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MATCHINGS IN HYPERGRAPHS

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To my school math teachers
Ms Ewa Ciesielska
Ms Maria Gawrych
Ms Aleksandra Olech

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Abstract

In 1965 Erdős asked what is the maximum number of edges in k-uniform hypergraphs on n vertices in which the largest matching has s edges. He conjectured that it is maximized either for cliques, or for graphs which consist of all edges intersecting a set of s vertices. Neither construction is uniformly better than the other in the whole range of parameter s ($1 \leq s \leq n/k$), so the conjectured bound is the maximum of these two possibilities.

In this thesis we present results obtained while working on this problem. In particular, we confirm Erdős' conjecture in a general k-uniform case for $n \geq 2k^2 s/\log k$, and, more importantly, settle it in the affirmative for $k = 3$ and n large enough. We also derive a stability result which shows that in order to verify Erdős' conjecture it is enough to prove it in an asymptotic form.

In the last chapter, we discuss new conjectures and results obtained while working on Erdős' problem. In particular, we formulate a structural conjecture that might be considered as an asymptotic generalization of Tutte's Theorem for hypergraphs, and, if true, may bring us closer to solve the Erdős' matching problem. Moreover, we state a new probabilistic conjecture on small deviation inequalities, of a similar flavour as Samuels' conjecture stated in 1965. We confirm it in a few instances, by proving that it is asymptotically equivalent to the fractional version of Erdős' matching problem.

Streszczenie

W 1965 roku Erdős badał rodzinę k-jednostajnych hipergrafów na n wierzchołkach, w których największe skojarzenie zawiera dokładnie s hiperkrawędzi. Zapytał wtedy, jaką największą liczbę krawędzi może posiadać hipergraf z takiej rodziny, wskazując przy tym dwóch naturalnych kandydatów na hipergrafy, które tę liczbę maksymalizują. Jednym z nich jest hipergraf, którego wszystkie krawędzie zawierają się w pewnym ustalonym $(ks + k - 1)$-elementowym podzbiorze wierzchołków; inny gęsty przedstawiciel tej rodziny to hipergraf składający się ze wszystkich krawędzi przecinających ustalony zbiór s wierzchołków. Gdy s jest małe drugi z tych hipergrafów ma więcej krawędzi, gdy s jest bliskie n/k zachodzi sytuacja odwrotna. Erdős postawił hipotezę, że dla każdej wartości parametru s $(1 \leq s \leq n/k)$, w rodzinie hipergrafów na n wierzchołkach, w których największe skojarzenie wynosi s, nie ma grafu gęstszego od powyższych dwóch hipergrafów.

Główną część rozprawy stanowią wyniki dotyczące sformułowanej powyżej hipotezy Erdősa. Pokazujemy, że jest ona prawdziwa dla hipergrafów k-jednostajnych jeśli tylko $n \geq 2k^2 s/\log k$ i, co ważniejsze, dowodzimy jej dla hipergrafów 3-jednostajnych dla $n > n_0$. Prócz tego podajemy również szereg wyników dotyczących struktury grafów, których gęstość jest zbliżona do grafów najgęstszych. Pokazują one w szczególności, że aby zweryfikować hipotezę Erdősa wystarczy pokazać prawdziwość jej słabszej, asymptotycznej wersji.

W ostatnim rozdziale omawiamy nowe hipotezy i wyniki związane z hipotezą Erdősa. Między innymi stawiamy pewną hipotezę strukturalną, która może być postrzegana jako asymptotyczne uogólnienie Twierdzenia Turána na hipergrafy, a której rozwiązanie może przybliżyć nas do udowodnienia hipotezy Erdősa. Ponadto formułujemy hipotezę dotyczącą rozkładu sumy pewnych niezależnych zmiennych losowych, podobną do hipotezy Samuelsa z roku 1965, pokazując, że jest ona asymptotycznie równoważna ułamkowej wersji hipotezy Erdősa.

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Chapter 1

Introduction

The thesis makes a contribution to the field of extremal combinatorics. We study a number of problems concerning matchings in hypergraphs related to Erdős conjecture from 1965. We also show how to apply our results to address some questions on small deviation inequalities for sums of independent random variables.

Extremal graph theory deals with problems, when we are to minimize or maximize some of the graph parameters over families of graphs satisfying certain properties. The fundamental result in this field is Mantel's theorem, proved in 1907, which states that any graph on n vertices with no triangles contains at most $n^2/4$ edges. In 1941 Turán [24] generalized this theorem, having determined the maximum number of edges in a graph on n vertices that does not contain a clique of a fixed size as a subgraph. This result inspired the development of the theory of Turán's type problems, which is now a substantial field of research in extremal graph theory. For a general graph F the maximum number of edges in a graph without copies of F is still not determined exactly although a celebrated result of Erdős and Stone [8] states that it asymptotically depends on the chromatic number of F, provided F is not bipartite. For bipartite graphs, determining the order of magnitude of this number is still a major open problem.

Similar extremal questions can be studied for k-uniform hypergraphs, i.e. families of k-element sets. More precisely, in a general hypergraph Turán's type problem, given a k-uniform hypergraph F, we want to determine the maximum number of edges $ex(n, k; F)$ in a k-uniform hypergraph on n vertices that contains no copies of F. It is well known that, typically, hypergraph problems are qualitatively much more difficult than their graphs analogues. This is also the case with hypergraph Turán's type problems, since there are very few hypergraphs for which the problem has been solved exactly, or even asymptotically, and most of these results are quite recent.

One example of a result of this type is the celebrated Erdős-Ko-Rado Theorem from 1961 [7], which bounds the number of edges in an intersecting k-uniform hypergraph, i.e. a hypergraph in which every two edges share a vertex. Note that, indeed, this is an instance of a Turán's type problem with two disjoint edges as a forbidden subgraph. The main problem considered in this thesis is the natural generalization of Erdős, Ko and Rado result, where instead of two disjoint edges we consider a matching of a given size as an excluded configuration. This is actually the well-known and long-standing open problem of Erdős, who in 1965 asked what is the maximum number of edges in a k-uniform hypergraph on n vertices whose matching number is exactly s. He conjectured that it is maximized either for cliques, or for graphs which consist of all edges intersecting a set of s vertices. Neither construction is uniformly better than the other in the whole range of parameter s ($1 \leq s \leq n/k$), so the conjectured bound is the maximum of these two possibilities.

Erdős Conjecture. *Every k-uniform hypergraph G on n vertices with matching number $\nu(G) = s \leq \frac{n}{k}$ satisfies*

$$e(G) \leq \max\left\{\binom{n}{k} - \binom{n-s}{k}, \binom{ks+k-1}{k}\right\}. \tag{1.1}$$

Although this problem has been extensively studied for the last fifty years, in its full generality, it still remains widely open and only some partial results have been obtained so far. Erdős Conjecture is known to be true for $s = 1$, as in this special case, the problem is equivalent to the Erdős-Ko-Rado Theorem [7]. In 1959, few years before the conjecture was stated in the whole generality, Erdős and Gallai [6] proved it for graphs, i.e. for $k = 2$. In Chapter 6 of this thesis we give an alternative proof of this result, based on Tutte's Theorem (see Theorem 6.4). For 3-uniform hypergraphs the conjecture has been verified just recently. First, Frankl, Rödl and Ruciński [15] confirmed it for $n \geq 4s$. In this range the conjectured maximum is still achieved by the first term in (1.1). In the main result of this thesis, by Łuczak and the author [19], we settled the conjecture in the affirmative for 3-uniform hypergraphs and $n > n_0$, having also shown that the only extremal 3-graphs are of the conjectured form (see Theorem 5.3). Eventually Frankl [12] got rid of the condition $n \geq n_0$ and confirmed Erdős Conjecture in the case of 3-graphs for every n. As for general case $k \geq 4$, there have been series of results which state that the conjecture holds for $n \geq g(k)s$, where $g(k)$ is some function of k. The existence of such $g(k)$ was shown by Erdős [5], then Bollobás, Daykin and Erdős [3] proved that the conjecture holds whenever $g(k) \geq 2k^3$, and Huang, Loh, and Sudakov [16] verified it for $g(k) \geq 3k^2$. The main result of author's joint paper with Frankl and Łuczak [14] slightly improved these bounds and confirmed the conjecture in a wider range for $g(k) \geq 2k^2/\log k$ (see Theorem 5.1). Currently, the best published bound for $g(k)$ is due to Frankl [13] who showed that the conjecture holds whenever $g(k) \geq 2k - 1$.

The asymptotic fractional version of Erdős Conjecture states that every k-uniform hypergraph G on n vertices with fractional matching number $\nu^*(G) = xn$, where $0 < x < 1/k$, satisfies

$$e(G) \leq (1 + o(1)) \max \left\{1 - (1 - x)^k, (kx)^k\right\} \binom{n}{k}.$$

This conjecture follows from Erdős Conjecture, and thus, it is true for $k = 2$ and $k = 3$ for every x, as a consequence of the results from [6], [12] and [19]. In a general case, the best bound is due to Frankl [13] and confirms the conjecture for $x \leq 1/(2k-1)$. For $k = 4$ and $x \leq 1/5$, the conjecture in its fractional version was proved by Alon *et al.* [2], who observed that it is closely related to an old probabilistic conjecture of Samuels on the behavior of the sum of independent random variables. This conjecture, if true, would imply fractional version of Erdős Conjecture for $x \leq 1/(k + 1)$, but for bigger values of x this is not the case anymore, and using Samuels' conjecture, one gets a bound on the extremal number of edges larger than the conjectured one. Together with Łuczak and Šileikis we state a new conjecture, of a similar flavour as Samuels', that is actually equivalent to the asymptotic fractional version of Erdős Conjecture and so, if proved, implies it for every $x \leq 1/k$ (see Theorem 6.9).

Conjecture (Łuczak, Mieczkowska, Šileikis)**.** *Let $X_1, \ldots, X_k$ be independent, identically distributed, nonnegative random variables with mean $\mathbb{E}(X_1) = x$. Then,*

$$\mathbb{P}(X_1 + \ldots X_k \geq 1) \leq \max\{1 - (1 - x)^k, (kx)^k\}. \tag{1.2}$$

This conjecture is a generalization of a result by Hoeffding and Shrinake [17] from 1955, in which they proved it for a sum of two random variables. In a general case, the conjecture holds for $x < 1/(k+1)$ whenever Samuels' conjecture is true.

Here, thanks to the equivalence result from Theorem 6.9, we confirm it in a few new instances, i.e. for $k = 3$ and $k \geq 5$ for $x \leq 1/(2k-1)$.

Let us remark that there has been a sudden increase of an interest in Erdős Conjecture for the last few years. In addition to being important in its own rights, it is mostly because of the results, obtained in Alon *et al.* [1] and Alon *et al.* [2], which revealed close connections of Erdős' problem to several important and seemingly unrelated questions. For instance, it is known (see Daykin and Häggkvist [4]) that it can be used to study Dirac's type problems on the minimum degree that guarantees the existence of a perfect matching in a uniform hypergraph. Moreover, it turned out (see Alon *et al.* [1]) that the fractional version of Erdős' problem might be used to attack an old number-theoretical conjecture of Manickam, Miklós and Singhi about non-negative sums. Furthermore, Erdős' problem has some interesting applications in information theory. For instance, results on Erdős Conjecture determine the optimal data allocation for the problem of data recovery in a uniform model of a distributed storage system studied by Sardari *et al.* [23], as it has been recently discovered to be asymptotically equivalent to the fractional version of Erdős' problem (see Alon *et al.* [2]).

The complete solution of a Turán's type problem usually consists of two steps. First, we need to show that any k-uniform hypergraph with at least $ex(n,k;F)+1$ edges contains a copy of F, and then to construct an F-free k-uniform hypergraph with n vertices and exactly $ex(n,k;F)$ edges. Therefore, consideration of this kind of extremal problem usually leads to the study of the structure of extremal hypergraphs, i.e. the largest F-free hypergraphs on n vertices. In Chapter 4 we present a number of results on families of k-uniform hypergraphs with a given matching number, and satisfying additional properties, e.g. being maximal, or shifted. In particular, we derive some stability results, which allow us to restrict the subject of the studies on Erdős Conjecture to the asymptotic properties of matchings in hypergraphs. Most of all we study a structure of hypergraphs which are extremal for our problem, what has led us to a new conjecture, stated in Section 6.1, that might be considered as an asymptotic generalization of Tutte's Theorem for hypergraphs, and, if true, might be helpful in proving Erdős Conjecture, for large n.

The structure of the thesis is as follows.

In the next chapter we recall the definitions and notions we shall use later. In particular, Section 2.3 introduces the main tool we will be using in the proofs – the shifting technique, heavily used in extremal set theory. Then, in Chapter 3, we state Erdős Conjecture in its exact and fractional version.

Chapter 4 contains preliminary results on families of k-uniform hypergraphs with a given matching number, and satisfying additional properties, e.g. being maximal, or shifted. They play an important role in our main results on Erdős Conjecture, which we introduce in Chapter 5. In Section 5.1, we present the result for a general k-uniform case, proving that Erdős Conjecture holds for k-uniform hypergraphs on n vertices whenever $n \geq 2k^2 s/\log k$ (see Theorem 5.1). In Section 5.2, we show that Erdős Conjecture is true for 3-uniform hypergraphs on n vertices, for $n > n_0$ (see Theorem 5.3).

In Chapter 6 we consider new conjectures and results obtained while working on Erdős Conjecture. In Section 6.1 we give a new proof of Erdős Conjecture for graphs based on Tutte's Theorem. We also formulate a structural conjecture that might be considered as an asymptotic generalization of Tutte's result for hypergraphs. In Section 6.2 we state a new probabilistic conjecture on small deviation inequalities and confirm it in a few new instances, by proving that it is asymptotically equivalent to the fractional version of Erdős' matching problem.

Chapter 2

Preliminaries

The aim of this chapter is to give an overview of the basic notions that are frequently used in extremal hypergraph theory and in this thesis.

We first introduce some notation we shall use throughout the thesis. Then we give basic definitions related to hypergraphs, paying particular attention to the notions concerning matchings in hypergraphs, as well as to their fractional analogues. In particular, we recall the idea of duality of linear programming, using which we can consider fractional vertex covers instead of fractional matchings, whenever convenient. Finally, in the last section, we discuss the shifting technique which is a widely used tool in extremal set theory, and which is crucial for most of the arguments presented in the thesis.

2.1 Notation

Here, we collect a list of frequently used notation and terminology.

- $\mathbb{R}$ denotes the set of real numbers.
- If $a, b \in \mathbb{R}$ and $a < b$, then $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$.

- $[n] := \{1, 2, \ldots, n\}$.
- $\emptyset$ denotes the *empty* set.
- $|X|$ denotes the *cardinality* of X.
- When writing $X = \{x_1, \ldots, x_n\}$, elements x_i are assumed to be distinct.
- $\{X_1, \ldots, X_k\}$ is a *partition* of X if $X = \bigcup_{i=1}^{k} X_i$ and $X_i \cap X_j = \emptyset$ for every $1 \le i < j \le k$.
- 2^X is the family of all subsets of X.
- $Y \subset 2^X$ denotes a family of subsets of X.
- $\binom{X}{k}$ denotes the family of all k-element subsets of X. Note that for $|X| = n$ we have $\left|\binom{X}{k}\right| = \binom{n}{k}$.

