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Algebra of smooth operators

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Abstract

The aim of this dissertation is to investigate the properties of the noncommutative Fréchet algebra with involution, called the algebra of smooth operators. This algebra is isomorphic as a Fréchet space to the commutative algebra s of rapidly decreasing sequences (isomorphic also to the well-known Schwartz space of smooth rapidly decreasing functions), and thus it is a kind of noncommutative analogue of the algebra s.

A significant part of the dissertation is devoted to the description and classification of the closed commutative *-subalgebras of the algebra of smooth operators. For instance, we show that such a subalgebra is isomorphic to a closed *-subalgebra of the algebra s if and only if it is isomorphic (as a Fréchet space) to a complemented subspace of s. We also find the multplier algebra of the algebra of smooth operators, prove theorems on spectral and Schmidt representations of elements of this algebra and show that there is a Hölder continuous functional calculus for normal smooth operators. Most of the proofs are based on the theory of bounded and unbounded operators on a Hilbert space and the theory of nuclear Fréchet spaces.

Streszczenie

Celem rozprawy jest zbadanie własności nieprzemiennej algebry Frécheta z inwolucją, zwanej algebrą operatorów gładkich. Algebra ta jest izomorficzna jako przestrzeń Frécheta z przemienną algebrą s ciągów szybko malejących do zera (izomorficzną także z dobrze znaną przestrzenią Schwartza gładkich funkcji szybko malejących) i w ten sposób jest pewnego rodzaju nieprzemiennym odpowiednikiem algebry s.

Znaczna część rozprawy jest poświęcona opisie i klasyfikacji domkniętych przemiennych *-podalgebr algebry operatorów gładkich. Na przykład, pokazujemy, że taka podalgebra jest izomorficzna z domkniętą *-podalgebrą algebry s wtedy, i tylko wtedy, gdy jest izomorficzna (jako przestrzeń Frécheta) z pewną dopełnialną podprzestrzenią s. Ponadto znajdujemy algebrę multiplikatorów algebry operatorów gładkich, dowodzimy twierdzeń o reprezentacji spektralnej i reprezentacji Schmidta elementów tej algebry oraz pokazujemy, że istnieje hölderowsko ciągły rachunek funkcyjny dla gładkich operatorów normalnych. Większość dowodów jest oparta na teorii ograniczonych i nieograniczonych operatorów na przestrzeni Hilberta oraz teorii nuklearnych przestrzeni Frécheta.

Contents

Introduction

The aim of this dissertation is to investigate the properties of some specific noncommutative Fréchet algebra with involution, called the algebra of smooth operators. The most important features of this algebra are the following:

- it is isomorphic as a Fréchet space to the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth rapidly decreasing functions on the real line;
- it has several representations as algebras of operators acting between natural spaces of distributions and functions;
- it is a dense *-subalgebra of the C^*-algebra $\mathcal{K}(\ell_2)$ of compact operators on ℓ_2;
- it is even contained in the class of Hilbert-Schmidt operators, and thus it is a unitary space;
- the operator C^*-norm $||\cdot||_{\ell_2\to\ell_2}$ is so-called dominating norm on that algebra (the dominating norm property is a key notion in the structure theory of nulcear Fréchet spaces – see discussion below).

From the philosophical point of view, the algebra of smooth operators can be seen as a noncommutative analogue of the commutative algebra s of rapidly decreasing sequences (isomorphic as a Fréchet space to $\mathcal{S}(\mathbb{R})$). Its structure (a Fréchet algebra with a natural noncommutative multiplication, the hermitian adjoint and the Hilbert-Schmidt scalar product) is essentially richer than the structure of s (a commutative Fréchet algebra with pointwise multiplication and conjugation, scalar product inherited from ℓ_2) and it involves many natural and interesting problems.

The algebra of smooth operators is defined as a Fréchet *-algebra $\mathcal{L}(s', s)$ of continuous linear operators from the LB-space (an inductive limit of Banach spaces)

$$s' := \left\{\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_q' := \left(\sum_{j=1}^{\infty} |\xi_j|^2 j^{-2q}\right)^{1/2} < \infty \text{ for some } q \in \mathbb{N}_0\right\}$$

of slowly increasing sequences to the Fréchet space

$$s := \left\{\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_q := \left(\sum_{j=1}^{\infty} |\xi_j|^2 j^{2q}\right)^{1/2} < \infty \text{ for every } q \in \mathbb{N}_0\right\}$$

of rapidly decreasing sequences. The space s' is isomorphic to the strong dual of the Fréchet space s (i.e. the space of all continuous linear functionals on s with the topology of uniform convergence on bounded subsets of s) and the isomorphism is defined via the "scalar product"

$$\langle \xi, \eta\rangle := \sum_{j=1}^{\infty} \xi_j \overline{\eta_j},$$

where $\xi \in s$ and $\eta \in s'$ (see [20, Ch. 22-25] for the general theory of Fréchet spaces and their duals). It turns out that $\mathcal{L}(s', s)$ with the topology of uniform convergence on bounded sets in s' is a Fréchet space and it is isomorphic (as a Fréchet space) to s. Moreover, one can easily show that $(||\cdot||_q)_{q\in\mathbb{N}_0}$,

$$||x||_q := \sup_{|\xi|'_q \leq 1} |x\xi|_q,$$

is a fundamental system of norms on $\mathcal{L}(s', s)$ (see Proposition 1.9).

It is worth mentioning that s is a nuclear space (i.e. every unconditionally convergent series of elements of s is absolutely convergent, see also [20, Def. on p. 344]) and, moreover, by the Kōmura-Kōmura theorem (see e.g. [20, Cor. 29.9]), it is universal in the class of all nuclear Fréchet spaces: more precisely, a Fréchet space is nuclear if and only if it is isomorphic to some closed subspace of $s^{\mathbb{N}}$. We will see later that s is isomorphic (as a Fréchet space) to many important classical spaces of analysis.

As we have seen above, from the point of view of Fréchet spaces, there is no difference between $\mathcal{L}(s', s)$ and s. Things dramatically change when we endow $\mathcal{L}(s', s)$ and s with additional algebraic operations: multiplication and involution. Clearly, s is a Fréchet *-algebra (i.e. a Fréchet space with involution and jointly continuous multiplication) when equipped with pointwise multiplication and termwise conjugation. Let us introduce multiplication and involution on $\mathcal{L}(s', s)$. First observe that $\mathcal{L}(s', s)$ is embedded in the C^*-algebra $\mathcal{L}(\ell_2)$ of continuous linear operators on ℓ_2 via the (continuous, linear, injective) map

$$\iota\colon \mathcal{L}(s', s) \hookrightarrow \mathcal{L}(\ell_2), \quad \iota(x) := j_1 \circ x \circ j_2,$$

where $j_1\colon s \hookrightarrow \ell_2$ and $j_2\colon \ell_2 \hookrightarrow s'$ are (continuous) identity maps. Now, multiplication and involution on $\mathcal{L}(s', s)$ are inherited from $\mathcal{L}(\ell_2)$ as the composition of operators (note that since $s \hookrightarrow s'$, we can compose operators in $\mathcal{L}(s', s)$) and the hermitian adjoint. With these operations $\mathcal{L}(s', s)$ becomes a Fréchet *-algebra. Moreover, the algebras s and $\mathcal{L}(s', s)$ are both locally m-convex, i.e. they admit fundamental systems of submultiplicative seminorms (see e.g. [25, Lemma 2.2]); in fact, $(|\cdot|_q)_{q\in\mathbb{N}_0}$ and $(||\cdot||_q)_{q\in\mathbb{N}_0}$ are submultiplicative systems of norms on s and $\mathcal{L}(s', s)$, respectively. Clearly, $\mathcal{L}(s', s)$, being noncommutative, is not isomorphic as a Fréchet *-algebra to s. Nevertheless, there are many ways to embed s into $\mathcal{L}(s', s)$ (as a closed *-subalgebra), e.g. as the algebra of diagonal operators:

$$\left\{ \sum_{k=1}^{\infty} \xi_k \langle \cdot, e_k \rangle e_k \colon (\xi_k)_{k\in\mathbb{N}} \in s \right\},$$

here e_k denotes the vector in $\mathbb{C}^{\mathbb{N}}$ whose k-th coordinate equals 1 and the others equal 0.

It appears that the embedding $\iota\colon \mathcal{L}(s', s) \hookrightarrow \mathcal{L}(\ell_2)$ acts in fact into the C^*-algebra $\mathcal{K}(\ell_2)$ of compact operators on ℓ_2 and $\iota(\mathcal{L}(s', s))$ is dense in $\mathcal{K}(\ell_2)$. Thus $\mathcal{L}(s', s)$ can be seen as a dense *-subalgebra of $\mathcal{K}(\ell_2)$. We can show even more: $\mathcal{L}(s', s)$ is (properly) contained in the intersection of all Schatten classes $\mathcal{S}_p(\ell_2)$ over $p > 0$. In particular, $\mathcal{L}(s', s)$ is contained in the Hilbert space $\mathcal{HS}(\ell_2)$ of Hilbert-Schmidt operators with the scalar product defined by

$$\langle x, y \rangle_{\mathcal{HS}} := \sum_{k=1}^{\infty} \langle xe_k, ye_k \rangle,$$

and thus $\mathcal{L}(s', s)$ is a unitary space.

It is worth comparing the algebras mentioned above with their commutative prototypes; this is done by the following diagram with the horizontal continuous embeddings of algebras:

$$\begin{tikzcd} s \arrow[r, hook] \arrow[d, leftrightsquigarrow] & \bigcap_{p>0} \ell_p \arrow[r, hook] \arrow[d, leftrightsquigarrow] & \ell_2 \arrow[r, hook] \arrow[d, leftrightsquigarrow] & c_0 \arrow[d, leftrightsquigarrow] \\ \mathcal{L}(s', s) \arrow[r, hook] & \bigcap_{p>0} \mathcal{S}_p(\ell_2) \arrow[r, hook] & \mathcal{HS}(\ell_2) \arrow[r, hook] & \mathcal{K}(\ell_2). \end{tikzcd}$$

The "vertical correspondences", mean, for example, that every monotonical element of the commutative algebras from the first row is a sequence of singular numbers of some element of their noncommutative analogues, and vice versa. Moreover, algebras from the first row are embedded into the corresponding algebras from the second row (e.g. as the algebras of diagonal operators). Let us also recall that $\mathcal{L}(s', s) \cong s$ (as Fréchet spaces) and $\mathcal{HS}(\ell_2)$ is unitarily isomorphic to ℓ_2.

The algebra $\mathcal{L}(s', s)$ is also called the *algebra of smoothing (compact) operators*. In order to explain why this name is suitable, let us recall that the space s can be represented in many ways by function spaces (usually spaces of smooth functions) which appear naturally in analysis. For example, the space s is isomorphic as a Fréchet space to:

- the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of smooth rapidly decreasing functions on $\mathbb{R}^n$,
- the space $C^\infty(M)$ of smooth functions on an arbitrary compact C^∞-manifold M,
- the space $C^\infty[0,1]$ of smooth functions on the interval $[0,1]$,
- the space $A^\infty(\mathbb{D})$ of holomorphic functions on the unit disc with the smooth extension to the boundary,

all equipped with their natural topologies. Note that the space s and all of the spaces above are also commutative Fréchet algebras (with pointwise multiplication), sometimes with involution (conjugation of functions), but they are not isomorphic as algebras to s (see Corollary 4.6).

Representations of the Fréchet space s above lead to natural representations of the Fréchet *-algebra $\mathcal{L}(s', s)$. More precisely, $\mathcal{L}(s', s)$ is isomorphic as a Fréchet *-algebra to the following algebras of continuous linear operators:

- $\mathcal{L}(\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$,
- $\mathcal{L}(\mathcal{E}'(M), C^\infty(M))$,
- $\mathcal{L}(\mathcal{E}'[0,1], C^\infty[0,1])$,
- $\mathcal{L}(A^{-\infty}(\mathbb{D}), A^\infty(\mathbb{D}))$,

where

- $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions on $\mathbb{R}^n$,
- $\mathcal{E}'(M)$ is the space of distributions on a compact C^∞-manifold M,
- $\mathcal{E}'[0,1]$ is the space of distributions with support in $[0,1]$,
- $A^{-\infty}(\mathbb{D})$ is the space of holomorphic functions on the unit disc with polynomial growth, i.e.
$$A^{-\infty}(\mathbb{D}) := \{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f(z)|(1-|z|)^q < \infty \text{ for some } q \in \mathbb{N}_0\}.$$

In order to define multiplication and involution on the spaces of operators above, we proceed like in the case of $\mathcal{L}(s',s)$ – we just have to find an appropriate Hilbert space lying between a Fréchet space and its dual, for example in the case of $\mathcal{S}(\mathbb{R}^n)$ we can choose the Hilbert space $L(\mathbb{R}^n)$ (for details, see Theorem 1.10, Example 1.13 and [11, Th. 2.1]). Now, it is clear that operators from $\mathcal{L}(s',s)$ are smoothing in the following sense: they map (in some representations of $\mathcal{L}(s',s)$) distributions (which may be highly irregular) to smooth functions. We use the term "smooth" for short; indeed, this term seems to be more popular than the term "smoothing" (see, for instance, [12, Th. 2, Ex. 2.6], [30, p. 301]).

Taking all the above into account, we can treat $\mathcal{L}(s',s)$ as a "noncommutative" analogue of the very important space of analysis: s.

We shall also mention two extra representations of the Fréchet *-algebra $\mathcal{L}(s',s)$: the algebra

$$\mathcal{K}_\infty := \{(a_{j,k})_{j,k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}^2} : \sup_{j,k\in\mathbb{N}} |a_{j,k}| j^q k^q < \infty \text{ for all } q \in \mathbb{N}_0\}$$

of rapidly decreasing matrices (with matrix multiplication and matrix complex involution) and its "continuous analogue": the algebra $\mathcal{S}(\mathbb{R}^2)$ of Schwartz functions on $\mathbb{R}^2$ with the Volterra convolution

$$(f \cdot g)(x,y) := \int_{\mathbb{R}} f(x,z)g(z,y)\mathrm{d}z$$

as multiplication and the involution

$$f^*(x,y) := \overline{f(y,x)}.$$

In these forms, the algebra $\mathcal{L}(s',s)$ usually appears and plays a significant role in K-theory of Fréchet algebras (see Bhatt & Inoue [1, Ex. 2.12], Cuntz [8, p. 144], [9, p. 64-65], Glöckner & Langkamp [14], Phillips [25, Def. 2.1]) and in C^*-dynamical systems (Elliot, Natsume & Nest [12, Ex. 2.6]).

As we have already seen, the algebra $\mathcal{L}(s',s)$ is also an example of a dense *-subalgebra of a C^*-algebra (namely, it is a dense subalgebra of $\mathcal{K}(\ell_2)$). Such algebras are of great importance in noncommutative geometry (see, for instance, Bhatt & Inoue [1], Blackadar & Cuntz [2], Connes [6, pp. 23, 183-184]) as they introduce differential structure on a noncommutative manifold. From the philosophical point of view C^*-algebras corespond to analogues of topological spaces whereas some of their dense smooth subalgebras play the role of smooth structures.

As we already said, the algebra $\mathcal{L}(s',s)$ has a lot of natural "structure": its first natural norm is a dominating C^*-norm (see the discussion below), it has a natural unitary space structure inherited from the space $\mathcal{HS}(\ell_2)$ of Hilbert-Schmidt operators, its spectral properties are closely related to those of $\mathcal{K}(\ell_2)$, etc. Therefore, it seems that $\mathcal{L}(s',s)$ is a very special dense *-subalgebra of $\mathcal{K}(\ell_2)$, and hence it might be the best candidate for a "differential structure" there. In spite of the role played by $\mathcal{L}(s',s)$ as explained above, very little is known, for example, about its algebraic structure. The main goal of the presented dissertation is to find initial results in this direction.

The dissertation is divided into 5 chapters. In the first chapter we establish notation and present some fundamental, well-known by now, facts concerning nuclear Fréchet spaces, operator theory, the space s and the algebra $\mathcal{L}(s',s)$.

Chapter 2 is devoted to the so-called algebra of multipliers of $\mathcal{L}(s',s)$, which can be seen as the largest (in some sense) *-algebra of operators acting on $\mathcal{L}(s',s)$, i.e. the largest "resonable" *-algebra in which $\mathcal{L}(s',s)$ is an ideal. Multiplier algebras of C^*-algebras are usually described by the so-called double centralizers (see [4] and Def. 2.2). In particular, the algebra of multipliers (i.e. the algebra of double centralizers) of $\mathcal{K}(\ell_2)$ is $\mathcal{L}(\ell_2)$ (see [23, pp. 38–39, 81–83]). Using similar techniques, we show in the main result of Chapter 2 that the *-algebra of unbounded operators on ℓ_2

$$\mathcal{L}^*(s) := \{x\colon s \to s : x \text{ is linear}, s \subset \mathcal{D}(x^*) \text{ and } x^*(s) \subset s\},$$

where

$$\mathcal{D}(x^*) := \{\eta \in \ell_2 : \exists \zeta \in \ell_2\ \forall \xi \in s \quad \langle x\xi, \eta\rangle = \langle \xi, \zeta\rangle\}$$

and $x^*\eta := \zeta$ for $\eta \in \mathcal{D}(x^*)$, is isomorphic as a *-algebra to the *-algebra of double centralizers of $\mathcal{L}(s', s)$ (Th. 2.7). This fact connects our considerations to the theory of *-algebras of unbounded operators on Hilbert spaces (the so-called O^*-algebras) developed e.g. by G. Lassner (see e.g. [18]) and K. Schmüdgen (see [29]).

In Section 3.1, we prove, using the fact that the norm $||\cdot||_{\ell_2\to\ell_2}$ is a dominating norm on $\mathcal{L}(s', s)$ (Proposition 3.2), the crucial for the whole dissertation theorem on the spectral representation of normal elements in $\mathcal{L}(s', s)$ (Theorem 3.1). As a by-product we obtain a kind of spectral description of normal elements of $\mathcal{L}(s', s)$ among those of $\mathcal{K}(l_2)$ (Corollary 3.6). We also present in Section 3.2 a theorem on the Schmidt representation of an arbitrary operator in $\mathcal{L}(s', s)$ (Theorem 3.8) and give a corresponding description of smooth operators among compact operators (Corollary 3.9).

The aim of Chapter 4 is to describe and classify closed commutative *-subalgebras of $\mathcal{L}(s', s)$. In Section 4.1, we show that every such algebra A is isomorphic as a Fréchet *-algebra to the Köthe algebra

$$\lambda^\infty(||P_k||_q) := \left\{(\xi_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sup_{k\in\mathbb{N}}(|\xi_k|\ ||P_k||_q) < \infty \text{ for every } q \in \mathbb{N}_0\right\}$$

with pointwise multiplication and conjugation (Theorem 4.9), where $(P_k)_{k\in\mathbb{N}}$ is the set of nonzero minimal (self-adjoint) projections in A. To prove this, we show that $(P_k)_{k\in\mathbb{N}}$ is a Schauder basis of A, called the canonical Schauder basis (Lemma 4.4). In particular, we prove that the algebra A is generated by a single operator x and also by the set of spectral projections of x (see again Theorem 4.9).

Section 4.2 is devoted to closed maximal commutative *-subalgebras of $\mathcal{L}(s', s)$, i.e. those closed commutative *-subalgebras of $\mathcal{L}(s', s)$ which are not properly contained in any larger closed commutative *-subalgebra of $\mathcal{L}(s', s)$. It appears that the canonical Schauder bases of such algebras consist of one-dimensional (pairwise orthogonal) projections P_k forming a sequence which is complete in the following sense: there is no nonzero projection P belonging to $\mathcal{L}(s', s)$ such that $P_kP = 0$ for every $k \in \mathbb{N}$ (Theorem 4.11). Consequently, algebra A is isomorphic to a closed maximal commutative *-subalgebra of $\mathcal{L}(s', s)$ if and only if

$$A \cong \lambda^\infty(|f_k|_q) := \left\{(\xi_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sup_{k\in\mathbb{N}}(|\xi_k|\ |f_k|_q) < \infty \text{ for every } q \in \mathbb{N}_0\right\}$$

as a Fréchet *-algebra, where $(f_k)_{k\in\mathbb{N}} \subset s$ is the orthonormal sequence corresponding to the canonical Schauder basis of A (Corollaries 4.16 and 4.21). Therefore, since every closed commutative *-subalgebra of $\mathcal{L}(s', s)$ is contained in some closed maximal commutative *-subalgebra of $\mathcal{L}(s', s)$ (Proposition 4.12), the class of closed commutative *-subalgebras of $\mathcal{L}(s', s)$ coincides (in the sense of Fréchet *-algebra isomorphism) with the class of closed commutative *-subalgebras of $\lambda^\infty(|f_k|_q)$, $(f_k)_{k\in\mathbb{N}} \subset s$ being an orthonormal sequence (see Corollary 4.22).

In Section 4.3 we show a surprising fact that a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ is isomorphic as Fréchet *-algebra to some closed *-subalgebra of s if and only if it is isomorphic as a Fréchet space to some complemented subspace of s (Theorem 4.25), i.e. if it has the so-called property (Ω) (see Definition 0.2 below). We also give an example of a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ which is not isomorphic to any closed *-subalgebra of s (Theorem 4.32).

In the last section of Chapter 4, we focus on a very specific class of closed commutative *-subalgebras of $\mathcal{L}(s', s)$ with the property (Ω), namely orthogonally complemented subalgebras. A subalgebra A of $\mathcal{L}(s', s)$ is orthogonally complemented in $\mathcal{L}(s', s)$, if there is an orthogonal projection $\widetilde{\pi}$ on the Hilbert space $\mathcal{HS}(\ell_2)$ such that $\widetilde{\pi}(\mathcal{L}(s', s)) = A$ (see also Definition 4.33). In Proposition 4.36 we characterize closed commutative orthogonally complemented *-subalgebras in terms of their canonical Schauder bases. The case of closed maximal commutative orthogonally complemented *-subalgebras of $\mathcal{L}(s', s)$ isomorphic (as Fréchet *-algebras) to s is of special interest: it turns out, for instance, that the set of orthonormal sequences correspondending to the canonical Schauder bases of algebras from this class coincides with the set of orthonormal sequences which are at the same time Schauder bases of s (Theorem 4.37). Moreover, it turns out that the closed maximal commutative orthogonally complemented *-subalgebras A of $\mathcal{L}(s', s)$ isomorphic to s are exactly those for which there exists an algebra isomorphism $T\colon \mathcal{L}(s', s) \to \mathcal{L}(s', s)$ preserving orthogonality which maps A onto the subalgebra of diagonal operators (Corollary 4.38). We finish Section 4.4 with an example of a closed maximal commutative *-subalgebra of $\mathcal{L}(s', s)$ isomorphic as a Fréchet *-algebra to s which is not orthogonally complemented in $\mathcal{L}(s', s)$ (Theorem 4.39).

In Chapter 5 we establish functional calculus for normal elements of $\mathcal{L}(s', s)$. In particular, $f(x)$ belongs to $\mathcal{L}(s', s)$ for each normal operator $x \in \mathcal{L}(s', s) \subset \mathcal{L}(\ell_2)$ and each Hölder continuous function f vanishing at zero and defined on the spectrum of x (Theorem 5.1). For instance, positive elements in $\mathcal{L}(s', s)$ have positive square roots in $\mathcal{L}(s', s)$. Another functional calculus (only C^∞ one) on dense subalgebras of C^*-algebras has been developed by Blackadar and Cuntz in [2] (see Prop. 6.4 and p. 277) under some additional assumptions on the algebra. Unfortunately, it seems that $\mathcal{L}(s', s)$ does not satisfy the required conditions.

Most of the results from Sections 3.1, 4.1, 4.2 and Chapter 5 have been already published in [5].

The results contained in this dissertation are mostly derived from and inspired by the theory of nuclear Fréchet spaces [20], the theory of compact operators on ℓ_2 ([7, 20]), the theory of unbounded operators [29] and the theory of double centralizers of C^*-algebras ([4, 15]). Probably the main novelty of the methods used in the dissertation is an application of the so-called properties (DN) and (Ω) of Vogt and Zahariuta.

Definition 0.1. (see [20, Def. on p. 359 and Lemma 29.10]) A Fréchet space $(X, (||\cdot||_q)_{q\in\mathbb{N}_0})$ has the property (DN) if there is a continuous norm $||\cdot||$ on X such that for any $q \in \mathbb{N}_0$ there is $r \in \mathbb{N}_0$ and $C > 0$ such that for all $x \in X$

$$||x||_q^2 \leq C||x||\,||x||_r.$$

The norm $||\cdot||$ is called a dominating norm.

Definition 0.2. (see [20, p. 367]) A Fréchet space E with a fundamental sequence $(||\cdot||_q)_{q\in\mathbb{N}_0}$ of seminorms has the property (Ω) if the following condition holds:

$$\forall p\ \exists q\ \forall r\ \exists \theta \in (0,1)\ \exists C > 0\ \forall y \in E' \quad ||y||_q' \leq C||y||_p'^{1-\theta}||y||_r'^{\theta},$$

where E' is the topological dual of E and $||y||_p' := \sup\{|y(x)| : ||x||_p \leq 1\}$.

Clearly, $\mathcal{L}(s', s)$ has the properties (DN) and (Ω) as isomorphic to s.

The properties (DN) and (Ω) with their several modifications are very important topological invariants in the theory of nuclear Fréchet spaces. For example, Vogt and Wagner (see [31, 32, 33, 35] and [20, Ch. 30]) proved the following splitting theorem for nuclear Fréchet spaces.

Theorem 0.3. *Let E, F, G be nuclear Fréchet spaces and let*

$$0 \longrightarrow E \xrightarrow{j} F \xrightarrow{q} G \longrightarrow 0$$

be a short exact sequence of continuous linear maps. If the space G has the property (DN) *and the space E has the property* (Ω)*, then the sequence splits, i.e. the map q has a continuous linear right inverse and the map j has a continuous linear left inverse.*

As a further consequence of the last theorem, one gets a characterization of subspaces and quotients of the space s in terms of the properties (DN) and (Ω) (Vogt and Wagner [33, 35], [20, Ch. 31]). Recall that a subspace F of a Fréchet space E is called complemented (in E) if there is a continuous projection $\pi\colon E \to E$ with $\operatorname{im}\pi = F$.

Theorem 0.4. *A Fréchet space is isomorphic to*

(i) a closed subspace of the space s if and only if it is nuclear and it has the property (DN)*;*

(ii) a quotient of the space s if and only if it is nuclear and it has the property (Ω)*;*

(iii) a complemented subspace of the space s if and only if it is nuclear and it has the properties (DN) *and* (Ω).

As mentioned above, the operator C^*-norm $||\cdot||_{\ell_2\to\ell_2}$ is a dominating norm on $\mathcal{L}(s', s)$ (compare with [27, Th. 4] and see Proposition 3.2 for the straightforward proof). This result will be of great importance in our considerations; it will lead to unexpected connections of the property (DN) and (Ω) to spectral properties of the elements of $\mathcal{L}(s', s)$ and algebraic properties of the algebra $\mathcal{L}(s', s)$.

Investigations of the algebra $\mathcal{L}(s', s)$ in the context of the theory of Fréchet spaces were proposed some years ago by Leonhard Frerick (University of Trier). Some results have been already known (see, for instance, a survey in [11]). For example, it is known that not only $\mathcal{L}(s', s)$ is contained in the intersetion of all Schatten classes $\mathcal{S}_p(\ell_2)$, $p > 0$, but also that the sequence of eigenvalues (non-increasing in modulus, counting geometric multiplicity) of an operator from $\mathcal{L}(s', s)$ belongs to s ([11, Cor. 2.5]). Moreover, it follows from [3, Prop. A.2.8] and [30, Lemma 5.7] that an operator belonging to the algebra with unit

$$\mathcal{L}(s', s)_{\mathbf{1}} := \{x + \lambda\mathbf{1} : x \in \mathcal{L}(s', s), \lambda \in \mathbb{C}\}$$

($\mathbf{1}$ is here the identity operator on ℓ_2) is invertible in $\mathcal{L}(s', s)_{\mathbf{1}}$ if and only if it is invertible in $\mathcal{L}(\ell_2)$ (see also [11, Th. 2.3]). Consequently, $\mathcal{L}(s', s)_{\mathbf{1}}$ is a Q-algebra (i.e. the set of invertible elements in $\mathcal{L}(s', s)_{\mathbf{1}}$ is open) so $\mathcal{L}(s', s)$ is a Q-algebra as well (i.e. the set of quasi-invertible elements in $\mathcal{L}(s', s)$ is open, see [13, Prop. 4.14] and [14]).

Piszczek proved that every positive functional on $\mathcal{L}(s', s)$ and every derivation from the algebra $\mathcal{L}(s', s)$ to an arbitrary bimodule over $\mathcal{L}(s', s)$ are continuous [28, Th. 11 and Th. 13]. Next he proved that the algebra $\mathcal{L}(s', s)$ is not boundedly approximately amenable but it is approximately amenable and approximately contractibile [28, Cor. 19, Th. 21 and Cor. 22].

