Adam Mickiewicz University, Poznań
Faculty of Mathematics and Computer Science
Uniwersytet im. Adama Mickiewicza w Poznaniu
Wydział Matematyki i Informatyki

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Structure constants of Jack characters

Stałe strukturalne charakterów Jacka

Adam Burchardt

Supervisor
Promotor
Prof. dr hab. Piotr Śniady

Auxiliary supervisor
Promotor pomocniczy
Dr Maciej Dołęga

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Abstract

In 1996 Goulden and Jackson introduced a family of coefficients $(c^{\lambda}_{\mu,\nu})$ indexed by triples of partitions which arise in the power sum expansion of some Cauchy sum for Jack symmetric functions $J^{(\alpha)}_{\pi}$. Goulden and Jackson suggested that there is a combinatorics of matchings hidden behind the coefficients $c^{\lambda}_{\mu,\nu}$. This *Matchings-Jack Conjecture* remains open.

Jack characters are a generalization of the characters of the symmetric groups, they provide a kind of dual information about the Jack polynomials. We investigate the structure constants $g^{\lambda}_{\mu,\nu}$ for Jack characters. They are a generalization of the connection coefficients for the symmetric groups. We give formulas for the top-degree part of $g^{\lambda}_{\mu,\nu}$ and $c^{\lambda}_{\mu,\nu}$. We present those results in context of Matchings-Jack Conjecture of Goulden and Jackson.

We adapt the probabilistic concept of cumulants to the setup of a linear space equipped with two multiplication structures. We present an algebraic formula which expresses a given nested product with respect to those two multiplications as a sum of products of the cumulants. This formula leads to some conclusions about the structure constants of Jack characters. We also show that our formula may be understood as an analogue of Leonov–Shiraev's formula.

Streszczenie

W 1996 r. Goulden i Jackson wprowadzili rodzinę współczynników $(c^{\lambda}_{\mu,\nu})$ indeksowaną trójkami partycji, która pojawia się w rozwinięciu w szereg potęgowy pewnej sumy Cauchy'ego symetrycznych wielomianów Jacka $J^{(\alpha)}_{\pi}$. Goulden i Jackson przypuszczali, że za współczynnikami $c^{\lambda}_{\mu,\nu}$ ukryta jest kombinatoryka związana ze skojarzeniami. Postawiona przez nich hipoteza „O Skojarzeniach Jacka" pozostaje do dzisiaj otwarta.

Charaktery Jacka są uogólnieniem charakterów grup symetrycznych oraz obiektami dualnymi do wielomianów Jacka. W rozprawie doktorskiej badamy stałe strukturalne $g^{\lambda}_{\mu,\nu}$ charakterów Jacka. Są one uogólnieniem stałych strukturalnych grup symetrycznych. Podajemy wzory na współczynniki najwyższych stopni wielomianów $g^{\lambda}_{\mu,\nu}$ i $c^{\lambda}_{\mu,\nu}$. Prezentujemy te rezultaty w kontekście hipotezy „O Skojarzeniach Jacka".

Adaptujemy probabilistyczne pojęcie kumulanty do struktury przestrzeni liniowej z dwoma mnożeniami. Prezentujemy formułę, która wyraża pewien mieszany iloczyn jako sumę kumulant. Znalezione wyrażenie prowadzi do wniosków na temat stałych strukturalnych charakterów Jacka. Pokazujemy również, że nasza formuła może zostać uznana za odpowiednik formuły Leonova i Shiraeva.

Podziękowania

Autor jest niezmiernie wdzięczny wszystkim tym, którzy pomogli mu przy przygotowaniu tej pracy.

Przede wszystkim pragnę złożyć podziękowania profesorowi Piotrowi Śniademu. Wprowadzenie do świata kombinatoryki algebraicznej, poświęcony przy tym czas i energia, okazana pomoc przy kształtowaniu pracy doktorskiej, wszystko to trudno przecenić.

Jestem wdzięczny Maciejowi Dołędze za inspirujące rozmowy. Dziękuję Leonowi Gondelmanowi za wsparcie i za pomoc przy korekcie tej pracy. Dziękuję Tomaszowi Godlewskiemu za wiele lat prawdziwej przyjaźni. Ostatnie, ale nie mniejsze, wyrazy wdzięcznośći kieruje ku mojej wspaniałej rodzinie.

Contents

Chapter 1

Introduction

Jack polynomials

Jack polynomials $\left(J_\pi^{(\alpha)}\right)$ are a family of symmetric functions that depend on a parameter $\alpha > 0$ and is indexed by an integer partition π. They were introduced by Henry Jack in his seminal paper [Jac71]. For certain values of α, Jack polynomials coincide with various well-known symmetric polynomials. For instance, up to multiplicative constants, Jack polynomials coincide with Schur polynomials for $\alpha = 1$; with the zonal polynomials, for $\alpha = 2$; with the symplectic zonal polynomials, for $\alpha = 1/2$; with the elementary symmetric functions, for $\alpha = 0$; and in some sense with the monomial symmetric functions, for $\alpha = \infty$. Since it has been shown that several results concerning Schur and zonal polynomials can be generalized in a rather natural way to Jack polynomials [Mac15, Section (VI.10)], Jack polynomials can be viewed as a natural interpolation between several interesting families of symmetric functions.

Connections of Jack polynomials with various fields of mathematics and physics were established: it turned out that they play a crucial role in understanding Ewens random permutations model [DH92], generalized β-ensembles and some statistical mechanics models [OO97], Selberg-type integrals [Kan93], certain random partition models [Ker00], and some problems of the algebraic geometry [Nak96, Oko03], among many others. Better understanding of Jack polynomials is also very desirable in the context of generalized β-ensembles and their discrete counterpart model [OO97]. Jack polynomials are a special case of the *Macdonald polynomials* [Sta89, Mac15].

Connection coefficients

In 1996 Goulden and Jackson [GJ96] introduced two families of coefficients $(c^{\lambda}_{\mu,\nu})$ and $(h^{\lambda}_{\mu,\nu})$ depending on the parameter α and indexed by triples of partitions which arise in the power sum expansion of some Cauchy sum for Jack symmetric functions, Section 2.1.3 explains some details more in depth. As Jack polynomials can be viewed as a natural interpolation between several interesting families of symmetric functions, so the coefficients $(c^{\lambda}_{\mu,\nu})$ can be viewed as an interpolation between the structure constants of the class algebra and the double coset algebra.

Combinatorics of maps and matchings

A *map* [LZ04] is classically defined as a connected graph G (possibly, with multiple edges) drawn on a surface Σ, *i.e.* a compact connected 2-dimensional manifold without boundary. The notion of a map is well established in enumerative combinatorics. A vertex two-coloured map is called *bipartite* if each edge connects vertices of different colors. For such a map we can canonically assign three partitions describing the distributions of both kinds of vertices and the distribution of the numbers of edges in the faces.

Matchings (also known as *perfect matchings* and *pair partitions*) are another class of combinatorial objects appearing naturally in enumerative combinatorics. There is a close relation between them and maps, see Section 2.2.3.

The Matchings-Jack Conjecture

Goulden and Jackson [GJ96] observed that some specialisations of the aforementioned coefficients $c^{\lambda}_{\mu,\nu}$ and $h^{\lambda}_{\mu,\nu}$ may be interpreted in terms of matchings and maps respectively. Moreover, they observed that both coefficients seem to be polynomials in the variable $\beta := \alpha - 1$ with non-negative integer coefficients. This supposition was expressed in two conjectures known today as the *Matchings-Jack Conjecture* and the *b-Conjecture*. After more then 20 years both still remain open.

Goulden and Jackson in their Matchings-Jack Conjecture suggested that the quantity $c^{\lambda}_{\pi,\sigma}$ can be expressed as

$$c^{\lambda}_{\pi,\sigma}(\beta) = \sum_{\delta} \beta^{\mathrm{wt}_{\lambda}(\delta)},$$

where wt_{λ} is some hypothetical combinatorial statistic and the sum runs over some special set of matchings. There are two easy specialisations of $c^{\lambda}_{\pi,\sigma}$

which indeed may be expressed in terms of matchings. Those specialisations coincide with the connection coefficients of two commutative subalgebras of the group algebra of the symmetric group: the class algebra and the double coset algebra.

Except for some special cases there are no closed formulas for the coefficients $c^{\lambda}_{\pi,\sigma}$. Bédard and Goupil [BG92] found a formula for $c^{(n)}_{\pi,\sigma}$ in the restricted case. Goulden and Jackson [GJ92] gave a bijective proof of this result. Goupil and Schaeffer [GS98] provide some formulas for $c^{(n)}_{\pi,\sigma}$ in the general case. Morales and Vassilieva [MV13, Vas15] found closed formulas for the expansion of the generating series of $c^{(n)}_{\pi,\sigma}$ using bijective methods. There are also some other results about the coefficients $c^{\lambda}_{\pi,\sigma}$ [Bia05, Irv06].

Parallel to the studies on the form of the coefficients $c^{\lambda}_{\pi,\sigma}$, the suitable statistic wt_{λ} is being sought. Goulden and Jackson constructed some statistics wt_{λ} for $\lambda = [1^n]$ and $\lambda = [2, 1^{n-1}]$ and proved the conjecture in those cases [GJ96]. Later on, the Matchings-Jack Conjecture has been proved [KV16] in the case $\pi = \sigma = (n)$ of the partitions with exactly one part. Recently, this result was strengthened [KVP18] to the case when one of the three partitions is equal to (n) (under some assumptions).

The question of existence of the suitable statistic wt_{λ} still remains open. However, it seems that the appropriate candidate η for a similar statistic which appears in the b-Conjecture was found [La 09] as a measure of non-orientability on a class of *rooted maps*.

The Matchings-Jack Conjecture and the b-Conjecture are ones of major open questions in enumerative algebraic combinatoric. They relate aspects like: symmetric functions, the representation theory of the symmetric groups, combinatorics of maps and matchings.

The first result

It was proved [DF16] that $c^{\lambda}_{\pi,\sigma}$ are polynomials in the variable $\beta := \alpha - 1$ and a satisfactory bound on their degrees was given. We investigate the leading coefficient of the polynomials $c^{\lambda}_{\pi,\sigma}$. More precisely, in Theorem 2.5 we give a sufficient and necessary condition for the polynomial $c^{\lambda}_{\pi,\sigma}$ to achive the bound on its degree given in [DF16]. We show that the leading coefficient of such $c^{\lambda}_{\pi,\sigma}$ is a positive integer.

We present our result in the context of Matchings-Jack Conjecture of Goulden and Jackson. Indeed, in Section 2.3.4 we construct the statistic stat_{η} on a special class of matchings. We show that the leading coefficient of the polynomials $c^{\lambda}_{\pi,\sigma}$ coincides with the leading coefficient relevant to the statistic stat_η.

In fact, we adapt the statistic η of La Croix [La 09] to the case of *lists of maps*. In some special cases it may be translated into the field of matchings, however generally significant difficulties appear. Such attempts have already been made [KVP18]; it seems that the difficulty increases with the generality. In Section 2.3.4 we present briefly the problem of transferring the statistic η into a satisfactory statistic which measures non-bipartiteness of a matching.

The approach we use to prove Theorem 2.5 is fundamentally different from previous attempts of other authors to prove the Matchings-Jack Conjecture and the b-Conjecture. We investigate the *structure constants* for *Jack characters*, which are kind of *dual* objects to Jack polynomials. Although our research allows us so far to recover only the leading coefficients of the structure constants and the connection coefficients, in Section 4.3 we introduce a method linked with the theory of cumulants, which in the future may give information about the remaining coefficients of the polynomials $c^{\lambda}_{\pi,\sigma}$.

Unnormalized Jack characters

By expanding Jack polynomial in the basis of power-sum symmetric functions:

$$J^{(\alpha)}_{\lambda} = \sum_{\mu} \theta^{(\alpha)}_{\mu}(\lambda)\ p_{\mu}$$

we get coefficients $\theta^{(\alpha)}_{\mu}(\lambda)$ called *unnormalized Jack characters*. Jack characters $\theta^{(\alpha)}_{\mu}$ provide a kind of *dual* information about the Jack polynomials.

The coefficients appearing in the expansion of a pointwise product of two unnormalized Jack characters in the unnormalized Jack character basis coincide with the connection coefficients [DF16], namely

$$\theta^{(\alpha)}_{\pi} \cdot \theta^{(\alpha)}_{\sigma} = \sum_{\mu \vdash n} c^{\mu}_{\pi,\sigma} \theta^{(\alpha)}_{\mu}.$$

This observation encourages us to look more deeply into the field of connection coefficients via the context of Jack characters.

This kind of *dual* approach may be traced back to the work of Kerov and Olshanski [KO94] on the characters of the symmetric groups. The usual way of viewing the characters of the symmetric groups is to fix the representation λ and to consider the character as a function of the conjugacy class π. Kerov and Olshanski suggested to do roughly the opposite. This idea was adapted by Lassalle [Las08, Las09] to the framework of Jack characters.

Normalized Jack characters

In order for this dual approach to be successful one has to choose the most convenient normalization constants. We define *(normalized) Jack characters* Ch_π as

$$\mathrm{Ch}_\pi(\lambda) := \begin{cases} \frac{1}{\sqrt{\alpha}}^{|\pi|+\ell(\pi)} \binom{|\lambda| - |\pi| + m_1(\pi)}{m_1(\pi)} \; z_\pi \; \theta^{(\alpha)}_{\pi \cup 1^{|\lambda|-|\pi|}}(\lambda) & \text{if } |\lambda| \geq |\pi|, \\ 0 & \text{if } |\lambda| < |\pi|, \end{cases}$$

where z_π is the standard numerical factor, and $\cup$ denotes concatenation of two partitions, see Section 2.1.1. At the first glance this choice of the normalization factors may be confusing, however it is the appropriate one if we look for the asymptotic behaviour of Jack characters or for a convenient alternative description of them. It is worth mentioning that the character Ch_π is a function on the set of *all* Young diagrams, which turns out to be a powerful tool of this approach.

We would like to notice that there are conjectures similar to the *b*-Conjecture and the Matchings-Jack Conjecture which involves objects *dual* to Jack polynomials. Lassalle proved that the characters Ch_π are polynomials in terms of *multirectangular coordinates* and conjectured that the coefficients of this expression are positive integers and posses some interpretation in terms of *free cumulants* [Las08, Las09]. Dołęga, Féray and Śniady conjectured that any given Jack character Ch_π may be expressed as a weighted sum of some simple functions indexed by maps [DFS13].

Structure constants of symmetric groups

Ivanov and Kerov [IK99] established the notion of *partial permutations* in some suitable inverse limit of the symmetric group algebras $\mathbb{C}[\mathfrak{S}(n)]$. For a given partition they were adding an appropriate number of units to obtain the partition of a relevant size. They observed that by a simple normalization of the conjugacy classes $A_{\pi,n}$ of the symmetric groups $\mathfrak{S}(n)$ the following convolution formula

$$A_{\pi,n} \cdot A_{\sigma,n} = \sum_\mu \widehat{g}^\mu_{\pi,\sigma} A_{\mu,n}$$

holds for any sufficiently large n. The integers $\widehat{g}^\mu_{\pi,\sigma}$ appearing in the formula above are independent of the size of the group $\mathfrak{S}(n)$. The group of finite permutations acts naturally on the inverse limit of the semigroups of partial permutations. The numbers $\widehat{g}^\mu_{\pi,\sigma}$ arise as the multiplication structure constants in the algebra of orbits of this action.

Structure constants

Structure constants $g^{\mu}_{\pi,\sigma}$ of Jack characters are defined by the expansion of the pointwise product of two Jack characters in the basis of Jack characters:

$$\mathrm{Ch}_\pi \cdot \mathrm{Ch}_\sigma = \sum_{\mu} g^{\mu}_{\pi,\sigma}(\delta)\,\mathrm{Ch}_\mu$$

with the parameter $\delta := \sqrt{\alpha} - \dfrac{1}{\sqrt{\alpha}}$. Explicit motivation for studying such quantities comes from the special choice of the deformation parameter $\delta = 0$, when Jack polynomials coincide with Schur polynomials. In this case, Frobenius duality ensures that the structure constants $g^{\mu}_{\pi,\sigma}(0)$ coincide with the *structure constants* $\widehat{g}^{\mu}_{\pi,\sigma}$ *for the symmetric groups* introduced by Ivanov and Kerov. It timidly suggests existence of some deformation of the symmetric group algebra $\mathbb{C}[\mathfrak{S}(n)]$ depending on the deformation parameter δ in which $g^{\mu}_{\pi,\sigma}(\delta)$ are the structure constants for the hypothetical conjugacy class indicators.

It has been observed [Śn16] that the quantities $g^{\mu}_{\pi,\sigma}(\delta)$ seem to be polynomials with non-negative integer coefficients. Since the notion of the connection coefficients $c^{\mu}_{\pi,\sigma}$ and the notion of the structure constants $g^{\mu}_{\pi,\sigma}$ are closely related, one can get an impression of triviality of this remark. Indeed, the structure constants $g^{\mu}_{\pi,\sigma}$ are some weighted decompositions of the connection coefficients [DF16] (see (3.3)) thus the polynomiality of both families is equivalent. The conjectures that the coefficients of these two polynomials are non-negative integers seem to be closely related, however they are not equivalent.

The second result

In Theorem 3.3 we give a necessary and sufficient condition for the polynomial $g^{\lambda}_{\pi,\sigma}$ to achieve the maximal degree provided by the bound of Dołęga and Féray [DF16], we also show that the leading coefficient is a positive integer and we present it in terms of oriented maps. This is the key tool in the proof of Theorem 2.5.

We were looking for combinatorial objects which may enumerate the coefficients of the polynomials $g^{\lambda}_{\mu,\nu}$. There was some evidence that there is a combinatorics of maps hidden behind them. Firstly we found a combinatorial formula for the top-degree part of the character Ch_π, see Proposition 3.9. This formula expresses $\left[A^{\mathrm{top}}\right] \mathrm{Ch}_\pi(\lambda)$ in terms of injective embeddings of a graph G_π assigned to the character Ch_π into the Young diagram λ. This formula brought us closer to discovering good candidates for the top-degree part of structure constants, which was one of the difficulties. It turned out that performing the "hands-shaking procedure", see Section 3.3.3, provides such candidates.

In order to prove that those candidates for the top-degree part of the structure constants $g^{\mu}_{\pi,\sigma}$ are truly them we adapted the probabilistic concept of cumulants to the setup of Jack characters. We present this approach in Section 4.3. Later on, we found a more direct proof of Theorem 3.3.

It is worth mentioning that the polynomiality of $c^{\mu}_{\pi,\sigma}$ and the bound on its degree have been proved by Dołęga and Féray [DF16] via the polynomiality and the bound on the degree of $g^{\mu}_{\pi,\sigma}$. As understanding of the combinatorics of Jack characters may lead to a better understanding of Jack polynomials themselves, so the combinatorial formulas for the structure constants may lead to some combinatorial formulas for the connection coefficients. Our result may be seen as an evidence for this general statement.

Cumulants

One of the classical problems in the probability theory is to describe the joint distribution of a family (X_i) of random variables in the most convenient way. The most common solution of this problem is to use the family of moments, *i.e.* the expected values of products of the form $\mathbb{E}\left(X_{i_1} \cdots X_{i_l}\right)$. It has been observed that in many problems it is more convenient to make use of the *cumulants* [Hal81, Fis28], defined as the coefficients of the expansion of the logarithm of the multidimensional Laplace transform around zero. For example, the Gaussian distribution may be characterized by the vanishing of all cumulants except the first two (*i.e.* , other than mean and variance).

Cumulants allow also a combinatorial description. One can show that the definition of cumulants is equivalent to the following system of equations, called *the moment-cumulant formula*:

$$\mathbb{E}\left(X_1 \cdots X_n\right) = \sum_{\nu} \prod_{b \in \nu} \kappa\left(X_i : i \in b\right)$$

which should hold for any choice of the random variables $X_1, \dots, X_n$ whose moments are all finite.

Generalized cumulants

The notion of cumulants was established also in a more general probabilistic setup. One may consider the *conditional expected value* defined as a unital linear map $\mathbb{E} : \mathcal{A} \longrightarrow \mathcal{B}$ between two commutative unital algebras and define *conditional cumulants* similarly to the classical ones.

The notion of cumulants was transfered into the field of *noncommutative probability theory*, where the *noncommutative expectation* is defined as a unital linear map ϕ on a unital algebra $\mathcal{A}$ [Spe94, Spe98]. Usually some other conditions, such as the *bimodule map property*, are required. Roland Speicher introduced the free cumulant functional [Spe94] in the free probability theory. It is related to the lattice of *noncrossing partitions* of the set $[n]$ in the same way as the classic cumulant functional is related to the lattice of all partitions of that set.

Leonov–Shiryaev's formula

In 1959 Leonov and Shiryaev [LS59] presented a formula for a cumulant of products of random variables:

$$\kappa\Big((X_{1,1} \cdots X_{k_1,1}) , \dots , (X_{1,n} \cdots X_{k_n,n}) \Big)$$

in terms of simple cumulants. This formula was first proved by Leonov and Shiryaev [LS59], a more direct proof was given by Speed [Spe83]. The technique of Leonov and Shiryaev was used in many situations [SSR88, Leh04] and was further developed in other papers: Krawczyk and Speicher [KS00, MST07] found the free analogue of the formula; the formula was further generalized to the partial cumulants [NS06].

Nested cumulants

We investigate the following particular case of the conditional cumulants. We assume that $\mathcal{A}$ is a linear space equipped with two commutative multiplication structures, which correspond to two products: $\cdot$ and $*$. Together with each multiplication $\mathcal{A}$ forms a commutative algebra. We call such structure an *algebra with two multiplications*. As a mapping $\mathbb{E}$ we take the identity $\mathrm{id} : (\mathcal{A}, \cdot) \longrightarrow (\mathcal{A}, *)$.

In this case the cumulants measure the discrepancy between these two multiplication structures on $\mathcal{A}$. This situation arises naturally in many branches of algebraic combinatorics, for example in the case of Macdonald cumulants [Doł17a, Doł17b] and cumulants of Jack characters [DF17, Śn16].

Since the mapping $\mathbb{E}$ is the identity, we can define cumulants of cumulants and further compositions of them. The terminology of cumulants of cumulants was introduced in [Spe83] and further developed in [Leh13] (called there *nested cumulants*) in a slightly different situation of an inclusion of algebras $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ and conditional expectations $\mathcal{A} \xrightarrow{\mathbb{E}_1} \mathcal{B} \xrightarrow{\mathbb{E}_2} \mathcal{C}$.

The third result

In Theorem 4.6 we present an algebraic formula for a nested product which involves two multiplications on the linear space $\mathcal{A}$ as a sum of $*$-products of cumulants:

$$\left(a_1^1 * \cdots * a_{k_1}^1\right) \cdots \left(a_1^n * \cdots * a_{k_n}^n\right) = \sum_{F \in \overline{\mathcal{F}}(A)} (-1)^{w_F}\, \kappa_F,$$

where the sum runs over *reduced forests* with leaves labelled by elements of the algebra $\mathcal{A}$ and satisfying some additional properties. Such forests are assigned to the nested cumulants and are markers of the way they are nested.

A simple reformulation of our formula, see Theorem 4.16, may be seen as an analogue of Leonov–Shiryaev's formula. Indeed, Leonov–Shiryaev's formula relates a cumulant of products with some products of cumulants. In the framework of an algebra with two multiplications we can define two types of cumulants according to each multiplication separately. For each of them we have Leonov–Shiryaev's formula. In Theorem 4.16 we present the third formula, which is a mix of those two.

With the *pointwise product* and the, so called, *disjoint product* Jack characters span an algebra with two multiplications. Observe that in this context our formula expresses the pointwise product of two Jack characters as a sum of disjoint products of cumulants. Together with the approximate factorisation property of cumulants [Śn16] our formula turns out to be a tool for capturing the structure constants $g^{\mu}_{\pi,\sigma}$, see Section 4.3.

At the initial stage of the research, this formula was an indispensable element in the proof of Theorem 3.3. Later on, we found a more direct proof of Theorem 3.3 based on the formula on $\left[A^{\mathrm{top}}\right] \mathrm{Ch}_\pi(\lambda)$ for *all* Young diagrams λ, see Proposition 3.16. Originally we used a formula for $\left[A^{\mathrm{top}}\right] \mathrm{Ch}_\pi(\lambda)$ just for *one-row* Young diagrams. This phenomenon suggests that the cumulant formula given in Theorem 4.6 may be useful while looking for combinatorial formulas for subdominant parts of the structure constants $g^{\mu}_{\pi,\sigma}$. We believe that in the future the cumulant approach will bring new results about the structure constants and indirectly about the connection coefficients from the Matchings-Jack Conjecture of Goulden and Jackson.

Structure of dissertation

The three next chapters of the dissertation are basically devoted to proving and discussing the three aforementioned main results.

Chapter 2 introduces the notions of Jack polynomials ($J_\pi^{(\alpha)}$) and their connection coefficients $c_{\pi,\sigma}^\mu$. In Section 2.1.5 we present the Matchings-Jack Conjecture. Theorem 2.5 presents the leading coefficient of the connection coefficients in the context of Matchings-Jack Conjecture of Goulden and Jackson. In Section 2.2 we introduce the terminology of maps and we investigate relations between maps and matchings. In Section 2.3 we present a measure of non-orientability in the context of b-Conjecture. We present the problem of transferring it into the satisfactory statistic which measures non-bipartiteness of a matching. We discuss recent results of Dołęga [Doł17c] about the top-degree part in b-Conjecture. Later, in Appendix A, we show that our result about the top-degree part in the Matchings-Jack Conjecture presented in Theorem 2.5 is equivalent to the result of Dołęga.

Chapter 3 is devoted to the dual approach, *i.e.* Jack characters Ch_π and their structure constants $g_{\pi,\sigma}^\mu$. In Section 3.1 we show the relation between the structure constants $g_{\pi,\sigma}^\mu$ for Jack characters and the connection coefficients $c_{\pi,\sigma}^\mu$ for Jack symmetric functions. We give a formula for the top-degree part of $g_{\pi,\sigma}^\mu$ and translate this result into the field of connection coefficients $c_{\pi,\sigma}^\mu$. We prove this formula in Section 3.3.

Chapter 4 introduces the notion of cumulants in the classical probability theory and adapts this probabilistic concept to the setup of algebras with two multiplications. Theorem 4.6 gives an algebraic formula which involves those two multiplications as a sum of products of cumulants. In Section 4.2 we present this formula as an analogue of Leonov–Shiraev's formula. We finish with some conclusions about the structure constants of Jack characters presented in Section 4.3.

Chapter 2

Connection coefficients of Jack polynomials

2.1. Jack Polynomials and connection coefficients

In this section we introduce Jack polynomials and their connection coefficients. Theorem 2.5 presents the leading coefficient of connection coefficients in the context of Matchings-Jack Conjecture of Goulden and Jackson.

2.1.1. Partitions

A *partition* λ of n (denoted by $\lambda \vdash n$) is a non-increasing list $(\lambda_1, \dots, \lambda_l)$ of positive integers of sum equal to n. Number n is called the *size* of λ and is denoted by $|\lambda|$, the number l is the *length* of the partition, denoted by $\ell(\lambda)$. Finally,

$$m_i(\lambda) := \big|\{k : \lambda_k = i\}\big|,$$

is the *multiplicity* of $i \geq 1$ in the partition λ.

There are many orders on the set of partitions. Beside the one shown in Definition 2.3 we introduce the *dominance order*. We say that $\lambda \leq \mu$ if and only if

$$\sum_{i \leq j} \lambda_i \leq \sum_{i \leq j} \mu_i$$

holds for any positive integer j.

For given two partitions λ and μ we construct their *concatenation* (denoted $\lambda \cup \mu$) by merging all parts from λ and μ and ordering them in a decreasing fashion.

