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The classical operators on the space of real analytic functions

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Klasyczne operatory na przestrzeni funkcji analitycznych zmiennej rzeczywistej

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Abstract

The aim of this thesis is to investigate three classes of operators on the space of real analytic functions $\mathscr{A}(\mathbb{R})$: Hadamard multiplier operators, Hankel operators and Toeplitz operators. The study of the Hadamard multipliers concentrates on the problem of generating strongly continuous semigroup by these operators. Based on the theory developed by P. Domański and M. Langenbruch, we give a generation theorem for Hadamard multipliers on the space $\mathscr{A}(\mathbb{R})$ and apply it to the classical examples of multipliers.

Next, we study the Hankel operators. We give the integral representation of these operators and prove that the space of Hankel operators is isomorphic to the space of entire functions $H(\mathbb{C})$. We also investigate the spectra and other properties of Hankel operators on $\mathscr{A}(\mathbb{R})$.

Finally, we study Toeplitz operators acting on $\mathscr{A}(\mathbb{R})$. We give a characterization of left-sided invertible Toeplitz operators, which together with the result of M. Jasiczak on right-side invertibility solves completely the problem of one-sided invertibility of Toeplitz operators. The other result which we provide is the characterization of finite rank commutators of Toeplitz operators.

Streszczenie

Celem rozprawy jest zbadanie trzech klas operatorów na przestrzeni funkcji analitycznych zmiennej rzeczywistej $\mathscr{A}(\mathbb{R})$: operatorów mnożnikowych Hadamarda, operatorów Hankela i operatorów Toeplitza. Badając operatory mnożnikowe Hadamarda skupimy się na problemie generowania silnie ciągłej półgrupy przez te operatory. W oparciu o teorię rozwiniętą przez P. Domańskiego i M. Langenbrucha, podajemy twierdzenie o generatorach silnie ciągłej półgrupy dla mnożników Hadamarda stosujemy je do klasycznych przykładów mnożników.

Następnie badamy operatory Hankela. Podajemy reprezentację całkową operatorów Hankela i dowodzimy, że przestrzeń operatorów Hankela jest izomorficzna z przestrzenią funkcji całkowitych. Ponadto badamy spektrum oraz inne własności operatorów Hankela na $\mathscr{A}(\mathbb{R})$.

Ostatnim tematem roprawy sa operatory Toeplitza na $\mathscr{A}(\mathbb{R})$. Podajemy pełną klasyfikację lewostronnie odwracalnych operatorów Toeplitza, co wraz z wynikiem M. Jasiczaka dotyczącym prawostronnie odwracalnych operatorów Toeplitza, rozwiązuje problem jednostronnej odwracalności operatorów Toeplitza. Ponadto podajemy opis skończenie wymiarowych komutatorów operatorów Toeplitza.

Contents

1 Introduction

There is an extensive literature on the various classes of continuous linear operators on Banach and Hilbert spaces. Much less is known about the continuous operators on arbitrary non-metrizable locally convex spaces, including the space of real analytic functions. Let $\mathscr{A}(\mathbb{R})$ be the space of all complex-valued real analytic functions, i.e., every function $f \in \mathscr{A}(\mathbb{R})$ develops into a Taylor series convergent to f at every point $x \in \mathbb{R}$. Thus, every real analytic function can be extended to some complex neighbourhood of $\mathbb{R}$. Therefore it is natural to equip the space $\mathscr{A}(\mathbb{R})$ with the topology of the inductive system

$$\mathscr{A}(\mathbb{R}) = \operatorname{ind}_U H(U),$$

where U runs over all open complex neighbourhoods of $\mathbb{R}$ and $H(U)$ denotes the Fréchet space of all holomorphic functions on U with the compact-open topology. The classical results on the topology on $\mathscr{A}(\mathbb{R})$ is due to Martineau [38] and it was motivated by the study of the so-called hyperfunctions [55].

The space $\mathscr{A}(\mathbb{R})$ is a complete, seperable, ultrabornological and webbed nuclear locally convex space. This allows for many tools from classical functional anaysis to work in this setting, even though $\mathscr{A}(\mathbb{R})$ is very far from being metrizable. Although polynomials are sequentially dense in $\mathscr{A}(\mathbb{R})$ and for any analytic function $f \in \mathscr{A}(\mathbb{R})$ we have that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ for z small enough, the monomials do not form a Schauder basis of $\mathscr{A}(\mathbb{R})$. In fact, it was shown by Domański and Vogt ([21]) that the space $\mathscr{A}(\mathbb{R})$ has no Schauder basis. Nevertheless for every linear continuous operator A on $\mathscr{A}(\mathbb{R})$ there is an associated matrix uniquely determining A, i.e. the matrix $(a_{ij})_{i,j\in\mathbb{N}}$ such that

$$Ax^n(\xi) = \sum_{i=0}^{\infty} a_{in}\xi^i$$

around zero.

The space $\mathscr{A}(\mathbb{R})$ is a natural object in analysis with great relevance to the theory of partial differential equations, which has recently attracted more attention. There is an extensive literature on linear partial differential operators with constant coefficients on $\mathscr{A}(\mathbb{R})$ or $\mathscr{A}(\Omega)$, $\Omega \subset \mathbb{R}^d$, as well as on convolution operators on $\mathscr{A}(\mathbb{R})$ (see e.g., Hörmander [30], Napalkov-Rudakov [43], Langenbruch [37]). Recent studies of Domański and Vogt improves our knowledge about the topological structure and the properties of the space $\mathscr{A}(\mathbb{R})$ and its subspaces (e.g.,[12],[21],[19],[20]). Not much is known about the structure of the space of continuous operators $L_b(\mathscr{A}(\mathbb{R}))$. Recently, there have been some research on three classes of operators on $\mathscr{A}(\mathbb{R})$: composition operators (e.g., [5],[4],[14]), so called Hadamard multipliers operators (e.g., [17],[15],[18]) and Toeplitz operators ([13],[32]). The purpose of this thesis is to further investigate operators on the space $\mathscr{A}(\mathbb{R})$. To be more precise, we study the properties of Toeplitz and Hankel operators and investigate the strongly continuous semigroups generated by Hadamard multipliers.

The first class of operators we consider is the class of Hadamard multipliers. Let G_1, G_2 be domains in $\mathbb{C}$ containing zero and let $f\colon G_1 \to \mathbb{C}$, $g\colon G_2 \to \mathbb{C}$ be holomorphic functions with

Taylor series at zero given by $f(z) = \sum_{n=0}^{\infty} f_n z^n$ and $f(z) = \sum_{n=0}^{\infty} g_n z^n$. The Hadamard product of f anf g is defined by $(f \star g)(z) = \sum_{n=0}^{\infty} f_n g_n z^n$. In 1899 Jacques Hadamard published his famous multiplication theorem stating that $f \star g$ extends to a holomorphic function on a domain G_3 which is the complement of the set $G_1^c \cdot G_2^c$. Hadamard's multiplication theorem leads to the definition of a coefficient multiplier given in [42],[41]: Let G_1, G_2 be domains containing zero. A power series $g(z) = \sum_{n=0}^{\infty} g_n z^n$ is a coefficient multiplier if $g \star f \in H(G_2)$ for all $f \in H(G_1)$, i.e., $T_g(f) = g \star f$ defines a linear mapping $T_g: H(G_1) \to H(G_2)$.