Moreover, Piszczek showed that $\mathcal{L}(s', s)$ has no bounded approximate identity ([28, Prop. 2]), i.e. there is no bounded net $(u_\lambda)_{\lambda\in\Lambda} \subset \mathcal{L}(s', s)$ such that $xu_\lambda \to x$ and $u_\lambda x \to x$ for all

$x \in \mathcal{L}(s', s)$. Let us also mention here, that $\mathcal{L}(s', s)$ is not a locally C^*-algebra, i.e. there is no sequence of C^*-norms defining the topology on $\mathcal{L}(s', s)$. Otherwise, since $\mathcal{L}(s', s)$ is a Q-algebra, it would be automatically a C^*-algebra so a Banach space (see [13, Cor. 8.2]); this gives a contradiction as $\mathcal{L}(s', s) \cong s$ is not a Banach space. In the book of Fragoulopoulou [13] it is developed a theory of topological algebras A with involution if either A has a bounded approximative identity or A is a locally C^*-algebra. In view of the above negative results the mentioned theory cannot be applied to $\mathcal{L}(s', s)$, and therefore we need new ideas. We hope that our dissertation contributes in this direction.

Chapter 1

Preliminaries

Throughout the thesis, $\mathbb{N}$ will denote the set of natural numbers $\{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

By a *projection* on the complex separable Hilbert space

$$\ell_2 := \left\{ \xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : ||\xi||_{\ell_2} := \left(\sum_{j=1}^{\infty} |\xi_j|^2 \right)^{1/2} < \infty \right\}$$

we always mean a continuous orthogonal (i.e. self-adjoint) projection.

By a *Fréchet space* we mean a complete metrizable locally convex space over $\mathbb{C}$ (we will not use locally convex spaces over $\mathbb{R}$). A *Fréchet algebra* is a Fréchet space which is an algebra with continuous multiplication. A *Fréchet* **-algebra* is a Fréchet algebra with an involution.

For locally convex spaces E, F, we denote by $\mathcal{L}(E, F)$ the space of all continuous linear operators from E to F. To shorten notation, we write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$.

We use the standard notation and terminology. All the notions from functional analysis are explained in [7] or [20] and those from topological algebras in [13], [19] or [36].

§1. The space s and its dual: We define the *space of rapidly decreasing sequences* as the Fréchet space

$$s := \left\{ \xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_q := \left(\sum_{j=1}^{\infty} |\xi_j|^2 j^{2q} \right)^{1/2} < \infty \text{ for all } q \in \mathbb{N}_0 \right\}$$

with the topology corresponding to the system $(|\cdot|_q)_{q\in\mathbb{N}_0}$ of norms. We may identify the strong dual of s (i.e. the space of all continuous linear functionals on s with the topology of uniform convergence on bounded subsets of s, see e.g. [20, Def. on p. 267]) with the *space of slowly increasing sequences*

$$s' := \left\{ \xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|'_q := \left(\sum_{j=1}^{\infty} |\xi_j|^2 j^{-2q} \right)^{1/2} < \infty \text{ for some } q \in \mathbb{N}_0 \right\}$$

equipped with the inductive limit topology given by the system $(|\cdot|'_q)_{q\in\mathbb{N}_0}$ of norms (note that for a fixed q, $|\cdot|'_q$ is defined only on a subspace of s'). More precisely, every $\eta \in s'$ corresponds to the continuous linear functional on s:

$$\xi \mapsto \langle \xi, \eta \rangle := \sum_{j=1}^{\infty} \xi_j \overline{\eta_j}.$$

Please note the conjugation on the second variable! These functionals are continuous, because, by the Cauchy-Schwartz inequality, for all $q \in \mathbb{N}_0$, $\xi \in s$ and $\eta \in s'$ we have

$$|\langle \xi, \eta \rangle| \le |\xi|_q |\eta|'_q. \tag{1.1}$$

Conversely, one can show that for each continuous linear functional y on s there is $\eta \in s'$ such that $y = \langle \cdot, \eta \rangle$.

Similarly, we identify $\xi \in s$ with the continuous linear functional on s':

$$\eta \mapsto \langle \eta, \xi \rangle := \sum_{j=1}^{\infty} \eta_j \overline{\xi_j}.$$

In particular, for each continuous linear functional y on s' there is $\xi \in s$ such that $y = \langle \cdot, \xi \rangle$.

We emphasize that the "scalar product" $\langle \cdot, \cdot \rangle$ is well-defined on $s \times s' \cup s' \times s$ and, of course, on $\ell_2 \times \ell_2$.

§2. Nuclear Fréchet spaces and the property (DN): Recall that a Fréchet space E is *nuclear* if every unconditionally convergent series of elements of E is absolutely convergent (see also [20, Def. on p. 344]). Nuclear Fréchet spaces share many nice properties with finite-dimensional spaces, e.g. every closed bounded set in a nuclear Fréchet space is compact (see [20, Lemma 24.19 and Cor. 28.5]). However, there are pathological examples of nuclear Fréchet spaces which do not behave like finite-dimensional spaces, e.g. there are nuclear Fréchet spaces without Schauder basis (the first example is due to Mityagin and Zobin [22]).

In the class of all Fréchet spaces the space s is, in some sense, universal. More precisely, the Kōmura-Kōmura theorem (see e.g. [20, Cor. 29.9]) gives the following characterization of nuclear spaces.

Theorem 1.1. *A Fréchet space is nuclear if and only if it is isomorphic to some closed subspace of* $s^{\mathbb{N}}$.

Closed subspaces of the space s can be characterized by the so-called property (DN) (see Theorem 1.3 below).

Definition 1.2. A Fréchet space $(X, (||\cdot||_q)_{q \in \mathbb{N}_0})$ has the *property* (DN) (see [20, Def. on p. 359]) if there is a continuous norm $||\cdot||$ on X such that for all $q \in \mathbb{N}_0$ there is $r \in \mathbb{N}_0$ and $C > 0$ such that

$$||x||_q^2 \le C ||x|| \, ||x||_r$$

for all $x \in X$. The norm $||\cdot||$ is called a *dominating norm.*

The following Theorem is due to Vogt (see [33] and [20, Ch. 31]).

Theorem 1.3. *A Fréchet space is isomorphic to a closed subspace of s if and only if it is nuclear and it has the property* (DN).

The (DN) condition for the space s reads as follows.

Proposition 1.4. *For every $p \in \mathbb{N}_0$ and $\xi \in s$ we have*

$$|\xi|_p^2 \le ||\xi||_{\ell_2} |\xi|_{2p}.$$

In particular, the norm $||\cdot||_{\ell_2}$ is a dominating norm on s.

Proof. Fix $p \in \mathbb{N}_0$ and $\xi \in s$. Then, from the Cauchy-Schwartz inequality, we obtain

$$|\xi|_p^2 = \sum_{j=1}^{\infty} |\xi_j|^2 j^{2p} = \sum_{j=1}^{\infty} |\xi_j| \cdot |\xi_j| j^{2p} \leq \left(\sum_{j=1}^{\infty} |\xi_j|^2\right)^{1/2} \left(\sum_{j=1}^{\infty} |\xi_j|^2 j^{4p}\right)^{1/2} = ||\xi||_{\ell_2} |\xi|_{2p}.$$

$\square$

We will usually use (see Lemma 3.3 and its proof) the condition (DN) in the following equivalent form (see [20, Lemma 29.10]): there is a continuous norm $||\cdot||$ on X such that for any $q \in \mathbb{N}_0$ and $\theta \in (0,1)$ there is $r \in \mathbb{N}_0$ and $C > 0$ such that

$$||x||_q \leq C||x||^{1-\theta} ||x||_r^{\theta} \tag{1.2}$$

for all $x \in X$.

§3. Köthe spaces: We say that a matrix $(a_{j,q})_{j\in\mathbb{N},q\in\mathbb{N}_0}$ of non-negative numbers is a *Köthe matrix* if the following conditions hold:

(i) for each $j \in \mathbb{N}$ there is $q \in \mathbb{N}_0$ such that $a_{j,q} > 0$;

(ii) $a_{j,q} \leq a_{j,q+1}$ for $j \in \mathbb{N}$ and $q \in \mathbb{N}_0$.

For $1 \leq p < \infty$ and a Köthe matrix $(a_{j,q})_{j\in\mathbb{N},q\in\mathbb{N}_0}$ we define the *Köthe space*

$$\lambda^p(a_{j,q}) := \left\{\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_{p,q} := \left(\sum_{j=1}^{\infty} |\xi_j a_{j,q}|^p\right)^{1/p} < \infty \text{ for all } q \in \mathbb{N}_0\right\}$$

and for $p = \infty$

$$\lambda^\infty(a_{j,q}) := \left\{\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_{\infty,q} := \sup_{j\in\mathbb{N}} |\xi_j| a_{j,q} < \infty \text{ for all } q \in \mathbb{N}_0\right\}$$

with the topology generated by the norms $(|\cdot|_{p,q})_{q\in\mathbb{N}_0}$ (see e.g. [20, Def. p. 326]).

It is well-known (see [20, Lemma 27.1]) that the spaces $\lambda^p(a_{j,q})$ are Fréchet spaces and sometimes they are Fréchet *-algebras with pointwise multiplication and conjugation, e.g. if $a_{j,q} \geq 1$ for all $j \in \mathbb{N}$ and $q \in \mathbb{N}_0$.

By definition, s is just the Köthe space $\lambda^2(j^q)$. Moreover, since the matrix $(j^q)_{j\in\mathbb{N},q\in\mathbb{N}_0}$ satisfies the so-called Grothendieck-Pietsch condition (see e.g. [20, Prop. 28.16 item 6]), s is a nuclear space, and thus it has also other Köthe space representations (see again [20, Prop. 28.16 & Ex. 29.4(1)]).

Proposition 1.5. *For all $1 \leq p \leq \infty$, $s = \lambda^p(j^q)$ as a Fréchet space. In particular, $\xi \in s$ if and only if*

$$\sup_{j\in\mathbb{N}} |\xi_j| j^q < \infty$$

for every $q \in \mathbb{N}_0$.

We use ℓ_2-norms in the definition of s to clarify our ideas, for example we have $|\xi|_0 = ||\xi||_{\ell_2}$ for $\xi \in s$ and $|\eta|_0' = ||\eta||_{\ell_2}$ for $\eta \in \ell_2$. However, in some situations the supremum norms $|\cdot|_{\infty,q}$ (as they are relatively easy to compute) will be more convenient. For instance, we use them in the proof of the following well-known facts.

Proposition 1.6. *If $(\xi_j)_{j\in\mathbb{N}} \in s$, then $(|\xi_j|^\theta)_{j\in\mathbb{N}} \in s$ for every $\theta > 0$.*

Proof. For $\xi \in s$, $\theta > 0$ and $q \in \mathbb{N}_0$ we get

$$\sup_{j\in\mathbb{N}} |\xi_j|^\theta j^q = (\sup_{j\in\mathbb{N}} |\xi_j| j^{q/\theta})^\theta < \infty.$$

□

Proposition 1.7. *We have*

$$s \subset \bigcap_{\theta>0} \ell_\theta.$$

Proof. Take $\xi \in s$ and $\theta > 0$. By Proposition 1.6, we obtain

$$\sum_{j=1}^{\infty} |\xi_j|^\theta \leq \sup_{j\in\mathbb{N}} |\xi_j|^\theta j^2 \cdot \sum_{j=1}^{\infty} j^{-2} < \infty.$$

□

§4. The algebra $\mathcal{L}(s', s)$: Let E, F be locally convex spaces. Recall that $\mathcal{L}(E, F)$ denotes the space of all continuous linear operators from E to F and, to shorten notation, we write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$.

It is a simple matter to show that $\mathcal{L}(s', s)$ with the topology of uniform convergence on bounded sets in s' is a Fréchet space and it is isomorphic to $s\widehat{\otimes}s$, the completed tensor product of s (see [16, §41.7 (5)] and note that, s being nuclear, there is only one tensor topology), and thus $\mathcal{L}(s', s) \cong s$ as a Fréchet space (see e.g. [20, Lemma 31.1]). Moreover, it is easily seen that $(||\cdot||_q)_{q\in\mathbb{N}_0}$,

$$||x||_q := \sup_{|\xi|'_q\leq 1} |x\xi|_q,$$

is a fundamental system of norms on $\mathcal{L}(s', s)$.

Let us introduce multiplication and involution on $\mathcal{L}(s', s)$. First observe that s is a dense subspace of ℓ_2, ℓ_2 is a dense subspace of s', and, moreover, the embedding maps $j_1\colon s \hookrightarrow \ell_2$, $j_2\colon \ell_2 \hookrightarrow s'$ are continuous. Hence,

$$\iota\colon \mathcal{L}(s', s) \hookrightarrow \mathcal{L}(\ell_2), \quad \iota(x) := j_1 \circ x \circ j_2, \tag{1.3}$$

is a well-defined (continuous) embedding of $\mathcal{L}(s', s)$ into the C^*-algebra $\mathcal{L}(\ell_2)$, and thus we may define a multiplication on $\mathcal{L}(s', s)$ by

$$xy := \iota^{-1}(\iota(x) \circ \iota(y)),$$

i.e.

$$xy = x \circ j \circ y,$$

where $j := j_2 \circ j_1\colon s \hookrightarrow s'$. Similarly, an involution on $\mathcal{L}(s', s)$ is defined by

$$x^* := \iota^{-1}(\iota(x)^*),$$

where $\iota(x)^*$ is the hermitian adjoint of $\iota(x)$. The following Proposition makes these definitions correct.

Proposition 1.8. *For all $x, y \in \mathcal{L}(s', s)$ we have:*

(i) $\iota(x) \circ \iota(y) \in \iota(\mathcal{L}(s', s))$;

(ii) $\iota(x)^* \in \iota(\mathcal{L}(s', s))$.

Proof. (i): This is clear.

(ii): Let $x \in \mathcal{L}(s', s)$ and let $(e_k)_{k\in\mathbb{N}}$ be the canonical orthonormal basis of ℓ_2. Then

$$\langle \iota(x)e_k, \eta\rangle = \langle e_k, \iota(x)^*\eta\rangle$$

for all $k \in \mathbb{N}$ and $\eta \in \ell_2$, hence $\iota(x)^*\eta = (\overline{\langle \iota(x)e_k, \eta\rangle})_{k\in\mathbb{N}}$ for $\eta \in \ell_2$.

Consider the operator $z\colon s' \to s$, $z\eta := (\overline{\langle xe_k, \eta\rangle})_{k\in\mathbb{N}}$. Fix $\eta \in s'$ and choose $r \in \mathbb{N}_0$ so that $|\eta|'_r < \infty$. Then for all $q \in \mathbb{N}_0$ there is a constant $C > 0$ such that

$$|z\eta|_q^2 = \sum_{k=1}^{\infty} |\langle xe_k, \eta\rangle|^2 k^{2q} \le (|\eta|'_r)^2 \sum_{k=1}^{\infty} |xe_k|_r^2 k^{2q} \le C(|\eta|'_r)^2 \sum_{k=1}^{\infty} k^{-2} < \infty,$$

the second inequality being a consequence of the continuity of x. This means that the operator z is well-defined, continuous, and clearly $z\,|_{\ell_2}= \iota(x)^*$. Hence, $\iota(x)^* = \iota(z) \in \iota(\mathcal{L}(s', s))$. □

In future, we just identify $x \in \mathcal{L}(s', s)$ and $\iota(x) \in \mathcal{L}(\ell_2)$ so we omit ι in the notation.

A Fréchet algebra E is called *locally m-convex* if E has a fundamental system of submultiplicative seminorms. It is well-known that $\mathcal{L}(s', s)$ is locally m-convex (see e.g. [25, Lemma 2.2]); we give a simple proof that the norms $||\cdot||_q$ are submultiplicative, which shows simultaneously that the multiplication introduced above is separately continuous, and thus, by [36, Th. 1.5], it is jointly continuous.

Proposition 1.9. *For every $x, y \in \mathcal{L}(s', s)$ and $q \in \mathbb{N}_0$ we have $||xy||_q \le ||x||_q||y||_q$.*

Proof. Let $x, y \in \mathcal{L}(s', s)$ and let B_q, B'_q denote the closed unit ball for the norms $|\cdot|_q$, $|\cdot|'_q$, respectively. Clearly, $y(B'_q) \subseteq ||y||_q B_q$ and $B_q \subseteq B'_q$. Hence,

$$\begin{aligned}||xy||_q &= \sup_{|\xi|'_q\le 1} |x(y(\xi))|_q = \sup_{\eta\in y(B'_q)} |x(\eta)|_q \le \sup_{\eta\in ||y||_q B_q} |x(\eta)|_q = ||y||_q \sup_{\eta\in B_q} |x(\eta)|_q \\ &\le ||y||_q \sup_{\eta\in B'_q} |x(\eta)|_q = ||x||_q||y||_q.\end{aligned}$$

□

We may summarize this paragraph by saying that $\mathcal{L}(s', s)$ is a locally m-convex Fréchet *-algebra. It is sometimes called the *algebra of smooth operators* or the *algebra of smoothing operators*.

§5. Representations of $\mathcal{L}(s', s)$: Let E be a Fréchet space, E' be its strong dual (i.e. the space of all continuous linear functionals on E with the topology of uniform convergence on bounded sets in E) and let $\mathcal{H}$ be a Hilbert space with a scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Assume that E is dense in $\mathcal{H}$ and

$$||\cdot||_{\mathcal{H}}\colon E \to [0, \infty), \quad ||\xi||_{\mathcal{H}} := \sqrt{\langle \xi, \xi\rangle_{\mathcal{H}}},$$

is a continuous norm on E. We call $(E, \mathcal{H}, E')$ a *Gelfand triple* or a *rigged Hilbert space* (see e.g. [29, Remark 2 on p. 47]).

Let $j_1\colon E \hookrightarrow \mathcal{H}$ denote the embedding map and let $j_2\colon \mathcal{H} \hookrightarrow E'$ be the adjoint of j_1, i.e.

$$(j_2(\eta))(\xi) = \langle \xi, \eta \rangle_{\mathcal{H}}$$

for $\xi \in E$ and $\eta \in \mathcal{H}$. Since E is dense in $\mathcal{H}$, j_2 is injective. Define $j := j_2 \circ j_1\colon E \hookrightarrow E'$.

Assume also that:

($\bigstar$) E has a Schauder basis which is orthonormal with respect to the scalar product of $\mathcal{H}$ and such that the corresponding coefficients space is s. In particular, by the closed graph theorem, E is isomorphic as a Fréchet space to s.

Then, repeating arguments from the previous paragraph, we may prove the following (see also [11, Th. 2.1]).

Theorem 1.10. *Under the conditions stated above,*

(i) *the map*
$$\iota\colon \mathcal{L}(E', E) \hookrightarrow \mathcal{L}(\mathcal{H}), \quad \iota(x) := j_1 \circ x \circ j_2,$$
is a well-defined continuous embedding of $\mathcal{L}(E', E)$ (with the topology of uniform convergence on bounded subsets of E') into the C^*-algebra $\mathcal{L}(\mathcal{H})$;

(ii) *$\mathcal{L}(E', E)$ with multiplication*
$$xy := \iota^{-1}(\iota(x) \circ \iota(y)) = x \circ j \circ y,$$
and involution
$$x^* := \iota^{-1}(\iota(x)^*),$$
$\iota(x)^*$ being the hermitian adjoint of $\iota(x)$, is a locally m-convex Fréchet *-algebra;

(iii) *$\mathcal{L}(E', E) \cong \mathcal{L}(s', s)$ as a Fréchet *-algebra.*

The following result is due Vogt.

Theorem 1.11. *[34, Cor. 7.7] Let E be a nuclear Fréchet space. If $||\cdot||_0$ is a dominating Hilbert norm on E and $E \cong s$ as a Fréchet space, then the isomorphism can be chosen so that it is unitary between E_0 and ℓ_2 (here E_0 is the completion of $(E, ||\cdot||_0)$).*

By Theorem 1.11 (with $||\cdot||_0 = ||\cdot||_{\mathcal{H}}$), assuming $||\cdot||_{\mathcal{H}}$ to be a dominating norm on $E \cong s$, we easily show the condition ($\bigstar$), and thus, by Theorem 1.10, $\mathcal{L}(E', E)$ is Fréchet *-algebra representation of $\mathcal{L}(s', s)$.

Theorem 1.12. *Let $(E, \mathcal{H}, E')$ be a Gelfand triple. If $||\cdot||_{\mathcal{H}}$ is a dominating norm on E and $E \cong s$ as a Fréchet space, then there is a unitary map $U\colon \mathcal{H} \to \ell_2$ such that $U_{|E}\colon E \to s$ is an isomorphism of Fréchet spaces. In particular, the condition ($\bigstar$) is satisfied, i.e. E has a Schauder basis which is orthonormal with respect to the scalar product of $\mathcal{H}$ and such that the corresponding coefficients space is s.*

Proof. The first statement of the theorem trivially follows from Theorem 1.11. We will show that $(U^{-1}(e_k))_{k\in\mathbb{N}}$ is a Schauder basis of s with the desired properties (here e_k denotes the vector in $\mathbb{C}^{\mathbb{N}}$ whose k-th coordinate equals 1 and the others equal 0).

Since $U^{-1}\colon \ell_2 \to \mathcal{H}$ is unitary and $U_{|s}^{-1}\colon s \to E$ is an isomorphism of Fréchet spaces, $(U^{-1}(e_k))_{k\in\mathbb{N}}$ is a Schauder basis of E which is orthonormal with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Moreover, the corresponding coefficients space is

$$\{(\langle \xi, U^{-1}(e_k)\rangle_{\mathcal{H}})_{k\in\mathbb{N}}\colon \xi\in E\} = \{(\langle U\xi, e_k\rangle)_{k\in\mathbb{N}}\colon \xi\in E\} = \{(\langle \eta, e_k\rangle)_{k\in\mathbb{N}}\colon \eta\in s\} = s,$$

which completes the proof. $\square$

Following [11, Th. 2.1], we now give some examples of Gelfand triples satisfying the condition ($\bigstar$).

Example 1.13. For the following Gelfand triples $(E, \mathcal{H}, E')$, $\mathcal{L}(E', E)$ are isomorphic as Fréchet *-algebras to $\mathcal{L}(s', s)$ (in the proof of ($\bigstar$) we may indicate – as in [11, Th. 2.1] – an appropriate Schauder basis or, alternatively, we can show that $||\cdot||_{\mathcal{H}}$ is a dominating norm on E and then apply Theorem 1.12):

(1)

$$(\mathcal{S}(\mathbb{R}^n), L^2(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)),$$

where $\mathcal{S}(\mathbb{R}^n)$ is the space of rapidly decreasing smooth functions on $\mathbb{R}^n$, $L^2(\mathbb{R}^n)$ is the Hilbert space of square integrable functions on $\mathbb{R}^n$ with the scalar product

$$\langle f, g\rangle := \int_{\mathbb{R}^n} f(t)\overline{g(t)}\mathrm{d}t$$

and $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions on $\mathbb{R}^n$;

(2)

$$(C^\infty(M), L^2(M), \mathcal{E}'(M)),$$

where M is a compact smooth manifold, $C^\infty(M)$ is the space of smooth functions on M, $L^2(M)$ is the space of square integrable functions on M with the scalar product

$$\langle f, g\rangle := \int_M f(t)\overline{g(t)}\mathrm{d}\mu(t),$$

μ being a measure which is strictly positive and absolutely continuous with respect to the Lebesgue measure on every element of the atlas of M, and $\mathcal{E}'(M)$ is the space of distributions on M;

(3)

$$(C^\infty[0,1], L^2[0,1], \mathcal{E}'[0,1]),$$

where $C^\infty[0,1]$ is the space of smooth functions on $[0,1]$, $L^2[0,1]$ is the Hilbert space of square integrable functions on $[0,1]$ with the scalar product

$$\langle f, g\rangle := \int_0^1 f(t)\overline{g(t)}\frac{\mathrm{d}t}{\sqrt{1-t^2}}$$

and $\mathcal{E}'[0,1]$ is the space of distributions on $[0,1]$ with compact support (here the orthogonal basis is given by the Chebyshev polynomials);

(4)

$$(A^\infty(\mathbb{D}), H^2(\mathbb{D}), A^{-\infty}(\mathbb{D})),$$

where $\mathcal{A}^\infty(\mathbb{D})$ is the space of holomorphic functions on the open unit disc $\mathbb{D}$ which admit the C^∞-extension to $\overline{\mathbb{D}}$, $H^2(\mathbb{D})$ is the Hardy space on $\mathbb{D}$ with the scalar product

$$\langle f, g\rangle := \lim_{r\to 1^-}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{it})\overline{g(re^{it})}\mathrm{d}t$$

and $A^{-\infty}(\mathbb{D}) \cong (\mathcal{A}^{\infty}(\mathbb{D}))'$ (as locally convex spaces) is the space of holomorphic functions on $\mathbb{D}$ of polynomial growth, i.e. such that

$$\sup_{z\in\mathbb{D}} |f(z)|(1-|z|)^q < \infty$$

for some $q \in \mathbb{N}_0$.

We shall also mention two extra representations of $\mathcal{L}(s', s)$.

Example 1.14. The following Fréchet *-algebras are isomorphic to $\mathcal{L}(s', s)$ (for the proof, see [11, Th. 2.1]):
(1) the algebra $\mathcal{K}_\infty$ of the so-called *rapidly decreasing matrices*:

$$\mathcal{K}_\infty := \{(a_{j,k})_{j,k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}^2} : \sup_{j,k\in\mathbb{N}} |a_{j,k}| j^q k^q < \infty \text{ for all } q \in \mathbb{N}_0\}$$

with matrix multiplication and matrix conjugate transpose as involution (see e.g. [14], [25, Def. 2.1]));
(2) the algebra $\mathcal{S}(\mathbb{R}^2)$ of rapidly decreasing smooth functions on $\mathbb{R}^2$ with the Volterra convolution

$$(f \cdot g)(x, y) := \int_{\mathbb{R}} f(x, z) g(z, y) \mathrm{d}z$$

as multiplication and involution

$$f^*(x, y) := \overline{f(y, x)}$$

(see e.g. [1, Ex. 2.12]).

§6. $\mathcal{L}(s', s)$ as a class of compact operators on ℓ_2: Let $\mathcal{K}(\ell_2)$ denote the space of all compact operators on ℓ_2. Recall that each $x \in \mathcal{K}(\ell_2)$ has a *Schmidt representation* of the form

$$x = \sum_{k=1}^{\infty} s_k(x) \langle \cdot, f_k \rangle g_k,$$

where $(s_k(x))_{k\in\mathbb{N}} \subset [0, \infty)$ – the so-called *sequence of singular numbers* – is a non-increasing null sequence, $(f_k)_{k\in\mathbb{N}}$, $(g_k)_{k\in\mathbb{N}}$ are orthonormal sequences in ℓ_2 and the series converges in the norm $||\cdot||_{\ell_2\to\ell_2}$ (see e.g. [20, Prop. 16.3]). It appears (see [11, Cor. 3.2]) that the canonical embedding $\iota\colon \mathcal{L}(s', s) \hookrightarrow \mathcal{L}(\ell_2)$ acts in fact into the space

$$\bigcap_{p>0} \mathcal{S}_p(\ell_2),$$

where

$$\mathcal{S}_p := \{x \in \mathcal{K}(\ell_2) : (s_k(x))_{k\in\mathbb{N}} \in \ell_p\}.$$

is the p-th *Schatten class*; in particular, every smooth operator is compact (as an operator on ℓ_2), and therefore $\mathcal{L}(s', s)$ can be regarded as some class of compact operators on ℓ_2. Since, every non-increasing sequence in $\bigcap_{p>0} \ell_p$ is already in s (see [26, 8.5.5]), this means the following.

Proposition 1.15. *The sequence of singular numbers of an element in $\mathcal{L}(s', s)$ belongs to s.*

Proposition 1.16. $\overline{\mathcal{L}(s', s)}^{||\cdot||_{\ell_2\to\ell_2}} = \mathcal{K}(\ell_2)$.

Proof. Let $\mathcal{F}(\ell_2)$ denote the space of finite operators on ℓ_2. It suffices to show that

$$\mathcal{F}(\ell_2) \subset \overline{\mathcal{L}(s', s)}^{||\cdot||_{\ell_2\to\ell_2}},$$

because

$$\overline{\mathcal{K}(\ell_2)}^{||\cdot||_{\ell_2\to\ell_2}} = \mathcal{K}(\ell_2) = \overline{\mathcal{F}(\ell_2)}^{||\cdot||_{\ell_2\to\ell_2}}$$

(see e.g. [20, Cor. 16.4]).

If x is a one-dimensional operator on ℓ_2, then $x = \langle\cdot,\xi\rangle\eta$ for some $\xi,\eta \in \ell_2$. Since the space s is dense in ℓ_2, one can find sequences $(\xi_k)_{k\in\mathbb{N}}$, $(\eta_k)_{k\in\mathbb{N}}$ of elements in s tending to ξ and η, respectively. It is easy to see that each $x_k := \langle\cdot,\xi_k\rangle\eta_k$ belongs to $\mathcal{L}(s', s)$, and moreover $||x - x_k||_{\ell_2\to\ell_2} \to 0$ as $k \to \infty$. Hence every one-dimensional operator on ℓ_2 is in $\overline{\mathcal{L}(s', s)}^{||\cdot||_{\ell_2\to\ell_2}}$, and thus $\mathcal{F}(\ell_2) \subset \overline{\mathcal{L}(s', s)}^{||\cdot||_{\ell_2\to\ell_2}}$, which completes the proof. □

§7. Spectral properties of $\mathcal{L}(s', s)$: Finally, we shall recall some basic spectral properties of the algebra $\mathcal{L}(s', s)$. For the sake of convenience, we state the following definition.