2.1.2. Jack polynomials

Jack polynomials $\left(J_\pi^{(\alpha)}\right)$ are a family of symmetric functions that depend on a parameter $\alpha > 0$ and is indexed by an integer partition π. They were introduced by Henry Jack in his seminal paper [Jac71]. Jack polynomials can be viewed as a natural interpolation between several interesting families of symmetric functions. For instance, up to multiplicative constants, Jack polynomials coincide with Schur polynomials for $\alpha = 1$; with the zonal polynomials, for $\alpha = 2$; with the symplectic zonal polynomials, for $\alpha = 1/2$; with the elementary symmetric functions, for $\alpha = 0$; and in some sense with the monomial symmetric functions, for $\alpha = \infty$. There are many ways to define Jack polynomials, we present one of them [Mac15, Section VI.10].

Let us consider the vector space $\Lambda_{\mathbb{Q}(\alpha)}$ of the *symmetric functions* [DKB66] over the field of rational functions $\mathbb{Q}(\alpha)$ and its basis $(p_\lambda)_\lambda$ of *power-sum symmetric functions*, *i.e.* the symmetric functions given by

$$p_\lambda(\mathbf{x}) = \prod_i p_{\lambda_i}(\mathbf{x}), \qquad p_k(\mathbf{x}) = x_1^k + x_2^k + \cdots .$$

The following scalar product on $\Lambda_{\mathbb{Q}(\alpha)}$ is defined on the power-sum basis by the formula

$$\langle p_\lambda, p_\mu \rangle_\alpha := \alpha^{\ell(\lambda)} z_\lambda \delta_{\lambda,\mu},$$

where

$$z_\lambda = \prod_i i^{m_i(\lambda)}\ m_i(\lambda)!$$

and further extended by bilinearity. This is a classical deformation of the *Hall inner product*, which corresponds to α =1 [Jac71].

Jack polynomials are the only family of symmetric functions $\left(J_\pi^{(\alpha)}\right)$ which satisfies the following three criteria:

1. $J_\lambda^{(\alpha)} = \sum_{\mu \leq \lambda} a_\mu^\lambda m_\mu$, where $a_\mu^\lambda \in \mathbb{Q}[\alpha]$,
2. $\left[m_{1^{|\lambda|}}\right] J_\lambda^{(\alpha)} := a_{1^{|\lambda|}}^\lambda = |\lambda|!$,
3. $\langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)} \rangle_\alpha = 0$ for $\lambda \neq \mu$,

where m_λ denotes the monomial symmetric function associated with λ.

2.1.3. Connection coefficients for Jack symmetric functions

Goulden and Jackson [GJ96] defined two families of coefficients $\left(c^{\lambda}_{\pi,\sigma}\right)$ and $\left(h^{\lambda}_{\pi,\sigma}\right)$ depending implicitly on the deformation parameter α and indexed by triples of integer partitions $\pi, \sigma, \lambda \vdash n$ of the same integer n. These coefficients are given by expansions of the left-hand sides in terms of the power-sum symmetric functions:

$$\sum_{\theta \in \mathcal{P}} \frac{1}{\langle J_\theta, J_\theta \rangle_\alpha} J^{(\alpha)}_\theta(\mathbf{x}) J^{(\alpha)}_\theta(\mathbf{y}) J^{(\alpha)}_\theta(\mathbf{z}) t^{|\theta|} = \sum_{n \geq 1} t^n \sum_{\lambda,\pi,\sigma \vdash n} \frac{c^{\lambda}_{\pi,\sigma}}{\alpha^{\ell(\lambda)}} z_\lambda^{-1} p_\pi(\mathbf{x}) p_\sigma(\mathbf{y}) p_\lambda(\mathbf{z}), \quad (2.1)$$

and

$$\alpha t \frac{\partial}{\partial t} \log \left(\sum_{\theta \in \mathcal{P}} \frac{1}{\langle J_\theta, J_\theta \rangle_\alpha} J^{(\alpha)}_\theta(\mathbf{x}) J^{(\alpha)}_\theta(\mathbf{y}) J^{(\alpha)}_\theta(\mathbf{z}) t^{|\theta|} \right) = \sum_{n \geq 1} t^n \sum_{\lambda,\pi,\sigma \vdash n} h^{\lambda}_{\pi,\sigma} p_\pi(\mathbf{x}) p_\sigma(\mathbf{y}) p_\lambda(\mathbf{z}), \quad (2.2)$$

see [GJ96, Equations (1),(5) and Equations (2),(4)].

Dołęga and Féray showed that the connection coefficients $(c^{\lambda}_{\pi,\sigma})$ are polynomials in the variable $\beta := \alpha - 1$ with rational coefficients and proved the following upper bound on the degrees of these polynomials [DF16, Proposition B.2.]:

$$\deg_\beta c^{\lambda}_{\pi,\sigma} \leq d(\pi, \sigma; \lambda), \quad (2.3)$$

where

$$d(\pi, \sigma; \lambda) := \Big(|\pi| - \ell(\pi)\Big) + \Big(|\sigma| - \ell(\sigma)\Big) - \Big(|\lambda| - \ell(\lambda)\Big).$$

One may wonder of the use of the new variable β, but this shift seems to be the adequate one in order to look at the connection coefficients from the combinatorial point of view.

2.1.4. Matchings

We present the well established terminology of matching given in [GJ96]. For a given integer n we consider the following set

$$\mathcal{N}_n = \left\{1, \hat{1}, \ldots, n, \hat{n}\right\}.$$

We denote by $\mathcal{F}_n$ the set of all matchings (partitions on two-elements sets) on $\mathcal{N}_n$. For matchings $\delta_1, \delta_2, \ldots \in \mathcal{F}_n$ we denote by $G(\delta_1, \delta_2, \ldots)$ the multigraph with the vertex set $\mathcal{N}_n$ whose edges are formed by the pairs in $\delta_1, \delta_2, \ldots$. For given matchings δ_1, δ_2 the corresponding graph $G(\delta_1, \delta_2)$ consists of disjoint even cycles, since each vertex has degree 2 and around each cycle the edges alternate between δ_1 and δ_2. Denote by $\Lambda(\delta_1, \delta_2)$ the partition of n which specifies halves the lengths of the cycles in $G(\delta_1, \delta_2)$. More generally, denote by $\Lambda(\delta_1, \ldots, \delta_s)$ the partition of n which specifies halves of the number of vertices in each connected component of $G(\delta_1, \delta_2, \ldots)$ (it is an easy observation that such numbers form a partition of n).

We call the sets $\{1, \ldots, n\}$ and $\{\hat{1}, \ldots, \hat{n}\}$ *classes* of $\mathcal{N}_n$. A pair in a matching is called a *between-class* pair if it contains elements of different classes. A matching δ in which every pair is a between-class pair is called a *bipartite* matching (in this case $G(\delta)$ is a bipartite graph on the vertex-sets given by the two classes of $\mathcal{N}_n$).

We introduce two specific bipartite matchings in the set $\mathcal{F}_n$. First, let

$$\epsilon := \Big\{ \{1, \hat{1}\}, \ldots, \{n, \hat{n}\} \Big\};$$

second, for a given partition $\mu \vdash n$, let

$$\delta_\lambda := \Big\{ \{1, \hat{2}\}, \{2, \hat{3}\}, \ldots, \{\lambda_1 - 1, \hat{\lambda_1}\}, \{\lambda_1, \hat{1}\}, \\ \{\lambda_1 + 1, \lambda_1 \hat{+} 2\}, \ldots, \{\lambda_1 + \lambda_1 - 1, \lambda_1 \hat{+} \lambda_1\}, \{\lambda_1 + \lambda_1, \lambda_1 \hat{+} 1\}, \ldots \Big\},$$

see Figure 2.1. Observe that both matchings: ϵ and δ_λ are bipartite and $\Lambda(\epsilon, \delta_\lambda) = \lambda$.

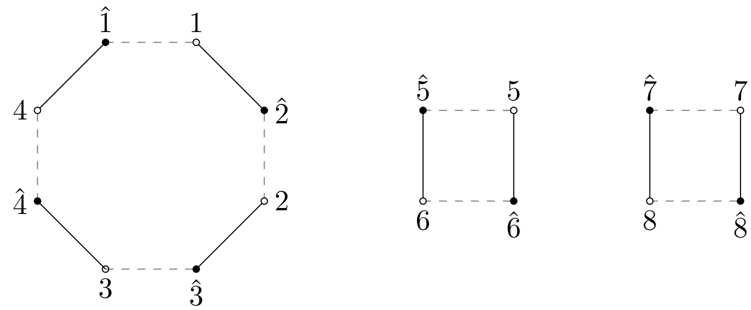
2.1.5. Matchings-Jack Conjecture

Definition 2.1. For given three partitions $\pi, \sigma, \lambda \vdash n$, we denote by $\mathcal{G}^\lambda_{\pi,\sigma}$ the set of all matchings $\delta \in \mathcal{F}_n$, for which $\Lambda(\delta, \epsilon) = \pi$ and $\Lambda(\delta, \delta_\lambda) = \sigma$.

Goulden and Jackson observed that the specializations of $c^\lambda_{\pi,\sigma}(\beta)$ for $\beta \in \{0, 1\}$ may be expressed in terms of matchings, namely

$$c^\lambda_{\pi,\sigma}(0) = \Big| \{\delta \in \mathcal{G}^\lambda_{\pi,\sigma} : \delta \text{ is bipartite}\} \Big|,$$
$$c^\lambda_{\pi,\sigma}(1) = \Big| \{\delta \in \mathcal{G}^\lambda_{\pi,\sigma}\} \Big|.$$

In fact, those specialisations coincide with the connection coefficients of two commutative subalgebras of the group algebra of the symmetric group: the



$$\lambda = (4,2,2)$$

$$\delta_\lambda := \Big\{ \underbrace{\{1,\hat{2}\},\{2,\hat{3}\},\{3,\hat{4}\},\{4,\hat{1}\}}_{\lambda_1=4}, \underbrace{\{5,\hat{6}\},\{6,\hat{5}\}}_{\lambda_2=2}, \underbrace{\{7,\hat{8}\},\{8,\hat{7}\}}_{\lambda_3=2} \Big\}$$

Figure 2.1 – An example of matchings ϵ (dotted line) and δ_λ (continuous line) for $\lambda = (4,2,2)$. Observe that both matchings: ϵ and δ_λ are bipartite and $\Lambda(\epsilon,\delta_\lambda) = \lambda$.

class algebra and the *double coset algebra* ($\beta = 0$ and $\beta = 1$ respectively) [HSS92].

Based on this observation Goulden and Jackson conjectured that the family $\left(c^\lambda_{\pi,\sigma}\right)$ of polynomials may have a combinatorial interpretation. The conjecture is known as the *Matchings-Jack Conjecture*.

Conjecture 2.2 (Matchings-Jack Conjecture)**.** *For any partitions $\pi, \sigma, \lambda \vdash n$ the quantity $c^\lambda_{\pi,\sigma}$ can be expressed as*

$$c^\lambda_{\pi,\sigma}(\beta) = \sum_{\delta \in \mathcal{G}^\lambda_{\pi,\sigma}} \beta^{\mathrm{wt}_\lambda(\delta)},$$

where $\mathrm{wt}_\lambda : \mathcal{G}^\lambda_{\pi,\sigma} \longrightarrow \mathbb{N}_0$ is some hypothetical combinatorial statistic, which vanishes if and only if δ is bipartite.

Clearly, it seems that the statistic wt_λ should be a marker of non-bipartiteness for matchings. Matchings-Jack Conjecture remains still open in the general case, however some special cases have been settled. Goulden and Jackson constructed some statistics wt_λ for $\lambda = [1^n]$ and $\lambda = [2, 1^{n-1}]$ and proved the conjecture in those cases [GJ96]. Later on, the Matchings-Jack Conjecture has been proved by Kanunnikov and Vassiliveva [KV16] in the case $\pi = \sigma = (n)$ of the partitions with exactly one part. Recently, in

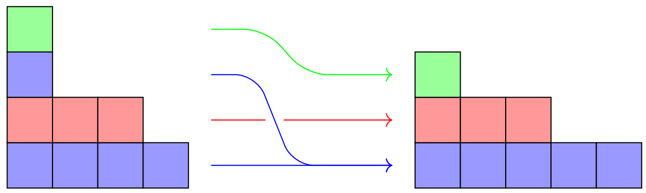


Figure 2.2 – A pair of partitions $\lambda = (4,3,1,1)$ and $\mu = (5,3,1)$ presented as Young diagrams. Partition λ is sub-partition of μ; indeed, each part of μ is given as a sum of different parts of λ.

a joint paper with Promyslov [KVP18], they proved the conjecture in the case when one of the three partitions is equal to (n). They made use of the *measure of non-orientability* θ defined by La Croix in his PhD thesis [La 09]. The measure of non-orientability θ is a statistic defined on a class of *rooted maps*. In some special cases it may be translated into the field of matchings, however generally significant difficulties appear. We also shall use the same statistic.

2.1.6. The first result

In this dissertation we give a necessary and sufficient condition for the polynomial $c^{\lambda}_{\pi,\sigma}$ to achieve the maximal degree given by (2.3). Moreover, we show that the leading coefficient of $c^{\lambda}_{\pi,\sigma}$ of this maximal degree is a non-negative integer and we present it in the context of Matchings-Jack Conjecture.

Definition 2.3. Consider two integer partitions λ and μ of the same integer n, let $k = \ell(\lambda)$ and $m = \ell(\mu)$ be the lengths of the partitions. We say that λ is a *subpartition* of μ (denoted $\lambda \preceq \mu$) if there exists a set-partition ν of $[k]$, such that

$$\mu_i = \sum_{j \in \nu_i} \lambda_j$$

for any $i \in [m]$, see Figure 2.2. We denote $\lambda \prec \mu$ if $\lambda \preceq \mu$ and $\lambda \neq \mu$.

Definition 2.4. For given partitions $\pi, \sigma, \lambda, \mu \vdash n$, we denote by $\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}$ the set of all matchings $\delta \in \mathcal{G}^{\lambda}_{\pi,\sigma}$ which are μ-*connected*, *i.e.* $\Lambda(\delta, \epsilon, \delta_\lambda) = \mu$.

The class $\mathcal{G}^{\lambda}_{\pi,\sigma}$ splits naturally into the classes $\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}$, namely

$$\mathcal{G}^{\lambda}_{\pi,\sigma} = \bigsqcup_{\mu : \lambda \preceq \mu} \mathcal{G}^{\lambda;\mu}_{\pi,\sigma}.$$

Contrary to previous works on the Matchings-Jack Conjecture we do not attempt to define the statistic wt_λ on $\mathcal{G}^{\lambda}_{\pi,\sigma}$ for a particular class of partitions λ, π or σ. We define the statistic "stat_η" on the class $\mathcal{G}^{\lambda;\lambda}_{\pi,\sigma}$.

Theorem 2.5 (The first result)**.** *For any triple of partitions $\pi, \sigma, \lambda \vdash n$ the corresponding polynomial $c^{\lambda}_{\pi,\sigma}(\beta)$ achieves the upper bound on the degree given in* (2.3) *if and only if π and σ are sub-partitions of μ. For such partitions, the leading coefficient of $c^{\lambda}_{\pi,\sigma}(\beta)$ may be expressed in two different manners:*

$$\left[\beta^{d(\pi,\sigma;\lambda)}\right] c^{\lambda}_{\pi,\sigma} = \left|\delta \in \mathcal{G}^{\lambda;\lambda}_{\pi,\sigma} : \delta \text{ is unhandled } \right| = \sum_{\nu:\nu\preceq\lambda} \frac{z_\lambda}{z_\nu}\left|\delta \in \mathcal{G}^{\nu;\lambda}_{\pi,\sigma} : \delta \text{ is bipartite } \right|,$$

for notion of unhandled matchings see Definition 2.28. Moreover, there exists a statistic $stat_\eta : \mathcal{G}^{\lambda;\lambda}_{\pi,\sigma} \longrightarrow \mathbb{N}_0$, which satisfies

$$\left[\beta^{d(\pi,\sigma;\lambda)}\right] c^{\lambda}_{\pi,\sigma} = \left[\beta^{d(\pi,\sigma;\lambda)}\right] \sum_{\delta\in\mathcal{G}^{\lambda;\lambda}_{\pi,\sigma}} \beta^{stat_\eta(\delta)}$$

and for $\delta \in \mathcal{G}^{\lambda;\lambda}_{\pi,\sigma}$ the statistic $stat_\eta(\delta)$ vanishes if and only if δ is bipartite.

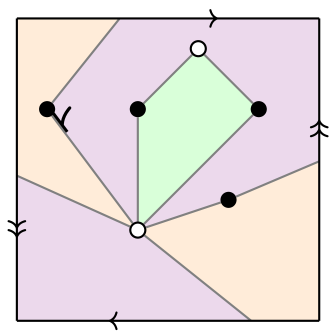
2.2. Matchings and maps

In this section we shall present the notion of maps. We also show relations between them and matchings.

2.2.1. Maps

In the literature a *map* [LZ04] is classically defined as a connected graph G (possibly, with multiple edges) drawn on a surface Σ, *i.e.* a compact connected 2-dimensional manifold without boundary. We assume that a collection of *faces* (*i.e.* $\Sigma \setminus \mathcal{E}$) is homeomorphic to a collection of open discs. A choice of an edge-side and one of its endpoints is called a *root* of the map, see Figure 2.3. A map together with a choice of a root is called a *rooted map*.

A vertex two-coloured map is called *bipartite* if each edge connects vertices of different colors; for simplicity we set that there are *white* and *black* vertices, we denote by $\mathcal{W}$ ($\mathcal{B}$) the set of white (black) vertices. By convention, from a *rooted bipartite* map we require that the rooted vertex is black.



$$\Lambda_{\mathcal{F}}(M) = (4,2,2)$$
$$\Lambda_{\mathcal{W}}(M) = (6,2)$$
$$\Lambda_{\mathcal{B}}(M) = (2,2,2,2)$$

Figure 2.3 – Example of a *rooted bipartite* map M on a projective plane. The left side of the square should be glued to the right side, as well as bottom to top, as indicated by the arrows. We present also the face, white and black vertex distributions.

Figure 2.3 presents an example of a rooted bipartite map M. For a given bipartite map M with n edges we establish two integer partitions of n:

$$\Lambda_{\mathcal{W}}(M) \quad \text{and} \quad \Lambda_{\mathcal{B}}(M),$$

given by the degrees of white/black vertices, For such a map we assign also the third partition

$$\Lambda_{\mathcal{F}}(M)$$

of n, which describes the face structure of M; it is specified by reading halves of the numbers of edges fencing each face (since the map M is bipartite, for each face there is an even number of edges adjacent to the face). The partition $\Lambda_{\mathcal{F}}(M)$ is called the *face-type* of the map M.

Definition 2.6. For three given partitions $\pi, \sigma, \lambda \vdash n$ we denote by $M^{\bullet}_{\pi,\sigma}$ the set of all bipartite, rooted maps M with n edges, for which $\Lambda_{\mathcal{W}}(M) = \pi$ and $\Lambda_{\mathcal{B}}(M) = \sigma$. Moreover we denote by $M^{\lambda}_{\pi,\sigma}$ the set of all such a maps M which additionally have the face-type λ, *i.e.* $\Lambda_{\mathcal{F}}(M) = \lambda$, see Figure 2.3.

Due to the nature of our result we extend this definition slightly, namely we waive the assumption of connectedness in the definition of a map. There are two natural ways to generalize the notion of connected maps to non-connected ones: either we consider *lists* of connected maps or we consider *collections* of them.

2.2.2. Lists and collections of maps

Definition 2.7. Let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition of an integer n. A list of maps $(M_1, \ldots, M_k)$ is called a *μ-list of maps* if the map M_i has μ_i edges for

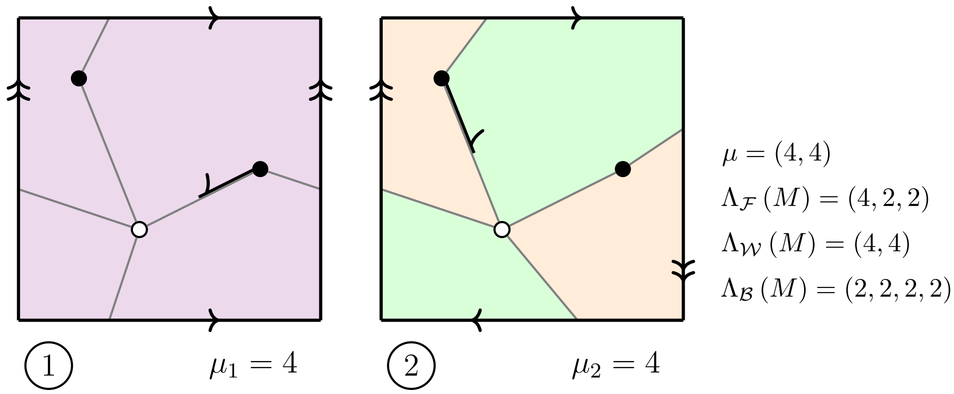


Figure 2.4 – Example of a *rooted, bipartite* μ*-list* of maps for $\mu = (4, 4)$. The first map is drawn on a torus, the second one on a projective plane. We present also the face, white vertices and black vertices distributions. By erasing the roots and the numbering of the connected components we obtain a *bipartite* μ*-collection* of maps.

each $i \in [k]$. We say that such a list is *rooted*, respectively *bipartite* if each map M_i is so. For a bipartite μ-list of maps we associate three partitions describing the black vertex, the white vertex and the face structures

$$\Lambda_{\mathcal{W}}(M) := \bigcup_{i=1}^{k} \Lambda_{\mathcal{W}}(M_i), \quad \Lambda_{\mathcal{B}}(M) := \bigcup_{i=1}^{k} \Lambda_{\mathcal{B}}(M_i), \quad \Lambda_{\mathcal{F}}(M) := \bigcup_{i=1}^{k} \Lambda_{\mathcal{F}}(M_i),$$

where $\bigcup$ denotes the concatenation of partitions.

Definition 2.8 (Extension of Definition 2.6). For given partitions $\pi, \sigma, \mu \vdash n$, we denote by $M_{\pi,\sigma}^{\bullet;\mu}$ the set of all bipartite rooted μ-lists of maps M which satisfy

$$\Lambda_{\mathcal{W}}(M) = \pi \qquad \text{and} \qquad \Lambda_{\mathcal{B}}(M) = \sigma,$$

see Figure 2.4. Moreover, for a given partition $\lambda \vdash n$ we denote by $M_{\pi,\sigma}^{\lambda;\mu}$ the set of all μ-lists of maps $M \in M_{\pi,\sigma}^{\bullet;\mu}$ which have face-type λ, *i.e.* $\Lambda_{\mathcal{F}}(M) = \lambda$.

Definition 2.9. Let $\mu = (\mu_1, \dots, \mu_k)$ be a partition of an integer n. A set of maps $\{M_1, \dots, M_k\}$ is called a μ*-collection of maps* if the map M_i has μ_i edges for each $i \in [k]$. We say that such a collection is *rooted* or *bipartite* if each map M_i is so. For such a collection of maps we associate three partitions describing black, white and face structures as in Definition 2.7.

Roughly speaking, a μ-collection of maps could be created from a μ-list of maps by erasing the numbering of the connected components, *i.e.* the order on the connected components (see Figure 2.4).

2.2.3. Matchings and maps

Matching and maps are closely related notions. Roughly speaking, a bipartite matching can be treated as a (possibly non-connected) bipartite map with rooted and numbered faces. We shall discuss relations between matchings and rooted list of maps with the same face, black vertices and white vertices distribution.

Definition 2.10. Consider two partitions $\lambda, \mu \vdash n$. We say that a bipartite μ-collection M of maps with the face distribution given by λ has *rooted and numbered faces* if all faces of M are rooted (*i.e.* on each face there is one marked edge-side) and the face labelled by the number i is surrounded by $2\lambda_i$ edges, for each i, see Figure 2.5. The set of such collections of maps with the face, black vertices and white vertices distributions given by the partitions $\lambda, \pi, \sigma \vdash n$ is denoted by $M\left(\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right)$.

Remark 2.11. Observe that rooting a face is nothing else but choosing one of the face corners adjacent to some black vertex and *orienting* the face. Through a map (or a list/a collection of maps) with rooted faces we can understand a map with oriented faces and chosen black corners for each of the faces, see Figure 2.5. Similarly, rooting a map is choosing one corner of a black vertex and orienting the face adjacent to this corner.

We consider four partitions: $\pi, \sigma, \lambda, \mu \vdash n$. To a given matching $\delta \in \mathcal{G}^{\lambda;\mu}_{\pi,\sigma}$ we associate a bipartite μ-collection $M_\delta \in M\left(\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right)$ given by the following procedure.

1. The matchings ϵ and δ_λ determine the polygons with the *vertices* labelled by $\mathcal{N}_n$, see Figure 2.1. We take theirs *duals*, *i.e.* the polygons with the *edges* labelled by $\mathcal{N}_n$, see Figure 2.6. The consecutive polygons have $2\lambda_1, 2\lambda_2, \dots$ edges. Observe that the parts of ϵ (respectively δ_λ) can be identified with the black (respectively white) vertices as it is shown on Figure 2.6;

2. The matching δ determines the unique way of gluing the edges of the polygons in such a way that black (white) vertices are glued with black (white) ones. Figure 2.7 presents such a gluing for the matching

$$\delta = \{\{\hat{1}, \hat{6}\}, \{1, 5\}, \{\hat{2}, \hat{8}\}, \{2, 7\}, \{\hat{3}, \hat{7}\}, \{3, 8\}, \{\hat{4}, 6\}, \{4, \hat{5}\}\}.$$

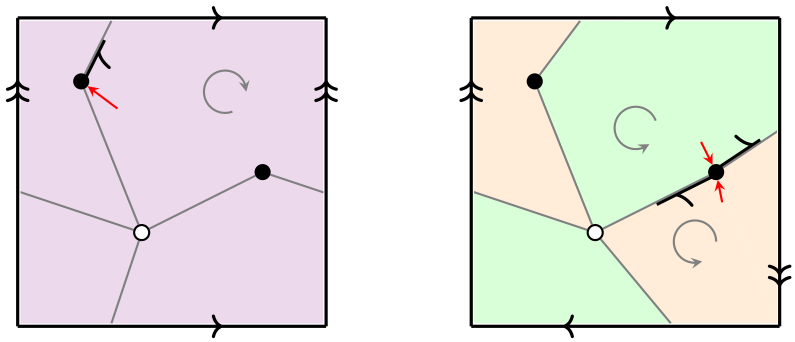


Figure 2.5 – Example of a bipartite $(4,4)$-collection of maps with rooted faces. By *rooting faces* we understand choosing one edge-side of each face (drawn as a black half-arrow going from a black vertex) or, equivalently, orienting each face (the rounded arrows) and choosing one black vertex for each face (the red arrows).

Observe that the distribution of black (respectively white) vertices is given by $\Lambda(\delta,\epsilon)$ (respectively by $\Lambda(\delta,\delta_\lambda)$). Moreover, $\mu = \Lambda(\delta,\epsilon,\delta_\lambda)$.

3. Each face is canonically numbered by an integer s related to the polygon λ_s, *i.e.* the edge-sides of this face are labelled by the elements

$$\sum_{i=1}^{s-1}\lambda_i + 1, \dots, \sum_{i=1}^{s-1}\lambda_i + \lambda_s, \widehat{\sum_{i=1}^{s-1}\lambda_i + 1}, \dots \widehat{\sum_{i=1}^{s-1}\lambda_i + \lambda_s}.$$

Such a face is canonically rooted by selecting the edge-side labelled by the number $\sum_{i=1}^{s-1}\lambda_i + 1$, see Figure 2.7.

4. We remove the labelling by the elements from $\mathcal{N}_n$.

Corollary 2.12. *The procedure described above gives a bijection $\delta \mapsto M_\delta$ between the set of matchings $\mathcal{G}_{\pi,\sigma}^{\lambda;\mu}$ and the set of collections of maps $M\left(\mathcal{G}_{\pi,\sigma}^{\lambda;\mu}\right)$.*

We compare the terminologies of matchings and maps in the table below.