In the recent years Domański and Langenbruch developed the corresponding theory of Hadamard multipliers on the space of real analytic functions $\mathscr{A}(\mathbb{R})$ and $\mathscr{A}(I)$ for any $I \subset \mathbb{R}$. The continuous linear operator $M: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is called a multiplier if every monomial is its eigenvector, i.e. $Mx^n = m_n x^n$ for all $n \in \mathbb{N}$ and some sequence $(m_n)_{n=0}^{\infty} \in \mathbb{C}$. If $f \in \mathscr{A}(\mathbb{R})$ develops into the Taylor series $f(z) = \sum_{n=0}^{\infty} f_n z^n$ at zero, then $Mf(z) = \sum_{n=0}^{\infty} m_n f_n z^n$ near zero. Hence the relation to classical (complex) coefficient multiplier is clear. In [16] the authors state the representation theorem for Hadamard multipliers on $\mathscr{A}(\mathbb{R})$: each multiplier corresponds to an analytic functional and by Köthe-Grothendieck-da Silva duality corresponds to a holomorphic function. In [15] the authors describe the multiplier sequences $(m_n)_{n\in\mathbb{N}}$ both in the "matricial language" and via interpolation properties of holomorphic functions with restricted growth. Papers [16],[15] together answer the question about the invertibility and surjectivity of the Hadamard multiplier on the space $\mathscr{A}(I)$ for an open subset $I \subset \mathbb{R}$.

In the thesis we investigate when a Hadamard multiplier $M: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ generates a strongly continuous semigroup. The C_0-semigroups arise naturally when studying the abstract Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t} u(t) &= Mu(t), \\ u(0) &= f, \end{aligned}$$

where M is a linear operator on $\mathscr{A}(\mathbb{R})$ and $f \in \mathscr{A}(\mathbb{R})$. We restrict ourselves to the case when M is a Hadamard multiplier. Given a Banach space, every continuous and linear operator from the space into itself generates a C_0-semigroup given by an exponential series representation. Hence, in the Banach space case the continuous generator is considered to be trivial situation. In a non-Banach locally convex space the exponential series $\exp(A)(x) = \sum_{n=0}^{\infty} A^k x/k!$ need not converge for a continuous operator, and a continuous linear operator does not always generate a strongly continuous semigroup (see for instanse [25, Example 4]). In the case of sequentially complete locally convex space not much is known. The first general generation theorem was the one mentioned in the book [58] of Yosida, which states that every power bounded operator always generates a C_0-semigroup. Recently, Wegner and the author of the thesis showed that m-topologizable operator on a sequentially complete locally convex space generates a uniformly continuous semigroup of operators [28, Thm. 1]. In chapter 3 we state a generation theorem for Hadamard multipliers on the space $\mathscr{A}(\mathbb{R})$. Next, we apply it to the classical examples of multipliers.

The first ones are finite order Euler differential operators, $E = \sum_{n=0}^{N} a_n \theta^n$, $\theta f(x) = xf'(x)$, $a_0, \ldots, a_n \in \mathbb{C}$. We prove that the first order Euler differential operator $E = a\theta + bI$ generates a strongly continuous semigroup if and only if $a \in \mathbb{R}$. Next we show the other cases when finite order Euler differential operator is not a ganerator of a C_0-semigroup. Unfortunately, we were not able to obtain a full characterization of Euler differtial generators. The other multiplier which we consider is the Hardy operator $Hf(x) = \frac{1}{x}\int_0^\infty f(y)dy$. We prove that every operator of the form $M = \sum_{n=0}^N a_n H^n$, $a_0, \dots, a_n \in \mathbb{C}$ generates a C_0-semigroup.

Let us note, that presented in the thesis results on the C_0-semigroups generated by the Hadamard multipliers was already published (see [26]).

The second class of operators which we are interested in are Hankel operators. An infinite matrix is called a Hankel matrix if it is of the form

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_1 & a_2 & a_3 & a_4 & \dots \\ a_2 & a_3 & a_4 & a_5 & \dots \\ \\ a_3 & a_4 & a_5 & a_6 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

where $(a_n)_{n\geq 0}$ is a sequence of complex numbers. A Hankel operator is a continuous operator for which the associated matrix is a Hankel matrix. Hankel operators are one of the most important classes of operators on the spaces of analytic functions and thus they are the object of interest of several domains of analysis such as operator theory, polynomial and rational approximation, interpolation by analytic or meromorphic functions.

The most classical setting for studying Hankel operators is the Hardy space $H^2(\mathbb{T})$. In that case Hankel operator H acts from the space $H^2(\mathbb{T})$ into $H^2_-(\mathbb{T}) = L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$. One of the earliest results on Hankel operators on $H^2(\mathbb{T})$ is the Kronecker's theorem which states that Hankel operator with associated matrix $[a_{i+j}]_{i,j\geq 0}$ is of finite rank if and only if $\sum_{n=0}^\infty a_n z^n$ is a rational function. The fundamental result in the theory of Hankel operator is the Nehari theorem [44] which states that for every bounded Hankel operator $H\colon H^2(\mathbb{T}) \to H^2(\mathbb{T})$ there exists a bounded function φ (a symbol of H) such that $Hf = H_\varphi f = P_-(\varphi f)$, $f \in H^2(\mathbb{T})$, where P_- denotes the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2_-(\mathbb{T})$. The next important result was a complete characterization of compact Hankel operators given by Hartman [29].

In the following years the spectral properties of the Hankel operators and the inverse spectral problem were studied, see for instance [47],[1],[39]. For more information on Hankel operators see the two recent monographs [45],[48] or the classical books [46],[53].

Lately Hankel operators have been studied on other Hilbert spaces of analytic functions – Bergman and Fock spaces and many analogues of the classical theorems were obtained. Hankel operators on the Hardy and Bergman spaces of several variables have been also studied recently, mainly on the unit disc and ball, but also on arbitrary strongly pseudoconvex domains.

In the thesis we study the class of Hankel operators on the space of real analytic functions $\mathscr{A}(\mathbb{R})$. We study operators for which the associated matrix is a Hankel matrix, but since the space $\mathscr{A}(\mathbb{R})$ does not have a basis the formal definition is the following

Definition. We say that a continuous operator $\Gamma\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is a *Hankel operator* if there exist complex numbers $a_0, a_1, a_2, \dots \in \mathbb{C}$ such that for all monomials x^n and for ξ near zero

$$\Gamma x^n(\xi) = \sum_{k=0}^\infty a_{n+k}\xi^k.$$

We call the function $\Gamma x^0 \in \mathscr{A}(\mathbb{R})$, $\Gamma x^0(\xi) = \sum_{k=0}^{\infty} a_k \xi^k$ around zero, the symbol of the operator H.

