

CRITICAL SOUND PROPAGATION IN MAGNETS

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ABSTRACT

The critical dynamics of sound is a very interesting field in which we can test modern concepts of the phase transition theory such as the universality of critical exponents, scaling or the crossover to another universality class etc. It is the aim of the study to present a general theory of critical sound propagation, which takes also into account some important nonasymptotic effects. In metallic magnets the critical anomalies in the sound attenuation coefficient are of different types than in magnetic insulators. The difference in the critical exponents used to be explained by the occurrence of different kinds of magnetoelastic coupling in the two classes of magnets mentioned. We will show in this chapter that one should assume coexistence of both types of coupling in all magnets. A very important role is played by the ratio of the spin-lattice relaxation time to the characteristic time of spin fluctuations. It is a crucial parameter determining whether the sound attenuation coefficient reveals a strong or a weak singularity in a given material.

After a short introduction the fundamental concepts of the phase transition theory such as critical exponents, the scaling and universality hypothesis etc are reviewed in Section 2 of this chapter. Section 3 presents the idea of critical slowing down, dynamic scaling as well as the presentation of the basic dynamic universality classes. In Section 4, the model describing the static behavior of acoustic degrees of freedom is investigated. The expressions for the adiabatic and the isothermal sound velocity are also derived. The phenomenological theory of critical sound propagation is presented in very intuitive way in Section 5, while Section 6 contains a detailed description of the dynamic model based on the coupled nonlinear Langevin equations of motion. Three basic regimes characterized by different critical exponents and scaling functions are distinguished in the sound attenuation coefficient. Crossover effects from the insulator-type regime to the metallic-type regime and to the high-frequency regime are demonstrated on the example of the ultrasonic data for MnF_2 . The concept of the effective sound attenuation exponent is introduced using the data reported for FeF_2 and RbMnF_3 . The frequency dependent longitudinal sound velocity and its relation to the static quantities are discussed. Finally, the unsolved questions and future prospects in this field are outlined.

1 Introduction

The sound attenuation coefficient and the sound velocity show anomalous behavior near the critical point of the magnetic systems. The singular behavior of these quantities is connected with very strong fluctuations of the magnetic order parameter near the critical temperature. These fluctuations give rise to a characteristic attenuation peak whose position is correlated with that of the minimum in the sound velocity. In Fig. 1 we show the temperature dependence of the longitudinal ultrasonic attenuation and changes in the sound velocity in Gd (Moran and Luthi [1]). The problem of strongly interacting fluctuations cannot be reduced to the

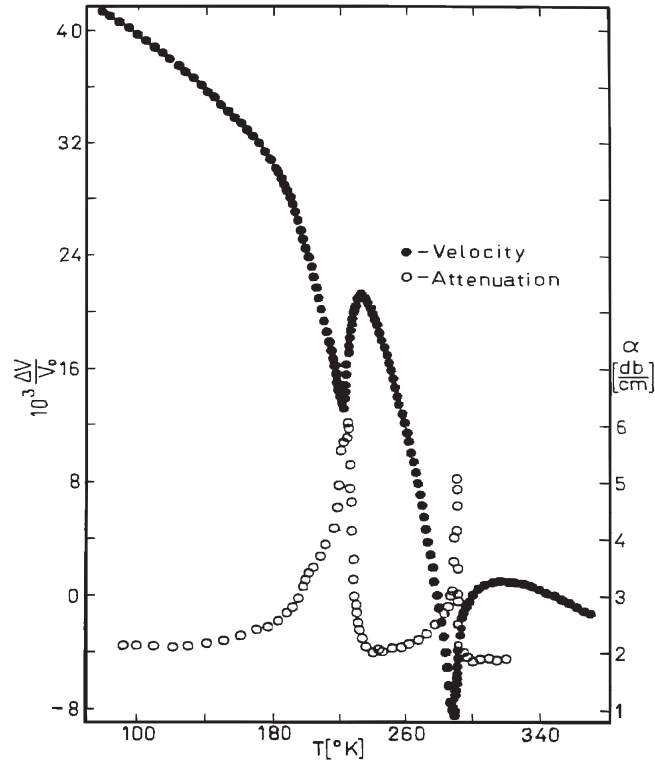


Figure 1 Temperature dependence of the ultrasonic attenuation and changes in the sound velocity for the longitudinal waves along c -axis with $f=50$ MHz (Moran and Luthi [1]).

problem of ideal gas even in the lowest order approximation. The general method of treating such issues has been shown by Wilson [2, 3] to be the renormalization group theory. Using this method we can find not only the critical exponents and the scaling functions but we can also study the nonasymptotic effects as the crossover from one universality class to another (crossover phenomena). Later it was possible to generalize the renormalization group formalism to the dynamic phenomena [4] such as transport coefficients and the relaxation rates. The studies of the critical dynamics of sound is a very important field where we can test the modern concepts

of phase transition theory such as scaling, universality of the critical exponents or the crossover to another universality class. Moreover, the measurements of the sound attenuation coefficient and the sound velocity permit determination of the phase diagram or the symmetry of the coupling between the order parameter and the elastic degrees of freedom. It is the class of magnetic materials which is especially important from this point of view (although it is still not fully recognized in many details), being a prototype for many other systems. In magnets we meet in general three types of magneto-elastic coupling (in this paper they will also be called the spin-phonon couplings) [5] but usually only one called the volume magnetostriction dominates. In spite of this ostensible simplification we observe there a whole variety of possible behaviors which sometimes cannot be explained. This fact is connected with the coexistence in magnets of many different spin interactions of different symmetry and range and with very rich and complicated dynamics in some systems [6, 7]. These factors can be manifested over different temperature ranges, which sometimes makes it impossible to describe the system's dynamics with the aid of one set of critical exponents. In the magnets being electric insulators the acoustic singularities encountered are different than in those showing metallic properties [6, 7]. In insulators we usually observe a weak singularity characterized by a small sound attenuation exponent. Sometimes the singularity is even not observed in a given experimental frequency range. In magnetic metals the singularity is much more noticeable and the critical exponent is much higher (usually higher than one). It was initially explained by the fact that in insulators the spin exchange interactions are of short range nature and in this case the spin-phonon Hamiltonian which arises mostly via the strain modulation of the exchange interaction [6] is proportional to the exchange Hamiltonian¹. This mechanism was proposed by Kawasaki [8] who noticed also that the energy fluctuations should decay only by the spin lattice relaxation. In this case we say that the sound wave couples to the energy fluctuations contrary to the metallic magnetic systems in which the long range exchange interactions generate a more general spin-phonon interaction which is linear in the sound mode and bilinear in the order parameter (spin) fluctuations. The different couplings should lead to different sound attenuation exponents. However, it was a simplified point of view as it was later shown [9, 10] that the energy fluctuations couple to the same bilinear combination of the order parameter fluctuations as for the magnetic metals. The general theory [9] which takes into account both types of magnetoelastic couplings as well as the proper coupling of energy to the order parameter fluctuations shows that both singularities: typical of the metallic as well as insulating systems appear in the acoustic self energy with the same effective coupling constant and the parameter which distinguishes the two types of behavior is the ratio of the spin-lattice relaxation time to the characteristic time of spin fluctu-

¹However, it is true only when we can neglect the next nearest neighbor exchange coupling and only in the case of propagation along some symmetry directions. In general the sound mode couples only to the part of the spin energy density [6].

ations². For insulators this ratio is very high as the spin-lattice relaxation times are much longer than for metals. The long spin-lattice relaxation time favors the weak singularity. If these times are comparable, the strong singularity dominates. We will show also the existence of another high-frequency regime which is expected for some materials. The nonasymptotic effects showing the crossover from insulator-type regime to the metallic-type regime and to the high-frequency regime will be demonstrated on the example of the ultrasonic data for MnF_2 . We will also show the usefulness of the concept of the effective sound attenuation exponent which is introduced using the experimental data for FeF_2 and RbMnF_3 . Finally a summary of the sound attenuation exponents in magnetic metals and insulators will be given as well as an outlook for the future progress in this field will be outlined.

2 The fundamental concepts of the phase transition theory

There is a huge variety of physical³ systems which undergo phase transitions. The most interesting class of phase transformations seems to be that of the continuous phase transitions which show no latent heat but at which many physical quantities diverge to infinity or tend to zero when approaching the critical temperature T_c . The behavior of the specific heat of a ferromagnet near the critical temperature is shown in Fig. 2. The free energy in such systems is a nonanalytical function

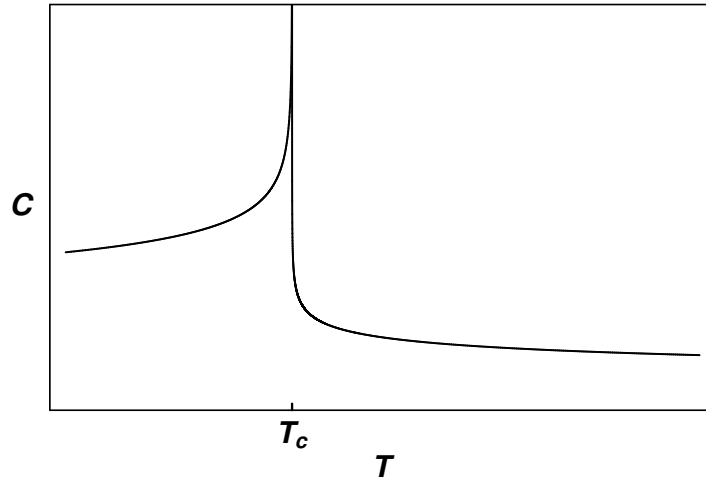


Figure 2 Specific heat C vs. temperature T in a ferromagnet.

²It will be shown explicitly in Section 6 of this chapter.

³In general the phase transitions can be found in economic, biological, social and many other systems. For example the collective motion of large groups of biological organisms like flocks of birds or fish schools (self-driven organisms) can develop a kinetic phase transition from an ordered to chaotic motion [13, 14].

of its arguments which is a manifestation of very strong fluctuations of a quantity called the order parameter. Usually, we define the order parameter as the quantity which is space and time dependent. It will be denoted by $S(\mathbf{x}, t)$ for anisotropic ferromagnets, and sometimes we will refer to it as the spin. The prototype of the continuous phase transition is that from the paramagnetic phase (disordered spins) to the ferromagnetic phase with nonzero average magnetization. In this case, the order parameter is the local magnetic moment whose average (magnetization) tends to zero when approaching the Curie temperature as shown in Fig. 3. For antiferro-

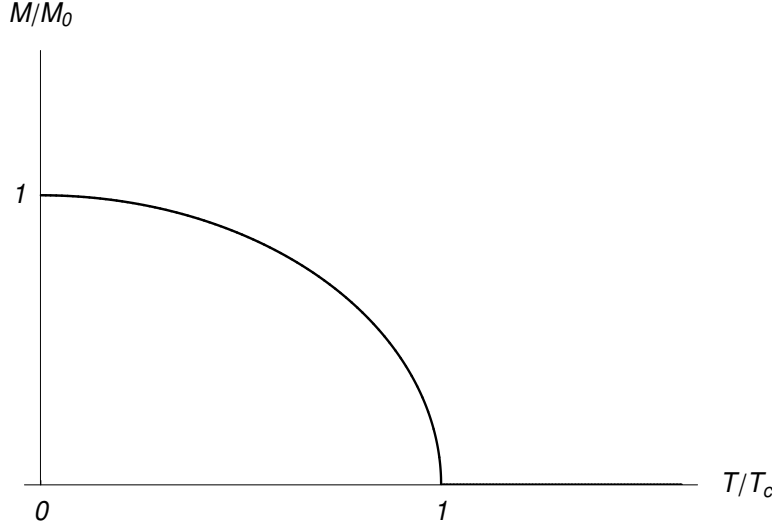


Figure 3 Magnetization as a function of temperature.

magnets the order parameter is given by the staggered local magnetization; in the case of gas-liquid transition it is proportional to the deviation of the mass density from its critical value, and for the superconducting transition it is a wave function of the Cooper pairs [12]. The order parameter can have more than one components as for example for isotropic ferromagnets in which it is a vector with three components. We say that in this case the order parameter dimension is three: $n = 3$. If there is an anisotropy in the system such that the magnetization (staggered magnetization) vector is forced to lie within a given plane we deal with the XY ferromagnet (antiferromagnet) for which $n = 2$. For a magnet with only one easy axis $n = 1$ and we talk about the scalar order parameter. The order parameter can have much more components and a nature more complicated than a vector as for example in liquid He^3 [12]. In the theory of phase transitions and critical phenomena the key problem is the identification of the order parameter since the same system of atoms may exhibit in different temperature ranges the liquid-gas transition, many structural and/or liquid crystals transitions, paramagnet-ferromagnet transition etc. The physical intuition plays here a very important role indicating the most important

features of a given phase transformation.

2.1 Critical exponents

The rate at which physical quantities diverge to infinity or converge to zero when approaching a critical point is described by critical exponents. If the distance from the critical point is measured by the reduced temperature

$$t = \frac{T - T_c}{T_c}, \quad (1)$$

than the critical exponent, describing the quantity $F(t)$ is defined by:

$$x_F = - \lim_{t \rightarrow 0^+} \frac{\ln F(t)}{\ln t}. \quad (2)$$

We say that for $t \rightarrow 0^+$ the function $F(t)$ diverges (with a positive exponent x_F) as t^{-x_F} . We can also define the low-temperature exponent

$$x'_F = - \lim_{t \rightarrow 0^-} \frac{\ln F(t)}{\ln t}, \quad (3)$$

which corresponds to the ordered (low-temperature phase), or other critical exponents describing the power-law behavior with respect to the other thermodynamic quantities, distance or the wave vector etc. For some quantities like the average of the order parameter the corresponding exponent is defined with the minus sign in Eq. 3. We define the basic static critical exponents on the example of the Ising type ferromagnet ($n = 1$). In such a simple system the critical behavior of all thermodynamic quantities is controlled by only two parameters: the reduced temperature t and the magnetic field h . Let us consider the following quantities:

1. The specific heat C_h under constant magnetic field. Near T_c it is described by the relations:

$$C_h \approx A^+ t^{-\alpha} + B, \quad t > 0, \quad h = 0, \quad (4)$$

$$C_h \approx A^- |t|^{-\alpha'} + B, \quad t < 0, \quad h = 0. \quad (5)$$

In the case of two dimensional Ising model $\alpha = 0$ and the specific heat diverges logarithmically

$$C_h \approx -A^\pm \ln |t|. \quad (6)$$

The coefficients A^+ i A^- are called the critical amplitudes and α i α' are known as the specific heat critical exponents.

2. Susceptibility χ (the derivative of the magnetization with respect to the magnetic field). We observe the following power-law behavior:

$$\chi \approx C^+ t^{-\gamma}, \quad t > 0, \quad h = 0, \quad (7)$$

$$\chi \approx C^- |t|^{-\gamma'}, \quad t < 0, \quad h = 0. \quad (8)$$

For the systems with vector order parameter ($n \geq 2$) below T_c the susceptibility is infinite in agreement with the famous Goldstone theorem [15] which says that for the system with broken continuous symmetry $n - 1$ transversal modes appear whose frequencies tend to zero as the wave vector goes to zero. These massless modes imply that the transversal susceptibility diverges for a vanishing external field.

3. The order parameter

$$M \approx B'(-t)^\beta, \quad t < 0, \quad h = 0. \quad (9)$$

Another interesting critical exponent is the one connected with approaching the critical point for $T = T_c$ with $h \rightarrow 0$. Then the order parameter is described by the following scaling law:

$$M \approx B^c h^{1/\delta}, \quad t = 0. \quad (10)$$

4. Two-point correlation function

$$C(x) = \langle S(\mathbf{x})S(\mathbf{0}) \rangle - \langle S(\mathbf{0}) \rangle^2, \quad (11)$$

where $\langle \dots \rangle$ denotes an average and $S(\mathbf{x})$ is a local value of the order parameter at point \mathbf{x} . At the critical point ($T = T_c$) it is characterized by the power-law behavior at large distances:

$$C(x) \propto x^{-d+2-\eta}, \quad t = 0, \quad h = 0, \quad (12)$$

where d is the space dimension and η is an anomalous critical exponent which measures the deviation from the classical Ornstein-Zernike behavior where $\eta = 0$.

In the neighborhood of the critical point (but not exactly at it) the correlation function decays exponentially

$$C(x) \propto \exp(-x/\xi), \quad (13)$$

where ξ denotes a correlation length of the system which diverges when approaching the critical temperature:

$$\xi \approx \xi_0^+ t^{-\nu}, \quad t > 0, \quad h = 0, \quad (14)$$

$$\xi \approx \xi_0^- (-t)^{-\nu'}, \quad t < 0, \quad h = 0. \quad (15)$$

2.2 Scaling hypothesis

Already at very early stage of development of the phase transition theory, it was realized that the critical exponents are not fully independent of each another and fulfill a number of relations called the „scaling laws” [17]. These relations can be derived from the scaling hypothesis which says that near the critical point the

correlation length ξ is the only characteristic length scale in terms of which all other quantities with dimensions of length are to be measured. In general a system has usually many intrinsic length scales, as for example the length of a system or the mean distance between nearest lattice points in a crystal. We say that the system near a critical point shows a *scale invariance*. Using the scaling hypothesis one can derive the above mentioned scaling laws which are in very good agreement with experiment. A mathematical manifestation of the scaling hypothesis is that the singular parts of the thermodynamic potentials or the correlation function etc. are generalized homogeneous⁴ functions of their arguments [17–19]. For example the free energy of the magnetic system $F_{\text{sing}}(T, h)$ obeys the relation:

$$F_{\text{sing}}(\lambda^{x_t} t, \lambda^{x_h} h) = \lambda F_{\text{sing}}(t, h), \quad (16)$$

where λ is a rescaling factor (any real number) and x_t and x_h are the characteristic exponents of the phase transition. Choosing $\lambda = t^{-1/x_t}$ we obtain $F_{\text{sing}}(t, h) = t^{1/x_t} \phi(h/t^{x_h/x_t})$ where ϕ is a scaling function. Also a derivative of one homogeneous function is another homogeneous function. Thus differentiating expression (16) with respect to the reduced temperature or magnetic field and comparing it with the corresponding definitions of critical exponents we can express the critical exponents α, β, γ and δ by only two independent ones x_t and x_h . Analogous considerations applied to the correlation function [19] shows that also the exponents η and ν can be obtained from the two mentioned independent ones. A consequence of the scaling hypothesis is also the equality of low-temperature and high-temperature exponents: $\alpha = \alpha', \gamma = \gamma'$ and $\nu = \nu'$. Eliminating x_t and x_h from the relations between the critical exponents one can obtain a number of exponent identities called the scaling laws [17]:

$$\alpha + 2\beta + \gamma = 2, \quad \text{Rushbrooke's law}, \quad (17)$$

$$\alpha + \beta(\delta + 1) = 2, \quad \text{Griffiths' law}, \quad (18)$$

$$\gamma = (2 - \eta)\nu, \quad \text{Fisher's law}, \quad (19)$$

$$\alpha = 2 - d\nu, \quad \text{Josephson's law}. \quad (20)$$

The Josephson's identity is the only one which involves the space dimension. Such identities are known as *hyperscaling relations*. They are true only for $d < d_c$ where d_c is the upper critical dimension ($d_c = 4$ for models with n -vector order-parameter) above which the mean-field critical exponents are exact:

$$\alpha = 0, \quad \gamma = 1, \quad \nu = \frac{1}{2}, \quad \eta = 0, \quad \beta = \frac{1}{2}, \quad \delta = 3. \quad (21)$$

⁴In general, a function $f(y_1, y_2, \dots)$ is homogeneous if $f(b^{x_1} y_1, b^{x_2} y_2, \dots) = b^{x_f} f(y_1, y_2, \dots)$ for any b . By a proper choice of the rescaling factor b one of the arguments of f can be removed, leading to a scaling forms used in this subsection. An important consequence of the scaling ideas is that the critical system has an additional dilatation symmetry.

The scaling laws were confirmed in many experiments, whereas the theoretical explanation was given by the renormalization group theory [3, 17, 23]. Moreover, this theory provided us also with the efficient tools for calculating the critical exponents and the scaling functions. Within this formalism one can also calculate the corrections to the asymptotic ($t \rightarrow 0$) power laws and assess their magnitude [22, 24]. As will be shown in the next section it is possible to generalize the scaling hypothesis onto the dynamic phenomena.

2.3 Universality hypothesis

The main goal of the theory of phase transitions is to permit the calculations of the scaling exponents and the scaling functions. According to the universality hypothesis, diverse physical systems that share the same essential symmetry properties will exhibit the same physical behavior close to their critical points and the values of their critical exponents do not depend on the thermodynamic parameters, the strength of interactions, atomic structure of the system and other microscopic details of the interactions. For example a uniaxial ferromagnet is characterized by the same set of critical exponents as the liquid-gas phase transition and the planar ferromagnet's exponents are the same as for the liquid helium near the transition to superfluid phase. Very close to the critical point the most of the detailed information about the interactions in the system becomes irrelevant and even highly idealized models (and much simpler than the real system), which possess the important symmetries of the real system, can be used to describe real systems accurately. These symmetries determine the type of critical behavior (values of the critical exponents) known as the universality class. The fact that every system undergoing a continuous phase transition belongs to one of such universality classes and that the universality classes constitute relatively not numerous set is probably the most unusual feature of the phase transitions. The renormalization group theory predicts that the universality classes are determined by the spatial dimensionality d , dimension of the order parameter n or more generally its symmetry, and the range of interactions. In some systems the presence of some kinds of impurities may influence the critical exponents leading to a new universality class. Besides the critical exponents also the scaling functions and some combinations of critical amplitudes like A^+/A^- or ξ_0^+/ξ_0^- are universal i.e. are the same for different sometimes quite dissimilar systems. The critical amplitudes alone are nonuniversal quantities and depend on a given system. In Table 1 we present the theoretical estimations of the most important (static) critical exponents and some universal amplitude ratios for three dimensional systems with n -vector order parameter and short range interactions. From the analysis of these data we can see that the change in the critical exponents from one class to another is not very impressive. Much greater variability is observed in the critical amplitude ratios and sometimes these ratios are better suited to identify the universality classes. Also the investigation of dynamic properties of the system as will be shown in the next section may be useful in solving this issue.