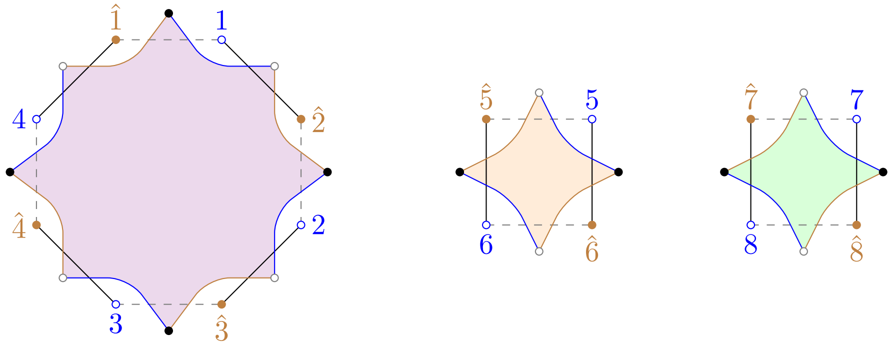
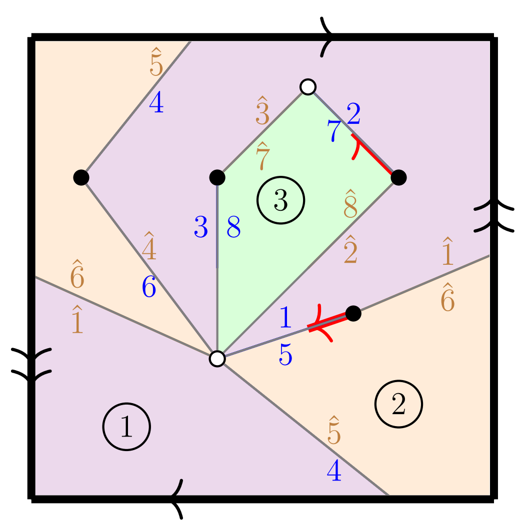


Figure 2.6 – Duals of the polygons created by the matchings ϵ and δ_λ presented on Figure 2.1. Black (respectively white) vertices of such polygons are labelled by the elements of ϵ (respectively δ_λ), the edges by the elements from $\mathcal{N}_8$.



$$\delta = \Big\{\{\hat{1}, \hat{6}\}, \{1, 5\}, \{\hat{2}, \hat{8}\}, \{2, 7\}, \{\hat{3}, \hat{7}\}, \{3, 8\}, \{\hat{4}, 6\}, \{4, \hat{5}\}\Big\}$$

Figure 2.7 – Matching δ on the set $\mathcal{N}_8$ describes the way of gluing the sides of the polygons from Figure 2.6. Labels from $\mathcal{N}_8$ determine the way of numbering and rooting faces of such a map (in general it could be a collection of maps), the roots (presented as half-arrows) correspond to the labels $1, 5, 7$. Numbers of faces are presented in black circles.

	Matching δ	μ-list of maps M
face-type	λ	$\Lambda_{\mathcal{F}}(M)$
distribution of black vertices	$\Lambda(\delta, \epsilon)$	$\Lambda_{\mathcal{B}}(M)$
distribution of white vertices	$\Lambda(\delta, \delta_\lambda)$	$\Lambda_{\mathcal{W}}(M)$
connected components	$\Lambda(\delta_\lambda, \epsilon, \delta)$	μ
μ-collections of maps with given faces, black and white vertices distribution and with rooted and numbered faces	$\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}$	$M\left(\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right)$

2.2.4. Matchings and lists of rooted maps

We showed that matchings are equivalent to collections of maps with rooted and numbered *faces*. However, collections of maps with rooted and numbered *connected components* (*i.e.* lists of rooted maps) are much more natural objects. We give a relation between those two ways of numbering and rooting collections of maps. More precisely, we present a relation between the set $M\left(\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right)$ and the set $M^{\lambda;\mu}_{\pi,\sigma}$.

What is common for those two classes is the fact that by rooting and numbering faces or connected components, the group of automorphisms becomes trivial.

Definition 2.13. For a given μ-collection of maps with rooted and numbered faces $M \in M\left(\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right)$ we define the set $\mathcal{R}(M)$ of all numberings of the connected components and rooting each of them in such a way that with respect to them M becomes a *μ-list* of maps from $M^{\lambda;\mu}_{\pi,\sigma}$. We call $\mathcal{R}(M)$ the *set of components-labellings* of M. For a given $r \in \mathcal{R}(M)$ we denote $(M, r) \in M^{\lambda;\mu}_{\pi,\sigma}$.

Similarly, for a given μ-list of maps $M \in M^{\lambda;\mu}_{\pi,\sigma}$ we define the set $\mathcal{L}(M)$ of all numberings of the faces and rooting each of them in such a way that M becomes an element from $M\left(\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right)$. We call $\mathcal{L}(M)$ the *set of faces-labellings* of M. For a given $l \in \mathcal{L}(M)$ we denote $(M, l) \in M\left(\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right)$.

Observation 2.14. *Let us fix partitions $\pi, \sigma, \mu, \lambda \vdash n$. For each $M_1 \in M\left(\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right)$ and $M_2 \in M^{\lambda;\mu}_{\pi,\sigma}$ we have*

$$\left|\mathcal{R}(M_1)\right| = 2^{\ell(\mu)} z_\mu \quad \textit{and} \quad \left|\mathcal{L}(M_1)\right| = 2^{\ell(\lambda)} z_\lambda.$$

Proof. Let us take $M \in M\left(\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right)$. There is $\prod_i m_i(\mu)!$ ways of numbering the connected components and $\prod_i (2i)^{m_i(\mu)}$ ways of rooting each of them. We may carry out a similar deduction for $M \in M^{\lambda;\mu}_{\pi,\sigma}$. $\square$

Observation 2.15. *For given partitions $\pi, \sigma, \mu, \lambda \vdash n$ we have*

$$\left|\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right| = \left|M\left(\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right)\right| = \frac{z_\lambda 2^{\ell(\lambda)}}{z_\mu 2^{\ell(\mu)}}\left|M^{\lambda;\mu}_{\pi,\sigma}\right|.$$

Proof. The first equation follows from Corollary 2.12. We investigate the second one. Each collection of maps from $M\left(\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right)$ has rooted and numbered faces, each collection of maps from $M^{\lambda;\mu}_{\pi,\sigma}$ has rooted and numbered components. From each of them we can get a collection of maps which have rooted and numbered both: faces and components. The number of ways of doing it is given in Observation 2.14. We use the double counting method and conclude the second equation. $\square$

2.2.5. Orientable maps and bipartite matchings

By an *orientable map* we understand a map which is drawn on an orientable surface. An *orientation* of a map is given by orienting each face in such a way, that the two edge-sides forming the same edge are oriented in the opposite way. We say that such an orientation of faces is *coherent*. Orienting any face is equivalent to orienting a map. Observe that a rooted map possesses the canonical orientation given by the root, see Remark 2.11. By a *rooted orientable* map we understand an orientable map together with the *orientation* given by the root, see Figure 2.8.

Definition 2.16. We use the following notation:

$$\widetilde{M}^{\lambda;\mu}_{\pi,\sigma} := \left\{M \in M^{\lambda;\mu}_{\pi,\sigma} : M \text{ is orientable}\right\},$$
$$\widetilde{M}^{\bullet;\mu}_{\pi,\sigma} := \left\{M \in M^{\bullet;\mu}_{\pi,\sigma} : M \text{ is orientable}\right\},$$
$$\widetilde{\mathcal{G}}^{\lambda;\mu}_{\pi,\sigma} := \left\{\delta \in \mathcal{G}^{\lambda;\mu}_{\pi,\sigma} : \delta \text{ is bipartite}\right\}.$$

The notion of bipartiteness of a matching is closely related to the notion of orientability.

Observation 2.17. *For given partitions $\pi, \sigma, \mu, \lambda \vdash n$, we have*

$$\left|\widetilde{\mathcal{G}}^{\lambda;\mu}_{\pi,\sigma}\right| = \frac{z_\lambda}{z_\mu}\left|\widetilde{M}^{\lambda;\mu}_{\pi,\sigma}\right|.$$

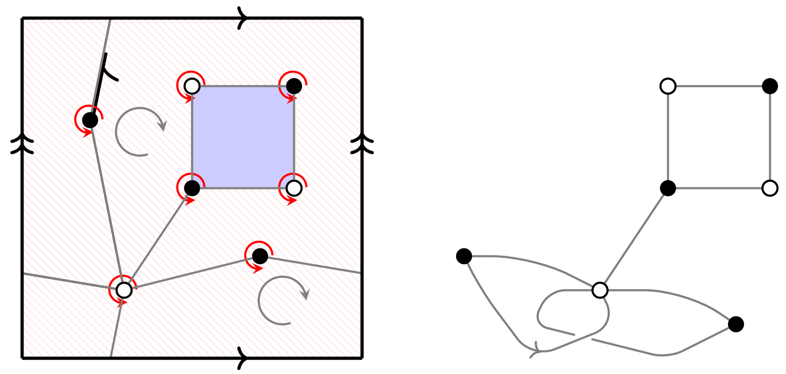


Figure 2.8 – Example of a *rooted oriented* map M drawn as a graph on a torus (on the left). There is the canonical orientation (grey arrows) given by the root. We are going to present oriented maps in such a way that their orientation is consistent with the *clockwise* orientation of the page (grey arrows) or, equivalently, the *counter-clockwise* orientation around each vertex (red arrows). The distinction between chosen orientations of the page and the vertices may seem awkward. However, it is more convenient for the purpose of Section 2.3.2. With this convention we can present the root of a map (similarly roots of lists of maps) by an *arrow* going out from a black vertex. Since M is oriented, it can be recovered from a graphical representation on the plane as a graph with a fixed cyclic order of outgoing edges around each vertex together with a choice of the root (on the right).

Proof. We identify a matching $\delta \in \mathcal{G}^{\lambda;\mu}_{\pi,\sigma}$ with a collection of maps $M_\delta \in M\left(\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right)$ with rooted and numbered faces by the procedure described in Section 2.2.3. Observe that a *bipartite* matching corresponds to a collection of *oriented* maps. Indeed, the orientations of faces given by the edge-sides: $1, \lambda_1+1, \dots$ are coherent. Observation 2.15 gives a relation between collections of maps with rooted and numbered faces and collections of maps with rooted and numbered components (lists of maps). An analysis similar to the one given in Observation 2.15 convinces us that the quantity $2^{\ell(\mu)} \prod_i i^{m_i(\lambda)}$ specifies the number of manners of rooting the faces in a *coherent* way and $\prod_i m_i(\lambda)!$ specifies the number of manners of numbering the faces. On the other hand, the quantity $z_\mu 2^{\ell(\mu)}$ is relevant for numbering and rooting the connected components. We use the double counting method and conclude the statement. □

2.3. Measures of non-orientability and non-bipartiteness

In this section we shall present a measure of non-orientability η in the context of b-Conjecture. We present a problem of transferring η into a satisfactory statistic which measures non-bipartiteness of a matching. We discuss the result of Dołęga [Doł17c] about the top-degree part in b-Conjecture.

2.3.1. The b-Conjecture

Equations (2.1) and (2.2) define two families of coefficients $\left(c^\lambda_{\mu,\nu}\right)$ and $\left(h^\lambda_{\mu,\nu}\right)$. Goulden and Jackson [GJ96] discussed some specialisations of the family $\left(c^\lambda_{\mu,\nu}\right)$ and hypothetical combinatorial interpretations of the polynomials $c^\lambda_{\mu,\nu}$ in terms of matchings known as the *Matchings-Jack Conjecture*, see Section 2.1.5. In the same paper they observed that specializations of $h^\lambda_{\pi,\sigma}(\beta)$ for $\beta = 0, 1$ may be expressed in terms of rooted maps, namely

$$\begin{aligned} h^\lambda_{\pi,\sigma}(0) &= \left|\{M \in M^\lambda_{\pi,\sigma} : M \text{ is orientable}\}\right|, \\ h^\lambda_{\pi,\sigma}(1) &= \left|\{M \in M^\lambda_{\pi,\sigma}\}\right|. \end{aligned}$$

Based on this observation Goulden and Jackson conjectured that the family $\left(h^\lambda_{\pi,\sigma}\right)$ of polynomials may have a combinatorial interpretation. The conjecture is known as the *b-Conjecture*.

Conjecture 2.18 (b-Conjecture)**.** *For any partitions $\pi, \sigma, \lambda \vdash n$ the quantity $h^{\lambda}_{\pi,\sigma}$ can be expressed as*

$$h^{\lambda}_{\pi,\sigma}(\beta) = \sum_{M \in M^{\lambda}_{\pi,\sigma}} \beta^{\eta(M)},$$

where $\eta : M^{\lambda}_{\pi,\sigma} \longrightarrow \mathbb{N}_0$ is some hypothetical combinatorial statistic such that $\eta(M) = 0$ if and only if M is orientable.

2.3.2. Root-deletion procedure and a measure of non-orientability

The statistic η from b-Conjecture should be a marker of non-orientability of maps. We shall present the definition of the measure of non-orientability introduced by La Croix [La 09, Definition 4.1], which seems to be a good candidate for the hypothetical statistic conjectured by Goulden and Jackson. We adapt the statistic given by La Croix to the case of lists of maps.

Definition 2.19 (Root-deletion procedure)*.* Denote by e the root edge of the map M. By deleting e from M we create either a new map, or two new maps. We give the canonical procedure of rooting it or them. By rooting a map we will understand choosing an oriented corner, see Figure 2.9. Denote by c the root corner of M.

Suppose that $M \setminus e$ is connected. Observe that c is contained in the unique oriented corner of $M \setminus e$, we define such an oriented corner as the root of $M \setminus e$.

Suppose that $M \setminus e$ has two connected components. One of them can be rooted as above. Observe that the first corner in the root face of M following c is contained in the unique oriented corner of the second component of $M \setminus e$, see Figure 2.9. We define such an oriented corner as the root of this component.

Remark 2.20*.* The Root-deletion procedure is defined for all maps, not necessary bipartite. In particular, we do not require that the rooted vertex is black.

We classify the root edges of maps. Let f be the number of faces of a map M with the root vertex e;

1. e is called a *bridge* if $M \setminus e$ is not connected,
2. otherwise $M \setminus e$ is connected and e is called
    - a *border* if the number of faces in $M \setminus e$ is equal to $f - 1$,

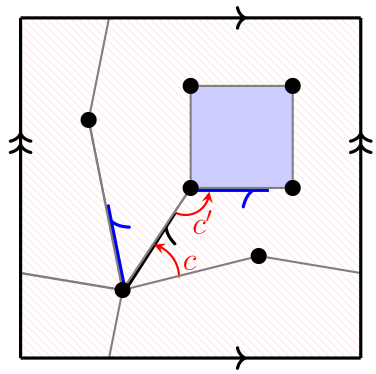


Figure 2.9 – The oriented corner c (red arrow) equivalent to the root (the black arrow) of a map. The first corner in the root face of the map following c is labelled by c' (red arrow). By deleting the root edge the map splits into two new maps. The oriented corners c and c' are contained in two oriented corners of the new maps. They give the roots of those maps (the blue arrows).

- a *twisted edge* if the number of faces in $M \setminus e$ is equal to f,
- a *handle* if the number of faces in $M \setminus e$ is equal to $f + 1$.

Remark 2.21. A leaf (*i.e.* an edge connecting a vertex of degree 1) is considered as a bridge.

Definition 2.22. [La 09, Definition 4.1] For a rooted map M, an invariant $\eta(M)$ is defined inductively as follows.

1. If M has no edges then $\eta(M) = 0$.

2. Otherwise M has the root edge e,

   - $\eta(M) = \eta(M_1) + \eta(M_2)$ if e is a bridge, while M_1 and M_2 are the connected components of $M \setminus e$,
   - $\eta(M) = \eta(M \setminus e)$ if e is a border,
   - $\eta(M) = \eta(M \setminus e) + 1$ if e is a twisted edge,
   - if e is a handle, there exists a unique map M' with the root edge e' constructed by *twisting* the edge e in M, in such a way that e' is a handle and the maps $M \setminus e$, $M' \setminus e$ are equal. In this case we require that

$$\{\eta(M), \eta(M')\} = \{\eta(M \setminus e), \eta(M \setminus e) + 1\}\,.$$

At most one of the maps M, M' is orientable. For such a map M we require $\eta(M) = \eta(M \setminus e)$.

We call such an invariant a *measure of non-orientability*.

Observe that the above definition introduces the whole family of measures of non-orientability η and among of them there is no *canonical measure of non-orientability*.

Remark 2.23. For a given rooted map M

$$\eta\,(M) = 0 \text{ if and only if } M \text{ is orientable.}$$

Indeed, removing twisted edges or handles during the root-deletion procedure are the only possibilities of increasing the recursively-defined statistic η. An orientable map does not have any twisted edges (a map with a twisted edge is embedded in a surface which contains the Möbius strip, hence is nonorientable). The recursive definition of η guarantees that removing handles from an orientable map does not increase the statistic η. Hence for an orientable map M, we have $\eta(M) = 0$. A reverse analysis or a simple induction on the number of edges provides the reverse implication.

Definition 2.24. For a rooted μ-list of maps $M = M_1, \dots M_k$ we define a *measure of non-orientability* η of M by

$$\eta\,(M) := \eta_1\,(M_1) + \cdots + \eta_k\,(M_k)$$

for any measures of non-orientability η_i from Definition 2.22.

2.3.3. Unhandled and unicellular maps

Definition 2.25. The rooted map M is called *unhandled* if by iteratively performing the root-deletion process (see Definition 2.19) it does not have any handles. The map M is called *unicellular* if it has only one face.

From now on we fix one of measures of non-orientability η of the class of maps. Dołęga [Doł17c, Section 4] showed that for such a measure η the polynomial H_η given by the sum

$$(H_\eta)^{\lambda}_{\pi,\sigma} := \sum_{M \in M^{\lambda}_{\pi,\sigma}} \beta^{\eta(M)}$$

has degree at most equal to $n + 1 - \ell(\pi) - \ell(\sigma)$ and the leading coefficient is enumerated by *unhandled unicellular* maps. In particular, $(H_\eta)^{\lambda}_{\pi,\sigma}$ may achieve this bound of the degree only if $\lambda = (n)$. He also showed that the aforementioned leading coefficient is also enumerated by *oriented* maps with arbitrary face-type, namely

$$\left| M \in M^{\bullet}_{\pi,\sigma} : M \text{ is orientable } \right| = \left| M \in M^{(n)}_{\pi,\sigma} : M \text{ is unhandled } \right|.$$

In fact, there is an explicit bijection between those two families of maps. Dołęga proved [Doł17c, Theorem 1.4] that for the statistic η

$$h^{(n)}_{\pi,\sigma}(\beta) = \sum_{M \in M^{(n)}_{\pi,\sigma}} \beta^{\eta(M)}$$

holds true for $\beta \in \{-1, 0, 1\}$, moreover for $M \in M^{(n)}_{\pi,\sigma}$ the statistic $\eta(M) = 0$ vanishes if and only if M is *orientable*; furthermore $\eta(M) = n+1-\ell(\pi)-\ell(\sigma)$ if and only if M is *unhandled* and *unicellular*.

The result of Dołęga is easily transferable to the context of μ-lists of maps. Let us choose the measures of non-orientability η_i for $i \in [k], k = \ell(\mu)$, which form the measure η as it is described in Definition 2.24.

Lemma 2.26. *For the statistic η, the polynomial $(H_\eta)^{\lambda;\bullet}_{\pi,\sigma}$ given by the sum*

$$(H_\eta)^{\lambda;\bullet}_{\pi,\sigma} := \sum_{\mu:\lambda \preceq \mu} (H_\eta)^{\lambda;\mu}_{\pi,\sigma} \tag{2.4}$$

where

$$(H_\eta)^{\lambda;\mu}_{\pi,\sigma}(\beta) = \sum_{M \in M^{\lambda;\mu}_{\pi,\sigma}} \beta^{\eta(M)} \tag{2.5}$$

is of degree at most $d(\pi, \sigma; \lambda)$. Moreover, a μ-lists of maps M contributes to the ground term if and only if M is a list of orientable maps. The μ-lists of maps M contributes to the leading coefficient if and only if M is a list of unicellular and unhandled maps, in particular $\mu = \lambda$.

Proof. Each $M = (M_1, \dots, M_k) \in M^{\lambda;\mu}_{\pi,\sigma}$ decompose into a list of maps $M_i \in M^{\lambda_{|\mu_i}}_{\pi_{|\mu_i},\sigma_{|\mu_i}}$ for some partitions $\pi_{|\mu_i}, \sigma_{|\mu_i}, \lambda_{|\mu_i} \vdash \mu_i$ satisfying

$$\bigcup_{i=1}^{k} \pi_{|\mu_i} = \pi, \quad \bigcup_{i=1}^{k} \sigma_{|\mu_i} = \sigma, \quad \bigcup_{i=1}^{k} \lambda_{|\mu_i} = \lambda.$$

We denote by $\mathcal{P}^{\mu}_{\pi}$ the set of lists of partitions $\left(\pi_{|\mu_1}, \dots, \pi_{|\mu_k}\right)$, where $\pi_{|\mu_i} \vdash \mu_i$ and

$$\bigcup_{i=1}^{k} \pi_{|\mu_i} = \pi.$$

Observe, that (2.4) can be rewritten in such a way:

$$\sum_{M \in M^{\lambda;\mu}_{\pi,\sigma}} \beta^{\eta(M)} = \sum_{\substack{(\pi^1,\dots,\pi^k)\in\mathcal{P}^\mu_\pi \\ (\sigma^1,\dots,\sigma^k)\in\mathcal{P}^\mu_\sigma \\ (\lambda^1,\dots,\lambda^k)\in\mathcal{P}^\mu_\lambda}} \prod_{i=1}^{k} \sum_{M \in M^{\lambda^i}_{\pi^i,\sigma^i}} \beta^{\eta_i(M)}.$$

We use the result of Dołęga for each most right side sum separately. Each such a sum has degree at most equal to $\mu_i + 1 - \ell(\pi^i) - \ell(\sigma^i)$ and the top-degree coefficient is enumerated by unhandled unicellular maps. Since

$$n + \ell(\mu) - \ell(\pi) - \ell(\sigma) = \sum_{i=1}^{k} \Big(\mu_i + 1 - \ell(\pi^i) - \ell(\sigma^i)\Big),$$

we conclude that (2.4) has degree at most equal to $d(\pi, \sigma; \lambda)$ and the top-degree coefficient is enumerated by μ-lists of unhandled unicellular maps. □

Corollary 2.27. *For three given partitions $\pi, \sigma, \lambda \vdash n$ we have*

$$\Big| M \in M^{\bullet;\mu}_{\pi,\sigma} : M \text{ is orientable } \Big| = \Big| M \in M^{\mu;\mu}_{\pi,\sigma} : M \text{ is unhandled } \Big|.$$

Proof. Fix a list $M \in M^{\mu;\mu}_{\pi,\sigma}$ of unhandled and unicellular maps. For each connected component of M we use the aforementioned bijection between such maps and oriented maps with arbitrary face-type given by Dołęga [Doł17c, Corollary 3.10]. We get a μ-list of orientable maps with arbitrary face type. □

2.3.4. Measure of non-bipartiteness for matchings

The hypothetical statistic wt_λ from the Matchings-Jack Conjecture should be a marker of non-bipartiteness for matchings. Naturally, matchings correspond to lists of maps, in particular bipartite matching to lists of oriented maps.

The naive thought how the statistic wt_λ should be defined is to adapt the measure of non-orientability introduced by La Croix by the correspondence between matchings and collections of maps given by Corollary 2.12. Regretfully, the measure introduced by La Croix is defined for lists of rooted maps, however there is no canonical way to create such a list from an element of $M\left(\mathcal{G}^{\lambda;\mu}_{\pi,\sigma}\right)$.

However, there is one special class of matchings, which may be identified with lists of rooted maps, namely $\mathcal{G}^{\lambda;\lambda}_{\pi,\sigma}$. When the number of faces is equal to the number of connected components, numbering and rooting *faces* overlap with numbering and rooting *components*. For a fixed measure of non-orientability η we define

$$\begin{array}{rccl} \mathrm{stat}_\eta : & \mathcal{G}^{\lambda;\lambda}_{\pi,\sigma} & \longrightarrow & [d(\pi,\sigma;\lambda)] \\ & \delta & \longmapsto & \mathrm{stat}_\eta(\delta) := \eta(M_\delta) \end{array}$$

For given partitions $\lambda, \pi, \sigma \vdash n$ we define the following polynomial

$$(G_\eta)^{\lambda;\lambda}_{\pi,\sigma} := \sum_{\delta \in \mathcal{G}^{\lambda;\lambda}_{\pi,\sigma}} \beta^{\mathrm{stat}_\eta(\delta)}. \tag{2.6}$$

Definition 2.28. We say that a matching $\delta \in \mathcal{G}^{\lambda;\lambda}_{\pi,\sigma}$ is *unhandled* if the corresponding map $M_\delta \in M^{\lambda;\lambda}_{\pi,\sigma}$ is so.

Lemma 2.29. *For any triple of partitions $\pi, \sigma, \lambda \vdash n$ the corresponding polynomial $(G_\eta)^{\lambda;\lambda}_{\pi,\sigma}$ is of degree at most $d(\pi,\sigma;\lambda)$. Moreover, the matching δ contributes to the ground term if and only if δ is bipartite. The matching δ contributes to the leading coefficient if and only if δ is an unhandled matching.*

Moreover, the top-degree coefficient may be enumerated in two different manners:

$$\left|\delta \in \mathcal{G}^{\lambda;\lambda}_{\pi,\sigma} : \delta \text{ is unhandled } \right| = \sum_{\nu : \nu \preceq \lambda} \frac{z_\lambda}{z_\nu} \left|\delta \in \mathcal{G}^{\nu;\lambda}_{\pi,\sigma} : \delta \text{ is bipartite } \right|.$$

Proof. Observe that for fixed measure of non-orientability η polynomials $(G_\eta)^{\lambda;\lambda}_{\pi,\sigma}$ and $(H_\eta)^{\lambda;\lambda}_{\pi,\sigma}$ are equal. The first statement follows immediately from Lemma 2.26. The second statement is an easy conclusion of Corollary 2.27 and relation given in Observation 2.17. □

Chapter 3

Structure constants of Jack characters

3.1. Jack characters and structure constants

In this section we present the notion of Jack characters and their structure constants.

3.1.1. Jack characters

We expand Jack polynomial in the basis of power-sum symmetric functions:

$$J_\lambda^{(\alpha)} = \sum_\mu \theta_\mu^{(\alpha)}(\lambda)\ p_\mu. \tag{3.1}$$

The above sum runs over partitions μ such that $|\mu| = |\lambda|$. The coefficient $\theta_\mu^{(\alpha)}(\lambda)$ is called *unnormalized Jack character*.

Jack characters $\theta_\mu^{(\alpha)}$ provide a kind of *dual* information about the Jack polynomials. Better understanding of the combinatorics of Jack characters may lead to a better understanding of Jack polynomials themselves. This kind of approach may be traced back to the work of Kerov and Olshanski [KO94]. For a fixed conjugacy class μ they considered characters of the symmetric group evaluated on μ. This is opposite to the usual way of viewing the characters of the symmetric groups, namely to fix the representation λ and to consider the character as a function of the conjugacy class μ. Lassalle [Las08, Las09] adapted idea of Kerov and Olshanski to the framework of Jack characters.

As Jack symmetric functions $\left(J_\lambda^{(\alpha)}\right)_\lambda$ form a basis of the symmetric functions, the functions $\left(\theta_\mu^{(\alpha)}\right)_{\mu\vdash n}$ form a basis of the algebra of functions on Young diagrams with n boxes [Fér12, Proposition 4.1]. Dołęga and Féray [DF16, Appendix B.2] showed that the coefficients appearing in the expansion of a pointwise product of two unnormalized Jack characters in the unnormalized Jack character basis coincide with the connection coefficients from (2.1), namely

$$\theta_\pi^{(\alpha)} \cdot \theta_\sigma^{(\alpha)} = \sum_{\mu\vdash n} c_{\pi,\sigma}^{\mu} \theta_\mu^{(\alpha)}.$$

for all triples of partitions $\pi, \sigma, \mu \vdash n$. This observation encourages us to look more closely into the field of connection coefficients via the context of Jack characters.