In chapter 4 we find the integral representation of Hankel operators and prove a representation theorem, which states that the space of all Hankel operators on $\mathscr{A}(\mathbb{R})$ with the topology induced from $L_b(\mathscr{A}(\mathbb{R}))$ is topologically isomorphic to the space of entire functions $H(\mathbb{C})$. Next, we study properties of the Hankel operators. The Kronecker theorem for finite rank Hankel operators can be applied in the setting of $\mathscr{A}(\mathbb{R})$. In the thesis we try to find the spectrum of a Hankel operators. In [47] Peller proved that Hankel operator $H_\varphi\colon H^2(\mathbb{T}) \to H^2(\mathbb{T})$ with $\varphi \in H(\mathbb{D})$ is in p- Schatten-von Neumann ideal if and only if φ is in a Besov class $B_p^{1/p}$. Using this result we prove that the spectrum of Hankel operator $\Gamma\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is equal to the point spectrum and the sequence of eigenvalues belongs to the space of rapidly decreasing sequences s.

The last class of operators investigated in the thesis is the class of Toeplitz operators. An infinite Toeplitz matrix is a matrix that is constant on each line parallel to the main diagonal, i.e. a matrix of the form

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \dots \\ a_1 & a_0 & a_{-1} & a_{-2} & \dots \\ a_2 & a_1 & a_0 & a_{-1} & \dots \\ a_3 & a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

where $(a_n)_{n\geq 0}$ is a sequence of complex numbers.

Similarly to Hankel operators, the most natural setting for studying Toeplitz operators is the Hardy space $H^2(\mathbb{T})$, where they are probably the most-studied and best-known class of operators. The simple, but very important examples are forward and backward shift. Toeplitz operators on $H^2(\mathbb{T})$ are defined to be the compressions of multiplication operators to the space $H^2(\mathbb{T})$, i.e. for each function $\varphi \in L^\infty(\mathbb{T})$ the Toeplitz operator with symbol φ is given by $T_\varphi f = P(\varphi f)$ for each $f \in H^2(\mathbb{T})$, where P denotes the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. The fundamental theorem of Brown and Halmos [7] states that the Toeplitz operators on $H^2(\mathbb{T})$ are exactly these operators whose matrices with respect to the basis $(e^{in\theta})_{n=0}^\infty$ of $H^2(\mathbb{T})$ are Toeplitz matrices. The study of spectra of Toeplitz operators led to many interesting results. Although there is no known way of expressing the spectrum of the Toeplitz operator in terms of the symbol for general $\varphi \in L^\infty(\mathbb{T})$, there are some special cases which are quite well understood. Let us mentioned here just two of them. There is a complete description of the spectrum of Toeplitz operators in case the symbol φ is a continuous function and in the case of self-adjoint Toeplitz operators.

Naturally, in the following years the Toeplitz operators were also considered on other function spaces like other Hardy spaces $H^p(\mathbb{T})$, Bergman spaces $A^2(\mathbb{D})$ or Fock spaces and the corresponding theories are well developed now. Recently, Domański and Jasiczak established the theory of Toeplitz operators on the space of real analytic functions $\mathscr{A}(\mathbb{R})$ [13],[33],[32]. It turned out that this theory is quite similar to the classical one. A Toeplitz operator on $\mathscr{A}(\mathbb{R})$ is the operator whose associated matrix is a Toeplitz matrix, but once again, since $\mathscr{A}(\mathbb{R})$ does not have a basis we use the following precise definition

Definition. We say that a continuous operator $T: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is a *Toeplitz operator* if there exist complex numbers $\ldots, a_{-1}, a_0, a_1, \ldots \in \mathbb{C}$ such that for all $n \in \mathbb{N}$ locally near zero

$$Tx^n(\xi) = a_{-n} + a_{-n+1}\xi + a_{-n+2}\xi^2 + \ldots.$$

In [13] Domański and Jasiczak proved that a Toeplitz operator on $\mathscr{A}(\mathbb{R})$, similarly to the classical one, is a compressions of a multiplication operator. More precisely they proved that an operator $T: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is a Toeplitz operator if and only if there exists a function $F \in \mathcal{X}$ such that $T = \mathcal{C}M_F$. The symbol space $\mathcal{X}$ is defined as the inductive limit of Fréchet spaces:

$$\mathcal{X} = \operatorname{ind}_{U,K} H(U \setminus K)$$

where U runs over all open complex neighbourhoods of $\mathbb{R}$ and K runs through all compact sets of $\mathbb{R}$. The symbol $\mathcal{C}$ denotes the appropriate Cauchy transform which is also a projection from $\mathcal{X}$ onto $\mathscr{A}(\mathbb{R})$. The operator M_F is the operator of multiplication by F. In the same paper the authors give a full characterization of Fredholm Toeplitz operators on $\mathscr{A}(\mathbb{R})$.

A Teoplitz operator on $\mathscr{A}(\mathbb{R})$ is invertible if and only if it is a Fredholm operator with index zero [32]. In chapter 5 of the thesis we study the problem of one-sided invertibility of Toeplitz operators on $\mathscr{A}(\mathbb{R})$ and give the complete characterization of one-sided invertible Toeplitz operators. This section is a part of the joint work with M. Jasiczak.

In [9] Ding and Zeng described when the commutator of two Toeplitz operators on Hardy space $H^2(\mathbb{T})$ has finite rank. Based on their work we give a complete characterization of the finite rank commutators of Teoplitz operators on $\mathscr{A}(\mathbb{R})$.

In the next Chapter 2 we will recall the definitions and notions necessary in this thesis.

2

The aim of this chapter is to introduce necessary notions and to collect basic facts which will be used further on. We use the standard notation and terminology. All the notions from functional analysis that are not defined here are explained in [40] and those from complex analysis in [8], [54].

2.1 The space of real analytic functions $\mathscr{A}(\mathbb{R})$

A complex-valued function $f\colon \mathbb{R} \to \mathbb{C}$ is real analytic at $x_0 \in \mathbb{R}$ if it develops into Taylor series centered at x_0 which converges to f on a small neigbourhood of x_0. It is called a real analytic function on $\mathbb{R}$ if it is real analytic at every point of $\mathbb{R}$. We denote by $\mathscr{A}(\mathbb{R})$ the space of all real analytic functions. Since $f \in \mathscr{A}(\mathbb{R})$ develops into a Taylor series at every point, it can be extended to a holomorphic function on some open neigbourhood U of $\mathbb{R}$. Therefore it is natural to equip the space $\mathscr{A}(\mathbb{R})$ with the inductive system topology,

$$\mathscr{A}(\mathbb{R}) = \operatorname{ind}_U H(U),$$

where U runs over all open neigbourhood of $\mathbb{R}$ and $H(U)$ denotes the Fréchet space of functions holomorphic on U with the topology of uniform convergence on compact subsets of U. The topology on $\mathscr{A}(\mathbb{R})$ is then the finest locally convex topology for which all the restriction maps $r_U\colon H(U) \to \mathscr{A}(\mathbb{R})$, $r_U(f) = f|_{\mathbb{R}}$, are continuous. For more information on the construction of the inductive limit of an inductive system of locally convex spaces we refer to [24, §23].