2.2 Matchings in hypergraphs

A *hypergraph* is a pair $G = (V, E)$, where $V = V(G)$ is a finite set of *vertices* and $E = E(G) \subset 2^V$ is a family of subsets of V, which are called *edges* of G. We say that a hypergraph $G = (V, E)$ is a *k-uniform hypergraph*, or briefly, a *k-graph*, if every edge of G consists of precisely k elements. By $v(G) = |V|$ and $e(G) = |E|$ we denote the number of vertices and edges of $G = (V, E)$, respectively. In this thesis we usually assume that $V = [n]$ and identify a hypergraph G with its edge set E. Therefore, whenever we write $|G|$, we mean $e(G)$, and by $e \in G$ we mean $e \in E$. For a given set $S \subset V$ the number of edges containing S is the *degree* of the set S and is denoted by $\deg_G(S)$. A *complete k-uniform hypergraph*, or a *clique*, is a k-graph on a vertex set V with all possible edges, i.e. for which $E = \binom{V}{k}$. Note that whenever we say a *graph*, we mean a 2-uniform hypergraph, i.e. a hypergraph in which every edge consists of exactly two vertices.

We say that a k-uniform hypergraph G is *maximal* with respect to property A, if G has property A, and for every k-tuple $e \notin E$ a k-graph $G \cup \{e\}$ does not have A. A k-uniform hypergraph G is the *maximum* k-graph with respect to property A, if no k-graph with property A has more edges than G.

A *matching* in G is a family of disjoint edges of G. We say that a matching M *saturates* vertex $v \in V$ if one of its edges contains v. A matching M is *maximal* if there is no edge $e \in E$ such that $M \cup \{e\}$ is a matching in G, and it is a *maximum* matching, if G contains no matching of a bigger size. The size of the largest matching contained in G is denoted by $\nu(G)$, and is called the *matching number* of G. We say that a matching is *perfect* if it is of size $v(G)/k$. A hypergraph

G in which each two edges intersect, i.e. for which $\nu(G) = 1$, is an *intersecting* hypergraph.

A *fractional matching* in a k-uniform hypergraph $G = (V, E)$ is a function

$$w : E \to [0, 1] \text{ such that}$$
$$\sum_{e \ni v} w(e) \le 1 \text{ for every vertex } v \in V.$$

Then, $\sum_{e \in E} w(e)$ is the *size* of matching w and the size of the largest fractional matching in G, denoted by $\nu^*(G)$, is the *fractional matching number* of G. Observe that if $w(e) \in \{0, 1\}$ for every edge e, then w is just a matching, or more precisely, the indicator function of a matching. Thus, every integral matching is also a fractional matching and hence, $\nu^*(G)$ is always greater or equal than $\nu(G)$.

Finding the fractional matching number is clearly a linear programming problem. Its dual problem is to find the size of the minimum fractional vertex cover. A *fractional vertex cover* in a k-uniform hypergraph $G = (V, E)$ is a function

$$w : V \to [0, 1] \text{ such that}$$
$$\text{for each } e \in E \text{ we have } \sum_{v \in e} w(v) \ge 1.$$

Then, $\sum_{v \in V} w(v)$ is the *size* of w and the size of the smallest fractional vertex cover in G is denoted by $\tau^*(G)$. By Duality Theorem (see, for instance, Nering *et al.* [20]), for every k-uniform hypergraph G we have $\nu^*(G) = \tau^*(G)$. Let us also recall that a *vertex cover* of G is a set of vertices $S \subset V$ such that each edge of G has at least one vertex in S, i.e. it is a solution of the above system of inequalities with the restriction that all vertex weights are either 0 or 1. Let $\tau(G)$ denote the minimum number of vertices in a vertex cover of G. Note that since any integral vertex cover is also a fractional vertex cover, $\tau^*(G)$ is always smaller or equal than $\tau(G)$, so we have

$$\nu(G) \le \nu^*(G) = \tau^*(G) \le \tau(G).$$

2.3 Shifting technique

The shifting technique, also known as compression, is used in all of our results on Erdős Conjecture. The method was introduced by Erdős, Ko, and Rado [7] and is one of the most important and widely-used tools in extremal set theory (see an extensive survey of Frankl [11] on this subject).

Generally speaking, the shift operator transforms our original hypergraph G into a more structured one, which still preserves a lot of properties of G.

In many cases such an attempt makes the argument much simpler and shorter. In particular, the basic fact we shall use about the shift operator is that it does not change the size of a hypergraph and does not increase its matching number. Therefore, in the main problem considered in this thesis: to maximize $e(G)$, given $v(G)$ and $\nu(G)$, it is enough to work with shifted hypergraphs. We start with sketching the main ideas which the method is based on.

Let $G = (V, E)$, $V = [n]$ be a k-graph. For vertices $i < j$, the graph $\mathbf{sh}_{ij}(G)$, called the (i, j)-shift of G, is obtained from G by replacing each edge $e \in E$, such that $j \in e$, $i \notin e$, and $f = (e \setminus \{j\}) \cup \{i\} \notin E$, by f. From the definition of the (i, j)-shift it is clear that this operation preserves the number of edges of a hypergraph and the following holds.

Proposition 2.1. *For any n-vertex hypergraph G and $1 \leq i < j \leq n$ we have*

$$|G| = |\,\boldsymbol{sh}_{ij}(G)|.$$

The following is another simple and well known result (see Frankl [11]), the proof of which we give here for the completeness of the argument.

Proposition 2.2. *For any n-vertex hypergraph G and $1 \leq i < j \leq n$ we have*

$$\nu(\boldsymbol{sh}_{ij}(G)) \leq \nu(G).$$

Proof. Let us assume that $M = \{e_1, \dots, e_\ell\}$ is a matching in $\mathbf{sh}_{ij}(G)$ but not in G. Then, one of the edges of M, let say e_1, is not an edge in G. Clearly, we must have $i \in e_1$, $j \notin e_1$, and $f = (e_1 \setminus \{i\}) \cup \{j\} \in E$. We distinguish two cases. If $j \notin \bigcup_r e_r$, then $M' = \{(e_1 \setminus \{i\}) \cup \{j\}, e_2, \dots, e_\ell\}$ is a matching in G. If vertex j is saturated by M, say $j \in e_2$, then $M'' = \{(e_1 \setminus \{i\}) \cup \{j\}, (e_2 \setminus \{j\}) \cup \{i\}, e_3, \dots, e_\ell\}$ is a matching in G. Hence $\nu(\mathbf{sh}_{ij}(G)) \leq \nu(G)$. □

Now let us define $\mathbf{Sh}(G)$ as a hypergraph which is obtained from G by a series of shifts and which is invariant under all possible shifts, i.e. $\mathbf{sh}_{ij}(\mathbf{Sh}(G)) = \mathbf{Sh}(G)$ for all $1 \leq i < j \leq n$. A graph G is called *shifted* if $G = \mathbf{Sh}(G)$. Although it is not hard to construct examples where the order of shifting can affect the structure of the final shifted hypergraph, one can check that $\binom{n}{2}$ shifts are sufficient to make a hypergraph shifted, if we do it in the right order. The following is a straightforward consequence of the definition of a shifted hypergraph that we use in our argument. It states that any k-tuple that precedes some edge of G in the lexicographical order is also an edge in G.

Proposition 2.3. *Let $G = (V, E)$ be a shifted k-graph, i.e. $\boldsymbol{Sh}(G) = G$, and let $\{v_1, \dots, v_k\} \in E$. For any $f = \{w_1, \dots, w_k\}$, such that $w_i \in V$ and $w_i \leq v_i$ for every $i = 1, \dots, k$, we have $f \in E$.* □

Due to the above fact, instead of G itself very often we can work with a highly structured shifted hypergraph obtained from G, which typically greatly simplifies the whole argument.

Chapter 3

Number of edges in hypergraphs with a given matching number

In this chapter we state Erdős Conjecture which, as we have already mentioned, inspired most of the results presented in this thesis. First, we describe the problem for integral matchings, as it was originally stated by Erdős in 1965, then we discuss its fractional relaxation.

3.1 Integral matchings

The main problem considered in this thesis is to determine the maximum number of edges in a k-uniform hypergraph on n vertices whose matching number is exactly s. More formally, let $\mathcal{H}_k(n,s)$ denote the set of all k-graphs $G=(V,E)$ such that $v(G)=n$ and $\nu(G)=s$; moreover let

$$\mu_k(n,s)=\max\{e(G):\ G\in\mathcal{H}_k(n,s)\},\tag{3.1}$$

and let

$$\mathcal{M}_k(n,s)=\{G\in\mathcal{H}_k(n,s):\ e(G)=\mu_k(n,s)\}\tag{3.2}$$

be the family of the extremal hypergraphs for this question. In 1965 Erdős [5] stated the following problem.

Problem 3.1. *For every k, s, and $n \geq k(s+1)$ determine $\mu_k(n,s)$ and $\mathcal{M}_k(n,s)$.*

Note that for $n \leq ks+k-1$ the question is trivial, as then, the only hypergraphs in $\mathcal{M}_k(n,s)$ are cliques.

Let us describe two types of k-graphs from $\mathcal{H}_k(n,s)$ which are natural candidates for members of $\mathcal{M}_k(n,s)$. By $Cov_k(n,s)$ we denote the family of k-graphs $G_1 = (V_1, E_1)$ such that $|V_1| = n$ and for some subset $S \subseteq V_1$, $|S| = s$, we have

$$E_1 = \{e \subseteq V_1 : \ e \cap S \neq \emptyset \text{ and } |e| = k\}.$$

Note that every edge of G_1 contains at least one vertex from the set S, and hence $\nu(G_1) \leq s$. Thus, clearly, if $n \geq ks$, then $Cov_k(n,s) \subseteq \mathcal{H}_k(n,s)$. Notice also that every hypergraph in $Cov_k(n,s)$ is actually a k-graph obtained from a clique on n vertices by removing edges of a smaller clique on $n-s$ vertices. Hence,

$$e(G_1) = \binom{n}{k} - \binom{n-s}{k}.$$

Now let $Cl_k(n,s)$ be the family of all k-graphs $G_2 = (V_2, E_2)$ which consist of a complete subgraph on $ks+k-1$ vertices and some isolated vertices, i.e. if for some subset $T \subseteq V_2$, $|T| = ks+k-1$, we have

$$E_2 = \{e \subseteq T : \ \ |e| = k\}.$$

It is easy to see that $Cl_k(n,s) \subseteq \mathcal{H}_k(n,s)$ and

$$e(G_2) = \binom{ks+k-1}{k}.$$

In 1965 Erdős [5] conjectured that, indeed, the function $\mu_k(n,s)$ is fully determined by the behavior of k-graphs from families $Cov_k(n,s)$ and $Cl_k(n,s)$. Since neither construction is uniformly better than the other in the whole range of parameter s ($1 \leq s \leq (n-k+1)/k$), the conjectured bound is the maximum of these two possibilities.

Erdős Conjecture. *For every k, s, and $n \geq ks+k-1$, the following holds*

$$\mu_k(n,s) = \max\left\{\binom{n}{k} - \binom{n-s}{k}, \binom{ks+k-1}{k}\right\}. \tag{3.3}$$

Moreover, for $n \geq 2k+1$, we have

$$\mathcal{M}_k(n,s) = Cov_k(n,s) \cup Cl_k(n,s).$$

Note that the second part of the statement does not hold when $n = 2k$, $k \geq 3$, and $s = 1$. Indeed, in such a case one can pair every k-tuple with its complement. Thus, every maximal graph contains exactly one k-tuple from each of $\frac{1}{2}\binom{2k}{k}$ such pairs, so $\mu_k(2k, 1) = \frac{1}{2}\binom{2k}{k}$ and

$$|\mathcal{M}_k(2k, 1)| = 2^{\frac{1}{2}\binom{2k}{k}},$$

while

$$|\,Cov_k(2k, 1)| = |\,Cl_k(2k, 1)| = 2k\,.$$

Let us now comment briefly on the formula (3.3). If by $s_0(n, k)$ we define the smallest s for which

$$\binom{n}{k} - \binom{n-s}{k} \leq \binom{ks+k-1}{k},$$

then it is easy to see that

$$\lim_{n\to\infty} \frac{s_0(n, k)}{n} = \alpha_k\,,$$

where $\alpha_k \in (0, 1/k)$ is the solution of the equation

$$1 - (1 - \alpha_k)^k = k^k \alpha_k^k\,.$$

One can check that for all $k \geq 2$ we have

$$\frac{1}{k} - \frac{1}{2k^2} < \alpha_k \leq \frac{1}{k} - \frac{2}{5k^2}; \tag{3.4}$$

in fact, $(1-k\alpha_k)k \to -\ln(1-e^{-1}) = 0.4586...$ as $k \to \infty$. Note that, in particular, for $k \geq 2$ and $n \geq (k+1)s$ we have

$$\binom{n}{k} - \binom{n-s}{k} > \binom{ks+k-1}{k}.$$

Although Problem 3.1 has been extensively studied for the last fifty years, in its full generality, it still remains widely open. However, a few partial results have been obtained in this direction and we list them briefly below.

Erdős Conjecture is known to be true for $s = 1$. Note that then, family $Cov_k(n, 1)$ consists of intersecting k-graphs and $\binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1}$. In this special case, the problem is equivalent to the celebrated Erdős-Ko-Rado Theorem [7], proved already in 1961.

Theorem 3.2 (Erdős-Ko-Rado)**.** *Let $n \geq 2k$. Every intersecting k-uniform hypergraph on n vertices satisfies*

$$|G| \leq \binom{n-1}{k-1}.$$

Moreover, if $n > 2k$ then the equality holds if and only if G consists of all k-element subsets containing i, for some $i \in V$.

In 1959, few years before the conjecture was stated in the whole generality, Erdős and Gallai [6] proved it in a graph case, i.e. for $k = 2$. In the last part of the thesis we give an alternative proof of this result for n large enough (see Theorem 6.4).

For 3-uniform hypergraphs the conjecture has been verified just recently. First, Frankl, Rödl and Ruciński [15] confirmed it for $n \geq 4s$. In this range the conjectured maximum is achieved by the first term in (3.3). Some time later, Łuczak and Mieczkowska [19] settled the conjecture in the affirmative for 3-uniform hypergraphs and $n > n_0$, having also shown that the only extremal 3-graphs are of the conjectured form.

Theorem. *Erdős Conjecture holds for* 3*-uniform hypergraphs, for* $n > n_0$*.*

We give the proof of this result in Section 5.2. Eventually Frankl [12] got rid of the condition $n \geq n_0$ and confirmed Erdős Conjecture in the case of 3-graphs for every n.

As for general case $k \geq 4$, there have been series of results, dealing mostly with the case when n is large compared to s, and proving that

$$\mu_k(n, s) = \binom{n}{k} - \binom{n-s}{k} \quad \text{for} \quad n \geq g(k)s, \tag{3.5}$$

where $g(k)$ is some function of k. The existence of such $g(k)$ was shown by Erdős [5], then Bollobás, Daykin and Erdős [3] proved that (3.5) holds whenever $g(k) \geq 2k^3$, and Huang, Loh, and Sudakov [16] verified (3.5) for $g(k) \geq 3k^2$. The main result of author's joint paper with Frankl and Łuczak [14] slightly improved these bounds and confirmed the conjecture in a wider range for $g(k) \geq 2k^2/\log k$.

Theorem. *Erdős Conjecture holds for k-uniform hypergraphs, for* $n \geq \frac{2k^2 s}{\log k}$*.*

Currently, the best published bound for $g(k)$ is due to Frankl [13] who showed that (3.5) holds whenever $g(k) \geq 2k - 1$.