3.1.2. Normalized Jack characters

We define Jack characters Ch_π by a choice of the normalization of $\theta_\pi^{(\alpha)}$. We will use the normalization introduced by Dołęga and Féray [DF16] which offers some advantages over the original normalization of Lassalle. Therefore, with the right choice of the multiplicative constant, the unnormalized Jack character $\theta_\lambda^{(\alpha)}(\pi)$ from (3.1) becomes the *normalized Jack character* $\mathrm{Ch}_\pi^{(\alpha)}(\lambda)$, defined as follows.

Definition 3.1. For a given number $\alpha > 0$ and a partition π, the *normalized Jack character* $\mathrm{Ch}_\pi^{(\alpha)}(\lambda)$ is defined by:

$$\mathrm{Ch}_\pi^{(\alpha)}(\lambda) := \begin{cases} \dfrac{1}{\sqrt{\alpha}}^{|\pi|+\ell(\pi)} \dbinom{|\lambda| - |\pi| + m_1(\pi)}{m_1(\pi)} \, z_\pi \, \theta_{\pi\cup 1^{|\lambda|-|\pi|}}^{(\alpha)}(\lambda) & \text{if } |\lambda| \geq |\pi|, \\ 0 & \text{if } |\lambda| < |\pi|, \end{cases}$$

where z_π is the standard numerical factor, and $\cup$ denotes concatenation of two partitions, see Section 2.1.1. The choice of an empty partition $\pi = \emptyset$ is acceptable; in this case $\mathrm{Ch}_\emptyset^{(\alpha)}(\lambda) = 1$.

3.1.3. The deformation parameters

In order to avoid dealing with the square root of the variable α, we introduce an indeterminate A such that $A^2 := \alpha$. Jack characters are usually defined in terms of the deformation parameter α. After the substitution $\alpha := A^2$, each Jack character becomes a function of A. In order to keep the notation light, we will make this dependence implicit and we will simply write $\mathrm{Ch}_\pi(\lambda)$.

The algebra of Laurent polynomials in the indeterminate A will be denoted by $\mathbb{Q}\left[A, A^{-1}\right]$. For an integer d we will say that a Laurent polynomial

$$f = \sum_{k \in \mathbb{Z}} f_k A^k \in \mathbb{Q}\left[A, A^{-1}\right]$$

is of *degree at most* d if $f_k = 0$ holds for each integer $k > d$.

The quantity

$$\gamma := -A + \frac{1}{A} \in \mathbb{Q}\left[A, A^{-1}\right]$$

and its opposite

$$\delta := A - \frac{1}{A} \in \mathbb{Q}\left[A, A^{-1}\right].$$

will play a special role in our setting.

3.1.4. Structure constants

Structure constants $g^{\mu}_{\pi,\sigma}$ of Jack characters are defined by expansion of the *pointwise product* of two Jack characters in the basis of Jack characters:

$$\mathrm{Ch}_\pi \cdot \mathrm{Ch}_\sigma = \sum_{\mu} g^{\mu}_{\pi,\sigma}(\delta)\, \mathrm{Ch}_\mu .$$

Explicit motivation for studying such quantities comes from a special choice of the deformation parameter $\alpha = 1$, when Jack polynomials coincide with Schur polynomials. In this case, Frobenius duality ensures that the structure constants coincide with the *connection coefficients for the symmetric groups* [IK99].

Dołęga and Féray proved [DF16, Theorem 1.4] that each structure constant $g^{\mu}_{\pi,\sigma}$ is a polynomial in the variable $\delta := \sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}$ of degree bounded as follows:

$$\deg_\delta g^{\mu}_{\pi,\sigma} \leq \min_{i=1,2,3} \Big(n_i(\pi) + n_i(\sigma) - n_i(\mu) \Big), \tag{3.2}$$

where

$$\begin{aligned} n_1(\pi) &= |\pi| + \ell(\pi), \\ n_2(\pi) &= |\pi| - \ell(\pi), \\ n_3(\pi) &= |\pi| - \ell(\pi) + m_1(\pi). \end{aligned}$$

For example, we have

$$\begin{aligned}\mathrm{Ch}_3\,\mathrm{Ch}_2 =& 6\delta\,\mathrm{Ch}_3 + \mathrm{Ch}_{3,2} + 6\,\mathrm{Ch}_{2,1} + 6\,\mathrm{Ch}_4,\\ \mathrm{Ch}_3\,\mathrm{Ch}_3 =& (6\delta^2+3)\,\mathrm{Ch}_3 + 9\delta\,\mathrm{Ch}_{2,1} + 18\delta\,\mathrm{Ch}_4 + 3\,\mathrm{Ch}_{1,1,1} + \\ & + 9\,\mathrm{Ch}_{3,1} + 9\,\mathrm{Ch}_{2,2} + 9\,\mathrm{Ch}_5 + \mathrm{Ch}_{3,3}\,.\end{aligned}$$

The numerical computations, such as the ones above, suggest that the structure constants of Jack characters might have some algebraic and combinatorial structure, which was proposed in the following conjecture [Śn16, Conjecture 0.1].

Conjecture 3.2 (Structure constants of Jack characters)**.** *For any partitions π, σ, μ, the corresponding structure constant*

$$g^{\mu}_{\pi,\sigma}(\delta) \in \mathbb{Q}[\delta]$$

is a polynomial with non-negative integer coefficients.

3.2. The top-degree part of structure constants and connection coefficients

In this section we present Theorem 3.3 which gives us a formula for the top-degree part of structure constants for Jack characters. Furthermore, in Section 3.2.2 we show the relation between the structure constants for Jack characters and the connection coefficients for Jack symmetric functions. We translate Theorem 3.3 into the field of connection coefficients, more precisely, we give a proof of Theorem 2.5.

3.2.1. The second result

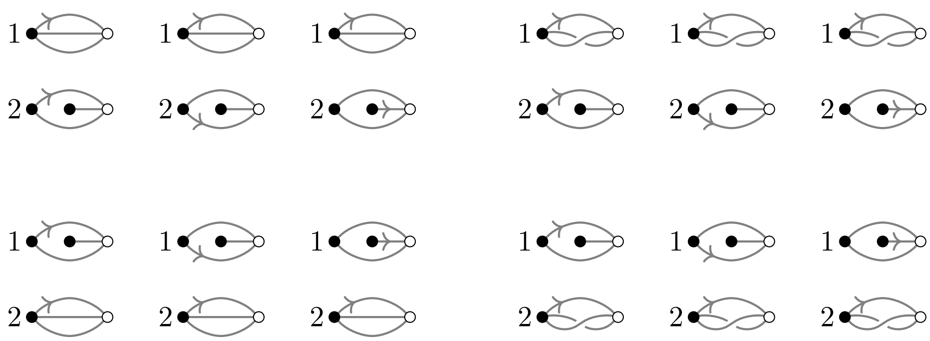
We present an explicit formula for the top-degree part of structure constants of Jack characters.

Let us recall that we present an oriented map as a graph on the plane with a fixed cyclic order of outgoing edges together with a choice of the root, see Figure 2.8. By convention we fixed the counter-clockwise orientation around vertices or, equivalently, the clockwise orientation of the page, see Figure 2.8. Similarly, we will present a μ-collections of maps.

Let us recall that $\widetilde{M}^{\bullet;\mu}_{\pi,\sigma}$ denotes the set of all μ-lists of bipartite rooted and oriented maps which satisfy

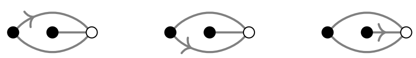
$$\Lambda_{\mathcal{W}}(M) = \pi \qquad \text{and} \qquad \Lambda_{\mathcal{B}}(M) = \sigma,$$

see Figure 3.1.



(a) All lists of maps in the set $\widetilde{M}^{\bullet;\mu}_{\pi,\sigma\cup 1}$ for partitions $\pi = (3,3), \sigma = (3,2)$, and $\mu = (3,3)$. Those lists of maps consist of two connected components which are numbered by 1 and 2, each has 3 edges. The vertex structure is given by π and σ.

(b) Maps from the set $\widetilde{M}^{\bullet;(3)}_{(3),(3)}$. Each of them can be rooted in the unique way.



(c) The only map from the set $\widetilde{M}^{\bullet;(3)}_{(3),(2,1)}$. It could be rooted in three different ways.

Figure 3.1 – There are twelve lists of maps in a set $\widetilde{M}^{\bullet;\mu}_{\pi,\sigma\cup 1}$ for partitions $\pi = (3,3), \sigma = (3,2)$, and $\mu = (3,3)$, see Figure 3.1a. Each of them consists of a map from $\widetilde{M}^{\bullet;(3)}_{(3),(3)}$ and $\widetilde{M}^{\bullet;(3)}_{(3),(2,1)}$ presented on Figure 3.1b and Figure 3.1c respectively.

Theorem 3.3 (The second result)**.** *For any triple of partitions π, σ, μ, the corresponding polynomial $g^{\mu}_{\pi,\sigma}(\delta)$ achieves one of the upper bounds on the degree given in* (2.3)*, namely*

$$d(\pi,\sigma;\lambda) := \Big(|\pi| - \ell(\pi)\Big) + \Big(|\sigma| - \ell(\sigma)\Big) - \Big(|\mu| - \ell(\mu)\Big)$$

if and only if $|\mu| \geq |\pi|, |\sigma|$, and both partitions $\pi \cup 1^{|\mu|-|\pi|}$ and $\sigma \cup 1^{|\mu|-|\sigma|}$ are sub-partitions of μ, see Definition 2.3. For such partitions, the leading coefficient of $g^{\mu}_{\pi,\sigma}(\delta)$ is a positive integer expressed in the following way:

$$\left[\delta^{d(\pi,\sigma;\mu)}\right] g^{\mu}_{\pi,\sigma} = C(\pi,\sigma;\mu) \cdot \frac{z_\pi z_\sigma}{z_\mu} \left| \widetilde{M}^{\bullet;\mu}_{\pi\cup 1^{|\mu|-|\pi|},\sigma\cup 1^{|\mu|-|\sigma|}} \right|,$$

where

$$C(\pi,\sigma;\mu) = \sum_{k=0}^{m_1(\mu)} \binom{m_1(\mu)}{k} \binom{m_1(\pi) + |\mu| - |\pi| - m_1(\mu)}{m_1(\pi) - k} \cdot \binom{m_1(\sigma) + |\mu| - |\sigma| - m_1(\mu) + k}{m_1(\sigma) - m_1(\pi) + k},$$

which is equal to

$$\binom{m_1(\pi) + |\mu| - |\pi|}{m_1(\pi)} \binom{m_1(\sigma) + |\mu| - |\sigma|}{m_1(\sigma)}$$

if $m_1(\mu) = 0$ and is equal to 1 if π, σ, μ are partitions of the same integer.

Section 3.3 is devoted to the proof of above theorem.

Example 3.4. Let us consider three partitions $\pi = (3,2), \sigma = (3,3)$, and $\mu = (3,3)$. In Figure 3.1 we have shown that $\widetilde{M}^{\bullet;\mu}_{\pi,\sigma\cup 1} = 12$. Using the theorem above, the $d(\pi,\sigma;\mu)$-coefficient is equal to

$$\left[\delta^{d(\pi,\sigma;\mu)}\right] g^{\mu}_{\pi,\sigma} = \frac{6 \cdot 18}{18} \binom{1}{0} \binom{0}{0} 12 = 72.$$

3.2.2. Relations between the structure constants $g^{\mu}_{\pi,\sigma}$ and the connection coefficients $c^{\mu}_{\pi,\sigma}$

It is worth mentioning that the coefficients $c^{\mu}_{\pi,\sigma}$ are indexed by three partitions of the same size, while the quantities $g^{\mu}_{\pi,\sigma}$ are indexed by triples of *arbitrary* partitions. Dołęga and Féray investigated the relationship between these two families of coefficients and showed [DF16, Equation (19)] that for $\mu, \pi, \sigma \vdash n$,

$$c^{\mu}_{\pi,\sigma} = \sqrt{\alpha}^{d(\pi,\sigma;\mu)} \frac{z_{\tilde{\mu}}}{z_{\tilde{\pi}} z_{\tilde{\sigma}}} \sum_{i=0}^{m_1(\pi)} g^{\tilde{\mu}\cup 1^i}_{\tilde{\pi},\tilde{\sigma}} \cdot i! \binom{n - |\tilde{\mu}|}{i}, \tag{3.3}$$

where $\widetilde{\pi}$ is constructed from the partition π by deleting all units.

Dołęga and Féray [DF16] proven the polynomiality and the bound on the degree of $g^{\mu}_{\pi,\sigma}$. Using (3.3) they deduced the polynomiality and the bound of the degree of connection coefficients $c^{\mu}_{\pi,\sigma}$. We establish other relations between those two families of coefficients.

Corollary 3.5. *For three given partitions $\mu, \nu, \lambda \vdash n$, each of the polynomials $c^{\lambda}_{\mu,\nu}(\beta)$ and $g^{\tilde{\mu}}_{\tilde{\pi},\tilde{\sigma}}(\delta)$ is of degree at most $d(\pi, \sigma; \mu)$, and their leading coefficients coincide up to a normalizing constant, namely*

$$\left[\beta^{d(\pi,\sigma;\mu)}\right] c^{\lambda}_{\mu,\nu} = \frac{z_{\tilde{\mu}}}{z_{\tilde{\pi}} z_{\tilde{\sigma}}} \cdot \left[\delta^{d(\pi,\sigma;\mu)}\right] g^{\tilde{\mu}}_{\tilde{\pi},\tilde{\sigma}}.$$

Proof. Fix three partitions $\mu, \nu, \lambda \vdash n$. Observe that for each $i \geq 0$, the third estimation shown in (3.2) gives us

$$\deg_{\delta} g^{\tilde{\mu}\cup 1^i}_{\tilde{\pi},\tilde{\sigma}} \leq d(\pi, \sigma; \mu) - i.$$

Let us recall that $\delta = \sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}$, hence the right-hand side of (3.3) is of α-degree at most equal to $2d(\pi, \sigma; \mu)$, and in the sum over i, the only contribution to the $2d(\pi, \sigma; \mu)$-degree coefficient comes from $g^{\tilde{\mu}}_{\tilde{\pi},\tilde{\sigma}}$. We have

$$\left[\sqrt{\alpha}^{2d(\pi,\sigma;\mu)}\right] \left(\sqrt{\alpha}^{d(\pi,\sigma;\mu)} \frac{z_{\tilde{\mu}}}{z_{\tilde{\pi}} z_{\tilde{\sigma}}} \sum_{i=0}^{m_1(\pi)} g^{\tilde{\mu}\cup 1^i}_{\tilde{\pi},\tilde{\sigma}} \cdot i! \binom{n - |\tilde{\mu}|}{i}\right) = \frac{z_{\tilde{\mu}}}{z_{\tilde{\pi}} z_{\tilde{\sigma}}} \left[\delta^{d(\pi,\sigma;\mu)}\right] g^{\tilde{\mu}}_{\tilde{\pi},\tilde{\sigma}}.$$

Since $\beta = \alpha - 1$, the $2d(\pi, \sigma; \mu)$-degree coefficient of $c^{\lambda}_{\mu,\nu}$ in variable $\sqrt{\alpha}$ coincides with $d(\pi, \sigma; \mu)$-degree coefficient in variable β. Hence (3.3) finishes the proof. □

Assuming Theorem 3.3 we are ready to prove the main result of the previous chapter. The proof may seem intricate, it combines different facts which have been proven so far.

Proof of Theorem 2.5. Fix partitions $\pi, \sigma, \lambda \vdash n$. We investigate the polynomial $c^{\lambda}_{\pi,\sigma}(\beta)$. By Corollary 3.5 we have

$$\left[\beta^{d(\pi,\sigma;\lambda)}\right] c^{\lambda}_{\pi,\sigma} = \frac{z_{\tilde{\lambda}}}{z_{\tilde{\pi}} z_{\tilde{\sigma}}} \cdot \left[\delta^{d(\pi,\sigma;\lambda)}\right] g^{\tilde{\lambda}}_{\tilde{\pi},\tilde{\sigma}},$$

and by Theorem 3.3 we know that the polynomial $g^{\tilde{\lambda}}_{\tilde{\pi},\tilde{\sigma}}$ achieves the $d(\pi, \sigma; \lambda)$-degree part if and only if $|\tilde{\lambda}| \geq |\tilde{\pi}|, |\tilde{\sigma}|$, $\tilde{\pi} \preceq \tilde{\lambda}$, and $\tilde{\sigma} \preceq \tilde{\lambda}$. Observe that this condition is equivalent to $\pi \preceq \lambda$ and $\sigma \preceq \lambda$. Hence the condition on partitions π, σ, λ for achieving by $c^{\lambda}_{\pi,\sigma}$ the $d(\pi, \sigma; \lambda)$-degree.

Thus, we have

$$\left[\beta^{d(\pi,\sigma;\lambda)}\right] c^{\lambda}_{\pi,\sigma} \overset{\text{Corollary 3.5}}{=} \frac{z_{\tilde{\lambda}}}{z_{\tilde{\pi}} z_{\tilde{\sigma}}} \cdot \left[\delta^{d(\pi,\sigma;\lambda)}\right] g^{\tilde{\lambda}}_{\tilde{\pi},\tilde{\sigma}} \overset{\text{Theorem 3.3}}{=} \left|\widetilde{M}^{\bullet;\tilde{\lambda}}_{\tilde{\pi},\tilde{\sigma}}\right|.$$

Since there is only one map $M_1 \in \widetilde{M}^{(1)}_{(1),(1)}$, we have

$$\left|\widetilde{M}^{\bullet;\tilde{\lambda}}_{\tilde{\pi},\tilde{\sigma}}\right| = \left|\widetilde{M}^{\bullet;\lambda}_{\pi,\sigma}\right|.$$

Indeed, from any λ-list of maps $M \in \widetilde{M}^{\bullet;\lambda}_{\pi,\sigma}$ we can canonically create a $\tilde{\lambda}$-list of map $\widetilde{M} \in \widetilde{M}^{\bullet;\tilde{\lambda}}_{\tilde{\pi},\tilde{\sigma}}$ by erasing the last $|\lambda| - |\tilde{\lambda}|$ components. This procedure is reversible, since we can add new M_1 components to $\widetilde{M}$. Then we have

$$\left|\widetilde{M}^{\bullet;\lambda}_{\pi,\sigma}\right| = \sum_{\nu:\nu \preceq \lambda} \left|\widetilde{M}^{\nu;\lambda}_{\pi,\sigma}\right| \overset{\text{Observation 2.17}}{=} \frac{z_\lambda}{z_\nu} \sum_{\nu:\nu \preceq \lambda} \left|\widetilde{\mathcal{G}}^{\nu;\lambda}_{\pi,\sigma}\right|.$$

Hence

$$\left[\beta^{d(\pi,\sigma;\lambda)}\right] c^{\lambda}_{\pi,\sigma} = \frac{z_\lambda}{z_\nu} \sum_{\nu:\nu \preceq \lambda} \left|\widetilde{\mathcal{G}}^{\nu;\lambda}_{\pi,\sigma}\right|.$$

From Lemma 2.29 we conclude that the leading coefficient of $c^{\lambda}_{\pi,\sigma}$ overlaps with the leading coefficient of the polynomial

$$(G_\eta)^{\lambda;\lambda}_{\pi,\sigma} := \sum_{\delta \in \mathcal{G}^{\lambda;\lambda}_{\pi,\sigma}} \beta^{\text{stat}_\eta(\delta)},$$

see (2.6), and that both are of the same degree. From Lemma 2.29 we also get the second expression for the leading coefficient of the polynomial $c^{\lambda}_{\pi,\sigma}$. □

3.3. Proof of Theorem 3.3

This section is devoted to the proof of Theorem 3.3. Firstly, we present some basic computations leading to the exact formulas for the top-degree part of Jack characters. We present those formulas in terms of injective embeddings into Young diagrams. Secondly, we consider a particular class of collections of bipartite maps $P^{\mu}_{\pi,\sigma}$ which constitute a good candidate for the top-degree parts of the structure constants $g^{\mu}_{\pi,\sigma}$. Finally, we prove that those candidates for the top-degree part of structure constants $g^{\mu}_{\pi,\sigma}$ (see Proposition 3.16) are indeed them.

3.3.1. Embeddings of bicolored graphs

A bicolored graph G is a bipartite graph together with a choice of the colouring of its vertex set $\mathcal{V}$; we denote by $\mathcal{V}_{\bullet}$ and $\mathcal{V}_{\circ}$ respectively the sets of black and white vertices of G.

Definition 3.6. An *injective embedding* F of a bicolored graph G to a Young diagram λ is a function which maps $\mathcal{V}_{\circ}$ to the set of columns of λ, maps $\mathcal{V}_{\bullet}$ to the set of rows of λ, and maps injectively the set of edges $\mathcal{E}$ to the set of boxes of λ, see Figure 3.2. We also require that F preserves the relation of incidence, *i.e.* each vertex $v \in \mathcal{V}$ should be mapped to a row or a column $F(v)$ which contains the box $F(e)$, for every edge $e \in \mathcal{E}$ incident to v. We denote by $N_G(\lambda)$ the number of such embeddings of G into λ.

It is also useful to consider injective embeddings of a graph G into a Young diagram λ, with the roles of black and white vertices reversed (*i.e.* black vertices are mapped into *columns*, white vertices into the *rows*). We refer to such embeddings as *negative injective embeddings* and denote the number of such embeddings as $\overline{N_G(\lambda)}$.

Definition 3.7. For any partition $\pi = (\pi_1, \dots, \pi_r)$ we define the graph G_π as the unique bicoloured graph consisting of r black vertices of degrees $\pi_1, \dots, \pi_r$ respectively and $|\pi|$ white vertices, each of degree one (see Figure 3.2). Similarly, we define $\overline{G_\pi}$ as the unique bicoloured graph consisting of r white vertices of degrees $\pi_1, \dots, \pi_r$ respectively and $|\pi|$ black vertices, each of degree one.

Remark 3.8. The number $N_{G_\pi}(\lambda)$ of injective embeddings of the graph G_π into the Young diagram λ is equal to the number $\overline{N_{\overline{G_\pi}}(\lambda)}$ of negative injective embeddings of the graph $\overline{G_\pi}$ into the Young diagram λ.

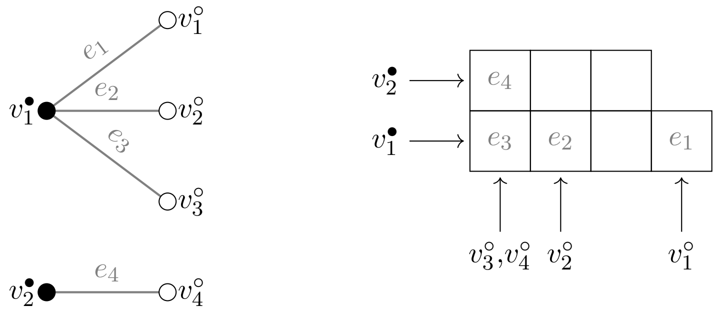


Figure 3.2 – The graph G_π associated with the partition $\pi = (3, 1)$. On the right, an example of its injective embedding into the Young diagram $\lambda = (4, 3)$.

3.3.2. Exact formulas for top-degree part of Jack characters

Śniady proved [Śn15, Proposition 3.5] that each Jack character is a function on the set $\mathbb{Y}$ of Young diagrams

$$\mathbb{Y} \ni \lambda \quad \longmapsto \quad \mathrm{Ch}_\pi(\lambda) \in \mathbb{Q}\left[A, A^{-1}\right]_{|\pi|-\ell(\pi)}$$

with values in the set $\mathbb{Q}\left[A, A^{-1}\right]_{|\pi|-\ell(\pi)}$ of Laurent polynomials in the variable A of degree at most $|\pi| - \ell(\pi)$. We denote by

$$\left[A^{\mathrm{top}}\right] \mathrm{Ch}_\pi(\lambda) := \left[A^{|\pi|-\ell(\pi)}\right] \mathrm{Ch}_\pi(\lambda)$$

the leading part of this Laurent polynomial. We shall express this quantity in terms of injective embeddings of G_π into λ.

Proposition 3.9. *For any Young diagram $\lambda \in \mathbb{Y}$ and partition π, we have that*

$$\left[A^{\mathrm{top}}\right] \mathrm{Ch}_\pi(\lambda) = N_{G_\pi}(\lambda).$$

That is, the leading part of $\mathrm{Ch}_\pi(\lambda)$ is equal to the number of injective embeddings of the graph G_π into the Young diagram λ.

Example 3.10. Let us consider the partition $\pi = (3, 1)$ and the Young diagram $\lambda = (\lambda_1, \lambda_2)$. We have

$$\left[A^{\mathrm{top}}\right] \mathrm{Ch}_{(3,1)}(\lambda_1, \lambda_2) = N_{G_\pi}(\lambda) = \lambda_1^{\underline{4}} + \lambda_1^{\underline{3}} \cdot \lambda_2^{\underline{1}} + \lambda_1^{\underline{1}} \cdot \lambda_2^{\underline{3}} + \lambda_2^{\underline{4}}.$$

One of embeddings which contributes to $N_{G_\pi}(\lambda)$ is presented on Figure 3.2.

Before proving Proposition 3.9 we introduce the notion of α-shifted symmetric functions (see more in [Las08, Section 2.2] or [AF17, Definition 2.2]) and present Jack characters in this context.

Definition 3.11. An α-shifted symmetric function $F = (F_N)_{N\geq 1}$ is a sequence of polynomials F_N such that

- for each $N \geq 1$, F_N is a polynomial in N variables $x_1, \dots, x_N$ with coefficients in the field of rational functions $\mathbb{Q}(\alpha)$ in some indeterminate α that is symmetric in the variables

$$\xi_1 := x_1 - \frac{1}{\alpha}, \quad \xi_2 := x_2 - \frac{2}{\alpha}, \dots, \quad \xi_N := x_N - \frac{N}{\alpha},$$

- for each $N \geq 1$, $F_{N+1}(x_1, \dots, x_N, 0) = F_N(x_1, \dots, x_N)$ (the stability property),
- $\sup_{N\geq 1} \deg(F_N) < \infty$.

The degree of a shifted-symmetric function F is defined as maximum of the degrees of the corresponding polynomials $F_N(x_1, \dots, x_N)$.

Śniady and Féray gave some abstract characterizations of Jack characters [Śn15, Theorem 1.7, Theorem A.2]. We present the one given by Féray, which can be traced back to the earlier work of Knop and Sahi [KS96].

Theorem 3.12. *[Śn15, Theorem A.2] Let π be a partition and A be a complex number such that $-\frac{1}{\alpha} = -\frac{1}{A^2}$ is not a positive integer. There exists a unique shifted-symmetric function F such that:*

(J1) F is a shifted-symmetric function of degree $|\pi|$, and its top-degree homogeneous part is equal to

$$A^{|\pi|-\ell(\pi)}\ p_\pi(\lambda_1, \dots, \lambda_m),$$

where p_π is the power-sum symmetric polynomial *given by the formula*

$$p_\pi(\lambda) = \prod_r \sum_i \lambda_i^{\pi_r}.$$

(J2) $F(\lambda) = 0$ holds for each Young diagram λ such that $|\lambda| < |\pi|$ (the vanishing property).