Table 1 The theoretical estimations of the critical exponents and the universal amplitude ratios for three dimensional Ising ($n = 1$), XY ($n = 2$) and Heisenberga ($n = 3$) systems.

n	1	2	3
α	0.110(1)	$-0.0146(8)^*$	$-0.133(6)^*$
β	0.3265(3)	$0.348\,5(2)^*$	$0.3689(3)^*$
γ	1.2372(5)	1.3177(5)	1.3960(9)
δ	4.789(2)	$4.780\,(7)^*$	$4.783(3)^*$
η	0.0364(5)	0.0380(4)	0.0375(5)
ν	0.6301(4)	0.67155(27)	0.7112(5)
A^+/A^-	0.532(3)	1.062(4)	1.56(4)
ξ_0^+/ξ_0^-	1.956(7)	0.33	0.38
$\alpha A^+ C^+/B^2$	0.0567(3)	0.127(6)	0.185(10)
References	[22]	[25]	[26]

The star (*) denotes the estimations obtained from the scaling laws $\alpha = 2 - 3\nu$, $\beta = \nu(1 + \eta)/2$ and $\delta = (\beta + \gamma)/\gamma$

3 Critical dynamics

We recall in this section the basic ideas which have contributed to the development of the modern theory of dynamic critical phenomena.

3.1 Critical slowing down

In description of the critical anomalies which are met in dynamic characteristics of the system like the linear response functions, we need an equation of motion describing the order parameter field. The most simple equation used in irreversible thermodynamics is that describing the rate of change in the quantity relaxing to its equilibrium state

$$\dot{\psi} = -L \frac{d\Phi}{d\psi}, \quad (22)$$

where the dot over ψ denotes the time derivative and L is a kinetic coefficient. The function $\Phi[\psi]$ is an increase in the corresponding thermodynamic potential related to the deviation of ψ from the equilibrium value ($\psi_{eq} = 0$). The probability of fluctuation ψ is proportional to

$$p_{eq} \propto \exp \{ -\Phi[\psi] / k_B T \} \quad (23)$$

If we assume that the probability distribution is Gaussian than we have

$$\Phi[\psi] = \frac{\psi^2}{2\chi}, \quad (24)$$

where

$$\chi = \frac{\langle \psi^2 \rangle}{k_B T} \quad (25)$$

is a susceptibility. The solution of (22) is given by

$$\psi(t) = \psi(0)e^{-t/\tau}, \quad (26)$$

where $\tau = \chi/L$ is known as the relaxation time of quantity ψ . In this section the symbol t refers to the time not to the reduced temperature and the distance to the critical point will be denoted by $(T - T_c)$. We have seen in the last section that an increase in the fluctuations near the critical point leads to the divergence of the susceptibility $\chi \propto (T - T_c)^{-\gamma}$. According the conventional theory of critical dynamics proposed by Van Hove [31] the kinetic coefficients stay finite at the critical point so the relaxation time goes to infinity at the critical point. We say that the system needs more and more time to get back to equilibrium. This phenomenon is known as the critical slowing down. The Van Hove's theory turns to be incorrect in most cases and the kinetic or transport coefficients diverge to infinity (or go to zero in some cases) but in no case does the kinetic coefficient diverge so strongly as the susceptibility [4] so the critical slowing down appears in all cases.

3.2 Dynamical scaling

In the physics of dynamic phenomena another critical exponent known as dynamic critical exponent z must be defined. In the dynamic scaling hypothesis we assume that the characteristic frequency (known also as the critical frequency) of the order parameter mode $S_{\mathbf{k}}$ scales as

$$\omega_c(k) = k^z f(k\xi), \quad (27)$$

where k is the wave vector and f is the scaling function. The characteristic frequency is defined as the half width of the dynamic correlation function $C_S(k, \omega)$

$$\int_{-\omega_c(k)}^{\omega_c(k)} \frac{d\omega}{2\pi} C_S(k, \omega) = \frac{1}{2} C_S(k) \quad (28)$$

where

$$C_S(k, \omega) = \int d^d x \int dt e^{-i(\mathbf{k} \cdot \mathbf{x} - \omega t)} [\langle S(\mathbf{x}, t) S(\mathbf{0}, 0) \rangle - \langle S(\mathbf{x}, t) \rangle \langle S(\mathbf{0}, 0) \rangle]. \quad (29)$$

The characteristic frequency may be also defined [4] by

$$\omega_c(k) = \frac{2C_S(k)}{C_S(k, 0)} = \frac{L_S(k)}{\chi_S(k)}. \quad (30)$$

$C(k)$ denotes here the Fourier transform of the static correlation function and $\chi_S(k) = C_S(k)/k_B$ is the susceptibility, $L_S(k)$ is the effective kinetic coefficient

$$\frac{1}{L_S(k)} = \frac{i\partial\chi_S^{-1}(k, \omega)}{\partial\omega}|_{\omega=0}, \quad (31)$$

where $\chi_S(k, \omega)$ is the linear response (dynamic susceptibility) to the infinitesimal field. We define the dynamic susceptibility $\chi_S(k, \omega)$ by the relation

$$\delta \langle S(k, \omega) \rangle_h = \chi_S(k, \omega) h(k, \omega). \quad (32)$$

Fourier transforms in wave vector and frequency are given by

$$S(\mathbf{x}, t) = \int \frac{d^d k}{(2\pi)^d} \int \frac{d\omega}{2\pi} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} S(\mathbf{k}, \omega).$$

The fluctuation-dissipation theorem for the classical systems says that the linear response and the correlation function are not independent:

$$C_S(k, \omega) = \frac{2k_B T}{\omega} \text{Im} \chi_S(k, \omega). \quad (33)$$

If the correlation function has a Lorentzian peak centered about $\omega = 0$ the definitions (28) and (30) are equivalent. If there is a propagating mode in the system which is reflected in the correlation function $C(k, \omega)$ as a sharp peak at a finite frequency, the definition (30) is not appropriate [4].

Within the dynamical scaling hypothesis [32, 33] we assume that the linear response function is homogenous:

$$\chi_S(k, \omega; T - T_c) = b^{2-\eta} \chi_S(bk, b^z \omega; b^{1/\nu} (T - T_c)).$$

With a proper substitution for b we obtain

$$\chi_S(k, \omega; T - T_c) = (T - T_c)^{-\gamma} Y(k\xi, \frac{\omega \xi^z}{\Omega_0}), \quad (34)$$

where Y is a scaling function and Ω_0 is a constant setting the time scale in the system. It is assumed that the wave vector and the frequency are much smaller than the inverse of the microscopic length (e.g. the lattice constant) and the microscopic relaxation time.

In the simple model of relaxational dynamics described in last subsection we have

$$\omega_c(k \rightarrow 0) \rightarrow L/\chi \propto |T - T_c|^{-\gamma} \propto \xi^{-\gamma/\nu}, \quad (35)$$

as in this model the kinetic coefficient L does not depend on k for $k \rightarrow 0$. The scaling function f from Eq. (27) has to behave as

$$f(x) \propto x^{-z}, \quad (36)$$

in order that $\omega_c(k \rightarrow 0)$ was wave vector independent. It gives the relation $\omega_c \propto \xi^{-z}$ which can be compared with the Van Hove's dynamic exponent $z = \gamma/\nu = 2 - \eta$ where we have exploited the Fisher identity (19).

3.3 Mode coupling and the equations of motion

Moreover, if we would like to take into account the fast movement associated with the other modes, we should add to Eq. (22) a stochastic Gaussian noise $\zeta(t)$ which mimics the thermal excitation

$$\dot{\psi} = -L \frac{d\Phi}{d\psi} + \zeta, \quad (37)$$

where the correlation function of the noise obeys the Einstein relation

$$\langle \zeta(t) \zeta(t') \rangle = 2k_B T L \delta(t - t'), \quad (38)$$

and $\delta(t)$ is the Dirac function. We assume that

$$\langle \zeta(t) \rangle = 0.$$

Both ψ and ζ should be regarded as the stochastic processes. Eq. (37) is known as a linear Langevin equation [30].

Generalizing Eq. (37) to include n -vector non uniform processes \mathbf{S} and taking into consideration also (static) nonlinear couplings present in the thermodynamic potential as a term proportional to S^4 , we obtain a model of dissipative dynamics known as the time-dependent Landau-Ginzburg model [34]

$$\dot{S}_i(\mathbf{x}) = -\Gamma_i \frac{\delta H}{\delta S_i(\mathbf{x})} + \zeta_i(\mathbf{x}), \quad (39)$$

where the potential (called the Landau-Ginzburg Hamiltonian or free energy)

$$H = \frac{1}{2} \int d^d x \{ r_0 S^2 + (\nabla S)^2 + \frac{u}{2} S^4 \}, \quad (40)$$

includes also nonlinear couplings between modes. In an external magnetic field \mathbf{h} the term $-\int d^d x \mathbf{h} \cdot \mathbf{S}(\mathbf{x})$ should be added to (40). In Eq. (40) we used the following abbreviations:

$$S^2 = \sum_{i=1}^n S_i^2(\mathbf{x}),$$

$$(\nabla S)^2 = \sum_{i=1}^n (\nabla S_i(\mathbf{x}))^2,$$

$$S^4 = (S^2)^2.$$

The symbol $\frac{\delta H}{\delta S_i(\mathbf{x})}$ denotes a functional derivative [12, 35]. The noises fulfill the relations

$$\langle \zeta_i(\mathbf{x}, t) \zeta_j(\mathbf{x}', t') \rangle = 2\Gamma_i \delta_{ij} \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'). \quad (41)$$

Usually it is assumed that the kinetic coefficients for different components of the order-parameter are equal: $\Gamma_i = \Gamma$.

The equation of motion (39) does not contain the nonlinear mode-coupling terms which in a crucial way decide on whether the kinetic coefficients diverge or tend to zero when approaching the critical point. For an isotropic ferromagnet for example, Eq. (39) should be modified by a term describing the precession of the spins around the local magnetic field:

$$\dot{\mathbf{S}} = \lambda \mathbf{S} \times \frac{\delta H}{\delta \mathbf{S}} + D \nabla^2 \frac{\delta H}{\delta \mathbf{S}} + \zeta, \quad (42)$$

where the Onsager kinetic coefficient Γ was replaced by the term $-D \nabla^2$ which guarantees that the total spin $\mathbf{S}_{\mathbf{k}=0}(t) = \int d^d x \mathbf{S}(\mathbf{x}, t)$ is a conserved quantity⁵ (which does not change its value during the motion) analogously to microscopic models and to hydrodynamics [60]. The first term in Eq. (42) describes the precession around the local field $\mathbf{h}_{loc} = -\frac{\delta H}{\delta \mathbf{S}}$ and is known in literature as the *mode-coupling* term or the *streaming term*. It is a non dissipative term i.e. it does not change the total energy of the system when the noise and the damping force are absent

$$\frac{dH}{dt} = \int d^d x \frac{\delta H}{\delta \mathbf{S}(\mathbf{x}, t)} \cdot \frac{\partial \mathbf{S}(\mathbf{x}, t)}{\partial t} = \int d^d x \frac{\delta H}{\delta \mathbf{S}(\mathbf{x}, t)} \cdot \left[\lambda \mathbf{S}(\mathbf{x}, t) \times \frac{\delta H}{\delta \mathbf{S}(\mathbf{x}, t)} \right] = 0. \quad (43)$$

When we investigate the critical dynamics we are not usually interested in complicated microscopic descriptions of the evolution of the system. Usually we tend to obtain the equations of motion for long-wavelength components of the so-called slow variables such as the conserved quantities, Goldstone modes and the order parameter. The fast variables are eliminated by a projection procedure on the subspace of slow variables [36, 37]. The reader can find the description of this procedure in the works of Mori et al. [38, 39]. The effective equations of motion for slow variables ϕ_α are reduced to nonlinear Langevin equations [38, 39]

$$\frac{d}{dt} \phi_\alpha(t) = V_\alpha(\{\phi_\alpha(t)\}) - \sum_\beta \Gamma_{\alpha\beta} \frac{\delta H(\{\phi_\alpha(t)\})}{\delta \phi_\beta^*(t)} + \zeta_\alpha(t), \quad (44)$$

⁵It is easy to see this performing a Fourier transform $\mathbf{S}(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k} < \Lambda} e^{i\mathbf{k} \cdot \mathbf{x}} \mathbf{S}_{\mathbf{k}}(t)$. Then

$D \nabla^2 \rightarrow -D k^2$ and the damping coefficient for the total spin tends to zero for $k \rightarrow 0$. In this equation Λ is a cutoff parameter which is usually chosen in such a way that Λ^{-1} is much larger than the lattice constant and simultaneously much shorter than the correlation length. Models which describe the fluctuations in such a scale are known as mesoscopic models.

where

$$H(\{\phi_\alpha(t)\}) = -k_B T \ln(P_{eq}(\{\phi_\alpha(t)\})), \quad (45)$$

and $P_{eq}(\{\phi_\alpha(t)\})$ is an equilibrium distribution function. The first term in Eq. (44) is the streaming term

$$V_\alpha(\{\phi_\alpha(t)\}) = -\lambda \sum_\beta \left[\frac{\delta}{\delta\phi_\beta} Q_{\alpha\beta}(\{\phi_\alpha\}) - Q_{\alpha\beta}(\{\phi_\alpha\}) \frac{\delta H(\{\phi_\alpha\})}{\delta\phi_\beta^*} \right], \quad (46)$$

with λ being a constant; and $Q_{\alpha\beta} = -Q_{\beta\alpha}$ are some functions constructed from the Poisson brackets or the commutators of slow variables $\{\phi_\alpha\}$. The second contribution describes the dumping and the last term is the stochastic force representing the effect of fast variables. The white noises have zero means and variations:

$$\langle \zeta_\alpha(t) \zeta_\beta(t') \rangle = 2\Gamma_{\alpha\beta} \delta(t - t').$$

It can be shown that the equations of motion have the correct stationary state (the long-time limits of the one-time correlation functions are the same as the static quantities obtained from the equilibrium distribution function).

It should be noted that the Poisson brackets and the detailed form of the damping coefficients are not determined by the functional H , which implies that with each static universality class usually a few dynamic universality classes (determined by the static critical exponents as well the dynamic one) can be associated. In general, we can say that the dynamic universality class is determined by the number of slow variables and the structure of the Poisson brackets.

3.4 Dynamic universality classes

In this section we consider a few of the most important models used in the study of critical dynamics. We begin with the models describing the relaxational dynamics. The description of other models can be found in the excellent reviews [4, 42, 43, 85, 86].

3.4.1 Model A

In this model there is no conserved quantity and the only slow variable is the order parameter of n components. It is described by simple equations [44, 45]:

$$\dot{S}_i(\mathbf{x}) = -\Gamma \frac{\delta H}{\delta S_i(\mathbf{x})} + \zeta_i(\mathbf{x}), \quad (47)$$

where H is the Ginzburg-Landau Hamiltonian of the form (40). The static exponents are determined in all dynamic models by the spatial dimensionality d and the order-parameter dimension. The dynamic critical exponent differs only slightly from that from the Van Hove's theory. The renormalization group theory gives the value:

$$z = 2 + c\eta, \quad (48)$$

where η is the correlation function exponent (12) and c is a function of d and n . One of the tools of this theory is the ϵ -expansion [3] of the critical exponents (and the scaling functions) where $\epsilon = 4-d$. It was obtained that $c = 0.7261(1-1.687\epsilon)+O(\epsilon^2)$ [46, 47]. Another useful expansion is that in $1/n$ (exact for $n \rightarrow 0$), it gives the following estimate $c = 1/2$ for $d = 3$ [46]. According to Eq. (48) the renormalized (by interactions) kinetic coefficient Γ_R ⁶ goes slowly to zero [4] as $t \rightarrow 0$ and it is actually finite to first order in ϵ .

3.4.2 Model B

It is a simple modification of Model A with the order parameter being a conserved quantity. In Eq. (47) the kinetic coefficient Γ is replaced by $-\lambda\nabla^2$ and the same replacement is made in Eq. (41). We find for this model

$$\omega_c(k) = \lambda k^2 / \chi_S,$$

as the transport coefficient λ is not renormalized by the nonlinear interactions present in the Hamiltonian [4]. Thus we obtain the classical result from the Van Hove's theory

$$z = z_{cl} = 4 - \eta.$$

The characteristic frequency $\omega_c(k)$ tends to zero for a given k as a result of the critical slowing down.

3.4.3 Model C

The nonconserved order parameter can be coupled to a conserved (noncritical) quantity such as the energy (or the magnetization in the case of uniaxial antiferromagnet). The static Landau-Ginzburg functional depends then on two quantities: the order parameter S and the additional conserved quantity m :

$$H = \frac{1}{2} \int d^d x \{ r_0 S^2 + (\nabla S)^2 + \frac{u}{2} S^4 + \chi_m^{-1} m^2 + f m S^2 \}, \quad (49)$$

where f is a new coupling constant and χ_m is the bare (i.e. without taking into considerations the effect of interactions which means for $f = 0$) susceptibility of m . Because m is a noncritical quantity it can be eliminated from statics by an integration over m [44]. This procedure leads to an effective Hamiltonian with the renormalized coupling constant u . Thus the static critical exponents are the same as for Hamiltonian (40).

The dynamics is described by two coupled equations:

$$\dot{S}_i(\mathbf{x}) = -\Gamma \frac{\delta H}{\delta S_i(\mathbf{x})} + \zeta_i(\mathbf{x}), \quad (50)$$

⁶The renormalized kinetic coefficient is usually defined by the relation $\Gamma_R = \omega_c \chi$ where χ is the susceptibility of the system.

$$\dot{m}(\mathbf{x}) = \lambda_m \nabla^2 \frac{\delta H}{\delta m(\mathbf{x})} + \xi_m(\mathbf{x}), \quad (51)$$

where the additional noise obeys the relation

$$\langle \xi_m(\mathbf{x}, t) \xi_m(\mathbf{x}', t') \rangle = -2\lambda_m \nabla^2 \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'). \quad (52)$$

The coupling with the conserved quantity leads to a new dynamic critical exponent for the systems with positive specific heat critical exponent (which happens for the Ising like systems with $n = 1$)

$$z = 2 + \frac{\alpha}{\nu} \quad (53)$$

where α is the specific heat critical exponent. The renormalized kinetic coefficient goes to zero more rapidly than in model A, and then $\Gamma_R \propto t^\alpha$. For $\alpha < 0$ (or $n > 1$) the coupling with the conserved quantity is irrelevant and this model is equivalent to model A. It is worth noting that the exponent α/ν is rather small so it is difficult to distinguish the predictions of models A and C experimentally, but for the tricritical point⁷ we have $\alpha_t = 1/2$ and $\nu_t = 1/2$ so the dynamic tricritical exponent $z_t = 3$ for model C is significantly different from model A where $z_t = 2$ [11, 52].

3.4.4 Model D

In this model the conserved order parameter is coupled with the conserved non-critical quantity. The dynamics of this model is described by Eqs. (50) and (51) where $\Gamma = -\lambda_S \nabla^2$. The model's dynamics is reduced to that of model B (independently of the order parameter dimensionality n) with $z = z_{cl} = 4 - \eta$.

3.4.5 Models E and F

Let's consider a planar magnet described by the following equations [48] (model F)

$$\dot{\psi} = -2\Gamma \frac{\delta H}{\delta \psi^*} - ig\psi \frac{\delta H}{\delta m} + \theta, \quad (54)$$

$$\dot{m} = \lambda_m \nabla^2 \frac{\delta H}{\delta m} + 2g \text{Im}(\psi^* \frac{\delta H}{\delta \psi^*}) + \xi_m, \quad (55)$$

where ψ is a complex order parameter representing $S_x - iS_y$ and m is the z -th component of magnetization (the z -axis is chosen to be perpendicular to the easy plane). The Landau-Ginzburg functional is given by

$$H = \frac{1}{2} \int d^d x \{ r_0 |\psi|^2 + |\nabla \psi|^2 + \frac{u}{2} |\psi|^4 + \chi_m^{-1} m^2 + f m |\psi|^2 - h m \}.$$

⁷At the tricritical point a change from the continuous to the first order transition occurs [51]. The tricritical exponents are classical ones for $d > 3$ with logarithmic corrections for $d = 3$.

In easy plane ferromagnets the order parameter is not conserved but the z -th component of magnetization is the conserved quantity which is also the generator of rotations in the order parameter space. So there is a non-vanishing Poisson bracket

$$\{\psi, M\} = ig\psi, \quad (56)$$

where g is the mode-coupling and $M = \int d^d x m(\mathbf{x})$.

The static properties of this model are the same as those of model C but the dynamic behavior is different due to the nondissipative coupling g and the complex value of the damping coefficient Γ . It was shown by Halperin and Hohenberg [49] that below T_c there is a spin-wave of the frequency $c_{sw}k$. Model F is significantly simplified in the situation when the external magnetic field vanishes. In such a case the total magnetization $\langle M \rangle$ also vanishes and we have the symmetric planar model denoted as model E. In this model we put $f = 0$ and assume a real Γ .

The propagating mode below T_c permits determination of the dynamic exponent z only by means of the static exponents and the spatial dimension. For the antisymmetric planar model (F) which also describes the liquid helium transition we obtain

$$z = \frac{d}{2} + \frac{\tilde{\alpha}}{2\nu}, \quad (57)$$

where $\tilde{\alpha} \equiv \max(\alpha, 0)$ and α is the specific-heat exponent. For model E

$$z = \frac{d}{2}, \quad (58)$$

thus $z = 3/2$ for $d = 3$. In both models the kinetic coefficient Γ diverges for $T \rightarrow T_c^+$ (but not so strongly as the susceptibility, so the critical slowing down takes place).

3.4.6 Model G

The isotropic antiferromagnet is also the system with the mode coupling. We have there the nonconserved order parameter (the staggered magnetization) which is a three-dimensional vector. The second field describes the local magnetization. The equations of motion can be written as

$$\dot{\mathbf{N}} = -\Gamma \frac{\delta H}{\delta \mathbf{N}} + g \mathbf{N} \times \frac{\delta H}{\delta \mathbf{m}} + \boldsymbol{\theta}, \quad (59)$$

$$\dot{\mathbf{m}} = \lambda \nabla^2 \frac{\delta H}{\delta \mathbf{m}} + g \mathbf{N} \times \frac{\delta H}{\delta \mathbf{N}} + g \mathbf{m} \times \frac{\delta H}{\delta \mathbf{m}} + \boldsymbol{\zeta}, \quad (60)$$

$$H = \frac{1}{2} \int d^d x \{ r_0 N^2 + (\nabla N)^2 + \frac{u}{2} N^4 + \chi_m^{-1} m^2 \}, \quad (61)$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\zeta}$ are white noises. There are non-vanishing Poisson brackets:

$$\{N_i, M_j\} = g \varepsilon_{ijk} N_k, \quad (62)$$

$$\{M_i, M_j\} = g\varepsilon_{ijk}M_k, \quad (63)$$

where $\mathbf{M} = \int d^d x \mathbf{m}(\mathbf{x})$ and ε_{ijk} is the antisymmetric tensor.