3.2 Fractional matchings

In this section we formulate the fractional version of Erdős Conjecture, in which the matching number $\nu(G)$ is replaced by the fractional matching number $\nu^*(G)$. In order to do that, for a real number $0 \leq s \leq (n - k + 1)/k$, let us define the fractional equivalent of function $\mu_k(n, s)$ as follows

$$\mu_k^*(n, s) = \max\left\{e(G) : V = [n], E \subset \binom{[n]}{k}, \nu^*(G) < s + 1\right\}.$$

Note that since $\nu(G) \le \nu^*(G)$, we get trivially that for $\lceil s \rceil \le (n-k+1)/k$ we have

$$\mu_k^*(n,s) \le \mu_k(n, \lceil s \rceil). \tag{3.6}$$

In order to get the lower bound for $\mu_k^*(n,s)$ let us consider again the families of k-graphs $Cov_k(n, \lceil s \rceil)$ and $Cl_k(n, \lceil ks \rceil/k)$.

Proposition 3.3. *Let $G \in Cov_k(n, \lceil s \rceil) \cup Cl_k(n, \lceil ks \rceil/k)$. Then, $\nu^*(G) < s+1$.*

Proof. Since, by duality, we have $\nu^*(G) = \tau^*(G)$, we can switch to the dual problem, and thus, it suffices to find a fractional vertex cover in G of size smaller than $s+1$. Let $G_1 \in Cov_k(n, \lceil s \rceil)$ and let $S \subset V_1$, $|S| = \lceil s \rceil$ be the set which covers all edges of G_1. Observe that a function $w : V_1 \to [0,1]$ such that $w(i) = 1$ if $i \in S$, and $w(i) = 0$ otherwise, is a fractional vertex cover in G_1 of size $\lceil s \rceil$. Thus, $\nu^*(G_1) = \tau^*(G_1) \le \lceil s \rceil < s+1$.

Let now $G_2 \in Cl_k(n, \lceil ks \rceil/k)$ and let $W \subset V_2$, $|W| = \lceil ks \rceil + k - 1$ be the vertex set of the largest clique in G_2. Note again that a function $w : V_2 \to [0,1]$ such that $w(i) = 1/k$ for every $i \in W$, and $w(i) = 0$ otherwise, is a fractional vertex cover in G_2 of size $(\lceil ks \rceil + k - 1)/k < s+1$. Thus, again $\nu^*(G_2) = \tau^*(G_2) < s+1$. $\square$

The above fact, together with (3.6), shows that for integer values of s we have

$$\mu_k^*(n,s) = \mu_k(n,s),$$

and thus, the following is conjectured to be the right bound on $\mu_k^*(n,s)$.

Conjecture 3.4. *Let $k \ge 2$, n, and s be integers such that $0 < s \le (n-k+1)/k$. Then,*

$$\mu_k^*(n,s) = \max\left\{\binom{n}{k} - \binom{n - \lceil s \rceil}{k}, \binom{\lceil ks \rceil + k - 1}{k}\right\}.$$

It is not hard (see Leong *et al.* [18]) to give an example of a (small) hypergraph which shows that Conjecture 3.4 does not hold for fractional s. Note that, in such a case, for $G \in Cl_k(n, \lceil ks \rceil/k)$ we have

$$|G| = \binom{\lceil ks \rceil + k - 1}{k} \le \binom{k\lceil s \rceil + k - 1}{k},$$

which does not always match the upper bound, following from (3.6) and Erdős Conjecture. It also seems that there is no obvious guess what should be the right bound in the precise version of this conjecture for non-integer s. Note however, that then, from Conjecture 3.4 it follows that the bound should be asymptotically true, whenever n is large enough. Thus, the asymptotic version of the fractional matching conjecture can be stated as follows (here and below by $o(1)$ we denote the quantity which tends to 0 as $n \to \infty$).

Conjecture 3.5. *Every k-uniform hypergraph G on n vertices with fractional matching number $\nu^*(G) = xn$, where $0 < x < 1/k$, satisfies*

$$|G| \leq (1 + o(1)) \max\left\{1 - (1-x)^k, (kx)^k\right\} \binom{n}{k}.$$

Since Conjecture 3.5 is weaker than Erdős Conjecture, it is true for $k = 2$ and $k = 3$, for every x, as a consequence of the Erdős-Gallai theorem from [6] and the latest results on 3-uniform hypergraphs of Frankl [12], and Łuczak and Mieczkowska [19]. For $k = 4$, Conjecture 3.5 was confirmed for $x \leq 1/5$ by Alon *et al.* [2]. In a general k-uniform case, the best bound on $\mu_k^*(n, s)$ follows from the result of Frankl [13], and confirms the conjecture for $x \leq 1/(2k-1)$. For this range of parameters the maximum is achieved by the first term.

Alon *et al.* [2] observed that the fractional version of Erdős Conjecture is closely related to an old conjecture of Samuels on the behavior of the sum of independent random variables. This conjecture, if proved, would imply Conjecture 3.5 for $x \leq 1/(k+1)$. In this range of parameters, the maximum is still achieved by the first term. It is also known (see [2]) that for bigger values of x this is not the case anymore, and using Samuels' conjecture, one gets a bound on $\mu_k^*(n, s)$ larger than that in Conjecture 3.5. In Section 6.2 we state a new conjecture, of a similar flavour as Samuels', that is actually equivalent to Conjecture 3.5, and so, if true, implies it for every $x \leq 1/k$.

Chapter 4

Preliminary results

In this chapter we discuss some results on families of k-graphs with matching number s satisfying additional properties, e.g. being maximal, or shifted. They will play an important role in the proofs of our results on Erdős Conjecture (see Chapter 5).

4.1 Degrees of maximal hypergraphs $\mathcal{L}_k(n, s)$

Let $\mathcal{L}_k(n, s)$ be the family of k-graphs G on n vertices which are maximal with respect to the property $\nu(G) = s$, i.e. G cannot be extended without increasing $\nu(G)$. More precisely, let

$$\mathcal{L}_k(n, s) = \left\{ G \in \mathcal{H}_k(n, s) : \forall_{e \in \binom{V}{k}, e \notin E} \; \nu(G \cup \{e\}) = s + 1 \right\}.$$

We start with the following result on degrees of maximal graphs, which is a generalization of a similar result by Łuczak and Mieczkowska from [19].

Lemma 4.1. *Let $1 \leq i < k$ and $G \in \mathcal{L}_k(n, s)$. If for a subset $f \subset V$, $|f| = i$, we have $\deg(f) > \binom{n-i}{k-i} - \binom{n-ks-i}{k-i}$, then $\deg(f) = \binom{n-i}{k-i}$.*

Proof. First let us observe that for $n \leq ks + k - 1$ the only graphs in $\mathcal{L}_k(n, s)$ are cliques, so in such a case the statement follows easily. Thus, we may assume that $n \geq k(s + 1)$. Let $f \subset V$ be a subset of a large degree in G and let us suppose that e is a k-subset of V such that $f \subsetneq e$ and $e \notin E$. Then, by the definition of $\mathcal{L}_k(n, s)$, graph $G \cup \{e\}$ contains a matching M of size $s + 1$, where, clearly, $e \in M$. However, the degree of f is chosen to be so large that for some $(k - i)$-element subset $g \subseteq V \setminus (\bigcup M \setminus e)$ the set $e' = f \cup g$ is an edge of G. But then, $M' = (M \setminus \{e\}) \cup \{e'\}$ is a matching of size $s + 1$ in G. This contradiction shows that each k-element subset of V which contains f is an edge of G and thus, $\deg(f) = \binom{n-i}{k-i}$. □

For shifted hypergraphs $G \in \mathcal{L}_k(n, s)$, a similar result can be derived from a bit weaker assumption.

Lemma 4.2. *Let $1 \leq i < k$ and let $G \in \mathcal{L}_k(n, s)$ be such that $\boldsymbol{Sh}(G) = G$. If for some $f \subset [ks + i]$, $|f| = i$ there exists $e \in G$ such that $e \cap [ks + i] = f$, then* $\deg(f) = \binom{n-i}{k-i}$.

Proof. As in the previous proof, for $n \leq ks + k - 1$ the statement holds trivially. Let now $n \geq k(s + 1)$. Take a subset $f \subset V$ such that $f = [ks + i] \cap e$ for some $e \in G$, and let us suppose that $\deg(f) < \binom{n-i}{k-i}$. Then, as G is shifted, $e' = f \cup \{n - k + i + 1, \dots, n\}$ is not an edge in G. Since G is maximal, the graph $G \cup \{e'\}$ contains a matching $M \cup \{e'\}$ of size $s + 1$. Observe that $M \subset E$ is contained in $[n] \setminus e'$. Thus, since G is shifted, there is a matching M' of size s in G, which is contained in the set $[ks + i] \setminus f$. But then, $M' \cup \{e\}$ is a matching of size $s + 1$ in G, contradicting the fact that $\nu(G) = s$. Thus, we must have $\deg(f) = \binom{n-i}{k-1}$. □

4.2 Families of hypergraphs $\mathcal{A}_k(n, s; l)$

In Section 3.1 we have discussed two families of k-graphs: $Cov_k(n, s)$ and $Cl_k(n, s)$. Here we come up with a more generalized notion of hypergraphs, that emerges naturally while asking about k-uniform hypergraphs with a given matching number. They were first defined in the Ph.D. dissertation of Frankl in 1976. For $l = 1, 2, \dots, k$ by $\mathcal{A}_k(n, s; l)$ we denote the family of k-graphs $G_l = (V_l, E_l)$ such that $|V_l| = n$ and for some subset $S_l \subseteq V_l$, $|S_l| = l(s + 1) - 1$, called the *ℓ-center* of G_l, we have

$$E_l = \left\{ e \in \binom{V_l}{k} : |e \cap S_l| \geq l \right\}.$$

Observe that then, for $n \geq ks$,

$$\nu(G_l) = s,$$

and if $n \geq k(s+1)$, then G_l are maximal. Notice also that $\mathcal{A}_k(n,s;l)$ is a common generalization of families $Cov_k(n,s)$ and $Cl_k(n,s)$, since

$$\mathcal{A}_k(n,s;1) = Cov_k(n,s) \text{ and } \mathcal{A}_k(n,s;k) = Cl_k(n,s).$$

Since most of the arguments in this work are done for the shifted k-graphs, let $A_k(n,s;l)$ denote the shifted representative for the family $\mathcal{A}_k(n,s;l)$, i.e. for $l = 1,\ldots,k$ we set

$$A_k(n,s;l) = \left\{ e \in \binom{[n]}{k} : |e \cap [l(s+1)-1)]| \geq l \right\}.$$

Whenever k, s, and n are fixed, we denote $A_k(n,s;l)$ just by A_l.

4.3 Shifting properties of $\mathcal{M}_k(n,s)$ and $\mathcal{A}_k(n,s;l)$

Let us start with the following easy consequence of Propositions 2.1 and 2.2.

Proposition 4.3. *Let $1 \leq i < j \leq n$. If $G \in \mathcal{M}_k(n,s)$, then $\boldsymbol{sh}_{ij}(G) \in \mathcal{M}_k(n,s)$.*

Proof. Since the operator $\mathbf{sh}_{ij}$ does not change the size of a hypergraph, we have that $|\mathbf{sh}_{ij}(G)| = |G| = \mu_k(n,s)$. Moreover, for fixed k and n the function $\mu_k(n,s)$ is strictly increasing, hence $\nu(\mathbf{sh}_{ij}(G)) \geq \nu(G) = s$. On the other hand, from Proposition 2.2 we know that $\nu(\mathbf{sh}_{ij}(G)) \leq \nu(G) = s$, and thus we have $\mathbf{sh}_{ij}(G) \in \mathcal{M}_k(n,s)$. $\square$

The main result of this section states that the families $\mathcal{A}_k(n,s;l)$ are invariant under the shift operator, and moreover, that the only k-graphs the shifts of which are members of $\mathcal{A}_k(n,s;l)$ are hypergraphs from $\mathcal{A}_k(n,s;l)$. The following theorem is a generalization of a similar results from [19], obtained for families $Cov_k(n,s)$ and $Cl_k(n,s)$.

Theorem 4.4. *Let $l \in [k]$, $s \geq 2$, $n \geq k(s+1)$ and $G \in \mathcal{H}_k(n,s)$. Then, for every $1 \leq i < j \leq n$,*

$$G \in \mathcal{A}_k(n,s;l) \text{ if and only if } \boldsymbol{sh}_{ij}(G) \in \mathcal{A}_k(n,s;l).$$

The following simple fact will be useful in our further argument.

Claim 4.5. *Let $G_l = (V_l, E_l)$ be a $(k-1)$-graph with a vertex set $V_l = V_1 \cup V_2$, where $|V_1| \geq 3(l-1)$, $|V_2| \geq 3(k-l)$ and the edge set is defined as follows*

$$E_l = \{e \subset V : |e \cap V_1| = l-1, |e \cap V_2| = k-l\}.$$

If we color all edges of G_l with two colors, then either we find two disjoint edges colored with different colors, or all of them are of the same color.

Proof. Let us color edges of G_l with two colors and let us assume that not all of them are of the same color, i.e. there exist $e, f \in E$ colored with different colors. If $e \cap f = \emptyset$, then the claim holds. Otherwise, consider an edge $g \in E$ such that it is disjoint with e and f. Observe that since both V_1 and V_2 are large such an edge always exists. Then, g is of a different color than one of the edges e and f. Since g is disjoint with both of them, the assertion follows. □

Proof of Theorem 4.4. Let us first observe that if $G \in \mathcal{A}_k(n, s; l)$ then the operator $\mathbf{sh}_{ij}(\cdot)$ clearly transform the l-center of G into l-center of $\mathbf{sh}_{ij}(G)$ and so $\mathbf{sh}_{ij} \in \mathcal{A}_k(n, s; l)$. Thus it is enough to show the implication in the opposite direction.

To this end let $\mathbf{sh}_{ij}(G) \in \mathcal{A}_k(n, s; l)$ and let S be the l-center of $\mathbf{sh}_{ij}(G)$. If either $i \notin S$ or $j \in S$, then clearly S is an l-center for G and so $G \in \mathcal{A}_k(n, s; l)$. Thus, let us assume that $i \in S$ and $j \notin S$. Note also that all edges e of $\mathbf{sh}_{ij}(G)$ which contain neither i nor j remain invariant under $\mathbf{sh}_{ij}(\cdot)$ operation; in particular all of them intersect $S \setminus \{i, j\}$ on at least l vertices. In order to deal with the remaining edges of $\mathbf{sh}_{ij}(G)$ let us color all $(k-1)$-element subsets f of $V \setminus \{i, j\}$ for which $|f \cap (S \setminus \{i\})| = l-1$ with two colors: red if $\{i\} \cup f \in G$ and blue if $\{j\} \cup f \in G$. Observe first that each such $(k-1)$-element subset is colored with exactly one color. Indeed, if it is not the case, then both $\{i\} \cup f$ and $\{j\} \cup f$ are edges of G and hence also $\{j\} \cup f \in \mathbf{sh}_{ij}(G)$. But then $|(\{j\} \cup f) \cap S| = l-1$, contradicting the fact that S is the l-center of $\mathbf{sh}_{ij}(G)$. Furthermore, if for a pair of disjoint subsets f' and f'', f' is red and f'' is blue, then the edges $\{i\} \cup f'$ and $\{j\} \cup f''$ can be completed to a matching of size $s+1$ in G, contradicting the fact that $\nu(G) = s$. Indeed, since $n \geq k(s+1)$ and $|S \setminus (\{i\} \cup f' \cup f'')| \geq l(s-1)$, it is easy to see that one can find $s-1$ disjoint edges $e \in E$, contained in $V \setminus (\{i\} \cup f' \cup \{j\} \cup f'')$, and such that $|e \cap (S \setminus (\{i\} \cup f' \cup f''))| = l$. Observe also that for $s \geq 2$ we have $|S \setminus \{i\}| \geq l(s+1) - 2 > 3(l-1)$ and $|V \setminus (S \cup \{j\})| \geq n - l(s+1) \geq (k-l)(s+1) \geq 3(k-l)$. Thus, by Claim 4.5, all such sets are colored with one color and either S or $(S \setminus \{i\}) \cup \{j\}$ is the l-center of G. Consequently, $G \in \mathcal{A}_k(n, s; l)$. □

Let us recall that $\mathbf{Sh}(G)$ is a hypergraph obtained from G by a series of shifts and is invariant under all possible shifts, i.e. $\mathbf{sh}_{ij}(\mathbf{Sh}(G)) = \mathbf{Sh}(G)$ for all $1 \le i < j \le n$. Thus, from Proposition 4.3 and Theorem 4.4 we get the following result (see [19]).