There is a second natural topology that we can equip the space $\mathscr{A}(\mathbb{R})$ with, namely the topology of a projective system of locally convex spaces. Let K be a compact subset of $\mathbb{R}$. Recall that in the set of all functions that are holomorphic on some neighborhood of K we can define an equivalence relation: $f \sim_K g$ if and only if there exists an open neigbourhood U of K such that $f|_U = g|_U$. Equivalence classes of this relation are called germs. By $H(K)$ we denote the space of germs of holomorphic functions on K. For more information on the space $H(K)$ we refer to [3, p. 63]. Each real analytic function $f \in \mathscr{A}(\mathbb{R})$ is holomorphic in some open complex neigbourhood of $\mathbb{R}$ and so it is also holomorphic in some open complex neigbourhood of K. Hence f defines the germ $[f]_{\sim K}$ which belongs to $H(K)$. We consider the space $\mathscr{A}(\mathbb{R})$ with the topology of the projective system,

$$\mathscr{A}(\mathbb{R}) = \operatorname{proj}_{K \Subset \mathbb{R}} H(K)$$

i.e., the coarsest (locally convex) topology for which all the maps $\pi_K\colon \mathscr{A}(\mathbb{R}) \to H(K)$, $\pi_K(f) = [f]_{\sim K}$ are continuous.

The following deep theorem is due to Marineau

Theorem 2.1.1. *[38] The topology of the inductive system $(r_U\colon H(U) \to \mathscr{A}(\mathbb{R}))_{U \supset \mathbb{R}}$ coincides with the topology of the projective system $(\pi_K\colon \mathscr{A}(\mathbb{R}) \to H(K))_{K \Subset \mathbb{R}}$.*

Recall that from the inductive system topology it follows

Proposition 2.1.2. *A sequence $(f_n) \in \mathscr{A}(\mathbb{R})$ converges to f in $\mathscr{A}(\mathbb{R})$ if and only if there exists an open neigbourhood U of $\mathbb{R}$ such that $f_n \in H(U)$ for all n and f_n converges to f in $H(U)$.*

The space of real analytic functions is not a Banach space, not even a Fréchet space, but it still has some useful properties. We start with a short introduction to the notions used in the thoery of general locally convex spaces.

Definition 2.1.3. A locally convex space E is said to be *barrelled*, if each absolutely convex, closed and absorbing set in E is a zero neighborhood.

Definition 2.1.4. A locally convex space E is said to be *nuclear*, if for each continuous seminorm p on E there exists a continuous seminorm q with $q \geq p$ such that the canonical linking map $i_q^p \colon E_q \to E_p$ between local Banach spaces is nuclear.

Nuclear spaces are an important class of the locally convex spaces because many natural non-normable locally convex spaces are in fact nuclear. For more information on the nuclear locally convex spaces we refer to [40, Chapter 28], [31, Chapter 21].

Definition 2.1.5. A locally convex space E is said to be *ultrabornological* if it has the topology of some inductive system $(j_i \colon E_i \to E)_{i \in I}$ of Banach spaces. E is said to be an *LB-space* if it has the topology of a countable inductive system of Banach spaces.

Definition 2.1.6. A web $\{C_{n_1,\dots,n_k}\}$ in a locally convex space E is a family $C_{n_1,\dots,n_k}, n_1, \dots, n_k \in \mathbb{N}$, $k \in \mathbb{N}$, of absolutely convex subsets of E with the following properties:

1. $\bigcup_{n=1}^{\infty} C_n = E$
2. $\bigcup_{n=1}^{\infty} C_{n_1,\dots,n_k,n} = C_{n_1,\dots,n_k}$ for all $n_1, \dots, n_k \in \mathbb{N}$, $k \in \mathbb{N}$
3. For each sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $(0,\infty)$ so that for every sequence $(x_k)_{k\in\mathbb{N}}$ in E with $x_k \in C_{n_1,\dots,n_k}$ for all $k \in \mathbb{N}$ the series $\sum_{k=1}^{\infty} \lambda_k x_k$ converges in E.

Lemma 2.1.7. *[40, 24.28] Let E be a locally convex space and F be a closed subspace of E. If E admits a web then so do F and E/F.*

The notion of webbed and ultrabornological space allows to generalize the open mapping theorem and closed graph theorem.

Theorem 2.1.8. *[40, 24.30] Let E and F be locally convex spaces. If E has a web and F is ultrabornological, then every continuous, linear, surjective map $A \colon E \to F$ is open.*

Theorem 2.1.9. *[40, 24.31] let E and F be locally convex spaces. If E has a web and F is ultrabornological, then every linear map $A \colon F \to E$ with a closed graph is continuous.*

Proposition 2.1.10. *[11, 1.28][20, 1.6] The space $\mathscr{A}(\mathbb{R})$ has the following properties*

- *it is barrelled, ultrabornological, nuclear and reflexive,*
- *it is seperable with polynomials with rational coefficients forming a dense subset,*

- *it is webbed,*
- *bounded sets are compact.*

Recall that a subset B of a locally convex space E is called *bounded*, if for each zero neighborhood U there exists $\varepsilon > 0$ such that $\varepsilon B \subset U$.

Although polynomials are dense in $\mathscr{A}(\mathbb{R})$ and for any analytic function $f \in \mathscr{A}(\mathbb{R})$ we have that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ around zero, the monomials do not form a Schauder basis of $\mathscr{A}(\mathbb{R})$. In fact, the space $\mathscr{A}(\mathbb{R})$ has no Schauder basis ([21, Thm. 4.1]).

We will now describe the dual space $\mathscr{A}(\mathbb{R})'_b$ equipped with the strong topology, i.e. the topology of uniform convergence on bounded subsets of $\mathscr{A}(\mathbb{R})$. Let $\mathbb{C}_\infty$ denote the Riemann sphere. For a compact set $K \subset \mathbb{R}$ we denote by $H_0(\mathbb{C}_\infty \setminus K)$ the Fréchet space of functions holomorphic in $\mathbb{C}_\infty \setminus K$ which vanish at infinity. We will always assume that K is connected.

By the Köthe-Grothendieck-da Silva duality (see [36, pp. 372-378] or [2, Thm. 1.3.5]) we can identify the dual space of $H(K)$ with the space $H_0(\mathbb{C}_\infty \setminus K)$. More precisely, for every functional $T \in H(K)'$ there exists a unique function $g_T \in H_0(\mathbb{C}_\infty \setminus K)$ such that for every germ $[f]_{\sim K} \in H(K)$

$$T([f]_{\sim K}) = \langle [f]_{\sim K}, g_T \rangle = \frac{1}{2\pi} \int_\gamma f(z) g_T(z) dz,$$

where $f \in H(U)$ for some simply connected neighborhood U of K, γ is a C^∞ smooth Jordan curve lying in $U \setminus K$ such that $\mathrm{Ind}_\gamma(x) = 1$ for $x \in K$.