Definition 1.17. We say that a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ is a *sequence of eigenvalues* of an infinite dimensional compact operator x on ℓ_2 if it satisfies the following conditions:

(i) $\{\lambda_n\}_{n\in\mathbb{N}}$ is the set of eigenvalues of x without zero;

(ii) $|\lambda_1| \geq |\lambda_2| \geq \ldots > 0$ and if two eigenvalues have the same absolute value then we can ordered them in an arbitrary way;

(iii) the number of occurrences of the eigenvalue λ_n is equal to its geometric multiplicity (i.e. the dimension of the space $\ker(\lambda_n\mathbf{1} - x)$).

Let us also introduce the algebra with a unit

$$\mathcal{L}(s', s)_{\mathbf{1}} := \{x + \lambda\mathbf{1} : x \in \mathcal{L}(s', s), \lambda \in \mathbb{C}\},$$

where $\mathbf{1}$ is the identity operator on ℓ_2. We endow the algebra $\mathcal{L}(s', s)_{\mathbf{1}}$ with the product topology.

Proposition 1.19 below is well-known (see e.g. [14] and [13, Prop. 4.14]) and it is a simple consequence of Proposition 1.18. However, Propositions 1.18 and 1.19 also follow from [3, Prop. A.2.8]. Straightforward proofs of Propositions 1.18 and 1.20 can be found in [11, Th. 3.3, Cor. 3.5].

Proposition 1.18. *An operator in $\mathcal{L}(s', s)_{\mathbf{1}}$ is invertible if and only if it is invertible in $\mathcal{L}(\ell_2)$.*

Proposition 1.19. *The algebra $\mathcal{L}(s', s)_{\mathbf{1}}$ is a Q-algebra, i.e. the set of invertible elements in $\mathcal{L}(s', s)_{\mathbf{1}}$ is open. Consequently, $\mathcal{L}(s', s)$ is a Q-algebra as well, i.e. the set of quasi-invertible elements in $\mathcal{L}(s', s)$ is open.*

Proposition 1.20. *The spectrum of x in $\mathcal{L}(s', s)_{\mathbf{1}}$ equals the spectrum of x in $\mathcal{L}(\ell_2)$ and it consists of zero and the set of all eigenvalues of x. If, moreover, x is infinite-dimensional, then the sequence of eigenvalues of x (see Definition 1.17) belongs to s.*

Chapter 2

Multiplier algebra of $\mathcal{L}(s', s)$

In this chapter we want to describe the so-called multiplier algebra of $\mathcal{L}(s', s)$, which is, in some sense, the largest algebra of operators acting on $\mathcal{L}(s', s)$. The algebra

$$\mathcal{L}(s) \cap \mathcal{L}(s') := \{x \in \mathcal{L}(s) : x = \widetilde{x}\,|_s \text{ for some } \widetilde{x} \in \mathcal{L}(s')\} \tag{2.1}$$

seems to be a good candidate, because if $x \in \mathcal{L}(s', s)$ and $y \in \mathcal{L}(s) \cap \mathcal{L}(s')$, then clearly $xy, yx \in \mathcal{L}(s', s)$. Now, using heuristic arguments, we will show that the algebra $\mathcal{L}(s) \cap \mathcal{L}(s')$ is optimal. Assume that $y \in \mathcal{L}(E, F)$ for some locally convex spaces E, F. If $xy \in \mathcal{L}(s', s)$ for every $x \in \mathcal{L}(s', s)$ then, in particular, $(\langle \cdot, \xi\rangle\xi)y \in \mathcal{L}(s', s)$ for all $\xi \in s$, and therefore $\langle y(\eta), \xi\rangle$ has to be well-defined for every $\xi \in s$ and $\eta \in s'$, which shows that $y \colon s' \to s'$. Similarly, we show that if $yx \in \mathcal{L}(s', s)$ for every $x \in \mathcal{L}(s', s)$ then $y \colon s \to s$. Hence, $y \in \mathcal{L}(s) \cap \mathcal{L}(s')$.

The algebra $\mathcal{L}(s) \cap \mathcal{L}(s')$ can also be seen as the algebra of unbounded operators on ℓ_2 (see Proposition 2.1):

$$\mathcal{L}^*(s) := \{x \colon s \to s : x \text{ is linear}, s \subset \mathcal{D}(x^*) \text{ and } x^*(s) \subset s\}, \tag{2.2}$$

where

$$\mathcal{D}(x^*) := \{\eta \in \ell_2 : \exists \zeta \in \ell_2 \; \forall \xi \in s \quad \langle x\xi, \eta\rangle = \langle \xi, \zeta\rangle\}$$

and $x^*\eta := \zeta$ for $\eta \in \mathcal{D}(x^*)$ (one can show that ζ is unique), which defines a natural involution on $\mathcal{L}^*(s)$. This result follows e.g. from [17, Prop. 2.2]; the proof we propose here involves basic theory of locally convex spaces, including properties of continuous linear functionals on s and s' (see Preliminaries).

Proposition 2.1. *$\mathcal{L}^*(s) = \mathcal{L}(s) \cap \mathcal{L}(s')$ as sets.*

Proof. Take $x \in \mathcal{L}^*(s)$. Let $(\xi_j)_{j\in\mathbb{N}} \subset s$ and assume that $\xi_j \to 0$ and $x\xi_j \to \eta$ as $j \to \infty$. Then, for every $\zeta \in s$, we have

$$\langle x\xi_j, \zeta\rangle = \langle \xi_j, x^*\zeta\rangle \to 0$$

and, on the other hand,

$$\langle x\xi_j, \zeta\rangle \to \langle \eta, \zeta\rangle.$$

Hence $\langle \eta, \zeta\rangle = 0$ for every $\zeta \in s$, and therefore $\eta = 0$. By the closed graph theorem for Fréchet spaces (see e.g. [20, Th. 24.31]), $x \colon s \to s$ is continuous. The continuity of $x^* \colon s \to s$ can be obtained in a similar way.

Now, we shall show that x can be extended to a continuous linear operator from s' to s'. Take $\xi \in s'$ and define a linear functional $\varphi_\xi \colon s \to \mathbb{C}$, $\varphi_\xi(\eta) := \langle x^*\eta, \xi\rangle$. From the continuity of $x^* \colon s \to s$, it follows that for every $q \in \mathbb{N}_0$ there is $r \in \mathbb{N}_0$ and $C > 0$ such that $|x^*\eta|_q \le C|\eta|_r$ for all η in s. Hence, with the same quantifiers, we get

$$|\varphi_\xi(\eta)| = |\langle x^*\eta, \xi\rangle| \le |x^*\eta|_q \cdot |\xi|'_q \le C|\eta|_r \cdot |\xi|'_q \tag{2.3}$$

so φ_ξ is continuous. Consequently, for each $\xi \in s'$ we can find a unique $\zeta \in s'$ such that

$$\langle \eta, \zeta\rangle = \varphi_\xi(\eta) = \langle x^*\eta, \xi\rangle$$

for all $\eta \in s$ and we may define $\widetilde{x} \colon s' \to s'$ by $\widetilde{x}\xi := \zeta$. Clearly, $\widetilde{x}$ is a linear extension of x, and moreover $\widetilde{x}$ is continuous. In fact, by (2.3), for every $q \in \mathbb{N}_0$ there is $r \in \mathbb{N}_0$ and $C > 0$ such that

$$|\widetilde{x}\xi|'_r = \sup_{|\eta|_r \le 1} |\langle \eta, \widetilde{x}\xi\rangle| = \sup_{|\eta|_r \le 1} |\langle x^*\eta, \xi\rangle| \le C|\xi|'_q$$

for all $\xi \in s'$, i.e. $\widetilde{x}$ is continuous.

Now, let $x \in \mathcal{L}(s) \cap \mathcal{L}(s')$. For each $\eta \in s$ we define a linear functional $\psi_\eta \colon s' \to \mathbb{C}$, $\psi_\eta(\xi) := \langle \widetilde{x}\xi, \eta\rangle$, where $\widetilde{x} \colon s' \to s'$ is the continuous extension of x. By the continuity of the operator $\widetilde{x}$ on the LB-space s', it follows that for every $r \in \mathbb{N}_0$ there is $q \in \mathbb{N}_0$ and $C > 0$ such that $|\widetilde{x}\xi|'_q \le C|\xi|'_r$ for $\xi \in s'$. Hence, for $\xi \in s'$, we have

$$|\psi_\eta(\xi)| = |\langle \widetilde{x}\xi, \eta\rangle| \le |\widetilde{x}\xi|'_q \cdot |\eta|_q \le C|\eta|_q \cdot |\xi|'_r$$

which shows that ψ_η is continuous, and therefore there exists $\zeta \in s$ such that $\psi_\eta(\cdot) = \langle \cdot, \zeta\rangle$, i.e. $\langle \widetilde{x}\xi, \eta\rangle = \langle \xi, \zeta\rangle$ for $\xi \in s'$. Consequently, $\langle x\xi, \eta\rangle = \langle \xi, \zeta\rangle$ for $\xi \in s$, hence $s \subset \mathcal{D}(x^*)$ and $x^*(s) \subset s$, i.e. $x \in \mathcal{L}^*(s)$. □

The algebras $\mathcal{L}^*(\mathcal{D})$ (here $\mathcal{D}$ is a dense subspace of a complex Hilbert space $\mathcal{H}$ and in the definition of $\mathcal{L}^*(s)$ we replace s with $\mathcal{D}$ and ℓ_2 with $\mathcal{H}$) and its *-subalgebras – called O^*-algebras or Op^*-algebras – were introduced by Lassner in [18]. In particular, $\mathcal{L}^*(s)$ and $\mathcal{L}(s', s)$ are O^*-algebras. For more information we refer the reader to the book of Schmüdgen [29].

Another, more abstract, approach to multipliers goes through the so-called double centralizers (see Definition 2.2) and it goes back to Johnson [15]. We will show that, in our case, both approaches give the same algebra of multipliers (Theorem 2.7). The theory of double centralizers of C^*-algebras was developed by Busby (see [4] and also [23, pp. 38–39, 81–83]); this exposition will be also very useful in the case of $\mathcal{L}(s', s)$.

Definition 2.2. Let A be a *-algebra (over $\mathbb{C}$). A pair (L, R) of maps from A to A (neither linearity nor continuity is required) such that $xL(y) = R(x)y$ for $x, y \in A$ is called a *double centralizer* on A. We denote the set of all double centralizers on A by $\mathcal{DC}(A)$. Moreover, for a map $T \colon A \to A$ we define $T^* \colon A \to A$ by $T^*(x) := (T(x^*))^*$.

Lemma 2.3. *If* $(L, R) \in \mathcal{DC}(A)$*, then* $(R^*, L^*) \in \mathcal{DC}(A)$*.*

Proof. For all $x, y \in A$ we have

$$xR^*(y) = x(R(y^*))^* = (R(y^*)x^*)^* = (y^*L(x^*))^* = L(x^*)^*y = L^*(x)y,$$

which completes the proof. □

Now, let $(L_1, R_1), (L_2, R_2) \in \mathcal{DC}(A)$, $\lambda \in \mathbb{C}$. We define:

(i) $(L_1, R_1) + (L_2, R_2) := (L_1 + L_2, R_1 + R_2)$;

(ii) $\lambda(L_1, R_1) := (\lambda L_1, \lambda R_1)$;

(iii) $(L_1, R_1) \cdot (L_2, R_2) := (L_1 L_2, R_2 R_1)$;

(iv) $(L_1, R_1)^* := (R_1^*, L_1^*)$.

A straightforward computation shows that $\mathcal{DC}(A)$ with the operations defined above is a *-algebra. The elements of A correspond to the elements of $\mathcal{DC}(A)$ via the map, called the *double representation* of A (see [15, p. 301]),

$$\varrho\colon A \to \mathcal{DC}(A), \quad \varrho(x) := (L_x, R_x),$$

where $L_x(y) := xy$ and $R_x(y) := yx$ are the right and left multiplication maps, respectively. One can easily show that ϱ is a homomorphism of *-algebras. Our main Theorem 2.7 states that the double representation of $\mathcal{L}(s', s)$ can be extended to a *-isomorphism of $\mathcal{L}^*(s)$ and $\mathcal{DC}(\mathcal{L}(s', s))$.

In general the double representation does not have to be even injective; algebras for which this is true are called faithful.

Definition 2.4. Let A be an algebra over $\mathbb{C}$. We say that A is *left faithful* (*right faithful*, resp.) if $xz = yz$ ($zx = zy$, resp.) for all $z \in A$ implies $x = y$. If A is left and right faithful, then A is said to be *faithful*.

It is easy to verify that every C^*-algebra is faithful (see [4, Cor. 2.4] and [10, 1.3.5]). In the case of $\mathcal{L}(s', s)$ we are able to prove more.

Proposition 2.5. *If $z \in \mathcal{L}(s) \cap \mathcal{L}(s')$ and $\widetilde{z}\colon s' \to s'$ is the continuous extension of z, then*

(i) *if $z\mathcal{L}(s', s) = 0$, then $z = 0$;*

(ii) *if $\mathcal{L}(s', s)\widetilde{z} = 0$, then $z = 0$.*

In particular, $\mathcal{L}(s', s)$ is faithful.

Proof. (i) Assume that $zx = 0$ for all $x \in \mathcal{L}(s', s)$. Then, in particular for $x := \langle \cdot, \xi \rangle \xi$ (here $\xi \in s$), we get

$$\langle \cdot, \xi \rangle z(\xi) = zx = 0.$$

Thus $z(\xi) = 0$ for all $\xi \in s$, i.e $z = 0$.

(ii) Let $x\widetilde{z} = 0$ for all $x \in \mathcal{L}(s', s)$. Then, for all $\xi \in s$, $(\langle \cdot, \xi \rangle \xi)\widetilde{z} = 0$, i.e. $\xi \circ \widetilde{z} = 0$ (we treat ξ as a functional on s'). Hence $\widetilde{z} = 0$. □

The following results are well-known (see [15, Th. 7, Th. 14]). For the convenience of the reader, we present the proofs. We follow the proof of [4, Prop. 2.5] (the case of C^*-algebras).

Proposition 2.6. *Let A be a faithful Fréchet algebra and let $(L, R) \in \mathcal{DC}(A)$. Then*

(i) *L and R are linear continuous maps on A;*

(ii) *$L(xy) = L(x)y$ for every $x, y \in A$;*

(iii) *$R(xy) = xR(y)$ for every $x, y \in A$.*

Proof. (i) Let $x, y, z \in A$, $\alpha, \beta \in \mathbb{C}$. Then

$$zL(\alpha x + \beta y) = R(z)(\alpha x + \beta y) = \alpha R(z)x + \beta R(z)y = z(\alpha L(x) + \beta L(y)),$$

hence, by the assumption, $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$ and so L is linear.

Now, let $(x_j)_{j\in\mathbb{N}} \subset A$ and assume that $x_j \to 0$ and $L(x_j) \to y$ (convergence in the topology of A). Let $(||\cdot||_q)_{q\in\mathbb{N}_0}$ be a fundamental system of seminorms on A. Then

$$\begin{aligned}||zy||_q &\le ||zy - zL(x_j)||_q + ||zL(x_j)||_q = ||z(y - L(x_j))||_q + ||R(z)x_j||_q \\ &\le ||z||_q \cdot ||y - L(x_j)||_q + ||R(z)||_q \cdot ||x_j||_q \to 0,\end{aligned}$$

as $j \to \infty$, so $||zy||_q = 0$ for every $q \in \mathbb{N}_0$, and therefore $zy = 0$. Hence, by the assumption, $y = 0$. Now, by the closed graph theorem for Fréchet spaces (see e.g. [20, Th. 24.31]), L is continuous.

Analogous arguments work for the map R.

(ii) Let $x, y, z \in A$. Then

$$zL(xy) = R(z)xy = (R(z)x)y = (zL(x))y = z(L(x)y),$$

and therefore, by the assumption, $L(xy) = L(x)y$.

(iii) Analogously as in (ii) (here we need the assumption that A is left faithful). □

For $z \in \mathcal{L}(s) \cap \mathcal{L}(s')$ we define $L_z, R_z \colon \mathcal{L}(s', s) \to \mathcal{L}(s', s)$, $L_z(x) := zx$, $R_z(x) := x\widetilde{z}$, where $\widetilde{z} \colon s' \to s'$ is the extension of z according to the definition of $\mathcal{L}(s) \cap \mathcal{L}(s')$.

Theorem 2.7. *The map $\widetilde{\varrho} \colon \mathcal{L}^*(s) \to \mathcal{DC}(\mathcal{L}(s', s))$, $z \mapsto (L_z, R_z)$ is a *-isomorphism between *-algebras.*

Proof. Throughout the proof, for $\xi, \eta \in s$, $\xi \otimes \eta$ denotes the one-dimensional operator $\langle \cdot, \eta \rangle \xi$.

By Proposition 2.1, $\mathcal{L}^*(s) = \mathcal{L}(s) \cap \mathcal{L}(s')$ so for $z \in \mathcal{L}^*(s)$ the left and right multiplication maps $L_z, R_z \colon \mathcal{L}(s', s) \to \mathcal{L}(s', s)$ are well defined. Moreover, it is easy to see that $xL_z(y) = R_z(x)y$ for $x, y \in \mathcal{L}(s', s)$ and $z \in \mathcal{L}^*(s)$. Hence, $(L_z, R_z) \in \mathcal{DC}(\mathcal{L}(s', s))$ for every $z \in \mathcal{L}^*(s)$, i.e. $\widetilde{\varrho}$ is well defined.

The proof of the fact that $\widetilde{\varrho}$ is a *-algebra homomorphism is straightforward and the injectivity of $\widetilde{\varrho}$ follows directly from Proposition 2.5. We will show that $\widetilde{\varrho}$ is surjective.

Let $(L, R) \in \mathcal{DC}(\mathcal{L}(s', s))$ and fix $e \in s$ with $||e||_{\ell_2} = 1$. We define a linear continuous map (use Propositions 2.5 and 2.6) $u \colon s \to s$ by

$$u\xi := L(\xi \otimes e)(e).$$

For $\xi, \eta \in s$ we have

$$\begin{aligned}\langle u\xi, \eta \rangle &= \langle L(\xi \otimes e)(e), \eta \rangle = \langle L(\xi \otimes e)(e), (\eta \otimes e)(e) \rangle = \langle (e \otimes \eta)[L(\xi \otimes e)(e)], e \rangle \\ &= \langle [(e \otimes \eta)L(\xi \otimes e)](e), e \rangle = \langle [R(e \otimes \eta)(\xi \otimes e)](e), e \rangle = \langle R(e \otimes \eta)[(\xi \otimes e)(e)], e \rangle \\ &= \langle R(e \otimes \eta)(\xi), e \rangle = \langle \xi, (R(e \otimes \eta))^*(e) \rangle.\end{aligned}$$

This means that $u^*\eta = (R(e \otimes \eta))^*(e) \in s$ for $\eta \in s$. Hence, $s \subset \mathcal{D}(u^*)$ and $u^*(s) \subset s$, i.e. $u \in \mathcal{L}^*(s)$, and thus, by Proposition 2.1, u has the continuous extension $\widetilde{u} \colon s' \to s'$. We have also shown that

$$\langle u\xi, \eta \rangle = \langle R(e \otimes \eta)(\xi), e \rangle. \tag{2.4}$$

Next, by Propositions 2.5 and 2.6, for $\zeta \in s$ we obtain

$$\begin{aligned}
L_u(\xi \otimes \eta)(\zeta) &= (u\xi \otimes \eta)(\zeta) = [L(\xi \otimes e)(e) \otimes \eta](\zeta) = \langle \zeta, \eta \rangle L(\xi \otimes e)(e) \\
&= L(\xi \otimes e)(\langle \zeta, \eta \rangle e) = L(\xi \otimes e)[(e \otimes \eta)(\zeta)] = [L(\xi \otimes e)(e \otimes \eta)](\zeta) \\
&= L((\xi \otimes e)(e \otimes \eta))(\zeta) = L(\xi \otimes \eta)(\zeta),
\end{aligned}$$

hence $L_u(\xi \otimes \eta) = L(\xi \otimes \eta)$. Since $\{\xi \otimes \eta : \xi, \eta \in s\}$ is a Schauder basis in $\mathcal{L}(s', s)$, it follows that $L_u = L$.

Likewise, (2.4) implies for $\zeta \in s$

$$\begin{aligned}
R_u(\xi \otimes \eta)(\zeta) &= [(\xi \otimes \eta)\widetilde{u}](\zeta) = \langle u\zeta, \eta \rangle \xi = \langle R(e \otimes \eta)(\zeta), e \rangle \xi = (\xi \otimes e)((R(e \otimes \eta)(\zeta)) = \\
&= [(\xi \otimes e)R(e \otimes \eta)](\zeta) = R((\xi \otimes e)(e \otimes \eta))(\zeta) = R(\xi \otimes \eta)(\zeta),
\end{aligned}$$

and therefore $R_u = R$. Hence $\widetilde{\varrho}(u) = (L_u, R_u) = (L, R)$, and thus $\widetilde{\varrho}$ is surjective. $\square$

Chapter 3

Spectral and Schmidt representations

As we have already seen in Preliminaries §6, elements of $\mathcal{L}(s', s)$ can be regarded as compact operators on ℓ_2, and therefore every infinite-dimensional normal operator $x \in \mathcal{L}(s', s)$ has the *spectral representation*

$$x = \sum_{k=1}^{\infty} \lambda_k P_k,$$

where $(\lambda_k)_{k\in\mathbb{N}}$ is a non-increasing (in modulus) null sequence of nonzero pairwise different complex numbers, $(P_k)_{k\in\mathbb{N}}$ is a sequence of nonzero pairwise orthogonal finite-dimensional projections and the series converges in the operator norm $||\cdot||_{\ell_2\to\ell_2}$ (see e.g. [7, Th. 7.6]). Moreover, every operator $x \in \mathcal{L}(s', s)$ has a Schmidt representation of the form

$$x = \sum_{k=1}^{\infty} s_k \langle \cdot, f_k \rangle g_k,$$

where $(s_k)_{k\in\mathbb{N}} \subset [0, \infty)$ is a non-increasing null sequence, $(f_k)_{k\in\mathbb{N}}$, $(g_k)_{k\in\mathbb{N}}$ are orthonormal sequences in ℓ_2 and the series converges in the norm $||\cdot||_{\ell_2\to\ell_2}$ (see e.g. [20, Prop. 16.3]).

In this chapter we derive necessary and sufficient conditions on these representations for a compact operator to belong to $\mathcal{L}(s', s)$. In both representations a crucial role is played by the property (DN) (see [20, Def. p. 359] and Definition 1.2); to be more precise, it is important that the operator norm $||\cdot||_{\ell_2\to\ell_2}$ is a dominating norm on $\mathcal{L}(s', s)$ (see Proposition 3.2). To my best knowledge the property (DN) has not been used yet in investigations of $\mathcal{L}(s', s)$.

3.1 Spectral representation of normal operators

In this section we prove the following theorem on the spectral representation of normal elements in $\mathcal{L}(s', s)$ which leads to a spectral characterization of normal elements in $\mathcal{L}(s', s)$ (see Corollary 3.6 below).

Theorem 3.1. *Every infinite-dimensional normal operator x in $\mathcal{L}(s', s)$ has a unique spectral representation $x = \sum_{k=1}^{\infty} \lambda_k P_k$, where $(\lambda_k)_{k\in\mathbb{N}}$ is a non-increasing (in modulus) sequence in s of nonzero pairwise different elements, $(P_k)_{k\in\mathbb{N}}$ is a sequence of nonzero pairwise orthogonal finite-dimensional projections belonging to $\mathcal{L}(s', s)$ and the series converges absolutely in $\mathcal{L}(s', s)$. Moreover, $(|\lambda_k|^{\theta}||P_k||_q)_{k\in\mathbb{N}} \in s$ for all $q \in \mathbb{N}_0$ and all $\theta \in (0, 1]$.*

Since $\mathcal{L}(s', s) \cong s$ as a Fréchet space, from Proposition 1.4, it follows that $\mathcal{L}(s', s)$ has the property (DN). The following result, which is closely related to the result of K. Piszczek [27, Th. 4], shows much more. For convenience, we give a more straightforward proof.

Proposition 3.2. *The norm* $||\cdot||_{\ell_2\to\ell_2}$ *is a dominating norm on* $\mathcal{L}(s', s)$.

Proof. Clearly, $||x||_{\ell_2\to\ell_2} = ||x||_0$ for $x \in \mathcal{L}(s', s)$. By [33, Th. 4.3] (see the proof), the conclusion is equivalent to the condition

$$\forall q \in \mathbb{N}_0, \theta > 0\ \exists r \in \mathbb{N}_0, C > 0\ \forall h > 0 \quad ||\cdot||_q \le C\Big(h^\theta ||\cdot||_r + \frac{1}{h}||\cdot||_0\Big).$$

By Proposition 1.4, the norm $|\cdot|_0$ is a dominating norm on s. Hence, again by [33, Th. 4.3], we get

$$\forall q \in \mathbb{N}_0, \eta > 0\ \exists r \in \mathbb{N}_0, D_0 > 0\ \forall k > 0 \quad |\cdot|_q \le D_0\Big(k^\eta|\cdot|_r + \frac{1}{k}|\cdot|_0\Big).$$

Now, by the bipolar theorem (see e.g. [20, Th. 22.13]), we obtain (following the proof of [20, Lemma 29.13]) an equivalent condition

$$\forall q \in \mathbb{N}_0, \eta > 0\ \exists r \in \mathbb{N}_0, D > 0\ \forall k > 0 \quad U_q^\circ \subset D\Big(k^\eta U_r^\circ + \frac{1}{k}U_0^\circ\Big), \tag{3.1}$$

where $U_q := \{\xi \in s : |\xi|_q \le 1\}$ and U_q° is its polar. If $\theta > 0$ and $h \in (0, 1]$ are given, we define $\eta := 2\theta + 1$ and $k := \sqrt{h}$. Since $k^{2\eta} \le k^{\eta-1}$, we obtain

$$\begin{aligned} U_q^\circ \otimes U_q^\circ := \{x \otimes y : x, y \in U_q^\circ\} &\subset D\Big(k^\eta U_r^\circ + \frac{1}{k}U_0^\circ\Big) \otimes D\Big(k^\eta U_r^\circ + \frac{1}{k}U_0^\circ\Big) \\ &\subset D^2\Big(k^{2\eta}U_r^\circ \otimes U_r^\circ + 2k^{\eta-1}U_r^\circ \otimes U_r^\circ + \frac{1}{k^2}U_0^\circ \otimes U_0^\circ\Big) \\ &\subset 3D^2\Big(k^{\eta-1}U_r^\circ \otimes U_r^\circ + \frac{1}{k^2}U_0^\circ \otimes U_0^\circ\Big) = 3D^2\Big(h^\theta U_r^\circ \otimes U_r^\circ + \frac{1}{h}U_0^\circ \otimes U_0^\circ\Big). \end{aligned}$$

Since r and D in the condition (3.1) can be choosen so that $q \le r$ and $D \ge 1$, we obtain

$$U_q^\circ \otimes U_q^\circ \subset U_r^\circ \otimes U_r^\circ \subset 3D^2\Big(h^\theta U_r^\circ \otimes U_r^\circ + \frac{1}{h}U_0^\circ \otimes U_0^\circ\Big)$$

for $h > 1$, whence

$$\forall q \in \mathbb{N}_0, \theta > 0\ \exists r \in \mathbb{N}_0, C > 0\ \forall h > 0 \quad U_q^\circ \otimes U_q^\circ \subset C\Big(h^\theta U_r^\circ \otimes U_r^\circ + \frac{1}{h}U_0^\circ \otimes U_0^\circ\Big).$$

Therefore,

$$\begin{aligned} \sup_{z \in U_q^\circ \otimes U_q^\circ} |z(x)| &\le C \sup\Big\{|z(x)| : z \in h^\theta U_r^\circ \otimes U_r^\circ + \frac{1}{h}U_0^\circ \otimes U_0^\circ\Big\} \\ &= C \sup\Big\{|(z' + z'')(x)| : z' \in h^\theta U_r^\circ \otimes U_r^\circ, z'' \in \frac{1}{h}U_0^\circ \otimes U_0^\circ\Big\} \\ &\le C \sup\Big\{|z'(x)| + |z''(x)| : z' \in h^\theta U_r^\circ \otimes U_r^\circ, z'' \in \frac{1}{h}U_0^\circ \otimes U_0^\circ\Big\} \\ &= C\Big(h^\theta \sup_{z \in U_r^\circ \otimes U_r^\circ} |z(x)| + \frac{1}{h} \sup_{z \in U_0^\circ \otimes U_0^\circ} |z(x)|\Big) \end{aligned}$$

for all $x := \sum_{j=1}^n x_j \otimes y_j \in s \otimes s$.