In this model we have also a propagating spin-wave mode and the dynamic critical exponent is the same as in model E: $z = d/2$ but some universal amplitude ratios are different [50].

3.4.7 Model J

The dynamics of this model is determined by spin precession and the conservation of the total magnetization:

$$\dot{\mathbf{S}} = \lambda \mathbf{S} \times \frac{\delta H}{\delta \mathbf{S}} + D \nabla^2 \frac{\delta H}{\delta \mathbf{S}} + \boldsymbol{\zeta},$$

where the Hamiltonian is given by (40). Below T_c we have the spin wave of the frequency $c_{sw}^{FM} k^2$ and the transport coefficient D diverges above T_c as

$$D \propto \xi_+^{(6-d-\eta)/2}, \quad (64)$$

revealing also the upper dynamic critical dimension $d_c^{dyn} = 6$ [53] which is the dimension above which the Van Hove theory applies. Below $d_c^{dyn} = 6$ the dynamic fluctuations become important and the kinetic coefficients diverge or vanish when approaching T_c . Thus in model J the dynamic critical dimension d_c^{dyn} differs from the static one d_c^{stat} , which equals four for the statics described by the Ginzburg–Landau model [3].

The dynamic critical exponent is determined only by the static exponent and the spatial dimension

$$z = \frac{1}{2} (d + 2 - \eta). \quad (65)$$

In three dimensions $z \simeq 5/2$. According to the renormalization group theory $\eta = 0$ for $d \geq 4$ so the critical exponent takes its classical value $z_{cl} = 4 - \eta$ for $d = 6$.

3.4.8 Summary of the universality classes

In Table 2 the basic information about the dynamic universality classes is given. As shown [4] by the renormalization group theory, the addition of any number of nonconserved fields (which do not change the structure of the Poisson brackets) to the models specified in Table 2 does not change the critical dynamics in that sense that it does not change the critical exponents and other universal quantities. Sometimes it may be difficult to decide which dynamic universality class the real magnetic system belongs to. Many factors matter. For example in the real magnet also phonons contribute to the spin dynamics and model A with nonconserved energy may be a better description than model C. If however, the spin-lattice relaxation rate

Table 2 Summary of the dynamic universality classes in magnets.

Model	Magnetic system	Order parameter dimension	Non-conserved fields	Conserv. fields	Poisson brackets	z
A	anisotropic magnets, uniaxial antiferromagnets	n	S	none	none	$2 + c\eta$
B	uniaxial ferromagn.	n	none	S	none	$4 - \eta$
C	anisotropic magnets, uniaxial antiferromagnets	1	S	m	none	$2 + \frac{\alpha}{\nu}$
D	uniaxial ferromagnets	n	none	S, m	none	$4 - \eta$
E	easy plane magnets $h_z = 0$	2	ψ	m	$\{\psi, m\}$	$\frac{d}{2}$
F	easy plane magnets $h_z \neq 0$	2	ψ	m	$\{\psi, m\}$	$\frac{d}{2} + \frac{\tilde{\alpha}}{2\nu}$
G	isotropic antiferromagnets	3	N	m	$\{N, m\}$	$\frac{d}{2}$
J	isotropic ferromagnets	3	none	S	$\{S, S\}$	$\frac{d+2-\eta}{2}$

is low compared to the spin exchange frequency model C which is an idealization of thermally isolated spins is a better description of the system [44]. Moreover, in real spin systems there is always anisotropy. In this case one or more terms should be added to the Hamiltonian (40) and the crossover effects from the isotropic behavior ($n = 3$) to that described by anisotropic models ($n = 2$ or $n = 1$) should be studied.

In such crossovers we generally observe the so-called effective⁸ critical exponents [54]. The universality classes which take into account the dipolar interactions are not included in Table 2.

4 Isothermal and adiabatic elastic moduli

The sound velocity exhibits sharp dip near the critical temperature. Fig. 4 presents exemplary sound velocities for rare earth metals: Gd, Tb, Dy and Ho (Luthi et al. [62]). It is well known that the static isothermal and adiabatic elastic moduli are related to the corresponding sound velocities in the zero frequency limit. Let us

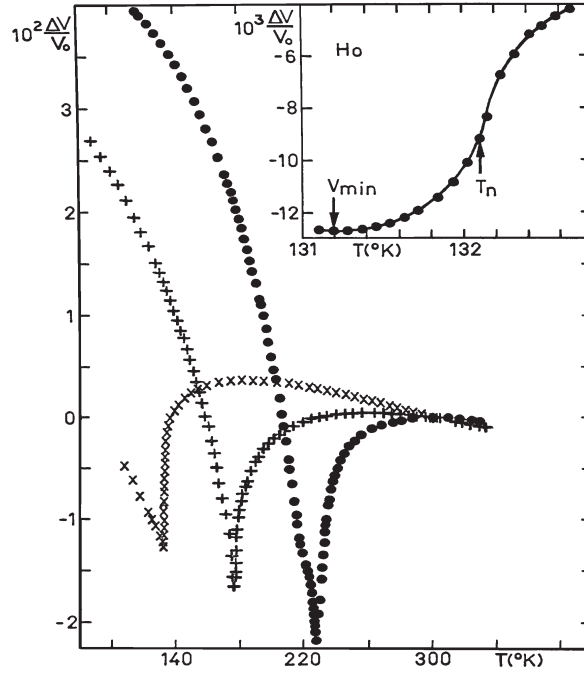


Figure 4 Temperature dependence of the sound velocity changes for rare earth metals. \times Ho; $+$ Dy; \bullet Tb. The inset shows an expanded plot near T_N in Ho (Luthi et al. [62]).

assume that the elastic medium is isotropic and nondissipative. The equation of motion for an elastic wave has a simple form [63]

$$\rho_0 \ddot{\mathbf{u}} = (C_{11} - C_{44}) \nabla (\nabla \cdot \mathbf{u}) + C_{44} \Delta \mathbf{u}, \quad (66)$$

where \mathbf{u} is a local displacement vector, ρ_0 is the mass density of the system and C_{11} and C_{44} are elastic constants, Δ denotes the Laplacian and ∇ the Nabla operator.

⁸The effective exponent depends on the reduced temperature or magnetic field. It will be discussed in Section 6.

Decomposing \mathbf{u} into a longitudinal part \mathbf{u}_L for which $\nabla \times \mathbf{u}_L = 0$ and a transverse part \mathbf{u}_T for which $(\nabla \cdot \mathbf{u}_T = 0)$, Eq. (66) splits into two independent wave equations:

$$\rho_0 \ddot{\mathbf{u}}_L = C_{11} \Delta \mathbf{u}_L, \quad \rho_0 \ddot{\mathbf{u}}_T = C_{44} \Delta \mathbf{u}_T. \quad (67)$$

The solution of each equation is a planar wave $\mathbf{u} = \mathbf{u}_0 \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t)$ with the wave vector \mathbf{k} and the frequency ω related by the dispersion relation

$$\rho_0 \omega^2 = C_{eff} k^2, \quad (68)$$

where C_{eff} is an effective elastic constant for a given mode. The phase velocity $c = \omega/k$ is equal $\sqrt{C_{11}/\rho_0}$ for the longitudinal mode and $\sqrt{C_{44}/\rho_0}$ for the transverse modes. In the general case of anisotropic crystal, the sound velocity is given by a similar formula $c = \sqrt{C_{eff}/\rho_0}$ where the effective elastic constant is a linear combination of the elastic constants C_{ij} . We can take for C_{ij} both the isothermal as well as the adiabatic elastic constants depending on the conditions of propagation of the elastic mode. The adiabatic elastic constant is greater than the isothermal one. It is evident from the last equation that the static isothermal and adiabatic elastic moduli are related to the sound velocities in the zero frequency limit, so we can find the sound velocity singularities directly from the thermodynamics. In this section we will investigate the relations between adiabatic or isothermal moduli (or equivalently the sound velocities) and some correlation functions appearing in our model.

4.1 Model

All thermodynamic quantities can be obtained from the corresponding thermodynamic potential

$$F(T, P, h) = F^0(T, P, h) - \frac{k_B T}{V} \ln Z, \quad (69)$$

where T is temperature, P - pressure and h - an external magnetic field. $F^0(T, P, h)$ is the background part which is assumed to be smooth in the temperature and the magnetic field and

$$Z = \int \mathfrak{D}[S_\alpha, e_{\alpha\beta}, q] \exp(-H) \quad (70)$$

is the sum over the states which in our case is the sum over all paths $\{S_\alpha(\mathbf{x}), e_{\alpha\beta}(\mathbf{x}), q(\mathbf{x})\}$ which can be written as a functional integral. The fields $S_\alpha(\mathbf{x}), e_{\alpha\beta}(\mathbf{x})$ and $q(\mathbf{x})$ are the complete set of slow variables in our problem [49, 61]. In addition to the order parameter $S_\alpha(\mathbf{x})$, which for the magnetic phase transition is the local magnetization (or staggered one), we have the strain tensor: $e_{\alpha\beta}(\mathbf{x}) = \frac{1}{2}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha)$, connected with the displacement field $\mathbf{u}(\mathbf{x})$ [61] and the fluctuations of entropy per mass $q(\mathbf{x})$. The functional H determines the probability distribution of equilibrium fluctuations $p \propto \exp(-H)$ and for a magnetoelastic

system of the Ising type ($n = 1$) it can be written as

$$H = H_S + H_{el} + H_q + H_{int}, \quad (71)$$

where

$$H_S = \frac{1}{2} \int d^d x \{ r_0 S^2 + (\nabla S)^2 + \frac{\tilde{u}_o}{2} S^4 \}$$

is the Landau-Ginzburg Hamiltonian for the order parameter fluctuations with $r_0 \propto T - T_c^0$, where T_c^0 is the mean field transition temperature. The elastic part

$$H_{el} = \frac{1}{2} \int d^d x \{ C_{12}^0 (\sum_{\alpha} e_{\alpha\alpha})^2 + 4C_{44}^0 \sum_{\alpha\beta} e_{\alpha\beta}^2 + 2(P - P_0) e_{\alpha\alpha} \}$$

describes the elastic energy in the harmonic approximation (the bare elastic constants $C_{\alpha\beta}^0$ contain the factor $(k_B T)^{-1}$). We assume that the crystal is isotropic so only two elastic constants appear C_{12}^0 and C_{44}^0 . P_0 is the pressure of a referential equilibrium state with respect to which we determine the strain. We assume that entropy fluctuations are Gaussian

$$H_q = \frac{1}{2C_V^0} \int d^d x q^2$$

where C_V^0 is bare specific heat. The last term in (71) describes the interactions

$$H_{int} = \int d^d x \left\{ g_0 \sum_{\alpha} e_{\alpha\alpha} S^2 + f_0 q S^2 + w_0 q \sum_{\alpha} e_{\alpha\alpha} \right\}$$

where the first term is the volume magnetostriction [6] with the coupling constant g_0 . The second term is responsible for the divergence of the specific heat and last term mimics the mentioned coupling of sound mode to energy fluctuations proposed by Kawasaki [8].

The first step in analysis of such a system is the decomposition of a given elastic configuration into a uniform part and a phonon part which is a periodic function of the position [64]

$$e_{\alpha\beta}(x) = e_{\alpha\beta}^0 + \frac{1}{\sqrt{\rho_0 V}} \sum_{\mathbf{k} \neq 0, \lambda} k_{\beta} e_{\alpha}(\mathbf{k}, \lambda) Q_{\mathbf{k}, \lambda} \exp(i\mathbf{k} \cdot \mathbf{x}), \quad (72)$$

where $Q_{\mathbf{k}, \lambda}$ is the normal coordinate⁹ of the sound mode with the polarization λ , wave vector \mathbf{k} and the polarization vector $\mathbf{e}(\mathbf{k}, \lambda)$. $e_{\alpha\beta}^0$ is the uniform deformation. For simplicity we will assume that the mass density is equal unity. In the new variables the elastic Hamiltonian takes the form

$$H_{el} = H_{el}(e_{\alpha\beta}^0) + \frac{1}{2} \sum_{\mathbf{k} \neq 0, \lambda} k^2 c_0^2(\hat{\mathbf{k}}, \lambda) |Q_{\mathbf{k}, \lambda}|^2, \quad (73)$$

⁹The factor i was incorporated into the variable Q .

where $c_0(\hat{\mathbf{k}}, \lambda)$ is the bare sound velocity for polarization λ and the vector $\hat{\mathbf{k}}$. Analogously for the interaction Hamiltonian we obtain

$$H_{int} = H_{int}(e_{\alpha\beta}^0) + \sum_{\mathbf{k}} \left\{ f_0 q_{\mathbf{k}} S_{-\mathbf{k}}^2 + \sum_{\lambda} [\mathbf{k} \cdot \mathbf{e}(\mathbf{k}, \lambda)] Q_{\mathbf{k}, \lambda} (g_0 S_{-\mathbf{k}}^2 + w_0 q_{-\mathbf{k}}) \right\}, \quad (74)$$

where $S_{\mathbf{k}}^2 = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}_1} S_{\mathbf{k}_1} S_{\mathbf{k}-\mathbf{k}_1}$ is the Fourier transform of the square of the order parameter. For the isotropic system only the longitudinal sound modes are coupled to the order parameter and the entropy fluctuations and $\mathbf{k} \cdot \mathbf{e}(\mathbf{k}, \lambda = L) = k$. As a result the transverse modes do not show any critical anomaly in this model. This is what one observes normally in experiment [62] at least for high-symmetry propagation directions. It is clearly manifested in Fig. 5 where the results for longitudinal

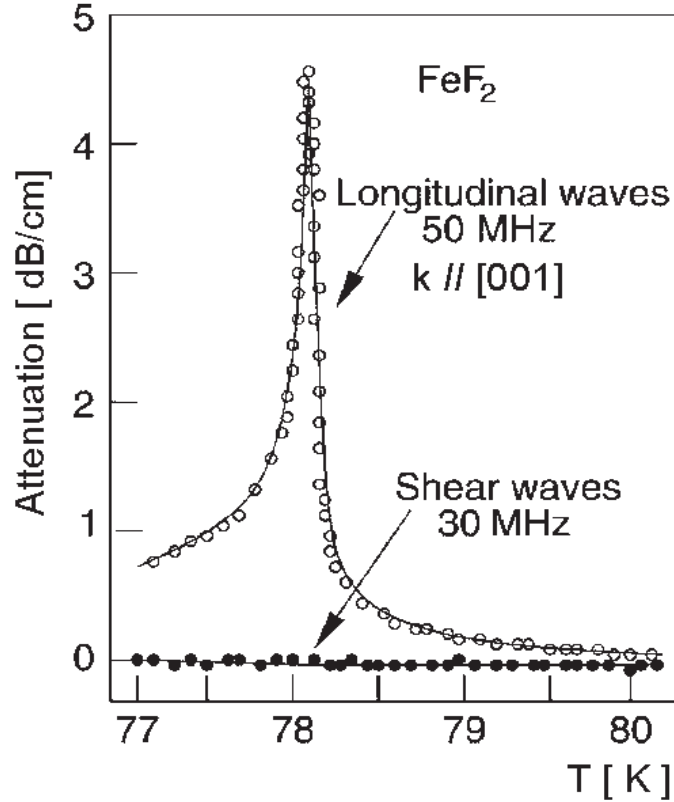


Figure 5 Ultrasonic attenuation of longitudinal and shear waves propagating along the tetragonal axis (symmetry direction) near the Neel temperature (Ikushima and Feigelson [120])

and shear sound attenuation for FeF_2 in the vicinity of Neel temperature are shown (Ikushima and Feigelson [120]).

The next step is the integration over the homogenous deformations and the

transverse modes

$$\exp[-H(S_{\mathbf{k}}, Q_{\mathbf{k}}, q_{\mathbf{k}})] = \int \mathfrak{D}[e_{\alpha\beta}^0, Q_{\mathbf{k}}, T] \exp[-H(S_{\mathbf{k}}, Q_{\mathbf{k}}, \lambda, q_{\mathbf{k}}, e_{\alpha\beta}^0)], \quad (75)$$

where index T refers to the transverse modes. From this point on we can forget about the transverse modes and homogenous deformations. The effect of homogenous deformations is only the renormalization of the parameters \tilde{u}, f_0 and C_V^0 [71]. It leads in principle to a non-analyticity (with respect to the wave vector) in the couplings $u(k), f(k), C_V(k)$ but because in magnetic systems the coupling constants g and w are usually very small we can argue that the first order phase transition expected for the Ising systems with positive specific-heat exponent [65] can be seen only extremely close to the critical point and in the experimentally accessible temperature range the transition is continuous, which is in perfect agreement with experimental observations. The problem was thoroughly investigated in the 1970s and the reasonable conclusion is to neglect the additional contributions generated by the homogenous deformations [54, 65–67]. So our effective Hamiltonian expressed by the Fourier components of the fields is

$$H = \frac{1}{2} \sum_{\mathbf{k}} \left\{ (r_0 + k^2) |S_{\mathbf{k}}|^2 + k^2 c_{0L}^2 |Q_{\mathbf{k}}|^2 + \frac{1}{C_V^0} |q_{\mathbf{k}}|^2 \right\} + H_{int}, \quad (76)$$

with

$$\begin{aligned} H_{int} = & w_0 \sum_{\mathbf{k}} k Q_{\mathbf{k}} q_{-\mathbf{k}} + \sum_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2} (f_0 q_{\mathbf{k}} + g_0 k Q_{\mathbf{k}}) S_{\mathbf{k}_1} S_{-\mathbf{k}-\mathbf{k}_1} + \\ & + \frac{u_0}{2V} \sum_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2} S_{\mathbf{k}} S_{\mathbf{k}_1} S_{\mathbf{k}_2} S_{-\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} \end{aligned} \quad (77)$$

where $c_{0L} = \sqrt{C_{11}^0}$ is the bare sound velocity of the longitudinal modes and the normal coordinate Q refers only to the longitudinal modes.

4.2 Isothermal sound velocity

The isothermal elastic constant or equivalently the isothermal sound velocity of the longitudinal modes can be determined from the corresponding correlation function

$$\langle Q_{\mathbf{k}} Q_{-\mathbf{k}} \rangle = \frac{1}{c_{is}^2 k^2}, \quad (78)$$

where $k \neq 0$ is assumed. It is easy to calculate this correlation function by a simple separation of variables in the Hamiltonian

$$q_{\mathbf{k}} = q'_{\mathbf{k}} - w_0 k C_V^0 Q_{\mathbf{k}} - f_0 C_V^0 (S^2)_{\mathbf{k}},$$

$$Q_k = Q'_k - (g_0 - w_0 f_0 C_V^0) c_{0r}^{-2} k^{-1} (S^2)_k, \quad (79)$$

where $c_{0r}^2 = c_0^2(1 - r^2)$ and $r^2 = w_0^2 C_V^0 c_0^{-2}$. In these new variables the Hamiltonian takes a form

$$H = \frac{1}{2} \sum_{\mathbf{k}} \left\{ k^2 c_{0r}^2 |Q'_k|^2 + \frac{1}{C_V^0} |q'_k|^2 \right\} + H_{eff}^T(S), \quad (80)$$

with the effective spin Hamiltonian of the Landau-Ginzburg form

$$H_{eff}^T(S) = \frac{1}{2} \left\{ \sum_{\mathbf{k}} (r_0 + k^2) |S_{\mathbf{k}}|^2 + \frac{u_0^T}{V} \sum_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2} S_{\mathbf{k}} S_{\mathbf{k}_1} S_{\mathbf{k}_2} S_{-\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} \right\}, \quad (81)$$

where $u_0^T = u_0 - v_T^{ph} - v_T^q$, $v_T^{ph} = \bar{g}^2 c_{0r}^{-2}$, $\bar{g}_0 = (g_0 - w_0 f_0 C_V^0)$ and $v_T^q = f_0^2 C_V^0$. The non-analyticity mentioned earlier is neglected here.

With such Hamiltonian we can write

$$\langle Q_{\mathbf{k}} Q_{-\mathbf{k}} \rangle = \langle Q'_k Q'_{-k} \rangle + v_T^{ph} c_{0r}^{-2} k^{-2} \langle S_{\mathbf{k}}^2 S_{-\mathbf{k}}^2 \rangle_{H_{eff}^T}. \quad (82)$$

or

$$c_{is}^2 = \frac{c_0^2(1 - r^2)}{1 + v_T^{ph} \langle S_{\mathbf{k}}^2 S_{-\mathbf{k}}^2 \rangle_{H_{eff}^T}}. \quad (83)$$

The index H_{eff}^T at the second average in Eq. (82) means that this average does not contain the elastic and entropic variables. The scaling behavior of $\langle S_{\mathbf{k}}^2 S_{-\mathbf{k}}^2 \rangle_{H_{eff}^T}$ is well known [12, 35, 54]. This function behaves as the specific-heat

$$\langle S_{\mathbf{k}}^2 S_{-\mathbf{k}}^2 \rangle_{H_{eff}^T(S)} \propto A t^{-\alpha} \Phi(k\xi) - B, \quad (84)$$

where A and B are the some nonuniversal constants and Φ is a scaling function (usually we assume that $\Phi(0) = 1$); ξ is the correlation length and t is the reduced temperature. In ultrasonic experiments the wavelength is much greater than the correlation length so we can take $k\xi = 0$. The specific-heat exponent α is positive for the Ising universality class and equal to about 0.11 so in this case the denominator in Eq. (83) tends to infinity and the isothermal sound velocity must go to zero

$$c_{is} \propto t^{\alpha/2} \searrow 0. \quad (85)$$

as we approach the critical temperature. We can say that the isothermal sound mode is softening at the critical point of Ising type systems. Otherwise, for the Heisenberg universality class $n = 3$, we have $\alpha < 0$ and the isothermal sound velocity stays finite at T_c .