From the description of the topology as a projective limit it follows that algebraically

$$\mathscr{A}(\mathbb{R})' = \mathrm{ind}_K H(K)' = \mathrm{ind}_K H_0(\mathbb{C}_\infty \setminus K),$$

where K runs over all compact subsets of $\mathbb{R}$. In fact, $\mathscr{A}(\mathbb{R})'_b$ with the strong topology is topologically isomorphic to the space $\mathrm{ind}_K H(K)'$ [20, Prop. 1.7]. In the thesis we use the notation

$$H_0(\mathbb{C}_\infty \setminus \mathbb{R}) = \mathrm{ind}_K H_0(\mathbb{C}_\infty \setminus K).$$

Notice that it is enough to take the inductive limit $\mathrm{ind}\, H_0(\mathbb{C} \setminus K)$, where K is connected. Further on we will always assume that K is connected.

By $L_b(\mathscr{A}(\mathbb{R}))$ we denote the space of all continuous operators on $\mathscr{A}(\mathbb{R})$ equipped with the topology of uniform convergence on bounded sets of $\mathscr{A}(\mathbb{R})$.

Proposition 2.1.11. *[17, Proof of Theorem 2.6] The space $L_b(\mathscr{A}(\mathbb{R}))$ admits a web.*

Let $T\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ be a linear continuous operator such that

$$T(z^k)(x) = \sum_{j=0}^{\infty} a_{jk} x^j \text{ around zero.}$$

We call the matrix $A = [a_{jk}]_{j,k \in \mathbb{N}}$ the *matrix corresponding to the operator T*.

We use the following definition of a bounded operator.

Definition 2.1.12. Let X be a locally convex space. An operator $T \in L_b(X)$ is called *bounded* if it maps some 0-neighbourhood of X into a bounded set.

We will need the following fact concerning bounded operators on LB-spaces

Proposition 2.1.13. *[40, Ex. 25.9] Every continuous operator from an LB-space into a Fréchet space is bounded.*

A bounded operator $T: X \to Y$ between locally convex spaces is called *strictly singular* if the restriction of T to any closed infinite-dimensional subspace of X is not an isomorphism. By an isomorphism we mean a continuous bijection with a continuous inverse.

Proposition 2.1.14. *Let X be an infinite dimensional nuclear, webbed locally convex space. Every linear and continuous operator on X which is bounded is strictly singular.*

Proof. Let $T \in L(X)$ be bounded. Then T can be written as a product of two operators $T = R \circ S$, such that $S: X \to Y$, $R: Y \to X$, and Y is a Banach space ([57, Satz 1]). Assume that T is not strictly singular, i.e. there exists a closed subspace $E \subset X$, $\dim E = \infty$ such that $T|_E$ is an isomorphism. Let $T^{-1}: T(E) \to E$ be a continuous inverse of T. We can extend it to the continuous operator $T^{-1}: \overline{T(E)} \to E$. It follows that $T(E)$ is closed in X. Indeed, let (z_α) be a net in $T(E)$ such that $z_\alpha \to z$. Then $z \in \overline{T(E)}$ and $T^{-1}z_\alpha \to T^{-1}z$. Hence $TT^{-1}z_\alpha = z_\alpha \to z$ and $TT^{-1}z_\alpha \to TT^{-1}z$. Hence $z = TT^{-1}z$ and $z \in T(E)$.

We claim that also the image $S(E)$ is closed in Y. Indeed, take a net $(x_\alpha)_{\alpha \in I} \subset E$ such that $Sx_\alpha \to y$. It follows that $Tx_\alpha = RSx_\alpha \to Ry$. Since $T(E)$ is closed in X there exists $z \in E$ such that $Tx_\alpha \to Tz$. We apply the inverse operator T^{-1} to both sides and get that $x_\alpha \to z$. Hence $Sx_\alpha \to Sz$ and $S(E)$ is closed subspace of Y. Since $S(E)$ is a Banach space and E has a web we can use the open mapping theorem to conclude that $S|_E$ is an isomorphism. Since the space X is nuclear it follows that $S(E)$ is nuclear. Since there are no nuclear infinite dimensional normed spaces ([31, 16.1.4]) we get a contradiction. □

2.2 C_0-semigroups of operators

In this section we introduce notation and collect definitions and facts from the theory of strongly continuous semigroups of operators which will be used in the thesis. Throughout this section X will always denote an arbitrary locally convex space and $L(X)$ the space of continuous operators on X. For more information on strongly continuous semigroups on locally convex spaces we refer to [35].

A family M in $L(X)$ is said to be *equicontinuous*, if for any neigbourhood U of zero in X, there exists a neighbourhood V of zero such that $T(V) \subset U$ for all $T \in M$.

Definition 2.2.1. A one-parameter family $(T_t)_{t \geq 0}$ in $L(X)$ is called a C_0-*semigroup* (or *strongly continuous semigroup*), if it satisfies the following conditions:

1. $T_t T_s = T_{t+s}$ for all $t, s \geq 0$
2. $T_0 = \mathrm{id}_X$
3. $\lim_{t \to s} T_t x = T_s x$ for any $s \geq 0$ and $x \in X$.

A C_0-semigroup $(T_t)_{t\geq 0}$ is said to be *locally equicontinuous*, if for any $0 < s < \infty$ the subfamily $\{T_t : 0 \leq t \leq s\}$ is equicontinuous in $L(X)$.

If the above properties hold for $t, s \in \mathbb{R}$ instead of $t, s \in \mathbb{R}_+ := [0, \infty)$, we call $(T_t)_{t\in\mathbb{R}}$ a *C_0-group*.

Proposition 2.2.2. *[35, Prop. 1.1] If X is barrelled, then every C_0-semigroup is locally equicontinuous.*

The *infinitesimal generator* $(A, D(A))$ of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on X is the (not neccessarily continuous) linear operator

$$Ax = \lim_{t\to 0} \frac{T_t x - x}{t} = \left.\frac{\partial T_t x}{\partial t}\right|_{t=0}$$

defined for every x in its domain

$$D(A) = \{x \in X : \lim_{t\to 0} \frac{T_t x - x}{t} \text{ exists}\}.$$

Proposition 2.2.3. *[35, Prop. 1.4] For every locally equicontinuous semigroup on X, its infinitesimal generator is closed.*

Since the space $\mathscr{A}(\mathbb{R})$ is barrelled, by Proposition 2.2.2, it follows that the infinitesimal generator of a C_0-semigroup of operators on $\mathscr{A}(\mathbb{R})$ is always closed.

Proposition 2.2.4. *[35, Prop. 1.2] Let $(T_t)_{t\geq 0}$ be a C_0-semigroup in a locally convex space X.*

1. *If $x \in D(A)$, then $T_t x \in D(A)$ for any $t \geq 0$ and $T_t x$ is continuously differentiable in t relative to the topology of X, and*

$$\frac{d}{dt} T_t x = AT_t x = T_t Ax \quad \text{for every } t \geq 0.$$

2. *An element $x \in X$ belongs to $D(A)$ and $Ax = y$ if and only if*

$$T_t x - x = \int_0^t T_s y ds \quad \text{for every } t \geq 0.$$

In the case of Banach spaces the well known spectral inclusion theorem holds ([22, 2.5]). For general locally convex space a similar property holds for the point spectrum.