Let $\chi\colon s \otimes s \to \mathcal{L}(s', s)$, $\chi\big(\sum_{j=1}^n x_j \otimes y_j\big)(z) := \sum_{j=1}^n z(y_j)x_j$. We have, for all $p \in \mathbb{N}_0$,

$$\begin{aligned}
\sup_{z \in U_p^\circ \otimes U_p^\circ} \left| z\Big(\sum_{j=1}^n x_j \otimes y_j\Big)\right| &= \sup\left\{\left|\sum_{j=1}^n z_1(x_j)z_2(y_j)\right| : z_1, z_2 \in U_p^\circ\right\} \\
&= \sup\left\{\left|z_1\Big(\sum_{j=1}^n z_2(y_j)x_j\Big)\right| : z_1, z_2 \in U_p^\circ\right\} \\
&= \sup\left\{\left|\sum_{j=1}^n z(y_j)x_j\right|_p : z \in U_p^\circ\right\} \\
&= \sup\left\{\left|\chi\Big(\sum_{j=1}^n x_j \otimes y_j\Big)(z)\right|_p : z \in U_p^\circ\right\} \\
&= \left|\left|\chi\Big(\sum_{j=1}^n x_j \otimes y_j\Big)\right|\right|_p.
\end{aligned}$$

Hence

$$\left|\left|\chi\Big(\sum_{j=1}^n x_j \otimes y_j\Big)\right|\right|_q \leq C\bigg(h^\theta\left|\left|\chi\Big(\sum_{j=1}^n x_j \otimes y_j\Big)\right|\right|_r + \frac{1}{h}\left|\left|\chi\Big(\sum_{j=1}^n x_j \otimes y_j\Big)\right|\right|_0\bigg).$$

Finally, since the set $\{\chi\big(\sum_{j=1}^n x_j \otimes y_j\big) : x_j, y_j \in s, k \in \mathbb{N}\}$ is dense in $\mathcal{L}(s', s)$, we obtain

$$||x||_q \leq C\Big(h^\theta ||x||_r + \frac{1}{h}||x||_0\Big)$$

for all $x \in \mathcal{L}(s', s)$. □

Lemma 3.3. *Let* $(E, (||\cdot||_q)_{q\in\mathbb{N}_0})$ *be a Fréchet space with the property* (DN) *and let* $||\cdot||_p$ *be a dominating norm. If* $(x_k)_{k\in\mathbb{N}} \subset E$, $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ *satisfy the conditions*

(i) $\sup_{k\in\mathbb{N}} ||x_k||_p < \infty$,

(ii) $\forall q \in \mathbb{N}_0 \ \sup_{k\in\mathbb{N}} |\lambda_k|\, ||x_k||_q < \infty$,

then

$$\forall q \in \mathbb{N}_0 \ \forall \theta \in (0,1] \quad \sup_{k\in\mathbb{N}} |\lambda_k|^\theta ||x_k||_q < \infty.$$

Moreover, for any other sequence $(y_k)_{k\in\mathbb{N}} \subset E$ *satisfying conditions (i) and (ii) we have*

$$\forall q \in \mathbb{N}_0 \ \forall q' \in \mathbb{N}_0 \ \forall \theta \in (0,1] \quad \sup_{k\in\mathbb{N}} |\lambda_k|^\theta ||x_k||_q ||y_k||_{q'} < \infty.$$

Proof. Fix $q \in \mathbb{N}_0$ and $\theta \in (0,1)$. Since $||\cdot||_p$ is a dominating norm on E, there are $C > 0$ and $r \in \mathbb{N}_0$ such that

$$||x_k||_q \leq C||x_k||_p^{1-\theta}||x_k||_r^\theta \tag{3.2}$$

for all $k \in \mathbb{N}$ (see Preliminaries §2, condition 1.2). Let $C_1 := \sup_{k\in\mathbb{N}} ||x_k||_p < \infty$, $C_2 := \sup_{k\in\mathbb{N}} |\lambda_k|\, ||x_k||_q < \infty$. Then by (3.2),

$$|\lambda_k|^\theta ||x_k||_q \leq C||x_k||_p^{1-\theta}(|\lambda_k|\, ||x_k||_r)^\theta \leq CC_1^{1-\theta}C_2^\theta =: C_3,$$

where C_3 does not depend on k.

To prove the second assertion we also fix $q' \in \mathbb{N}_0$ and let $(y_k)_{k\in\mathbb{N}} \subset E$ satisfy conditions (i) and (ii). We have

$$|\lambda_k|^\theta ||x_k||_q ||y_k||_{q'} = (|\lambda_k|^{\theta/2}||x_k||_q)(|\lambda_k|^{\theta/2}||y_k||_{q'})$$

and from the first part of the proof,

$$\sup_{k\in\mathbb{N}} |\lambda_k|^{\theta/2}||x_k||_q < \infty \quad \text{and} \quad \sup_{k\in\mathbb{N}} |\lambda_k|^{\theta/2}||y_k||_{q'} < \infty,$$

so we are done. □

Proposition 3.4. *Let $\mathcal{N}$ be a finite set or $\mathbb{N}$. If $(P_k)_{k\in\mathcal{N}}$ is a sequence of pairwise orthogonal finite-dimensional projections on ℓ_2, $(\lambda_k)_{k\in\mathcal{N}} \subset \mathbb{C}\setminus\{0\}$ and $x := \sum_{k\in\mathcal{N}} \lambda_k P_k \in \mathcal{L}(s', s)$ (the series converging in the norm $||\cdot||_{\ell_2\to\ell_2}$), then $(P_k)_{k\in\mathcal{N}} \subset \mathcal{L}(s', s)$.*

Proof. Since, $P_k = \frac{1}{\lambda_k} x \circ P_k$, it follows that $P_k\colon \ell_2 \to s$. On the other hand, $P_k = P_k \circ \frac{1}{\lambda_k} x$, so P_k extends to $P_k\colon s' \to \ell_2$. Hence $P_k = P_k \circ P_k\colon s' \to s$. □

Lemma 3.5. *Let $(\lambda_k)_{k\in\mathbb{N}}$ be a decreasing (in modulus) sequence of nonzero complex numbers and let $(P_k)_{k\in\mathbb{N}}$ be a sequence of nonzero pairwise orthogonal finite-dimensional projections on ℓ_2. Moreover, assume that the series $\sum_{k=1}^\infty \lambda_k P_k$ converges in the norm $||\cdot||_{\ell_2\to\ell_2}$ and its limit belongs to $\mathcal{L}(s', s)$. Then $(\lambda_k)_{k\in\mathbb{N}} \in s$, $(P_k)_{k\in\mathbb{N}} \subset \mathcal{L}(s', s)$ and the series converges absolutely in $\mathcal{L}(s', s)$. Moreover, $(|\lambda_k|^\theta ||P_k||_q)_{k\in\mathbb{N}} \in s$ for all $q \in \mathbb{N}_0$ and $\theta \in (0, 1]$.*

Proof. By Proposition 1.20, the sequence of eigenvalues of the operator $x := \sum_{k=1}^\infty \lambda_k P_k$ belongs to s. Clearly, each λ_k is an eigenvalue of $\sum_{k=1}^\infty \lambda_k P_k$ and the number of its occurrences is less than or equal to the geometric multiplicity, so $(\lambda_k)_{k\in\mathbb{N}}$ is, likewise, in s. Moreover, by Proposition 3.4, $P_k \in \mathcal{L}(s', s)$ for $k \in \mathbb{N}$.

Fix $q \in \mathbb{N}_0$ and $\theta \in (0, 1]$. We will show that $(|\lambda_k|^\theta ||P_k||_q)_{k\in\mathbb{N}} \in s$, which implies that the series $\sum_{k=1}^\infty \lambda_k P_k$ converges absolutely in $\mathcal{L}(s', s)$. For this purpose, consider the operator $T_x\colon \mathcal{L}(\ell_2) \to \mathcal{L}(s', s)$ which sends $z \in \mathcal{L}(\ell_2)$ to the following composition (in $\mathcal{L}(s', s)$):

$$s' \xrightarrow{x} s \hookrightarrow \ell_2 \xrightarrow{z} \ell_2 \hookrightarrow s' \xrightarrow{x} s.$$

By the closed graph theorem for Fréchet spaces (see e.g. [20, Th. 24.31]), T_x is continuous and since the sequence of operators $(P_k)_{k\in\mathbb{N}}$ is bounded in $\mathcal{L}(\ell_2)$, the sequence $(\lambda_k^2 P_k)_{k\in\mathbb{N}} = (T_x P_k)_{k\in\mathbb{N}}$ is bounded in $\mathcal{L}(s', s)$, hence

$$\sup_{k\in\mathbb{N}} |\lambda_k|^2 ||P_k||_q < \infty.$$

Therefore, since $||\cdot||_{\ell_2\to\ell_2}$ is a dominating norm on $\mathcal{L}(s', s)$ and $||P_k||_{\ell_2\to\ell_2} = 1$ for $k \in \mathbb{N}$, Lemma 3.3 (applied to the sequences $(\lambda_k^2)_{k\in\mathbb{N}}$ and $(P_k)_{k\in\mathbb{N}}$) implies that

$$\sup_{k\in\mathbb{N}} |\lambda_k|^{\theta/2} ||P_k||_q < \infty.$$

Hence, by Proposition 1.6, we get

$$\sup_{k\in\mathbb{N}} |\lambda_k|^\theta ||P_k||_q k^{q'} \le \sup_{k\in\mathbb{N}} |\lambda_k|^{\theta/2} ||P_k||_q \cdot \sup_{k\in\mathbb{N}} |\lambda_k|^{\theta/2} k^{q'} < \infty$$

for every $q' \in \mathbb{N}_0$, which completes the proof. □

Now, it is not hard to prove the main theorem of this section.

Proof of Theorem 3.1. Let x be a normal infinite-dimensional operator in $\mathcal{L}(s', s)$. The operator x (as an operator on ℓ_2) is compact (see [11, Prop. 3.1]), thus by the spectral theorem for normal compact operators (see e.g. [7, Th. 7.6]), $x = \sum_{k=1}^{\infty} \lambda_k P_k$, where $(\lambda_k)_{k\in\mathbb{N}}$ is a decreasing null sequence of nonzero pairwise different elements, $(P_k)_{k\in\mathbb{N}}$ is a sequence of nonzero pairwise orthogonal finite-dimensional projections and the series converges in the norm $||\cdot||_{\ell_2\to\ell_2}$. Now, the conclusion follows from Lemma 3.5. □

As a corollary, we get a characterization of normal operators in $\mathcal{L}(s', s)$ among compact operators on ℓ_2 (remember that we identify elements of $\mathcal{L}(s', s)$ with some compact operators on ℓ_2).

Corollary 3.6. *Let x be a compact infinite-dimensional normal operator on ℓ_2 with spectral representation $x = \sum_{k=1}^{\infty} \lambda_k P_k$, i.e. $(\lambda_k)_{k\in\mathbb{N}}$ is a non-increasing in modulus null sequence of nonzero pairwise different complex numbers, $(P_k)_{k\in\mathbb{N}}$ is a sequence of nonzero pairwise orthogonal finite-dimensional projections and the series converges in the norm $||\cdot||_{\ell_2\to\ell_2}$. Then the following assertions are equivalent:*

(i) $x \in \mathcal{L}(s', s)$;

(ii) $P_k \in \mathcal{L}(s', s)$ *for* $k \in \mathbb{N}$ *and* $(|\lambda_k|^{\theta}||P_k||_q)_{k\in\mathbb{N}} \in s$ *for all* $q \in \mathbb{N}_0$ *and* $\theta \in (0, 1]$;

(iii) $P_k \in \mathcal{L}(s', s)$ *for* $k \in \mathbb{N}$, $(\lambda_k)_{k\in\mathbb{N}} \in s$ *and* $\sup_{k\in\mathbb{N}} |\lambda_k|\, ||P_k||_q < \infty$ *for all* $q \in \mathbb{N}_0$;

(iv) $P_k \in \mathcal{L}(s', s)$ *for* $k \in \mathbb{N}$ *and* $\sum_{k=1}^{\infty} |\lambda_k|\, ||P_k||_q < \infty$ *for all* $q \in \mathbb{N}_0$.

Moreover, if $x = \sum_{k=1}^{N} \lambda_k P_k$ is a finite-dimensional operator on ℓ_2, then $x \in \mathcal{L}(s', s)$ if and only if $P_k \in \mathcal{L}(s', s)$ for $k = 1, \ldots, N$.

Proof. The implication (i)⇒(ii) follows directly from Theorem 3.1. The implications (ii)⇒(iii), (iv)⇒(i) are trivial.

(iii)⇒(iv): We have

$$\sum_{k=1}^{\infty} |\lambda_k|\, ||P_k||_q \leq \sup_{k\in\mathbb{N}} |\lambda_k|^{1/2}||P_k||_q \cdot \sum_{k=1}^{\infty} |\lambda_k|^{1/2} < \infty,$$

because, by Lemma 3.3, $\sup_{k\in\mathbb{N}} |\lambda_k|^{1/2}||P_k||_q < \infty$ and, by Proposition 1.7, $s \subset \ell_{1/2}$.

The finite-dimensional case is an immediate consequence of Proposition 3.4. □

3.2 Schmidt representation

The technique used in the previous section can be applied to get the Schmidt representation of arbitrary operators belonging to $\mathcal{L}(s', s)$.

Proposition 3.7. *Let $\mathcal{N}$ be a finite set or $\mathbb{N}$. If $(f_k)_{n\in\mathcal{N}}$ and $(g_k)_{n\in\mathcal{N}}$ are orthonormal sequences in ℓ_2, $(\lambda_k)_{n\in\mathcal{N}} \subset \mathbb{C}\setminus\{0\}$ and $x := \sum_{n\in\mathcal{N}} \lambda_k\langle\cdot, f_k\rangle g_k \in \mathcal{L}(s', s)$ (the series converging in the norm $||\cdot||_{\ell_2\to\ell_2}$), then $(f_k)_{k\in\mathbb{N}}, (g_k)_{k\in\mathcal{N}} \subset s$.*

Proof. Since $x\colon s' \to s$, it follows that $g_k = \frac{1}{\lambda_k} x f_k \in s$ for every $k \in \mathcal{N}$. Moreover, $\sum_{n\in\mathcal{N}} \overline{\lambda_k}\langle\cdot, g_k\rangle f_k = x^* \in \mathcal{L}(s', s)$ (see Proposition 1.8), hence $f_k = \frac{1}{\lambda_k} x^* g_k \in s$ for every $k \in \mathcal{N}$. □

Theorem 3.8. *Let x be an infinite-dimensional operator in $\mathcal{L}(s',s)$ with a Schmidt representation $x=\sum_{k=1}^{\infty}s_k\langle\cdot,f_k\rangle g_k$, i.e. $(s_k)_{k\in\mathbb{N}}\subset[0,\infty)$ is a non-increasing null sequence, $(f_k)_{k\in\mathbb{N}}$, $(g_k)_{k\in\mathbb{N}}$ are orthonormal sequences in ℓ_2 and the series converges in the norm $||\cdot||_{\ell_2\to\ell_2}$. Then $(s_k)_{k\in\mathbb{N}}\in s$, $(f_k)_{k\in\mathbb{N}},(g_k)_{k\in\mathbb{N}}\subset s$ and the series converges absolutely in $\mathcal{L}(s',s)$. Moreover, $(s_k^{\theta}|f_k|_q|g_k|_q)_{k\in\mathbb{N}}\in s$ for all $q\in\mathbb{N}_0$ and every $\theta\in(0,1]$.*

Proof. By Proposition 1.15, $(s_k)_{k\in\mathbb{N}}\in s$ and, by Proposition 3.7, $(f_k)_{k\in\mathbb{N}},(g_k)_{k\in\mathbb{N}}\subset s$.

Take $q\in\mathbb{N}_0$ and $\theta\in(0,1]$. We claim that $(s_k^{\theta}|f_k|_q|g_k|_q)_{k\in\mathbb{N}}\in s$; this will imply that the series $\sum_{k=1}^{\infty}s_k\langle\cdot,f_k\rangle g_k$ converges absolutely in $\mathcal{L}(s',s)$.

As in the proof of Lemma 3.5, we consider the continuous operator $T_x\colon\mathcal{L}(\ell_2)\to\mathcal{L}(s',s)$ mapping $z\in\mathcal{L}(\ell_2)$ to the composition:

$$s'\xrightarrow{x}s\hookrightarrow\ell_2\xrightarrow{z}\ell_2\hookrightarrow s'\xrightarrow{x}s.$$

Since the sequence of one-dimensional operators $(\langle\cdot,g_k\rangle f_k)_{k\in\mathbb{N}}$ is bounded in $\mathcal{L}(\ell_2)$, it follows that $(s_k^2\langle\cdot,f_k\rangle g_k)_{k\in\mathbb{N}}=(T_x(\langle\cdot,g_k\rangle f_k))_{k\in\mathbb{N}}$ is bounded in $\mathcal{L}(s',s)$. Hence,

$$\sup_{k\in\mathbb{N}}s_k^2|f_k|_q|g_k|_q<\infty,$$

because

$$||\langle\cdot,f_k\rangle g_k||_q=|f_k|_q|g_k|_q.$$

Therefore, since $||\cdot||_{\ell_2\to\ell_2}$ is a dominating norm on $\mathcal{L}(s',s)$ and $||\langle\cdot,f_k\rangle g_k||_{\ell_2\to\ell_2}=1$ for $k\in\mathbb{N}$, Lemma 3.3 (applied to the sequences $(s_k^2)_{k\in\mathbb{N}}$ and $(\langle\cdot,f_k\rangle g_k)_{k\in\mathbb{N}}$) implies that

$$\sup_{k\in\mathbb{N}}|s_k|^{\theta/2}|f_k|_q|g_k|_q<\infty.$$

If we combine this with Proposition 1.6, we get

$$\sup_{k\in\mathbb{N}}|s_k|^{\theta}|f_k|_q|g_k|_q k^{q'}=\sup_{k\in\mathbb{N}}|s_k|^{\theta/2}|f_k|_q|g_k|_q\cdot\sup_{k\in\mathbb{N}}|s_k|^{\theta/2}k^{q'}<\infty$$

for every $q'\in\mathbb{N}_0$, which completes the proof. □

Corollary 3.9. *Let x be a compact infinite-dimensional normal operator on ℓ_2 with a Schmidt representation $x=\sum_{k=1}^{\infty}s_k\langle\cdot,f_k\rangle g_k$. Then the following assertions are equivalent:*

(i) $x\in\mathcal{L}(s',s)$ *(as an operator on ℓ_2);*

(ii) $f_k,g_k\in s$ *for* $k\in\mathbb{N}$ *and* $(s_k^{\theta}|f_k|_q|g_k|_q)_{k\in\mathbb{N}}\in s$ *for all* $q\in\mathbb{N}_0$ *and* $\theta\in(0,1]$*;*

(iii) $f_k,g_k\in s$ *for* $k\in\mathbb{N}$, $(s_k)_{k\in\mathbb{N}}\in s$ *and* $\sup_{k\in\mathbb{N}}s_k|f_k|_q|g_k|_q<\infty$ *for all* $q\in\mathbb{N}_0$*;*

(iv) $f_k,g_k\in s$ *for* $k\in\mathbb{N}$ *and* $\sum_{k=1}^{\infty}s_k|f_k|_q|g_k|_q<\infty$ *for all* $q\in\mathbb{N}_0$.

Moreover, if $x=\sum_{k=1}^{N}s_k\langle\cdot,f_k\rangle g_k$ is a Schmidt representation of a finite-dimensional operator on ℓ_2, then $x\in\mathcal{L}(s',s)$ if and only if $f_k,g_k\in s$ for $k=1,\dots,N$.

Proof. The implication (i)⇒(ii) follows directly from Theorem 3.8 and the implication (ii)⇒(iii) is trivially satisfied.

(iii)⇒(iv): We have

$$\sum_{k=1}^{\infty}s_k|f_k|_q|g_k|_q\le\sup_{k\in\mathbb{N}}s_k^{1/2}|f_k|_q|g_k|_q\cdot\sum_{k=1}^{\infty}s_k^{1/2}<\infty,$$

because, by Lemma 3.3 (applied to the space s and sequences $(s_k)_{k\in\mathbb{N}}$, $(f_k)_{k\in\mathbb{N}}$, $(g_k)_{k\in\mathbb{N}}$), $\sup_{k\in\mathbb{N}} s_k^{1/2}|f_k|_q|g_k|_q < \infty$ and, by Proposition 1.7, $s \subset \ell_{1/2}$.

(iv)⇒(i): By inequality (1.1) from Preliminaries §1, each $\langle\cdot, f_k\rangle g_k$ belongs to $\mathcal{L}(s', s)$. Hence, by assumption, the series $\sum_{k=1}^{\infty} s_k\langle\cdot, f_k\rangle g_k$ is absolutely convergent in $\mathcal{L}(s', s)$, and thus $x \in \mathcal{L}(s', s)$.

The finite-dimensional case is an immediate consequence of Proposition 3.7. □

Chapter 4

Closed commutative *-subalgebras of $\mathcal{L}(s', s)$

This chapter is devoted to the study of closed commutative *-subalgebras of $\mathcal{L}(s', s)$. We apply the spectral representation theorem for normal smooth operators (Theorem 3.1) to show that such algebras are isomorphic (as Fréchet *-algebras) to Köthe algebras

$$\lambda^\infty((||P_k||_q)_{k\in\mathbb{N},q\in\mathbb{N}_0})$$

($\lambda^\infty(||P_k||_q)$ for short) for an appropriate sequence $(P_k)_{k\in\mathbb{N}}$ of pairwise orthogonal projections belonging to $\mathcal{L}(s', s)$. Conversely, for every sequence $(P_k)_{k\in\mathbb{N}} \subset \mathcal{L}(s', s)$ of pairwise orthogonal projections, the algebra $\lambda^\infty(||P_k||_q)$ is isomorphic as a Fréchet *-algebra to a closed commutative *-subalgebra of $\mathcal{L}(s', s)$. Observe that, since $||P_k||_q \geq ||P_k||_{\ell_2} = 1$, the Köthe space $\lambda^\infty(||P_k||_q)$ is really a Fréchet *-algebra with pointwise multiplication and conjugation as involution.

Next, it turns out that every algebra $\lambda^\infty(||P_k||_q)$ is isomorphic (as a Fréchet *-algebra) to some algebra

$$\lambda^\infty((\max_{j\in\mathcal{N}_k} |f_j|_q)_{j\in\mathbb{N},q\in\mathbb{N}_0})$$

(again, to simplify notation, we write $\lambda^\infty(\max |f_j|_q)$) for some orthonormal sequence $(f_j)_{j\in\mathbb{N}} \subset s$ and a family $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ of finite nonempty pairwise disjoint sets of natural numbers, i.e. they are isomorphic to closed *-subalgebras of $\lambda^\infty(|f_j|_q)$. That is why we can reduce investigations of closed commutative *-subalgebras of $\mathcal{L}(s', s)$ to the study of orthonormal sequences whose elements belong to the space s. Developing this idea, we give a characterization of some special types of closed commutative *-subalgebras of $\mathcal{L}(s', s)$: so-called maximal subalgebras, subalgebras which are isomorphic to some closed *-subalgebras of s (note that s is a commutative Fréchet *-algebra with pointwise multiplication and conjugation) and orthogonally complemented subalgebras. In particular, we show that every orthogonally complemented commutative *-subalgebra of $\mathcal{L}(s', s)$ is isomorphic to a *-subalgebra of s. We also provide some examples showing that not every commutative *-subalgebra of $\mathcal{L}(s', s)$ can be embedded isomorphically into s as a *-subalgebra as well as that not every closed commutative *-subalgebra of $\mathcal{L}(s', s)$ embeddable in s is orthogonally complemented.

In this chapter e_k denotes the vector in $\mathbb{C}^\mathbb{N}$ whose k-th coordinate equals 1 and the others equal 0.

4.1 Köthe algebra representation of closed commutative *-subalgebras of $\mathcal{L}(s', s)$

In the first section we show, applying the spectral representation theorem for normal smooth operators (Theorem 3.1), that minimal projections $(P_k)_{k\in\mathbb{N}}$ of a closed commutative *-subalgebra A of $\mathcal{L}(s', s)$ form in A a Schauder basis (the so-called canonical Schauder basis of A, see Lemma 4.4). Several consequences of this fact are derived; in particular, we prove that A is isomorphic as a Fréchet *-algebra to the corresponding Köthe algebra $\lambda^\infty(||P_k||_q)$ (Theorem 4.9). This will be also the starting point for the next sections.

Lemma 4.1. *Let A be a subalgebra of the algebra $\widetilde{A}$ over $\mathbb{C}$. Let $N \in \mathbb{N}$, $a_1, \ldots, a_N \in \widetilde{A}$, $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, $a_j \neq 0$, $a_j^2 = a_j$, $a_j a_k = 0$ for $j \neq k$, $\lambda_j \neq 0$ and $\lambda_j \neq \lambda_k$ for $j \neq k$. Then $a_1, \ldots, a_N \in A$ whenever $\lambda_1 a_1 + \ldots + \lambda_N a_N \in A$.*

Proof. We use induction on N. The case $N = 1$ is trivial.

Assume that the conclusion holds for all $M < N$. Let $a := \lambda_1 a_1 + \ldots + \lambda_N a_N \in A$. We have

$$\lambda_1^2 a_1 + \ldots + \lambda_N^2 a_N = a^2 \in A,$$

and, on the other hand,

$$\lambda_N \lambda_1 a_1 + \ldots + \lambda_N^2 a_N = \lambda_N a \in A$$

so

$$(\lambda_1^2 - \lambda_N \lambda_1) a_1 + \ldots + (\lambda_{N-1}^2 - \lambda_N \lambda_{N-1}) a_{N-1} = a^2 - \lambda_N a \in A.$$

Since $\lambda_j \neq 0$ and $\lambda_j \neq \lambda_N$ for $j \in \{1, \ldots, N-1\}$, we have $\lambda_j^2 - \lambda_N \lambda_j = \lambda_j(\lambda_j - \lambda_N) \neq 0$ for $j \in \{1, \ldots, N-1\}$. If $\lambda_j^2 - \lambda_N \lambda_j$ are pairwise different then, from the inductive assumption, $a_1, \ldots, a_{N-1} \in A$ so $a_N \in A$ as well.

Assume that these numbers are not pairwise different. Then we define an equivalence relation $\mathcal{R}$ on the set $\{1, \ldots, N-1\}$ in the following way:

$$j\mathcal{R}k \Leftrightarrow \lambda_j(\lambda_j - \lambda_N) = \lambda_k(\lambda_k - \lambda_N).$$

Let $I_1, \ldots, I_{N_1}$ denote the equivalence classes which contain no less than two elements and let $I_0 := \{i_1, \ldots, i_{N_0}\}$ be the remaining indices. From our assumption, $I_1 \neq \emptyset$. For $j \in \{1, \ldots, N_1\}$ and $k \in I_j$ let

$$\lambda'_j := \lambda_k(\lambda_k - \lambda_N)$$

and let

$$a'_j := \sum_{n \in I_j} a_n.$$

We also define

$$\lambda'_{N_1+1} := \lambda_{i_1}(\lambda_{i_1} - \lambda_N), \lambda'_{N_1+2} := \lambda_{i_2}(\lambda_{i_2} - \lambda_N), \ldots, \lambda'_{N_1+N_0} := \lambda_{i_{N_0}}(\lambda_{i_{N_0}} - \lambda_N)$$

and

$$a'_{N_1+1} := a_{i_1}, a'_{N_1+2} := a_{i_2}, \ldots, a'_{N_1+N_0} := a_{i_{N_0}}.$$

Clearly, $1 \leq N' := N_1 + N_0 < N$, $a'_j \neq 0$, $a'^2_j = a'_j$, $a'_j a'_k = 0$, $\lambda'_j \neq 0$, $\lambda'_j \neq \lambda'_k$ for $j, k \in \{1, \ldots, N'\}$, $j \neq k$ and

$$\lambda'_1 a'_1 + \ldots + \lambda'_{N'} a'_{N'} = a^2 - \lambda_N a \in A.$$

From the inductive assumption, $a_1' \in A$, hence

$$\sum_{n\in I_1} \lambda_n a_n = \sum_{n\in I_1} a_n \cdot \sum_{n=1}^{N} \lambda_n a_n = a_1' a \in A.$$

Again, from the inductive assumption, $a_n \in A$ for $n \in I_1$, and therefore $\sum_{n\in\{1,\ldots,N\}\setminus I_1} \lambda_n a_n \in A$. Once again, from the inductive assumption, $a_n \in A$ for $n \in \{1,\ldots,N\}\setminus I_1$. Thus $a_1,\ldots,a_N \in A$, which completes the proof. $\square$

Recall that, by Theorem 3.1, every infinite-dimensional normal operator $x \in \mathcal{L}(s', s)$ has a unique spectral representation

$$x = \sum_{k=1}^{\infty} \lambda_k P_k,$$

where $(\lambda_k)_{k\in\mathbb{N}}$ is a non-increasing (in modulus) sequence in s of nonzero pairwise different elements, $(P_k)_{k\in\mathbb{N}}$ is a sequence of nonzero pairwise orthogonal finite-dimensional projections belonging to $\mathcal{L}(s', s)$ (i.e., P_k as an operator on ℓ_2, is a projection) and the series converges absolutely in $\mathcal{L}(s', s)$.