The experimental observation of the relation (85) is extremely difficult for the two reasons. The first is that the critical exponent of the sound velocity, $\alpha/2$, is very small of an order of 0.05 and we must be very close to the critical temperature

in order to observe a significant changes in the sound velocity. The second is that the coupling constant $v_T^{ph} = (g_0 - w_0 f_0 C_V^0)^2 c_{0r}^{-2}$, which precedes the singular term in the denominator of Eq.(83), depends on the coupling constants g_0 and w_0 which are usually very small in magnets (contrary to e.g. the structural phase transitions [11]). Because v_T^{ph} is a small quantity it is reasonable to expand the expression (83) obtaining

$$c_{is}^2 \simeq c_{0r}^2 - A_T t^{-\alpha}. \quad (86)$$

This expression is a very good approximation to the experimentally observed measurements of isothermal sound velocity [7, 62, 116] (isothermal elastic moduli C_{11}^T). The expressions (85) and (86) were given independently by Dengler and Schwabl [69] and by the author [54, 70].

4.3 Adiabatic sound velocity

Another static quantity of interest is the adiabatic elastic constant. In our case it is the modulus C_{11}^{ad} or the related quantity c_{ad} . From the theory of fluctuations [29] we know that the adiabatic compressibility is given by the correlation function of pressure fluctuations. The pressure fluctuations are defined as the quantity which is orthogonal to the entropy fluctuations. The orthogonality is understood as vanishing of the corresponding correlation function $\langle \mathcal{P}_k q_{-k} \rangle = 0$ (where \mathcal{P}_k is a fluctuation of pressure). Looking at the Hamiltonian (76) we see that the variables Q_k and the entropy fluctuations q_k are not orthogonal as a result of the coupling between these quantities in H_{int} . In order to get a quantity which is orthogonal to q_k we have to perform the Schmidt orthogonalization procedure (choosing as the first variable the entropy fluctuations)

$$\mathcal{P}_k = Q_k - \langle Q_k q_{-k} \rangle \frac{q_k}{\langle q_k q_{-k} \rangle}. \quad (87)$$

or in other words we must subtract from the acoustic variable Q_k a part linear in q_k . Immediately we get that

$$\langle \mathcal{P}_k \mathcal{P}_{-k} \rangle = \frac{1}{c_{ad}^2 k^2} = \langle Q_k Q_{-k} \rangle - \frac{\langle Q_k q_{-k} \rangle^2}{\langle q_k q_{-k} \rangle}. \quad (88)$$

By a shift of variables Q_k and q_k we can separate these variables in the Hamiltonian obtaining

$$H = \frac{1}{2} \sum_{\mathbf{k}} \left\{ k^2 c_0^2 |Q''_{\mathbf{k}}|^2 + \frac{1}{\bar{C}_V^0} |q''_{\mathbf{k}}|^2 \right\} + H_{eff}^{ad}(S),$$

where q'' and $Q''_{\mathbf{k}}$ are the shifted variables, $\bar{C}_V^0 = C_V^0 (1 - r^2)^{-1}$ and

$$H_{eff}^{ad}(S) = \frac{1}{2} \left\{ \sum_{\mathbf{k}} (r_0 + k^2) |S_{\mathbf{k}}|^2 + \frac{u_0^{ad}}{V} \sum_{\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2} S_{\mathbf{k}} S_{\mathbf{k}_1} S_{\mathbf{k}_2} S_{-\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} \right\}, \quad (89)$$

is effective adiabatic Hamiltonian with $u_0^{ad} = \tilde{u}_0 - v_{ad}^{ph} - v_{ad}^q$, $v_{ad}^{ph} = g_0^2 c_0^{-2}$ and $v_{ad}^q = \bar{f}_0^2 \bar{C}_V^0$, where $\bar{f}_0 = (f_0 - w_0 g_0 c_0^{-2})$.

Now we can find the correlation functions in (88). A simple algebra shows that

$$c_{ad}^2 = c_0^2 \frac{1 + v_{ad}^q \langle S_{\mathbf{k}}^2 S_{-\mathbf{k}}^2 \rangle_{H_{eff}^{ad}}}{1 + v_+^{ad} \langle S_{\mathbf{k}}^2 S_{-\mathbf{k}}^2 \rangle_{H_{eff}^{ad}}}, \quad (90)$$

where $v_+^{ad} = v_{ad}^q + v_{ad}^{ph}$.

Straightforward calculations show that

$$v_{ad}^{ph} + v_{ad}^q = v_T^{ph} + v_T^q \equiv v_+, \quad (91)$$

therefore $u_0^T = u_0^{ad} \equiv u_0$ and the effective spin Hamiltonians H_{eff}^{ad} and H_{eff}^T are identical. As a consequence, the correlation function $\langle S_{\mathbf{k}}^2 S_{-\mathbf{k}}^2 \rangle_{H_{eff}^{ad}}$ is identical with the function $\langle S_{\mathbf{k}}^2 S_{-\mathbf{k}}^2 \rangle_{H_{eff}^T}$ which shows a specific-heat singularity as was discussed earlier (84). The expression (90) can be given in more transparent form

$$c_{ad}^2 = c_0^2 \left(1 - \frac{v_{ad}^{ph}}{v_+}\right) + c_0^2 \frac{v_{ad}^{ph}}{v_+} \frac{1}{1 + v_+ \langle S_{\mathbf{k}}^2 S_{-\mathbf{k}}^2 \rangle}, \quad (92)$$

where we have omitted the Hamiltonian index. It is seen from this equation that there is a constant term and a correction which tends to zero at T_c

$$c_{ad}^2 = c_0^2 \left(1 - \frac{v_{ad}^{ph}}{v_+}\right) + A_{ad} t^\alpha. \quad (93)$$

The critical amplitude of the singular term $A_{ad} = c_0^2 \frac{v_{ad}^{ph}}{v_+} A^{-1}$, where A is the critical amplitude of the specific heat from Eq. (84), is very small for magnets because $v_{ad}^{ph} \ll v_{ad}^q \simeq v_+$, which explains small sound velocity changes near the magnetic phase transition [6].

It should be noted that this result for the adiabatic sound velocity obtained by the author [71] differs from that obtained by Drossel and Schwabl [72] who obtained for the adiabatic sound velocity a result similar to Eq. (83) for the isothermal velocity (only v_T^{ph} should be replaced by v_{ad}^{ph} in this equation). The reason for this discrepancy is a different choice of the pressure variable. Drossel and Schwabl took for the pressure a variable which is orthogonal to entropy only in the Gaussian approximation and it leads to non-vanishing correlation function of pressure-entropy. The correlation function of such „pressure” containing a non-zero entropy component is similar to that obtained for isothermal sound.

On the other hand, the expression (92) shows a close analogy to the adiabatic sound velocity in liquid He^4 , obtained by Pankert and Dohm [73, 74]. A similar result was also obtained by Folk and Moser [75] for binary liquids.

5 Phenomenological theory of sound attenuation and dispersion

The sound wave propagation through the medium disturbs the existing balance as a consequence of the temperature (or pressure) changes in a wave of successive compressions and dilatations. Let the molecular equilibrium of the system be described by a parameter ψ called the reaction coordinate (or the degree of advance) [77] which can correspond to the extent of the chemical reaction or to the temperature of some internal degrees of freedom. For gases such internal degrees of freedom are the rotational or vibrational modes of many-atomic molecules. The parameter ψ does not follow the temperature and pressure changes and this delay is described by the relaxation equation

$$\dot{\psi} = -\frac{\psi - \bar{\psi}}{\tau}, \quad (94)$$

where τ is the relaxation time characterizing the rate at which the coordinate ψ approaches the equilibrium value $\bar{\psi}(p, T)$ determined by temporary pressure and temperature in the ultrasonic wave. The lag between the oscillations of the temperature and pressure, and the excitation of a given mode leads to the dynamic hysteresis, to dissipation of energy and dispersion of the sound wave.

The Eq. (94) is the simplest equation of the irreversible thermodynamics. Historically the method of irreversible thermodynamics was first applied to the sound dynamics by Herzfeld and Rice [78] in 1928. They postulated that the medium through which the sound waves propagates is characterized by two temperatures: one of them is called the external temperature and determines the energy distribution of translational degrees of freedom of molecules and the other one is the internal temperature connected with the energy distribution in internal degrees of freedom (e.g. the rotational or vibrational modes in a gas of many-atom molecules). Herzfeld and Rice assumed that the rate at which the internal temperature changes is proportional to the difference between these temperatures and the coefficient of proportionality is the inverse of the relaxation time. They noticed that each process in which the energy is transferred with some delay from translational motion (the sound wave) to other (internal) degrees of freedom, is connected with a dissipation of acoustic energy or in other words to the attenuation of the sound wave. As a result we obtain a complex effective elastic constant (and sound velocity) in the dispersion relation \hat{C}_{eff} , where for the single relaxational process we obtain

$$\hat{C}_{eff} = C_{eff}^{\infty} - \frac{\Delta'}{1 - i\omega\tau}, \quad (95)$$

The constant C_{eff}^{∞} is the high frequency limit of (95), where the reaction coordinate does not follow the stress changes. The symbol τ stands for the relaxation time and Δ' is a parameter describing the coupling of the sound to the relaxing variable known as the relaxation strength. In the ultrasonic experiments the sound frequency

is a real-valued quantity and for the propagation wave vector we assume a complex value $k = k_r + i\alpha$, where α is the sound attenuation coefficient. A single relaxational process results in a frequency dependent sound velocity

$$c^2(\omega) = \frac{\omega^2}{k_r^2} = c_\infty^2 - \frac{\Delta}{1 + \omega^2\tau^2} = c^2(0) + \frac{\Delta\omega^2\tau^2}{1 + \omega^2\tau^2} \quad (96)$$

and the sound attenuation

$$\alpha(\omega) = \frac{\Delta}{2c_\infty^3} \frac{\omega^2\tau}{1 + \omega^2\tau^2} = B \frac{\omega^2\tau}{1 + \omega^2\tau^2}, \quad (97)$$

where $c_\infty = \sqrt{C_{eff}^\infty/\rho_0}$ is the high-frequency limit of (96) and $\Delta = \Delta'/\rho_0$ and $B = \Delta/2c_\infty^3$. It was assumed that the frequency dependence of the sound velocity is weak.

Figure (6) shows these dependencies in the logarithmic scale for frequency. It should be noted that the sound velocity increases from the value $c(0)$ for $\omega\tau = 0$ to the value c_∞ for $\omega\tau \rightarrow \infty$. The velocity $c(0)$ corresponds to the situation when the temperature and the pressure of the sound wave change so slowly that system remains at the thermodynamic equilibrium (the reaction coordinate has the same phase as the pressure applied). The sound attenuation coefficient $\alpha(\omega)$ increases from zero for low frequencies to the „saturation” value B/τ for very high frequencies. In the low-frequency regime the attenuation coefficient is proportional to the square of frequency and to the relaxation time: $\alpha(\omega) = B\omega^2\tau$. For many relaxational processes with the relaxation times τ_j and the relaxation strengths Δ_j the equations (96) and (97) change into

$$c^2(\omega) = c_\infty^2 - \sum_j \frac{\Delta_j}{1 + \omega^2\tau_j^2} \quad (98)$$

and

$$\alpha(\omega) = \omega^2 \sum_j \frac{B_j\tau_j}{1 + \omega^2\tau_j^2}. \quad (99)$$

In Fig. 7 the dependences described by Eqs. (98) and (99) are shown for two relaxational processes with the relaxation times τ_1 and τ_2 . In the classical theory, the relaxational processes do not interact and if the relaxation times are well separated from each other one can see something like a staircase (with slightly rounded stairs). The height of the j -th stair for the sound velocity is Δ_j , and $\Delta_j/2c_\infty^3\tau_j$ for the sound attenuation.

At the magnetic phase transition we have a quasi continuum¹⁰ of the relaxation times. They are attributed to the internal degrees of freedom which are the Fourier

¹⁰The index j at τ_j in the case of phase transitions denotes the wave vector which is a quasi continuous variable for the finite volume of the crystal.

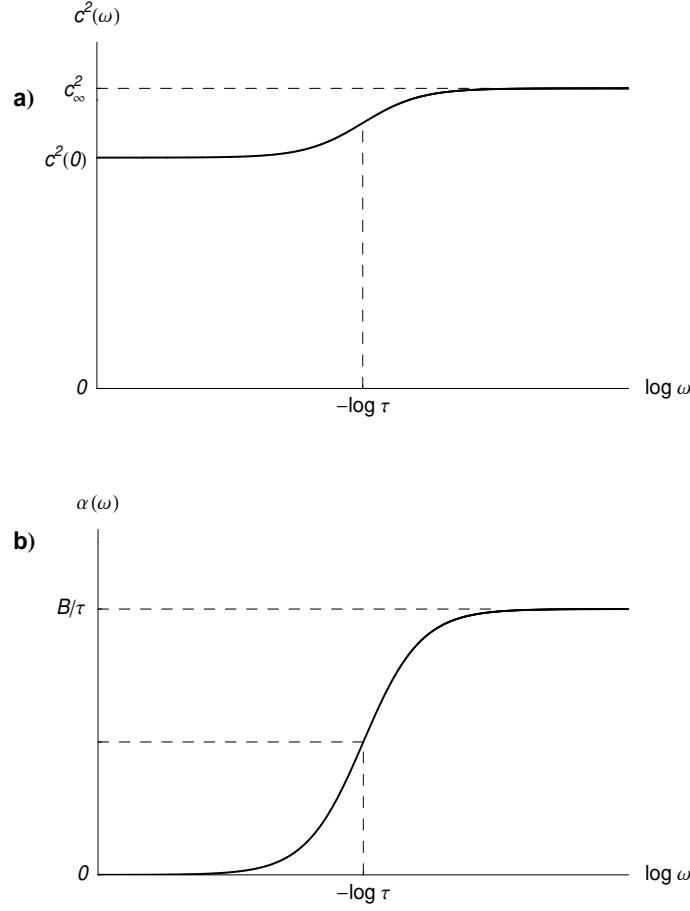


Figure 6 The sound velocity (a) and the attenuation coefficient (b) for a single relaxational process; $c^2(0) = c_\infty^2 - \Delta$, $\alpha(\infty) = B/\tau$.

components of the order parameter¹¹. Their relaxation times are typically i.e. far from the critical point, very short of the order of 10^{-12} s, so their inverses are much lower than the ultrasound frequencies used in the study of the acoustic properties of solids. They are typically in the range from 1 MHz to 1000 MHz. Due to the critical slowing down the relaxation times are getting longer and longer and some of them may become comparable with the period of the ultrasonic wave. The longest of the relaxation times may diverge even to infinity. It is illustrated in Fig. (7). Also the relaxation strengths increase when approaching the critical point. It is easily seen if we consider the contribution to the acoustic linear response function in the Gaussian¹² approximation for the order parameter fluctuations. In this approxima-

¹¹To be precise they are usually the Fourier components of the square of the order parameter field for the magnetostrictive coupling.

¹²The Gaussian approximation assumes that the Fourier components of the order parameter do

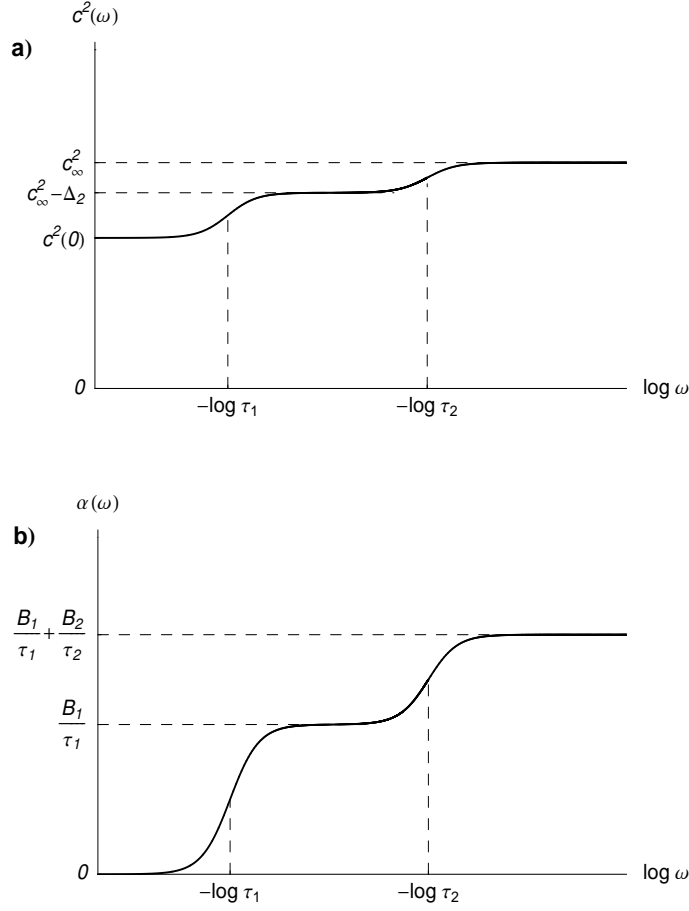


Figure 7 The sound velocity (a) and the attenuation coefficient (b) for two relaxational processes with the relaxation times τ_1 and τ_2 .

tion the complex sound velocity has a simple form [79]

$$\hat{c}^2(\omega) = c_\infty^2 - g^2 \int \frac{d^3 p}{(\xi^{-2} + p^2)^2 [1 - i\omega\tau(p)]}, \quad (100)$$

where $\tau(p) = [2\Gamma(\xi^{-2} + p^2)]^{-1}$ is a relaxation time of the product of two Fourier components of the order parameter $S_{\mathbf{p}}S_{-\mathbf{p}}$, and the integration is over the wave vectors inside a sphere $|\mathbf{p}| \leq \Lambda$. The coefficients g^2 and Γ are some constants and Λ is a cutoff wave-vector. $\xi^{-2} \propto T - T_c$ is the correlation length in the Gaussian approximation. It is evident that the summation over j in Eqs. (98) and (99) is replaced by the integration over the wave vector \mathbf{p} in Eq. (100). The relaxation strength Δ_j corresponds to $(\xi^{-2} + p^2)^{-2}$ which is a contribution of the mode $S_{\mathbf{p}}$ to the

not interact and there is only the first term in the spin Hamiltonian (89).

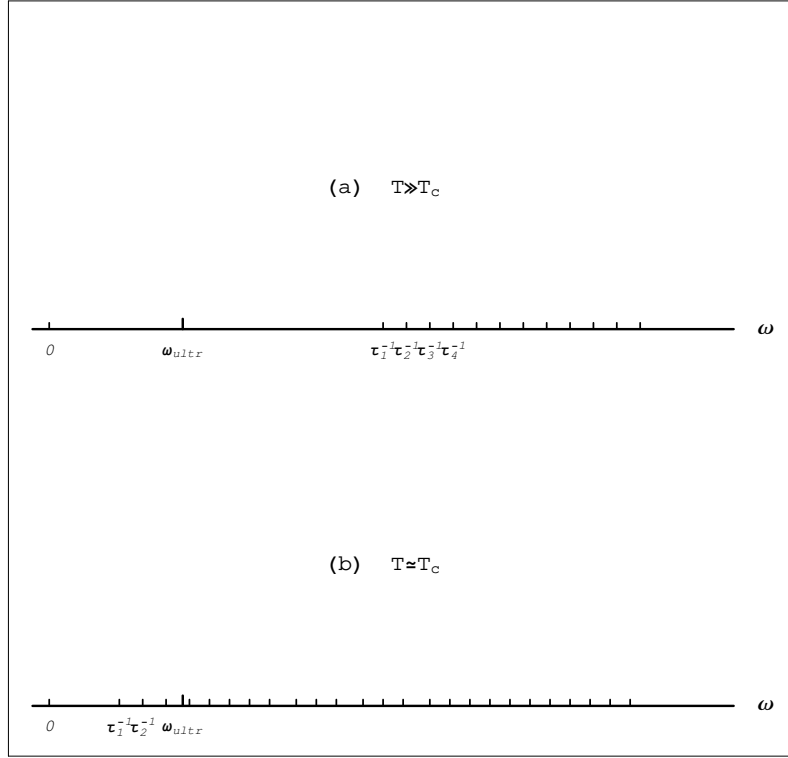


Figure 8 The effect of the critical slowing down on the relaxation times of the order parameter.

specific heat in the Gaussian approximation [34]. The correspondence $\tau_j \longleftrightarrow \tau(p)$ is also obvious so Eq. (100) is a continuous version of Eqs. (98) and (99) written in the complex form. From the expression for $\tau(p)$ we find that the relaxation times for small wave vectors will diverge roughly as ξ^2 when $T \rightarrow T_c$. It is a quite strong divergence which is the principal cause of the divergence of the attenuation coefficient in the so-called hydrodynamic regime i.e. for $\omega\tau(p) \ll 1$. In the hydrodynamic regime we have $\alpha(\omega) \propto \int_p \Delta(p)\tau(p)$ and the sound attenuation strongly increases when we come near the critical temperature. It is visualized in Fig. 9. Very far from the critical temperature all modes are in the hydrodynamic regime, as illustrated in Fig. 9a and the attenuation is very small. As we approach the critical temperature, the attenuation starts to increase due to an increase in $\tau(p)$ (the main factor) and $\Delta(p)$. Only when the slowest modes' relaxation times become longer than the period of the sound wave or in other words the inverses of the relaxation times „get across” to the left of the ultrasonic frequency, only then the slowest modes with $\omega_{ultr}\tau(p) > 1$ get into the „saturated” state and the curve of the sound attenuation coefficient starts to level off. It is illustrated in Fig. 9b where for the sake of the figure transparency, only the frequencies close to the ultrasonic frequency are shown. Not all the relaxational modes could get across ω_{ultr} , because the relaxation times $\tau(p)$

depend not only on temperature (through ξ^2) but also on the square of the wave vector which does not change for $T \rightarrow T_c$. For $p^2 > \xi^{-2}$ the relaxation time is not very sensitive to temperature changes, similarly as the relaxation strength $\Delta(p)$. The crossover of the slowest modes (and simultaneously those giving the largest contribution to $\alpha(\omega, T)$) into the saturated state and the smaller sensibility (to the temperature change) of the other modes leads ultimately to the levelling off the sound attenuation coefficient.

