F_1 and $(f_{k_n^1}^2) \subset F_1$, then $(f_{k_n^1}^2)$ is a Cauchy sequence in (F, q_2). In this way we can choose in turn for every $v \in \mathbb{N}$ a subsequence (k_n^v) of (k_n^{v-1}) with $q_v(f_{k_n^v}^v - f_{k_{n+1}^v}^v) < n^{-1}, n \in \mathbb{N}$, and a sequence $(f_{k_n^v}^{v+1}) \subset F_v$ with $q_v(f_{k_n^v}^v - f_{k_n^v}^{v+1}) < n^{-1}, n \in \mathbb{N}$.

For any $n \in \mathbb{N}$ there exists $s \in \mathbb{N}$ with $s > n$ such that $k_{n+1}^{n+1} = k_s^n$. Since $q_n(f_{k_i^n}^n - f_{k_{i+1}^n}^n) < i^{-1}, n \leq i \leq s-1$, then $q_n(f_{k_n^n}^n - f_{k_{n+1}^{n+1}}^n) < n^{-1}$. Moreover

$$q_n(f_{k_{n+1}^{n+1}}^n - f_{k_{n+1}^{n+1}}^{n+1}) = q_n(f_{k_s^n}^n - f_{k_s^n}^{n+1}) < s^{-1} < n^{-1}.$$

Hence $q_n(f_{k_n^n}^n - f_{k_{n+1}^{n+1}}^{n+1}) < n^{-1}, n \in \mathbb{N}$. This follows that $q_i(f_{k_n^n}^n - f_{k_{n+1}^{n+1}}^{n+1}) \to 0$ for any $i \in \mathbb{N}$. Thus $(f_{k_n^n}^n)$ is a Cauchy sequence in F. Let f be the limit of $(f_{k_n^n}^n)$ in F. Since

$$\begin{aligned} q_1(f_{k_n^n}^1 - f_{k_n^n}^n) &\leq \max_{1 \leq i \leq n-1} q_1(f_{k_n^n}^i - f_{k_n^n}^{i+1}) \leq \max_{1 \leq i \leq n-1} q_i(f_{k_n^n}^i - f_{k_n^n}^{i+1}) \\ &< n^{-1}, n \in \mathbb{N}, \end{aligned}$$

and $q_1(f_{k_n^n}^n - f) \to 0$, then $q_1(f_{k_n^n}^1 - f) \to 0$. Hence $q_1(f_k^1 - f) \to 0$, because (f_k^1) is a Cauchy sequence in (F, q_1). Thus we have proved that the normable space (F, q_1) is complete.

By the open mapping theorem ([4], Corollary 2.74) the Fréchet space F is normable. □

Finally, we obtain the following

Proposition 10. *Let E be an infinite-dimensional metrizable lcs whose completion X is not isomorphic to any of the following spaces: $c_0, c_0 \times \mathbb{K}^{\mathbb{N}}, \mathbb{K}^{\mathbb{N}}, c_0^{\mathbb{N}}$. Then E contains a closed subspace without an orthogonal Schauder basis.*

Proof. It is enough to consider the case when E has an orthogonal Schauder basis (y_n). Then (y_n) is an orthogonal Schauder basis of X. Let (q_k) be a non-decreasing base in $\mathcal{P}(X)$ with $q_1 = 0$ such that (y_n) is (1)-orthogonal with respect to (q_k). Put $D_k = \{n \in \mathbb{N} : y_n \in (\ker q_{k-1} \setminus \ker q_k)\}, k \in \mathbb{N}$. Denote by X_k the closed linear span of $\{y_n : n \in D_k\}$ in X. As in the proof of Lemma 5 we obtain that X is isomorphic to $\prod_{n=1}^{\infty} X_n$. If all the Fréchet spaces $X_k, k \in \mathbb{N}$, are normable, then X is isomorphic to one of the following spaces: $c_0, c_0 \times \mathbb{K}^{\mathbb{N}}, \mathbb{K}^{\mathbb{N}}, c_0^{\mathbb{N}}$, contrary to our assumption. Thus, for some $m \in \mathbb{N}$ the space X_m is non-normable. Clearly, $\{y_n : n \in D_m\}$ is an orthogonal Schauder basis of X_m and $q_m | X_m$ is a continuous norm on X_m. Hence, by the proofs of Lemma 1 and Theorem 2, there exists a linear subspace V of $\text{lin}\{y_n : n \in D_m\}$ such that the closure V_1 of V in X_m has no Schauder basis. Then the closure V_2 of V in E has no orthogonal Schauder basis. □

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