Proposition 4.2. *Let A be a closed *-subalgebra of $\mathcal{L}(s', s)$ (not necessarily commutative) and let x be an infinite-dimensional normal operator in $\mathcal{L}(s', s)$ with spectral representation $x = \sum_{k=1}^{\infty} \lambda_k P_k$. Then $x \in A$ if and only if $P_k \in A$ for all $k \in \mathbb{N}$.*

Proof. Let $N_0 := 0$, $N_1 := \sup\{k \in \mathbb{N} : |\lambda_k| = |\lambda_1|\}$ and for $j = 2, 3, \ldots$ let $N_j := \sup\{k \in \mathbb{N} : |\lambda_k| = |\lambda_{N_{j-1}+1}|\}$. Since $(|\lambda_k|)_{k\in\mathbb{N}}$ is a null sequence, we have $N_j < \infty$ for all $j \in \mathbb{N}$.

By Theorem 3.1, if $P_k \in A$ for all $k \in \mathbb{N}$ then $x \in A$. To prove the converse assume that $x \in A$. Then $x^* = \sum_{k=1}^{\infty} \overline{\lambda_k} P_k \in A$ so $xx^* = \sum_{k=1}^{\infty} |\lambda_k|^2 P_k \in A$, whence

$$y_n := \sum_{k=1}^{\infty} \left(\frac{|\lambda_k|}{|\lambda_1|}\right)^{2n} P_k = \left(\frac{xx^*}{|\lambda_1|^2}\right)^n \in A$$

for all $n \in \mathbb{N}$. Hence for arbitrary q and n we get

$$\begin{aligned}
||y_n - (P_1 + \ldots + P_{N_1})||_q &= \left|\left| \sum_{k=1}^{\infty} \left(\frac{|\lambda_k|}{|\lambda_1|}\right)^{2n} P_k - (P_1 + \ldots + P_{N_1}) \right|\right|_q \\
&= \left|\left| \sum_{k=N_1+1}^{\infty} \left(\frac{|\lambda_k|}{|\lambda_1|}\right)^{2n} P_k \right|\right|_q \\
&\leq \sum_{k=N_1+1}^{\infty} \left(\frac{|\lambda_k|}{|\lambda_1|}\right)^{2n} ||P_k||_q \\
&\leq \frac{1}{|\lambda_1|} \left(\frac{|\lambda_{N_1+1}|}{|\lambda_1|}\right)^{2n-1} \sum_{k=N_1+1}^{\infty} |\lambda_k| \, ||P_k||_q.
\end{aligned}$$

By Theorem 3.1, $\sum_{k=N_1+1}^{\infty} |\lambda_k| \, ||P_k||_q < \infty$, and moreover $\frac{|\lambda_{N_1+1}|}{|\lambda_1|} < 1$. Thus

$$||y_n - (P_1 + \ldots + P_{N_1})||_q \to 0$$

as $n \to \infty$. Therefore, since A is closed, we conlude that $P_1 + \ldots + P_{N_1} \in A$. Consequently,

$$\sum_{k=N_1+1}^{\infty} |\lambda_k|^2 P_k = xx^* - |\lambda_1|^2 (P_1 + \ldots + P_{N_1}) \in A;$$

hence, proceeding by induction, $P_{N_j+1} + \ldots + P_{N_{j+1}} \in A$ for $j \in \mathbb{N}_0$, so

$$\sum_{k=N_j+1}^{N_{j+1}} \lambda_k P_k = (P_{N_j+1} + \ldots P_{N_{j+1}})x \in A.$$

Finally, by Lemma 4.1, $P_k \in A$ for $k \in \mathbb{N}$. □

Proposition 4.3. *For every othonormal system $(f_k)_{k\in\mathbb{N}}$ in ℓ_2 and a sequence $(\lambda_k)_{k\in\mathbb{N}} \in c_0$, the series $\sum_{k=1}^{\infty} \lambda_k \langle\cdot, f_k\rangle f_k$ converges in the norm $||\cdot||_{l_2\to l_2}$.*

Proof. This is a simple consequence of the Pythagorean theorem and the Bessel inequality. □

Lemma 4.4. *Let A be a commutative subalgebra of $\mathcal{L}(s', s)$. Let $\mathcal{P}$ denote the set of nonzero (self-adjoint) projections belonging to A and let $\mathcal{M}$ be the set of minimal elements in $\mathcal{P}$ with respect to the order relation*

$$\forall P, Q \in \mathcal{P} \quad P \preceq Q \Leftrightarrow PQ = QP = P.$$

Then

(i) *$\mathcal{M}$ is an at most countable family of pairwise orthogonal projections belonging to $\mathcal{L}(s', s)$ such that*
$$\forall P \in \mathcal{P}\ \exists P'_1, \ldots, P'_m \in \mathcal{M} \quad P = P'_1 + \cdots + P'_m.$$

(ii) *If A is also a closed *-subalgebra of $\mathcal{L}(s', s)$, then $\mathcal{M}$ is a Schauder basis in A.*

For a closed commutative *-subalgebra A of $\mathcal{L}(s', s)$ the Schauder basis $\mathcal{M}$ from Lemma 4.4 will be called the *canonical Schauder basis* (of A).

Proof. (i) By definition

$$\mathcal{M} = \{P \in \mathcal{P} : \forall Q \in \mathcal{P} \quad (Q \preceq P \Rightarrow Q = P)\}.$$

Firstly, we will show that

$$\forall P \in \mathcal{P}\ \exists P'_1, \ldots, P'_m \in \mathcal{M} \quad P = P'_1 + \cdots + P'_m. \tag{4.1}$$

Take $P \in \mathcal{P}$. If $P \in \mathcal{M}$, then we are done. Otherwise, there is $Q \in \mathcal{P}$ such that $Q \preceq P$, $Q \neq P$. Of course, $P - Q \in \mathcal{P}$. If $Q, P - Q \in \mathcal{M}$, then $P = Q + (P - Q)$ is the desired decomposition. Otherwise, we decompose Q or $P - Q$ into smaller projections as was done above for P. Since P is finite-dimensional, after finitely many steps we finish our procedure.

Next, we shall prove that projections in $\mathcal{M}$ are pairwise orthogonal. Let $P, Q \in \mathcal{M}$, $P \neq Q$ and suppose, to derive a contradiction, that $PQ \neq 0$. Since A is commutative,

$$(PQ)^2 = P^2Q^2 = PQ \quad \text{and} \quad (PQ)^* = PQ$$

and thus $PQ \in \mathcal{P}$. Moreover,

$$P(PQ) = P^2Q = PQ$$

so $PQ \preceq P$. Now, $PQ \neq P$ implies that $P \notin \mathcal{M}$ and if $PQ = P$ then $Q \notin \mathcal{M}$, which is a contradiction.

Finally, since projections in $\mathcal{M}$ are pairwise orthogonal (as projections on ℓ_2), $\mathcal{M}$ is at most countable.

(ii) Let $x \in A$. If x is finite-dimensional, then x has spectral decomposition of the form $\sum_{k=1}^{N} \mu_k Q_k$. Hence by Lemma 4.1 and (i), x is a linear combination of projections in $\mathcal{M}$.

Assume that x is infinite-dimensional and let $x = \sum_{k=1}^{\infty} \mu_k Q_k$ (spectral representation of x). Since A is a closed commutative *-subalgebra of $\mathcal{L}(s', s)$, by Proposition 4.2, $Q_k \in A$ for $k \in \mathbb{N}$. Next, from (i),

$$\forall k \in \mathbb{N}\ \exists Q_1^{(k)}, \ldots, Q_{l_k}^{(k)} \in \mathcal{M} \quad Q_k = \sum_{j=1}^{l_k} Q_j^{(k)}.$$

Hence

$$x = \sum_{k=1}^{\infty} \sum_{j=1}^{l_k} \mu_k Q_j^{(k)}.$$

For $l_0 = 0$, $j = l_0 + l_1 + \ldots + l_{k-1} + n$, $1 \leq n \leq l_k$ let $P_j := Q_n^{(k)}$ and let $\lambda_j := \mu_k$. Consider the series $\sum_{k=1}^{\infty} \lambda_k P_k$. Clearly, if the series converges in $\mathcal{L}(s', s)$ (or in $||\cdot||_{\ell_2 \to \ell_2}$) then its limit is x. We shall first show that the series converges in the norm $||\cdot||_{\ell_2 \to \ell_2}$.

Since P_k is a (self-adjoint) projection of finite dimension d_k, we have $P_k = \sum_{j=1}^{d_k} \langle \cdot, e_j^{(k)} \rangle e_j^{(k)}$ for every orthonormal basis $(e_j^{(k)})_{j=1}^{d_k}$ of the image of P_k. For $d_0 = 0$, $j = d_0 + d_1 + \ldots + d_{k-1} + n$, $1 \leq n \leq d_k$ let $e_j := e_n^{(k)}$ and let $\lambda'_j := \lambda_k$. By Proposition 4.3, the series $\sum_{j=1}^{\infty} \lambda'_j \langle \cdot, e_j \rangle e_j$ converges in the norm $||\cdot||_{\ell_2 \to \ell_2}$. Hence $\sum_{k=1}^{\infty} \lambda_k P_k$ converges in the norm $||\cdot||_{\ell_2 \to \ell_2}$ because $(\sum_{k=1}^{N} \lambda_k P_k)_{N \in \mathbb{N}}$ is a subsequence of the sequence of partial sums of the series $\sum_{j=1}^{\infty} \lambda'_j \langle \cdot, e_j \rangle e_j$.

Now, by Lemma 3.5, $x = \sum_{k=1}^{\infty} \lambda_k P_k$ and the series converges absolutely in $\mathcal{L}(s', s)$. This shows that every operator in A is represented by an absolutely convergent series $\sum_{k=1}^{\infty} \lambda''_k P''_k$ with $P''_k \in \mathcal{M}$. To prove the uniqueness of this representation assume that $\sum_{k=1}^{\infty} \lambda''_k P''_k = 0$. Then

$$\lambda''_m P''_m = P''_m \sum_{k=1}^{\infty} \lambda''_k P''_k = 0$$

so $\lambda''_m = 0$ for $m \in \mathbb{N}$. This shows that the sequence of coefficients is unique, hence $\mathcal{M}$ is a Schauder basis in A. □

If A is an arbitrary algebra, then $\widehat{A}$ denotes the set of nonzero *-multiplicative functionals on A (the so-called Gelfand space of A).

Corollary 4.5. *The set $\widehat{A}$ of nonzero *-multiplicative functionals on a closed commutative *-subalgebra A of $\mathcal{L}(s', s)$ is exactly the set of coefficient functionals with respect to the canonical Schauder basis of A.*

Proof. Clearly, every coefficient functional is *-multiplicative. Conversely, if φ is a nonzero *-multiplicative functional on A and $\{P_n\}_{n \in \mathbb{N}}$ is the canonical Schauder basis then $\varphi(P_n) = \varphi(P_n^2) = (\varphi(P_n))^2$, thus $\varphi(P_n) = 0$ or $\varphi(P_n) = 1$. Suppose that $\varphi(P_n) = \varphi(P_m) = 1$ for $n \neq m$. Then

$$\begin{aligned} 2 &= \varphi(P_n) + \varphi(P_m) = \varphi(P_n + P_m) = \varphi((P_n + P_m)^2) = (\varphi(P_n + P_m))^2 \\ &= (\varphi(P_n) + \varphi(P_m))^2 = 4, \end{aligned}$$

a contradiction. Hence, there is at most one $n \in \mathbb{N}$ such that $\varphi(P_n) = 1$. If $\varphi(P_n) = 0$ for all $n \in \mathbb{N}$ then, since $\{P_n\}_{n \in \mathbb{N}}$ is a basis, $\varphi = 0$, a contradiction. Thus, there is exactly one $n \in \mathbb{N}$ such that $\varphi(P_n) = 1$ and $\varphi(P_m) = 0$ for $m \neq n$, i.e. φ is a coefficient functional. □

Corollary 4.6. *Let A be one of the following Fréchet *-algebras with pointwise multiplication (without a unit):*

(i) *the algebra $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions;*

(ii) *the algebra $\mathcal{D}(K)$ of test functions with support in a compact set $K \subset \mathbb{R}^n$ such that* $\operatorname{int}(K) \neq \emptyset$*;*

(iii) *the algebra $C_a^\infty(M)$ of smooth functions on a compact smooth manifold M vanishing at $a \in M$;*

(iv) *the algebra $C_a^\infty(\overline{\Omega})$ of smooth functions on $\overline{\Omega}$ vanishing at $a \in \Omega$, where $\Omega \neq \emptyset$ is an open bounded subset of $\mathbb{R}^n$ with C^1-boundary;*

(v) *the algebra $\mathcal{E}_a(K)$ of Withney jets on a compact set $K \subset \mathbb{R}^n$ with the extension property, flat at $a \in K$ and such that* $\operatorname{int}(K) \neq \emptyset$.

Then A is isomorphic to s as a Fréchet space but it is not isomorphic to any closed commutative *-subalgebra of $\mathcal{L}(s', s)$ as a Fréchet *-algebra.

Proof. It is well-known that the spaces in (i)–(v) are isomorphic to s as Fréchet spaces (see e.g. [20, Ch. 31], [32, Satz 4.1]).

To prove the second assertion let us compare the relevant sets of *-multiplicative functionals. If A is one of the spaces from items (i)–(v), then every point evaluation functional on A is *-multiplicative and since the underlying space has the cardinality $\mathfrak{c}$ of the continuum, the cardinality of the set of *-multiplicative functionals on A is no less than $\mathfrak{c}$. On the other hand, by Corollary 4.5, the set of *-multiplicative functionals on any infinite dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$ is at most countable, hence none of the spaces from (i)–(v) is isomorphic to A. □

It is clear that the algebra s with pointwise multiplication and conjugation is a *-subalgebra of $\mathcal{L}(s', s)$ (consider, for example, diagonal operators). The previous corollary shows that it is not the case for many other interesting Fréchet *-algebras isomorphic to s (as Fréchet spaces).

For a subset Z of $\mathcal{L}(s', s)$ we will denote by $\operatorname{alg}(Z)$ the closed *-subalgebra of $\mathcal{L}(s', s)$ generated by Z.

Proposition 4.7. *If $\{P_k\}_{k\in\mathcal{N}}$ is a family of pairwise orthogonal projections belonging to $\mathcal{L}(s', s)$, then*

$$\operatorname{alg}(\{P_k\}_{k\in\mathcal{N}}) = \overline{\operatorname{lin}}(\{P_k\}_{k\in\mathcal{N}})$$

and $\operatorname{alg}(\{P_k\}_{k\in\mathcal{N}})$ is a commutative *-algebra.

Proof. Clearly, $\overline{\operatorname{lin}}(\{P_k\}_{k\in\mathcal{N}}) \subseteq \operatorname{alg}(\{P_k\}_{k\in\mathcal{N}})$ and $\operatorname{lin}(\{P_k\}_{k\in\mathcal{N}})$ is a commutative *-algebra. By the continuity of multiplication and involution, $\overline{\operatorname{lin}}(\{P_k\}_{k\in\mathcal{N}})$ is a commutative *-algebra as well. Hence, $\overline{\operatorname{lin}}(\{P_k\}_{k\in\mathcal{N}}) = \operatorname{alg}(\{P_k\}_{k\in\mathcal{N}})$. □

Proposition 4.8. *Every sequence $\{P_k\}_{k\in\mathcal{N}} \subset \mathcal{L}(s', s)$ of nonzero pairwise orthogonal projections is the canonical Schauder basis of the algebra $\operatorname{alg}(\{P_k\}_{k\in\mathcal{N}})$. In particular, $\{P_k\}_{k\in\mathcal{N}}$ is a basic sequence in $\mathcal{L}(s', s)$, i.e. it is a Schauder basis of the Fréchet space $\overline{\operatorname{lin}}(\{P_k\}_{k\in\mathcal{N}})$.*

Proof. Let $\mathcal{M}$ be the canonical Schauder basis of $A := \mathrm{alg}(\{P_k\}_{k\in\mathcal{N}})$ which consists of all projections which are minimal with respect to the order relation described in Lemma 4.4. We shall show that $\{P_k\}_{k\in\mathcal{N}} = \mathcal{M}$, and then the second statement follows from Proposition 4.7.

Fix $k \in \mathcal{N}$ and assume that $Q \preceq P_k$ for some nonzero projection $Q \in A$, i.e. $QP_k = Q$. Since $A = \overline{\mathrm{lin}}(\{P_k\}_{k\in\mathcal{N}})$, we have

$$Q = \lim_{j\to\infty} \sum_{n=1}^{M_j} \lambda_n^{(j)} P_n$$

for some $M_j \in \mathbb{N}$ and $\lambda_n^{(j)} \in \mathbb{C}$. From the continuity of algebra multiplication and scalar multiplication, we get

$$\begin{aligned} Q = QP_k &= \Big(\lim_{j\to\infty} \sum_{n=1}^{M_j} \lambda_n^{(j)} P_n\Big)P_k = \lim_{j\to\infty} \Big(\sum_{n=1}^{M_j} \lambda_n^{(j)} P_n P_k\Big) = \lim_{j\to\infty} \lambda_k^{(j)} P_k \\ &= (\lim_{j\to\infty} \lambda_k^{(j)})P_k = \lambda_k P_k, \end{aligned}$$

where $\lambda_k := \lim_{j\to\infty} \lambda_k^{(j)} \in \mathbb{C}$. Since Q is a nonzero projection, we deduce that $\lambda_k = 1$ and $Q = P_k$. Hence $\{P_k\}_{k\in\mathcal{N}} \subseteq \mathcal{M}$.

Now, suppose that there is a projection Q in $\mathcal{M} \setminus \{P_k\}_{k\in\mathcal{N}}$. We have already proved that $\{P_k\}_{k\in\mathcal{N}} \subseteq \mathcal{M}$, hence by Lemma 4.4(i), $Qx = 0$ for all $x \in \mathrm{lin}(\{P_k\}_{k\in\mathcal{N}})$. By continuity of multiplication, $Qx = 0$ for every $x \in \overline{\mathrm{lin}}(\{P_k\}_{k\in\mathcal{N}}) = A$. In particular, $Q = Q^2 = 0$, a contradiction. Hence, $\{P_k\}_{k\in\mathcal{N}} = \mathcal{M}$. □

Closed commutative *-subalgebras of $\mathcal{L}(s', s)$ are, in some sense, quite simple: each of them is generated by a single operator and also by its spectral projections. From nuclearity we also get the following sequence space representations.

Theorem 4.9. *Let A be a closed commutative infinite-dimensional *-subalgebra of $\mathcal{L}(s', s)$ and let $\{P_k\}_{k\in\mathbb{N}}$ be the canonical Schauder basis of A (see Lemma 4.4 and the definition below). Then*

$$A = \mathrm{alg}(\{P_k\}_{k\in\mathbb{N}}) \cong \lambda^\infty(||P_k||_q)$$

as Fréchet *-algebras and the isomorphism is given by $P_k \mapsto e_k$ for $k \in \mathbb{N}$. Moreover, there is an operator $x \in A$ with spectral representation $x = \sum_{k=1}^\infty \lambda_k P_k$ such that $A = \mathrm{alg}(x)$.

Proof. By Proposition 4.7, $A = \overline{\mathrm{lin}}(\{P_k\}_{k\in\mathbb{N}}) = \mathrm{alg}(\{P_k\}_{k\in\mathbb{N}})$, and, from the nuclearity of the space $\mathcal{L}(s', s) \cong s$ (see Preliminaries §1),

$$A \cong \lambda^1(||P_k||_q) = \lambda^\infty(||P_k||_q)$$

as Fréchet spaces (see e.g. [20, Cor. 28.13, Prop. 28.16]), where the isomorphism is given by $P_k \mapsto e_k$ for $k \in \mathbb{N}$. Moreover, since on the linear span of $\{P_k\}_{k\in\mathbb{N}}$ multiplication (resp. involution) corresponds to pointwise multiplication (resp. conjugation) in $\lambda^1(||P_k||_q)$, the isomorphism is also a *-algebra isomorphism, where the Köthe space is equipped with pointwise multiplication.

Now, we shall show that there is a decreasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of positive numbers such that the series $\sum_{k=1}^\infty \lambda_k P_k$ is absolutely convergent in $\mathcal{L}(s', s)$. To do so, choose a sequence $(C_q)_{q\in\mathbb{N}}$ such that $C_q \geq \max_{1\leq k\leq q} ||P_k||_q$. Clearly, $C_q/||P_k||_q \geq 1$ for $q \geq k$, so

$$\inf_{q\in\mathbb{N}} \frac{C_q}{||P_k||_q} \geq \min\left\{\frac{C_1}{||P_k||_1}, \frac{C_2}{||P_k||_2}, \ldots, \frac{C_{k-1}}{||P_k||_{k-1}}, 1\right\} > 0$$

for $k \in \mathbb{N}$. Let $\lambda_1 := 1$ and let

$$\lambda_k := \min\left\{\frac{1}{k^2}\inf_{q\in\mathbb{N}}\frac{C_q}{||P_k||_q}, \frac{\lambda_{k-1}}{2}\right\}.$$

Then $\lambda_k > 0$, the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is strictly decreasing and

$$\sum_{k=1}^{\infty}\lambda_k||P_k||_q \leq \sum_{k=1}^{\infty}\frac{1}{k^2}\inf_{r\in\mathbb{N}}\frac{C_r}{||P_k||_r}||P_k||_q \leq C_q\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty.$$

Consequently, $x := \sum_{k=1}^{\infty}\lambda_k P_k \in \mathcal{L}(s', s)$ and this series is the spectral representation of x. Moreover, since $P_k \in A$ for $k \in \mathbb{N}$ and A is closed, we have $x \in A$. Finally, the equality $\mathrm{alg}(x) = \mathrm{alg}(\{P_k\}_{k\in\mathbb{N}})$ is a consequence of Proposition 4.2. □

4.2 Closed maximal commutative *-subalgebras of $\mathcal{L}(s', s)$

A closed commutative *-subalgebra of $\mathcal{L}(s', s)$ is said to be *maximal commutative* if it is not properly contained in any larger closed commutative *-subalgebra of $\mathcal{L}(s', s)$. We begin this section with a characterization of canonical Schauder bases of closed maximal commutative *-subalgebras of $\mathcal{L}(s', s)$ (Theorem 4.11). Then we show that every such algebra A is isomorphic to a Köthe algebra $\lambda^\infty(|f_k|_q)$, where $(f_k)_{k\in\mathbb{N}} \subset s$ is an orthonormal sequence corresponding to the canonical Schauder basis of A (Corollaries 4.16, 4.21).

By the Kuratowski-Zorn lemma, every closed commutative *-subalgebra of $\mathcal{L}(s', s)$ is contained in some closed maximal commutative *-subalgebra of $\mathcal{L}(s', s)$ (Proposition 4.12), and therefore the class of closed commutative *-subalgebras of $\mathcal{L}(s', s)$ is (in the sense of Fréchet *-algebra isomorphism) the class of closed commutative *-subalgebras of $\lambda^\infty(|f_k|_q)$, $(f_k)_{k\in\mathbb{N}} \subset s$ being an orthonormal sequence (see Theorems 4.17, 4.20 and Corollaries 4.22, 4.23). We provide formulas for the corresponding isomorphisms; the Kuratowski-Zorn lemma is used only in the proofs of Propositions 4.12 and 4.13.

For a subset Z of $\mathcal{L}(s', s)$, the set

$$\mathrm{comm}(Z) := \{x \in \mathcal{L}(s', s) : xy = yx \text{ for all } y \in Z\}$$

is called the *commutant* of Z. Let us note that the commutant of $Z \subset \mathcal{L}(s', s)$ differs from the classical commutant of Z as a subset of $\mathcal{L}(\ell_2)$. We will also show the relation between closed maximal commutative *-subalgebras of $\mathcal{L}(s', s)$ and their commutants (see Theorem 4.11 and compare with the case of C^*-algebras [24, 2.8.1]).

Proposition 4.10. *For every self-adjoint subset Z of $\mathcal{L}(s', s)$, the commutant* $\mathrm{comm}(Z)$ *is a closed *-subalgebra of $\mathcal{L}(s', s)$.*

Proof. Clearly, if x, y commute with every $z \in Z$ then λx, $x + y$, xy and x^* commute as well. Hence, from the continuity of the algebra operations and the involution, $\mathrm{comm}(Z)$ is a closed *-subalgebra of $\mathcal{L}(s', s)$. □

We say that a sequence $\{P_k\}_{k\in\mathbb{N}}$ of nonzero pairwise orthogonal projections belonging to $\mathcal{L}(s', s)$ is *complete* if there is no nonzero projection P belonging to $\mathcal{L}(s', s)$ such that $P_kP = 0$ for every $k \in \mathbb{N}$. We say that an orthonormal system $(f_k)_{k\in\mathbb{N}}$ of ℓ_2 is *s-complete*, if every f_k belongs to s and for every $\xi \in s$ the following implication holds: if $\langle\xi, f_k\rangle = 0$ for every $k \in \mathbb{N}$, then $\xi = 0$.

Theorem 4.11. *For a closed commutative *-subalgebra A of $\mathcal{L}(s', s)$ the following assertions are equivalent:*

(i) *A is maximal commutative;*

(ii) *the canonical Schauder basis $\{P_k\}_{k\in\mathcal{N}}$ of A is a complete sequence of pairwise orthogonal one-dimensional projections belonging to $\mathcal{L}(s', s)$;*

(iii) *there is an s-complete sequence $(f_k)_{k\in\mathbb{N}}$ such that $(\langle\cdot, f_k\rangle f_k)_{k\in\mathbb{N}}$ is the canonical Schauder basis of A.*

(iv) *$A = \mathrm{comm}(A)$.*

Proof. (i)$\Rightarrow$(ii): Suppose that for some $m \in \mathbb{N}$ the projection P_m is not one-dimensional. Then there are two nonzero pairwise orthogonal projections $Q_1, Q_2 \in \mathcal{L}(s', s)$ such that $P_m = Q_1 + Q_2$. By Proposition 4.7, $\overline{\mathrm{lin}}(\{P_k : k \neq m\} \cup \{Q_1, Q_2\})$ is a closed commutative *-subalgebra of $\mathcal{L}(s', s)$, and clearly

$$A = \overline{\mathrm{lin}}(\{P_k\}_{k\in\mathbb{N}}) \subseteq \overline{\mathrm{lin}}(\{P_k : k \neq m\} \cup \{Q_1, Q_2\}).$$

By Proposition 4.8, $\{P_k\}_{k\in\mathbb{N}}$ is the canonical Schauder basis of A, and $\{P_k : k \neq m\} \cup \{Q_1, Q_2\}$ is the canonical Schauder basis of $\overline{\mathrm{lin}}(\{P_k : k \neq m\} \cup \{Q_1, Q_2\})$, so

$$A \neq \overline{\mathrm{lin}}(\{P_k : k \neq m\} \cup \{Q_1, Q_2\}).$$

Thus, A is not maximal, a contradiction.

If $P \in \mathcal{L}(s', s)$ is a nonzero projection orthogonal to all P_k, then, using similar arguments, we find that $\overline{\mathrm{lin}}(\{P_k\}_{k\in\mathbb{N}} \cup \{P\})$ is a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ properly containing A, a contradiction.

(ii)$\Leftrightarrow$(iii): One can easily show that an orthonormal system $(f_k)_{k\in\mathbb{N}}$ is s-complete if and only if the sequence of projections $(\langle\cdot, f_k\rangle f_k)_{k\in\mathbb{N}}$ is complete in $\mathcal{L}(s', s)$.

(ii)$\Rightarrow$(iv): Since A is commutative, we get $A \subset \mathrm{comm}(A)$. Now, suppose that there is $x \in \mathrm{comm}(A) \setminus A$. By Proposition 4.10, $x^* \in \mathrm{comm}(A)$ so $x + x^*, i(x - x^*) \in \mathrm{comm}(A)$, and moreover $x^* \notin A$. Since $x = \frac{x+x^*}{2} + \frac{i(x-x^*)}{2i}$, we have $x + x^* \notin A$ or $i(x - x^*) \notin A$. Without loss of generality assume that $x + x^* \notin A$. The operator $x + x^*$ is self-adjoint, hence it has a spectral representation $\sum_{m=1}^{\infty} \mu_m Q_m$. Then, by Propositions 4.2 and 4.10, $Q_m \in \mathrm{comm}(A)$ for all $m \in \mathbb{N}$ and there exists m_0 for which $Q_{m_0} \notin A$ (otherwise $x + x^* \in A$). Let $J := \{k : P_k \preceq Q_{m_0}\}$ (see the definition of $\preceq$ in Lemma 4.4). Since Q_{m_0} is finite-dimensional, J is finite. One can easily check that $Q_{m_0} - \sum_{j\in J} P_j$ is a projection (if $J = \emptyset$, then $\sum_{j\in J} P_j := 0$). Moreover,

$$\Big(Q_{m_0} - \sum_{j\in J} P_j\Big) P_n = 0 \tag{4.2}$$

for all $n \in \mathbb{N}$. Indeed, if $n \in J$, then from the definition of $\preceq$, $Q_{m_0} P_n = P_n$, so

$$\Big(Q_{m_0} - \sum_{j\in J} P_j\Big) P_n = Q_{m_0} P_n - P_n = 0.$$

Let $n \notin J$. We have $Q_{m_0} P_n = P_n Q_{m_0}$ because $Q_{m_0} \in \mathrm{comm}(A)$. This implies that $Q_{m_0} P_n$ is a projection and $\mathrm{im}\, Q_{m_0} P_n = \mathrm{im}\, Q_{m_0} \cap \mathrm{im}\, P_n$. Therefore, since the projections P_n are one-dimensional, we have $Q_{m_0} P_n = P_n$ or $Q_{m_0} P_n = 0$. By our assumption, $Q_{m_0} P_n \neq P_n$, so $Q_{m_0} P_n = 0$. Now,

$$\Big(Q_{m_0} - \sum_{j\in J} P_j\Big) P_n = Q_{m_0} P_n = 0.$$

Since the sequence $(P_k)_{k\in\mathbb{N}}$ is complete, (4.2) implies that $Q_{m_0} - \sum_{j\in J} P_j = 0$. Hence $Q_{m_0} \in A$, a contradiction.