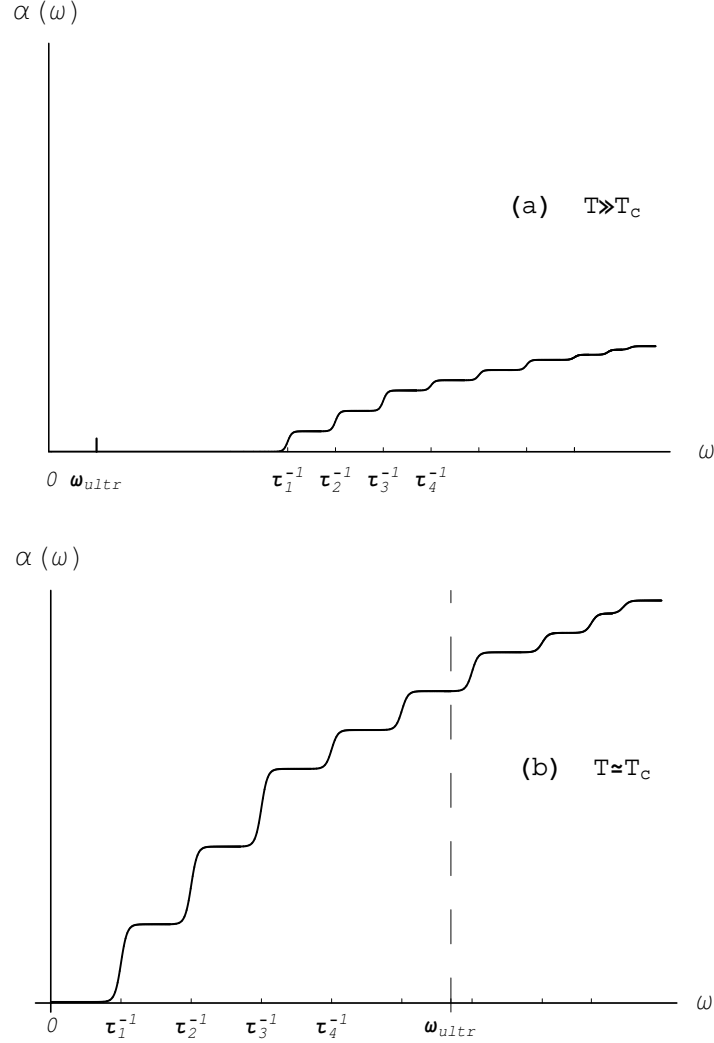


Figure 9 It demonstrates in intuitive way how the sound attenuation changes when approaching the critical temperature. In Fig. 9(b) the frequency was rescaled in order to show more precisely the situation around the ultrasonic frequency.

For the sound velocity in the hydrodynamic regime we have $c^2(\omega) = c_\infty^2 - \int_p \Delta(p)$.

Thus we observe a weaker singularity as the integral $\int_p \Delta(p)$ is less singular than the corresponding integral $\int_p \Delta(p)\tau(p)$ for the sound attenuation coefficient. It is well known that it is a singularity of the specific heat type. As shown in Fig. 10, some of the relaxational modes get into the saturated state as the critical temperature is approached. At the first sight it would seem that the sound velocity should increase (for a given frequency) as we get closer T_c because more and more „stairs” (contributions) from saturated modes are added to $c^2(0)$. It is however, not so as

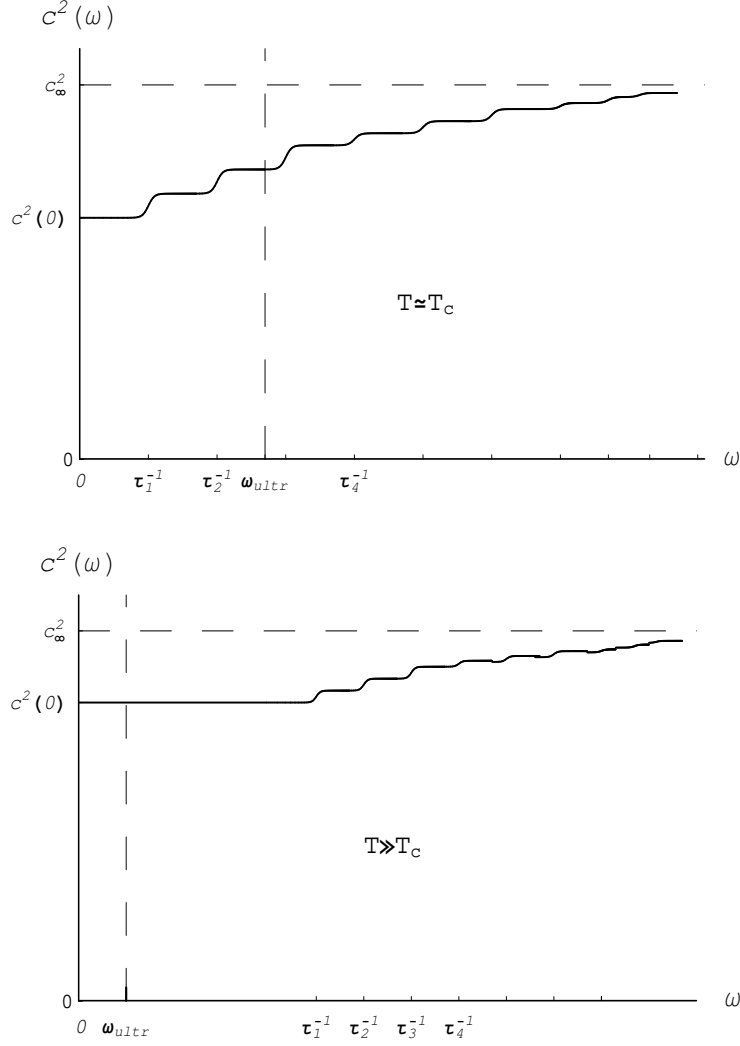


Figure 10 The sound velocity changes when approaching the critical temperature.

equation (96) only describes the way the sound velocity changes relatively to $c^2(0)$ or c_∞^2 . It gives no clue about the temperature dependence of these parameters near T_c . We know from the previous section that the sound velocity in the limit of zero

frequency is a thermodynamic quantity which is singular near the critical point so it is not a good reference level to measure the critical sound velocity changes. Such a good reference level turns out to be c_∞^2 . It is a quantity corresponding to the infinite frequency so it is measured far from the critical point which corresponds to $\xi^{-1} = k = \omega = 0$ as we know from the phase transition theory. Such uncritical quantity does not depend strongly on temperature and can to a good approximation be taken as a constant near T_c . Having decided which quantity c_∞^2 or $c^2(0)$ is a constant near the phase transition we see that the sound velocity can only decrease with $T \rightarrow T_c$. Of course this decrease is eventually stopped by the crossover of the slowest modes in the saturated state like for the sound attenuation. The higher the sound frequency the larger is the sound velocity as more modes will get across this frequency. It should be also noted that the difference $c_\infty^2 - c^2(0)$ increases as $T \rightarrow T_c$ because the sum of the relaxational strengths increases as the specific heat.

The presented here phenomenological theory of sound attenuation and dispersion has only a qualitative character as it does not take into account the interaction between the relaxational processes (modes). It is well known [3, 4] that these very interactions between the modes are the origin of the nontrivial singularities encountered in many physical quantities near the critical point. We need a more elaborated theory of the dynamic phenomena to obtain a precise form of the sound attenuation coefficient and dispersion. It will be presented in the next section.

6 Model of critical sound propagation

In order to build a detailed theory of sound propagation near the critical point we need the equations of motion. We use the phenomenological hydrodynamic description in terms of nonlinear Langevin equations for slow variables described in Sect. 3. Our choice of the slow variables depends mainly on the nature of the physical system we want to describe (e.g. whether it is a ferro- or antiferromagnet, isotropic or anisotropic system, etc.) and on the quantity of interest. It depends additionally on the frequency range (the time scale used to investigate the system) and in some cases we should add also some fast variables important in the time scale considered to the system of slow variables. In practice in each universality class we would need a different system of equations. Fortunately, some of the aspects of critical attenuation and dispersion presented in this chapter can be easily generalized over other magnetic dynamic universality classes.

6.1 Anisotropic magnet with the spin-lattice relaxation

As mentioned in the Introduction, in magnetic materials which are also insulators we observe a weak divergence of the sound attenuation coefficient. The critical exponent characterizing this initial increase (in the hydrodynamic region) is usually very small of an order of 0.2. It was postulated by Kawasaki [8] that the origin of

such a weak divergence (or even the lack of any singularity) is connected with the linear coupling of the sound mode with the spin energy density which decays through spin-lattice relaxation process. With such assumptions the sound attenuation can be written as a quantity proportional to the square of the specific heat

$$\alpha(\omega, T) \propto \omega^2 C^2 \propto \omega^2 t^{-2\alpha}.$$

The other group of magnets, these which are conductors of the electric current such as for example the rare earth metals, we observe a much stronger increase in $\alpha(\omega, T)$ for $T \rightarrow T_c$, with the sound attenuation critical exponent $\rho_s > 1$ [80–82]. It was recognized [62] that for such a strong singularity another coupling which involves one acoustic mode and two fluctuations of the order parameter is responsible. As we noted in the Introduction, both types of the coupling arise from the dependence of the exchange integral on the distance between the interacting magnetic ions (we call this general interaction a volume magnetostriction) and both are present in all magnets. Only when the interactions with the next nearest neighbors are neglected and only for some sound propagation directions the other part of the coupling typical of metals can be neglected. Therefore, a general model which involves both couplings was proposed [9]. The Hamiltonian comprising all important interactions is of the form Eq. (71) and the dynamics is given by the Langevin equations

$$\dot{S}_{\mathbf{k}} = -\Gamma \frac{\delta H}{\delta S_{-\mathbf{k}}} + \zeta_{\mathbf{k}}, \quad (101)$$

$$\ddot{Q}_{\mathbf{k}} = -\frac{\delta H}{\delta Q_{-\mathbf{k}}} - \Theta k^2 \dot{Q}_{\mathbf{k}} + \xi_{\mathbf{k}}, \quad (102)$$

$$\dot{q}_{\mathbf{k}} = -\gamma \frac{\delta H}{\delta q_{-\mathbf{k}}} + \varphi_{\mathbf{k}}, \quad (103)$$

where $S_{\mathbf{k}}$, $Q_{\mathbf{k}}$ and $q_{\mathbf{k}}$ denote similarly as in Sect. 4 the Fourier components of the order parameter, longitudinal acoustic mode and entropy. $\zeta_{\mathbf{k}}$, $\xi_{\mathbf{k}}$ and $\varphi_{\mathbf{k}}$ are white Gaussian noises simulating the thermal agitation forces. They have zero means and their variances are connected with the bare damping terms by relations

$$\langle \zeta_{\mathbf{k}}(t) \zeta_{-\mathbf{k}}(t') \rangle = 2\Gamma \delta(t - t'), \quad (104)$$

$$\langle \xi_{\mathbf{k}}(t) \xi_{-\mathbf{k}}(t') \rangle = 2\Theta k^2 \delta(t - t'), \quad (105)$$

$$\langle \varphi_{\mathbf{k}}(t) \varphi_{-\mathbf{k}}(t') \rangle = 2\gamma \delta(t - t'), \quad (106)$$

where now t denotes a time not the reduced temperature¹³. Equation (101) describes the relaxation of the order parameter fluctuation which is a non-conserved quantity

¹³Unfortunately, it is well established manner to denote time and the reduced temperature by the same symbol t but it is easy to find out from the context whether t refers to time or to the temperature. We will need the first meaning only in Sect. 6.1 for the definition of the dynamic model and in Sect. 6.2 in formulation of the functional representation of the equations of motion.

and this equation corresponds to the model A dynamic universality class [4, 44, 45]. This class comprises e.g. the anisotropic magnets. Eq. (102) describes the longitudinal sound mode and the damping coefficient Θk^2 is responsible for all other interactions of the sound mode except those with the long-wavelength fluctuations of the order parameter. The last equation (103) characterizes the non-conserved quantity $q(\mathbf{x})$ which decays by the spin-lattice relaxation. The rate of the spin-lattice relaxation is γ/C_V^0 . In this equation the heat conduction process is neglected. It describes an idealized situation when the lattice is characterized by infinite thermal capacity or infinite thermal conductivity. The problem of the heat conduction will be touched in Sect. 6.2 It can be shown that Eqs. (102) and (103) do not change the dynamic critical exponent z which corresponds to the model A universality class $z = 2 + c\eta$.

6.1.1 Functional form of the equations of motion

There are a few ways of constructing the perturbation expansion for the dynamic model [84–87]. One of the most frequently used is the path-integral formalism called also the functional form of the equations of motion which uses the dynamic functional known also as Lagrangian or Onsager-Machlup functional [84]. This functional determines the probability of the whole trajectory $\{S_{\mathbf{k}}, Q_{\mathbf{k}}, q_{\mathbf{k}}\}_{t \in [-t_0, t_0]}$ in some time interval. Let us forget for a moment about the fields Q and q and the indexes \mathbf{k} . The stochastic process e.g. $\zeta(t)$ is characterized by the set of the probability densities

$$\begin{aligned} P_1(\zeta_1, t_1), \\ P_2(\zeta_1, t_1; \zeta_2, t_2), \\ \dots\dots\dots P_n(\zeta_1, t_1; \zeta_2, t_2, \dots, \zeta_n, t_n), \end{aligned}$$

with P_i having the sense of the probability density to find the system in the state ζ_1 at the time moment t_1 , ζ_2 at time t_2 etc. The probability density $P_\infty[\zeta]$ for the whole trajectory $\{\zeta(t)\}_{t \in [-t_0, t_0]}$ is obtained in the limit where the differences between the successive times go to zero. It is a function of the infinite number of variables $\zeta(t)$, for all times in the interval $[-t_0, t_0]$. For the Gaussian noise it has the form

$$\begin{aligned} P_\infty[\zeta] \mathcal{D}\zeta &= \frac{1}{Z} \exp \left\{ - \int_{-t_0}^{t_0} dt \left[\frac{\zeta^2(t)}{4\Gamma} \right] \right\} \mathcal{D}\zeta \equiv \\ &\equiv \frac{1}{Z} \lim_{\Delta t \rightarrow 0} \exp \left\{ - \sum_{\sigma} \Delta t \left[\frac{\zeta_{\sigma}^2}{4\Gamma} \right] \right\} \prod_{\sigma=0}^N \left(\frac{\Delta t}{4\Gamma} \right)^{\frac{1}{2}} d\zeta_{\sigma}, \end{aligned} \tag{107}$$

where the time interval was divided into $N = 2t_0/\Delta t$ sub-intervals, and Z is a normalization factor. The second part of this expression defines the functional differential $\mathcal{D}\zeta$. We can interpret $P_\infty[\zeta]\mathcal{D}\zeta$ as the probability density of a given trajectory passing through the infinite set of time windows at times $t_\sigma = -t_0 + \sigma\Delta t$ of the width $d\zeta_\sigma$.

If we want to determine the time-dependent correlation function which depends on S not on the noise, it is favorable to have the functional $P_\infty[S]$, instead of $P_\infty[\zeta]$. The transformation which allow us to change the variables in $P_\infty[S]$ is the equation of motion (101) which in the discrete form is given by

$$\frac{1}{\Delta t} (S_\sigma - S_{\sigma-1}) + \Gamma \frac{\delta H}{\delta S_\sigma} = \zeta_\sigma. \quad (108)$$

By eliminating ζ_σ we obtain

$$P_\infty[S] = \frac{1}{Z} \exp \left\{ - \int_{-t_0}^{t_0} dt \left[\frac{\left| \dot{S} + \Gamma \frac{\delta H}{\delta S} \right|^2}{4\Gamma} \right] \right\}, \quad (109)$$

where the Jacobian of the transformation, $\frac{1}{2}\Gamma \frac{\delta^2 H}{\delta S^2}$, was omitted as it is compensated in the perturbation expansion by some acausal terms which are also usually omitted [84, 85, 87]. The exponent in (109) is known as the Onsager-Machlup functional. Its form is not very useful for the two reasons. The first is that it involves the interactions of the high order and the second is that for the conserved quantities we have $\Gamma \rightarrow Dk^2$ so the functional becomes infinite for $k \rightarrow 0$. In order to avoid such problems we perform a Gaussian transformation

$$P_\infty[S] = \frac{1}{Z} \int \mathcal{D}[i\tilde{S}] \exp \mathcal{J}\{\tilde{S}, S\}, \quad (110)$$

with

$$\mathcal{J}\{\tilde{S}, S\} = \int_{-t_0}^{t_0} dt \mathcal{L}(\tilde{S}, S) = \int_{-t_0}^{t_0} dt \left[\tilde{S} \Gamma \dot{S} - \tilde{S} \left(\dot{S} + \Gamma \frac{\delta H}{\delta S} \right) \right], \quad (111)$$

where a new functional $\mathcal{L}(\tilde{S}, S)$ known as the dynamic functional or Lagrangian (by analogy to the quantum field theory) is introduced. It is a function of an artificial imaginary field \tilde{S} known as the response field because an additional term related to the external magnetic field hS in the Hamiltonian, gives the contribution $\Gamma h \tilde{S}$ in the Lagrangian and the linear response function which is defined as the derivative $\langle S(t) \rangle$ over the external field $h(t')$ is given by

$$\frac{\delta \langle S(t) \rangle}{\delta h(t')} = \Gamma \langle S(t) \tilde{S}(t') \rangle. \quad (112)$$

The angular brackets denote nonequilibrium average

$$\langle O(S, \tilde{S}) \rangle = \frac{1}{Z} \int \mathcal{D}[i\tilde{S}] \mathcal{D}[S] O[S, \tilde{S}] \exp \mathcal{J}\{\tilde{S}, S\}. \quad (113)$$

In this formalism all the correlation and response functions are obtained as path integrals weighted with the favorite exponential factor $\exp(\mathcal{J})$ which permits expressing the dynamics in a form analogous to that in statics [85, 86]. The dynamic diagram technique is a simple generalization of the diagram technique used in statics. We have two kinds of propagators. In the Fourier representation the free response propagator $G_0^S(k, \omega)$ has the form

$$\Gamma \langle S_{\mathbf{k}, \omega} \tilde{S}_{-\mathbf{k}', \omega'} \rangle_0 = \delta_{\mathbf{k}, \mathbf{k}'} \delta(\omega + \omega') \frac{\Gamma}{-i\omega + \Gamma(r_0 + k^2)} = \delta_{\mathbf{k}, \mathbf{k}'} \delta(\omega + \omega') G_0^S(k, \omega), \quad (114)$$

and the free two-point correlation function $K_0^S(k, \omega)$ is given by

$$\langle S_{\mathbf{k}, \omega} S_{-\mathbf{k}', \omega'} \rangle_0 = \delta_{\mathbf{k}, \mathbf{k}'} \delta(\omega + \omega') \frac{2\Gamma}{\omega^2 + \Gamma^2(r_0 + k^2)^2} = \delta_{\mathbf{k}, \mathbf{k}'} \delta(\omega + \omega') K_0^S(k, \omega). \quad (115)$$

The lower index 0 denotes that the average is taken over the free Lagrangian with $u_0 = g_0 = f_0 = w_0 = 0$. The upper index indicates the field which the propagator is referred to. Another important advantage of this formalism is the possibility to carry out the Gaussian transformations decoupling different modes in the Lagrangian and a simple generalization of the renormalization group into the dynamics.

6.1.2 The acoustic response function

In the studies of the sound propagation we are interested primarily in the acoustic response function

$$\langle Q_{\mathbf{k}, \omega} \tilde{Q}_{-\mathbf{k}', \omega'} \rangle = \delta_{\mathbf{k}, \mathbf{k}'} \delta(\omega + \omega') G^Q(k, \omega). \quad (116)$$

The imaginary part of $G^Q(k, \omega)$ determines the sound damping and the real part - the sound dispersion. The perturbation expansion for $G^Q(k, \omega)$ can be represented by the Dyson equation

$$G^{-1}(k, \omega) = G_0^{-1}(k, \omega) - \Sigma(k, \omega), \quad (117)$$

where the index Q was omitted and

$$G_0^{-1}(k, \omega) = -\omega^2 - i\Theta k^2 \omega + c_0^2 k^2,$$

with c_0 being the bare longitudinal sound velocity. The self-energy $\Sigma(k, \omega)$ is an infinite sum of one-particle irreducible Feynman diagrams [84–87]. Performing the dynamic Gaussian transformations [9, 10] and extracting an irreducible, with respect

to the acoustic propagators as well as to the entropy propagators, part of the four-spin response function we are able to preserve also relevant nonasymptotic effects in the critical sound attenuation coefficient. The acoustic self-energy after these transformations can be written as [9]

$$\Sigma(k, \omega) = \frac{c_0^2 k^2 \left[(v_T^{ph} - i\tilde{\omega} v_{ad}^{ph}) \Pi^{(A)}(k, \omega) - r^2 \right]}{1 + v_T^{ph} \Pi^{(A)}(k, \omega) - i\tilde{\omega} [1 + v_+ \Pi^{(A)}(k, \omega)]}, \quad (118)$$

where the coefficients v_T^{ph} , v_{ad}^{ph} , v_+ and r^2 are defined in Sect. 4, $\tilde{\omega} = \omega C_V^0 / \gamma$ is the ratio of the sound frequency to the bare spin-lattice relaxation time. The function

$$\Pi^{(A)}(k, \omega) = 2\Gamma \int_{-\infty}^{\infty} dt e^{i\omega t} \left\langle S_{\mathbf{k}}^2(t) \widetilde{S}_{-\mathbf{k}}^2(0) \right\rangle_{\mathcal{L}_S^{(A)}(S, \tilde{S})} \quad (119)$$

is the response function of the square of the order parameter. $\widetilde{S}_{\mathbf{k}}^2(t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}_1} \tilde{S}_{\mathbf{k}_1}(t) S_{\mathbf{k}-\mathbf{k}_1}(t)$ is the reaction field coupled to $S_{\mathbf{k}}^2$. The index $\mathcal{L}_S^{(A)}(S, \tilde{S})$ denotes the effective spin Lagrangian of the model A. The function $\Pi^{(A)}(k, \omega)$ depends on the spin variables only and does not contain any irrelevant parameters, so it can be relatively easily calculated by the perturbation technique [54, 69, 88]. In the limit $\omega = 0$ the function $\Pi^{(A)}$ becomes the static correlation function $\langle S_{\mathbf{k}}^2 S_{-\mathbf{k}}^2 \rangle_{H^{(A)}}$ which is well known from the Sect.4. For $k = 0$ the last quantity is proportional to the specific heat so we can interpret $\Pi^{(A)}(k, \omega)$ as the frequency and wave-vector dependent specific heat of the system. It is worth noting at this point that the structure of the self-energy described by Eq.(118) does not depend on the order-parameter dimension nor on the detailed form of the equation of motions so it has a quite universal character. The dynamic universality class, i.e. the fact whether we study an isotropic antiferromagnet or a planar ferromagnet, influences only the form of the function $\Pi^{(A)}(k, \omega)$. Also the fact whether the system is below or above the critical temperature influences only the shape of the function $\Pi^{(A)}(k, \omega)$. Of essential importance is the fact if the entropy is a conserved quantity or not. The case of conserved entropy¹⁴ will be discussed later. As already mentioned, the sound attenuation coefficient and the sound dispersion are easily obtained from self energy:

$$\alpha(\omega) = \frac{1}{2c(\omega)} \text{Im} \frac{\Sigma(k, \omega)}{\omega}, \quad (120)$$

$$c^2(\omega) - c^2(0) = \text{Re} [\Sigma(k, \omega) - \Sigma(k, 0)]. \quad (121)$$

¹⁴It is well known that in all irreversible processes, entropy must increase. However, the terms describing the entropy production are of higher order and can be shown to be irrelevant (in the language of renormalization group theory) parameters in the equation of motion.