Moreover, if α is a positive real number, the function $F = (F_N)_{N\geq 1}$ satisfies $\mathrm{Ch}_\pi(\lambda) = F_r(\lambda_1, \dots, \lambda_r)$ *for each Young diagram $\lambda = (\lambda_1, \dots, \lambda_r)$.*

To keep notation short, we introduce the following symmetric function

$$\widehat{p_\pi}(\lambda) := \mathop{\ast}_{r} \sum_i \lambda_i^{\underline{\pi_r}},$$

where

$$\lambda_i^{\underline{l_i}} * \lambda_j^{\underline{l_j}} = \begin{cases} \lambda_i^{\underline{l_i+l_j}} & \text{if } i = j, \\ \lambda_i^{\underline{l_i}} \cdot \lambda_j^{\underline{l_j}} & \text{otherwise,} \end{cases}$$

and

$$\lambda^{\underline{l}} = \underbrace{\lambda \cdot (\lambda - 1) \cdots (\lambda - l + 1)}_{l \text{ factors}}.$$

Proof of Proposition 3.9. Observe that

$$\widehat{p_\pi}(\lambda) = N_{G_\pi}(\lambda).$$

We will show that

$$\left[A^{\mathrm{top}}\right] \mathrm{Ch}_\pi(\lambda) = \widehat{p_\pi}(\lambda).$$

Let F be an α-shifted symmetric function associated to π by Definition 3.11. Let us choose a sufficiently large integer N, *e.g.* $N > |\pi|$. Let us treat the coefficients of the polynomial F_N as variables. The equality

$$F_N(\lambda) = \mathrm{Ch}_\pi(\lambda) \in \mathbb{Q}\left[A, A^{-1}\right]_{|\pi|-\ell(\pi)},$$

which holds for each $\lambda \in \mathbb{Y}$, becomes a system of equations with coefficients in $\mathbb{N}_{\geq 0}$. This system is large enough to conclude that each coefficient of a polynomial F_N is a linear combination of the quantities $\mathrm{Ch}_\pi(\lambda)$ over $\mathbb{Q}$, hence F_N is a polynomial in N variables with coefficients in $\mathbb{Q}\left[A, A^{-1}\right]_{|\pi|-\ell(\pi)}$.

Notice that formally we have equality for all $\alpha > 0$. However, the rational function from $\mathbb{Q}(\alpha)$ is uniquely determined by its *values* for $\alpha \geq 0$.

Since F_N is a shifted-symmetric function with coefficients in the set $\mathbb{Q}\left[A, A^{-1}\right]_{|\pi|-\ell(\pi)}$, its A-top degree

$$\left[A^{\mathrm{top}}\right] F_N(\lambda_1, \dots, \lambda_N) := \left[A^{|\pi|-\ell(\pi)}\right] F_N(\lambda_1, \dots, \lambda_N)$$

is a *symmetric* function in the variables $\lambda_1, \dots, \lambda_N$. Indeed, for each permutation σ of $[N]$ we have

$$\left[A^{\mathrm{top}}\right] F_N(x_1, \dots, x_N) = \left[A^{\mathrm{top}}\right] F_N\left(x_1 - \frac{1}{A^2}, \dots, x_N - \frac{N}{A^2}\right) =$$
$$\left[A^{\mathrm{top}}\right] F_N\left(x_{\sigma(1)} - \frac{\sigma(1)}{A^2}, \dots, x_{\sigma(N)} - \frac{\sigma(N)}{A^2}\right) =$$
$$\left[A^{\mathrm{top}}\right] F_N\left(x_{\sigma(1)}, \dots, x_{\sigma(N)}\right).$$

Since F_N is of a degree $|\pi|$, the polynomial $\left[A^{\text{top}}\right] F_N$ has the same bound of the degree. Observe that the homogeneous top-degree part of $\widehat{p_\pi}(\lambda)$ is equal to $p_\pi(\lambda)$ and so does the homogeneous top-degree part of $\left[A^{\text{top}}\right] F_N$. Polynomials $\widehat{p_\pi}(\lambda)$ and $\left[A^{\text{top}}\right] F_N$ are both symmetric, hence

$$\left[A^{\text{top}}\right] F_N - \widehat{p_\pi}$$

is a symmetric polynomial in variables $(\lambda_1, \dots, \lambda_N)$ of a degree at most $|\pi| - 1$.

We use the following notation:

$$\mathcal{Y}_0 := \left\{(\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N : \lambda_1 \geq \dots \geq \lambda_N \geq 0 \text{ and } \lambda_1 + \dots + \lambda_N < |\pi|\right\}.$$

By the vanishing property we have

$$\left[A^{\text{top}}\right] F_N(\lambda) = \left[A^{\text{top}}\right] \mathrm{Ch}_\pi(\lambda) = 0$$

for all elements $\lambda \in \mathcal{Y}_0$. Since there are no injective embeddings of G_π into a Young diagram with the number of boxes smaller then the number of edges in G_π, we have

$$\widehat{p_\pi}(\lambda) = N_{G_\pi}(\lambda) = 0$$

for all elements $\lambda \in \mathcal{Y}_0$. From that we deduce that $\mathcal{Y}_0$ is a set of zeros of the polynomial $\left[A^{\text{top}}\right] F_N - \widehat{p_\pi}$. The appropriate set of zeros of a polynomial of sufficiently small degree determines the vanishing of the polynomial. In fact, we can use the characterisation given by Śniady [Śn15, Lemma 7.1] to conclude that the symmetric polynomial

$$W(x_1, \dots, x_N) := \left(\left[A^{\text{top}}\right] F_N - \widehat{p_\pi}\right)(x_1 - 1, \dots, x_N - 1),$$

which is of a degree at most $|\pi| - 1$, vanishes. Hence we conclude that

$$\left[A^{\text{top}}\right] F_N = \widehat{p_\pi}$$

which finishes the proof. □

3.3.3. Hands-shaking procedure

Let

$$\pi = (\pi_1, \dots, \pi_n), \quad \sigma = (\sigma_1, \dots, \sigma_l)$$

be two partitions. We define a class of collections of maps by the following procedure:

1. For each $i \in [n]$ we assign a white vertex with π_i outgoing half-edges. We label this vertex by the number i and we root it, *i.e.* we choose one of the outgoing half-edges and decorate it. Similarly, for each $j \in [l]$ we assign a black vertex with σ_j outgoing half-edges and we root it.

2. We match some of the half-edges going out from the white vertices with some of those going out from black vertices.

3. We close each of the non-closed half-edges by a white or a black vertex so that the graph remains bipartite.

We call the procedure described above the "hands-shaking procedure". The name provenance could be explained as follows: there are white and black vertices with hands; the number of hands is given by the partitions π and σ. They shake theirs hands in any way they like, but only black-white connections are allowed. On Figure 3.3 we present an example of applying this procedure.

Definition 3.13. For a given triple of partitions π, σ, μ we denote by $P^{\mu}_{\pi,\sigma}$ the set of all μ-collections of maps which may be obtain as an outcome of performing the above presented "hands-shaking procedure".

Each μ-collection of maps $M \in P^{\mu}_{\pi,\sigma}$ can be obtained in the *unique* way as an outcome of the presented procedure. The uniqueness follows from the fact that the position of each edge from M is uniquely determined by the labellings on the rooted vertices and the order of the outgoing half-edges.

Observation 3.14. *For given partitions π, σ, μ the set $P^{\mu}_{\pi,\sigma}$ is non-empty if and only if the following conditions holds*

1. $|\pi|, |\sigma| \le |\mu|$,

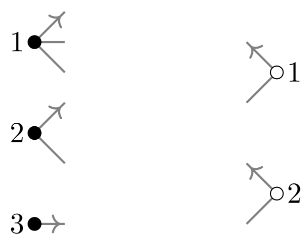
2. *both partitions $\pi \cup 1^{|\mu|-|\pi|}$ and $\sigma \cup 1^{|\mu|-|\sigma|}$ are sub-partitions of μ.*

Proof. Firstly, we will show the necessity of conditions. Observe that by performing "hands-shaking procedure", in which we obtain a μ-collection of map, the vertex set is given by

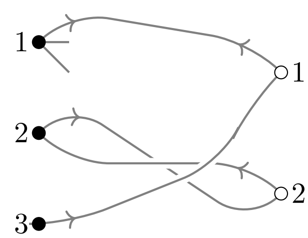
$$\Lambda_{\mathcal{W}}(M) = \pi \cup 1^{|\mu|-|\pi|} \qquad \text{and} \qquad \Lambda_{\mathcal{B}}(M) = \sigma \cup 1^{|\mu|-|\sigma|}.$$

The first condition follows immediately. Partitions describing white or black vertices distributions are sub-partitions of a partition describing face distribution. Hence the second condition has to be satisfied.

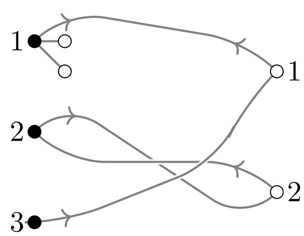
For partitions satisfying those two conditions one can exhibit a collection of maps from $P^{\mu}_{\pi,\sigma}$, which proves the sufficiency of those conditions. $\square$



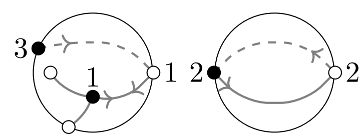
Step 1. Black and white vertices with outgoing half-edges of degrees (σ_i) and (π_j) respectively.



Step 2. Some of the outgoing half-edges were matched. The crossing of edges is not important.



Step 3. The rest of outgoing half-edges is closed. The collection of maps is bipartite.



As an outcome we obtain the following collection of two maps drawn on a pair of spheres.

Figure 3.3 – The three steps of “hands-shaking procedure”. As an output we obtain the $(4, 2)$-collection of bipartite maps. Vertices are labelled and rooted as as the “hands-shaking procedure” describes.

Observation 3.15. *For given partitions π, σ, μ: $|\pi|, |\sigma| \leq |\mu|$ we have*

$$\left|P^{\mu}_{\pi,\sigma}\right| = C(\pi,\sigma;\mu) \cdot \frac{z_\pi z_\sigma}{z_\mu} \left|\widetilde{M}^{\bullet;\mu}_{\pi\cup 1^{|\mu|-|\pi|},\sigma\cup 1^{|\mu|-|\sigma|}}\right|$$

where

$$C(\pi,\sigma;\mu) = \sum_{k=0}^{m_1(\mu)} \binom{m_1(\mu)}{k} \binom{m_1(\pi)+|\mu|-|\pi|-m_1(\mu)}{m_1(\pi)-k} \cdot \binom{m_1(\sigma)+|\mu|-|\sigma|-m_1(\mu)+k}{m_1(\sigma)-m_1(\pi)+k}, \quad (3.4)$$

which is equal to

$$\binom{m_1(\pi)+|\mu|-|\pi|}{m_1(\pi)} \binom{m_1(\sigma)+|\mu|-|\sigma|}{m_1(\sigma)}$$

if $m_1(\mu) = 0$ and is equal to 1 if π, σ, μ are partitions of the same integer.

Proof. Observe that the elements of $P^{\mu}_{\pi,\sigma}$ are μ-collection of bipartite orientable maps whose vertex set is given by

$$\Lambda_{\mathcal{W}}(M) = \pi \cup 1^{|\mu|-|\pi|} \qquad \text{and} \qquad \Lambda_{\mathcal{B}}(M) = \sigma \cup 1^{|\mu|-|\sigma|}.$$

Each such an element has the following labels and roots on the vertices and half-edges:

1. there are n white vertices of degrees $\pi_1, \dots, \pi_n$, each being labelled by a relevant natural number from $[n]$ and rooted, *i.e.* we choose one of the outgoing half-edges and decorate it by an arrow,
2. there are l black vertices of degrees $\sigma_1, \dots, \sigma_l$, each being labelled by a relevant natural number from $[l]$ and rooted.

Moreover, each connected component of an element from $P^{\mu}_{\pi,\sigma}$ has *at least one* decorated vertex.

We use the double counting method as in Observation 2.15. For each $M \in P^{\mu}_{\pi,\sigma}$ we can root and number the connected components in z_μ ways.

Let us choose $M \in \widetilde{M}^{\bullet;\mu}_{\pi\cup 1^{|\mu|-|\pi|},\sigma\cup 1^{|\mu|-|\sigma|}}$. The procedure of labelling and rooting the vertices is much more subtle. Firstly, we have to choose $m_1(\pi)$

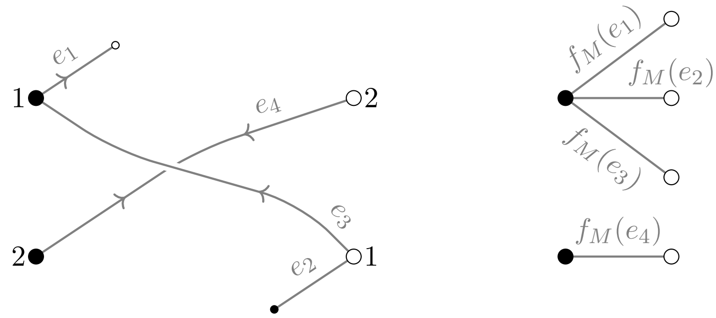


Figure 3.4 – Example of a collection of maps $M \in P^{(3,1)}_{(2,1),(2,1)}$ and an example of a bijection f^{μ}_M between the edges in M and the edges of the graph $G_{(3,1)}$.

white (respectively $m_1(\sigma)$ black) vertices and label them by adequate numbers. At the first sight, we could do this in

$$\binom{m_1(\pi) + |\mu| - |\pi|}{m_1(\pi)}\binom{m_1(\sigma) + |\mu| - |\sigma|}{m_1(\sigma)} z_\pi z_\sigma$$

ways (which is equal to $z_\pi z_\sigma$ if π, σ, μ are partitions of the same integer). However, in the definition of $P^{\mu}_{\pi,\sigma}$ we required to contain at least one labelled vertex from each connected component. This is trivially satisfied if $m_1(\mu) = 0$. This consideration yields the expression (3.4). We describe briefly the details.

There is $m_1(\mu)$ one-element connected components in M. Denote the set of those components by M_1. For each integer k: $0 \leq k \leq m_1(\mu)$ we can choose k white vertices from M_1 and we required that exactly those white vertices among all white vertices in M_1 are numbered. The number of possible ways of numbering vertices of M in such a way is the contribution to the sum in (3.4) relevant to k. We sum up over all $k = 0, \ldots, m_1(\mu)$. □

3.3.4. Proof of Theorem 3.3

We prove that candidates $p^{\mu}_{\pi,\sigma} := |P^{\mu}_{\pi,\sigma}|$ for top-degree part of structure constants $g^{\mu}_{\pi,\sigma}$ suit well for that role.

Proposition 3.16. *For any Young diagram $\lambda \in \mathbb{Y}$, the following equality holds:*

$$\left[A^{\text{top}}\right] \mathrm{Ch}_\pi(\lambda) \cdot \left[A^{\text{top}}\right] \mathrm{Ch}_\sigma(\lambda) = \sum_\mu p^{\mu}_{\pi,\sigma} \left[A^{\text{top}}\right] \mathrm{Ch}_\mu(\lambda). \tag{3.5}$$

Proof. According to Proposition 3.9, the two quantities

$$\left[A^{\text{top}}\right] \text{Ch}_\sigma(\lambda) = N_{G_\sigma}(\lambda) \qquad \text{and} \qquad \left[A^{\text{top}}\right] \text{Ch}_\mu(\lambda) = N_{G_\mu}(\lambda)$$

can be represented equivalently by the number of injective embeddings of G_σ and G_μ into λ. Similarly,

$$\left[A^{\text{top}}\right] \text{Ch}_\pi(\lambda) = \overline{N_{\overline{G_\pi}}(\lambda)}$$

is equal to the number of negative injective embeddings of $\overline{G_\pi}$ into λ (see Remark 3.8).

For each $M \in P^{\mu}_{\pi,\sigma}$ we choose some bijection f^{μ}_{M} between the edges of M and the edges of the graph G_μ (see Definition 3.7), which preserves the connected components, see Figure 3.4.

We shall construct a bijection between:

- a pair $\left(\overline{N_{\overline{G_\pi}}(\lambda)}, N_{G_\sigma}(\lambda)\right)$ consisting of negative injective embeddings and injective embeddings of G_σ and $\overline{G_\pi}$ into λ respectively;
- a pair $\left(P^{\mu}_{\pi,\sigma}, N_{G_\mu}(\lambda)\right)$ consisting of collections of maps from the class $P^{\mu}_{\pi,\sigma}$ and injective embeddings of G_μ into λ.

Construction of such bijection follows the statement of Proposition 3.16. We proceed analogously as in the "hands-shaking procedure" described in Section 3.3.3.

For each $i \in [n]$ we assign a white vertex with π_i outgoing half-edges. We label this vertex by a number i and root it, *i.e.* we choose one of outgoing half-edges and label it. We can choose a bijection between such half-edges and the edges in $\overline{G_\pi}$ which preserves the connected components. Similarly, for each $j \in [l]$ we assign a black vertex with σ_j outgoing half-edges and we root it. Then we choose a bijection between such half-edges and the edges in G_σ which preserves the connected components.

A reverse injective embedding of $\overline{G_\pi}$ and an injective embedding of G_σ into λ transfer into an injective embedding of above described half-edges going out from labelled and rooted black and white vertices.

We use the procedure described in Section 3.3.3 to connect *in the unique way* those outgoing half-edges which are embedded in the same box of Young diagram λ. We close each of non-closed half-edges by a white or a black vertex so that the graph remains bipartite.

In that way we obtain a list of maps $M \in P^{\mu}_{\pi,\sigma}$ injectively embedded into the Young diagram λ. Observe that all edges from any given connected component of M are embedded into the boxes of λ which are in the same row. Using the bijection f_M^{μ} between the edges of M and the edges of G_μ, we obtain the injective embedding of G_μ into λ.

The above procedure is reversible. Indeed, for a given collection of maps $M \in P^{\mu}_{\pi,\sigma}$ and an injective embedding of G_μ into diagram λ, we can easily construct the injective embedding of the edges of M into the diagram λ, for which all edges from any given connected component of M are embedded to the boxes from the same row. From such an object we can recover the elements from $N_{G_\sigma}(\lambda)$ and $\overline{N_{\overline{G_\pi}}(\lambda)}$. □

With Proposition 3.16 in hand, we are ready to present the proof of Theorem 3.3.

Proof of Theorem 3.3. The upper bound of a degree for polynomials $g^{\mu}_{\pi,\sigma}(\delta)$ is given in (3.2). Since $\delta = \frac{1}{A} - A$, we have the following estimation

$$\deg_A g^{\mu}_{\pi,\sigma} = \deg_\delta g^{\mu}_{\pi,\sigma} \leq d\left(\pi,\sigma;\mu\right).$$

Let us fix a Young diagram λ. Recall that the evaluation of Ch_π on any Young diagram λ is a Laurent polynomial in $\mathbb{Q}\left[A, A^{-1}\right]$ of a degree at most $n_2(\pi) := |\pi| - \ell(\pi)$. We investigate the $n_2(\pi) + n_2(\sigma)$ degree part of the pointwise product of two Jack characters, namely

$$\left[A^{n_2(\pi)+n_2(\sigma)}\right] \mathrm{Ch}_\pi\left(\lambda\right) \cdot \mathrm{Ch}_\sigma\left(\lambda\right) = \left[A^{n_2(\pi)+n_2(\sigma)}\right] \sum_{\mu} g^{\mu}_{\pi,\sigma} \mathrm{Ch}_\mu\left(\lambda\right).$$

By the estimations on the upper bounds of the A-degrees of Laurent polynomials $\mathrm{Ch}_\pi(\lambda)$ and $g^{\mu}_{\pi,\sigma}$ we have

$$\left[A^{\mathrm{top}}\right] \mathrm{Ch}_\pi\left(\lambda\right) \cdot \left[A^{\mathrm{top}}\right] \mathrm{Ch}_\sigma\left(\lambda\right) = \sum_{\mu} \left[A^{d(\pi,\sigma;\mu)}\right] g^{\mu}_{\pi,\sigma} \left[A^{\mathrm{top}}\right] \mathrm{Ch}_\mu\left(\lambda\right).$$

We compare the above equation with Proposition 3.16 and we get

$$\sum_{\mu} \left[A^{d(\pi,\sigma;\mu)}\right] g^{\mu}_{\pi,\sigma} \mathrm{Ch}_\mu\left(\lambda\right) = \sum_{\mu} p^{\mu}_{\pi,\sigma} \mathrm{Ch}_\mu\left(\lambda\right).$$

Recall that $\mathrm{Ch}_\mu(\lambda) = \widehat{p_\mu}(\lambda)$. We have

$$\sum_{\mu} \left[A^{d(\pi,\sigma;\mu)}\right] g^{\mu}_{\pi,\sigma} \widehat{p_\mu}\left(\lambda\right) = \sum_{\mu} p^{\mu}_{\pi,\sigma} \widehat{p_\mu}\left(\lambda\right), \tag{3.6}$$

The function $\widehat{p_\mu}(\lambda)$ is symmetric and its homogeneous top-degree part coincides with the power-sum symmetric polynomial p_μ. This coincidence together with the fact that power-sum symmetric functions form a basis of symmetric functions allows us to deduce that functions $\widehat{p_\mu}(\lambda)$ form also such a basis. We may look at (3.6) as on the equality of symmetric functions. Since the basis determines its coefficients in the unique way, we conclude that

$$\left[A^{d(\pi,\sigma;\mu)}\right] g^\mu_{\pi,\sigma} = p^\mu_{\pi,\sigma}.$$

The $d(\pi,\sigma;\mu)$-degree coefficients in variable A and δ of $g^\mu_{\pi,\sigma}$ are equal. We conclude

$$\delta^{(\pi,\sigma;\mu)} g^\mu_{\pi,\sigma} = p^\mu_{\pi,\sigma}.$$

Observation 3.15 and Observation 3.14 finish the proof. □

Chapter 4

Algebras with two multiplications and their cumulants

4.1. Algebras with two multiplications

In this section we shall present the notion of cumulants in probability theory. In Section 4.1.3 we define algebras with two multiplications and their cumulants. In Theorem 4.6 we present some cumulant formula which holds in algebras with two multiplications.

4.1.1. Cumulants in probability theory

One of classical problems in probability theory is to describe the joint distribution of a family (X_i) of random variables in the most convenient way. Common solution of this problem is to use the family of moments, *i.e.* the expected values of products of the form

$$\mathbb{E}\left(X_{i_1}\cdots X_{i_l}\right).$$

It has been observed that in many problems it is more convenient to make use of the *cumulants* [Hal81, Fis28], defined as the coefficients of the expansion of the logarithm of the multidimensional Laplace transform around zero:

$$\begin{aligned}\kappa\left(X_1,\dots,X_n\right) &:= [t_1\cdots t_n]\log\mathbb{E}e^{t_1X_1+\cdots+t_nX_n}\\ &= \left.\frac{\partial^n}{\partial t_1\cdots\partial t_n}\log\mathbb{E}e^{t_1X_1+\cdots+t_nX_n}\right|_{t_1=\ldots=t_n=0},\end{aligned} \tag{4.1}$$

where the terms on the right-hand side should be understood as a formal power series in the variables $t_1, \ldots, t_n$. Cumulant is a linear map with respect to each of its arguments.

There are some good reasons for claiming advantage of cumulants over the moments. One of them is that the convolution of measures corresponds to the product of the Laplace transforms or, in other words, to the sum of the logarithms of the Laplace transforms. It follows that the cumulants behave in a very simple way with respect to the convolution, namely cumulants linearize the convolution.

Cumulants allow also a combinatorial description. One can show that the expression (4.1) is equivalent to the following system of equations, called *the moment-cumulant formula*:

$$\mathbb{E}\left(X_1 \cdots X_n\right) = \sum_{\nu} \prod_{b \in \nu} \kappa\left(X_i : i \in b\right) \tag{4.2}$$

which should hold for any choice of the random variables $X_1, \ldots, X_n$ whose moments are all finite. The above sum runs over the set partitions ν of the set $[n] = \{1, \ldots, n\}$ and the product runs over the blocks of the partition ν.

Example 4.1. For three random variables the corresponding moment expands as follows:

$$\begin{aligned} \mathbb{E}\left(X_1 X_2 X_3\right) = \; & \kappa(X_1) \cdot \kappa(X_2) \cdot \kappa(X_3) + \kappa(X_1, X_2) \cdot \kappa(X_3) \\ & + \kappa(X_2, X_3) \cdot \kappa(X_1) + \kappa(X_1, X_3) \cdot \kappa(X_2) \\ & + \kappa(X_1, X_2, X_3). \end{aligned}$$

The moment-cumulant formula defines the cumulant $\kappa(X_1, \ldots, X_n)$ inductively according to the number of arguments n.

4.1.2. Conditional cumulants

Let $\mathcal{A}$ and $\mathcal{B}$ be commutative unital algebras and let $\mathbb{E} : \mathcal{A} \longrightarrow \mathcal{B}$ be a unital linear map. We say that $\mathbb{E}$ is a conditional expected value. For any tuple $x_1, \ldots, x_n \in \mathcal{A}$ we define their *conditional cumulant* as

$$\begin{aligned} \kappa\left(x_1, \ldots, x_n\right) &= [t_1 \cdots t_n] \log \mathbb{E} e^{t_1 x_1 + \cdots + t_n x_n} \\ &= \left. \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \log \mathbb{E} e^{t_1 x_1 + \cdots + t_n x_n} \right|_{t_1 = \ldots = t_n = 0} \in \mathcal{B} \end{aligned} \tag{4.3}$$

where the terms on the right-hand side should be understood as in Eq. (4.1). In this general approach, cumulants give a way of measuring the discrepancy between the algebraic structures of $\mathcal{A}$ and $\mathcal{B}$.

4.1.3. Framework

We are interested in a following particular case. We assume that $\mathcal{A}$ is a linear space equipped with two commutative multiplication structures, which correspond to two products: $\cdot$ and $*$. Together with each multiplication $\mathcal{A}$ form the commutative algebra. We call such structure an *algebra with two multiplications*. We also assume that the mapping $\mathbb{E}$ is the identity map on $\mathcal{A}$:

$$\mathbb{E} : (\mathcal{A}, \cdot) \xrightarrow{\text{id}} (\mathcal{A}, *) .$$

In this case the cumulants measure the discrepancy between these two multiplication structures on $\mathcal{A}$. This situation arises naturally in many branches of algebraic combinatorics, for example in the case of Macdonald cumulants [Doł17a, Doł17b] and cumulants of Jack characters [DF17, Śn16].

Since the mapping $\mathbb{E}$ is the identity, we can define cumulants of cumulants and further compositions of them. The terminology of cumulants of cumulants was introduced in [Spe83] and further developed in [Leh13] (called there *nested cumulants*) in a slightly different situation of an inclusion of algebras $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ and conditional expectations $\mathcal{A} \xrightarrow{\mathbb{E}_1} \mathcal{B} \xrightarrow{\mathbb{E}_2} \mathcal{C}$.

As we already mentioned in Section 4.1.1, cumulants allow also a combinatorial description via the moment-cumulant formula. When $\mathbb{E}$ is the identity map (4.3) is equivalent to the following system of equations:

$$a_1 * \cdots * a_n = \sum_{\nu} \prod_{b \in \nu} \kappa\left(a_i : i \in b\right), \tag{4.4}$$

for any $a_i \in \mathcal{A}$ (the product on the right-hand side is the $\cdot$-product). The above sum runs over the set partitions ν of the set $[n]$ and the product runs over the blocks of the partition ν.

Let A be a multiset consisting of elements of the algebra $\mathcal{A}$. To simplify notation, for any partition ν of a multiset A we introduce the corresponding cumulant κ_ν as the product:

$$\kappa_\nu = \prod_{b \in \nu} \kappa\left(a : a \in b\right).$$

We denote by $\mathcal{P}(A)$ the set of all partitions of A. With this notation, the moment-cumulant formula has the following form:

$$\mathop{\Large *}_{a \in A} a = \sum_{\nu \in \mathcal{P}(A)} \kappa_\nu. \tag{4.5}$$

Example 4.2. Given three elements $a_1, a_2, a_3 \in \mathcal{A}$, we have:

$$\begin{array}{rlcll} a_1 * a_2 * a_3 = & \kappa(a_1)\cdot\kappa(a_2)\cdot\kappa(a_3) & + & \kappa(a_1,a_2)\cdot\kappa(a_3) & + \\ & \kappa(a_2,a_3)\cdot\kappa(a_1) & + & \kappa(a_1,a_3)\cdot\kappa(a_2) & + \\ & \kappa(a_1,a_2,a_3). & & & \end{array}$$

4.1.4. The third result

We present an algebraic formula which involves two multiplications on linear space $\mathcal{A}$:

$$\left(a_1^1 * \cdots * a_{k_1}^1\right) \cdots \left(a_1^n * \cdots * a_{k_n}^n\right),$$

as a sum of products of only one type of multiplication.