Theorem 4.6.

(i) If $G \in \mathcal{M}_k(n,s)$ then $\boldsymbol{Sh}(G) \in \mathcal{M}_k(n,s)$.

(ii) If $n \ne 2k$, $G \in \mathcal{M}_k(n,s)$, and $\boldsymbol{Sh}(G) \in Cov_k(n,s)$, then $G \in Cov_k(n,s)$.

(iii) If $n \ne 2k$, $G \in \mathcal{M}_k(n,s)$, and $\boldsymbol{Sh}(G) \in Cl_k(n,s)$, then $G \in Cl_k(n,s)$.

Proof. Let us just remark that for $n \le ks+k-1$, the only hypergraphs in $\mathcal{M}_k(n,s)$ are cliques, and for $s = 1$ and $n \ge 2k+1$ we have $\mathcal{M}_k(n,1) = Cov_k(n,1)$ by the Erdős-Ko-Rado Theorem. Thus, for $n \le ks+k-1$ the assertion follows. Then, we may assume that $n \ge k(s+1)$, $s \ge 2$, and use Theorem 4.4 to derive all three statements. □

Note that in view of the last theorem, in order to confirm Erdős Conjecture it suffices to prove it for hypergraphs G for which $\mathbf{Sh}(G) = G$.

4.4 Structure of shifted $\mathcal{H}_k(n,s)$

The best general bound on $\mu_k(n,s)$, true for all k, s and $n \ge ks$, is due to Frankl [10].

Theorem 4.7. *Let $n \ge ks$, then*

$$\mu_k(n,s) \le s\binom{n}{k-1}. \tag{4.1}$$

We present the proof of the above statement below, as we use a similar ideas in the proof of Theorem 4.10. Let us recall that A_l is the only shifted graph in the family $\mathcal{A}(n,s;l)$. We start with the following observation.

Lemma 4.8. *If $G \in \mathcal{H}_k(n,s)$ is such that $\boldsymbol{Sh}(G) = G$, then*

$$G \subseteq A_1 \cup A_2 \cup \cdots \cup A_k\,.$$

Proof. First observe that for $n \le ks+k-1$ the assertion holds trivially, as then $G \subset A_k$. Thus let us assume that $n \ge k(s+1)$ and note that the set $e_0 = \{s+1, 2s+2, \ldots, ks+k\}$ is not an edge of G. Indeed, if it was the case, then each of the edges $\{i, i+s+1, \ldots, i+(k-1)(s+1)\}$, $i = 1, 2, \ldots, s+1$, belongs to G, due to the fact that $G = \mathbf{Sh}(G)$. Clearly, they form a matching of size $s+1$, contradicting the fact that $\nu(G) = s$. Thus, $e_0 \notin G$ and it is enough to observe that all k-tuples which do not dominate e_0 in a lexicographical order must belong to $\bigcup_{l=1}^{k} A_l$. $\square$

Let $A = \bigcup_{l=1}^{k} A_l$. It turns out that the size of A can be easily found.

Lemma 4.9. *If $n \geq k(s+1)$, then $|A| = s\binom{n}{k-1}$.*

Proof. We prove the statement using induction on k and n. For $k \geq 1$ and $n = k(s+1)-1$ we have clearly $|A| = \binom{n}{k} = s\binom{n}{k-1}$. Now let $k \geq 2$, $n \geq k(s+1)$ and split all the sets of A into those which contain n and those which do not, i.e. $A = A_{n\in} \cup A_{n\notin}$. Then, $\nu(A_{n\notin}) \leq s$ and for $A'_{n\in} = \left\{e \subset \binom{[n]}{k-1} : e \cup \{n\} \in G\right\}$ we have $|A'_{n\in}| = |A_{n\in}|$ and $\nu(A'_{n\in}) \leq s$. Thus, the inductional hypothesis gives

$$|A| = |A_{n\in}| + |A_{n\notin}| \leq s\binom{n-1}{k-2} + s\binom{n-1}{k-1} = s\binom{n}{k-1},$$

so the assertion follows. $\square$

Proof of Theorem 4.7. Note first that for $n \leq ks+k-1$ the only k-graphs in $\mathcal{M}_k(n,s)$ are cliques, so it is easy to check that then clearly $\binom{n}{k} \leq s\binom{n}{k-1}$. For $n \geq k(s+1)$ the assertion is an immediate consequence of Lemmas 4.8 and 4.9, and part (i) of Theorem 4.6. $\square$

In [14], together with Frankl and Łuczak, we derive the following numerical consequence of Lemmas 4.8 and 4.9 that is crucial for our argument in Section 5.1.

Theorem 4.10. *If $G \in \mathcal{H}_k(n,s)$ is such that $\boldsymbol{Sh}(G) = G$, then all except at most $\frac{s(s+1)}{2}\binom{n-1}{k-2}$ edges of G intersect $[s]$.*

Proof. Let $A = \bigcup_{l=1}^{k} A_l$. From Lemma 4.8 and the proof of Lemma 4.9 we know that $G \subset A$ and $|A| = s\binom{n}{k-1}$. Observe also that $\binom{n}{k} = \sum_{i=1}^{s}\binom{n-i}{k-1} + \binom{n-s}{k}$, which is a direct consequence of the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. Thus, the number of edges of G which do not intersect $\{1, 2, \ldots, s\}$ can be bounded from above by:

$$\begin{aligned}
|G| - |G \cap A_1| \leq |A| - |A_1| &= s\binom{n}{k-1} - \left[\binom{n}{k} - \binom{n-s}{k}\right] \\
&= s\left[\sum_{i=1}^{s}\binom{n-i}{k-2} + \binom{n-s}{k-1}\right] - \sum_{i=1}^{s}\binom{n-i}{k-1} \\
&= s\sum_{i=1}^{s}\binom{n-i}{k-2} - \sum_{i=1}^{s}\sum_{j=1}^{s-i}\binom{n-i-j}{k-2} \\
&= s\sum_{i=1}^{s}\binom{n-i}{k-2} - \sum_{i=2}^{s}(i-1)\binom{n-i}{k-2} \\
&= \sum_{i=1}^{s}(s-i+1)\binom{n-i}{k-2} \leq \sum_{i=1}^{s} i\binom{n-1}{k-2} \\
&= \frac{s(s+1)}{2}\binom{n-1}{k-2}.
\end{aligned}$$

□

4.5 Stability of $\boldsymbol{Cov}_k(n, s)$ and $\boldsymbol{Cl}_k(n, s)$

The aim of this section is to show that if a k-graph $G \in \mathcal{M}_k(n, s)$ is, in a way, similar to the hypergraphs from $Cov_k(n, s)$ [or $Cl_k(n, s)$], then in fact it belongs to this family. From such a stability result it follows that to prove Erdős Conjecture it is enough to show that the k-graphs from $\mathcal{M}_k(n, s)$ look similar to the conjectured ones. This fact allows us to restrict our studies on Erdős' problem to the asymptotic properties of matchings in hypergraphs.

In order to make it precise let us introduce families of graphs $Cov_k(n, s; \varepsilon)$ and $Cl_k(n, s; \varepsilon)$. Let us recall that if $G = (V, E)$ belongs to $Cov_k(n, s)$, then there exists a set $S \subseteq V$, $|S| = s$, which covers all edges of G. We say that $G \in Cov_k(n, s; \varepsilon)$ for some $\varepsilon > 0$, if there exists a set $S \subseteq V$, $|S| = s$, which covers all but at most $\varepsilon|E|$ edges of G. Moreover, we define $Cl_k(n, s; \varepsilon)$ as the set of all k-graphs G which contain a complete subgraph on at least $(1 - \varepsilon)ks$ vertices. Then the main result of this section, by Łuczak and Mieczkowska [19], can be stated as follows.

Theorem 4.11. *For every $k \geq 2$ there exists $\varepsilon > 0$ such that for every n and s such that $1 \leq s \leq n/k$, and $G \in \mathcal{M}_k(n, s)$, the following holds:*

(i) if $G \in Cov_k(n, s; \varepsilon)$, then $G \in Cov_k(n, s)$;

(ii) if $G \in Cl_k(n, s; \varepsilon)$, then $G \in Cl_k(n, s)$.

Proof. Let us start with the observation that for every n_0 we can choose $\varepsilon > 0$ small enough such that for every $n \le n_0$ and $1 \le s \le n/k$, we have $Cov_k(n, s; \varepsilon) = Cov_k(n, s)$ and $Cl_k(n, s; \varepsilon) = Cl_k(n, s)$. Consequently, we may and shall verify the assertion for $n \ge n_0$ for some sufficiently large n_0.

For the proof of (i), let us assume that $G = (V, E) \in \mathcal{M}_k(n, s)$ belongs to $Cov_k(n, s; \varepsilon)$ and let S be the set which covers all but at most $\varepsilon|E|$ edges of G. Let $T \subseteq S$ be the set of vertices which are not contained in $\binom{n-1}{k-1}$ edges of G and let $t = |T|$. We need to show that $t = 0$.

Observe first that, because of (3.4), we may and shall assume that for n large enough $s \le n(1/k - 2/(5k^2))$, since otherwise there exists a k-graph $G' \in Cl_k(n, s)$ with more edges than G, contradicting the fact that $G \in \mathcal{M}_k(n, s)$. Thus, by Lemma 4.1, each vertex $v \in T$ is contained in at most

$$\binom{n-1}{k-1} - \binom{n-ks-1}{k-1} \le \left(1 - \left(\frac{1}{5k}\right)^{k-1}\right)\binom{n-1}{k-1}$$

edges. Now let $\bar{G}$ denote the k-graph obtained from G by deleting all vertices from $S \setminus T$ and all edges intersecting them. It is easy to see that $\bar{G} \in \mathcal{M}_k(n-s+t, t)$. Now, for any k-graph $\hat{G} \in Cov_k(n-s+t, t)$, we have

$$|E(\hat{G}) \setminus E(\bar{G})| \ge \frac{t}{k}\left(\frac{1}{5k}\right)^{k-1}\binom{n-1}{k-1}.$$

Furthermore, from the assumption $G \in Cov_k(n, s; \varepsilon)$ for $\varepsilon \le \frac{1}{2}$ we get

$$|E(\bar{G}) \setminus E(\hat{G})| \le \frac{\varepsilon}{1-\varepsilon} e(\hat{G}) \le 2\varepsilon s\binom{n-1}{k-1}.$$

Hence, if $t \ge n/(10k^5) \ge s/(10k^4)$ we have

$$\begin{aligned} e(\hat{G}) - e(\bar{G}) &\ge \frac{t}{k}\left(\frac{1}{5k}\right)^{k-1}\binom{n-1}{k-1} - 2\varepsilon s\binom{n-1}{k-1} \\ &\ge \left(\left(\frac{1}{5k}\right)^{k-1} - 20k^5\varepsilon\right)\frac{t}{k}\binom{n-1}{k-1}. \end{aligned}$$

Thus, if $\varepsilon > 0$ is small enough, then $\hat{G}$ has more edges than $\bar{G}$ contradicting the fact that $\bar{G} \in \mathcal{M}_k(n-s+t, t)$. Hence, $t \le n/(10k^5) \le (n-s+t)/2k^3$. But in such a case, Erdős Conjecture holds by the result of Bollobás, Daykin, and Erdős [3], thus

$$\bar{G} \in \mathcal{M}_k(n-s+t, t) = Cov_k(n-s+t, t)$$

and, since by the definition no vertex of T has a full degree, $t = 0$. Consequently, $G \in Cov_k(n, s)$ and (i) follows.

Now assume that $G = (V, E) \in \mathcal{M}_k(n, s)$ belongs to $\mathit{Cl}_k(n, s; \varepsilon)$. Let U be the set of vertices of the largest complete k-subgraph of G such that $|U| \geq (1-\varepsilon)ks$. Furthermore, let M be a matching in G of size s which maximizes $|\bigcup M \cup U|$, and let $M' = \{e \in M \,:\, e \not\subseteq U\}$. Then, for n large enough, the following holds.

Claim 4.12.

(i) $|\bigcup M \cup U| = ks + k - 1$.

(ii) $|M'| \leq 2\varepsilon ks$.

(iii) each edge of G *either is contained in* U, *or intersects an edge of* M'.

Proof. Observe that at most $k-1$ vertices of U are unsaturated by M, thus $|\bigcup M \cup U| \leq ks + k - 1$. On the other hand, since U induces the largest clique in G, there exists a k-element subset $e \notin E$ such that $|e \cap U| = k - 1$. Then, since $G \in \mathcal{M}_k(n, s)$, the graph $G \cup \{e\}$ contains a matching $M^* \cup \{e\}$ of size $s+1$. Thus, M^* is a matching of size s, in which precisely $k-1$ vertices from U are unsaturated, so $|\bigcup M \cup U| \geq |\bigcup M^* \cup U| \geq ks + k - 1$, and (i) follows. To prove (ii), observe that $|M'| \leq |V(M') \setminus U| = |\bigcup M \cup U| - |U|$ and use (i), obtaining $|M'| \leq \varepsilon ks + k - 1 \leq 2\varepsilon ks$ for n large enough. Finally, (iii) is a direct consequence of the choice of M. $\square$

Let $G' = (V, E')$ denote the k-graph which consists of the clique with vertex set $\bigcup M \cup U$ and isolated vertices. Clearly, the size of the largest matching in G' is s. We shall show that G' has more edges than G provided $|M'| > 0$. Thus, we must have $M' = \emptyset$ and the assertion (ii) of Theorem 4.11 follows.

In order to show that $e(G') > e(G)$ we need to introduce some more definitions. We say that a subset $f \subseteq V$ of ℓ vertices is *thick* if it is contained in more than $3\varepsilon k^3 \binom{|U|}{k-\ell}$ edges $e \in G$ such that $e \subseteq U \cup f$; otherwise we call it *thin*.

Claim 4.13. *If a subset* f *of* ℓ *elements is thick, then each* k*-element subset of* $U \cup f$ *containing* f *is an edge of* G.