In the last equation the noncritical contribution to the sound attenuation $\alpha_{nc} = \Theta\omega^2/2c^3$ was omitted. Also the temperature dependence of these quantities was not displayed. The self-energy depends on the temperature through the four spin response function $\Pi^{(A)}$ which in the static limit turns into the specific-heat. The scaling theory predicts that

$$\Pi^{(A)}(k, \omega; t) = At^{-\alpha}\Phi'(k\xi, \omega\tau_c) + B, \quad (122)$$

where A and B are some constants and t denotes the reduced temperature, α and ν are the critical exponents of the specific-heat and the correlation length, respectively. $\tau_c = \frac{1}{2\Gamma}t^{-z\nu}$ is a characteristic relaxation time of the order parameter fluctuations, which diverges to infinity at the critical point. The dynamic critical exponent z determines the dynamic universality class. We will focus on the high-temperature phase of the Ising type magnets ($n = 1$) which are usually described by the universality class of model A with $z = z_A = 2 + c\eta$ where c is a constant of an order of unity and η is the critical exponent of the correlation function [4].

For the ultrasonic frequencies the wavelength is much longer than the correlation length, $k\xi \ll 1$, so we usually put $k\xi = 0$ in the function Φ' . The scaling function Φ' can be obtained by the exponentiation procedure [3] or by integration of the recursion relations of the renormalization group [54]. In the leading order in the expansion parameter $\epsilon = 4 - d$ it is given by

$$\Phi'(y) = [1 + (y/2)^2]^{-\alpha/2z\nu} \left\{ \frac{\nu}{\alpha} + \frac{i}{y} [i(1 - iy/2) \arctan(y/2) + \right. \\ \left. - \frac{1}{2} \ln(1 + (y/2)^2)] \right\} K_4, \quad (123)$$

where $y = \omega\tau_c$ is the reduced frequency and K_4 is a constant. Substituting (122) into Eq. ((120)) and neglecting the irrelevant terms we obtain

$$\frac{\alpha(\omega, t)c_0^3}{\omega^2} = \frac{W_1(\omega)t^{-(\alpha+z\nu)}\text{Im}(\Phi(y)/y) + W_2t^{-2\alpha}|\Phi(y)|^2}{|1 - i\tilde{\omega}[1 + t^{-\alpha}\Phi(y)]|^2}, \quad (124)$$

where: $W_1(\omega) = \frac{1}{2v_+\Gamma}(\bar{g}_0^2 + \tilde{\omega}^2g_0^2)$, $W_2 = \frac{\bar{g}_0^2C_V^0}{v_+\gamma}$ and $\Phi = Av_+\Phi'$.

Eqs. (120) and (121) show many different types of behavior depending on the relative size of the reduced temperature, frequency and the bare relaxation times: for the order parameter $\tau_c^0 = 1/(2\Gamma)$ and the spin lattice one $\tau_{SL}^0 = C_V^0/\gamma$.

6.1.3 Low-frequency regime

Sound Attenuation Let's assume at the beginning that the sound frequency is very low, so that $\omega\tau_{SL}^0 \ll 1$ is satisfied. It is true if the sound frequency is much lower than the spin-lattice relaxation rate. Than the denominator of Eq.(124) can

be approximated by unity and the term proportional to $\tilde{\omega}^2$ can be neglected so we obtain

$$\frac{\alpha(\omega, t)c_0^3}{\omega^2} = \frac{\bar{g}_0^2}{v_+} \left\{ \tau_c^0 t^{-(\alpha+z\nu)} \text{Im}(\Phi(y)/y) + \tau_{SL}^0 t^{-2\alpha} |\Phi(y)|^2 \right\}. \quad (125)$$

We can see two competing terms with different exponents but also with different amplitudes which are proportional to the relaxation times τ_c^0 and τ_{SL}^0 . It is worth noting here that the effective coupling constant \bar{g}_0 is identical for both terms, so the second term is present in the sound attenuation even if the initial coupling constant „entropy-sound” w_0 is equal zero. Asymptotically, i.e. for very small reduced temperature the first term dominates because of the larger the critical exponent $\rho_s = z\nu + \alpha$, as long as the sound frequency is sufficiently low.

$$\alpha(\omega, t) \propto \omega^2 t^{-(z\nu+\alpha)} g_1(y). \quad (126)$$

The scaling function for the sound attenuation coefficient $g_1(y) = \text{Im}(\Phi(y)/y)$ was introduced in Eq. (126). This strong singularity with the sound attenuation critical

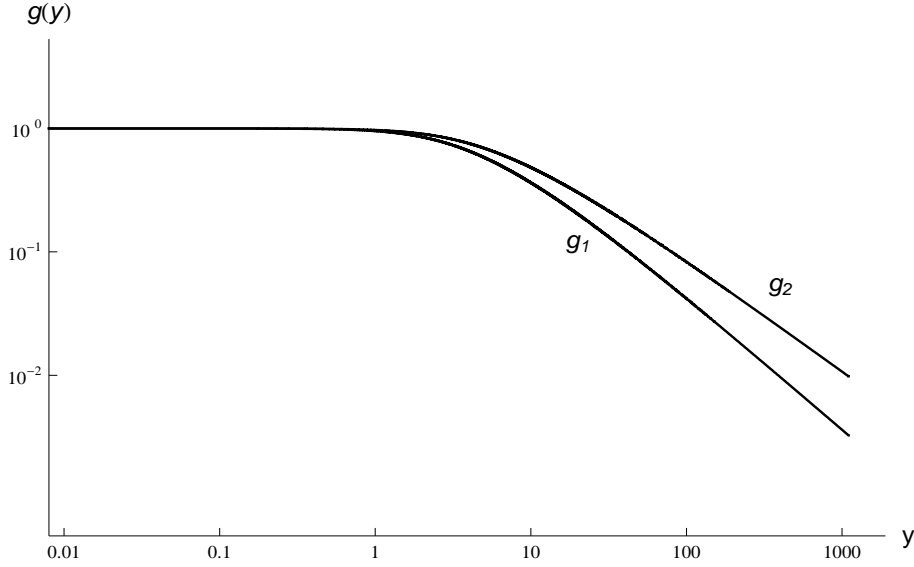


Figure 11 The scaling functions of the sound attenuation coefficient for the low frequency, g_1 , and high frequency, g_2 , regimes. The normalization $g_1(0) = g_2(0) = 1$ was used.

exponent $\rho_s = z\nu + \alpha$ was predicted for all systems with the magnetoelastic coupling of the type QS^2 by Murata [82] and Iro and Schwabl [88]. For the Ising type systems $\alpha \simeq 0.110$, $\nu \simeq 0.630$ and $z \simeq 2.013$ so $\rho_s \simeq 1.38$. The scaling function $g_1(y)$ (after the normalization to unity in the hydrodynamic region) is shown in Fig. 11.

The reduced frequency can vary from the values much lower to those much higher than unity. As mentioned above, the range of the reduced frequency $y \ll 1$ is known as the hydrodynamic region and that for which $y \gg 1$ is called the critical region.

In the hydrodynamic region $g_1(y) \simeq g_1(0) = \text{const}$ so $\alpha(\omega, t) \propto \omega^2 t^{-\rho_s}$. In the critical region $g_1(y \rightarrow \infty) \propto y^{-\rho_s/z\nu}$ thus $\alpha(\omega, t)$ reach a saturation value which is independent of temperature: $\alpha_{sat}(\omega) \propto \omega^{1-\alpha/z\nu}$. In Fig. 12 the temperature dependence of the sound attenuation coefficient given by Eq. (126) is shown for a few ultrasonic frequencies. The crossover from the strong increase region (hydrodynamic region) for not very small t to the saturation region (critical region) is seen. Note that in the double logarithmic scale used in this figure the power law behavior is seen as a straight line.

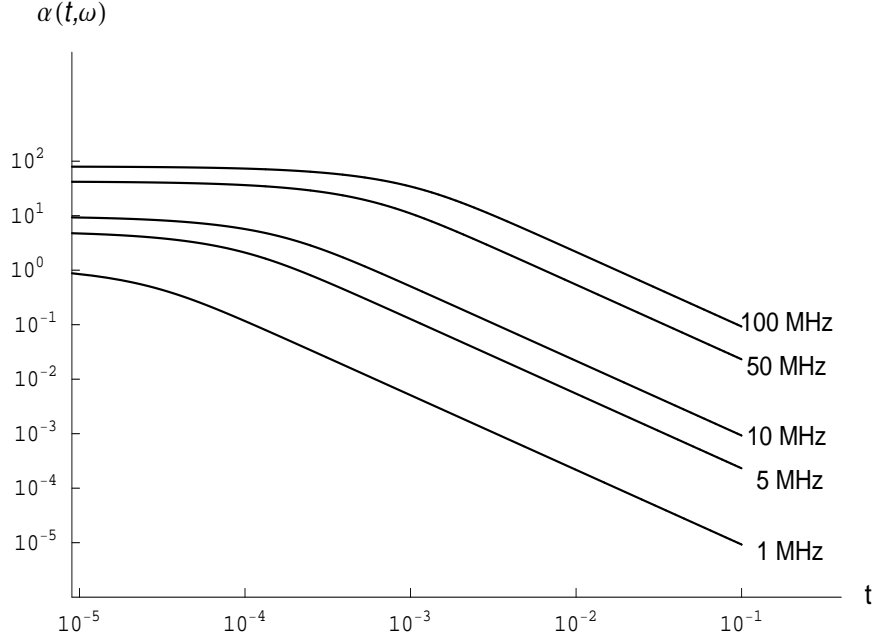


Figure 12 The sound attenuation coefficient as a function of the reduced temperature and frequency in the regime described by Eq. (126).

For $\tau_{SL}^0 \gg \tau_c^0$ despite the smaller exponent, the second term in (125) dominates as long as the reduced temperature is much higher than the crossover temperature $t_{\text{cross}} = (\frac{vK_4\tau_c^0}{4a^2\tau_{SL}^0})^{\frac{1}{z\nu-\alpha}}$ where $a = v_+/v_+^*$ and $v_+^*K_4 = \alpha/\nu + O(\epsilon^2)$ is a fixed point value of this coupling in the renormalization group analysis. In this regime known as the Kawasaki regime a weaker singularity is expected:

$$\alpha(\omega, t) \propto \omega^2 t^{-2\alpha}, \quad (127)$$

where for the scaling function suitable for this case we have taken a constant because we have here $y < \omega\tau_c^0 \ll 1$. The critical sound attenuation exponent $\rho_s = 2\alpha \simeq 0.22$ ($n = 1$) is relatively small in comparison with the one for the Murata-Iro-Schwabl (MIS) regime described previously. It should be noted that this analysis implies that the crucial factor which decides which singularity dominates in this low frequency

region is the ratio of the relaxation times τ_{SL}^0/τ_c^0 not the ratio of the coupling constants w_0/g_0 .

One may ask now why the measured sound attenuation exponents are usually small in insulators and large in metallic magnets. It seems that it is connected with the fact that in metals due to the coupling to the conduction electrons the spin-lattice relaxation times are generally shorter than in the insulators. For some dielectric magnets like MnF_2 , RbMnF_3 , Fe_5O_{12} and RbMnF_3 the spin-lattice relaxation time is known from the ultrasonic measurements. It is $\tau_{SL} = 3 \times 10^{-9}$ s for MnF_2 [89], whereas the critical relaxation time for order parameter fluctuations τ_c is of an order of 10^{-11} s for this compound. For RbMnF_3 a relatively short spin-lattice relaxation time was measured near the critical point [89] varying from 2×10^{-10} s to 4×10^{-10} s whereas τ_c measured in the inelastic neutron scattering varied from 0.08×10^{-10} s to 3×10^{-10} s [90]. For both these compounds the inequality $\tau_{SL}^0 \gg \tau_c^0$ held in the experimental temperature range which allowed the experimental observation of the Kawasaki singularity in the sound attenuation coefficient. Much longer spin-lattice relaxation time of an order of 10^{-8} s was observed for $\text{Y}_3\text{Fe}_5\text{O}_{12}$ [91]. The experimental values for MnF_2 and RbMnF_3 agree (to the order of magnitude) with the theoretical estimations of for insulating antiferromagnets of reported by Huber [92] and Itoh [93].

While for insulators the ultrasonic measurements seem to be a good tool in determining the spin-lattice relaxation time, for conducting magnets this method is less suitable as the high sound attenuation critical exponents measured in these magnets suggest that the sound attenuation is dominated by the first term in Eq.(125) which is connected with τ_c rather than τ_{SL} . Unfortunately, there are only very few methods permitting studies of the spin-lattice relaxation in metals. Recently, Vaterlaus et al. [95] were the first who measured τ_{SL} in rare earth metals. They used a pioneering technique of time resolved spin-polarized photoemission. Applying strong 10 ns laser heating pulses followed by 60 ps weak probe pulses they determined τ_{SL} in gadolinium. This result $\tau_{SL} = 100 \pm 80$ ps (averaged in the temperature interval $45 < T < 225$ K) is in satisfactory agreement with a theoretical estimation by Hübner and Bennemann [96] who obtained $\tau_{SL} = 48$ ps for Gd. Furthermore Bloembergen obtained $\tau_{SL} = 4 \times 10^{-11}$ s for nickel by extrapolating the magnetic resonance data to the Curie temperature. These results confirm the expectations that the spin-lattice relaxation times in metals can be even a few orders of magnitude shorter than in insulators and that this is the reason why the strong Murata-Iro-Schwabl singularity dominates in metallic magnets. However, it would be of a great interest to get more experimental data on the spin-lattice relaxation time in metals.

The sound velocity It is useful to write Eqs. (117) and (118) in terms of the complex sound velocity $\hat{c}^2(\omega) = \frac{1}{k^2} [G^{-1}(k, \omega) - \omega^2]$, where

$$\hat{c}^2(\omega) = c_0^2 \frac{(1 - r^2) - i\tilde{\omega} [1 + v_{ad}^q \Pi^{(A)}(k, \omega)]}{1 + v_T^{ph} \Pi^{(A)}(k, \omega) - i\tilde{\omega} [1 + v_+ \Pi^{(A)}(k, \omega)]}. \quad (128)$$

In the limit $\omega \rightarrow 0$ the last equation becomes the static relation

$$\hat{c}^2(0) = c^2(0) = c_0^2 \frac{(1 - r^2)}{1 + v_T^{ph} \langle S_{\mathbf{k}}^2 S_{-\mathbf{k}}^2 \rangle}, \quad (129)$$

which is exactly the isothermal sound velocity (83) obtained in Sect.4, if the effective spin Hamiltonians used in the calculations of the average $\langle S_{\mathbf{k}}^2 S_{-\mathbf{k}}^2 \rangle$ are identical. We have thus obtained a proof of the internal consistency of this theory.

In the low-frequency regime (but for the finite frequencies) Eq. (128) becomes the „isothermal” relation

$$\hat{c}^2(\omega) = c_0^2 \frac{(1 - r^2)}{1 + v_T^{ph} \Pi^{(A)}(k, \omega)} = c_0^2 \frac{(1 - r^2)}{1 + \frac{v_T^{ph}}{v_+} t^{-\alpha} \Phi(y)}, \quad (130)$$

whose structure is similar to Eq. (129). The complex sound velocity (130) depends on frequency through y . As a measure of dispersion we take $c^2(\omega) - c^2(0)$. In the low-frequency regime it is given by

$$c^2(\omega) - c^2(0) \propto t^{-\alpha} f_1(y), \quad (131)$$

where $f_1(y) = \text{Re } \Phi(0) - \text{Re } \Phi(y)$ is a new scaling function shown in Fig.13. In the hydrodynamic limit ($y \rightarrow 0$) this function behaves as $f_1(y) \propto y^2$ and then we can write $c^2(\omega) - c^2(0) \propto t^{-(2z\nu + \alpha)} \omega^2$. Thus the sound dispersion is characterized by a high critical exponent $2z\nu + \alpha$ and a quadratic sound frequency dependence. In the critical range $f_1(y)$ behaves as $1 - y^{-\alpha/z\nu}$, so it reaches a constant value for $T = T_c$.

The experiments on the critical sound velocity [62] reveal that the sound changes are usually very small so the coupling constants v_T^{ph} and v_{ad}^{ph} are expected also to be very small. That is the reason why the term $\frac{v_T^{ph}}{v_+} t^{-\alpha} \Phi(y)$ in the denominator of Eq. (124) was neglected in the paragraph concerning the sound attenuation. However, it is worth noting here that this term leads to qualitatively different behavior of the sound attenuation coefficient in the strong-coupling limit i.e. for such strong magnetoelastic couplings that $\frac{v_T^{ph}}{v_+} t^{-\alpha} \Phi(y)$ can be greater than one [11, 54]. For such strong-coupling regime the sound velocity tends to zero as $t^{\alpha/2}$ and the sound attenuation exponent $\rho_s = z\nu + \alpha/2$. The behavior of sound characteristics in the strong-coupling limit is very similar to the that of the sound velocity and attenuation

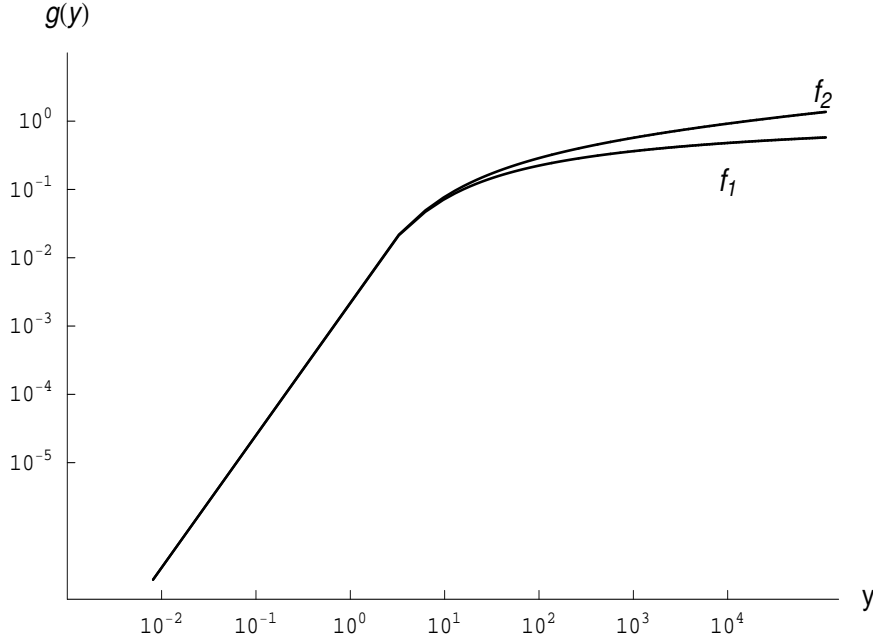


Figure 13 The plot of the scaling functions for the sound dispersion for low-frequency, f_1 , and high-frequency, f_2 , regimes. The normalization $f_1(0) = f_2(0)$ and $f_1(\infty) = 1$ was used.

in liquids near a critical point [54]. Although it is hard to be expected in magnets (because of the smallness of the coupling constants) it is believed that it takes place at some structural phase transitions as for example the order-disorder transition in ammonium halides [11]. In the tricritical points in NH_4Cl and NH_4Br at which the high tricritical specific-heat exponent $\alpha_t = 0.5$ favors the revealing of the term $\frac{v_T^{ph}}{v_+} t^{-\alpha} \Phi(y)$ the tricritical sound attenuation exponents 1.2 and 1.1 were found [97], to be compared with the theoretical strong-coupling tricritical value $\rho_s^t = z_t \nu_t + \alpha_t/2 = 1.25$.

6.1.4 High-frequency regime

Sound attenuation For the sound frequencies much higher than the spin-lattice relaxation rate ($\omega \tau_{SL}^0 \gg 1$), the denominator of Eq. (124) becomes singular and the term proportional to $\tilde{\omega}^2$ dominates the numerator so $\alpha(\omega, t)$ can be written as

$$\alpha(\omega, t) = \left(\frac{g_0^2}{c_0^3 v_+} \right) \omega \frac{t^{-\alpha} \text{Im} \Phi(y)}{|1 + t^{-\alpha} \Phi(y)|^2} \quad (132)$$

or in the form analogous to the Eq. (126) as

$$\alpha(\omega, t) \propto \omega^2 t^{-(z\nu - \alpha)} g_2(y), \quad (133)$$

where the scaling function $g_2(y) = \frac{\text{Im}\Phi(y)}{y|\Phi(y)|^2} = -\frac{\text{Im}\Phi^{-1}(y)}{y}$ is shown in Fig. 11.