(iv)⇒(i): Follows directly from the definition of the commutant of A. □

Proposition 4.12. *Every closed commutative *-subalgebra of $\mathcal{L}(s', s)$ is contained in some maximal commutative *-subalgebra of $\mathcal{L}(s', s)$.*

Proof. Let A be a closed commutative *-subalgebra of $\mathcal{L}(s', s)$. Clearly,

$$\mathcal{A} := \{B : B \text{ commutative } ^*\text{-subalgebra of } \mathcal{L}(s', s) \text{ and } A \subset B\}$$

with $\subseteq$ is a partially ordered set. Consider a chain $\mathcal{C}$ in $\mathcal{A}$ and let $\widetilde{A}_{\mathcal{C}} := \bigcup_{A\in\mathcal{C}} A$. It is easy to check that $\widetilde{A}_{\mathcal{C}} \in \mathcal{A}$, and, of course, $\widetilde{A}_{\mathcal{C}}$ is an upper bound of $\mathcal{C}$. Hence, by the Kuratowski-Zorn lemma, $\mathcal{A}$ has a maximal element; let us call it M. By the continuity of the algebra operations, $\overline{M}^{\mathcal{L}(s',s)}$ is a closed commutative *-subalgebra of $\mathcal{L}(s', s)$, hence from the maximality of M, we have $M = \overline{M}^{\mathcal{L}(s',s)}$, i.e. M is a (closed) maximal commutative *-subalgebra of $\mathcal{L}(s', s)$ containing A. □

Proposition 4.13. *Every sequence of pairwise orthogonal projections belonging to $\mathcal{L}(s', s)$ can be extended by one-dimensional projections to a complete sequence.*

Proof. Let $\{P_k\}_{k\in\mathcal{N}}$ be a sequence of pairwise orthogonal projections belonging to $\mathcal{L}(s', s)$. Then, by Proposition 4.7, $A := \operatorname{alg}(\{P_k\}_{k\in\mathcal{N}})$ is a closed commutative *-subalgebra of $\mathcal{L}(s', s)$, and, by Proposition 4.12, there is a maximal commutative *-subalgebra $\widetilde{A}$ of $\mathcal{L}(s', s)$ containing A. Hence, by Theorems 4.9 and 4.11, we have $\widetilde{A} = \operatorname{alg}(\{Q_k\}_{k\in\mathbb{N}})$ for some complete sequence $\{Q_k\}_{k\in\mathbb{N}}$ of one-dimensional projections belonging to $\mathcal{L}(s', s)$. By Lemma 4.4(i), for every $k \in \mathcal{N}$ there exist $Q_1^{(k)}, Q_2^{(k)}, \ldots, Q_{l_k}^{(k)} \in \{Q_m\}_{m\in\mathbb{N}}$ such that $P_k = \sum_{j=1}^{l_k} Q_j^{(k)}$, and, in consequence,

$$\{P_k\}_{k\in\mathcal{N}} \cup \left(\{Q_m\}_{m\in\mathbb{N}} \setminus \left\{Q_j^{(k)}\right\}_{k\in\mathcal{N}, 1\le j\le l_k}\right)$$

is a complete extension of the sequence $\{P_k\}_{k\in\mathcal{N}}$. □

The property (DN) for the space s gives us the following useful inequality.

Proposition 4.14. *For every $p, r \in \mathbb{N}_0$ there is $q \in \mathbb{N}_0$ such that for all $\xi \in s$ with $||\xi||_{\ell_2} = 1$ the following inequality holds*

$$|\xi|_p^r \le |\xi|_q.$$

Proof. Take $p, r \in \mathbb{N}_0$ and let $j \in \mathbb{N}_0$ be such that $r \le 2^j$. Applying iteratively (j-times) the inequality from Proposition 1.4 to $\xi \in s$ with $||\xi||_{\ell_2} = 1$ we get

$$|\xi|_p^r \le |\xi|_p^{2^j} \le |\xi|_{2^j p},$$

and thus the required inequality holds for $q = 2^j p$. □

Proposition 4.15. *Let $(f_k)_{k\in\mathbb{N}} \subset s$ be an orthonormal sequence. Then*

$$\Phi\colon \operatorname{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}}) \to \lambda^\infty(|f_k|_q), \quad \Phi\left(\sum_{k=1}^{\infty} \lambda_k \langle\cdot, f_k\rangle f_k\right) := (\lambda_k)_{k\in\mathbb{N}}$$

is a Fréchet *-algebra isomorphism.

Proof. By Proposition 4.8, $\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}}$ is the canonical Schauder basis of $\mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$. Hence, by Theorem 4.9,

$$\Phi_0\colon \mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}}) \to \lambda^\infty(||\langle\cdot, f_k\rangle f_k||_q), \quad \Phi_0\left(\sum_{k=1}^{\infty} \lambda_k\langle\cdot, f_k\rangle f_k\right) := (\lambda_k)_{k\in\mathbb{N}}$$

is a Fréchet *-algebra isomorphism. Moreover, we have

$$1 \le |f_k|_q \le |f_k|_q^2 = ||\langle\cdot, f_k\rangle f_k||_q = |f_k|_q^2 \le |f_k|_{2q}$$

for $q \in \mathbb{N}_0$, where the last inequality follows from Proposition 4.14. Consequently,

$$\lambda^\infty(||\langle\cdot, f_k\rangle f_k||_q) = \lambda^\infty(|f_k|_q)$$

as Fréchet *-algebras (notice that the algebra operations are the same in both algebras), which completes the proof. □

Let us recall that, by Theorem 4.11, the canonical Schauder basis of a maximal commutative closed *-subalgebra of $\mathcal{L}(s', s)$ is a sequence $(\langle\cdot, f_k\rangle f_k)_{k\in\mathbb{N}}$ for some s-complete sequence $(f_k)_{k\in\mathbb{N}}$. Hence, by Theorem 4.9 and Proposition 4.15, we immediately get the following:

Corollary 4.16. *Let A be a closed maximal commutative *-subalgebra of $\mathcal{L}(s', s)$ with the canonical Schauder basis $(\langle\cdot, f_k\rangle f_k)_{k\in\mathbb{N}}$. Then*

$$A \cong \lambda^\infty(|f_k|_q)$$

and the isomorphism is given by $\langle\cdot, f_k\rangle f_k \mapsto e_k$ for $k \in \mathbb{N}$.

Theorem 4.17. *Let A be an infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$ and let $(\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j)_{k\in\mathbb{N}}$ be its canonical Schauder basis (here $(f_j)_{j\in\mathbb{N}}$ is an orthonormal sequence in s). Then A is isomorphic as a Fréchet *-algebra to the closed *-subalgebra of $\lambda^\infty(|f_k|_q)$ generated by $\{\sum_{j\in\mathcal{N}_k} e_j\}_{k\in\mathbb{N}}$ and the isomorphism is given by $\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j \mapsto \sum_{j\in\mathcal{N}_k} e_j$ for $k \in \mathbb{N}$.*

Please note that $(\mathcal{N}_k)_{k\in\mathbb{N}}$ is a family of pairwise disjoint finite subsets of $\mathbb{N}$.
Proof. Let B be the closed *-subalgebra of $\lambda^\infty(|f_n|_q)$ generated by $\{\sum_{j\in\mathcal{N}_k} e_j\}_{k\in\mathbb{N}}$. Let

$$J\colon A = \mathrm{alg}\left(\left\{\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j\right\}_{k\in\mathbb{N}}\right) \to \mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathcal{N}})$$

be the identity map and define

$$\Phi\colon \mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathcal{N}}) \to \lambda^\infty(|f_k|_q)$$

by $\langle\cdot, f_k\rangle f_k \mapsto e_k$, where $\mathcal{N} := \bigcup_{k\in\mathbb{N}}\mathcal{N}_k$. We consider the map $\Psi := \Phi\circ J\colon A \to \mathrm{im}(\Phi\circ J)$.

Clearly, $\Psi(\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j) = \sum_{j\in\mathcal{N}_k} e_j$ for $k \in \mathbb{N}$. Moreover, by Proposition 4.15, Ψ is a Fréchet *-algebra isomorphism and $\mathrm{im}\,\Psi$ is a closed *-subalgebra of $\lambda^\infty(|f_k|_q)$. Hence, by Proposition 4.7,

$$\mathrm{im}\,\Psi = \overline{\mathrm{lin}}\left(\left\{\sum_{j\in\mathcal{N}_k} e_j\right\}_{k\in\mathbb{N}}\right) \subset B \subset \mathrm{im}\,\Psi$$

and so $\mathrm{im}\,\Psi = B$, which completes the proof. □

In the following proposition we characterize infinite-dimensional closed *-subalgebras of $\lambda^\infty(|f_k|_q)$ (here $(f_k)_{k\in\mathbb{N}} \subset s$ is an orthonormal sequence), and consequently we obtain a characterization of closed *-subalgebras of s (Corollary 4.19). It is possible to generalize this result for a larger class of Köthe algebras, which seems to be a more natural approach. Nevertheless, we will confine ourselves to the case $\lambda^\infty(|f_k|_q)$, for which we already have all the tools needed.

Proposition 4.18. *Let $(f_k)_{k\in\mathbb{N}} \subset s$ be an orthonormal sequence and let A be an infinite-dimensional closed *-subalgebra of $\lambda^\infty(|f_k|_q)$. Then*

(i) *there is a family $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ of finite nonempty pairwise disjoint sets of natural numbers such that $\{\sum_{j\in\mathcal{N}_k} e_j\}_{k\in\mathbb{N}}$ is a Schauder basis of A;*

(ii) *$A \cong \lambda^\infty\left(\max_{j\in\mathcal{N}_k} |f_j|_q\right)$ as Fréchet *-algebras and the isomorphism is given by $\sum_{j\in\mathcal{N}_k} e_j \mapsto e_k$ for $k \in \mathbb{N}$.*

Conversely, if $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ is a family of finite nonempty pairwise disjoint sets of natural numbers and B is the closed *-subalgebra of $\lambda^\infty(|f_k|_q)$ generated by the set $\{\sum_{j\in\mathcal{N}_k} e_j\}_{k\in\mathbb{N}}$, then

(iii) *$(\sum_{j\in\mathcal{N}_k} e_j)_{k\in\mathbb{N}}$ is a Schauder basis of B;*

(iv) *$B \cong \lambda^\infty(\max_{j\in\mathcal{N}_k} |f_j|_q)$ as Fréchet *-algebras and the isomorphism is given by $\sum_{j\in\mathcal{N}_k} e_j \mapsto e_k$ for $k \in \mathbb{N}$.*

Proof. In order to prove (i) and (ii) define

$$\Phi\colon \operatorname{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}}) \to \lambda^\infty(|f_k|_q)$$

by $\langle\cdot, f_k\rangle f_k \mapsto e_k$ for $k \in \mathbb{N}$. Then, by Proposition 4.15, $\Phi^{-1}(A)$ is a closed *-subalgebra of $\operatorname{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$, and $(P_k)_{k\in\mathbb{N}}$ – the canonical Schauder basis of $\Phi^{-1}(A)$ – consists of projections belonging to $\operatorname{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$. Hence, by Proposition 4.8 and Lemma 4.4(i), $P_k = \sum_{j\in\mathcal{N}_k} \langle\cdot, f_j\rangle f_j$ for some family $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ of finite nonempty pairwise disjoint sets of natural numbers, and therefore $(\sum_{j\in\mathcal{N}_k} e_j)_{k\in\mathbb{N}} = (\Phi(P_k))_{k\in\mathbb{N}}$ is a Schauder basis of A. Clearly, the q-th norm of $\sum_{j\in\mathcal{N}_k} e_j$ in the space $A \subset \lambda^\infty(|f_k|_q)$ equals $\max_{j\in\mathcal{N}_k} |f_j|_q$. Hence, since A is a nuclear space, we have (see e.g. [20, Cor. 28.13 and Prop. 28.16])

$$A \cong \lambda^\infty\left(\max_{j\in\mathcal{N}_k} |f_j|_q\right)$$

as Fréchet spaces, where the isomorphism is defined by $\sum_{j\in\mathcal{N}_k} e_j \mapsto e_k$ for $k \in \mathbb{N}$. Hence, $A \cong \lambda^\infty(\max_{j\in\mathcal{N}_k} |f_j|_q)$ also as a Fréchet *-algebra. Similar arguments apply to prove (iii) and (iv). □

Corollary 4.19. *Every infinite-dimensional closed *-subalgebra of s is isomorphic as a Fréchet *-algebra to $\lambda^\infty(n_k^q)$ for some strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers. Conversely, if $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, then $\lambda^\infty(n_k^q)$ is isomorphic as a Fréchet *-algebra to some infinite-dimensional closed *-subalgebra of s.*

Proof. We apply Proposition 4.18 for $f_k = e_k$. Let $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ be a family of finite nonempty pairwise disjoint sets of natural numbers. We have

$$\max_{j\in\mathcal{N}_k} |e_j|_q = \max_{j\in\mathcal{N}_k} j^q = (\max\{j : j \in \mathcal{N}_k\})^q$$

for all $q \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be the bijection for which $(\max\{j : j \in \mathcal{N}_{\sigma(k)}\})_{k\in\mathbb{N}}$ is (strictly) increasing and let $n_k := \max\{j : j \in \mathcal{N}_{\sigma(k)}\}$ for $k \in \mathbb{N}$. Then

$$\lambda^\infty\left(\max_{j\in\mathcal{N}_k} |e_j|_q\right) = \lambda^\infty(n_k^q),$$

and therefore the conclusion follows from Proposition 4.18. □

The following characterization of infinite-dimensional closed commutative *-subalgebras of $\mathcal{L}(s', s)$ is a straightforward consequence of Theorem 4.17 and Proposition 4.18(iv).

Theorem 4.20. *Let A be an infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$ and let $(\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j)_{k\in\mathbb{N}}$ be its canonical Schauder basis. Then*

$$A \cong \lambda^\infty\left(\max_{j\in\mathcal{N}_k} |f_j|_q\right)$$

as a Fréchet *-algebra and the isomorphism is given by $\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j \mapsto e_k$ for $k \in \mathbb{N}$.

Corollaries 4.21, 4.22 and 4.23 are reformulations of Corollary 4.16 and Theorems 4.17, 4.20, respectively. They summarize the content of the present section and give a full description of closed commutative *-subalgebras of $\mathcal{L}(s', s)$.

Corollary 4.21. *Every closed maximal commutative *-subalgebra of $\mathcal{L}(s', s)$ is isomorphic as a Fréchet *-algebra to the algebra $\lambda^\infty(|f_k|_q)$ for some s-complete orthonormal sequence $(f_k)_{k\in\mathbb{N}}$. Conversely, if $(f_k)_{k\in\mathbb{N}}$ is an s-complete orthonormal sequence, then $\lambda^\infty(|f_k|_q)$ is isomorphic as a Fréchet *-algebra to some closed maximal commutative *-subalgebra of $\mathcal{L}(s', s)$.*

Proof. Let A be a closed maximal commutative *-subalgebra of $\mathcal{L}(s', s)$. By Theorem 4.11, there is an s-complete sequence $(f_k)_{k\in\mathbb{N}}$ such that $\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}}$ is the canonical Schauder basis of A. Hence, by Proposition 4.15, $A \cong \lambda^\infty(|f_k|_q)$ as Fréchet *-algebras.

Now, let $(f_k)_{k\in\mathbb{N}}$ be an arbitrary s-complete orthonormal sequence. Then, by Proposition 4.15, $\mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$ is isomorphic as a Fréchet *-algebra to $\lambda^\infty(|f_k|_q)$, and, by Theorem 4.11, $\mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$ is maximal commutative. □

Corollary 4.22. *Every closed commutative *-subalgebra of $\mathcal{L}(s', s)$ is isomorphic as a Fréchet *-algebra to some closed *-subalgebra of the algebra $\lambda^\infty(|f_k|_q)$ for some orthonormal sequence $(f_k)_{k\in\mathbb{N}} \subset s$. Conversely, if $(f_k)_{k\in\mathbb{N}} \subset s$ is an orthonormal sequence, then every closed *-subalgebra of $\lambda^\infty(|f_k|_q)$ is isomorphic as a Fréchet *-algebra to some closed commutative *-subalgebra of $\mathcal{L}(s', s)$.*

Proof. The first assertion follows immediately from Theorem 4.17.

If now $(f_k)_{k\in\mathbb{N}} \subset s$ is an arbitrary orthonormal sequence, then according to Proposition 4.15, $\lambda^\infty(|f_k|_q)$ is isomorphic as a Fréchet *-algebra to $\mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$. Consequently, every closed *-subalgebra of $\lambda^\infty(|f_k|_q)$ is isomorphic as a Fréchet *-algebra to some closed *-subalgebra of $\mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$. □

Corollary 4.23. *Every infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$ is isomorphic as a Fréchet *-algebra to the algebra $\lambda^\infty(\max_{j\in\mathcal{N}_k} |f_j|_q)$ for some orthonormal sequence $(f_k)_{k\in\mathbb{N}} \subset s$ and some family $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ of finite nonempty pairwise disjoint sets of natural numbers. Conversely, if $(f_k)_{k\in\mathbb{N}} \subset s$ is an orthonormal sequence and $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ is a family of finite nonempty pairwise disjoint sets of natural numbers, then $\lambda^\infty(\max_{j\in\mathcal{N}_k} |f_j|_q)$ is isomorphic as a Fréchet *-algebra to some infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$.*

Proof. The assertions easily follow from Corollary 4.22 and Proposition 4.18. □

4.3 Closed commutative *-subalgebras of $\mathcal{L}(s', s)$ with the property (Ω)

In the present section we prove that a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ is isomorphic as a Fréchet *-algebra to some closed *-subalgebra of s if and only if it is isomorphic as a Fréchet space to some complemented subspace of s (Theorem 4.25), i.e. if it has the so-called property (Ω) (see Definition 4.24 below). We also give an example of a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ which is not isomorphic to any closed *-subalgebra of s (Theorem 4.32).

Definition 4.24. A Fréchet space E with a fundamental sequence $(||\cdot||_q)_{q\in\mathbb{N}_0}$ of seminorms has the property (Ω) if the following condition holds:

$$\forall p\ \exists q\ \forall r\ \exists \theta \in (0,1)\ \exists C > 0\ \forall y \in E' \quad ||y||_q' \leq C||y||_p'^{1-\theta}||y||_r'^{\theta},$$

where E' is the topological dual of E and $||y||_p' := \sup\{|y(x)| : ||x||_p \leq 1\}$.

The property (Ω) (together with the property (DN)) plays a crucial role in the theory of nuclear Fréchet spaces (for details, see [20, Ch. 29] and Introduction).

Recall that a subspace F of a Fréchet space E is called *complemented* (in E) if there is a continuous projection $\pi\colon E \to E$ with $\operatorname{im}\pi = F$. Since every subspace of $\mathcal{L}(s', s)$ has the property (DN) (and, by Proposition 3.2, the norm $||\cdot||_{\ell_2\to\ell_2}$ is already a dominating norm), Theorem [20, Prop. 31.7] implies that a closed *-subalgebra of $\mathcal{L}(s', s)$ is isomorphic to a complemented subspace of s if and only if it has the property (Ω). The class of complemented subspaces of s is still not well-understood (e.g. we do not know, whether every such subspace has a Schauder basis – the Pełczyński problem) and, on the other hand, the class of closed *-subalgebras of s has a simple description (see Corollary 4.19). Therefore, in view of Theorem 4.25, the property (Ω) seems to be very restrictive for closed commutative *-subalgebras of $\mathcal{L}(s', s)$.

Theorem 4.25. *Let A be an infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$ and let $(\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j)_{k\in\mathbb{N}}$ be its canonical Schauder basis. Then the following assertions are equivalent:*

(i) *A is isomorphic as a Fréchet *-algebra to some closed *-subalgebra of s;*

(ii) *A is isomorphic as a Fréchet space to some complemented subspace of s;*

(iii) *A has the property (Ω);*

(iv) $\exists p\ \forall q\ \exists r\ \exists C > 0\ \forall k \quad \max_{j\in\mathcal{N}_k}|f_j|_q \leq C\max_{j\in\mathcal{N}_k}|f_j|_p^r$.

In order to prove Theorem 4.25, we will need Propositions 4.26–4.28 and Lemma 4.30.

The following result is a consequence of nuclearity of closed commutative *-subalgebras of $\mathcal{L}(s', s)$.

Proposition 4.26. *Let $(f_k)_{k\in\mathbb{N}} \subset s$ be an orthonormal system in ℓ_2 and for $r \in \mathbb{N}_0$ let $\sigma_r\colon \mathbb{N} \to \mathbb{N}$ be a bijection such that the sequence $(|f_{\sigma_r(k)}|_r)_{k\in\mathbb{N}}$ is non-decreasing. Then*

(i) *for all $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$ there exists $r \in \mathbb{N}$ such that*

$$\sum_{k=1}^{\infty} \frac{|f_k|_q}{|f_k|_r^{1/p}} < \infty;$$

(ii) *for every $p \in \mathbb{N}_0$ there are $r_0 \in \mathbb{N}$ such that*

$$\lim_{k\to\infty} \frac{k^p}{|f_{\sigma_r(k)}|_r} = 0$$

for all $r \geq r_0$.

Proof. (i) Take $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$. By Proposition 4.8, $(\langle\cdot, f_k\rangle f_k)_{k\in\mathbb{N}}$ is a Schauder basis in the nuclear Fréchet space $\overline{\operatorname{lin}}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$. Hence, by the Grothendieck-Pietsch theorem (see e.g. [20, Th. 28.15]), there is $r_1 \in \mathbb{N}_0$ such that

$$\sum_{k=1}^{\infty} \frac{||\langle\cdot, f_k\rangle f_k||_q}{||\langle\cdot, f_k\rangle f_k||_{r_1}} < \infty.$$

Clearly, $|f_k|_q \leq |f_k|_q^2$ and, by Proposition 1.4, $|f_k|_{r_1}^2 \leq |f_k|_{2r_1}$. Hence for $r_2 := 2r_1$ we get

$$\sum_{k=1}^{\infty} \frac{|f_k|_q}{|f_k|_{r_2}} \leq \sum_{k=1}^{\infty} \frac{|f_k|_q^2}{|f_k|_{r_1}^2} = \sum_{k=1}^{\infty} \frac{||\langle\cdot, f_k\rangle f_k||_q}{||\langle\cdot, f_k\rangle f_k||_{r_1}} < \infty.$$

By Proposition 4.14, there is $r \in \mathbb{N}$ such that

$$|f_k|_{r_2} \leq |f_k|_r^{1/p},$$

and therefore

$$\sum_{k=1}^{\infty} \frac{|f_k|_q}{|f_k|_r^{1/p}} \leq \sum_{k=1}^{\infty} \frac{|f_k|_q}{|f_k|_{r_2}} < \infty.$$

(ii) Let $p \in \mathbb{N}$. From (i) (applied to $q = 0$) there is r_0 such that for $r \geq r_0$ we have

$$\sum_{k=1}^{\infty} \frac{1}{|f_{\sigma_r(k)}|_r^{1/p}} = \sum_{k=1}^{\infty} \frac{1}{|f_k|_r^{1/p}} < \infty.$$

Since $(|f_{\sigma_r(k)}|_r^{1/p})_{k\in\mathbb{N}}$ is non-decreasing, it follows from the elementary theory of number series that $\lim_{k\to\infty} \frac{k}{|f_{\sigma_r(k)}|_r^{1/p}} = 0$, whence $\lim_{k\to\infty} \frac{k^p}{|f_{\sigma_r(k)}|_r} = 0$. □

Proposition 4.27. *Let $(a_{j,q})_{j\in\mathbb{N},q\in\mathbb{N}_0}$, $(b_{j,q})_{j\in\mathbb{N},q\in\mathbb{N}_0}$ be Köthe matrices for which $\lambda^\infty(a_{j,q})$ and $\lambda^\infty(b_{j,q})$ are nuclear Fréchet *-algebras. Then the following assertions are equivalent:*

(i) *$\lambda^\infty(a_{j,q}) \cong \lambda^\infty(b_{j,q})$ as Fréchet *-algebras;*

(ii) *there is a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $\lambda^\infty(a_{\sigma(j),q}) = \lambda^\infty(b_{j,q})$ as Fréchet *-algebras;*

(iii) *there is a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $\lambda^\infty(a_{\sigma(j),q}) = \lambda^\infty(b_{j,q})$ as sets;*

(iv) *there is a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that*

(α) $\forall q \in \mathbb{N}_0\ \exists r \in \mathbb{N}_0\ \exists C > 0\ \forall j \in \mathbb{N}\quad a_{\sigma(j),q} \leq C b_{j,r}$,

(β) $\forall r' \in \mathbb{N}_0\ \exists q' \in \mathbb{N}_0\ \exists C' > 0\ \forall j \in \mathbb{N} \quad b_{j,r'} \leq C' a_{\sigma(j),q'}$.

Proof. Denote $A := \lambda^\infty(a_{j,q})$ and $B := \lambda^\infty(b_{j,q})$.

(i)⇒(ii): Assume that there is an isomorphism $\Phi\colon A \to B$ of Fréchet *-algebras. Clearly, if $\xi^2 = \xi$, then $\Phi(\xi) = \Phi(\xi^2) = (\Phi(\xi))^2$, and the same is true for Φ^{-1}, i.e. Φ maps the idempotents of A onto the idempotents of B. Note also that each idempotent of A and B is a sum of pairwise different idempotents e_k. Then, for a fixed $k \in \mathbb{N}$, there is a set $I \subset \mathbb{N}$ such that

$$\Phi(e_k) = \sum_{l \in I} e_l$$

On the other hand, if $\xi^{(l)} = \sum_{j \in J_l} e_j$ is the idempotent of A such that $\Phi(\xi^{(l)}) = e_l$, then

$$\Phi\left(\sum_{l \in I} \xi^{(l)}\right) = \sum_{l \in I} \Phi(\xi^{(l)}) = \sum_{l \in I} e_l$$

so, by the injectivity of Φ,

$$\sum_{l \in I} \sum_{j \in J_l} e_j = \sum_{l \in I} \xi^{(l)} = e_k,$$

which implies that $|I| = |J_l| = 1$. Hence $\Phi(e_k) = e_{l_k}$ for some $l_k \in \mathbb{N}$, i.e. for the bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ defined by $l_{\sigma(k)} := k$ we have $\Phi(e_{\sigma(k)}) = e_k$. Therefore, $\Phi((\xi_{\sigma(k)})_{k\in\mathbb{N}}) = (\xi_k)_{k\in\mathbb{N}}$ for $(\xi_k)_{k\in\mathbb{N}} \in B$ (note that $(e_{\sigma(k)})_{k\in\mathbb{N}}$ is a Schauder basis in A and $(e_k)_{k\in\mathbb{N}}$ is a Schauder basis in B), which shows (ii).

(ii)⇒(iii): Obvious.

(iii)⇒(iv): The proof follows from the observation that the identity map $\mathrm{Id}\colon \lambda^\infty(a_{\sigma(j),q}) \to \lambda^\infty(b_{j,q})$ is continuous (use the closed graph theorem).

(iv)⇒(i): It is easy to see that $\Phi\colon A \to B$ defined by $e_{\sigma(k)} \mapsto e_k$ is an isomorphism of Fréchet *-algebras. □

Proposition 4.28. *Let A be an infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$ and let $(\sum_{j\in\mathcal{N}_k} \langle\cdot, f_j\rangle f_j)_{k\in\mathbb{N}}$ be its canonical Schauder basis. Moreover, let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers and let B be the closed *-subalgebra of s generated by $\{e_{n_k}\}_{k\in\mathbb{N}}$. Then the following assertions are equivalent:*

(i) *A is isomorphic to B as a Fréchet *-algebra;*

(ii) *$\lambda^\infty(\max_{j\in\mathcal{N}_k} |f_j|_q) \cong \lambda^\infty(n_k^q)$ as Fréchet *-algebras;*

(iii) *there is a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $\lambda^\infty(\max_{j\in\mathcal{N}_{\sigma(k)}} |f_j|_q) = \lambda^\infty(n_k^q)$ as Fréchet *-algebras;*

(iv) *there is a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $\lambda^\infty(\max_{j\in\mathcal{N}_{\sigma(k)}} |f_j|_q) = \lambda^\infty(n_k^q)$ as sets;*

(v) *there is a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that*

(α) $\forall q \in \mathbb{N}_0\ \exists r \in \mathbb{N}_0\ \exists C > 0\ \forall k \in \mathbb{N} \quad \max_{j\in\mathcal{N}_{\sigma(k)}} |f_j|_q \leq C n_k^r$,

(β) $\forall r' \in \mathbb{N}_0\ \exists q' \in \mathbb{N}_0\ \exists C' > 0\ \forall k \in \mathbb{N} \quad n_k^{r'} \leq C' \max_{j\in\mathcal{N}_{\sigma(k)}} |f_j|_{q'}$.