Proposition 2.2.5. *Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ acting on a locally convex space X. If x is an eigenvector of A with an eigenvalue λ, then for every $t \geq 0$*

$$T_t x = e^{t\lambda} x.$$

Proof. For a fixed eigenvector x with an eigenvalue λ denote by $(S_t)_{t\geq 0}$ the rescaled semigroup, $S_t = e^{-t\lambda}T_t$. Clearly, the semigroup $(S_t)_{t\geq 0}$ is strongly continuous. Let us denote by B the generator of $(S_t)_{t\geq 0}$. For every $x \in X$ we have

$$\begin{aligned}\frac{S_t x - x}{t} = \frac{e^{-\lambda t}T_t x - x}{t} &= \frac{e^{-\lambda t}T_t x - T_t x + T_t x - x}{t}\\ &= \frac{e^{-\lambda t} - 1}{t}T_t x + \frac{T_t x - x}{t}.\end{aligned}$$

Since

$$\frac{e^{-\lambda t} - 1}{t}T_t x \xrightarrow{t\searrow 0} -\lambda x,$$

we observe that $D(B) = D(A)$ and $B = A - \lambda$.

For $x \in D(A-\lambda)$, by Proposition 2.2.4, we have

$$S_t x - x = \int_0^t S_s(A-\lambda)x ds.$$

Hence

$$e^{-\lambda t}T_t x - x = \int_0^t e^{-\lambda t}T_s(A-\lambda)x ds.$$

As $Ax = \lambda x$ by assumption, the right hand side equals 0 and we have $T_t x = e^{t\lambda}x$. □

Hadamard multiplier operators and C_0-semigroups

3

A continuous operator on $\mathscr{A}(\mathbb{R})$ is called a Hadamard multiplier, if every monomial is its eigenvector. The goal of this chapter is to investigate the problem of genarating the C_0-semigroup by the Hadamard operators. Note that on a non-Banach locally convex space a continuous linear operator does not always generate a strongly continuous semigroup (see [25]).

We start with a short introduction to the theory of multipliers on $\mathscr{A}(\mathbb{R})$, developed by Domański and Langenbruch [15],[16],[17]. We explain the basic definitions and state representation theorems for multipliers, which will be our main tool in the second part.

Next we present the criterion for a multiplier to be a generator of a C_0-semigroup. We use it for some Euler differential operators. Unfortunately, we were not able to obtain a full characterization of the generators of C_0-semigroups in this class of operators.

Finally, we introduce the concept of Mellin transform and prove that the Hardy operator, $Hf(x) = \frac{1}{x}\int_0^x f(y)dy$, generates a C_0-semigroup on $\mathscr{A}(\mathbb{R})$.

3.1 Algebra of Hadamard multipliers

A continuous operator $M\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is called a *Hadamard multiplier*, if every monomial is its eigenvector, i.e.

$$Mx^n = m_n x^n \quad \text{for all } n \in \mathbb{N}.$$

We call the sequence $(m_n)_{n\in\mathbb{N}}$ the *multiplier sequence*. Since monomials are linearly dense in the space $\mathscr{A}(\mathbb{R})$, a multiplier is uniquely determined by its multiplier sequence. We will use notation $(M,(m_n))$ for a multiplier M with multiplier sequence $(m_n)_{n\in\mathbb{N}}$.

The most common examples of multipliers are:

- the Euler differential operators

$$E(f) = \sum_{n=0}^{N} a_n \theta^n f, \quad \text{where } \theta f(x) = xf'(x),$$

- the dilation operator

$$D_a f(x) = f(ax), \quad \text{for } a \in \mathbb{R},$$

- the Hardy operator

$$Hf(x) = \frac{1}{x}\int_0^x f(y)dy.$$

We denote by $M(\mathbb{R})$ the set of all multipliers on $\mathscr{A}(\mathbb{R})$. The space $M(\mathbb{R})$, with the topology induced from $L_b(\mathscr{A}(\mathbb{R}))$, is a closed linear subspace of $L_b(\mathscr{A}(\mathbb{R}))$. Moreover, $M(\mathbb{R})$ is an algebra with composition as a seperately continuous multiplication. In [17] Domański and Langenbruch investigated the properties of the multipliers and the space $M(\mathbb{R})$.

The following theorem tells us that each multiplier can be represented as a special convolution with an analytic functional.

Theorem 3.1.1 (The first representation theorem). *[17, Thm. 2.6] The map $\mathcal{B}\colon \mathscr{A}(\mathbb{R})' \to M(\mathbb{R})$ given by*

$$\mathcal{B}(T)(g)(y) := \langle g(y\cdot), T\rangle$$

is a bijective continuous map and the multiplier sequence of $\mathcal{B}(T)$ is equal to the sequence of moments of the analytic functional T, i.e. to $(\langle z^n, T\rangle)_{n\in\mathbb{N}}$.

We will also need another representation of the space $M(\mathbb{R})$. Recall that for a compact set $K \subset \mathbb{R}$ we denote by $H(\mathbb{C}_\infty \setminus K)$ the Fréchet space of functions holomorphic on $\mathbb{C}_\infty \setminus K$ and by $H_0(\mathbb{C}_\infty \setminus K)$ its subspace consisting of functions that vanish at infinity. Put

$$H_0(\mathbb{C}_\infty \setminus \mathbb{R}) := \operatorname{ind}_{K\Subset\mathbb{R}} H_0(\mathbb{C}_\infty \setminus K).$$

The space $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ equipped with the Hadamard multiplication of Laurent series, i.e.

$$f * g(z) = \sum_{n=0}^{\infty} \frac{f_n g_n}{z^{n+1}} \quad \text{around infinity}$$

for

$$f(z) = \sum_{n=0}^{\infty} \frac{f_n}{z^{n+1}}, \quad g(z) = \sum_{n=0}^{\infty} \frac{g_n}{z^{n+1}} \quad \text{around infinity,}$$

forms an algebra. The algebra $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ is isomorphic to the algebra $H(\mathbb{C}_\infty \setminus \frac{1}{\mathbb{R}})$ of functions holomorphic at zero which extend to holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$ with Hadamard multiplication of Taylor series, i.e

$$f * g(z) = \sum_{n=0}^{\infty} f_n g_n z^n \quad \text{around zero}$$

for

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \quad g(z) = \sum_{n=0}^{\infty} g_n z^n \quad \text{around zero.}$$

The isomorphism $\varphi\colon H_0(\mathbb{C}_\infty \setminus \mathbb{R}) \to H(\mathbb{C}_\infty \setminus \frac{1}{\mathbb{R}})$ is given by $\varphi(f)(z) = \frac{1}{z} f(\frac{1}{z})$.

Now, we can state the second representation theorem

Theorem 3.1.2 (The second representation theorem). *[17, Thm. 2.8] The algebra of multipliers $M(\mathbb{R})$ is topologically isomorphic as an algebra with the following algebras of holomorphic functions:*

(1) *$H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ with Hadamard multiplication of Laurent series,*

(2) *$H(\mathbb{C}_\infty \setminus \frac{1}{\mathbb{R}})$ with Hadamard multiplication of Taylor series.*

The multiplier sequence of the given multiplier is equal to the Laurent (Taylor) coefficients at infinity (zero) (f_n) of the corresponding function f.