The new regime with the critical sound attenuation exponent $z\nu - \alpha \simeq 1.16$, is obtained. This exponent is about 0.22 smaller than that in the MIS regime. The coupling constant is also different: g_0^2 instead of \bar{g}_0^2 . The scaling function in the critical range behaves as $g_2(y) \propto y^{-\rho_s/z\nu}$, so the value at which the attenuation saturates is $\alpha_{sat}(\omega) \propto \omega^{1+\alpha/z\nu}$ a little different from $\alpha_{sat}(\omega) \propto \omega^{1-\alpha/z\nu}$ for the one in the low-frequency regime.

The sound attenuation exponent for the high-frequency regime is given by the same formula ($\rho_s = z\nu - \alpha$) as for binary liquids [69, 98, 99], where of course the numerical value of ρ_s is different from that for the Ising magnet considered here, because of different dynamic universality class ($z \simeq 3.06$ in the critical mixtures). It is related to the structure of the expression for in the high-frequency region. Eq. (132) resembles the well known expression for the sound attenuation coefficient introduced by Ferrell and Bhattacharjee

$$\alpha(\omega, t) \propto -\omega \text{Im}[C_{FB}(\omega)]^{-1} \simeq \frac{\omega \text{Im} C_{FB}(\omega)}{(\text{Re} C_{FB}(\omega))^2}, \quad (134)$$

where $C_{FB}(\omega)$ is a phenomenological frequency dependent specific heat¹⁵ [100, 101]. This expression applies also to the λ phase transition in the liquid helium [73, 74, 100, 101], to the binary mixtures [75] and to some extent to the liquid crystals [102]. The essential factor for the change of the attenuation critical exponent from $z\nu + \alpha$ to $z\nu - \alpha$ is the divergence of $\text{Re} C_p^{FB}$ near a critical temperature. In liquid helium the order parameter dimensionality is two and the specific heat exponent is very close to zero. For the Heisenberg systems ($n = 3$) this exponent is negative so there is no change in the sound attenuation exponent in the high-frequency regime. However, the singularity in the denominator of (124) will influence the nonasymptotic behavior of the sound attenuation coefficient. It may also happen that the background part of the specific heat will be much higher than its singular part and the high-frequency exponent $\rho_s = z\nu - \alpha$, will not be observed for some Ising type systems in experimentally accessible temperature range.

The sound velocity In the high-frequency regime the sound frequency takes a familiar „adiabatic” form

$$\hat{c}^2(\omega) = c_0^2 \frac{1 + v_{ad}^q \Pi^{(A)}(k, \omega)}{1 + v_+ \Pi^{(A)}(k, \omega)} = c_0^2 \left(1 - \frac{v_{ad}^{ph}}{v_+}\right) + c_0^2 \frac{v_{ad}^{ph}}{v_+} \frac{1}{1 + t^{-\alpha} \Phi(y)}. \quad (135)$$

In the limit $y \rightarrow 0$ the Eqs. (92) and (93) are recovered. The above should be expected because for the long spin-lattice relaxation times the fluctuations of temperature of the spin system do not decay by a fast process of energy relaxation to

¹⁵The correspondence between (132) and (134) is obtained if we interpret $C_V^0 [1 + v_+ \Pi^{(A)}(\omega)]$ as the Ferrell-Bhattacharjee specific heat. Note however that it is not exactly the specific heat even in $\omega \rightarrow 0$ limit as v_+ differs slightly from v_T^q .

the lattice. The sound dispersion is also easily obtained

$$c^2(\omega) - c^2(0) \propto t^\alpha f_2(y), \quad (136)$$

where the scaling function $f_2(y) = \text{Re } \Phi^{-1}(y) - \text{Re } \Phi^{-1}(0)$ is shown in Fig. 13. It behaves as $f_2(y) \propto y^2$ in the hydrodynamic range and as $f_2(y) \propto y^{\alpha/z\nu}$ in the critical range so in the hydrodynamic region $c^2(\omega) - c^2(0) \propto t^{-(2z\nu - \alpha)}\omega^2$, and the critical exponent for the sound dispersion is lowered by 2α in comparison to the low-frequency regime.

6.2 Low temperature phase and other dynamic models

In the low-temperature phase only the four-spin response function $\Pi(k, \omega)$ has to be obtained. The theory becomes more complicated as there is spontaneous polarization $\langle S \rangle$ and introducing $S = \langle S \rangle + \delta S$ where δS is the spin fluctuation we can write this function as [69]

$$\Pi = 2\langle(\delta S \delta S)(\Gamma \tilde{S} \delta S)\rangle + 4\langle S \rangle \langle(\delta S)(\Gamma \tilde{S} \delta S)\rangle + 2\langle S \rangle \langle(\delta S \delta S)(\Gamma \tilde{S})\rangle + 2\langle S \rangle^2 \langle \delta S \Gamma \tilde{S} \rangle \quad (137)$$

The last term is the analogue of the Landau-Khalatnikov sound damping [105] which is the only one which contributes to in the mean-field theory. The first term in (137) is known as a fluctuation contribution to the sound attenuation and the other two terms are sometimes called the mixing contribution [5]. Considered separately, the three contributions are characterized by different critical exponents and for example the Landau-Khalatnikov term diverges with the critical exponent equal to $2(\gamma - \beta)$ [5]. As was noted by Halperin and Hohenberg [4] in the scaling region there should be cancellations between different contributions and the critical sound attenuation exponent should be the same as in the disordered phase. It was shown explicitly by Dengler and Schwabl [69] that this was the case. They calculated also the scaling function $g_1^-(y)$ in the MIS low frequency regime to the second order in ϵ . The upper index indicates the low-temperature phase. Interestingly, this scaling function showing the same asymptotic properties (for $y \rightarrow 0$ and $y \rightarrow \infty$) as $g_1^+(y)$ exhibits a characteristic maximum below the transition temperature at $y \simeq 1$ as shown in Fig. (14). This maximum is due to the Landau-Khalatnikov term. The authors were also able to calculate the universal amplitude ratios for ultrasonic attenuation coefficient above and below the critical point [69]. For uniaxial magnets investigated in this section the critical amplitude ratio

$$\alpha^+/\alpha^- = \frac{\epsilon}{72} 2^{z\nu+\alpha} (1 + \frac{29}{27}\epsilon) + O(\epsilon^2) \quad (138)$$

is small, of an order of 0.05 as a reminiscent of the LK theory (for which this ratio is zero).

Another point is the calculations of the scaling functions in other universality classes. There a substantial progress has been made due to the renormalization

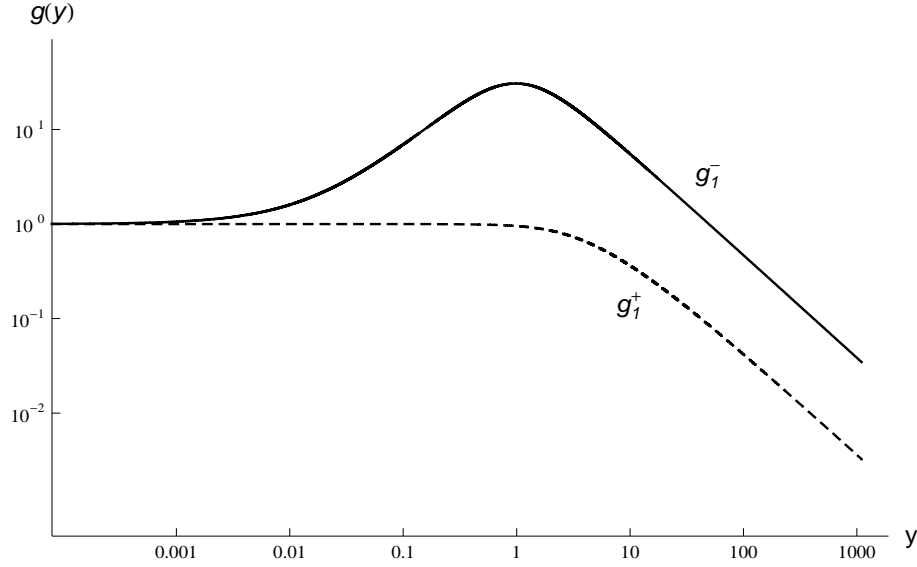


Figure 14 The scaling functions of the sound attenuation coefficient for the low temperature phase, g_1^- (continuous line), in comparison with that for the high temperature phase, g_1^+ (dashed line). The normalization $g_1^+(0) = 1$ was used.

group theory. Referring the reader to the original works [69, 106, 107] we focus only shortly on the model C with conserved energy field as it is a limiting case for the model with spin-lattice relaxation considered here. In the limit of very slow spin-lattice relaxation the spin energy is a conserved quantity. Assuming that the fluctuations of q decay due to diffusion the kinetic coefficient Γ in Eq. (103) should be replaced by κk^2 where k is the wave vector and κ is the thermal conductivity. Analysis of this model [108] gives the following expressions for the sound attenuation and dispersion

$$\alpha(\omega, t) \propto \omega^2 t^{-\rho_s} g_C(y), \quad (139)$$

$$c^2(\omega) - c^2(0) \propto t^\alpha f_C(y), \quad (140)$$

with the sound attenuation exponent $\rho_s = z\nu - \alpha = 2\nu \simeq 1.26$, and $g_C(y) = \text{Im } \Psi(y)/y$ and $f_C(y) = \text{Re } \Psi(y)$ as the scaling functions, where

$$\Psi(y) = \left[1 + \left(\frac{y}{2} \right)^2 \right]^{\frac{\alpha}{2z\nu}} \left\{ 1 + \frac{\alpha}{\nu} \left[\frac{1 - \frac{iy}{2}}{y} \arctan \frac{y}{2} + i \frac{\ln(1 + (\frac{y}{2})^2)}{2y} \right] \right\}. \quad (141)$$

The scaling functions in the one-loop approximation differ from g_2 and f_2 only by the value of the dynamic critical exponent ($z \simeq 2.175$ for the model C). We can also express the complex sound velocity as

$$\hat{c}(\omega)^2 = c_0^2 \left(1 - \frac{v_{ad}^{ph}}{v_+} \right) + \frac{\text{const}}{C_{FB}(\omega)}. \quad (142)$$

with a Ferrell–Bhattacharjee function¹⁶ $C_{FB}(\omega)$ which diverges as the static specific heat near a critical point. In the limit $y \rightarrow 0$, the adiabatic formula (90) for the sound velocity is recovered again. The sound attenuation behavior in model C is very similar to that in the high-frequency regime of model A and Eqs. (139) and (140) appear as the high-frequency regime expressions of model A (133) and (131) in which the crossover to another dynamic class has taken place (at least in the one-loop approximation).

The formula 142 seems to have quite general character (magnets, liquid helium, binary mixtures and liquid crystals). However, for the liquid-gas critical point we have different expression $\hat{c}(\omega)^2 = \text{const}/C_{FB}(\omega)$ which induces different behavior of the sound velocity and different sound attenuation exponent $\rho_s = z\nu + \alpha/2$ [69]. The difference originates from the different role played by the order parameter in this system [74]. In magnets, liquid helium, binary mixtures the order parameter couples to heat and sound modes by two static couplings f_0 and g_0 , whereas in the liquid-gas system there is only one static coupling between the heat (which is the order parameter) and the sound modes.

7 Eksperiment

There are many experimental works on the ultrasonic propagation near a critical point. It is beyond the scope of this review to discuss them all and we refer the reader to the excellent reviews devoted the experimental results [6, 7, 62, 91] focusing only on some questionable problems. First of all it is worth considering the question of how do the measured sound attenuation critical exponents compare with the theoretical estimations: $\rho_s \simeq 1.26$ for the model with conserved energy and the Kawasaki singularity $\rho_s \simeq 0.22$, MIS singularity $\rho_s \simeq 1.38$ as well as for the high-frequency singularity ($\rho_s \simeq 1.16$) in the Ising type model ($n=1$) with nonconserved energy (spin-lattice relaxation). In table 5 we present also the theoretical estimations for the other universality classes in the MIS regime. In real systems e.g. for isotropic Heisenberg ($n = 3$) magnets the cubic anisotropy destroys the conservation of the order parameter leading to the purely dissipative dynamics described by model A. Therefore, in Table 5 this possibility is also taken into account. Other anisotropies can induce the crossover to the Ising universality class ($n = 1$). In real systems it is not clear a priori whether the total spin energy is conserved or not. It is connected with the strength of the spin-lattice interactions and in some cases model A may be a better description of the magnetic system and in other cases model C will be more suitable.

¹⁶The Ferrell–Bhattacharjee function can be expressed ($C_{FB}(\omega) = C_V^0/\Gamma_{m_2m_2}(\omega)$) in terms of a vertex function of an idealised phonon-free model [108].

7.1 Insulators

For the most of insulators the exponent ρ_s takes small vales from zero for europium oxide (EuO) to 0.75 for FeF₂. In Table 3 the values of ρ_s defined for the hydrodynamic range are presented for magnetic insulators. A conspicuous exception is the chromium oxide (Cr₂O₃) for which $\rho_s = 1.3$. This value is very close to those observed in metals (see Table 4). This compound will be discussed later. First we discuss typical magnets from this group. The small values of the critical sound attenuation exponents in insulators are commonly interpreted as an evidence that the Kawasaki regime is realized there. Two questions arise here: how the exponent 0.77, observed in FeF₂ corresponds to the value $\rho_s^{KAW} = 2\alpha_I \simeq 0.22$ obtained for the Ising like magnets (here α_I denotes the specific heat exponent for the Ising systems). What is the connection of $\rho_s \simeq 0.32$ for the isotropic antiferromagnet RbMnF₃ with the negative value $2\alpha_H \simeq -0.27$ obtained from $\rho_s^{KAW} = 2\alpha_I$ by replacing α_I with α_H ? It could be explained by the inaccuracy in determining the critical temperature which can influence the values of the critical exponents measured [91]. In our

Table 3 Critical sound attenuation exponents for insulators.

Compound	Exponent ρ_s	Anisotropy parameter	Frequency range (MHz)	Reduced temperature range	Refer.
MnF ₂	0.14(1)	$1.4 \cdot 10^{-2}$	10 – 110	$10^{-4} - 10^{-1}$	[114, 115] [62, 116]
RbMnF ₃	0.32(2)	$5 \cdot 10^{-6}$	30 – 150	$10^{-4} - 6 \cdot 10^{-2}$	[117]
EuO	0	$4 \cdot 10^{-4}$	50, 170	$10^{-4} - 10^{-1}$	[62, 118]
Y ₃ Fe ₅ O ₁₂	0.5(1)	10^{-5}	5, 30	$2 \cdot 10^{-4} - 3 \cdot 10^{-2}$	[91]
Gd ₃ Fe ₅ O ₁₂	0.42(10)	10^{-5}	5, 30	$2 \cdot 10^{-4} - 5 \cdot 10^{-1}$	[91]
FeF ₂	0.77(7)	0.6	10 – 70	$3 \cdot 10^{-4} - 2 \cdot 10^{-2}$	[120]
Cr ₂ O ₃	1.3(1)	$3 \cdot 10^{-4}$	100 – 1500	$3 \cdot 10^{-5} - 10^{-3}$	[123]

opinion the cause of this apparent contradiction rests with the fact that in experiment only an effective critical exponent is measured from the slope of the sound attenuation curve vs. the reduced temperature (shown in the double-logarithmic scale) in the hydrodynamic range. The effective sound attenuation exponent can be

defined [54, 124] as

$$\rho_s(t) = -\frac{\partial \ln [\alpha(\omega, t)]_{\omega \rightarrow 0}}{\partial \ln t}. \quad (143)$$

It was shown in the last section that the Kawasaki singularity is proportional to the square of the specific heat so the asymptotic value for this exponent in Ising systems is $\rho_s^{Kaw} = 2\alpha_I$. It should be noted that this is only an asymptotic approximation because the specific heat behaves near T_c as

$$C = At^{-\alpha} + B, \quad (144)$$

The constant term B can be neglected for Ising systems only infinitesimally close to T_c . In the Heisenberg model where $A < 0$ this constant is necessary to assure the positivity of the specific heat. Only the two terms assure a peak in the specific heat [34]. In real systems we should take into account also the first correction to scaling [112, 113, 125] writing

$$C = At^{-\alpha}(1 + Et^{\Delta_1}) + B + Ft, \quad (145)$$

where Δ_1 is the exponent of the first correction to scaling equal about 0.52 [22] for the one-component order parameter systems. A regular term proportional to t is also added. The effective attenuation exponent can be found from (144) in the Kawasaki regime as

$$\rho_s(t) = 2\alpha \frac{At^{-\alpha}}{At^{-\alpha} + B}. \quad (146)$$

For the Heisenberg magnets ($n = 3$) we have $\alpha < 0$, $A < 0$ and $B > 0$ ($|A| < B$) so the effective exponent is positive because the product αA is positive. Moreover, its absolute value can be higher than $2|\alpha_H| \simeq 0.27$ which could explain the experimental data in ferrites $\text{Gd}_3\text{Fe}_5\text{O}_{12}$ and $\text{Y}_3\text{Fe}_5\text{O}_{12}$.

7.1.1 FeF₂

In this strongly anisotropic antiferromagnet, both the specific heat exponent and its amplitude are positive and the constant B is negative [125]. In Figure 15 the experimental data [120] in the symmetric phase are fitted to the general formula (145) and in Fig. 16 the effective attenuation exponent is shown for $B/A = -1.59$, $E = 1.2$, $F/A = 0.1$, $\alpha = 0.11$ and $\Delta_1 = 0.52$ (Pawlak and Fechner [119]). The solid line in Fig. (15) is the plot of the square of specific heat. The dotted line represents a simple power law behavior $\alpha(\omega, t) \propto t^{-\rho_s}$ with the exponent $\rho_s = 0.73$. Remembering about stochastic scatter of data it is not hard to imagine that the exponent ~ 0.73 could be obtained in the temperature range $10^{-2} - 10^{-3}$. The lower value of this exponent in the low-temperature phase of FeF_2 $\rho_s \simeq 0.5$ [120] should be connected to the greater amplitude A^- because for the Ising systems we have $A^+/A^- \simeq 0.53$. The greater amplitude A^- implies a lower value of the ratio B/A^- so also the smaller deviation of ρ_s from the value $2\alpha_I$.

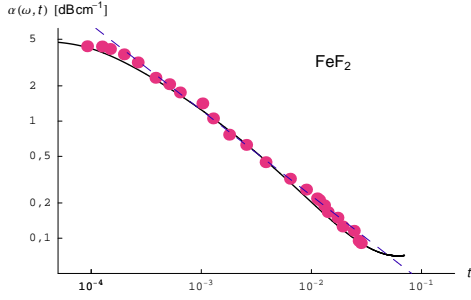


Figure 15 The ultrasonic attenuation vs the reduced temperature for $f = 50$ MHz in FeF_2 along $[0,0,1]$ direction (the data are from [120]). The solid curve is the plot of Eq. (145). The dotted line is the fit to the single power law with exponent $\rho_s = 0.73$ (Pawlak and Fechner [119]).

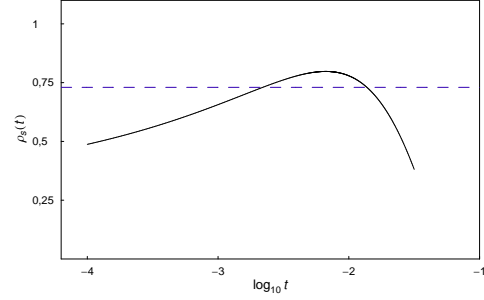


Figure 16 The effective sound attenuation exponent in the Kawasaki regime for the specific heat characterized by Eq.(145) (Pawlak and Fechner [119]). The dotted line corresponds to "experimental" value $\rho_s = 0.73$.

7.1.2 RbMnF_3

Because RbMnF_3 is the isotropic antiferromagnet with very small anisotropy its specific-heat exponent is negative [121]. The analysis is restricted to relatively low frequencies at which the saturation effects (appearing when the reduced frequency y is comparable to unity) can be neglected in the explored temperature range. In Fig. (17) the experimental data [93] in the high-temperature phase are fitted to the expression (145). The Heisenberg critical exponents $\alpha = -0.133$, $\Delta_1 = 0.5$ [22] and $A = -0.372$, $B = 0.292$, $E = -0.0273$, $F = 0.25$ are used (Pawlak and Fechner [119]). The critical amplitudes for the high temperature specific heat are consistent with the experimental estimations obtained by Marinelli et al. [121]. In Fig. (18) the effective sound attenuation exponent versus the reduced temperature is shown. It is seen that it is positive for the Heisenberg type antiferromagnet RbMnF_3 and its

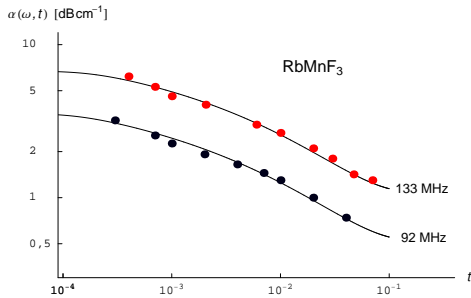


Figure 17 The high temperature ultrasonic attenuation vs the reduced temperature for low frequencies: $f = 92$ and 133 MHz in RbMnF_3 along $[1,0,0]$ direction. The solid curve is the plot of Eq. (145) with $\alpha = -0.133$, $\Delta_1 = 0.5$ and $A = -0.372$, $B = 0.292$, $E = -0.0273$, $F = 0.25$ (Pawlak and Fechner [119]).

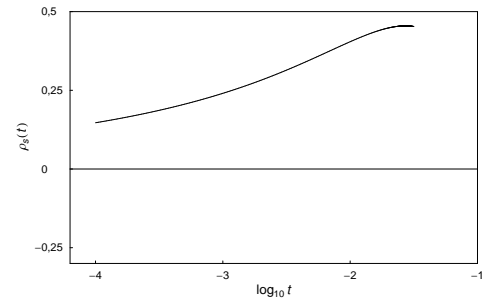


Figure 18 The effective sound attenuation critical exponent for RbMnF_3 (Pawlak and Fechner [119]).

average value in the experimental range of reduced temperature is about $0.25 - 0.35$ as is observed in experiment (see Table 4). That explains also the positive sound attenuation exponents in Heisenberg magnets like $\text{Y}_3\text{Fe}_5\text{O}_{13}$ and $\text{Gd}_3\text{Fe}_5\text{O}_{13}$.