Proof. Let us suppose that for a thick f there exists a k-element set e such that $f \subseteq e \subseteq U \cup f$ and $e \notin E$. Then, since $G \in \mathcal{M}_k(n, s)$, graph $G \cup e$ contains a matching M'' of size $s+1$, where $e \in M''$. Furthermore, at most $2\varepsilon k^3 s\binom{|U|}{k-l-1} \leq 3\varepsilon k^3 \binom{|U|}{k-l}$ of the $(k-\ell)$-element subsets of U intersect sets from M'' not contained in $U \cup f$. Since f is thick, there exists a $(k-\ell)$-subset h of U which intersects only the edges of M'' contained in U and such that $f \cup h \in E$. But then one can modify $(M'' \setminus \{e\}) \cup \{f \cup h\}$, replacing each edge g of M'' which intersects h by $(g \setminus \{h\}) \cup f_g$, where f_g is a subset of $e \setminus f$ of size $|g \cap h|$, and

f_g, $f_{g'}$ are disjoint for $g \neq g'$. The matching obtained in such a way is of size $s+1$, contradicting the fact that $G \in \mathcal{M}_k(n,s)$. Hence, all edges e for which $f \subseteq e \subseteq U \cup f$ must already belong to G. $\square$

Now let $W = \bigcup M \setminus U$, and $a = |W|$. Our aim is to show that $a = 0$. To this end we suppose that $a > 0$ and show that this assumption will lead to contradiction.

Observe first that no singleton from W is thick. Indeed, if $w \in W$ is thick, then due to Claim 4.13 all k-element subsets of $U \cup \{w\}$ belong to G, so $U \cup \{w\}$ is a clique larger than U, what contradicts the maximality of U. Consequently, all $\{w\}$ such that $w \in W$ are thin. Using this fact one can estimate from below the number of edges of G' which do not belong to G in the following way:

$$|E' \setminus E| \geq (1 - 3\varepsilon k^3) a \binom{|U|}{k-1}. \tag{4.2}$$

Now we estimate the number of edges in $|E \setminus E'|$. First, we bound the number $|E_1|$ of edges which have at least two vertices in $\bigcup M$ by

$$|E_1| \leq \binom{ka}{2}\binom{n-2}{k-2} \leq \varepsilon k^4 a \binom{n}{k-1}. \tag{4.3}$$

Now we look at the edges e such that $e \cap \bigcup M = \{w\}$ and the set $(e \setminus U) \cup \{w\}$ is thin. Then, for the number $|E_2|$ of such edges, we get

$$\begin{aligned} |E_2| &\leq \binom{ka}{1} \sum_{l=1}^{k-1} \binom{n - |U| - a}{l} 3\varepsilon k^3 \binom{|U|}{k-l-1} \\ &\leq 3\varepsilon k^4 a \left(\binom{n}{k-1} - \binom{|U|}{k-1} \right). \end{aligned} \tag{4.4}$$

Finally let us consider the set E_3 of all edges e such that $e \cap \bigcup M = \{w\}$ and the set $\bar{e} = (e \setminus U) \cup \{w\}$ is thick. Note that then, by Claim 4.13, all possible extensions of $\bar{e}$ to the k-element sets contained in $U \cup \bar{e}$ are edges of G. In particular, all edges e for which $|\bar{e}| = |\{w\}| = 1$ are contained in U, and thus do not belong to $E \setminus E'$. Moreover, it is easy to see that if the set E_3 contains a matching consisting of more than a edges, then it can be used to construct a matching of size larger than s in G and thus, it is impossible. Now let us consider two cases, depending on the size of the set $U \setminus \bigcup M'$.

Case 1. $n - |U \setminus \bigcup M'| \leq 2k^3 a$.

Then $|E_3|$ can be crudely bounded from above by

$$\begin{aligned}|E_3| &\le \binom{ka}{1}\sum_{\ell=1}^{k-1}\binom{n-|U|-a}{\ell}\binom{|U\setminus\bigcup M'|}{k-\ell-1}\\ &\le k^2a(n-|U\setminus\bigcup M'|)\binom{|U\setminus\bigcup M'|}{k-2}\\ &\le 2k^5a^2\binom{|U\setminus\bigcup M'|}{k-2}\\ &\le 8\varepsilon k^7a\binom{|U|}{k-1}.\end{aligned}\tag{4.5}$$

From (4.2), (4.3), (4.4), and (4.5), we get

$$e(G')-e(G)\ge a\left(\binom{|U|}{k-1}-8\varepsilon k^7\binom{|U|}{k-1}-6\varepsilon k^4\binom{n}{k-1}\right).\tag{4.6}$$

Case 2. $n-|U\setminus\bigcup M'|\ge 2k^3a$.

In this case we apply the result on Erdős Conjecture of Bollobás, Daykin and Erdős [3] for the subset of edges e for which $|\bar{e}|=\ell$, for each $\ell=2,\dots,k$. Each of these sets of edges is an l-uniform hypergraph on $n-|U\setminus\bigcup M'|$ vertices which contains no matchings of size larger than a. Thus, for ε small enough, we have

$$\begin{aligned}|E_3| &\le \sum_{\ell=2}^{k} a\binom{n-|U\setminus\bigcup M'|}{\ell-1}\binom{|U\setminus\bigcup M'|}{k-\ell}\\ &= a\left(\binom{n}{k-1}-\binom{|U\setminus\bigcup M'|}{k-1}\right)\\ &\le a\left(\binom{n}{k-1}-0.99\binom{|U|}{k-1}\right).\end{aligned}\tag{4.7}$$

From (4.2), (4.3), (4.4), and (4.7), we get

$$\begin{aligned}e(G')-e(G) &\ge (1-3\varepsilon k^3)a\binom{|U|}{k-1}-\varepsilon k^4a\binom{n}{k-1}\\ &-3\varepsilon k^4a\left(\binom{n}{k-1}-\binom{|U|}{k-1}\right)\\ &-a\left(\binom{n}{k-1}-0.99\binom{|U|}{k-1}\right)\\ &\ge a\left(1.99\binom{|U|}{k-1}-\binom{n}{k-1}-4\varepsilon k^4\binom{n}{k-1}\right).\end{aligned}\tag{4.8}$$

Now note that due to (3.4) we may assume that $|U|/n\ge 1-1/(2k)$ and so $\binom{|U|}{k-1}\ge 0.6\binom{n}{k-1}$. Hence, for $\varepsilon>0$ small enough and $a>0$, from (4.6) and (4.8) we infer that $e(G')>e(G)$. Thus we must have $a=0$ and, consequently, the assertion follows. □

Chapter 5

Results on Erdős Conjecture

In this chapter we present author's main results, in which we verify Erdős Conjecture in some special cases.

In Section 5.1, we present the result from [14], by Frankl, Łuczak and Mieczkowska, for a general k-uniform case, proving that $\mathcal{M}_k(n,s) = Cov_k(n,s)$ for $n \geq \frac{2k^2s}{\log k}$.

In Section 5.2, we show that Erdős Conjecture is true for 3-uniform hypergraphs whenever $n > n_0$, where n_0 is a constant independent of s.

5.1 k-uniform hypergraphs and $n \geq \frac{2k^2s}{\log k}$

The main result of this section slightly improves the best bound known at the time of its publishing, and confirms Erdős Conjecture for $n \geq \frac{2k^2s}{\log k}$. The following theorem is due to Frankl, Łuczak and the author [14]].

Theorem 5.1. *If* $k \geq 3$ *and*

$$n > \frac{2k^2s}{\log k}, \tag{5.1}$$

then $\mathcal{M}_k(n,s) = Cov_k(n,s)$.

The main idea of the proof of Theorem 5.1 is based on the numerical estimate obtained in Theorem 4.10 and Lemma 4.1 which says that that degrees of vertices in a maximal graph are either full or not too big.

Proof of Theorem 5.1. Let us assume that (5.1) holds for $G \in \mathcal{M}_k(n,s)$. Then, by part (i) of Theorem 4.6, the hypergraph $H = \mathbf{Sh}(G)$ belongs to $\mathcal{M}_k(n,s)$. We shall show that $H \in Cov_k(n,s)$ which, due to part (ii) of Theorem 4.6, would imply that $G \in Cov_k(n,s)$. Here and below by $\deg(i)$ we mean the degree of a vertex i, and by V and E we denote the sets of vertices and edges of H, respectively. Our argument is based on the following observation.

Claim 5.2. *If* $s \geq 2$, *then* $\deg(1) = \binom{n-1}{k-1}$.

Proof. Let us assume that the assertion does not hold. We shall show that then H has fewer edges than the graph $H' = (V, E')$ whose edge set consists of all k-subsets intersecting $\{1, 2, \ldots, s\}$. Since H is maximal, from Lemma 4.1 we know that $\deg(1) \leq \binom{n-1}{k-1} - \binom{n-ks-1}{k-1}$. Moreover, since $H = \mathbf{Sh}(H)$, we have $\deg(i) \leq \binom{n-1}{k-1} - \binom{n-ks-1}{k-1}$ for every $i \in [n]$. Thus,

$$\begin{aligned} |E' \setminus E| &\geq s\binom{n-ks-1}{k-1} \\ &\geq \frac{s(n-1)_{k-1}}{(k-1)!}\Big(1 - \frac{ks}{n-k+1}\Big)^{k-1}, \end{aligned} \tag{5.2}$$

while from Theorem 4.10 we get

$$\begin{aligned} |E \setminus E'| &\leq \frac{s(s+1)}{2}\binom{n-1}{k-2} = \frac{s(n-1)_{k-1}}{(k-1)!}\frac{(s+1)(k-1)}{2(n-k+1)} \\ &\leq \frac{s(n-1)_{k-1}}{(k-1)!}\frac{ks}{n-k+1}. \end{aligned} \tag{5.3}$$

Hence

$$e(H') - e(H) \geq \frac{s(n-1)_{k-1}}{(k-1)!}\bigg(\Big(1 - \frac{ks}{n-k+1}\Big)^{k-1} - \frac{ks}{n-k+1}\bigg).$$

Let $x = ks/(n-k+1)$. It is easy to check that for all $k \geq 3$ and $x \in (0, 0.7\log k/k)$ we have

$$(1-x)^{k-1} > x\,.$$

Thus, $e(H') - e(H) > 0$, provided $k^2 s < 0.7\log k(n-k+1)$, which holds whenever $n \geq 2sk^2/\log k$. Since clearly $\nu(H') = s$, we arrive at contradiction with the assumption that $H \in \mathcal{M}_k(n,s)$. □

Since $n \geq ks$, the hypergraph H^-, obtained from H by deleting vertex 1 together with all edges it is contained in, belongs to $\mathcal{M}_k(n-1, s-1)$. Now Theorem 5.3 follows easily from the observation that, since $\frac{s-1}{n-1} \leq \frac{s}{n}$, if (5.1) holds then it holds also when n is replaced by $n-1$ and s is replaced by $s-1$. Hence, we can reduce the problem to the case when $s = 1$ and use Erdős-Ko-Rado Theorem (note that then $n > 2k^2/\log k > 2k+1$). $\square$

5.2 3-uniform hypergraphs and $n > n_0$

This section is mostly an adjusted and somewhat simplified copy of the material as it appears in article [19], by Łuczak and Mieczkowska. Below we prove the following theorem.

Theorem 5.3. *There exists n_0 such that for $n \geq n_0$ large enough and each s, $1 \leq s \leq (n-2)/3$, we have*

$$\mu_3(n,s) = \max\left\{\binom{n}{3} - \binom{n-s}{3}, \binom{3s+2}{3}\right\}. \tag{5.4}$$

Furthermore, for such parameters n and s, we have

$$\mathcal{M}_3(n,s) \subseteq Cov_3(n,s) \cup Cl_3(n,s)\,.$$

The crucial part of the proof of Theorem 5.3 is the following lemma.

Lemma 5.4. *Let $\varepsilon > 0$. There exists n_0 such that for every $n \geq n_0$, $1 \leq s \leq n/3$, and $G \in \mathcal{M}_3(n,s)$ we have*

$$\boldsymbol{Sh}(G) \in Cov_3(n,s;\varepsilon) \cup Cl_3(n,s;\varepsilon)\,.$$

We shall show Lemma 5.4 by a detailed analysis of the structure of $\mathbf{Sh}(G)$ but before we do it we argue that it implies Theorem 5.3.

Proof of Theorem 5.3. Let $G \in \mathcal{M}_3(n,s)$. Then, by Theorem 4.6(i), we know that $\mathbf{Sh}(G) \in \mathcal{M}_3(n,s)$. Thus, using Theorem 4.11 and Lemma 5.4, for n large enough we get

$$\mathbf{Sh}(G) \in Cov_3(n,s) \cup Cl_3(n,s)\,,$$

and so, by Theorem 4.6(ii),(iii)

$$G \in Cov_3(n,s) \cup Cl_3(n,s)\,. \quad \square$$

Let us remark that in order to show Theorem 5.3, it is enough to show Lemma 5.4 for some given absolute constant $\varepsilon > 0$.

Proof of Lemma 5.4. Let $\varepsilon > 0$ and $G \in \mathcal{M}_3(n,s)$. By Theorem 4.6(i), we get $\mathbf{Sh}(G) \in \mathcal{M}_3(n,s)$. To simplify the notation, by writing (i,j,k) we always mean that an edge $\{i,j,k\}$ is such that $i < j < k$. Whenever a picture of the edge (i,j,k) appears, the lower vertex is meant to be i, j is in the middle, and k is the upper one. Hence, if we draw an edge, each edge which lies “below it" appears in the graph as well. Let $M = \{(i_l, j_l, k_l) : l = 1, \dots, s\}$ be the largest matching in

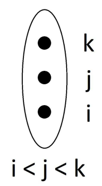

Fig. 1.

$\mathbf{Sh}(G)$, and let us partition its vertex set into three parts $V(M) = I \cup J \cup K$ such that for every edge $(i,j,k) \in M$ we have $i \in I$, $j \in J$, and $k \in K$. Moreover, let vertices of K be labeled in such a way that $k_l < k_m$ for every $l < m$, and denote $L = \{i_l, j_l, k_l : l \le (1-\varepsilon)s\}$. We shall show that for n large enough either I covers all but at most $\epsilon|E|$ edges of $\mathbf{Sh}(G)$ or $\{e \in \mathbf{Sh}(G) : e \subset L\}$ is a clique.

In order to study the structure of $\mathbf{Sh}(G)$ we introduce an auxiliary (non-uniform!) hypergraph H. Denote by V' the set of vertices which are not saturated by M. Obviously, none of the edges of $\mathbf{Sh}(G)$ is contained in V'. In this proof, we use $\deg_{V'}(v)$ to denote the number of unordered pairs $u, w \in V'$ such that $\{v,u,w\}$ is an edge in $\mathbf{Sh}(G)$. Similarly, the number of vertices $w \in V'$ such that $\{v,u,w\} \in \mathbf{Sh}(G)$ is denoted by $\deg_{V'}(v,u)$. Finally, we use $e(v)$ to denote the unique edge of M containing vertex v. Let $H = (W,F)$ be a hypergraph with vertices $W = V(M)$ and the edge set $F = M \cup F_1 \cup F_2 \cup F_3$, where

$$F_1 = \{v \in W : \deg_{V'}(v) \ge n - 3s\},$$
$$F_2 = \{\{v,w\} \in W^{(2)} : e(v) \neq e(w) \text{ and } \deg_{V'}(v,w) \ge 3\},$$
$$F_3 = \{\{v,w,u\} \in W^{(3)} : e(v), e(w) \text{ and } e(u) \text{ are pairwise different}\}.$$

Let us remark that due to Lemma 4.2, for all $v \in F_1$ we have $\deg_{V'}(v) = \binom{n-3s}{2}$ and for each $\{v,w\} \in F_2$ we have, in fact, $\deg_{V'}(v,w) = n - 3s$. Note also that since $\mathbf{Sh}(G)$ is shifted, hypergraphs F_1, F_2, F_3 are shifted as well. We shall call an edge e of $\mathbf{Sh}(G)$ *traceable* if $e \cap V(M) \in F$, and *untraceable* otherwise. Observe also that the number of untraceable edges of $\mathbf{Sh}(G)$ is bounded from above by $3s \cdot (n-3s) + \binom{s}{2} \cdot 9 \cdot 2 + s \cdot 3 \cdot n \le 2n^2$, so we can afford to ignore them.