We use the following notation. We denote by $A_1, \dots, A_n$ multisets consisting of elements of $\mathcal{A}$. We denote by $A = A_1 \cup \cdots \cup A_n$ the multiset, corresponding to the sum of all multisets A_i. We use also the following notation for elements of A_i:

$$A_i = \left\{a_1^i, \dots, a_{k_i}^i\right\},$$

hence the multiset A consists of the following elements:

$$A = \left\{a_1^1, \dots, a_{k_1}^1, \dots, a_1^n, \dots, a_{k_n}^n\right\}.$$

Due to a combinatorial nature of this result we introduce now the definitions of the mixing reduced forests and theirs cumulants. We begin with the following definition.

Definition 4.3. Consider a forest F whose leaves are labelled by elements of an algebra $\mathcal{A}$. We denote by A the multiset consisting of labels of all leaves. If each node (vertex which is not a leaf) of F, has at least two descendants, we call F a *reduced forest* with leaves in A. We denote the set of such forests by $\mathcal{F}(A)$.

For a reduced forest $F \in \mathcal{F}(A)$ we associate a cumulant κ_F in the following way:

Definition 4.4. Consider a reduced forest $F \in \mathcal{F}(A)$. Denote by a_v the label of a leaf v. For any vertex $v \in F$ we define inductively the quantities κ_v as follows:

$$\kappa_v := \begin{cases} a_v & \text{if } v \text{ is a leaf,} \\ \kappa(\kappa_{v_1}, \dots, \kappa_{v_n}) & \text{otherwise,} \end{cases}$$

where $v_1, \dots, v_n$ are the descendants of v. For the whole forest F, we define the cumulant κ_F to be the product:

$$\kappa_F := \mathop{*}_{i} \kappa_{V_i},$$

where V_i are the roots of all trees in F.

Finally, we introduce a class of the *mixing forests* and the associated quantity w_F.

Definition 4.5. Consider a multiset $A = A_1 \cup \dots \cup A_n$ and a reduced forest $F \in \mathcal{F}(A)$. We say that F is *mixing for a division* $A_1, \dots, A_n$ (or shortly *mixing*) if for each vertex v whose descendants are all leaves, those descendants are elements of at least two distinct multisets A_i and A_j. Denote by $\overline{\mathcal{F}}(A)$ the set of all *reduced mixing forests.*

For a reduced mixing forest F we define the quantity w_F to be the number of vertices in F minus the number of leaves (see Figure 4.1).

We are ready to formulate the main result of this chapter.

Theorem 4.6 (The third result)**.** *Let $A_1, \dots, A_n$ be multisets consisting of elements of $\mathcal{A}$. Let A be the sum of those multisets. Then:*

$$\left(a_1^1 * \dots * a_{k_1}^1\right) \cdots \left(a_1^n * \dots * a_{k_n}^n\right) = \sum_{F \in \overline{\mathcal{F}}(A)} (-1)^{w_F} \kappa_F.$$

Example 4.7. Figure 4.1 presents all reduced forests F on the multiset $A = \{a_1^1, a_2^1, a_1^2\}$. Six of them are mixing. By the statement of the theorem, we have

$$\begin{aligned}(a_1^1 * a_2^1) \cdot a_1^2 \quad = \quad & a_1^1 * a_2^1 * a_1^2 - \kappa(a_1^1, a_1^2) * \kappa(a_2^1) \\ & -\kappa(a_2^1, a_1^2) * \kappa(a_1^1) - \kappa(a_1^1, a_2^1, a_1^2) \\ & +\kappa\left(\kappa(a_1^1, a_2^2), \kappa(a_2^1)\right) + \kappa\left(\kappa(a_2^1, a_2^2), \kappa(a_1^1)\right).\end{aligned}$$

4.1.5. How to prove Theorem 4.6?

Theorem 4.6 is a straightforward conclusion from two propositions which we present in this section. In our opinion they are interesting themselves.

We introduce a gap-free vertex colouring on forests $F \in \mathcal{F}(A)$.

Definition 4.8. For a reduced forest F with leaves in a multiset $A = A_1 \cup \dots \cup A_n$ we say that c is a gap-free vertex colouring with length r if

- c is a coloured by the numbers $\{0, \dots, r\}$ and each colour is used at least once;

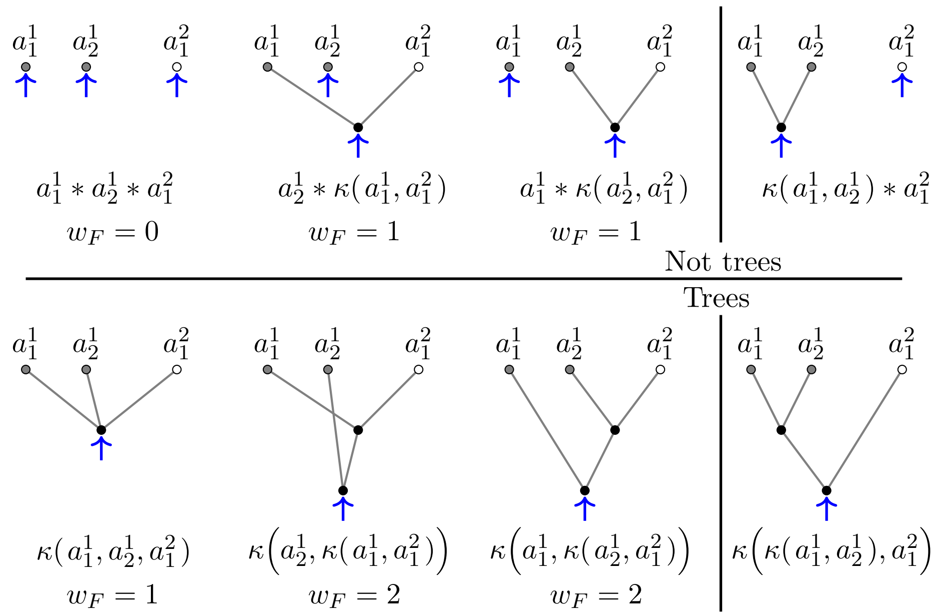


Figure 4.1 – All reduced forests on $A = A_1 \cup A_2 = \{a_1^1, a_2^1, a_1^2\}$. Six of them (on the right-hand side) are mixing; we present theirs w_F numbers. The remaining two elements (shown on the left-hand side) are not mixing. We also present the corresponding cumulants κ_F. Observe that, among all reduced forests $F \in \mathcal{F}(A)$, exactly half, presented on top, consists of a single tree (see Remark 4.39).

- each leaf is coloured by 0;
- the colours are strictly increasing on any path from the root to a leaf.

We denote by $|c| := r$ the length of c . We call such a colouring c *weakly-mixing* if it satisfies one of the following additional conditions:

1. either there exists a vertex coloured by 1 with at least two descendants, each of whom belongs to a distinct multiset A_i,
2. or colouring c does not use the colour 1 at all.

We denote by $\mathcal{C}_F$ the set of all gap-free and weakly-mixing colourings of a forest F.

The following result is a juggling of a concept of cumulants. We present its proof in Section 4.4.

Proposition 4.9. *Let $A_1, \ldots, A_n$ be multisets consisting of elements of $\mathcal{A}$. Let A be a sum of those multisets. Then*

$$\left(a_1^1 * \cdots * a_{k_1}^1\right) \cdots \left(a_1^n * \cdots * a_{k_n}^n\right) = \sum_{F \in \mathcal{F}(A)} \kappa_F \sum_{c \in \mathcal{C}_F} (-1)^{|c|}. \tag{4.6}$$

In Section 4.5, we will show that summing over all colourings $c \in \mathcal{C}_F$ for a reduced forest $F \in \mathcal{F}(A)$ gives a surprisingly simple number. This result is presented in proposition below.

Proposition 4.10. *Let $A_1, \ldots, A_n$ be multisets consisting of elements of $\mathcal{A}$. Let A be a sum of those multisets. Then, for any reduced forest $F \in \mathcal{F}(A)$, the following holds:*

$$\sum_{c \in \mathcal{C}_F} (-1)^{|c|} = \begin{cases} (-1)^{w_F} & \text{if } F \in \overline{\mathcal{F}}(A), \\ 0 & \text{otherwise.} \end{cases}$$

Observe that combing Proposition 4.9 and Proposition 4.10 we obtain the statement of Theorem 4.6.

4.2. Analogue of Leonov–Shiryaev's formula

In this section we present the well-known cumulant formula given by Leonov and Shiryaev in 1959. Theorem 4.16 is a reformulation of Theorem 4.6 and may be seen as an analogue of Leonov–Shiryaev's formula.

4.2.1. Leonov–Shiryaev's formula

In 1959 Leonov and Shiryaev [LS59, Equation IV.d] presented a formula for a cumulant of products of random variables:

$$\kappa\left(X_{1,1}\cdots X_{k_1,1},\ldots,X_{1,n}\cdots X_{k_n,n}\right)$$

in terms of simple cumulants. This formula was first proved by Leonov and Shiryaev [LS59], a more direct proof was given by Speed [Spe83]. The technique of Leonov and Shiryaev was used in many situations [SSR88, Leh04] and was further developed in other papers: Krawczyk and Speicher [KS00, MST07] found the free analogue of the formula; the formula was further generalized to the partial cumulants [NS06, Proposition 10.11].

We briefly present the original formula stated by Leonov and Shiryaev in the framework of an algebra with two multiplications. We use the same notation for multisets $A_1,\ldots,A_n$ and its sum $A = A_1 \cup \cdots \cup A_n$ as in Section 4.1.4.

We introduce a notion of a strongly-mixing partitions (called also *indecomposable* partitions).

Definition 4.11. Consider a multiset $A = A_1 \cup \cdots \cup A_n$ and any partition ν of A. A partition $\lambda = \{\lambda_1, \lambda_2\}$ is called a *row partition* if for each multiset A_i we have: either $A_i \subseteq \lambda_1$ or $A_i \subseteq \lambda_2$.

A partition $\nu = \{\nu_1,\ldots,\nu_q\}$ is called a *strongly-mixing partition for the division* $A = A_1 \cup \cdots \cup A_n$ (or shortly *strongly-mixing partition*), if there is no row partition λ such that for any i either $\nu_i \in \lambda_1$, or $\nu_i \in \lambda_2$ (see Figure 4.2).

We denote by $\hat{\mathcal{P}}(A)$ the set of all strongly-mixing partitions of a set A.

We can now express the Leonov–Shiryaev's formula in a framework relevant to this dissertation.

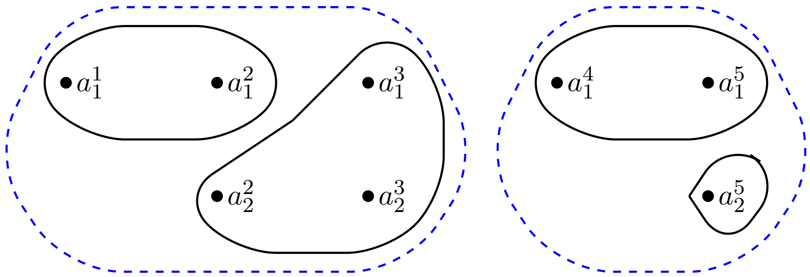
Theorem 4.12 (Leonov–Shiryaev's formula)**.**

$$\kappa\left(a_1^1\cdots a_{k_1}^1,\ldots,a_1^n\cdots a_{k_n}^n\right) = \kappa\left(\prod_{j=1}^{k_i} a_i^j : i\in[n]\right) = \sum_{\nu\in\hat{\mathcal{P}}(A)} \kappa_\nu, \tag{4.7}$$

where the sum on the right-hand side is running over all strongly-mixing partitions of a set A.

Example 4.13. By Leonov–Shiryaev's formula the cumulant $\kappa\left(a_1^1\cdot a_2^1, a_1^2\right)$ expresses as follows:

$$\begin{aligned}\kappa\left(a_1^1\cdot a_2^1, a_1^2\right) &= +\kappa(a_1^1,a_2^2)\cdot\kappa(a_2^1)+\kappa(a_2^1,a_2^2)\cdot\kappa(a_1^1)\\ &\quad+\kappa(a_1^1,a_2^1,a_1^2).\end{aligned}$$



$$\nu = \Big\{\{a_1^1, a_1^2\}, \{a_1^3, a_2^2, a_2^3\}, \{a_1^4, a_1^5\}, \{a_2^5\}\Big\}$$

$$\lambda = \Big\{\{a_1^1, a_1^2, a_2^2, a_1^3, a_2^3\}, \{a_1^4, a_1^5, a_2^5\}\Big\}$$

Figure 4.2 – The multiset $A = \{a_1^1, a_1^2, a_2^2, a_1^3, a_2^3, a_1^4, a_1^5, a_2^5\}$ and the set partition ν. There exists a row partition λ (dashed line) such that each part of ν is contained in one of the parts of λ. Hence the partition ν is not strongly-mixing.

4.2.2. Analogue of Leonov–Shiryaev's formula

Leonov–Shiryaev's formula relates a cumulant of products with some products of cumulants. In the situation we investigate, where the conditional expected value is the identity mapping, we can define two types of cumulants. For each of them we have Leonov–Shiryaev's formula. We present now the third formula, which is a mix of those two.

Consider the identity map:

$$(\mathcal{A}, \cdot) \xrightarrow{\text{id}} (\mathcal{A}, *)$$

between commutative unital algebras $(\mathcal{A}, \cdot)$ and $(\mathcal{A}, *)$. Equation (4.4) defined cumulants κ of the identity mapping. Observe that we can also consider the inverse mapping, namely the map:

$$(\mathcal{A}, *) \xrightarrow{\text{id}^{-1}} (\mathcal{A}, \cdot).$$

This mapping gives us a way to define cumulants (according to (4.4)), which we denote by κ^*.

We present below the Leonov–Shiryaev's formula for both mappings mentioned above:

$$\kappa\left(\prod_{j=1}^{k_i} a_i^j : i \in [n]\right) = \sum_{\nu \in \hat{\mathcal{P}}(A)} \mathop{\ast}_{b \in \nu} \kappa\,(a \in b)\,,$$

and

$$\kappa^*\left(\mathop{\ast}_{j=1}^{k_i} a_i^j : i \in [n]\right) = \sum_{\nu \in \hat{\mathcal{P}}(A)} \prod_{b \in \nu} \kappa^*\left(a \in b\right),$$

where the sums in both equalities run over all strongly-mixing partitions of a multiset $A = \{a_i^j : i \in [n], j \in [k_i]\}$. Observe that in each equality the cumulants on each side are of the same type but the multiplications are not. In our formula we will mix types of cumulants on both sides but keep the same multiplication.

To present our result we introduce a class of *strongly-mixing forests* $\hat{\mathcal{F}}(A)$.

Definition 4.14. Let $A_1, \ldots, A_n$ be multisets consisting of elements of $\mathcal{A}$. Consider a reduced mixing forest $F \in \overline{\mathcal{F}}(A)$ consisting of trees $T_1, \ldots, T_s$. Denote by $a_v \in A$ the label of a leaf $a \in A$. We define a partition ν_F of a set A as follows:

$$\nu_F = \left\{ \{a_v : v \in T_k\}_{k \in [s]} \right\}.$$

We say that a mixing reduced forest $F \in \mathcal{F}(A)$ is *strongly-mixing* if the partition ν_F is strongly-mixing partition. We denote the set of such forests by $\hat{\mathcal{F}}(A)$.

Remark 4.15. Observe that the class of all strongly-mixing forests $\hat{\mathcal{F}}(A)$ is a subclass of all mixing forests $\overline{\mathcal{F}}(A)$, which itself is a subclass of all reduced forests $\mathcal{F}(A)$, *i.e.*:

$$\hat{\mathcal{F}}(A) \subset \overline{\mathcal{F}}(A) \subset \mathcal{F}(A).$$

This is analogue to the natural order between classes of strongly-mixing partitions $\hat{\mathcal{P}}(A)$, mixing partitions $\overline{\mathcal{P}}(A)$ and partitions $\mathcal{P}(A)$:

$$\hat{\mathcal{P}}(A) \subset \overline{\mathcal{P}}(A) \subset \mathcal{P}(A).$$

We can reformulate Theorem 4.6 as follows.

Theorem 4.16 (Analogue of Leonov–Shiryaev's formula)**.** *Consider an algebra $\mathcal{A}$ with two multiplicative structures $\cdot$ and $\ast$. Denote by κ and κ^* cumulants related to the identity map on $\mathcal{A}$ as we described above. Then the following formula holds:*

$$\kappa^*\left(\mathop{\ast}_{j=1}^{k_i} a_i^j : i \in [n]\right) = \sum_{F \in \hat{\mathcal{F}}(A)} (-1)^{w_F} \kappa_F, \tag{4.8}$$

where $\hat{\mathcal{F}}(A)$ is a set consisting of strongly-mixing reduced forests.

Example 4.17. Figure 4.1 presents all reduced forests on the multiset $A = \{a_1^1, a_2^1, a_1^2\}$. Six of them are mixing, five of them are strongly mixing. Thus:

$$\begin{aligned}\kappa^*\left(a_1^1 * a_2^1, a_1^2\right) &= a_1^1 * a_2^1 * a_1^2 - \kappa(a_1^1, a_1^2) * \kappa(a_2^1)\\ &\quad -\kappa(a_2^1, a_1^2) * \kappa(a_1^1) - \kappa(a_1^1, a_2^1, a_1^2)\\ &\quad +\kappa\left(\kappa(a_2^1, a_2^2), \kappa(a_1^1)\right).\end{aligned}$$

Proof. In (4.4) we present the moment cumulant formula for cumulants κ related to the map $(\mathcal{A}, \cdot) \xrightarrow{\text{id}} (\mathcal{A}, *)$. Similar expression for cumulants κ^* related to the inverse map $(\mathcal{A}, *) \xrightarrow{\text{id}^{-1}} (\mathcal{A}, \cdot)$ is of the following form:

$$a_1 \cdots a_n = \sum_{\nu \in \mathcal{P}([n])} \mathop{\ast}_{b \in \nu} \kappa^*\left(a_i : i \in b\right).$$

We express the $\cdot$-product $\left(a_1^1 * \cdots * a_{k_1}^1\right) \cdots \left(a_1^n * \cdots * a_{k_n}^n\right)$ via the moment cumulant formula given by the equation above:

$$\left(a_1^1 * \cdots * a_{k_1}^1\right) \cdots \left(a_1^n * \cdots * a_{k_n}^n\right) = \sum_{\nu \in \mathcal{P}([n])} \mathop{\ast}_{b \in \nu} \kappa^*\left(\mathop{\ast}_{j=1}^{k_i} a_i^j : i \in [n]\right). \quad (4.9)$$

From Theorem 4.6 we can express the left-hand side of this equation in another way:

$$\left(a_1^1 * \cdots * a_{k_1}^1\right) \cdots \left(a_1^n * \cdots * a_{k_n}^n\right) = \sum_{F \in \overline{\mathcal{F}}(A)} (-1)^{w_F} \kappa_F.$$

Observe that we can split the summation of $(-1)^{w_F} \kappa_F$ over all mixing reduced forests $F \in \mathcal{F}(A)$ into $*$-product of summation over all strongly-mixing reduced forests:

$$\sum_{F \in \overline{\mathcal{F}}(A)} (-1)^{w_F} \kappa_F = \sum_{\nu \in \mathcal{P}([n])} \mathop{\ast}_{b \in \nu} \left(\sum_{F \in \hat{\mathcal{F}}\left(A^b\right)} (-1)^{w_F} \kappa_F \right),$$

where, for each partition ν, sets $A^b := \cup_{i \in b} A_i$ are division of a set A.

Observe, that quantities:

$$\sum_{F \in \hat{\mathcal{F}}\left(A^b\right)} (-1)^{w_F} \kappa_F$$

satisfy the system of equations given by the moment cumulant formula (4.9), which has a unique solution. This yields the statement of the theorem. □

Remark 4.18. The above equation is still valid when we replace κ (which is hidden in κ_F terms) with κ^* and replace $*$-products with $\cdot$-products simultaneously.

4.3. Cumulant formula for Jack characters

In this section we show that Jack characters forms an algebra with two multiplications. In consequence, in Section 4.3.2 we present some statements about structure constants which may be seen via this characterisation of Jack characters.

4.3.1. Approximate factorization property

In many cases cumulants are quantities of a very small degree. The following definition specifies this statement [Śn16, Definition 1.8].

Definition 4.19. Let $\mathcal{A}$ and $\mathcal{B}$ be filtered unital algebras and let $\mathbb{E} : \mathcal{A} \longrightarrow \mathcal{B}$ be a unital linear map. Let κ be the corresponding cumulants. We say that $\mathbb{E}$ has *approximate factorization property* if for all choices of $a_1, \dots, a_l \in \mathcal{A}$ we have that

$$\deg_{\mathcal{B}} \kappa\left(a_1, \dots, a_l\right) \leq \deg_{\mathcal{A}} a_1 + \cdots + \deg_{\mathcal{A}} a_l - 2\left(l - 1\right).$$

Observation 4.20. *Let us go back to the case, when $\mathbb{E}$ is the identity map on algebra $\mathcal{A}$ with two multiplications. Suppose that the identity map*

$$\left(\mathcal{A}, \cdot\right) \xrightarrow{id} \left(\mathcal{A}, *\right),$$

satisfies the approximate factorization property.

Let $A_1, \dots A_n$ be multisets consisting of elements of $\mathcal{A}$. Let A be the sum of those multisets. Then for any forest $F \in \mathcal{F}\left(A\right)$ consisting of f trees, there is the following restriction on the degree of cumulants:

$$\deg \kappa_F \leq \left(\sum_{a \in A} \deg a\right) - 2|A| + 2f,$$

where $|A|$ is the number of elements in A.

Proof. We analyse the definition of κ_F (Definition 4.4). For any vertex $v \in F$ we defined the quantities κ_v. Using the approximate factorization property, observe that:

$$\deg \kappa_v \leq \sum_{i=1}^{n} \deg \kappa_{v_i} - 2\left(n + 1\right).$$

Going from the root r to the leaves we obtain:

$$\deg \kappa_r \leq \sum_{i=1}^{n_r} \deg \kappa_{v_i} - 2\left(n_r + 1\right).$$

where n_r is the number of leaves in a tree rooted in r, and v_i for $i \in [n_r]$ are leaves of this tree.

The cumulants κ_F were defined as follows:

$$\kappa_F := \mathop{*}_{i} \kappa_{V_i},$$

where V_i are roots of all trees in F, hence $\deg \kappa_F \leq \sum \deg \kappa_{V_i}$. It is now easy to see the statement of this observation. □

4.3.2. Cumulants of Jack characters

Jack characters Ch_π form a natural family (indexed by partitions π) of functions on the set $\mathbb{Y}$ of Young diagrams. One can introduce two different multiplicative structures on the linear space spanned by Jack characters.

The $*$-product is given by concatenations of partitions:

$$\mathrm{Ch}_\pi * \mathrm{Ch}_\sigma = \mathrm{Ch}_{\pi \sqcup \sigma}.$$

For any partitions π and σ one can uniquely express the pointwise product of the corresponding Jack characters

$$\mathrm{Ch}_\pi \cdot \mathrm{Ch}_\sigma = \sum_\mu g^\mu_{\pi,\sigma} \, \mathrm{Ch}_\mu \, (\delta)$$

in the linear basis of Jack characters.

Śniady considers an algebra of Jack characters as a graded algebra, with gradation given by the notion of *α-polynomial functions* [Śn15, Section 1.7]. Jack characters are α-polynomial function of the following degrees

$$\deg \mathrm{Ch}_\pi = |\pi| + \ell(\pi).$$

Śniady gave explicit formulas for the top-degree homogeneous part of Jack characters. We sketch shortly how we use Theorem 4.6 in order to find the top-degree coefficients of the structure constants below.

Consider two integer partitions $\pi = (\pi_1, \dots, \pi_n)$ and $\sigma = (\sigma_1, \dots, \sigma_l)$ and the relevant multiset $A = A_1 \cup A_2$ given by:

$$A_1 = \{\mathrm{Ch}_{\pi_1}, \dots, \mathrm{Ch}_{\pi_n}\},$$
$$A_2 = \{\mathrm{Ch}_{\sigma_1}, \dots, \mathrm{Ch}_{\sigma_l}\}.$$

Together with the $\cdot$-product and the $*$-product described above, the linear space spanned by Jack characters becomes an algebra with two multiplications. We can introduce cumulants as a way of measuring the discrepancy between those two types of multiplications via (4.4). Recently the approximation factorisation property of cumulants was proven [Śn16].

Lemma 4.21 (reformulation of Theorem 4.6)**.** *Let $A_1, \dots, A_n$ be multisets consisting of elements of $\mathcal{A}$. Let A be the sum of those multisets. Then:*

$$\left(a_1^1 * \cdots * a_{k_1}^1\right) \cdots \left(a_1^n * \cdots * a_{k_n}^n\right) = \sum_{\nu \in \mathcal{P}(A)} \mathop{\Huge *}_{i=1}^{|\nu|} \sum_{T \in \overline{\mathcal{T}}(\nu_i)} (-1)^{w_T} \kappa_T.$$

where $*$ and $\cdot$ are two different multiplications on $\mathcal{A}$ and $\nu = \{\nu_1, \dots, \nu_{|\nu|}\}$ is a partition of A.

Proof. Theorem 4.6 presents $\left(a_1^1 * \cdots * a_{k_1}^1\right) \cdots \left(a_1^n * \cdots * a_{k_n}^n\right)$ as a sum over reduced mixing forests of cumulants associated to those forests. Observe that each reduced mixing forest F splits naturally into a collection of trees $T_1, \dots, T_k$. Each of T_i possesses the property of being reduced and mixing. Leaves of F are labelled by elements of $\mathcal{A}$, thus we denoted by A the multiset consisting of those labels. Division of F into $T_1, \dots, T_k$ determines a partition $\nu = \{\nu_1, \dots, \nu_k\}$ of a set A, namely $\nu_i \subset A$ consists of all labels of leaves of T_i. The cumulant κ_F is equal to:

$$\mathop{\Huge *}_{i=1}^{k} \kappa_{T_i}$$

by the definition. Moreover $(-1)^{w_F} = \prod_{i=1}^{k} (-1)^{w_{T_i}}$. □

Theorem 4.22. *With notation presented above, for any two partitions π and σ, the following decomposition is valid*

$$\mathrm{Ch}_\pi \cdot \mathrm{Ch}_\sigma = \sum_{\nu \in \mathcal{P}(A)} \mathop{\Huge *}_{i=1}^{|\nu|} \sum_{T \in \overline{\mathcal{T}}(\nu_i)} (-1)^{w_T} \kappa_T, \tag{4.10}$$

where $\nu = \{\nu_1, \dots, \nu_{|\nu|}\}$ is a partition of A and $\overline{\mathcal{T}}(\nu_i)$ denotes the set of all reduced mixing trees on $\nu_i \subseteq A$.

Moreover, there is the following restriction on the degree of products of cumulants:

$$\deg\left(\mathop{\Huge *}_{i=1}^{|\nu|} \sum_{T \in \overline{\mathcal{T}}(\nu_i)} (-1)^{w_T} \kappa_T \right) \leq |\pi| + |\sigma| + 2|\nu|,$$

where $|\nu|$ is the number of parts in partition ν.

Presented statement is based on Lemma 4.21, the bound of a degree follows immediately from Observation 4.20.