Proof. This is an immediate consequence of Theorem 4.20 and Proposition 4.27. □

Remark 4.29. In view of Corollary 4.19, every closed *-subalgebra of s is isomorphic as a Fréchet *-algebra to $\lambda^\infty(n_k^q)$ (i.e. the closed *-subalgebra of s generated by $\{e_{n_k}\}_{k\in\mathbb{N}}$) for some strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, hence Proposition 4.28 characterizes closed commutative *-subalgebras of $\mathcal{L}(s', s)$ which are isomorphic as Fréchet *-algebras to some *-subalgebra of s.

Lemma 4.30. *Let $(a_k)_{k\in\mathbb{N}} \subset [1, \infty)$ be a non-decreasing sequence such that $a_k \geq 2k$ for k big enough. Then there exist a strictly increasing sequence $(b_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and $C > 0$ such that*

$$\frac{1}{C} a_k \leq b_k \leq C a_k^2$$

for every $k \in \mathbb{N}$.

Proof. Let $k_0 \in \mathbb{N}$ be such that $a_k \geq 2k$ for $k > k_0$ and choose $C \in \mathbb{N}$ so that

$$\frac{1}{C} a_k \leq k \leq C a_k^2$$

for $k \in \mathcal{N}_0 := \{1, \ldots, k_0\}$. Denote also $\mathcal{N}_1 := \{k \in \mathbb{N} : a_k = a_{k_0+1}\}$ and, recursively, $\mathcal{N}_{j+1} := \{k \in \mathbb{N} : a_k = a_{\max \mathcal{N}_j + 1}\}$. Clearly, $\mathcal{N}_j$ are finite, pairwise disjoint, $\bigcup_{j\in\mathbb{N}_0} \mathcal{N}_j = \mathbb{N}$ and $k < l$ for $k \in \mathcal{N}_j$, $l \in \mathcal{N}_{j+1}$.

Let $b_k := k$ for $k \in \mathcal{N}_0$ and let

$$b_{m_j+l-1} := C\lceil \max\{a_{m_j-1}^2, a_{m_j}\}\rceil + l$$

for $j \in \mathbb{N}$ and $1 \leq l \leq |\mathcal{N}_j|$, where $m_j := \min \mathcal{N}_j$ and $\lceil x \rceil := \min\{n \in \mathbb{Z} : n \geq x\}$ stands for the ceiling of $x \in \mathbb{R}$. We will show inductively that $(b_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers such that

$$\frac{1}{C} a_k \leq b_k \leq C a_k^2 \tag{4.3}$$

for every $k \in \mathbb{N}$.

Clearly, the condition (4.3) holds for $k \in \mathcal{N}_0$. Assume that $(b_k)_{k\in\mathcal{N}_0\cup\ldots\cup\mathcal{N}_j}$ is a strictly increasing sequence of natural numbers for which the condition (4.3) holds. For simplicity, denote $m := \min \mathcal{N}_{j+1}$. By the inductive assumption, we obtain $b_{m-1} \leq C a_{m-1}^2$, hence

$$b_m - b_{m-1} \geq C\lceil \max\{a_{m-1}^2, a_m\}\rceil + 1 - C a_{m-1}^2 \geq C a_{m-1}^2 + 1 - C a_{m-1}^2 \geq 1$$

so $b_{m-1} < b_m$, and, clearly, $b_m < b_{m+1} < \ldots < b_{\max \mathcal{N}_{j+1}}$.

Fix $1 \leq l \leq |N_{j+1}|$. We have

$$b_{m+l-1} \geq C a_m = C a_{m+l-1} \geq \frac{1}{C} a_{m+l-1}$$

so the first inequality in (4.3) holds for $k \in \mathcal{N}_{j+1}$. Next, by assumption, we get

$$a_{m+l-1} \geq 2(m + l - 1), \tag{4.4}$$

whence

$$l \leq a_{m-l+1} - m + 1. \tag{4.5}$$

Consider two cases. If $a_m \geq a_{m-1}^2$, then, from (4.5)

$$\begin{aligned} b_{m-l+1} &= C\lceil a_m \rceil + l = C\lceil a_{m+l-1}\rceil + l \leq 2C a_{m+l-1} + a_{m+l-1} - m + 1 \\ &\leq (2C + 1) a_{m+l-1} \leq C a_{m+l-1}^2, \end{aligned}$$

where the last inequality holds because $C \geq 1$ and, from (4.4), we have

$$a_{m-l+1} \geq 2(m+l-1) \geq 2m \geq 2(k_0+1) \geq 4.$$

Finally, if $a_{m-1}^2 > a_m$, then, from (4.4), we obtain (note that, by the definition of $\mathcal{N}_j$ and $\mathcal{N}_{j+1}$, we have $a_{m-1} < a_m$)

$$\begin{aligned}
b_{m-l+1} &= C\lceil a_{m-1}^2 \rceil + l \\
&\leq C\lceil (a_m-1)^2 \rceil + l \\
&= C\lceil a_m^2 - 2a_m + 1 \rceil + l \\
&\leq C(a_m^2 - 2a_m + 2) + l \\
&\leq Ca_m^2 - 2Ca_m + 2C + Cl \\
&= Ca_{m+l-1}^2 - C(2a_{m+l-1} - 2 - l) \\
&\leq Ca_{m+l-1}^2 - C(4(m+l-1) - 2 - l) \\
&= Ca_{m+l-1}^2 - C(4m+3l-6) \leq Ca_{m+l-1}^2.
\end{aligned}$$

Hence we have shown that the second inequality in (4.3) holds for $k \in \mathcal{N}_{j+1}$, and the proof is complete. □

Now, we are ready to prove the main theorem of this section.

Proof of Theorem 4.25. (i)⇒(ii): By Corollary 4.19, each closed *-subalgebra of s is isomorphic to some complemented subspace of s (and one can prove that it is complemented in s).

(ii)⇔(iii): See e.g. [20, Prop. 31.7].

(iii)⇒(iv): By Theorem 4.20 and nuclearity (see e.g. [20, Prop. 28.16]),

$$A \cong \lambda^\infty\left(\max_{j\in\mathcal{N}_k} |f_j|_q\right) = \lambda^1\left(\max_{j\in\mathcal{N}_k} |f_j|_q\right)$$

as a Fréchet *-algebra. Next, by [20, Lemma 27.11, Lemma 27.12], we have

$$A' \cong \{\eta \in \mathbb{C}^{\mathbb{N}} : \sum_{k=1}^{\infty} |\xi_k| \cdot |\eta_k| < \infty \text{ for all } \xi \in \lambda^1(\max_{j\in\mathcal{N}_k} |f_j|_q)\},$$

where the Minkowski functional $||\cdot||_q'$ of the polar of $\{\xi \in \mathbb{C}^{\mathbb{N}} : \sum_{k=1}^{\infty} |\xi_k| \cdot \max_{j\in\mathcal{N}_k} |f_j|_q \leq 1\}$ is given by

$$||\eta||_q' := \sup\left\{\left|\sum_{k=1}^{\infty} \xi_k\eta_k\right| : \sum_{k=1}^{\infty} |\xi_k| \cdot \max_{j\in\mathcal{N}_k} |f_j|_q \leq 1\right\} = \sup_{k\in\mathbb{N}} \frac{|\eta_k|}{\max_{j\in\mathcal{N}_k} |f_j|_q}.$$

Hence, the property (Ω) applied to unit vectors gives

$$\forall l\ \exists m\ \forall n\ \exists 0<\theta<1\ \exists C>0\ \forall k \quad \frac{1}{\max_{j\in\mathcal{N}_k} |f_j|_m} \leq C\frac{1}{\max_{j\in\mathcal{N}_k} |f_j|_l^{1-\theta} \max_{j\in\mathcal{N}_k} |f_j|_n^{\theta}}.$$

In particular, taking $l = 0$, we get (iv).

(iv)⇒(i): By Proposition 4.26(ii), there is $p_1 \geq p$ and a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that the sequence $(|f_{\mu_{\sigma(k)}}|_{p_1})_{k\in\mathbb{N}}$ is non-decreasing and $\lim_{k\to\infty} \frac{k}{|f_{\mu_{\sigma(k)}}|_{p_1}} = 0$, where p is taken from the condition (iv) and $\mu_k \in \mathcal{N}_k$ is choosen so that $\max_{j\in\mathcal{N}_k} |f_j|_{p_1} = |f_{\mu_k}|_{p_1}$. Then for sufficiently large k we get $\frac{k}{|f_{\mu_{\sigma(k)}}|_{p_1}} \le \frac{1}{2}$, so

$$|f_{\mu_{\sigma(k)}}|_{p_1} \ge 2k. \tag{4.6}$$

Consequently, for k big enough we have

$$\max_{j\in\mathcal{N}_{\sigma(k)}} |f_j|_{p_1} = |f_{\mu_{\sigma(k)}}|_{p_1} \ge 2k$$

and the sequence $(\max_{j\in\mathcal{N}_{\sigma(k)}} |f_j|_{p_1})_{k\in\mathbb{N}}$ is non-decreasing. Hence, by Lemma 4.30, there is a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and $C_1 > 0$ such that

$$\frac{1}{C_1} \max_{j\in\mathcal{N}_{\sigma(k)}} |f_j|_{p_1} \le n_k \le C_1 \max_{j\in\mathcal{N}_{\sigma(k)}} |f_j|_{p_1}^2 \tag{4.7}$$

for every $k \in \mathbb{N}$. Now, by the conditions (iv) and (4.7), we get that for all q there is r and $C_2 := CC_1^r$ such that

$$\max_{j\in\mathcal{N}_{\sigma(k)}} |f_j|_q \le C \max_{j\in\mathcal{N}_{\sigma(k)}} |f_j|_{p_1}^r \le C_2 n_k^r$$

for all $k \in \mathbb{N}$, so the condition (α) from Proposition 4.28(v) holds. Finally, by (4.7) and Proposition 4.14 we obtain that for all r' there is q' and $C_3 := C_1^{r'}$ such that

$$n_k^{r'} \le C_3 \max_{j\in\mathcal{N}_{\sigma(k)}} |f_j|_{p_1}^{2r'} \le C_3 \max_{j\in\mathcal{N}_{\sigma(k)}} |f_j|_{q'}$$

for every $k \in \mathbb{N}$, so the condition (β) from Proposition 4.28(v) is satisfied, and therefore, by Proposition 4.28, A is isomorphic as a Fréchet *-algebra to the closed *-subalgebra of s generated by $\{e_{n_k}\}_{k\in\mathbb{N}}$. □

Lemma 4.31. *For every increasing sequence $(\alpha_j)_{j\in\mathbb{N}} \subset (0, \infty)$ and every $p \in \mathbb{N}$ we have*

$$\sup_{j\in\mathbb{N}} \left(\alpha_j^{p-j+1} \cdot \prod_{i=1}^{j-1} \alpha_i \right) = \prod_{i=1}^{p} \alpha_i.$$

Proof. For $j \ge p+1$ we get

$$\frac{\alpha_j^{p-j+1} \cdot \prod_{i=1}^{j-1} \alpha_i}{\prod_{i=1}^{p} \alpha_i} = \alpha_j^{p-j+1} \cdot \prod_{i=p+1}^{j-1} \alpha_i = \frac{\prod_{i=p+1}^{j-1} \alpha_i}{\alpha_j^{j-p-1}} \le 1$$

and, similarly, for $j \le p-1$ we obtain

$$\frac{\alpha_j^{p-j+1} \cdot \prod_{i=1}^{j-1} \alpha_i}{\prod_{i=1}^{p} \alpha_i} = \frac{\alpha_j^{p-j+1}}{\prod_{i=j}^{p} \alpha_i} \le 1.$$

Since $\alpha_p^{p-p+1} \cdot \prod_{i=1}^{p-1} \alpha_i = \prod_{i=1}^{p} \alpha_i$, the supremum is attained for $j = p$, and we are done. □

Theorem 4.32. *There is a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ which is not isomorphic to any closed *-subalgebra of s.*

Proof. Let m_k be the k-th prime number, $N_{k,1} := m_k$, $N_{k,j+1} := m_k^{N_{k,j}}$ for $j, k \in \mathbb{N}$. Denote $a_{k,1} := c_k$ and

$$a_{k,j} := c_k \frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}}$$

for $j \geq 2$, where the sequence $(c_k)_{k\in\mathbb{N}}$ is choosen so that $||(a_{k,j})_{j\in\mathbb{N}}||_{\ell_2} = 1$, i.e.

$$c_k := \left(\sum_{j=1}^{\infty} \left(\frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}} \right)^2 \right)^{-1/2}.$$

The numbers c_k are well-defined, because, by Lemma 4.31,

$$\sum_{j=1}^{\infty} \left(\frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}} \right)^2 = \sum_{j=1}^{\infty} \left(N_{k,j}^{-j+1} \cdot \prod_{i=1}^{j-1} N_{k,i} \right)^2 = \sum_{j=1}^{\infty} \frac{1}{N_{k,j}^2} \left(N_{k,j}^{1-j+1} \cdot \prod_{i=1}^{j-1} N_{k,i} \right)^2$$
$$\leq \sup_{j\in\mathbb{N}} \left(N_{k,j}^{1-j+1} \cdot \prod_{i=1}^{j-1} N_{k,i} \right)^2 \sum_{j=1}^{\infty} \frac{1}{N_{k,j}^2} = N_{k,1}^2 \sum_{j=1}^{\infty} \frac{1}{N_{k,j}^2} < N_{k,1}^2 \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty.$$

Finally, define an orthonormal sequence $(f_k)_{k\in\mathbb{N}}$ by

$$f_k := \sum_{j=1}^{\infty} a_{k,j} e_{N_{k,j}}.$$

We will show that $\mathrm{alg}(\{\langle \cdot, f_k \rangle f_k\}_{k\in\mathbb{N}})$ is a closed *-subalgebra of $\mathcal{L}(s', s)$ which is not isomorphic as an algebra to any closed *-subalgebra of s. By Theorem 4.25, it is enough to show that each f_k belongs to s and for every $p, r \in \mathbb{N}$ the following condition holds

$$\lim_{k\to\infty} \frac{|f_k|_{\infty,p+1}}{|f_k|_{\infty,p}^r} = \infty,$$

where $|\xi|_{\infty,q} := \sup_{j\in\mathbb{N}} |\xi_j| j^q$ (see Proposition 1.5).

Note first that $|f_k|_{\infty,p} = a_{k,p} N_{k,p}^p$. In fact, by Lemma 4.31, we get

$$|f_k|_{\infty,p} = \sup_{j\in\mathbb{N}} a_{k,j} N_{k,j}^p = c_k \sup_{j\in\mathbb{N}} \left(N_{k,j}^p \cdot \frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}} \right) = c_k \sup_{j\in\mathbb{N}} \left(N_{k,j}^{p-j+1} \cdot \prod_{i=1}^{j-1} N_{k,i} \right)$$
$$= c_k \prod_{i=1}^{p} N_{k,i} = c_k N_{k,p}^p \cdot \frac{\prod_{i=1}^{p-1} N_{k,i}}{N_{k,p}^{p-1}} = a_{k,p} N_{k,p}^p.$$

In particular, $f_k \in s$ for $k \in \mathbb{N}$. Next, for $j, k \in \mathbb{N}$, we have

$$\frac{a_{k,j+1} N_{k,j+1}^j}{a_{k,j}} = \frac{c_k N_{k,j+1}^j \cdot \frac{\prod_{i=1}^{j} N_{k,i}}{N_{k,j+1}^j}}{c_k \frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}}} = \frac{\prod_{i=1}^{j} N_{k,i}}{\frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}}} = N_{k,j}^j.$$

Moreover, for every $j, r \in \mathbb{N}$ we get

$$\frac{N_{k,j+1}}{N_{k,j}^r} = \frac{m_k^{N_{k,j}}}{N_{k,j}^r} \geq \frac{2^{N_{k,j}}}{N_{k,j}^r} \xrightarrow[k\to\infty]{} \infty,$$

and clearly $a_{k,j} \leq 1$ for $j, k \in \mathbb{N}$. Hence, for $p, r \in \mathbb{N}$ we obtain

$$\begin{aligned}\frac{|f_k|_{\infty,p+1}}{|f_k|^r_{\infty,p}} &= \frac{a_{k,p+1}N^{p+1}_{k,p+1}}{a^r_{k,p}N^{pr}_{k,p}} = \frac{a_{k,p+1}N^{p}_{k,p+1}}{a_{k,p}} \cdot \frac{1}{a^{r-1}_{k,p}} \cdot \frac{N_{k,p+1}}{N^{pr}_{k,p}} = N^p_{k,p} \cdot \frac{1}{a^{r-1}_{k,p}} \cdot \frac{N_{k,p+1}}{N^{pr}_{k,p}} \\ &\geq \frac{N_{k,p+1}}{N^{pr}_{k,p}} \xrightarrow[k\to\infty]{} \infty,\end{aligned}$$

which is the desired conclusion. □

4.4 Orthogonally complemented closed commutative *-subalgebras of $\mathcal{L}(s', s)$

It would be interesting to describe commutative *-subalgebras of $\mathcal{L}(s', s)$ which are complemented (in $\mathcal{L}(s', s)$). This problem does not seem to be so easy to solve because, in general, a projection on $\mathcal{L}(s', s)$ has nothing in common with the algebraic structure of $\mathcal{L}(s', s)$. In this section we shall consider commutative *-subalgebras of $\mathcal{L}(s', s)$ which are complemented in a very specific way – the so-called orthogonally complemented subalgebras (see Definition 4.33).

We first characterize closed commutative orthogonally complemented *-subalgebras in terms of their canonical Schauder bases (Proposition 4.36). Next, we consider the class of closed maximal commutative orthogonally complemented *-subalgebras of $\mathcal{L}(s', s)$ isomorphic as Fréchet *-algebras to s (Theorem 4.37). It appears that closed maximal commutative orthogonally complemented *-subalgebras A of $\mathcal{L}(s', s)$ isomorphic to s are exactly those for which there exists an algebra isomorphism $T\colon \mathcal{L}(s', s) \to \mathcal{L}(s', s)$ preserving orthogonality, which maps A onto the subalgebra of diagonal operators (Corollary 4.38). We also give an example of a closed maximal commutative *-subalgebra of $\mathcal{L}(s', s)$ isomorphic as a Fréchet *-algebra to s which is not orthogonally complemented in $\mathcal{L}(s', s)$ (Theorem 4.39).

Definition 4.33. A linear map $\pi\colon \mathcal{L}(s', s) \to \mathcal{L}(s', s)$ is said to be an *orthogonal projection* if there is a continuous projection $\widetilde{\pi}\colon \mathcal{HS}(\ell_2) \to \mathcal{HS}(\ell_2)$ which is orthogonal with respect to the Hilbert-Schmidt scalar product and satisfies $\widetilde{\pi}_{|\mathcal{L}(s',s)} = \pi$. A linear subspace A of $\mathcal{L}(s', s)$ is called *orthogonally complemented* (in $\mathcal{L}(s', s)$) if $A = \operatorname{im}\pi$ for some orthogonal projection $\pi\colon \mathcal{L}(s', s) \to \mathcal{L}(s', s)$.

Recall that $\mathcal{HS}(\ell_2)$ stands for the space of *Hilbert-Schmidt operators*, i.e.

$$\mathcal{HS}(\ell_2) := \mathcal{S}_2(\ell_2) := \{x \in \mathcal{K}(\ell_2) : (s_k(x))_{k\in\mathbb{N}} \in \ell_2\},$$

where $(s_k(x))_{k\in\mathbb{N}}$ is the sequence of singular numbers of x. The space $\mathcal{HS}(\ell_2)$ with a scalar product defined by

$$\langle x, y\rangle_{\mathcal{HS}} := \sum_{k=1}^{\infty} \langle xe_k, ye_k\rangle$$

becomes a Hilbert space. The corresponding hilbertian norm is denoted by ν_2, i.e.

$$\nu_2(x) := (\langle x, x\rangle_{\mathcal{HS}})^{1/2} = \Big(\sum_{k=1}^{\infty} ||xe_k||^2_{\ell_2}\Big)^{1/2}.$$

Remark 4.34. Clearly, an orthogonal projection π on $\mathcal{L}(s', s)$ is a (continuous) projection in the sense that $\pi^2 = \pi$. Hence, every orthogonally complemented *-subalgebra of $\mathcal{L}(s', s)$ has (Ω) (see Theorem 0.4) and can be embedded isomorphically as a closed *-subalgebra into the Fréchet *-algebra s (Theorem 4.25).

Remark 4.35. Let $(P_k)_{k\in\mathbb{N}}$ be a sequence of nonzero pairwise orthogonal finite dimensional (self-adjont) projections on ℓ_2 and let d_k denote the dimension of $\operatorname{im} P_k$ for $k \in \mathbb{N}$. Then $\left(d_k^{-1}P_k\right)_{k\in\mathbb{N}}$ is an orthonormal basis of the Hilbert space $\overline{\operatorname{lin}}^{\mathcal{HS}}(\{P_k\}_{k\in\mathbb{N}})$ (the closed linear span of $\{P_k\}_{k\in\mathbb{N}}$ in the topology of $(\mathcal{HS}(\ell_2), \nu_2)$), hence the map $\widetilde{\pi}\colon \mathcal{HS}(\ell_2) \to \mathcal{HS}(\ell_2)$,

$$\widetilde{\pi}x := \sum_{k=1}^{\infty} d_k^{-1}\langle x, P_k\rangle_{\mathcal{HS}} P_k, \tag{4.8}$$

is a continuous orthogonal projection onto $\overline{\operatorname{lin}}^{\mathcal{HS}}(\{P_k\}_{k\in\mathbb{N}})$. If, moreover, $\widetilde{\pi}(\mathcal{L}(s', s)) \subseteq \mathcal{L}(s', s)$, then, by the closed graph theorem for Fréchet spaces and Lemma 3.5, the map

$$\pi := \widetilde{\pi}|_{\mathcal{L}(s',s)} \colon \mathcal{L}(s', s) \to \mathcal{L}(s', s)$$

is a continuous orthogonal projection onto $A := \overline{\operatorname{lin}}^{\mathcal{L}(s',s)}(\{P_k\}_{k\in\mathbb{N}}) = \operatorname{alg}(\{P_k\}_{k\in\mathbb{N}})$, and thus A is orthogonally complemented in $\mathcal{L}(s', s)$.

On the other hand, if $\operatorname{alg}(\{P_k\}_{k\in\mathbb{N}})$ is orthogonally complemented in $\mathcal{L}(s', s)$, then one can easily check that the corresponding orthogonal projection $\widetilde{\pi}\colon \mathcal{HS}(\ell_2) \to \mathcal{HS}(\ell_2)$ is defined by (4.8).

In the following Proposition we characterize orthogonally complemented commutative *-subalgebras of $\mathcal{L}(s', s)$ among all closed commutative *-subalgebras of $\mathcal{L}(s', s)$.

Proposition 4.36. *Let A be an infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$ and let $(P_k)_{k\in\mathbb{N}} = (\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j)_{k\in\mathbb{N}}$ be its canonical Schauder basis. Then the following assertions are equivalent:*

(i) *A is orthogonally complemented;*

(ii) *for all $q \in \mathbb{N}_0$ and for all $x \in \mathcal{L}(s', s)$*

$$\sup_{k\in\mathbb{N}} \frac{|\langle x, P_k\rangle_{\mathcal{HS}}|}{|\mathcal{N}_k|} \cdot ||P_k||_q < \infty;$$

(iii) *for all $q \in \mathbb{N}_0$ and for all $x \in \mathcal{L}(s', s)$*

$$\sup_{k\in\mathbb{N}} \frac{1}{|\mathcal{N}_k|}\left|\sum_{j\in\mathcal{N}_k}\langle xf_j, f_j\rangle\right| \cdot \max_{j\in\mathcal{N}_k} |f_j|_q < \infty.$$

Proof. (i)⇒(ii): If A is orthogonally complemented in $\mathcal{L}(s', s)$ then, by Remark 4.35, the corresponding orthogonal projection $\pi\colon \mathcal{L}(s', s) \to \mathcal{L}(s', s)$ is given by

$$\pi x := \sum_{k=1}^{\infty} \frac{1}{|\mathcal{N}_k|}\langle x, P_k\rangle_{\mathcal{HS}} P_k.$$

Now, from Lemma 3.5 it follows that the series above is absolutely convergent for all $x \in \mathcal{L}(s', s)$, and therefore (ii) holds.

(ii)⇒(i): Assume that the condition (ii) is satisfied. Since the space $A = \operatorname{alg}(\{P_k\}_{k\in\mathbb{N}})$ is a nuclear Fréchet space (as a closed subspace of the nuclear Fréchet space $\mathcal{L}(s', s)$), the space

$\lambda^\infty(||P_k||_q) \cong \mathrm{alg}(\{P_k\}_{k\in\mathbb{N}})$ (see Theorem 4.9) is nuclear as well. Hence, by the Grothendieck-Pietsch theorem (see e.g. [20, Th. 28.15]), for given $q \in \mathbb{N}_0$ one can find $r \in \mathbb{N}_0$ such that

$$\sum_{k=1}^{\infty} \frac{||P_k||_q}{||P_k||_r} < \infty,$$

whence, by assumption,

$$\sum_{k=1}^{\infty} \frac{|\langle x, P_k\rangle_{\mathcal{HS}}|}{|\mathcal{N}_k|} ||P_k||_q \leq \sup_{k\in\mathbb{N}} \frac{|\langle x, P_k\rangle_{\mathcal{HS}}|}{|\mathcal{N}_k|} \cdot ||P_k||_r \cdot \sum_{k=1}^{\infty} \frac{||P_k||_q}{||P_k||_r} < \infty.$$

for all $x \in \mathcal{L}(s', s)$. This shows that for the orthogonal projection $\widetilde{\pi}\colon \mathcal{HS}(\ell_2) \to \mathcal{HS}(\ell_2)$ defined by (4.8) we have $\widetilde{\pi}(\mathcal{L}(s', s)) \subseteq \mathcal{L}(s', s)$, and thus, by Remark 4.35, A is orthogonally complemented in $\mathcal{L}(s', s)$.

(ii)$\Leftrightarrow$(iii): Take $x \in \mathcal{HS}(\ell_2)$ and fix an orthonormal basis $\{\widetilde{f}_j\}_{j\in\mathbb{N}} := \{f_j\}_{j\in\mathbb{N}} \cup \{g_j\}_{j\in\mathcal{N}}$ of ℓ_2 extending the orthonormal system $\{f_j\}_{j\in\mathbb{N}}$. First note that for $k \in \mathbb{N}$ we have

$$\langle x, P_k\rangle_{\mathcal{HS}} = \sum_{j=1}^{\infty} \langle x\widetilde{f}_j, P_k\widetilde{f}_j\rangle = \sum_{j\in\mathcal{N}_k} \langle xf_j, f_j\rangle,$$

because the Hilbert-Schmidt scalar product does not depend on the choice of an orthonormal basis (see e.g. [20, Prop. 16.16(2)]). Hence,

$$\frac{|\langle x, P_k\rangle_{\mathcal{HS}}|}{|\mathcal{N}_k|} = \frac{1}{|\mathcal{N}_k|} \left| \sum_{j\in\mathcal{N}_k} \langle xf_j, f_j\rangle \right|,$$

Moreover, by Theorems 4.9 and 4.20

$$\lambda^\infty(||P_k||_q) \cong \lambda^\infty(\max_{j\in\mathcal{N}_k} |f_j|_q),$$

as Fréchet *-algebras and the isomorphism is given by $e_k \mapsto e_k$. Therefore,

$$\lambda^\infty(||P_k||_q) = \lambda^\infty(\max_{j\in\mathcal{N}_k} |f_j|_q),$$

which completes the proof. □

Now, we shall focus on maximal commutative orthogonally complemented *-subalgebras of $\mathcal{L}(s', s)$ isomorphic to s as Fréchet *-algebras.

Theorem 4.37. *Let $(f_k)_{k\in\mathbb{N}}$ be an s-complete sequence such that $\lambda^\infty(|f_{\sigma(k)}|_q) = s$ as a Fréchet *-algebra for some bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ and let $u\colon \ell_2 \to \ell_2$ be given by $ue_k := f_{\sigma(k)}$ for $k \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) $u \in \mathcal{L}(s) \cap \mathcal{L}(s')$;

(ii) $\sup_{k\in\mathbb{N}} |\langle xf_k, f_k\rangle|\, |f_k|_q < \infty$ *for all* $x \in \mathcal{L}(s', s)$ *and all* $q \in \mathbb{N}_0$;

(iii) $(\langle \eta, f_{\sigma(k)}\rangle)_{k\in\mathbb{N}} \in s$ *for all* $\eta \in s$;

(iv) $u(s) = s$;

(v) $(f_k)_{k\in\mathbb{N}}$ *is a Schauder basis of* s;

(vi) *for all* $q \in \mathbb{N}_0$ *there is* $r \in \mathbb{N}_0$ *such that* $\sup_{k\in\mathbb{N}} |f_k|'_r |f_k|_q < \infty$;

(vii) $\mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$ *is orthogonally complemented in* $\mathcal{L}(s', s)$.