From Theorem 3.1.1 and Theorem 3.1.2 it follows

Corollary 3.1.3. *For every functional $T \in \mathscr{A}(\mathbb{R})'$ there exists a holomorphic function $f \in H(\mathbb{C} \setminus \frac{1}{\mathbb{R}})$, $f(\xi) = \sum_{n=}^{\infty} f_n \xi^n$ around zero, with Taylor coefficients equal to moments of the functional T, i.e. $f_n = \langle z^n, T \rangle$.*

We will also need the following simple fact

Proposition 3.1.4. *[15, Prop. 2.1] The following conditions are equivalent:*

1. *$M: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is a multiplier with a multiplier sequence $(m_n)_{n\in\mathbb{N}}$*
2. *for every function $f \in \mathscr{A}(\mathbb{R})$, $f(z) = \sum_{n=0}^{\infty} f_n z^n$ around zero, we have*

$$Mf(z) = \sum_{n=0}^{\infty} m_n f_n z^n$$

around zero.

3.2 C_0-semigroups generated by Euler differential operators

In this section we state the criterion for multipliers to generate a C_0- semigroup and apply it to the Euler differential operators.

Theorem 3.2.1. *Let $M: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ be a multiplier with the multiplier sequence $(m_n)_{n\in\mathbb{N}}$. The following assertions are equivalent:*

- (i) *The multiplier M generates a C_0-semigroup $(T_t)_{t\geq 0}$.*
- (ii) *For every $t \in \mathbb{R}_+$ there exists a multiplier $(T_t, (m_n^t))$ with the multiplier sequence $(m_n^t)_{n\in\mathbb{N}} = (\exp(tm_n))_{n\in\mathbb{N}}$ and the map $Tf: \mathbb{R}_+ \to \mathscr{A}(\mathbb{R})$, $Tf(t) = T_t f$ is continuous for every $f \in \mathscr{A}(\mathbb{R})$.*
- (iii) *For every $t \in \mathbb{R}_+$ there exists a multiplier $(T_t, (m_n^t))$ with the multiplier sequence $(m_n^t)_{n\in\mathbb{N}} = (\exp(tm_n))_{n\in\mathbb{N}}$ and the set $\{T_t f : t \in [0, t_0]\}$ is bounded in $\mathscr{A}(\mathbb{R})$ for every $f \in \mathscr{A}(\mathbb{R})$ and some (equivalently, every) $t_0 > 0$.*

Proof. $(i) \Rightarrow (ii)$: From Proposition 2.2.5 it follows that if a C_0-semigroup $(T_t)_{t\geq 0}$ is generated by a multiplier $(M, (m_n))$, then it is a semigroup of multipliers. Moreover, for every $t \in \mathbb{R}_+$ the multiplier sequence of $(T_t, (m_n^t))$ is given by $m_n^t = \exp(tm_n)$.

$(ii) \Rightarrow (iii)$: Clear.

$(iii) \Rightarrow (i)$: First we will show that the multipliers $(T_t, (m_n^t))$ form a semigroup. For every $t, s \geq 0$ and every monomial x^n we have

$$T_t T_s x^n = T_t e^{sm_n} x^n = e^{(t+s)m_n} x^n = T_{t+s} x^n.$$

Since polynomials are dense in $\mathscr{A}(\mathbb{R})$ we get that $T_t T_s = T_{t+s}$ for every $t, s \geq 0$ and $(T_t)_{t\geq 0}$ is indeed a semigroup.

Now we will show that $(T_t)_{t\geq 0}$ is a C_0-semigroup. We assume that the set $\{T_t f : t \in [0, t_0]\}$ is bounded in $\mathscr{A}(\mathbb{R})$ for arbitrary $f \in \mathscr{A}(\mathbb{R})$ and some $t_0 > 0$. Notice that for any $t_1 > 0$ we have $\{T_t f : t \in [0, t_1]\} \subset \bigcup_{n=0}^{N} \{T_t f : t \in [nt_0, (n+1)t_0]\} = \bigcup_{n=0}^{N} T_{t_0}^n(\{T_t f : t \in [0, t_0]\})$ for some $N \in \mathbb{N}$. Hence the set $\{T_t f : t \in [0, t_1]\}$ is bounded for arbitrary t_1.

We denote the natural topology on $\mathscr{A}(\mathbb{R})$ by τ. Recall that an operator

$$V: \mathscr{A}(\mathbb{R}) = \operatorname{ind}_{\mathbb{R}\subset U} H(U) \to \mathbb{C}$$

is continuous if and only if $V \circ r_U : H(U) \to \mathbb{C}$ is continuous for every complex neighbourhood U of $\mathbb{R}$ ([11, 1.25]). Denote by $\mathbb{C}^{\mathbb{N}}$ the space of all sequences with the topology of pointwise convergence. Then the linear map

$$\begin{aligned} B : \mathscr{A}(\mathbb{R}) &\longrightarrow \mathbb{C}^{\mathbb{N}} \\ f &\longmapsto \left(\frac{f^{(n)}(0)}{n!}\right)_n, \end{aligned}$$

is continuous. Indeed, from the Cauchy inequality we get $\left|\frac{f^{(n)}(0)}{n!}\right| \leq C_{K,n} \|f\|_{\infty,K}$ for any compact set $K \subset U$ with $0 \in \operatorname{Int} K$. Hence we can consider $\mathscr{A}(\mathbb{R})$ with the coarser topology induced by the map above i.e. $\tau' = B^{-1}(\tau_{\mathbb{C}^{\mathbb{N}}})$, where $\tau_{\mathbb{C}^{\mathbb{N}}}$ denotes the topology on $\mathbb{C}^{\mathbb{N}}$.

The multiplier sequence of T_t equals $(e^{tm_n})_{n\in\mathbb{N}}$. Hence, by Proposition 3.1.4, we have $(T_t f)^{(n)}(0) = e^{tm_n} f^{(n)}(0)$ and the map

$$\begin{aligned} C_f : \mathbb{R}_+ &\longrightarrow \mathbb{C}^{\mathbb{N}} \\ t &\longmapsto \left(\frac{(T_t f)^{(n)}(0)}{n!}\right)_n = \left(\frac{e^{tm_n} f^{(n)}(0)}{n!}\right)_n \end{aligned}$$

is continuous.

Consider the mapping $Tf : \mathbb{R}_+ \to (\mathscr{A}(\mathbb{R}), \tau)$, $Tf(t) := T_t f$. The map

$$Tf : \mathbb{R}_+ \to (\mathscr{A}(\mathbb{R}), B^{-1}(\tau_{\mathbb{C}^{\mathbb{N}}})))$$

is continuous. Indeed, take an open set $U \in B^{-1}(\tau_{\mathbb{C}^{\mathbb{N}}}))$. Hence, there exists an open set $V \in \mathbb{C}^{\mathbb{N}}$ such that $U = B^{-1}(V)$ and we have

$$(Tf)^{-1}(U) = (Tf)^{-1}(B^{-1}(V)) = (B \circ Tf)^{-1}(V) = C_f^{-1}(V).$$

Since by the assumption the set $\{T_t f : t \in [0, t_0]\}$ is bounded in $(\mathscr{A}(\mathbb{R}), \tau)$, hence compact and the compact Hausdorff topology is the minimal Hausdorff topology [23, 3.1.14] we get that $\tau = \tau'$ on $\{T_t f : t \in [0, t_0]\}$ and the map $Tf : [0, t_0] \to (\mathscr{A}(\mathbb{R}), \tau)$ is continuous for every $t_0 \geq 0$. Hence $(T_t)_{t\geq 0}$ is strongly continuous.