We call a triple T of edges from M *bad*, if in $\bigcup T$ there are three disjoint edges of H whose union intersects I on at most 2 vertices, and *good* otherwise. We show first that there are only few bad triples in M, since otherwise one could increase the matching number.

Claim 5.5. *No three disjoint triples are bad. Consequently, there exists a set B which consists of at most six edges from M such that each bad triple contains an edge from B.*

Proof. Let us suppose for a contradiction that there exist nine disjoint edges $\{(i_l, j_l, k_l) : l = 1, \dots, 9\} \subset M$ such that in $\{i_l, j_l, k_l : l = 1, \dots, 9\}$ one can find a set of nine disjoint edges $H' \subset H$, which do not cover vertices i_3, i_6 and i_9. One can easily see that for any ordering of the sets $\{j_3, j_6, j_9\}$ and $\{k_3, k_6, k_9\}$ there exists a permutation $\sigma(3)$, $\sigma(6)$, $\sigma(9)$ such that $j_{\sigma(9)} > j_{\sigma(6)}$ and $k_{\sigma(9)} > k_{\sigma(3)}$; to simplify the notation let us assume that $j_9 > j_6 > i_6$ and $k_9 > k_3 > i_3$. Replace in H' an edge e which contains j_9 by $e' = (e \setminus \{j_9\}) \cup \{i_6\}$ and the edge f containing k_9 by $f' = (e \setminus \{k_9\}) \cup \{i_3\}$; note that both e' and f' belong to H since $H = \mathbf{Sh}(H)$. Thus, we obtain the family of nine disjoint edges of $H'' \subseteq H$, all of which are contained in eight edges of M. Furthermore, since edges from $F_1 \cup F_2$ have large degrees in V', all edges from H'' which belong to $F_1 \cup F_2$ can be simultaneously extended to disjoint edges of $\mathbf{Sh}(G)$ by adding to them vertices from V'. But this would lead to a matching M' of size $s+1$ in $\mathbf{Sh}(G)$, contradicting the assumption $\mathbf{Sh}(G) \in \mathcal{M}_3(n, s)$. □

Now we study properties of good triples. We start with the following simple observation.

Claim 5.6. *Let T be a good triple.*

(i) $(F_1 \cap \bigcup T) \subset I$.

(ii) For any two edges of T there are at most 5 *edges in F_2 contained in their vertex set. Moreover, the only possible configuration with exactly* 5 *edges from F_2 is when all these edges intersect I (see Fig. 2).*

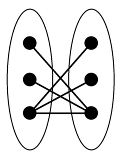

Fig. 2.

Proof. Let $T = \{(i_1, j_1, k_1), (i_2, j_2, k_2), (i_3, j_3, k_3)\}$ be a good triple.

(i) Let $j_1 < j_2 < j_3$ and assume that $(F_1 \cap \bigcup T) \not\subset I$. Then, since hypergraph F_1 is shifted, $\{j_1\} \in F_1$ and T is a bad triple because of the edges $\{j_1\}$, (i_2, j_2, k_2), (i_3, j_3, k_3), a contradiction.

(ii) Let us assume by contradiction that there are 6 edges from F_2 which are contained in $\{i_1, j_1, k_1, i_2, j_2, k_2\}$. Then $\{j_1, j_2\} \in F_2$ and at least one of the edges $\{i_1, k_2\}, \{i_2, k_1\}$ is in F_2. Let us assume that $\{i_1, k_2\} \in F_2$. Then, T is bad because of the edges $\{j_1, j_2\}, \{i_1, k_2\}, (i_3, j_3, k_3)$. □

For a triple $T \in M^{(3)}$ and for $i = 1, 2, 3$, let $f_i(T)$ be the number of edges of F_i contained in $\bigcup T$. Clearly, $f_1(T) \leq 9$, $f_2(T) \leq 27$ and $f_3(T) \leq 27$ for any triple T. However, if T is good, then, by Claim 5.6, we immediately infer that $f_1(T) \leq 3$ and $f_2(T) \leq 15$. Our next result shows how to estimate $f_1(T)$ and $f_2(T)$ more precisely for good triples for which $f_3(T)$ is large. We will later use it to bound the number of edges in $\mathbf{Sh}(G)$, using the facts that there are only few bad triples $T \in M^{(3)}$, and that $f_i(T)$ cannot be all too large for good triples.

Claim 5.7. *Let T be a good triple.*

(i) If $f_3(T) \geq 24$, then $f_1(T) = f_2(T) = 0$.

(ii) If $f_3(T) = 20$, then $f_1(T) \leq 1$ and $f_2(T) \leq 12$.

(iii) If $f_3(T) \leq 19$, then $f_1(T) \leq 3$ and $f_2(T) \leq 15$. Moreover, the only triples for which $f_3(T) = 19$, $f_2(T) = 15$, and $f_1(T) = 3$, are those in which each edge of H contained in $\bigcup T$ intersects I.

(iv) If $f_3(T) = 21$, then $f_1(T) \leq 1$ and $f_2(T) \leq 10$.

(v) If $22 \leq f_3(T) \leq 23$, then $f_1(T) = 0$ and $f_2(T) \leq 7$.

Proof. Let $T = \{(i_1, j_1, k_1), (i_2, j_2, k_2), (i_3, j_3, k_3)\}$ be a good triple.

(i) Observe that since $f_3(T) \geq 24$, one of the pairs of edges presented in Fig. 3 must be in H. Let $e, f \in F_3$ be disjoint edges such that $e, f \subset \{j_1, j_2, j_3, k_1, k_2, k_3\}$,

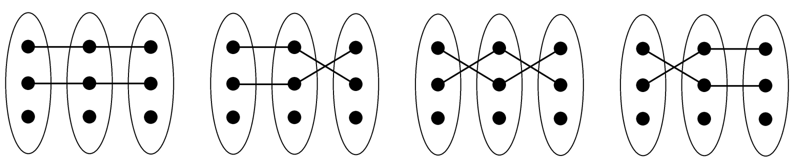

Fig. 3.

and let us assume that $i_1 < i_2 < i_3$. If $f_1(T) \neq 0$, then $i_1 \in F_1$ and so T is bad because of $\{i_1, e, f\}$. Similarly, if $f_2(T) \neq 0$, then $\{i_1, i_2\} \in F_2$ and again T is bad, while we assumed that T is good.

(ii) Observe that if $\{j_1, j_2, j_3\} \notin F_3$, then every edge in F_3 intersects I and we have $f_3(T) \leq 19$. Thus, if $f_3(T) \geq 20$, then $\{j_1, j_2, j_3\} \in F_3$, and because T is good, we must have $f_1(T) \leq 1$. Now assume by contradiction that $f_2(T) \geq 13$. Then, there are two edges in T, let say $(i_1, j_1, k_1), (i_2, j_2, k_2)$, such that at least five edges of F_2 are contained in their set of vertices. By Claim 5.6, we know that $\{i_1, k_2\}, \{k_1, i_2\} \in F_2$ and thus, T is bad because of the edges $\{i_1, k_2\}, \{k_1, i_2\}, \{j_1, j_2, j_3\}$.

(iii) It is a direct consequence of Claim 5.6. Furthermore, if $f_2(T) = 15$, then we must have $\{j_1, j_2, j_3\} \notin F_3$ (otherwise, as we argued in (ii) above, $f_2(T) \leq 12$).

(iv) Since $f_3(T) = 21$, we know that $\{j_1, j_2, j_3\} \in F_3$ and $\{k_1, k_2, k_3\} \notin F_3$. Therefore, at least one pair of edges from Fig. 4. and Fig. 5. is in F_3.

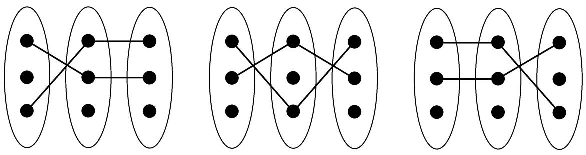

Fig. 4.

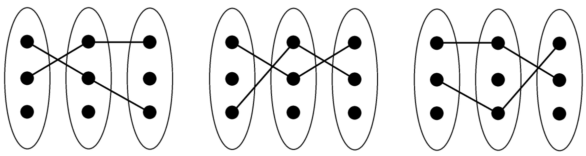

Fig. 5.

Since each such a pair saturates only one vertex from I, we have $f_1(T) \leq 1$. To estimate $f_2(T)$ let us assume that j_2 is not saturated by such a pair of edges. Then, $\{i_1, j_2\}, \{j_2, i_3\} \notin F_2$, because T is good. Consequently, none of the sets $\{i_1, k_2\}, \{j_1, j_2\}$, $\{j_2, j_3\}, \{k_2, i_3\}$ is in F_3, and thus at most six edges of F_2 are contained in $\{i_1, j_1, k_1, i_2, j_2, k_2\}$ or in $\{i_2, j_2, k_2, i_3, j_3, k_3\}$. Now, since $\{j_1, j_2, j_3\} \in F_3$, using the same argument as in (ii), we conclude that at most four edges of F_2 are contained in $\{i_1, j_1, k_1, i_3, j_3, k_3\}$. Hence, $f_2(T) \leq 10$.

(v) From (i) we know that if in T we can find one of the pairs of edges marked on Fig. 3, then $f_1(T) = f_2(T) = 0$. Thus, let us assume that for each of these pairs at least one edge is not in F_3 and $22 \leq f_3(T) \leq 23$. Hence $\{j_1, j_2, j_3\} \in F_3$ and $\{k_1, k_2, k_3\} \notin F_3$. Now consider $\{j_1, j_2, k_3\}$, $\{j_1, k_2, j_3\}$, $\{k_1, j_2, j_3\}$. It is easy to check that if at most one of them is in F_3, then $f_3(T) \leq 21$. Thus, we split our further argument into two cases.

Case 1. All three edges $\{j_1, j_2, k_3\}$, $\{j_1, k_2, j_3\}$, $\{k_1, j_2, j_3\}$ are in F_3.

Then, $\{j_1, k_2, k_3\}$, $\{k_1, j_2, k_3\}$, $\{k_1, k_2, j_3\}$, $\{k_1, k_2, k_3\} \notin F_3$. Therefore, as $f_3(T) \geq 22$, at least two pairs of edges shown on Fig. 4. are in F_3. Let say these are $\{i_1, k_2, k_3\}, \{k_1, j_2, j_3\}$ and $\{k_1, k_2, i_3\}, \{j_1, j_2, k_3\}$. Since T is good, edges $\{j_1, i_2\}$, $\{j_1, i_3\}$, $\{i_2, j_3\}$, $\{i_1, j_3\}$ are not in F_2, and because F_2 is shifted, the edges of F_2 contained in $\bigcup T$ are contained in the set $\{\{i_1, i_2\}, \{i_1, j_2\}, \{i_1, k_2\}, \{i_2, i_3\}, \{j_2, i_3\}, \{k_2, i_3\}, \{i_1, i_3\}\}$. Hence, $f_2(T) \leq 7$. It is also easy to observe that in that case $f_1(T) = 0$.

Case 2. Exactly two of the edges $\{j_1, j_2, k_3\}$, $\{j_1, k_2, j_3\}$, $\{k_1, j_2, j_3\}$ are in F_3.

Without loss of generality let $\{j_1, j_2, k_3\}$, $\{j_1, k_2, j_3\} \in F_3$. Then, $\{k_1, j_2, j_3\}$, $\{k_1, j_2, k_3\}$, $\{k_1, k_2, j_3\}$, $\{k_1, k_2, k_3\} \notin F_3$. Therefore, if $f_3(T) = 23$, then all other edges are in F_3, and so two pairs of edges shown on Fig. 4. are in F_3. Thus, as we have shown in the proof of Case 1, $f_2(T) \leq 7$. Let now consider the case when $f_3(T) = 22$. If both pairs of edges $\{j_1, k_2, j_3\}, \{k_1, i_2, k_3\}$ and $\{k_1, k_2, i_3\}$, $\{j_1, j_2, k_3\}$ are in F_3, then again $f_2(T) \leq 7$. Let now assume that only one of these pairs is in F_3, let say $\{j_1, k_2, j_3\}, \{k_1, i_2, k_3\} \in F_3$. Then also a pair $\{j_1, k_2, k_3\}, \{k_1, j_2, i_3\}$ is in F_3. Thus, $\{i_1, j_2\}$, $\{j_2, i_3\}$, $\{i_1, j_3\}$, $\{i_2, j_3\} \notin F_2$, and therefore, $f_2(T) \leq 7$. In that case we also have $f_1(T) = 0$. $\square$

Now we bound the number of edges in $\mathbf{Sh}(G)$. First of all let us remove from M six edges so that in the remaining matching $\bar{M}$ we have only good triples (see Claim 5.5). In this way we omit at most $9n^2$ edges of $\mathbf{Sh}(G)$. Let us recall also that the number of untraceable edges of $\mathbf{Sh}(G)$ is at most $2n^2$. Finally, since for each edge $f \in F_i$ there are $\binom{n-3s}{3-i}$ edges $e \in \mathbf{Sh}(G)$ such that $e \cap V(M) = f$, the number of edges in $\mathbf{Sh}(G)$ is given by

$$e(\mathbf{Sh}(G)) = |F_1|\binom{n-3s}{2} + |F_2|(n-3s) + |F_3| + O(n^2).$$

To bound $|F_i|$, let us sum $f_i(T)$ over all $T \in \bar{M}^{(3)}$. Observe that in such a sum each edge from F_i is counted exactly $\binom{s-i}{3-i}$ times. Thus,

$$e(\mathbf{Sh}(G)) = \sum_{T \in \bar{M}^{(3)}} \left(f_1(T)\frac{\binom{n-3s}{2}}{\binom{s-1}{2}} + f_2(T)\frac{n-3s}{s-2} + f_3(T) \right) + O(n^2).$$

Now we divide good triples into 27 groups, depending on $f_3(T)$. If

$$T_i = \{T \in \bar{M}^{(3)} : f_3(T) = i\}$$

for $i = 1, \ldots, 27$, then

$$e(\mathbf{Sh}(G)) = \sum_{i=1}^{27} \sum_{T \in T_i} \left(f_1(T)\frac{(n-3s)^2}{s^2} + f_2(T)\frac{n-3s}{s} + f_3(T) \right) + O(n^2).$$

Let us now denote $x_1 = \sum_{i=1}^{19} |T_i|$, $x_2 = |T_{20}|$, $x_3 = |T_{21}|$, $x_4 = |T_{22}| + |T_{23}|$, $x_5 = \sum_{i=24}^{27} |T_i|$. By Claim 5.7, we get the following bound.

$$\begin{aligned} e(\mathbf{Sh}(G)) \leq\ & (3x_1 + x_2 + x_3)\frac{(n-3s)^2}{s^2} \\ & + (15x_1 + 12x_2 + 10x_3 + 7x_4)\frac{n-3s}{s} \\ & + (19x_1 + 20x_2 + 21x_3 + 23x_4 + 27x_5) + O(n^2). \end{aligned}$$

Now it sufficies to maximize the above function under the conditions $\sum_{i=1}^{5} x_i \leq \binom{s-6}{3}$ and $x_i \geq 0$ for every $i = 1, \ldots, 5$. Then, we are to maximize a function

$$f_{s,n}(x_1, x_2, x_3, x_4, x_5) = \sum_{i=1}^{5} \alpha_i(s,n) x_i,$$

where

$$\begin{aligned} \alpha_1(s,n) &= 3(n-3s)^2/s^2 + 15(n-3s)/s + 19 \\ \alpha_2(s,n) &= (n-3s)^2/s^2 + 12(n-3s)/s + 20 \\ \alpha_3(s,n) &= (n-3s)^2/s^2 + 10(n-3s)/s + 21 \\ \alpha_4(s,n) &= 7(n-3s)/s + 23 \\ \alpha_5(s,n) &= 27\,, \end{aligned}$$

over the domain $\sum_{i=1}^{5} x_i \leq \binom{s-6}{3}$, $x_i \geq 0$ for $i = 1, \ldots, 5$. This is a particularly simple case of linear programming: it is easy to see that in order to maximize $f_{s,n}$ it is enough to check which of the coefficients $\alpha_i(s,n)$ is the largest one and set the variable x_i which corresponds to this coefficient to be maximum, while the rest of the variables should be equal to zero.