Proof. First note that, by Proposition 4.28, there is a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that

(α) $\forall q \in \mathbb{N}_0\ \exists r \in \mathbb{N}_0\ \exists C > 0\ \forall k \in \mathbb{N}\quad |f_{\sigma(k)}|_q \le Ck^r$;

(β) $\forall r' \in \mathbb{N}_0\ \exists q' \in \mathbb{N}_0\ \exists C' > 0\ \forall k \in \mathbb{N}\quad k^{r'} \le C'|f_{\sigma(k)}|_{q'}$.

(i)⇒(ii): Let $\widetilde{u}\colon s' \to s'$ be the continuous extension of u (see (2.1)) and let us recall that, by Proposition 2.1, $u^* \in \mathcal{L}(s)$. Hence for $x \in \mathcal{L}(s', s)$ we get

$$\langle xf_{\sigma(k)}, f_{\sigma(k)}\rangle = \langle x\widetilde{u}e_k, ue_k\rangle = \langle u^*x\widetilde{u}e_k, e_k\rangle,$$

and thus the condition (α) shows that for every $q \in \mathbb{N}_0$ there are $r \in \mathbb{N}_0$ and $C > 0$ such that

$$|\langle xf_{\sigma(k)}, f_{\sigma(k)}\rangle|\ |f_{\sigma(k)}|_q \le C|\langle u^*x\widetilde{u}e_k, e_k\rangle|k^r.$$

Now, since $u^*x\widetilde{u} \in \mathcal{L}(s', s)$, we obtain

$$\sup_{k\in\mathbb{N}} |\langle u^*x\widetilde{u}e_k, e_k\rangle|k^r \le \sup_{k\in\mathbb{N}} ||u^*x\widetilde{u}e_k||_{\ell_2} \cdot ||e_k||_{\ell_2} k^r \le C' \sup_{k\in\mathbb{N}} |e_k|'_r k^r = C'$$

for some $C' > 0$, which completes the proof.

(ii)⇒(iii): For $\eta = 0$ the conclusion is trivially satisfied. If $\eta \in s \setminus \{0\}$, then we define $x := \langle\cdot, \eta\rangle\eta$. Clearly, $x \in \mathcal{L}(s', s)$ and

$$|\langle xf_{\sigma(k)}, f_{\sigma(k)}\rangle|^{1/2} = |\langle\langle f_{\sigma(k)}, \eta\rangle\eta, f_{\sigma(k)}\rangle|^{1/2} = |\langle\eta, f_{\sigma(k)}\rangle|.$$

Therefore, by the condition (β), for every $r' \in \mathbb{N}_0$ there are $q' \in \mathbb{N}_0$ and $C' > 0$ such that

$$\begin{aligned} |\langle\eta, f_{\sigma(k)}\rangle|k^{r'} &\le C'|\langle\eta, f_{\sigma(k)}\rangle| \cdot |f_{\sigma(k)}|_{q'} = C'|\langle xf_{\sigma(k)}, f_{\sigma(k)}\rangle|^{1/2} \cdot |f_{\sigma(k)}|_{q'} \\ &\le C'(|\langle xf_{\sigma(k)}, f_{\sigma(k)}\rangle| \cdot |f_{\sigma(k)}|_{2q'})^{1/2}, \end{aligned}$$

where the last inequality follows from Proposition 1.4. Hence, by assumption, $(\langle\eta, f_{\sigma(k)}\rangle)_{k\in\mathbb{N}} \in s$.

(iii)⇒(iv): By the condition (α), we obtain $u(s) \subseteq s$, so we only need to prove that $s \subseteq u(s)$. To do this we first show that

$$\eta = \sum_{k=1}^{\infty} \langle\eta, f_{\sigma(k)}\rangle f_{\sigma(k)} \tag{4.9}$$

for all $\eta \in s$ and the series converges in the norm $||\cdot||_{\ell_2}$.

Take $\eta \in s$. Clearly, if $\{f_{\sigma(k)}\}_{k\in\mathbb{N}}$ is an orthonormal basis of ℓ_2, then (4.9) holds. If it is not the case, we may find an orthonormal system $\{g_k\}_{k\in\mathcal{N}}$ in ℓ_2 such that $\{f_{\sigma(k)}\}_{k\in\mathbb{N}} \cup \{g_k\}_{k\in\mathcal{N}}$ is an orthonormal basis of ℓ_2. Then

$$\eta = \sum_{k=1}^{\infty} \langle\eta, f_{\sigma(k)}\rangle f_{\sigma(k)} + \sum_{k\in\mathcal{N}} \langle\eta, g_k\rangle g_k$$

(the series converges in the norm $||\cdot||_{\ell_2}$). Notice that (iii) and (α) imply that the first series is absolutely convergent in s, and consequently $\gamma := \sum_{k\in\mathcal{N}}\langle\eta, g_k\rangle g_k \in s$. But γ is orthogonal to each $f_{\sigma(k)}$ and $\{f_{\sigma(k)}\}_{k\in\mathbb{N}}$ is s-complete, so $\gamma = 0$, and thus (4.9) holds.

Now, by assumption and (4.9), we get

$$\eta = u\left(\sum_{k=1}^{\infty}\langle \eta, f_{\sigma(k)}\rangle e_k\right), \quad \text{where} \quad \sum_{k=1}^{\infty}\langle \eta, f_{\sigma(k)}\rangle e_k \in s,$$

hence $s \subseteq u(s)$.

(iv)$\Rightarrow$(v): Since $u(s) = s$, $u\colon \ell_2 \to \ell_2$ has dense range. Moreover, u preserves the scalar product and thus u is unitary. Now the conclusion follows by [34, Remark 1]. We include the proof for completeness.

For $\eta \in s$ take $\xi \in s$ such that $\eta = u\xi$. Then

$$\eta = \sum_{k=1}^{\infty}\langle \eta, f_{\sigma(k)}\rangle f_{\sigma(k)}$$

(the series converges in the norm $||\cdot||_{\ell_2\to\ell_2}$) and, since u is unitary, we have $\langle \eta, f_{\sigma(k)}\rangle = \langle \xi, e_k\rangle = \xi_k$, so $(\langle \eta, f_{\sigma(k)}\rangle)_{k\in\mathbb{N}} \in s$. Hence, the condition (α) shows that for every $q \in \mathbb{N}_0$ there are $r \in \mathbb{N}_0$ and $C > 0$ such that

$$\sum_{k=1}^{\infty}|\langle \eta, f_{\sigma(k)}\rangle|\;|f_{\sigma(k)}|_q \le C\sum_{k=1}^{\infty}|\langle \eta, f_{\sigma(k)}\rangle| k^r < \infty,$$

and thus the series $\sum_{k=1}^{\infty}\langle \eta, f_{\sigma(k)}\rangle f_{\sigma(k)}$ is absolutely convergent in s. Moreover, by orthogonality, it is easily seen that there is no other representations $\sum c_k f_k$, so $(f_k)_{k\in\mathbb{N}}$ is a Schauder basis of s.

(v)$\Rightarrow$(i): First observe that the assumption implies that $(f_{\sigma(k)})_{k\in\mathbb{N}}$ is an orthonormal basis of ℓ_2, and therefore u is unitary.

By (α), for $\eta \in s$ and $q \in \mathbb{N}_0$ we obtain

$$\begin{aligned}|u(\eta)|_q &= \left|\sum_{k=1}^{\infty}\langle u(\eta), f_{\sigma(k)}\rangle f_{\sigma(k)}\right|_q = \left|\sum_{k=1}^{\infty}\langle \eta, u^*(f_{\sigma(k)})\rangle f_{\sigma(k)}\right|_q \le \sum_{k=1}^{\infty}|\langle \eta, e_k\rangle|\;|f_{\sigma(k)}|_q \\ &\le C|\eta|_{r+2}\sum_{k=1}^{\infty}|e_k|'_{r+2}k^r = C|\eta|_{r+2}\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty,\end{aligned}$$

hence $u(s) \subset s$.

Next, by assumption, for every $\eta \in s$ there is a unique sequence $(c_k)_{k\in\mathbb{N}}$ of complex numbers such that $\eta = \sum_{k=1}^{\infty} c_k f_{\sigma(k)}$, and, by the Dynin-Mityagin theorem (see e.g. [20, Th. 28.12]), the series is convergent absolutely in s. On the other hand, since $(f_{\sigma(k)})_{k\in\mathbb{N}}$ is an orthonormal basis of ℓ_2, it follows that

$$\eta = \sum_{k=1}^{\infty}\langle \eta, f_{\sigma(k)}\rangle f_{\sigma(k)}$$

(a priori, the series converges in the norm $||\cdot||_{\ell_2\to\ell_2}$), and thus $c_k = \langle \eta, f_{\sigma(k)}\rangle$. By the condition (β), for every $r' \in \mathbb{N}_0$ there are $q' \in \mathbb{N}_0$ and $C' > 0$ such that

$$\sum_{k=1}^{\infty}|\langle \eta, f_{\sigma(k)}\rangle| k^{r'} \le C'\sum_{k=1}^{\infty}|\langle \eta, f_{\sigma(k)}\rangle|\;|f_{\sigma(k)}|_{q'} < \infty,$$

i.e. $(\langle \eta, f_{\sigma(k)}\rangle)_{k\in\mathbb{N}} \in s$. Consequently, since

$$u^*(\eta) = u^{-1}(\eta) = u^{-1}\left(\sum_{k=1}^{\infty}\langle \eta, f_{\sigma(k)}\rangle f_{\sigma(k)}\right) = \sum_{k=1}^{\infty}\langle \eta, f_{\sigma(k)}\rangle e_k,$$

$u^*(s) \subset s$.

We have proved that $u \in \mathcal{L}^*(s)$ (see (2.2) for definition), and thus, by Proposition 2.1, $u \in \mathcal{L}(s) \cap \mathcal{L}(s')$.

(v)$\Leftrightarrow$(vi): Easily follows from the Dynin-Mityagin theorem (see [21, remark after Th. 9]).

(ii)$\Leftrightarrow$(vii): This is an immediate consequence of Proposition 4.36. □

Corollary 4.38. *Let A be a maximal commutative *-subalgebra of $\mathcal{L}(s', s)$ isomorphic (as a Fréchet *-algebra) to s. The following assertions are equivalent:*

(i) *A is orthogonally complemented;*

(ii) *there is a unitary map $u\colon \ell_2 \to \ell_2$, $u(s) = s$ such that*

$$T\colon \mathcal{L}(s', s) \to \mathcal{L}(s', s), \quad T(x) := u^*xu,$$

is a Fréchet *-algebra isomorphism preserving orthogonality (i.e. $\langle x, y\rangle_{\mathcal{HS}} = \langle Tx, Ty\rangle_{\mathcal{HS}}$) which maps A onto the *-subalgebra

$$D := \left\{ \sum_{k=1}^{\infty} \xi_k \langle \cdot, e_k\rangle e_k \colon (\xi_k)_{k\in\mathbb{N}} \in s \right\}$$

of diagonal operators in $\mathcal{L}(s', s)$;

(iii) *there is a Fréchet *-algebra isomorphism $T\colon \mathcal{L}(s', s) \to \mathcal{L}(s', s)$ preserving orthogonality such that $T(A) = D$.*

Proof. (i)$\Rightarrow$(ii): By assumption, Theorem 4.11 and Corollary 4.16, there is an s-complete sequence $(f_k)_{k\in\mathbb{N}}$ such that

$$s \cong A \cong \lambda^\infty(|f_k|_q)_{k\in\mathbb{N}})$$

as Fréchet *-algebras. Hence, by Proposition 4.27, we get $\lambda^\infty(|f_{\sigma(k)}|_q) = s$ (as Fréchet *-algebras) for some bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$. Therefore, by Theorem 4.37 (the implications (vii)$\Rightarrow$(i) and (vii)$\Rightarrow$(iv)), the map

$$u\colon \ell_2 \to \ell_2, \quad ue_k := f_{\sigma(k)}$$

is unitary and $u_{|s}\colon s \to s$ is a continuous automorphism of s. Now, it is easy to show that $T\colon \mathcal{L}(s', s) \to \mathcal{L}(s', s)$ defined by $T(x) := u^*xu$ has the desired properties.

(ii)$\Rightarrow$(iii): Obvious.

(iii)$\Rightarrow$(i): Since D is orthogonally complemented with the projection

$$\pi\colon \mathcal{L}(s', s) \to \mathcal{L}(s', s), \quad \pi(x) := \sum_{k=1}^{\infty} \langle xe_k, e_k\rangle \langle \cdot, e_k\rangle e_k,$$

it follows that $T^{-1} \circ \pi \circ T$ is the orthogonal projection onto A. □

Our next result shows that the algebra s can be embedded in $\mathcal{L}(s', s)$ in a non-orthogonally way. In particular, there are closed commutative *-subalgebras of $\mathcal{L}(s', s)$ isomorphic to a complemented subspace of s (i.e. with the property (Ω)) which are not orthogonally complemented in $\mathcal{L}(s', s)$.

Theorem 4.39. *There is a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ isomorphic as a Fréchet *-algebra to s which is not orthogonally complemented in $\mathcal{L}(s', s)$.*

Proof. By Corollary 4.16, Proposition 4.28 and Theorem 4.37, it is enough to find an orthonormal basis $(f_k)_{k\in\mathbb{N}} \subset s$ of ℓ_2 which is not a Schauder basis of s and such that $\lambda^\infty(|f_k|_q) \cong s$ as a Fréchet *-algebra.

Let

$$f_k := \begin{cases} \frac{1}{\sqrt{2}}(e_{2^{j!}} + e_{2^{(j+1)!}-1}) & \text{for } k = 2^{j!}, j \in \mathbb{N}, \\ \frac{1}{\sqrt{2}}(e_{2^{j!}} - e_{2^{(j+1)!}-1}) & \text{for } k = 2^{(j+1)!} - 1, j \in \mathbb{N}, \\ e_k & \text{otherwise.} \end{cases}$$

Clearly, $\{e_k\}_{k\in\mathbb{N}} \subset \operatorname{lin}(\{f_k\}_{k\in\mathbb{N}})$, so $\{f_k\}_{k\in\mathbb{N}}$ is linearly dense in ℓ_2, i.e. $(f_k)_{k\in\mathbb{N}}$ is an orthonormal basis of ℓ_2.

Suppose that $(f_k)_{k\in\mathbb{N}}$ is a Schauder basis of s. Then $\xi \mapsto \langle \xi, f_k\rangle$ for $k \in \mathbb{N}$ are coefficient functionals, and therefore, by the Dynin-Mityagin theorem (see [21, remark after Th. 9]), for every $q \in \mathbb{N}_0$ there is $r \in \mathbb{N}_0$ such that

$$\sup_{k\in\mathbb{N}} |f_k|'_{\infty,r}|f_k|_{\infty,q} < \infty, \tag{4.10}$$

where $|\xi|'_{\infty,q} := \sup_{j\in\mathbb{N}} |\xi_j| j^{-q}$ and $|\xi|_{\infty,q} := \sup_{j\in\mathbb{N}} |\xi_j| j^q$ (see also Proposition 1.5). For $q \in \mathbb{N}_0$ we easily compute

$$|f_{2^{j!}}|_{\infty,q} = \frac{1}{\sqrt{2}}(2^{(j+1)!} - 1)^q \quad \text{and} \quad |f_{2^{j!}}|'_{\infty,q} = \frac{1}{\sqrt{2}} 2^{-qj!}.$$

Hence for $q = 1$ and $r \in \mathbb{N}_0$ we obtain

$$|f_{2^{j!}}|'_{\infty,r}|f_{2^{j!}}|_{\infty,q} = \frac{1}{2}2^{-rj!} \cdot (2^{(j+1)!} - 1) \geq \frac{1}{4}2^{(j+1)!-rj!} = \frac{1}{4}2^{j!(j+1-r)} \to \infty$$

as $j \to \infty$. Thus the condition (4.10) is not satisfied, a contradiction. Therefore, $(f_k)_{k\in\mathbb{N}}$ is not a Schauder basis of s.

It remains to prove that $\lambda^\infty(|f_k|_q) \cong s$. Let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be a bijection for which the sequence $(|f_{\sigma(k)}|_{\infty,1})_{k\in\mathbb{N}}$ is non-decreasing. For $j \in \mathbb{N}$ let

$$A_j := \{|f_{\sigma(k)}|_{\infty,1} : 2^{j!} + 1 \leq k \leq 2^{(j+1)!}\}$$

and let

$$B_j := \{2^{j!} + 2, 2^{j!} + 3, \ldots, 2^{(j+1)!} - 2\} \cup \{\frac{1}{\sqrt{2}}(2^{(j+1)!} - 1)\} \cup \{2^{(j+1)!} + 1\}.$$

We will show inductively that $A_j = B_j$.

An easy computation shows that for $q \in \mathbb{N}$

$$|f_k|_{\infty,q} = \begin{cases} \frac{1}{\sqrt{2}}(2^{(j+1)!} - 1)^q & \text{for } k = 2^{j!} \text{ or } k = 2^{(j+1)!} - 1, \ j \in \mathbb{N}, \\ k^q & \text{otherwise.} \end{cases} \tag{4.11}$$

Hence $|f_{\sigma(1)}|_{\infty,1} = 1$, $|f_{\sigma(2)}|_{\infty,1} = |f_{\sigma(3)}|_{\infty,1} = \frac{3\sqrt{2}}{2}$, $|f_{\sigma(4)}|_{\infty,1} = 5$ so $A_1 = B_1 = \{\frac{3\sqrt{2}}{2}, 5\}$ and the conclusion holds for $j = 1$.

Now, let us assume that $A_{j-1} = B_{j-1}$. Then, in particular, $|f_{\sigma(2^{j!})}|_{\infty,1} = \max A_{j-1} = 2^{j!}+1$, and therefore, since for $j \geq 2$ we have

$$2^{j!} + 2 < \frac{1}{\sqrt{2}}(2^{(j+1)!} - 1) < 2^{(j+1)!} - 1, \tag{4.12}$$

it follows that $\min A_j = \min B_j = 2^{j!} + 2$. Clearly, $|B_j| = 2^{(j+1)!} - 2^{j!} - 1$ and $\frac{1}{\sqrt{2}}(2^{(j+1)!} - 1)$ occurs in the sequence $(|f_{\sigma(k)}|_{\infty,1})_{k\in\mathbb{N}}$ two times. But A_j is indexed by $2^{(j+1)!} - 2^{j!}$ numbers, so by (4.12) we have $\frac{1}{\sqrt{2}}(2^{(j+1)!} - 1) \in A_j$ and therefore $A_j = B_j$.

Now, comparing the elements of sets A_j and B_j, it is easy to see that $k-1 \le |f_{\sigma(k)}|_{\infty,1} \le k+1$ for all $k \in \mathbb{N}$. Moreover, by (4.11), for every $q \in \mathbb{N}_0$ there is a constant $C > 0$ such that $|f_k|_{\infty,q} \le C|f_k|^q_{\infty,1}$ for all $k \in \mathbb{N}$. Hence, for $k \in \mathbb{N}$ we have

$$|f_{\sigma(k)}|_{\infty,q} \le C|f_{\sigma(k)}|^q_{\infty,1} \le C(k+1)^q \le 2^q C k^q$$

and also for $k \ge 2$

$$k^{r'} \le 2^{r'}(k-1)^{r'} \le 2^{r'}|f_{\sigma(k)}|^{r'}_{\infty,1} \le 2^{r'}C'|f_{\sigma(k)}|_{\infty,q'},$$

where the last inequality follows from Proposition 4.14. The case $k = 1$ (i.e. $1 = |f_{\sigma(1)}|_{\infty,1}$) is trivial. Hence, by Propositions 1.5 and 4.28, $\lambda^\infty(|f_k|_q) \cong s$. $\square$

Chapter 5

Functional calculus in $\mathcal{L}(s', s)$

If x is a normal operator in $\mathcal{L}(s', s) \subset K(\ell_2)$ with spectral representation $x = \sum_{k\in\mathcal{N}} \lambda_k P_k$ and f is a continuous function on the spectrum $\sigma(x)$ of x vanishing at zero, then the continuous functional calculus for normal operators provides a uniquely determined operator $f(x) := \sum_{k\in\mathcal{N}} \lambda_k P_k \in \mathcal{K}(\ell_2)$ (see e.g. [20, Prop. 17.20]). Recall that, by Proposition 1.20, the spectrum of x (in $\mathcal{L}(s', s)_{\mathbf{1}}$) coincides with the spectrum of x in $\mathcal{L}(\ell_2)$ (and thus also in $\mathcal{K}(\ell_2)$).

In this chapter, we want to describe those functions f for which $f(x)$ is again in $\mathcal{L}(s', s)$ (see Theorem 5.2 and Corollary 5.3). Moreover, it turns out that for a normal operator $x \in \mathcal{L}(s', s)$ and a Hölder continuous at zero function $f\colon \sigma(x) \to \mathbb{C}$ with $f(0) = 0$, we have $f(x) \in \mathcal{L}(s', s)$ as well (Proposition 5.1).

From the general theory of Fréchet locally m-convex algebras we get the holomorphic functional calculus on $\mathcal{L}(s', s)$ (see Prop. 1.9 and [25, Lemma 1.3], [36, Th. 12.16]). More precisely, if x is an arbitrary operator in $\mathcal{L}(s', s)$ and f is a holomorphic function on an open neighborhood U of $\sigma(x)$ with $f(0) = 0$, then $f(x) \in \mathcal{L}(s', s)$, and moreover the map $\Phi\colon H_0(U) \to \mathcal{L}(s', s)$, $f \mapsto f(x)$, is a continuous algebra homomorphism ($H_0(U)$ stands for the space of holomorphic functions on U vanishing at zero).

It is worth mentioning that Blackadar and Cuntz developed a C^∞-functional calculus on some dense subalgebras of C^*-algebras (see [2, Prop. 6.4 and p. 277]). Unfortunately, it seems that $\mathcal{L}(s', s)$ does not fit to their theory.

Recall that a function $f\colon X \to \mathbb{C}$ ($X \subset \mathbb{C}$, $0 \in X$) is *Hölder continuous at zero* if there are $\theta \in (0, 1]$ and $C > 0$ such that $|f(t) - f(0)| \le C|t|^\theta$ for all t in a neighborhood of zero. Let us note that every function which is differentiable at zero is also Hölder continuous at zero. As an immediate consequence of Theorem 3.1, we get the following Hölder continuous functional calculus for normal operators in $\mathcal{L}(s', s)$.

Theorem 5.1. *If $x \in \mathcal{L}(s', s) \subset \mathcal{K}(\ell_2)$ is normal, then for every Hölder continuous at zero function $f\colon \sigma(x) \to \mathbb{C}$ with $f(0) = 0$, we have $f(x) \in \mathcal{L}(s', s)$ as well. In particular:*

(i) *if x is positive (i.e. $\sigma(x) \subset [0, \infty)$) and $\theta \in (0, \infty)$, then $x^\theta \in \mathcal{L}(s', s)$;*

(ii) *$|x| \in \mathcal{L}(s', s)$;*

(iii) *if x is self-adjoint (i.e. $x^* = x$), then $x_+ := (|x| + x)/2$, $x_- := (|x| - x)/2 \in \mathcal{L}(s', s)$.*

Proof. Let $x = \sum_{k\in\mathcal{N}} \lambda_k P_k \in \mathcal{L}(s', s)$ be normal and let $f\colon \sigma(x) \to \mathbb{C}$ be Hölder continuous at zero with $f(0) = 0$. Then $|f| \leq C|\cdot|^\theta$ for some $C > 0$ and $\theta \in (0, 1]$. Hence, by Corollary 3.6,

$$\sum_{k\in\mathcal{N}} ||f(\lambda_k)P_k||_q \leq C \sum_{k\in\mathcal{N}} |\lambda_k|^\theta ||P_k||_q < \infty.$$

So, again from Corollary 3.6, it follows that $f(x) = \sum f(\lambda_k)P_k \in \mathcal{L}(s', s)$.

To prove (i), observe that $f\colon [0,\infty) \to \mathbb{C}$, $f(x) := x^\theta$, is Hölder continuous for every $\theta \in (0,\infty)$. Then $|x| = \sqrt{x^*x} \in \mathcal{L}(s', s)$, since $x^*x \geq 0$. Finally the functions f_+, f_-, $f_+(x) := \max\{x, 0\}$, $f_-(x) := \max\{-x, 0\}$ are also Hölder continuous at zero. □

For a normal operator x in $\mathcal{L}(s', s)$ with spectral representation $x = \sum_{k=1}^\infty \lambda_k P_k$, we define the function space

$$C_s(\sigma(x)) := \{f : \sigma(x) \to \mathbb{C} : f(0) = 0, (f(\lambda_k))_{k\in\mathbb{N}} \in \lambda^\infty(||P_k||_q)\}.$$

It is easy to show that the space $C_s(\sigma(x))$ with the system $(c_q)_{q\in\mathbb{N}_0}$,

$$c_q(f) := \sup_{k\in\mathbb{N}} |f(\lambda_k)|\, ||P_k||_q,$$

of seminorms, pointwise multiplication and conjugation is a Fréchet *-algebra.

Theorem 5.2. *If x is an infinite-dimensional normal operator in $\mathcal{L}(s', s)$ with spectral representation $x = \sum_{k=1}^\infty \lambda_k P_k$, then the map*

$$\Phi : C_s(\sigma(x)) \longrightarrow \operatorname{alg}(x), \quad \Phi(f) := f(x) = \sum_{k=1}^\infty f(\lambda_k)P_k,$$

is a Fréchet *-algebra isomorphism such that $\Phi(\mathrm{id}) = x$.

Proof. By Theorem 4.9, Φ is well defined, and of course $\Phi(\mathrm{id}) = x$ and $\Phi(\overline{f}) = \Phi(f)^*$. The space $\operatorname{alg}(x)$ is a nuclear Fréchet space (as a closed subspace of the nuclear Fréchet space $\mathcal{L}(s', s)$) so $\lambda^\infty(||P_k||_q) \cong \operatorname{alg}(x)$ (see Theorem 4.9) is a nuclear Fréchet space as well. Thus, by the Grothendieck-Pietsch theorem (see e.g. [20, Th. 28.15]), for given $q \in \mathbb{N}_0$ one can find $r \in \mathbb{N}_0$ such that $C := \sum_{k=1}^\infty \frac{||P_k||_q}{||P_k||_r} < \infty$. Hence

$$\begin{aligned} ||\Phi(f)||_q &\leq \sum_{k=1}^\infty |f(\lambda_k)|\, ||P_k||_q = \sum_{k=1}^\infty |f(\lambda_k)|\, ||P_k||_r \frac{||P_k||_q}{||P_k||_r} \\ &\leq \sup_{k\in\mathbb{N}} |f(\lambda_k)|\, ||P_k||_r \cdot \sum_{k=1}^\infty \frac{||P_k||_q}{||P_k||_r} = Cc_r(f). \end{aligned}$$

Thus Φ is continuous.

Clearly, Φ is injective. To prove that it is also surjective, take $y \in \operatorname{alg}(x)$. By Theorem 4.9, $(P_k)_{k\in\mathbb{N}}$ is a Schauder basis, so there is a sequence $(\mu_k)_{k\in\mathbb{N}}$ such that $y = \sum_{k=1}^\infty \mu_k P_k$. Let $g(\lambda_k) := \mu_k$ for $k \in \mathbb{N}$. Then

$$\sup_{k\in\mathbb{N}} |g(\lambda_k)|\, ||P_k||_q = \sup_{k\in\mathbb{N}} |\mu_k|\, ||P_k||_q < \infty,$$

hence $g \in C_s(\sigma(x))$, and of course, $\Phi(g) = y$. □

For a normal operator $x \in \mathcal{K}(\ell_2)$ we define the C^*-algebra

$$C_0(\sigma(x)) := \{f : \sigma(x) \longrightarrow \mathbb{C} : f \text{ is continuous and } f(0) = 0\}.$$

Theorem 5.2 shows, in particular, that for a fixed normal operator $x \in \mathcal{L}(s', s)$, $f(x) \in \mathcal{L}(s', s)$ for all $f \in C_s(\sigma(x))$. The following result shows that $C_s(\sigma(x))$ is the biggest subspace of $C_0(\sigma(x))$ with this property.

Corollary 5.3. *Let x be a normal operator belonging to $\mathcal{L}(s', s)$ and let $f \in C_0(\sigma(x))$. Then $f(x) \in \mathcal{L}(s', s)$ if and only if $f \in C_s(\sigma(x))$.*

Proof. Let $\sum_{k=1}^{\infty} \lambda_k P_k$ be spectral representation of x. By Theorem 5.2, $f \in C_s(\sigma(x))$ implies that $f(x) \in \mathcal{L}(s', s)$. Conversely, if $f(x) = \sum_{k=1}^{\infty} f(\lambda_k) P_k \in \mathcal{L}(s', s)$, then by Corollary 3.6, $\sum_{k=1}^{\infty} |f(\lambda_k)| \; ||P_k||_q < \infty$ for every $q \in \mathbb{N}_0$. Hence, $\sup_{k \in \mathbb{N}} |f(\lambda_k)| \; ||P_k||_q < \infty$, and thus $f \in C_s(\sigma(x))$. □

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