The division given in (4.10) is a tool for capturing the structure constants $g^{\mu}_{\pi,\sigma}$. It opens a way for induction over the number $\ell(\sigma) + \ell(\pi)$. More precisely, we express κ_T in in the linear basis of Jack characters inductively, according to the number of leaves.

4.4. Proof of Proposition 4.9

In this section we shall prove Proposition 4.9. We use the same notation as in Section 4.1.4. We denote by $A_1, \dots, A_n$ multisets consisting of elements of $\mathcal{A}$. We denote by $A = A_1 \cup \dots \cup A_n$ the multiset, which is the sum of all multisets A_i. We use also the following notation for the elements of A_i:

$$A_i = \left\{a^i_1, \dots, a^i_{k_i}\right\},$$

hence the multiset A consists of the following elements:

$$A = \left\{a^1_1, \dots, a^1_{k_1}, \dots, a^n_1, \dots, a^n_{k_n}\right\}.$$

We denote additionally the set of all partitions of A by $\mathcal{P}(A)$. We denote by $\overline{\mathcal{P}}(A)$ a set of all *mixing partitions* of A, i.e. all partitions $\nu = \{\nu_1, \dots, \nu_l\}$ such that

$$\underset{i \in [l]}{\exists} \; \underset{j \in [n]}{\forall} \; \nu_i \not\subseteq A_j.$$

4.4.1. Outline of the proof

Firstly, we express the left-hand side of (4.6) as a sum of cumulants, where the sum runs over all mixing partitions $\nu \in \overline{\mathcal{P}}(A)$, see (4.11) below. By applying inductively the procedure (4.12) described below, we replace summation over all mixing partitions $\nu \in \overline{\mathcal{P}}(A)$ by a sum over all *nested upward sequences of partitions*, see the Definition 4.24. Then we construct a bijection between such sequences and reduced forests $F \in \mathcal{F}(A)$ equipped with gap-free, weakly-mixing colourings $c \in \mathcal{C}_F$ (see Definitions 4.3, 4.8). Later on we will prove that the weighted sum over all gap-free colourings for a fixed forest is either equal to 0 or to ± 1.

4.4.2. Cumulants of mixing partitions

Observe that the following equality of the sets holds:

$$\mathcal{P}(A) = \Big(\mathcal{P}(A_1) \times \dots \times \mathcal{P}(A_n)\Big) \;\cup\; \overline{\mathcal{P}}(A),$$

where the elements of the Cartesian product $\mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$ are understood as a partition of a multiset $A = A_1 \cup \cdots \cup A_n$.

We apply the moment-cumulant formula given in (4.4):

$$a_1^1 * \cdots * a_{k_1}^1 * \cdots * a_1^n * \cdots * a_{k_n}^n = \sum_{\nu \in \mathcal{P}(A)} \kappa_\nu.$$

We split all partitions $\mathcal{P}(A)$ into two categories: mixing partitions $\overline{\mathcal{P}}(A)$ and products of partitions $\mathcal{P}(A_i)$. In this way:

$$\begin{aligned}\sum_{\nu \in \mathcal{P}(A)} \kappa_\nu = \sum_{\nu \in \coprod_{i\in[n]} \mathcal{P}(A_i)} \kappa_\nu + \sum_{\nu \in \overline{\mathcal{P}}(A)} \kappa_\nu &= \prod_{i=1}^{n} \sum_{\nu \in \mathcal{P}(A_i)} \kappa_\nu + \sum_{\nu \in \overline{\mathcal{P}}(A)} \kappa_\nu \\ &= \left(a_1^1 * \cdots * a_{k_1}^1\right) \cdots \left(a_1^n * \cdots * a_{k_n}^n\right) + \sum_{\nu \in \overline{\mathcal{P}}(A)} \kappa_\nu.\end{aligned}$$

From the equations above we obtain the following formula:

$$\begin{aligned}&\left(a_1^1 * \cdots * a_{k_1}^1\right) \cdots \left(a_1^n * \cdots * a_{k_n}^n\right) \\ &\qquad = \left(a_1^1 * \cdots * a_{k_1}^1\right) * \cdots * \left(a_1^n * \cdots * a_{k_n}^n\right) - \sum_{\nu \in \overline{\mathcal{P}}(A)} \kappa_\nu.\end{aligned} \tag{4.11}$$

4.4.3. Cumulants of upward sequences of partitions

Each cumulant on the right-hand side of (4.11) is a $\cdot$-product of simple cumulants. We use the moment-cumulant formula in a form given below

$$a_1 \cdots a_k = a_1 * \ldots * a_k - \sum_{\substack{\nu \in \mathcal{P}([k]) \\ \nu \neq \{\{1\},\ldots,\{k\}\}}} \kappa_\nu(a_1, \ldots, a_k). \tag{4.12}$$

to replace $\cdot$-products by $*$-products and $\cdot$-products consisting of a strictly smaller number of components.

For each cumulant on the right-hand side in (4.11) we apply the procedure (4.12). As an output we get one term which is a $*$-product of cumulants and several terms of the form of a $\cdot$-product of cumulants. Observe that in each term of the second type the number of factors is strictly smaller than before applying the procedure. We apply to them this procedure iteratively as long as we have $\cdot$-terms in our extension. In the end we get a sum of the terms given by $*$-product and cumulants.

Example 4.23. Let us express $(a_1^1 * a_2^2) \cdot a_1^2$ using the procedure described above:

$$
\begin{aligned}
(a_1^1 * a_2^1) \cdot a_1^2 &\overset{(4.11)}{=} a_1^1 * a_2^1 * a_1^2 - \kappa(a_1^1, a_2^1, a_1^2) \\
&\quad -\kappa(a_1^1, a_1^2) \cdot \kappa(a_2^1) - \kappa(a_2^1, a_1^2) \cdot \kappa(a_1^1) \\
&\overset{(4.12)}{=} a_1^1 * a_2^1 * a_1^2 - \kappa(a_1^1, a_2^1, a_1^2) \\
&\quad -\kappa\left(\kappa(a_1^1, a_1^2), \kappa(a_2^1)\right) + \kappa\left(\kappa(a_2^1, a_1^2), \kappa(a_1^1)\right) \\
&\quad -\kappa(a_1^1, a_1^2) * \kappa(a_2^1) + \kappa(a_2^1, a_1^2) * \kappa(a_1^1).
\end{aligned}
$$

To formalize our idea we define nested upward sequences and theirs cumulants.

Definition 4.24. A sequence of partitions $\omega = (\nu^1 \nearrow \cdots \nearrow \nu^r)$ is said to be *upward* if

$$\nu^{i+1} \text{ is a partition of the set } \nu^i,$$

for any $1 \leq i \leq r-1$ and ν^1 is a partition of a multiset A. Moreover, if for each i the partition ν^{i+1} is non-trivial, *i.e.* $\nu^{i+1} \neq \{\nu^i\}$, it is said to be *nested*. We define the length of an upward sequence of partitions $\omega = (\nu^1 \nearrow \cdots \nearrow \nu^r)$ as the length of a sequence, and we denote $|\omega| = r$.

Let us provide a simple example.

Example 4.25. Consider a 5-element multiset $A = \{a_1, \ldots, a_5\}$ and the following nested upward sequences of partitions $\omega_1 = (\nu^1 \nearrow \nu^2)$ and $\omega_2 = (\nu^1 \nearrow \nu^2 \nearrow \nu^3)$, where:

$$\nu^1 = \Big\{\{a_1, a_4\}, \{a_2\}, \{a_3\}, \{a_5\}\Big\},$$

$$\nu^2 = \Big\{\Big\{\{a_1, a_4\}, \{a_2\}, \{a_3\}\Big\}, \Big\{\{a_5\}\Big\}\Big\},$$

$$\nu^3 = \Bigg\{\Big\{\Big\{\{a_1, a_4\}, \{a_2\}, \{a_3\}\Big\}, \Big\{\{a_5\}\Big\}\Big\}\Bigg\}.$$

We introduce the following technical notation (similar to the definition of a cumulant κ_ν for a partition ν).

$$\overline{\kappa}_\nu\{a_1, \ldots, a_n\} := \left\{\kappa(a_{i_1}, \ldots, a_{i_{|\nu_j|}})\right\}_{j=1}^{l}.$$

Definition 4.26. Let ν be a partition of a multiset $A = \{a_1, \ldots, a_n\}$. Consider an upward sequence of partitions $\omega = (\nu^1 \nearrow \cdots \nearrow \nu^r)$ such that

$$\nu^1 = \nu.$$

We define the cumulant associated to the sequence ω as follows

$$\kappa_\omega := \mathop{\Huge *}_{b \ \in \ \overline{\kappa}_{\nu^{r-1}}\left(\ldots \overline{\kappa}_{\nu^1}\left(a_1, \cdots, a_n\right)\right)} b.$$

Example 4.27. The cumulants κ_{ω_1} and κ_{ω_2} associated to the nested upward sequences of partitions ω_1 and ω_2 respectively from Example 4.25 are of the following forms:

$$\begin{aligned}
\kappa_{\omega_1} &= \kappa\Big(\kappa(a_1, a_4), \kappa(a_2), \kappa(a_3)\Big) * \kappa\Big(\kappa(a_5)\Big) \\
&= \kappa\Big(\kappa(a_1, a_4), a_2, a_3\Big) * a_5,
\end{aligned}$$

$$\begin{aligned}
\kappa_{\omega_2} &= \kappa\Bigg(\kappa\Big(\kappa(a_1, a_4), \kappa(a_2), \kappa(a_3)\Big), \kappa\Big(\kappa(a_5)\Big)\Bigg) \\
&= \kappa\Bigg(\kappa\Big(\kappa(a_1, a_4), a_2, a_3\Big), a_5\Bigg),
\end{aligned}$$

where we used the property $\kappa(x) = x$.

Definition 4.28. Consider a multiset $A = A_1 \cup \cdots \cup A_n$. Denote by $\mathcal{N}(A)$ the set of all nested upward sequences of partitions $\omega = (\nu^1 \nearrow \cdots \nearrow \nu^r)$ such that $\nu^1 \in \overline{\mathcal{P}}(A)$ is a mixing partition.

Proposition 4.29. *Consider a multiset $A = A_1 \cup \cdots \cup A_n$. Then*

$$\sum_{\nu \in \overline{\mathcal{P}}(A)} \kappa_\nu = - \sum_{\omega \in \mathcal{N}(A)} (-1)^{|\omega|} \kappa_\omega. \tag{4.13}$$

Proof. Apply procedure (4.12) iteratively to the left-hand side of Proposition 4.29. Observe that applying this iterative procedure is nothing else but summing over all nested upward sequences of partitions. The sign of the term is determined by the number of iterations. Partition ν_1 describes the first application of the procedure (this is why $\nu^1 \in \overline{\mathcal{P}}(A)$), partition ν_2 the second, and so on. □

Observe that different nested upward sequences of partitions ω may lead to the same cumulant κ_ω. The following example illustrates this phenomenon.

Example 4.30. Let $A_1 = \{a_1^1, a_2^1\}$ and $A_2 = \{a_1^2, a_2^2\}$. Consider $\omega_1, \omega_2, \omega_3 \in \mathcal{N}(A)$ given by $\omega_1 = (\nu_1^1 \nearrow \nu_1^2)$ and $\omega_2 = (\nu_2^1 \nearrow \nu_2^2 \nearrow \nu_2^3)$ and $\omega_3 = (\nu_3^1 \nearrow \nu_3^2 \nearrow \nu_3^3)$, where:

$$\nu_1^1 = \Big\{\{a_1^1, a_2^2\}, \{a_2^1, a_1^2\}\Big\}, \qquad \nu_1^2 = \Big\{\Big\{\{a_1^1, a_2^2\}, \{a_2^1, a_1^2\}\Big\}\Big\},$$

$$\nu_2^1 = \Big\{\{a_1^1, a_2^2\}, \{a_2^1\}, \{a_1^2\}\Big\}, \quad \nu_2^2 = \Big\{\Big\{\{a_1^1, a_2^2\}\Big\}, \Big\{\{a_2^1\}, \{a_1^2\}\Big\}\Big\},$$

$$\nu_2^3 = \Bigg\{\Big\{\Big\{\{a_1^1, a_2^2\}\Big\}, \Big\{\{a_2^1\}, \{a_1^2\}\Big\}\Big\}\Bigg\},$$

$$\nu_3^1 = \Big\{\{a_1^1\}, \{a_2^2\}, \{a_2^1, a_1^2\}\Big\}, \quad \nu_3^2 = \Big\{\Big\{\{a_1^1\}, \{a_2^2\}\Big\}, \Big\{\{a_2^1, a_1^2\}\Big\}\Big\},$$

$$\nu_3^3 = \Bigg\{\Big\{\Big\{\{a_1^1\}, \{a_2^2\}\Big\}, \Big\{\{a_2^1, a_1^2\}\Big\}\Big\}\Bigg\}.$$

Observe that all sequences $\omega_1, \omega_2, \omega_3$ lead to the same term up to the sign. Moreover, they are the only ones which lead to this cumulant. Observe that

$$(-1)^{|\omega_1|}\kappa_{\omega_1} + (-1)^{|\omega_2|}\kappa_{\omega_2} + (-1)^{|\omega_3|}\kappa_{\omega_3} = \kappa\Big(\kappa(a_1^1, a_2^2), \kappa(a_2^1, a_1^2)\Big).$$

With the weights given by $(-1)^{|\omega_i|}$, cumulants corresponding to sequences $\omega_1, \omega_2, \omega_3$ sum up to just one term. We will see that this is true in general.

4.4.4. Reduced forests and their colourings

To each upward nested sequence of partitions $\omega = (\nu^1 \nearrow \cdots \nearrow \nu^r)$ we shall assign a certain rooted forest with a colouring. We construct a bijection between the sequences from $\mathcal{N}(A)$ and relevant rooted forests equipped with the colourings.

Definition 4.31. Let $\omega = (\nu^1 \nearrow \cdots \nearrow \nu^r)$ be a nested sequence of partitions. Denote the elements of partition ν^i by $\nu^i = \{\nu_1^i, \ldots, \nu_{k_i}^i\}$. Let $\nu^1 = \{\nu_1^1, \ldots, \nu_{k-1}^1\}$ be a partition of $A = A_1 \cup \cdots \cup A_n$. We associate to ω a rooted forest with coloured vertices by the following procedure:

- The elements of A are leaves of the forest. We colour each of them by 0.

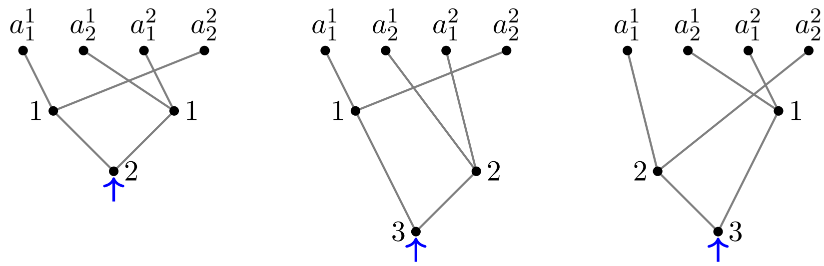


Figure 4.3 – The forests associated with $\omega_1, \omega_2, \omega_3$ from Example 4.30. All leaves are coloured by 0.

- For each element ν_j^i, where $1 \leq i \leq r$ and $1 \leq j \leq k_i$, we create a vertex and colour it by i.
- We join ν_j^{i-1} and ν_j^i if $\nu_j^{i-1} \subseteq \nu_j^i$. Similarly we join $a \in A$ and ν_j^1 if $a \in \nu_j^1$.
- We delete each vertex v which has only one descendant. We join the descendant and the parent of v.

We denote by $\Phi_1(\omega)$ the forest and by $\Phi_2(\omega)$ the colouring associated to ω.

Example 4.32. The coloured forests associated with $\omega_1, \omega_2, \omega_3$ from Ex. 4.30 are presented on Figure 4.3. Since $\omega_1, \omega_2, \omega_3$ start from one element partition, all three forests are trees.

The forest described in Definition 4.31 consists of k_r rooted trees, where k_r is a number of elements in ν^r, namely $\nu^r = \{\nu_1^r, \ldots, \nu_{k_r}^r\}$. The condition of nestedness of ω translates to the fact that each colour is used. Except for leaves, each vertex has at least two descendants. It leads to the definition of reduced forest and gap-free, weakly-mixing colouring. We mentioned their definitions in the introduction (see Definitions 4.3 and 4.8).

Lemma 4.33. *There exists a bijection Φ between the set $\mathcal{N}(A)$ of nested upward sequences starting with a mixing partition and the set of pairs (F, c) consisting of a reduced forest $F \in \mathcal{F}(A)$ of length $r \geq 1$ with a gap-free, weakly-mixing colouring $c \in \mathcal{C}_F$:*

$$\Phi : \omega \longmapsto (F, c) := (\Phi_1(\omega), \Phi_2(\omega))$$

For any nested upward sequence starting with a mixing partition $\omega \in \mathcal{N}(A)$, the following equality of cumulants holds

$$\kappa_\omega = \kappa_{\Phi_1(\omega)},$$

where $\kappa_{\Phi_1(\omega)}$ is a cumulant of a reduced forest $\Phi_1(\omega)$, see Definition 4.4.

Moreover $|\omega| = |\Phi_2(\omega)|$ i.e. *the length of the nested upward sequence is equal to the length of the corresponding colouring.*

Proof. Definition 4.31 shows already how to associate a reduced forest $F := \Phi_1(\omega)$ with the gap-free colouring $c := \Phi_2(\omega)$ to a nested upward sequence ω. The construction is done in such a way that $|c| = |\omega|$. For the reverse direction, the algorithm is easily reproducible. The condition that a nested upward sequence $\omega = (\nu^1 \nearrow \cdots \nearrow \nu^r) \in \mathcal{N}(A)$ starts with a mixing partition $\nu^1 \in \overline{\mathcal{P}}(A)$ translates to the condition of c being a weakly-mixing colouring (Definition 4.8).

In Definition 4.4 we introduced cumulant κ_F for a forest $F \in \mathcal{F}(A)$. There is an exact correspondence between this expression and the one, which is given in Definition 4.26. □

We are ready to prove Proposition 4.9 which is the purpose of this section. Let us recall its statement:

Proposition 4.9. *Let $A_1, \ldots, A_n$ be multisets consisting of elements of $\mathcal{A}$. Let A be a sum of those multisets. Then*

$$\left(a_1^1 * \cdots * a_{k_1}^1\right) \cdots \left(a_1^n * \cdots * a_{k_n}^n\right) = \sum_{F \in \mathcal{F}(A)} \kappa_F \sum_{c \in \mathcal{C}_F} (-1)^{|c|}. \tag{4.6}$$

Proof. Combining the formula (4.11) and the Proposition 4.29 lead to the following expression:

$$\left(a_1^1 * \cdots * a_{k_1}^1\right) \cdots \left(a_1^n * \cdots * a_{k_n}^n\right) = \\ \left(a_1^1 * \cdots * a_{k_1}^1\right) * \cdots * \left(a_1^n * \cdots * a_{k_n}^n\right) + \sum_{\omega \in \mathcal{N}(A)} (-1)^{|\omega|} \kappa_\omega.$$

We identify the product term on the right-hand side of the equation above with the only reduced forest of length $r = 0$. Indeed, there is just one reduced forest of length $r = 0$ and the only one gap-free, weakly-mixing vertex colouring c of it, namely the forest F consisting of separated vertices $a \in A$, each coloured by 0. The term $\left(a_1^1 * \cdots * a_{k_1}^1\right) * \cdots * \left(a_1^n * \cdots * a_{k_n}^n\right)$ is equal to the corresponding cumulant κ_F.

We replace the sum term on the right-hand side of the equation above, according to the bijection between sequences $\omega \in \mathcal{N}(A)$ and reduced forests of length $r \geq 1$ with gap-free, weakly-mixing colourings given in Lemma 4.33. □

4.5. Proof of Proposition 4.10

In this section we shall prove Proposition 4.10. For a given reduced forest F, we investigate the following sum

$$\sum_{c \in \mathcal{C}_F} (-1)^{|c|}$$

over all gap-free, weakly-mixing colourings of F, which occur in Proposition 4.9.

4.5.1. Parameter w_F of a reduced forest $F \in \mathcal{F}(A)$

We introduce an invariant w_F which determines the coefficient of κ_F. This definition was already mentioned in Section 4.1.4, we recall it below and next extend it slightly:

Definition 4.5. Consider a multiset $A = A_1 \cup \cdots \cup A_n$ and a reduced forest $F \in \mathcal{F}(A)$. We say that F is *mixing for a division* $A_1, \ldots, A_n$ (or shortly *mixing*) if for each vertex v whose descendants are all leaves, those descendants are elements of at least two distinct multisets A_i and A_j. Denote by $\overline{\mathcal{F}}(A)$ the set of all *reduced mixing forests*.

For a reduced mixing forest F we define the quantity w_F to be the number of vertices in F minus the number of leaves (see Figure 4.1).

If F is not mixing, we define $w_F := \infty$. We may also introduce number w_F inductively, according to the height of a forest.

Definition 4.34. Let F be a reduced forest. The height of a forest F is the maximum distance between one of the roots and one of its leaves. We denote this quantity by $h(F)$.

Definition 4.35 (Definition equivalent to Definition 4.5). Let F be a reduced forest and $F_1, \ldots, F_r$ its sub-forests obtained by deleting the roots of F. We define the number $w_F \in \mathbb{N} \cup \{\infty\}$ inductively on $h(F)$ as follows:

$$w_F = \begin{cases} \sum_{i=1}^{r} k_{F_i} + 1 & \text{if } h(F) \geq 2, \\ \infty & \text{if } h(F) = 1 \text{ and all descendants of some root} \\ & \text{belong to just one multiset } A_i \text{ for some } i \in [n], \\ 1 & \text{if } h(F) = 1 \text{ and for each root there are at least} \\ & \text{two descendants belonging to some two distinct} \\ & \text{multisets } A_i \text{ and } A_j, \\ 0 & \text{if } h(F) = 0. \end{cases}$$

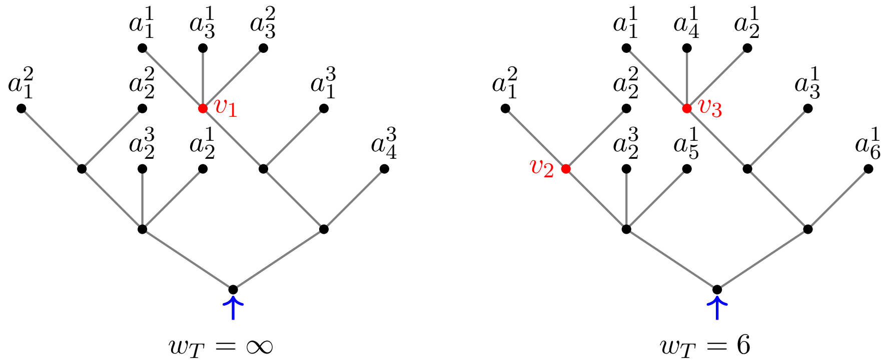


Figure 4.4 – The tree on the left-hand side is mixing. Indeed, descendants of the vertex v_1 belong to at least two multisets: A_1 and A_3. Observe that the tree on the right-hand side is not mixing. Indeed, there are two vertices of height equal to one: v_2 and v_3. All descendants of v_2 belong to A_2 and all descendants of v_3 belong to A_1.

Example 4.36. Let $A = A_1 \cup A_2 \cup A_3$. On Figure 4.4, we give an example of two forests (in particular trees) and we count the two corresponding w_F numbers. Observe that number w_F depends on the labels of the leaves in the forest F.

4.5.2. Proof of Proposition 4.10

Let us recall the statement of proposition.

Proposition 4.10. *Let $A_1, \dots, A_n$ be multisets consisting of elements of $\mathcal{A}$. Let A be a sum of those multisets. Then, for any reduced forest $F \in \mathcal{F}(A)$, the following holds:*

$$\sum_{c \in \mathcal{C}_F} (-1)^{|c|} = \begin{cases} (-1)^{w_F} & \text{if } F \in \overline{\mathcal{F}}(A), \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Proposition 4.10 is divided into two cases: either a forest F is not mixing, *i.e.* $w_F = \infty$ (Lemma 4.37), or a forest F is mixing, *i.e.* $w_F \neq \infty$ (Lemma 4.38). The next two subsections establish these two cases.

4.5.3. Proof of the not mixing case

Lemma 4.37. *Let $A_1, \dots, A_n$ be multisets consisting of elements of $\mathcal{A}$. Let A be a sum of those multisets. For any reduced forest F witch is not mixing,* *we have*

$$\sum_{c \in \mathcal{C}_F} (-1)^{|c|} = 0.$$

Proof. Since F is not mixing, there exists a vertex v such that all of its descendants are leaves, and all of them belong to just one multiset A_i for some $i \in [n]$. Consider the following partition of set $\mathcal{C}_F$:

$$\left\{\mathcal{C}_i^k\right\}_{i \in \mathbb{Z}}^{k=1,2}$$

where each $\mathcal{C}_i^1$ consists of all $c \in \mathcal{C}_F$ with $|c| = i$ and where the vertex v is coloured by its own colour; $\mathcal{C}_i^2$ consists of all $c \in \mathcal{C}_F$ with $|c| = i$ and where there is another vertex coloured by the same colour as the vertex v. We express the sum over all $c \in \mathcal{C}_F$ as follows:

$$\sum_{c \in \mathcal{C}_F} (-1)^{|c|} = \sum_{i \in \mathbb{Z}} \left(\sum_{c \in \mathcal{C}_i^1} (-1)^i + \sum_{c \in \mathcal{C}_i^2} (-1)^i \right) = \\ \sum_{i \in \mathbb{Z}} (-1)^i \left(\sum_{c \in \mathcal{C}_i^1} 1 - \sum_{c \in \mathcal{C}_{i-1}^2} 1 \right).$$

We will show the equipotency of the sets $\mathcal{C}_i^1$ and $\mathcal{C}_{i-1}^2$ from which it follows that the sum above is equal to 0 and the statement of the lemma is true.

Let us construct a bijection between $\mathcal{C}_i^1$ and $\mathcal{C}_{i-1}^2$. Take any $c \in \mathcal{C}_i^1$. Suppose that the vertex v is coloured by k. Observe that $k \geq 2$. Indeed, if v was coloured by 1, it would be the only vertex of this colour. Then, the only 1-coloured vertex would have descendants belonging to just one multiset A_i, which is in contradiction with the fact that $c \in \mathcal{C}_F$ (*i.e.* c is a weakly-mixing colouring). From $c \in \mathcal{C}_i^1$ we construct $c' \in \mathcal{C}_{i-1}^1$ as follows:

1. keep the colours of vertices coloured by $1, \ldots, k-1$ unchanged,
2. change the colours of vertices coloured by $k, \ldots, i$ to $k-1, \ldots, i-1$ respectively.

This procedure is reversible. Indeed, take $c' \in \mathcal{C}_{i-1}^1$ and suppose that vertex v is coloured by k for some $k \geq 1$. Then $c \in \mathcal{C}_i^2$ can be recovered by the following procedure:

1. do not change colours of the vertices coloured by $1, \ldots, k-1$,

2. do not change the colour of v,
3. change the colours of the vertices coloured by $k, \ldots, i-1$ to $k+1, \ldots, i$ respectively (excluding vertex v).

□

4.5.4. How to prove the mixing case?

We will prove the following lemma.

Lemma 4.38. *Let $A_1, \ldots, A_n$ be multisets consisting of elements of $\mathcal{A}$. Let A be a sum of those multisets. For any reduced mixing forest F, we have:*

$$\sum_{c \in \mathcal{C}_F} (-1)^{|c|} = (-1)^{w_F}.$$

To prove the lemma above we show a bijection between gap-free colourings of reduced trees $T \in \mathcal{T}(A)$ and gap-free colourings of reduced forests $F \in \mathcal{F}(A)$, which are not trees (see Remark 4.39). Using this bijection we can restrict proof of Lemma 4.38 just to trees. For reduced trees and their gap-free colourings we define a projection of this colourings (see Definition 4.40). We make use of the notion of projection in Lemma 4.41. Proof of Lemma 4.38 is done by induction on number of vertices in tree T and presented in Section 4.5.7.