It is easy to verify that if $s = an$ and $a < a_0$, where $a_0 = (\sqrt{321} - 3)/52$, then $\alpha_1(s,n)$ dominates, and so for $s = an$, $a < a_0$, we have

$$\begin{aligned} e(\mathbf{Sh}(G)) &\leq \left(3\frac{(1-3a)^2}{a^2} - 15\frac{15(1-3a)}{a} + 19 \right) \frac{(an)^3}{6} + O(n^2) \\ &= (3a - 3a^2 + a^3)\frac{n^3}{6} + O(n^2), \end{aligned}$$

which nicely matches the lower bound for $e(\mathbf{Sh}(G))$ given by

$$e(Cov_3(n,s)) = \binom{n}{3} - \binom{n-s}{3} = (3a - 3a^2 + a^3)\frac{n^3}{6} + O(n^2)\,.$$

Furthermore, in order to achieve this bound for all but $O(n^2)$ triples T we must have $f_3(T) = 19$, $f_2(T) = 15$, $f_1(T) = 3$, which is possible only if all edges of such triple intersect I (see Claim 5.7(iii)). Consequently, for this range of s, in $\mathbf{Sh}(G)$ there is a subset I, $|I| = s$, which covers all but at most $O(n^2)$ edges of $\mathbf{Sh}(G)$.

For $a > a_0$ the dominating coefficient is $\alpha_5(s,n) = 27$, which gives

$$e(\mathbf{Sh}(G)) \leq 27\frac{(an)^3}{6} + O(n^2) = \frac{9}{2}a^3n^3 + O(n^2)\,,$$

matched by the lower bound

$$e(\mathit{Cl}_k(n,s)) = \binom{3s+2}{3} = \frac{9}{2}a^3n^3 + O(n^2)\,.$$

Again, to achieve this bound for all but $O(n^2)$ triples we must have $f_3(T) = 27$. Let us now recall that $k_1 < \ldots < k_s$ and let $\{k_{i-2}, k_{i-1}, k_i\} \in \mathbf{Sh}(G)$ be such that $\{k_{i+1}, k_{i+2}, k_{i+3}\} \notin \mathbf{Sh}(G)$. Then, because of shifting, $\{k_{i+1}, \ldots, k_s\}$ is an independent set and thus, there are at least $\binom{s-i}{3}$ triples T of edges from M such that $f_3(T) < 27$. Since $\binom{s-i}{3} = O(n^2)$ we have $i = s - O(n^{2/3})$. But then, $\{k_1, \ldots, k_i\}$ is a clique of size $i = s - O(n^{2/3}) = s - O(s^{2/3})$ in $\mathbf{Sh}(G)$.

In order to complete the proof we need to consider the remaining case when $s = (a_0 + o(1))n$. Since $\alpha_1(a_0n, n) = \alpha_5(a_0n, n) > \alpha_i(a_0n, n)$ for $i = 2, 3, 4$, we infer that in $\mathbf{Sh}(G)$ all triples, except for at most $O(n^2)$, must be of one of two types: either for such a triple T we have $f_3(T) = 27$, $f_2(T) = f_1(T) = 0$, or $f_3(T) = 19$, $f_2(T) = 15$, $f_3(T) = 3$. It is easy to see that it is possible only when one of these two types of triples dominates. Indeed, let $M' \subseteq M$ denote the set of edges of M which contain a singleton edge from F_1. Since all but $O(s^2)$ triples must be of either of the two types, for the number of triples which are contained neither in M', nor in $M \setminus M'$ is $O(s^2)$, we get the following estimate

$$\binom{|M'|}{2}|M \setminus M'| + \binom{|M \setminus M'|}{2}|M'| = O(s^2)\,.$$

Consequently, $\min\{|M'|, |M \setminus M'|\} = O(s^{1/2})$, so

$$\begin{aligned}\mathbf{Sh}(G) &\in \mathit{Cov}_3(n,s;O(n^{-1})) \cup \mathit{Cl}_3(n,s;O(n^{-1/3})) \\ &\subseteq \mathit{Cov}_3(n,s;\varepsilon) \cup \mathit{Cl}_3(n,s;\varepsilon)\,,\end{aligned}$$

and the assertion follows. □

Chapter 6

Variations on Erdős Conjecture

In this chapter we discuss several possible approaches for solving Erdős Conjecture and present new conjectures and results, obtained while working on this problem.

In Section 6.1 we give a new proof of Erdős Conjecture for graphs based on Tutte's Theorem. We also state a new conjecture that might be considered as a weak version of Tutte's Theorem for hypergraphs, and, if true, might be helpful in proving Erdős Conjecture, for large n.

In Section 6.2 we discuss the connections between Erdős' matching problem to an old probabilistic conjecture of Samuels. Inspired with this relation, we state a new conjecture, which, as it turns out, is asymptotically equivalent to the fractional version of Erdős' matching problem. In particular, we prove the following result, which is due to Łuczak, Šileikis and the author.

Theorem 6.1. *Let X_1, X_2, X_3 be independent, identically distributed and non-negative random variables with expected value $\mathbb{E}(X_1) = x$. Then,*

$$\mathbb{P}(X_1 + X_2 + X_3 \geq 1) \leq \max\{1 - (1 - x)^3, (3x)^3\}.$$

6.1 Erdős Conjecture and Tutte's Theorem

Let us recall that Erdős Conjecture for graphs was proven first by Erdős and Gallai [6] in 1959. Since then, some other proofs, much shorter and more elegant, have been presented. Here we give a new proof of this result, the main part of which is based on a structural lemma of Figaj and Łuczak from [9].

In this section we need a few more definitions concerning graphs. Let us recall that a *graph* is a pair $G = (V, E)$, where $V = V(G)$ is a finite set of *vertices* and $E = E(G) \subset \binom{V}{2}$ is a family of *edges* of G. A *subgraph* $G' \subset G$ is a graph $G' = (V', E')$ such that $V' \subset V$ and $E' \subset E$. We say that $G' \subset G$ is an *induced* subgraph if for every edge $e \in E$ contained in V', we have $e \in E'$. By $G[V']$ we denote an induced subgraph $G' \subset G$ with vertex set $V' \subset V$. We say that two vertices $v, w \in V$ are *connected* if there exists a *path* between them, i.e. a sequence of vertices

$$(v_0, v_1, \ldots, v_{k-1}, v_k),$$

where $v = v_0$, $v_k = w$, $v_i \in V$, $i = 0, \ldots, k$, are such that $\{v_i, v_{i+1}\} \in E$ for every $i = 0, \ldots, k-1$. A graph G is *connected* if every two of its vertices are connected. A *component* C of a graph G is a maximal connected subgraph, i.e. it is not a proper subgraph of any other connected subgraph. The number of odd components of G, i.e. of components having an odd number of vertices, is denoted by $q(G)$. A vertex $w \in V$ is a *neighbor* of a vertex $v \in V$ if $\{v, w\} \in E$. The number of neighbors of vertex $v \in V$ is called the *degree* of vertex v, and the largest degree of G is denoted by $\Delta(G)$.

Let us first recall Tutte's Theorem which we shall use in the following form.

Theorem 6.2. *Let $G = (V, E)$ be a graph with n vertices such that $\nu(G) \leq s$. Then, there exists a set $S \subset V$ such that $q(G[V \setminus S]) \geq |S| + n - 2s$.*

As noticed by Figaj and Łuczak [9], one can easily derive from Theorem 6.2 the following structural result.

Lemma 6.3. *Let $G = (V, E)$ be a graph with n vertices such that $\nu(G) \leq s$. Then, there exists a partition $\{S_1, S_2, S_3\}$ of V such that:*

(i) *each vertex $v \in S_3$ has at most $\sqrt{n} - 1$ neighbors in S_3,*

(ii) *there are no edges between the sets S_2 and S_3,*

(iii) $2|S_1| + |S_2| < 2s + \sqrt{n}$.

Proof. Let $G = (V, E)$ be an n-vertex graph such that $\nu(G) \leq s$. Then, by Theorem 6.2, there exists a subset $S_1 \subset V$ such that

$$q(G[V \setminus S_1]) \geq |S_1| + n - 2s.$$

Let $V_1, V_2, \ldots, V_k$ be vertex sets of the odd components of a graph $G[V \setminus S_1]$ with at most $\sqrt{n}$ vertices, and let $S_3 = \bigcup_{i=1}^{k} V_i$ and $S_2 = V \setminus (S_1 \cup S_3)$. It is easy to see that for such a partition $\{S_1, S_2, S_3\}$ conditions (i) and (ii) clearly hold. Since there are fewer than $\sqrt{n}$ components in $G[V \setminus S_1]$ of size larger than $\sqrt{n}$, and because of Tutte's condition we have

$$|S_3| \geq k > q(G[V \setminus S_1]) - \sqrt{n} \geq |S_1| + n - 2s - \sqrt{n}.$$

Therefore, since $|S_2| = n - |S_1| - |S_3|$, we get

$$2|S_1| + |S_2| = n + |S_1| - |S_3| < 2s + \sqrt{n}$$

and (iii) holds. □

Now we can prove Erdős Conjecture for graphs, which is an immediate consequence of Lemma 6.3 and Theorem 4.11.

Theorem 6.4. *Let $s \geq 1$ and $n > n_0$. Then,*

$$\mathcal{M}_2(n, s) = Cov_2(n, s) \cup Cl_2(n, s).$$

Proof. Let $G = (V, E) \in \mathcal{M}_2(n, s)$. By Lemma 6.3, there exists a partition $\{S_1, S_2, S_3\}$ of V such that each vertex of S_3 has at most $\sqrt{n} - 1$ neighbors in S_3, there are no edges between the sets S_2 and S_3, and

$$2|S_1| + |S_2| < 2s + \sqrt{n}.$$

Thus,

$$|S_2| < 2s - 2|S_1| + \sqrt{n} \quad \text{and} \quad |S_1| < s + \sqrt{n}.$$

Now, the number of edges of G can be bounded from above by

$$\begin{aligned} |E| &\leq \binom{|S_1|}{2} + \binom{|S_2|}{2} + \frac{1}{2}|S_3|\Delta(G[S_3]) + |S_1||S_2| + |S_1||S_3| \\ &< \frac{|S_1|^2}{2} + \frac{(2s - 2|S_1| + \sqrt{n})^2}{2} + \frac{1}{2}|S_3|\sqrt{n} + |S_1|(n - |S_1|) \\ &= \frac{3}{2}|S_1|^2 + (n - 4s)|S_1| + 2s^2 + o(n^2). \end{aligned}$$

Since the last expression is a quadratic function of $|S_1|$, it achieves the maximum value either for $|S_1| = 0$ or $|S_1| = s$. Hence,

$$|E| \leq \max\left\{2s^2, ns - \frac{s^2}{2}\right\} + o(n^2),$$

and either $G \in Cl_2(n, s; \varepsilon)$ or $G \in Cov_2(n, s; \varepsilon)$ for some $\varepsilon > 0$ and $n > n_0$. Consequently, by Theorem 4.11, $G \in Cov_2(n, s) \cup Cl_k(n, s)$, for n big enough. □

The proof of Lemma 6.3 is based on Tutte's Theorem, but this result can be also shown in a different way, using the switching technique similar to the one employed in the proof of Lemma 5.4. Although the new proof is neither simpler nor shorter, we believe that it might be generalized and used to obtain analogous structural result for hypergraphs. Not being able to prove it in the whole generality, we state it below as a new conjecture. This conjecture describes a structure of k-uniform hypergraphs with a bounded matching number and might be considered as a generalization of Tutte's Theorem for k-uniform hypergraphs.

Conjecture 6.5. *Let $k \geq 2$, $s \geq 1$ and $n \geq k(s+1)$. Moreover, let $G = (V, E)$ be a k-uniform hypergraph with n vertices and $\nu(G) \leq s$. Then, one can remove $o(n^k)$ edges from G so that the graph $G' = (V, E')$ obtained in this way admits a partition $\{S_1, S_2, \dots, S_k, S_{k+1}\}$ of the vertex set V such that:*

(i) for every $e \in E'$ there exists $i \in [k]$ such that $|e \cap S_i| \geq i$,

(ii) $k|S_1| + \frac{k}{2}|S_2| + \ldots + \frac{k}{k-1}|S_{k-1}| + |S_k| \leq ks$.

The proof that Conjecture 6.5 implies Erdős Conjecture in a general case seems to be very technical. Note however that the problem is just to maximize the sum

$$\sum_{j=1}^{k} \sum_{i=j}^{k} \binom{|S_j|}{i} \binom{\sum_{l=j+1}^{k+1} |S_l|}{k-i},$$

which bounds the number of edges in a k-graph G, under the conditions

(i) $k|S_1| + \frac{k}{2}|S_2| + \frac{k}{3}|S_3| + \cdots + \frac{k}{k-1}|S_{k-1}| + |S_k| \leq ks$,

(ii) $|S_1| + |S_2| + \cdots + |S_{k+1}| = n$,

(iii) $|S_i| \geq 0$ for $i = 1, \dots, k+1$.

But even this task does not seem possible along these lines. Note just that, unless Erdős Conjecture is false, the maximum should occur either for $|S_1| = s$, or for $|S_k| = ks$.

6.2 Erdős Conjecture and probabilistic inequalities

Let us remind that the fractional version of Erdős Conjecture gives the bound on the number of edges of a k-uniform hypergraph in terms of fractional matching number. Alon *et al.* [2] observed that this conjecture is closely related to the following probabilistic conjecture, proposed by Samuels [21] in 1966.