7.1.3 EuO

Another source of concern is the lack of the critical attenuation observed in isotropic ferromagnet EuO ($\rho_s = 0$) although a singularity in the specific heat is observed in this compound [126]. It is however connected with a very long spin-lattice relaxation time which is of an order of milliseconds ($\tau_{SL}^{-1} \approx 1.5 \cdot 10^6 \text{ s}^{-1}$ [127]). The lack of singularity in the sound attenuation for EuO was commonly explained with the aid of simple formula (97) which for this case was usually written as

$$\alpha(\omega, t) \propto C_p \frac{\omega^2 \tau}{1 + \omega^2 \tau^2}, \quad (147)$$

where $\tau \propto C_p$ and C_p is the specific heat. For $\omega\tau \gg 1$ this relation leads to $\alpha(\omega, t) \propto \text{const}$. For the frequency $f = 50 \text{ MHz}$ and $\tau = \tau_{SL} \simeq 10^{-6} \text{ s}$ we have $\omega\tau \simeq 300$ so the assumption $\omega\tau \gg 1$ is well satisfied. However, there is a small problem with the formula (147) based on only one relaxation time. Namely, the sound velocity change $\Delta c \equiv c(\omega) - c_\infty$ on the grounds of Eq. (96) is given by

$$\Delta c \simeq \frac{C_p}{2c_\infty(1 + \omega^2 \tau^2)}. \quad (148)$$

For $\omega\tau \gg 1$ Δc it is very small and depends on the frequency as ω^{-2} . It is very difficult to reconcile with a small anomaly observed in the sound velocity in EuO which practically does not depend on the sound frequency [127]. The lack of the ω^{-2} dependence seems to be unquestionable. So the lack of the critical attenuation and the anomaly in the sound velocity cannot be simultaneously explained. It can be easily done with the aid of formula (118) if we assume that

$$\omega^2 \tau_{SL}^2 \frac{\tau_c}{\tau_{SL}} \ll \frac{v_T^{ph}}{v_{ad}^{ph}} \ll \omega^2 \tau_{SL}^2. \quad (149)$$

Then the imaginary part of (118) gives the equation analogous to (147)

$$\alpha(\omega, t) \propto \frac{\omega^2 \tau_{SL}^0 \text{Re} \Pi^{(A)}(\omega; t) (1 + v_+ \text{Re} \Pi^{(A)}(\omega; t))}{1 + \omega^2 (\tau_{SL}^0 (1 + v_+ \text{Re} \Pi^{(A)}(\omega; t)))^2}, \quad (150)$$

where $\Pi^{(A)}(k, \omega; t) = t^{-\alpha} \Phi'(\omega\tau_c) + B$ is the discussed four spin response function which turns into the static specific heat in the $\omega\tau_c \rightarrow 0$ limit. The role of the relaxation time τ in Eq. (147) is played here by the product $\tau_{SL}^0 (1 + v_+ \text{Re} \Pi^{(A)}(\omega; t))$ which for $\omega\tau_c \ll 0$ behaves as the specific heat. The real part of Eq. (118)

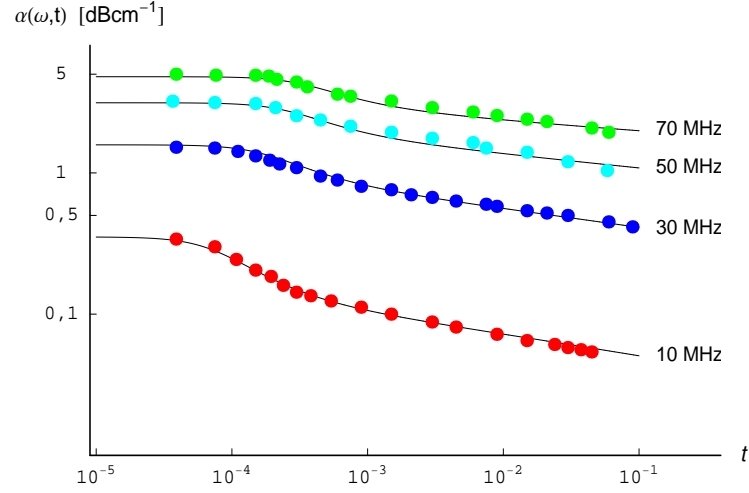


Figure 19 The temperature dependence of the attenuation coefficient for longitudinal waves in $[1, 1, 0]$ direction for MnF_2 ($T > T_N$) (Pawlak [122]). The continuous curves represent the imaginary part of Eq. (118).

becomes the adiabatic sound velocity described by Eq. (135) with the singular term proportional to the inverse of the specific heat. Thus both aspects of the sound propagation in EuO can be explained with the aid of formula (118).

Because the spin-lattice relaxation time in EuO is extremely long (much longer than τ_c) the inequality (149) is probably satisfied for very wide range of frequencies. It should be however expected that for very low as well as for very high frequencies anomalous sound attenuation should be observed. Actually, it was mentioned [127] that the critical attenuation was observed in vibrating reeds experiments EuO for the very low frequency range 0.4–3 kHz.

7.1.4 MnF_2

The critical sound attenuation in magnets was for the first time observed in the antiferromagnet MnF_2 [128]. As shown by the experimental investigation of the specific heat [129] the magnet shows a crossover from the isotropic Heisenberg behavior ($n = 3$) to the Ising type behavior at t of an order of 10^{-2} . The sound velocity and attenuation coefficient of manganese fluoride have been studied by many authors [62, 114–116, 123, 128, 130–132] and it is the best known (from the experimental point of view) magnetic compound so far. In Fig. 19 the longitudinal sound attenuation dependence on the reduced temperature is shown (Pawlak [122]). The experimental data are taken from Ikushima's work [114]. The continuous curves represent the expression (124) for the frequencies $f = 10, 30, 50$ and 70 MHz. The acoustic self energy is taken in the most general form (118) for the model with the spin-lattice relaxation, with $\tau_{SL}^0 = 3 \cdot 10^{-9}$ s, $\tau_c^0 = 10^{-13}$ s, $\frac{v_{ad}^{ph}}{v_T^{ph}} = 1.9$,

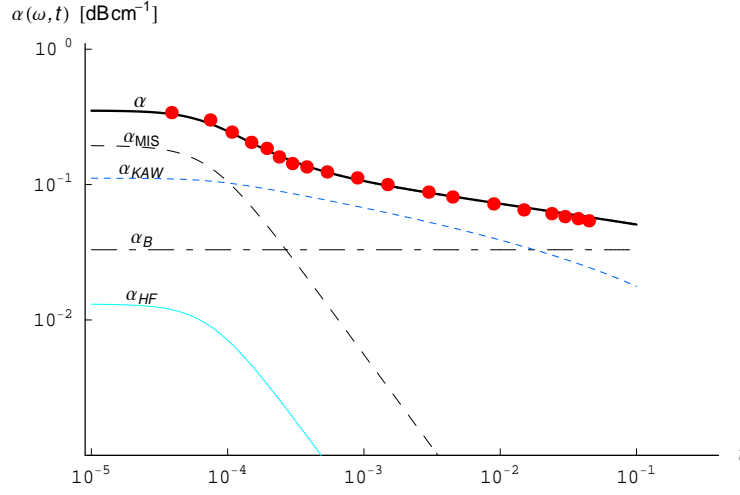


Figure 20 The contribution of different terms to the total attenuation of the longitudinal wave along the $[1, 1, 0]$ direction in MnF_2 for $f = 10$ MHz (Pawlak [122]). The MIS, Kawasaki, high-frequency and the background terms are denoted as α_{MIS} , α_{KAW} , α_{HF} and α_B , respectively.

$(A\nu K_4/\alpha B) = -1.03$. It can be seen from this plot that for low frequencies e.g. for $f = 10$ MHz, $\alpha(\omega, t)$ exhibits a typical Kawasaki behavior with a small slope of the curve and only for the reduced temperatures of an order of 10^{-4} the curve starts to climb up and soon saturates. It is connected with the fact that $\tau_{SL}^0 \gg \tau_c^0$ in MnF_2 , and although the critical relaxation time for the order parameter fluctuations

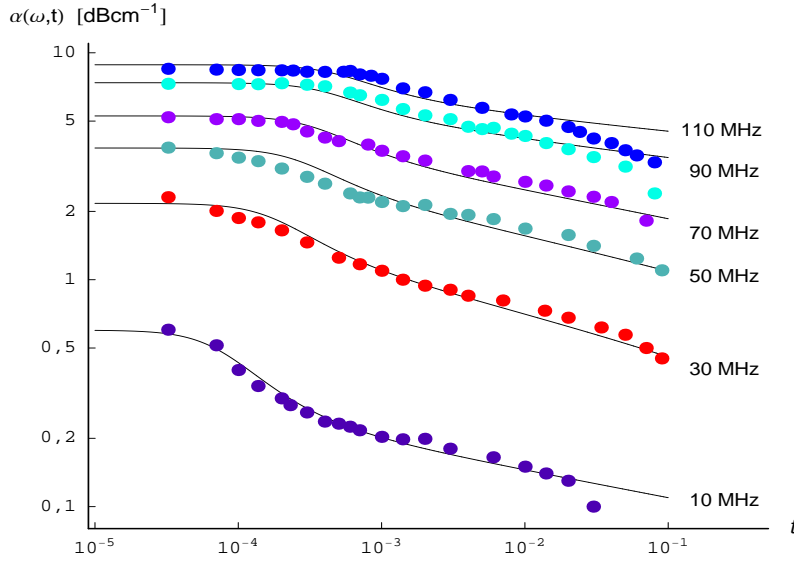


Figure 21 The temperature dependence of the attenuation coefficient for longitudinal waves in $[1, 0, 0]$ direction for MnF_2 ($T > T_N$). The continuous curves represent the imaginary part of Eq. (118). The experimental point are taken from [115].

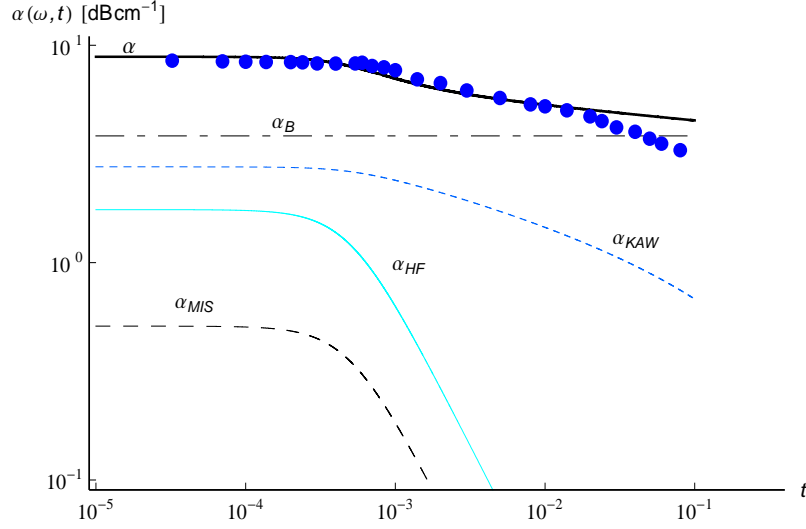


Figure 22 The contribution of different terms to the total attenuation of the longitudinal wave along the $[1, 0, 0]$ direction in MnF_2 for $f = 110$ MHz. The MIS, Kawasaki, high-frequency and the background terms are denoted as α_{MIS} , α_{KAW} , α_{HF} and α_B , respectively.

increases much faster than the spin-lattice relaxation time the MIS term exceeds the Kawasaki term only for t close to 10^{-4} . The crossover to the critical range ($y \gg 1$) is observed when ω^{-1} becomes comparable with τ_c .

Fig. 20 shows the contributions of different terms to the total sound attenuation coefficient (Pawlak [122]). The background noncritical term is also included. It should be noted that every contribution saturates at roughly the same temperature at which $\omega\tau_c \sim 1$. For higher frequencies the MIS term cannot exceed the Kawasaki term (α_{KAW}) because the saturation of both terms takes place before.

Fig. 21 and 22 present analogous plots for the longitudinal sound waves along the $[1, 0, 0]$ direction for $T > T_N$. The experimental points are from [115]. This time the range of the ultrasonic frequencies is wider (10-110 MHz). Generally, it is expected that in the systems without full isotropic (elastic) symmetry the effective coupling constants will depend on the direction of propagation and this time $\frac{v_{ad}^{ph}}{v_T^{ph}} = 0.8$ is assumed. The share of the individual terms in the total attenuation is seen in Fig. 22 for the highest frequency $f = 110$ MHz. This time α_{KAW} significantly exceeds α_{MIS} term and competes with α_{HF} for the small reduced temperatures. The background term is very high now (about 50% of the total attenuation) and suppresses the relative magnitude of the attenuation peak.

7.1.5 Cr_2O_3

From Table 3 it is seen that this antiferromagnet does not match the other insulators because of its high exponent $\rho_s = 1.3$. The experiment [113] shows

that this system is characterized by Ising specific-heat exponent for $t < 3 \cdot 10^{-3}$, despite only small anisotropy. As follows from our analysis, high sound attenuation exponent ρ_s can appear in three cases. One is typical of metals where $\tau_{SL}^0 \sim \tau_c^0$ and then due to the critical slowing down the MIS behavior is observed with the exponent $\rho_s = z\nu + \alpha$ where $z = z_A$ and the nonconserved energy is assumed. For this universality class another high-frequency regime is also characterized by large attenuation exponent $\rho_s = z\nu - \alpha$ and it is the second case. The third case is that the energy is conserved and the attenuation exponent is given by $\rho_s = z_C\nu - \alpha$ with $z_C = 2 + \alpha/\nu$ for the model C. In model A we have $\rho_s \simeq 1.38$ for the MIS regime and $\rho_s \simeq 1.16$ for the high-frequency regime. The experimental value 1.3 measured for Cr_2O_3 is closest to the value for model C $\rho_s \simeq 1.26$.

It should be noted that the sound attenuation was measured for very high frequencies, 100–1500 MHz, in this compound. Assuming even very short spin-lattice relaxation time of an order of one nanosecond we have $1 \leq \omega\tau_{SL} \leq 15$ which indicates the crucial role of the denominator in Eq. (124) which rather excludes the MIS regime. Further measurements of the sound attenuation coefficient in the low-frequency range as well as an experimental estimation of the spin-lattice relaxation time are desirable for this magnet in order to recognize the source of its exceptionality.

7.2 Metals

In Table 4 the critical exponents for conducting magnets are given. They are equal or greater than one, which confirms the hypothesis that the spin-lattice relaxation time are shorter than in insulators and the Kawasaki regime is less important in these compounds. Taking into account that the ultrasonic frequencies used were not very high i.e. such that $\omega\tau_{SL}^0 \sim 1$ could be met, we can suppose that in the investigated frequency interval the MIS term dominates with the exponent $\rho_s = z\nu + \alpha$. What is the source of so large differences in the attenuation exponents measured in metals? Of course, they follow from the fact that these compounds belong to different static (n) and dynamic universality classes as explains Table 5 [4].

7.2.1 Ni

The attenuation exponent observed in this isotropic ferromagnet $\rho_s = 1.4 \pm 0.2$ suggests that the earlier mentioned cubic anisotropy terms are important in this ferromagnet, which indicates the crossover to the relaxational dynamics with $z \simeq 2$. Below T_C a temperature interval was observed in which $\rho_s \simeq 0.3$ [139]. Such a small value of this exponent suggests that the Kawasaki regime is important here for not very small reduced temperature.

Table 4 Critical sound attenuation exponents for magnetic metals.

Metal	Exponent ρ_s	Anisotropy parameter	Frequency range (MHz)	Reduced temperature range	Refer.
Gd	1.2(1) 1.63(10) 1.8(2) 1.15(10)	$5 \cdot 10^{-4}$	30–180 10–70 5 5–30	$10^{-3}–10^{-1}$ $10^{-3}–10^{-1}$ $3.4 \cdot 10^{-3}–2.4 \cdot 10^{-2}$ $10^{-4}–10^{-1}$	[134] [62] [135] [91]
Tb	1.24(10)	–0.4	30–170	$7 \cdot 10^{-3}–7 \cdot 10^{-2}$	[136]
Dy	1.37(10) 1.26(10)	–0.3	30–170	$3 \cdot 10^{-3}–10^{-1}$ $3 \cdot 10^{-3}–10^{-1}$	[136] [62]
Ho	1.0(1)	$-7 \cdot 10^{-2}$	30–170	$3 \cdot 10^{-4}–10^{-1}$	[136]
MnP	1.1(1) 1.1(1)		30–210 10–520	$10^{-4}–10^{-2}$ $10^{-4}–10^{-2}$	[137] [138]
Ni	1.4(2)	$\sim 10^{-4}$	20, 60	$10^{-6}–3 \cdot 10^{-3}$	[139]

7.2.2 Gd

The most interesting metal is the ferromagnetic gadolinium. Is it possible to explain such great differences in the sound attenuation exponent shown in Table 4? It seems that yes. Firstly, the four experiments mentioned in Table 4 have been performed for different samples and the structure and the symmetry of the crystal depend on the number and character of impurities. Secondly, in Gd besides the anisotropy the dipole interactions are very important. It follows from the renormalization group analysis [140] that in such system there are four fixed points: Heisenberg (H), anisotropic Ising (I), anisotropic dipolar (AD) and isotropic dipolar (ID) fixed points. Depending on the ratio of the anisotropy to the parameter describing the strength of the dipolar interactions we have a series of the crossovers between these fixed points which can be seen as transitions from one set of critical

Table 5 Theoretical estimations of ρ_s in the MIS regime for different universality classes.

n	Magnetic system	Model	ρ_s
1	anisotropic magnets, uniaxial antiferromagnets	A	1.38
1	uniaxial ferromagnets	B	2.50
1	anisotropic magnets, uniaxial antiferromagnets	C	1.48
1	uniaxial ferromagnets	D	2.50
2	easy plane magnets, $h_z = 0$	A	1.35
		E	1.00
3	isotropic antiferromagnets	A	1.31
		G	0.94
3	isotropic ferromagnets	A	1.31
		J	1.65

exponents to another. One of the possible series is $H \rightarrow I \rightarrow AD$ and the second one is $H \rightarrow ID \rightarrow AD$. For the sequence $H \rightarrow I \rightarrow AD$ the behavior of the isotropic ferromagnet with $\rho_s = 1.65$ will be observed for high reduced temperatures. The exponents reported in [62] and [135] probably refer to this temperature range. For smaller reduced temperatures the Ising behavior with relaxational dynamics and $\rho_s = 1.38$ can be expected in such a sample (or in other sample in different temperature interval). Eventually, the system will be found in the temperature range controlled by the anisotropic dipolar fixed point where also the relaxational dynamics and the mean-field critical exponents with $\rho_s = 1$ are expected. Because the upper critical dimension for such system is $d = 3$ [20, 21] fractal powers of logarithms also appear [69, 141, 142]. For the sequence $H \rightarrow ID \rightarrow AD$ we should observe first $\rho_s = 1.65$ (for Heisenberg fixed point) then a crossover to the isotropic dipolar fixed point ($\rho_s = 1.31$) is expected with the static exponents only slightly different from the Heisenberg exponents. However, the dipolar interactions do not conserve the order parameter so $z_{ID} \simeq 2$. Smaller dynamic critical exponent implies smaller sound attenuation exponent $\rho_s = 1.31$ in this regime. The asymptotically stable fixed point for this sequence is again the anisotropic dipolar fixed point with $\rho_s = 1$ and the logarithmic factors [69, 141, 142]. As follows from this discussion gadolinium is such a complex system that it is very difficult to describe it with the aid of one fixed point (and one value of ρ_s). Instead a sequence of crossovers between different exponents is expected and the effective exponents are measured in different temperature ranges and for different samples.

7.3 Conclusion

This chapter deals with the theoretical and experimental studies of ultrasonic wave attenuation and velocity in magnets. It begins with a short description of the basic concepts of the statics of the phase transitions such as critical exponents, universality etc. Then a short introduction to the critical dynamics is presented, in which dynamical scaling, critical slowing down and main universality classes are described. It was shown that the adiabatic longitudinal sound velocity remains finite at the magnetic phase transition temperature. Additionally, an extensive discussion of the phenomenological theory of sound attenuation and dispersion is given. The effect of spin-lattice relaxation on the sound propagation is investigated on the basis of the stochastic model. Three asymptotic regimes for sound attenuation are discussed. Two of them: MIS and Kawasaki regime refer to the low-frequency range. The additional regime refers to the high-frequency range and corresponds to the adiabatic sound propagation. It transforms into the sound propagation in model C (with conserved energy) in the limit of vanishing spin-lattice relaxation frequency. An overview of experimental and theoretical sound attenuation exponents both for magnetic insulators as well as magnetic metals is given. The concept of the effective sound attenuation exponent is discussed and illustrated on the example of FeF_2 and RbMnF_3 . The crossovers between different regimes are shown for the antiferromagnet MnF_2 .

However some unsolved questions still remain. In future the experimental measurements should cover a wider frequency range. For example the measurements for frequencies lower than 10 MHz in MnF_2 would be extremely important in verification of the theory. The same applies to Cr_2O_3 for which only high frequency measurements are accessible so far. The critical sound propagation in low-dimensional systems is a very interesting topic both from theoretical and experimental point of view. Another challenge is the construction of critical sound attenuation in an external magnetic field. For the ferromagnets the magnetic field is coupled to the order parameter so the non-zero magnetic field destroys the phase transition and the scaling relation for the attenuation takes a form

$$\alpha(t, \omega, h) = \omega^2 t^{-\rho_s} f(\omega t^{-z\nu}, h t^{-\Delta})$$

or

$$\alpha(t, \omega, h) = \omega^2 h^{-\rho_s/\Delta} g(\omega h^{-z\nu/\Delta}, t h^{-1/\Delta})$$

where $\Delta = \gamma + \beta$ is the gap exponent and f, g are scaling functions [143]. Additional variable in the scaling function induces the existence of six asymptotic regions in the space (ω, t, h) instead of two (hydrodynamic and critical ones) in the space (ω, t) . The scaling functions need to be determined theoretically as well as experimentally. So far only some mean-field results have been obtained [144–147] and the measurements far from critical temperature were carried out in MnP [148]. In particular the sound attenuation and velocity exactly at the Curie temperature would be of

importance. In antiferromagnets the magnetic field does not destroy the continuous transition and only shifts the Neel temperature in a similar way as the transverse field shifts the Curie temperature in anisotropic ferromagnets. The effects of the external field on the critical sound propagation in antiferromagnets need a further theoretical and experimental studies.

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