4.5.5. Restriction to the trees

Remark 4.39. There is a natural bijection f between all reduced trees $T \in \mathcal{T}(A)$ and all reduced forests $F \in \mathcal{F}(A)$, which are not trees. This bijection is obtained by deleting the root of T (see Figure 4.1). Moreover, for a given reduced tree $T \in \mathcal{T}(A)$, there is an obvious bijection f_T between all gap-free colourings of T and all gap-free colourings of the corresponding reduced forest $f(T)$, obtained by keeping the colours of the non-deleted vertices, so that

$$|f_T(c)| = |c| - 1.$$

Additionally f_T preserves the property of being a weakly-mixing colouring.

The above statement allows us to prove Lemma 4.38 just for the case of trees $T \in \mathcal{T}(A)$ and conclude the statement for all forests $F \in \mathcal{F}(A)$. Indeed, suppose that the statement of Lemma 4.38 holds for trees. Consider a mixing forest $F \in \mathcal{F}(A)$ which is not a tree. Then, the tree $T := f^{-1}(F) \in \mathcal{T}(A)$ is also mixing, hence we can use the statement of

Lemma 4.38. Observe that $w_F = w_T - 1$. Using bijections f and f_T we get the following equality

$$\sum_{c\in\mathcal{C}_F} (-1)^{|c|} = \sum_{c\in\mathcal{C}_{f^{-1}(F)}} (-1)^{|f_T^{-1}(c)|+1} = - \sum_{c\in\mathcal{C}_T} (-1)^{|c|} = -(-1)^{w_T} = (-1)^{w_F},$$

which is the statement of Lemma 4.38 for the mixing forest $F \in \mathcal{F}(A)$.

4.5.6. Projection of a gap-free colouring

For any reduced tree $T \in \mathcal{T}(A)$ we consider sub-trees $T_1, \ldots, T_k$ formed by deleting the root of T. Number k is equal to the degree of the root. Every sub-tree T_i is also a reduced tree. Every gap-free colouring c induces also a sub-colourings $\bar{c}_1, \ldots, \bar{c}_k$ on $T_1, \ldots, T_k$. Observe that sub-colourings obtained this way are not necessarily gap-free. However, there is a canonical way to make them gap-free.

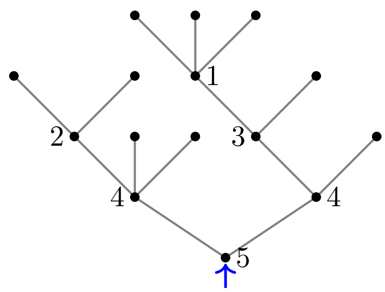
Definition 4.40. Let T be a reduced tree with a gap-free colouring c. Let $\bar{c}_1, \ldots, \bar{c}_k$ be the induced colourings on sub-trees $T_1, \ldots, T_k$ formed by deleting the root of T. For some $i \in [k]$, let $j_0^i < \cdots < j_l^i$ be the sequence of colours used in the colouring $\bar{c}_i$. By replacing each j_n^i by n in the colouring $\bar{c}_i$ we obtain a gap-free colouring, which we denote by c_i. We say that c_i as an *i-th projection of the colouring* c and denote it as $\mathrm{p}_i(c) := c_i$, see Figure 4.5.

Lemma 4.41. *Let T be a reduced mixing tree of height $h(T) \geq 2$. Denote by $T_1, \ldots, T_r$ all sub-trees obtained by deleting the root of T. Let $c_1, \ldots, c_r$ be gap-free colourings of $T_1, \ldots, T_r$ respectively. Then the following equality holds:*

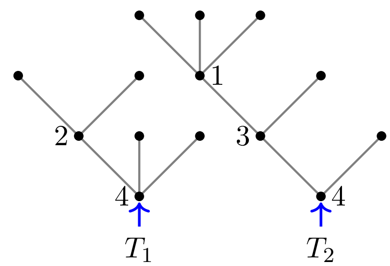
$$\sum_{\substack{c\in\mathcal{C}_T \\ \mathrm{p}_i(c)=c_i}} (-1)^{|c|} = -\prod_{i=1}^{r} (-1)^{|c_i|}.$$

Proof of Lemma 4.41. Observe that if T is the mixing tree, then any gap-free colouring c belongs to $\mathcal{C}_T$. Indeed, take any vertex v coloured by 1. Clearly, its descendants are leaves labelled by elements of at least two distinct multisets A_i and A_j (by assumption that T is mixing). The existence of such a vertex implies that $c \in \mathcal{C}_T$.

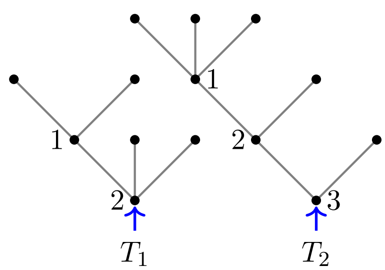
The proof is divided into three steps: first, we construct a bijection between the gap-free colouring c projecting onto $c_1, \ldots, c_r$ and some integer



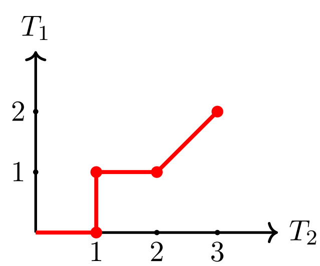
(a) Colouring c of a tree T uses the colours $\{0, 1, 2, 3, 4, 5\}$.



(b) Colouring $\bar{c}_1$ of the tree T_1 uses the colours $\{0, 2, 4\}$. Colouring $\bar{c}_2$ of the tree T_2 uses the colours $\{0, 1, 3, 4\}$.



(c) Projections $\mathrm{p}_1(c)$ and $\mathrm{p}_2(c)$ use colours $\{0, 1, 2\}$ and $\{0, 1, 2, 3\}$ respectively.



(d) Path ρ constructed from c in proof of Lemma 4.41.

Figure 4.5 – (a) A reduced tree T with a gap-free colouring c. (b) By deleting the root we obtain two reduced subtrees: T_1 and T_2 with inherited colourings: $\bar{c}_1$ and $\bar{c}_2$. Observe that they are not gap-free. (c) However, the procedure given in Definition 4.40 describes the canonical way of producing a gap-free colourings $\mathrm{p}_1(c)$ and $\mathrm{p}_2(c)$. (d) Moreover, for the colouring c we present an associated path ρ which will be introduced in a proof of Lemma 4.41.

paths in $\mathbb{N}^r$; second, we introduce a generating function of these paths and characterise it by recursion on the endpoints and some boundary condition; finally, we find a function satisfying those conditions.

Step 1. Let us recall that any gap-free colouring c of T induces colourings $\bar{c}_i$ on T_i, from which we deduce a gap-free colouring c_i of T_i (Definition 4.40). We shall construct a bijection between all gap-free colourings c of T projecting on $c_1, \dots, c_r$ and all integer paths ρ such that:

- ρ connects $(0, \dots, 0)$ and $(|c_1|, \dots, |c_r|) \in \mathbb{N}^r$,
- each step of ρ is of the following form: $(k_1^n, \dots, k_r^n) \in \{0,1\}^r \backslash (0, \dots, 0)$.

Denote the class of such paths by $\mathcal{P}_{|c_1|,\dots,|c_r|}$. Moreover the construction is done in such a way that $|c| = |\rho| + 1$, where by $|\rho|$ we denote the number of steps in ρ.

For a gap-free colouring c we construct a path ρ starting from $(0, \dots, 0) \in \mathbb{N}^r$ by the following procedure: the n-th step of ρ is of the form $(k_1^n, \dots, k_r^n)$ where:

$$k_i^n = \begin{cases} 1 & \text{if colour } n \text{ appears in } \bar{c}_i, \\ 0 & \text{if it does not.} \end{cases}$$

An example of such path is presented on Figure 4.5.

The procedure described above is reversible. Indeed, take a path ρ between $(0, \dots, 0)$ and $(|c_1|, \dots, |c_r|) \in \mathbb{N}^r$. Suppose that the n-th step is of the form:

$$(k_1^n, \dots, k_r^n) \in \{0,1\}^r \setminus (0, \dots, 0)\,.$$

We can assign to the path ρ a colouring c by the following procedure. Let $(x_1, \dots, x_r)$ be an endpoint of ρ after the n-th step. We colour each vertex $v \in F_i$ by n if v was coloured by x_i in colouring c_i and $k_i^n \neq 0$. We colour the root by $|\rho| + 1$.

Step 2. The bijection from *Step 1* was constructed in such a way that $|c| = |\rho| + 1$. Observe that

$$\sum_{\substack{c \in \mathcal{C}_T \\ \mathrm{p}_i(c) = c_i}} (-1)^{|c|} = \sum_{\rho \in \mathcal{P}_{|c_1|,\dots,|c_r|}} (-1)^{|\rho|+1}\,.$$

Let us define a function $F : \mathbb{Z}^r \longrightarrow \mathbb{Z}$:

$$F : (x_1, \dots, x_r) \longmapsto \sum_{\rho \in \mathcal{P}_{x_1,\dots,x_r}} (-1)^{|\rho|+1}\,.$$

Observe that:

I. for all $(x_1,\ldots,x_r)\notin\mathbb{N}^r$, $F(x_1,\ldots,x_r)=0$,

II. $F(0,\ldots,0)=-1$,

III. for all $(x_1,\ldots,x_r)\in\mathbb{N}^r\backslash(0,\ldots,0)$, the function F satisfies the following recursive formula:

$$F(x_1,\ldots,x_r)=-\sum_{\substack{X\subseteq[r]\\ X\neq\emptyset}} F(\overline{x}_1^X,\ldots,\overline{x}_r^X),$$

where $\overline{x}_i^X=\begin{cases} x_i & \text{if } i\notin X\\ x_i-1 & \text{if } i\in X\end{cases}$

Let us shortly comment on this observation. There are no paths connecting $(0,\ldots,0)$ with points $(x_1,\ldots,x_r)\notin\mathbb{N}^r$ using the set of steps which is non-negative (Observation I). There is just one path connecting point $(0,\ldots,0)$ to itself: the empty path. Its length is equal to 0 (Observation II). Consider all possibilities for the last step in path ρ. It is equivalent to choosing indices $X\subset[r]$, $X\neq\emptyset$ and summing over all paths ending in $(\overline{x}_1^X,\ldots,\overline{x}_r^X)$ multiplied by -1, because we count the sign of the path (Observation III).

Those three statements about function F define it uniquely. The recursive formula gives us the way to compute $F(x_1,\ldots,x_r)$ inductively according to $\sum_{i=1}^{r} x_i$. The first and the second observation give us the starting point for our induction, namely the values $F(x_1,\ldots,x_r)$ for $\sum_{i=1}^{r} x_i=0$.

Step 3. We show now that the function $G:\mathbb{Z}^r\longrightarrow\mathbb{Z}$:

$$G:(x_1,\ldots,x_r)\longmapsto\begin{cases} -\prod_{i=1}^{r}(-1)^{x_i} & \text{if for all } i: x_i\geq 0,\\ 0 & \text{otherwise.}\end{cases}$$

satisfies all three properties I, II, III mentioned in *Step 2.* Hence, those two functions F and G are equal. By connecting the results of each step, we get the statement of the lemma:

$$\begin{aligned}\sum_{\substack{c\in\mathcal{C}_T\\ p_i(c)=c_i}}(-1)^{|c|} &= \sum_{\rho\in\mathcal{P}_{(|c_1|,\ldots,|c_r|)}}(-1)^{|\rho|+1}=F(|c_1|,\ldots,|c_r|)=\\ &= G(|c_1|,\ldots,|c_r|)=-\prod_{i=1}^{r}(-1)^{|c_i|}.\end{aligned}$$

We shall show that function G satisfies three properties I, II, III. Clearly, it satisfies I, II. In order to show that the recursive formula also holds, take any $(x_1,\ldots,x_r)$: $\underset{i}{\forall}\, x_i \geq 0$ and $\sum\limits_{i=1}^{r} x_i > 0$.

Define the set Y consisting of boundary indices i of the point $(x_1,\ldots,x_r)$. More precisely, define $Y \subset [r]$ as follows: $\begin{cases} i \in Y & \text{if } x_i > 0 \\ i \notin Y & \text{if } x_i = 0 \end{cases}$.

In order to show that G satisfies III, we have to show the vanishing of the following sum:

$$\begin{aligned} G(x_1,\ldots,x_r) + \sum_{\substack{X\subset[r]\\ X\neq\emptyset}} G(\overline{x}_1^X,\ldots,\overline{x}_r^X) &= \sum_{X\subset[r]} G(\overline{x}_1^X,\ldots,\overline{x}_r^X) = \\ &= \sum_{X\subset Y} G(\overline{x}_1^X,\ldots,\overline{x}_r^X) + \sum_{X\not\subset Y} G(\overline{x}_1^X,\ldots,\overline{x}_r^X). \end{aligned} \tag{4.14}$$

Observe that summants of the sum over $X \not\subset Y$ are equal to 0. From $X \not\subset Y$ it follow that there exists $i \in [r]$ such that $i \in X$ and $i \notin Y$. It means that $\overline{x}_i^X = -1$ and by definition $G(\overline{x}_1^X,\ldots,\overline{x}_r^X) = 0$.

Observe that the summants in the sum over $X \subset Y$ are of the form $-\prod_{i=1}^{r}(-1)^{\overline{x}_i^X}$. Indeed, from $X \subset Y$ it follows that $\overline{x}_i^X \geq 0$ for all $i \in [r]$. We use this observation to show the vanishing of the sum in (4.14):

$$\begin{aligned} (4.14) = \sum_{X\subset Y} -\prod_{i=1}^{r}(-1)^{\overline{x}_i^X} &= -\sum_{X\subset Y}(-1)^{\sum\limits_{i=1}^{r} x_i - |X|} \\ &= (-1)^{\sum\limits_{i=1}^{r} x_i} \cdot \sum_{i=0}^{|Y|} \binom{|Y|}{i} \cdot (-1)^i = 0. \end{aligned}$$

□

4.5.7. Proof of Lemma 4.38

Remark 4.42. Let T be a reduced mixing tree such that $h(T) \leq 2$. Let $T_1,\ldots,T_r$ be sub-trees obtained from F by deleting the roots. For any colouring $c \in \mathcal{C}_F$ the projection $c_i := \mathrm{p}_i(c)$ is in $\mathcal{C}_{T_i}$ for each $i \in [r]$.

Indeed, by definition, $c_i = \mathrm{p}_i(c)$ are gap-free colourings of F_i. Take any vertex v coloured by 1 in c_i. Its descendants are leaves. Hence $w_F \neq \infty$, so also $k_{F_i} \neq \infty$. That means that descendants of v belong to at least two distinct multisets A_i. The existence of such vertex implies that $c \in \mathcal{C}_{T_i}$.

Proof of Proposition 4.38. We will use induction on the height of tree $T \in \mathcal{T}(A)$.

We cannot begin with a tree of height 0, namely a one-vertex tree $T = \bullet$, because there is no such s tree if $|A| \geq 2$.

Induction base. We begin from a tree T of height one, namely consisting just of the root and leaves. We have exactly one gap-free colouring of length 1, namely leaves are coloured by 0 and the root by 1. The claim follows immediately.

Induction step. Let $n \geq 2$. Suppose now that the statement of Lemma 4.38 is true for any tree T of the height $h(T) \leq n-1$. We will show that it is also true for any tree of height equal to $n \geq 2$. Take such a tree T. Denote by $T_1, \ldots, T_r$ its sub-trees obtained from T by deleting the root. Clearly for every T_i, the $h(T_i) \leq n-1$ and we can use the induction hypothesis for them. We have:

$$
\begin{aligned}
\sum_{c \in \mathcal{C}_T} (-1)^{|c|} &\overset{\text{Remark 4.42}}{=} \sum_{\substack{c_1 \in \mathcal{C}_{T_1} \\ \cdots \\ c_r \in \mathcal{C}_{T_r}}} \sum_{\substack{c \in \mathcal{C}_T \\ \mathrm{p}_i(c) = c_i}} (-1)^{|c|} \\
&\overset{\text{Lemma 4.41}}{=} - \sum_{\substack{c_1 \in \mathcal{C}_{T_1} \\ \cdots \\ c_r \in \mathcal{C}_{T_r}}} \prod_{i=1}^{r} (-1)^{|c_i|} = - \prod_{i=1}^{r} \sum_{c_i \in \mathcal{C}_{T_i}} (-1)^{|c_i|} \\
&\overset{\text{Induction}}{=} - \prod_{i=1}^{r} (-1)^{k_{T_i}} \\
&\overset{\text{Definition 4.35}}{=} (-1)^{k_T},
\end{aligned}
$$

which proves the statement for any tree of height equal to n. $\square$

Appendix A

Top-degree parts in the Matchings-Jack Conjecture and the b-Conjecture

We shall prove that our result about the top-degree part in the Matchings-Jack Conjecture presented in Theorem 2.5 and the result of Dołęga [Doł17c, Theorem 1.5] about the top-degree part in b-Conjecture are equivalent.

First note that the polynomials $c^\lambda_{\pi,\sigma}$ and $h^\lambda_{\pi,\sigma}$ are related as follows

$$\sum_{n\geq 1} t^n \sum_{\lambda,\pi,\sigma\vdash n} h^\lambda_{\pi,\sigma} p_\pi(\mathbf{x}) p_\sigma(\mathbf{y}) p_\lambda(\mathbf{z}) = \alpha t \frac{\partial}{\partial t} \log\left(\sum_{n\geq 1} t^n \sum_{\lambda,\pi,\sigma\vdash n} \frac{c^\lambda_{\pi,\sigma}}{\alpha^{\ell(\lambda)} z_\lambda} p_\pi(\mathbf{x}) p_\sigma(\mathbf{y}) p_\lambda(\mathbf{z})\right) \tag{A.1}$$

and

$$\sum_{n\geq 1} t^n \sum_{\lambda,\pi,\sigma\vdash n} \frac{c^\lambda_{\pi,\sigma}}{\alpha^{\ell(\lambda)} z_\lambda} p_\pi(\mathbf{x}) p_\sigma(\mathbf{y}) p_\lambda(\mathbf{z}) = \exp\left(\sum_{n\geq 1} \frac{1}{\alpha n} t^n \sum_{\lambda,\pi,\sigma\vdash n} h^\lambda_{\pi,\sigma} p_\pi(\mathbf{x}) p_\sigma(\mathbf{y}) p_\lambda(\mathbf{z})\right), \tag{A.2}$$

see (2.1) and (2.2).

In Theorem 2.5 we showed that the leading coefficient of $c^{\lambda}_{\pi,\sigma}$ can be expressed in the following way:

$$\left[\beta^{d(\pi,\sigma;\lambda)}\right] c^{\lambda}_{\pi,\sigma} = \left| M \in M^{\lambda;\lambda}_{\pi,\sigma} : M \text{ is unhandled } \right|$$

where $M^{\lambda;\lambda}_{\pi,\sigma}$ is the set of λ-*lists of unicellular maps* with the white and black vertices distribution given by π and σ respectively.

On the other hand, Dołęga [Doł17c, Theorem 1.5] showed that the leading coefficient of $h^{(n)}_{\pi,\sigma}$ can be expressed in the following way:

$$\left[\beta^{d(\pi,\sigma;(n))}\right] h^{(n)}_{\pi,\sigma} = \left| M \in M^{(n)}_{\pi,\sigma} : M \text{ is unhandled } \right|$$

where $M^{(n)}_{\pi,\sigma}$ is the set of *unicellular maps* with the white and black vertices distribution given by π and σ respectively.

Remark A.1. Observe that multiplication of power-sum symmetric functions expresses as follows

$$p_{\lambda_1}(\mathbf{z}) \cdot p_{\lambda_1}(\mathbf{z}) = p_{\lambda_1 \cup \lambda_1}(\mathbf{z})$$

in the terms of concatenations of relevant partitions.

We investigate the $\left[p_{(n)}(\mathbf{z})\right]$ coefficient in both sides of (A.1). We have

$$t^n \sum_{\pi,\sigma \vdash n} h^{(n)}_{\pi,\sigma} p_\pi(\mathbf{x}) p_\sigma(\mathbf{y}) = \alpha t \frac{\partial}{\partial t} \left(t^n \sum_{\pi,\sigma \vdash n} \frac{c^{(n)}_{\pi,\sigma}}{\alpha z_{(n)}} p_\pi(\mathbf{x}) p_\sigma(\mathbf{y}) \right) = \alpha n t^n \sum_{\pi,\sigma \vdash n} \frac{c^{(n)}_{\pi,\sigma}}{\alpha n} p_\pi(\mathbf{x}) p_\sigma(\mathbf{y})$$

hence $c^{(n)}_{\pi,\sigma}$ and $h^{(n)}_{\pi,\sigma}$ are equal.

Since $c^{(n)}_{\pi,\sigma} = h^{(n)}_{\pi,\sigma}$, it might seem that our result extends the result of Dołęga. However, a more subtle analysis of relationships between the coefficients of $c^{\lambda}_{\pi,\sigma}$ and $h^{\lambda}_{\pi,\sigma}$ shows that both results are equivalent.

The power series expansion of the exponent function in (A.2) gives us

$$\sum_{n \geq 1} t^n \sum_{\lambda,\pi,\sigma \vdash n} \frac{c^{\lambda}_{\pi,\sigma}}{\alpha^{\ell(\lambda)} z_\lambda} p_\pi(\mathbf{x}) p_\sigma(\mathbf{y}) p_\lambda(\mathbf{z}) = \sum_{k \geq 0} \frac{1}{k!} \left(\sum_{s \geq 1} \frac{1}{s} t^s \sum_{\lambda,\pi,\sigma \vdash s} \frac{h^{\lambda}_{\pi,\sigma}}{\alpha} p_\pi(\mathbf{x}) p_\sigma(\mathbf{y}) p_\lambda(\mathbf{z}) \right)^k . \quad \text{(A.3)}$$

We denote by $\mathcal{P}_k^{\lambda,\pi,\sigma}$ the set of triplets of lists of partitions

$$\left(\left(\lambda^1,\dots,\lambda^k\right),\left(\pi^1,\dots,\pi^k\right),\left(\sigma^1,\dots,\sigma^k\right)\right)$$

such that

$$\bigcup_{i=1}^k \lambda^i=\lambda, \qquad \bigcup_{i=1}^k \mu^i=\mu, \qquad \bigcup_{i=1}^k \sigma^i=\sigma$$

and for each i we have $|\lambda^i|=|\pi^i|=|\sigma^i|$.

Let us investigate the $[p_\pi(\mathbf{x})p_\sigma(\mathbf{y})p_\lambda(\mathbf{z})]$ coefficient in both sides of (A.3). We have

$$t^n\frac{c^\lambda_{\pi,\sigma}}{\alpha^{\ell(\lambda)}z_\lambda}=t^n\sum_{k:1\le k\le \ell(\lambda)}\frac{1}{k!}\sum_{\substack{\left((\lambda^1,\dots,\lambda^k),\right.\\(\pi^1,\dots,\pi^k),\\ \left.(\sigma^1,\dots,\sigma^k)\right)\in\mathcal{P}_k^{\lambda,\pi,\sigma}}}\prod_{i=1}^k\frac{1}{|\lambda^i|}\frac{h^{\lambda^i}_{\pi^i,\sigma^i}}{\alpha}. \tag{A.4}$$

Dołęga and Féray [DF17, Theorem 1.2] gave the following bound on the degree

$$\deg h^{\lambda^i}_{\pi^i,\sigma^i}\le|\lambda^i|+2-\ell(\lambda^1)-\ell(\pi^i)-\ell(\sigma^i).$$

Hence, each summand of the first sum on the right-hand side of (A.4) has degree equal to at most

$$n+k-\ell(\lambda)-\ell(\pi)-\ell(\sigma),$$

and the maximal bound may be achieved only for summands corresponding to $k=\ell(\lambda)$. For such a summand, its bound on the degree is the same as the bound on the degree for the left-hand side of (A.4) given by (2.3). We have

$$\frac{1}{z_\lambda}\left[\alpha^{d(\pi,\sigma;\lambda)}\right]c^\lambda_{\pi,\sigma}=$$
$$\frac{1}{\ell(\lambda)!}\sum_{\substack{\left((\lambda^1,\dots,\lambda^{\ell(\lambda)}),\right.\\(\pi^1,\dots,\pi^{\ell(\lambda)}),\\ \left.(\sigma^1,\dots,\sigma^{\ell(\lambda)})\right)\in\mathcal{P}_{\ell(\lambda)}^{\lambda,\pi,\sigma}}}\prod_{i=1}^{\ell(\lambda)}\frac{1}{|\lambda^i|}\left[\alpha^{|\lambda_i|+1-\ell(\pi)-\ell(\sigma)}\right]h^{\lambda_i}_{\pi^i,\sigma^i}.$$

For a partition $\lambda=(\lambda_1,\dots,\lambda_k)$, denote by $\mathcal{C}_\lambda$ the set of all *compositions* of a type λ, *i.e.* the set of all lists $(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(k)})$, for some $\sigma\in\mathfrak{S}(n)$. Observe that

$$|\mathcal{C}_\lambda|=\frac{\ell(\lambda)!}{\sum_i m_i(\lambda)!}.$$

Observe that for $k = \ell(\lambda)$ the first list in any triplet from $\mathcal{P}_k^{\lambda,\pi,\sigma}$ is a composition of a type of the Young diagram λ. We have

$$\frac{1}{z_\lambda}\left[\alpha^{d(\pi,\sigma;\lambda)}\right] c_{\pi,\sigma}^{\lambda} = \frac{1}{\ell(\lambda)!}|\mathcal{C}_\lambda| \sum_{\substack{\left(\left((\lambda_1),\dots,(\lambda_{\ell(\lambda)})\right),\\ \left(\pi^1,\dots,\pi^{\ell(\lambda)}\right),\\ \left(\sigma^1,\dots,\sigma^{\ell(\lambda)}\right)\right)\in\mathcal{P}_{\ell(\lambda)}^{\lambda,\pi,\sigma}}} \prod_{i=1}^{\ell(\lambda)} \frac{1}{\lambda_i}\left[\alpha^{\lambda_i+1-\ell(\pi)-\ell(\sigma)}\right] h_{\pi^i,\sigma^i}^{(\lambda_i)}$$

and hence

$$\left[\alpha^{d(\pi,\sigma;\lambda)}\right] c_{\pi,\sigma}^{\lambda} = \sum_{\substack{\left(\left((\lambda_1),\dots,(\lambda_{\ell(\lambda)})\right),\\ \left(\pi^1,\dots,\pi^{\ell(\lambda)}\right),\\ \left(\sigma^1,\dots,\sigma^{\ell(\lambda)}\right)\right)\in\mathcal{P}_{\ell(\lambda)}^{\lambda,\pi,\sigma}}} \prod_{i=1}^{\ell(\lambda)} \left[\alpha^{\lambda_i+1-\ell(\pi)-\ell(\sigma)}\right] h_{\pi^i,\sigma^i}^{(\lambda_i)}$$

Dołęga's result [Doł17c, Theorem 1.5] shows us that

$$\left[\alpha^{\lambda_i+1-\ell(\pi)-\ell(\sigma)}\right] h_{\pi^i,\sigma^i}^{(\lambda_i)} = \left|M \in M_{\pi^i,\sigma^i}^{(\lambda_i)} : M \text{ is unhandled}\right|.$$

Directly from the definition of $\mathcal{P}_{\ell(\lambda)}^{\lambda,\pi,\sigma}$ we obtain that

$$\left[\alpha^{d(\pi,\sigma;\lambda)}\right] c_{\pi,\sigma}^{\lambda} = \left|M \in M_{\pi,\sigma}^{\lambda;\lambda} : M \text{ is unhandled}\right|,$$

which allows us to conclude the equivalence of both results.

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