Conjecture 6.6. *Let $\lambda, x_1, \dots, x_k$ be real numbers, satisfying $0 \le x_1 \le \ldots \le x_k$ and $\sum_{i=1}^k x_i < \lambda$. Moreover, let $X_1, \dots, X_k$ be independent, nonnegative random variables with expectations $x_1, \dots, x_k$, respectively. Then,*

$$\mathbb{P}(X_1 + \ldots + X_k \ge \lambda) \le \max_{l=0,\dots,k-1} \{P_l(\lambda; x_1, \dots, x_k)\},$$

where

$$P_0(\lambda; x_1, \dots, x_k) = 1 - \prod_{i=1}^k \left(1 - \frac{x_i}{\lambda}\right),$$

and

$$P_l(\lambda; x_1, \dots, x_k) = 1 - \prod_{i=l+1}^k \left(1 - \frac{x_i}{\lambda - \sum_{j=1}^l x_j}\right) \text{ for } l = 1, \dots, k-1.$$

Observe that for $k = 1$ Conjecture 6.6 is equivalent to the Markov's inequality. It was also confirmed for $k \le 4$ by Samuels [21], [22], but for all $k \ge 5$ this problem is still widely open. Note also that the value $P_0(\lambda; x_1, \dots, x_k)$ is attained when $X_i \in \{0, \lambda\}$, for every i. For $l \ge 1$, the value $P_l(\lambda; x_1, \dots, x_k)$ is equal exactly to the value of $\mathbb{P}(X_1 + \ldots + X_k \ge \lambda)$ when X_i is identically x_i for all $i \le l$, and $X_i \in \left\{0, \lambda - \sum_{j=1}^l x_j\right\}$ for all $i \ge l + 1$.

In [2] it was proved that Conjecture 6.6, if true, implies asymptotically the fractional version of Erdős' problem for $x \le 1/(k+1)$.

Theorem 6.7. *If Conjecture 6.6 holds for $\lambda = 1$, $k \ge 3$ and $0 < x_1 = \ldots = x_k \le \frac{1}{k+1}$, then*

$$\mu_k^*(n, xn) = (1 + o(1)) \left(1 - (1-x)^k\right) \binom{n}{k}.$$

The proof of Theorem 6.7 is based on the fact that

$$\mu_k^*(n, xn) \le (1 + o(1)) \binom{n}{k} \{\mathbb{P}(X_1 + \ldots X_k \ge 1)\},$$

for some identically distributed random variables X_i, satisfying assumptions of Conjecture 6.6 for $\lambda = 1$ and $0 < x_1 = \ldots = x_k \le \frac{1}{k+1}$, and the observation that in such a case we have

$$\max_{l=0,\dots,k-1} P_l(1; x, \dots, x) = P_0(1; x, \dots, x) = 1 - (1-x)^k.$$

By combining Samuels' results for $k = 3, 4$ with Theorem 6.7, Alon *et al.* [2] confirmed Conjecture 3.5 for $k = 3$, $x < 1/4$ and for $k = 4$, $x < 1/5$. As noted in [2], Samuels' conjecture for $\frac{1}{k+1} < x \leq \frac{1}{k}$, gives a bound on the number of edges which is larger than that in Conjecture 3.5. Here, we state a new conjecture which, if true, should provide an appropriate bound for every $0 < x \leq 1/k$, since it is asymptotically equivalent to the fractional version of Erdős Conjecture.

Conjecture 6.8 (Łuczak, Mieczkowska, Šileikis)**.** *Let $X_1, \dots, X_k$ be independent, identically distributed, nonnegative random variables with a common mean $\mathbb{E}(X_1) = x$. Then,*

$$\mathbb{P}(X_1 + \dots X_k \geq 1) \leq \max\{1 - (1-x)^k, (kx)^k\}. \tag{6.1}$$

Let us mention that for $k = 2$, Conjecture 6.8 follows from an old result of Hoeffding and Shrikande [17]. For $x < 1/(k+1)$, it holds whenever Samuels' conjecture is true, as then the bound in Conjecture 6.8 is equal to the one from Samuels' conjecture. Here we confirm Conjecture 6.8 in a few new instances, i.e. for $k = 3$ and $k \geq 5$ for $x \leq 1/(2k-1)$. The following unpublished result is due to Łuczak, Šileikis and the author, and proves that the fractional Erdős Conjecture is asymptotically equivalent to Conjecture 6.8.

Theorem 6.9. *For $x \in [0, 1/k]$ Conjecture 3.5 holds if and only if Conjecture 6.8 holds as well.*

Before proving Theorem 6.9, for the sake of simplicity, let

$$M(x) = \max\{1 - (1-x)^k, (kx)^k\},$$

and observe that function M is continuous and increasing. We will use this observation while proving the following lemma, in which we show that it is enough to confirm Conjecture 6.8 just for some specific random variables. Here and below, by $\operatorname{supp}(X)$ we denote the support of random variable X, i.e. the set of values which X attains with a non-zero probability.

Lemma 6.10. *It suffices to prove Conjecture 6.8 for independent, identically distributed, nonnegative random variables X_i with a common mean $\mathbb{E}(X_i) = x$ such that*

(i) $x < \frac{1}{k}$;

(ii) $X_i \leq 1$*;*

(iii) $\operatorname{supp}(X_i) < \infty$*;*

(iv) $\min\{\operatorname{supp}(X_i)\} = 0$*;*

(v) $\mathbb{P}(X_i = a_j) \in \mathbb{Q}$ *for every* $a_j \in \operatorname{supp}(X_i)$*.*

Proof. (i) Note that for $x \geq 1/k$ we have $M(x) \geq 1$ and then the bound (6.1) holds trivially. Therefore, from now on, we assume that $x < 1/k$.

(ii) Let us assume that Conjecture 6.8 holds for random variables with values less or equal to 1. We shall show that then it also holds for every random variable X_i. In order to do this, let us define

$$Y_i = \min\{X_i, 1\}.$$

Then, $\mathbb{P}\left(\sum_{i=1}^{k} X_i \geq 1\right) = \mathbb{P}\left(\sum_{i=1}^{k} Y_i \geq 1\right)$ and $y = \mathbb{E}(Y_i) \leq \mathbb{E}(X_i)$. Since the function M is increasing, we get

$$\mathbb{P}\left(\sum_{i=1}^{k} X_i \geq 1\right) = \mathbb{P}\left(\sum_{i=1}^{k} Y_i \geq 1\right) \leq M(y) \leq M(x).$$

Thus, from now on, we assume that $X_i \leq 1$.

(iii) Let us assume that Conjecture 6.8 holds for random variables attaining finite number of values from the interval $[0,1]$. For every natural m we define

$$Y_i^{(m)} = \lceil mX_i \rceil / m.$$

Then, we have $X_i \leq Y_i^{(m)} \leq X_i + 1/m$. Since M is an increasing function, $y_m = \mathbb{E}(Y_i^{(m)}) \leq \mathbb{E}(X_i) + \frac{1}{m}$, and $Y_i^{(m)}$ attains finite number of values from the interval $[0,1]$, the following holds:

$$\mathbb{P}\left(\sum_{i=1}^{k} X_i \geq 1\right) \leq \mathbb{P}\left(\sum_{i=1}^{k} Y_i^{(m)} \geq 1\right) \leq M(y_m) \leq M(x + 1/m).$$

Now, as the previous inequalities hold for every natural m and function M is continuous, we get

$$\mathbb{P}\left(\sum_{i=1}^{k} X_i \geq 1\right) \leq M(x).$$

Thus, from now on, we assume that $\operatorname{supp}(X_i) = \{a_1, \ldots, a_m\}$ for some $a_j \in [0,1]$.

(iv) Let $a_1 = \min\{\operatorname{supp}(X_i)\}$. First observe that if $a_1 = x$ then $X_i \equiv x$ and then Conjecture 6.8 clearly holds. Thus, we may assume that $a_1 \neq x$. Suppose now that $a_1 \neq 0$ and let

$$Y_i = (X_i - x)\frac{x}{x - a_1} + x.$$

Then, $\mathbb{E}(Y_i) = x$ and since $0 \in \operatorname{supp}(Y_i)$, we have

$$\mathbb{P}\left(\sum_{i=1}^{k} X_i \geq 1\right) \leq \mathbb{P}\left(\sum_{i=1}^{k} X_i \geq 1 - \frac{a_1(1-kx)}{x}\right) = \mathbb{P}\left(\sum_{i=1}^{k} Y_i \geq 1\right) \leq M(x).$$

Thus, we may only consider random variables for which $a_1 = 0$.

(v) Let us now assume that Conjecture 6.8 holds for random variables attaining finite number of values from interval $[0,1]$, each with rational probability, and attaining value 0 with non-zero probability. Let $p_j = \mathbb{P}\left(X_i = a_j\right)$, $j = 1, \ldots, m$. For every n define random variable $Y_i^{(n)} \in \{a_1, \ldots, a_m\}$ such that

$$\mathbb{P}\left(Y_i^{(n)} = a_j\right) = p_j^{(n)},$$

where $p_j^{(n)} = \lceil np_j \rceil / n$ for $j = 2, \ldots, m$, and $p_1^{(n)} = 1 - \sum_{j=2}^{m} p_j^{(n)}$. Then, for $j \geq 2$ we have

$$p_j \leq p_j^{(n)} \leq p_j + 1/n,$$

and thus,

$$y_n = \mathbb{E}(Y_i^{(n)}) \leq \mathbb{E}(X_i) + 1/n.$$

Moreover, since $\mathbb{P}(Y_i^{(n)} \geq a) \geq \mathbb{P}(X_i \geq a)$ for every $a \in \mathbb{R}$, X_i is stochastically dominated by Y_i, and the same applies to their sums. Thus, we get

$$\mathbb{P}\left(\sum_{i=1}^{k} X_i \geq 1\right) \leq \mathbb{P}\left(\sum_{i=1}^{k} Y_i^{(n)} \geq 1\right) \leq M(y_n) \leq M(x + 1/n).$$

As the last inequalities hold for every n, by continuity of function M, we conclude that

$$\mathbb{P}\left(\sum_{i=1}^{k} X_i \geq 1\right) \leq M(x).$$

□

Proof of Thorem 6.9. First, let us assume that Conjecture 6.8 is true for some k and $x < 1/k$. Following the argument of Alon *et al.* [2], we will show that it implies the asymptotic fractional version of Erdős Conjecture.

Let G be a k-uniform hypergraph on a vertex set V, $|V| = n$ such that $\nu^*(G) = xn$. By duality we know that $\tau^*(G) = xn$, hence there exists a weight function $w : V \to [0,1]$ such that

$$\sum_{v \in V} w(v) = xn,$$

and $\sum_{v \in e} w(v) \geq 1$ for every $e \in G$. Let $(v_1, \ldots, v_k)$ be a sequence of random vertices, chosen independently and uniformly at random from the vertices of G.

For each $i = 1, \ldots, k$, define X_i to be the weight of the i-th chosen vertex, i.e. $X_i = w(v_i)$. Note that $X_1, \ldots, X_k$ are independent and identically distributed random variables, attaining value $w(v)$ with probability $1/n$ for each $v \in V$. By the definition, for every $i = 1, \ldots, k$ we have that

$$\mathbb{E}(X_i) = \frac{1}{n} \sum_{v \in V} w(v) = \frac{1}{n} xn = x.$$

Our goal now is to express the number of edges in hypergraph G in terms of the probability that such a randomly chosen sequence of k vertices corresponds to an edge in G. First of all, notice that for every edge $e = \{v_i, \ldots, v_k\} \in G$, there are $k!$ possible sequences corresponding to this edge that might have been obtained as a result of such a random choice. Therefore, if $(v_1, \ldots, v_k)$ is a random sequence obtained in our process, then

$$\mathbb{P}(\{v_1, \ldots, v_k\} \in G) = \frac{k! e(G)}{n^k}. \tag{6.2}$$

On the other hand we know that for every edge $\{v_1, \ldots, v_k\} \in G$ we have $\sum_{i=1}^k w(v_i) \geq 1$ and thus

$$\mathbb{P}(\{v_1, \ldots, v_k\} \in G) \leq \mathbb{P}\left(\sum_{i=1}^k X_i \geq 1 \right). \tag{6.3}$$

From (6.2), (6.3) and the assumption that Conjecture 6.8 is true, we eventually conclude that

$$\begin{aligned} e(G) \leq & \frac{n^k}{k!} \mathbb{P}\left(\sum_{i=1}^k X_i \geq 1 \right) \\ \leq & (1 + o(1)) \binom{n}{k} \max\{(1 - (1-x)^k, (kx)^k\}. \end{aligned}$$

To prove the equivalence of the conjectures now we need to prove the reverse implication. Therefore, let us assume that Conjecture 3.4 is valid for some k, $x < 1/k$ and n large enough. Let X_i, $i = 1, \ldots, k$ be independent and identically distributed nonnegative random variables such that

$$\mathbb{P}(X_i = a_j) = \frac{p_j}{q_j}$$

for some $a_j \in [0, 1]$ and positive co-prime integers p_j and q_j, $j = 1, \ldots, m$. Moreover, let n be the smallest common multiple of the numbers $\{q_1, \ldots, q_m\}$, $p'_j = np_j/q_j$ and let us assume that

$$x = \mathbb{E}(X_i) = \sum_{j=1}^m a_j \frac{p_j}{q_j} = \frac{1}{n} \sum_{j=1}^m a_j p'_j < \frac{1}{k}.$$

Now we will use variables X_i to define hypergraphs with a small fractional matching number. Let $V_r = [rn]$ and let $w_r : V_r \to [0,1]$ be a function such that

$$w_r(v) = a_j \text{ for every } v \in V_r \text{ such that } \sum_{l=1}^{j-1} rp'_l < v \leq \sum_{l=1}^{j} rp'_l.$$

We define $G_r = (V_r, E_r)$ to be a sequence of k-uniform hypergraphs with vertex sets V_r and edge sets defined as follows:

$$E_r = \left\{ e \in \binom{V_r}{k} : \sum_{v \in e} w_r(v) \geq 1 \right\}.$$

Note that, for such defined hypergraphs G_r, the function w_r is actually a fractional vertex cover of size $\sum_{v=1}^{rn} w_r(v) = \sum_{j=1}^{m} a_j r p'_j = xrn$. Therefore, by the duality, we claim that $\nu^*(G_r) = \tau^*(G_r) < xrn$. Now from the assumption that Conjecture 3.5 is true, we get the following bound on the number of edges of G_r:

$$e(G_r) \leq (1 + o(1)) \binom{rn}{k} \max\{1 - (1-x)^k, (kx)^k\}.$$

Let now N_r denote the number of k-element sequences $(v_1, \ldots, v_k)$ of vertices $v_i \in V_r$ with at least two equal elements and such that $\sum_{1=1}^{k} w_r(v_i) \geq 1$. As $N_r = o((rn)^k)$, we have

$$\begin{aligned}
\mathbb{P}(X_1 + \ldots + X_k \geq 1) =& \frac{\left| \left\{ (v_1, \ldots, v_k) \in V_r^{(k)} : \sum_{i=1}^{k} w(v_i) \geq 1 \right\} \right|}{(rn)^k} \\
=& \frac{k! e(G_r) + N_r}{(rn)^k} \\
\leq& (1 + o(1)) \max\{1 - (1-x)^k, (kx)^k\}.
\end{aligned}$$

Finally, since the probability on the left hand side depends neither on r, nor on n, we conclude that

$$\mathbb{P}(X_1 + \ldots + X_k \geq 1) \leq \max\{1 - (1-x)^k, (kx)^k\}.$$

□

The following theorem is an immediate consequence of Theorem 6.9, combined with the known results on Erdős Conjecture from [2, 6, 13, 12, 19].

Theorem 6.11. *Conjecture 6.8 holds for $k = 2, 3$, for $k = 4$ and $x \leq 1/5$, and finally when $k \geq 5$ and $x \leq 1/(2k-1)$.* □

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