Denote by A the generator of the semigroup $(T_t)_{t\geq 0}$. For every monomial x^n we have

$$Ax^n = \lim_{t\searrow 0} \frac{T_t x^n - x^n}{t} = \lim_{t\searrow 0} \frac{e^{tm_n} x^n - x^n}{t} = \lim_{t\searrow 0} \frac{e^{tm_n} - 1}{t} x^n = m_n x^n.$$

Hence, $A = M$ on the set of polynomials, which is dense in $\mathscr{A}(\mathbb{R})$. As the operator M is continuous, for any function $f \in \mathscr{A}(\mathbb{R})$ and a sequence of polynomials p_n converging to f, we have $Ap_n = Mp_n \to Mf$ in $\mathscr{A}(\mathbb{R})$. Because the generator A is closed (Proposition 2.2.3), we get that $f \in D(A)$ and $Af = Mf$.

□

The equivalence $(ii) \Leftrightarrow (iii)$ in Theorem 3.2.1 can be also proved by [35, 1.1].

From Theorems 3.1.2 and 3.2.1 follows

Corollary 3.2.2. *The following assertions are equivalent*

(1) *The multiplier $(M,(m_n))$ generates a C_0-semigroup $(T_t)_{t\geq 0}$ on $\mathscr{A}(\mathbb{R})$*

(2) *For every $t \geq 0$ the function f_t, $f_t(z) = \sum_{n=0}^{\infty} \exp(tm_n)z^n$, extends to a holomorphic function belonging to $H(\mathbb{C} \setminus \frac{1}{\mathbb{R}})$ and the set $\{f_t : t \leq t_0\}$ is bounded in $H(\mathbb{C} \setminus \frac{1}{\mathbb{R}})$ for some $t_0 \geq 0$.*

(3) *For every $t \geq 0$ the function $\widetilde{f_t}$, $\widetilde{f_t}(z) = \sum_{n=0}^{\infty} \frac{\exp(tm_n)}{z^{n+1}}$, extends to a holomorphic function belonging to $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ and the set $\{\widetilde{f_t} : t \leq t_0\}$ is bounded in $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ for some $t_0 \geq 0$.*

Proof. (1) ⇔ (2): By Theorem 3.2.1, the statement (1) is equivalent to the operators T_t being multipliers with multiplier sequences $(e^{tm_n})_{n\in\mathbb{N}}$ and $\{T_t f : t \leq t_0\}$ being bounded in $\mathscr{A}(\mathbb{R})$ for all $t_0 > 0$ and all $f \in \mathscr{A}(\mathbb{R})$. By Theorem 3.1.2, the operator $(T_t, (e^{tm_n})_{n\in\mathbb{N}})$ is a multiplier if and only if $f_t \in H(\mathbb{C}_\infty \setminus \frac{1}{\mathbb{R}})$, where $f_t(z) = \sum_{n=0}^{\infty} \exp(tm_n)z^n$ around zero. Since $\mathscr{A}(\mathbb{R})$ is barrelled we can use the uniform boundedness principle [49, Prop. 4.1.3] and get that the set $\{T_t f : t \leq t_0\}$ is bounded in $\mathscr{A}(\mathbb{R})$ if and only if $\{T_t : t \leq t_0\}$ is bounded in $L_b(\mathscr{A}(\mathbb{R}))$, which by Theorem 3.1.2 is equivalent to $\{f_t : t \leq t_0\}$ being bounded in $H(\mathbb{C}_\infty \setminus \frac{1}{\mathbb{R}})$.

(1) ⇔ (3): The proof is similar to the above. □

Lemma 3.2.3. *The set of multipliers generating a C_0-semigroup on $\mathscr{A}(\mathbb{R})$ is additive.*

Proof. Let the multipliers $(A,(a_n))$, $(B,(b_n))$ generate the C_0-semigroups $(T_t^A, (e^{ta_n}))_{t\geq 0}$ and $(T_t^B, (e^{tb_n}))_{t\geq 0}$ respectively and let $f_t, g_t \in H(\mathbb{C}_\infty \setminus \frac{1}{\mathbb{R}})$ be the corresponding (in view of Theorem 3.1.2) holomorphic functions. Take $t \geq 0$ and choose $0 < \varepsilon, \delta < 1$ such that $f_t \in H(\mathbb{C}_\infty \setminus ((-\infty, -\varepsilon] \cup [\varepsilon, \infty)))$ and $g_t \in H(\mathbb{C}_\infty \setminus ((-\infty, -\delta] \cup [\delta, \infty)))$. By the Hadamard multiplication theorem $f_t * g_t \in H(\mathbb{C}_\infty \setminus ((-\infty, -\varepsilon\delta] \cup [\varepsilon\delta, \infty)))$ [42, Th. H]. Hence by Theorem 3.1.2 there exists a multiplier T_t^{A+B} corresponding to $f_t * g_t$ with the multiplier sequence $(e^{t(a_n+b_n)})_{n\geq 0}$. Since for monomials we have $T_t^{A+B}x^n = e^{t(a_n+b_n)}x^n = T_t^A T_t^B x^n$ and monomials are linearly dense in $\mathscr{A}(\mathbb{R})$, we get that $T_t^{A+B} = T_t^A T_t^B$. Hence the map $T^{A+B}f : \mathbb{R}_+ \to \mathscr{A}(\mathbb{R})$, $T^{A+B}f(t) = T_t^{A+B}f$ is continuous for all $f \in \mathscr{A}(\mathbb{R})$. Thus by Theorem 3.2.1 the multiplier $(A+B, (a_n+b_n))$ generates a C_0-semigroup $(T_t^{A+B})_{t\geq 0}$.

□

Now we answer the question when the Euler differential operator generates a C_0-semigroup.

Theorem 3.2.4. *Let $E \in L(\mathscr{A}(\mathbb{R}))$ be a first order Euler differential operator,*

$$Ef(x) = axf'(x) + bf(x).$$

The multiplier E generates a C_0-semigroup if and only if $a \in \mathbb{R}$.

For $a \in \mathbb{R}$ the semigroup generated by E is given by

$$T_t f(x) = e^{bt} f(e^{at}x), \quad f \in \mathscr{A}(\mathbb{R}), x \in \mathbb{R}.$$

Proof. A multiplier $(M,(c))$ with a constant multiplier sequence generates the strongly continuous semigroup $(T_t)_{t\geq 0}$, $T_t f = e^{ct} f$. Hence by Lemma 3.2.3 without loss of generality we can assume that $b = 0$.

The multiplier sequence of E is $(m_n) = (an)$ and the corresponding functions are

$$f_t(z) = \sum_{n=0}^{\infty} e^{tan} z^n = \frac{1}{1 - ze^{ta}} \in H_0(C_\infty \setminus e^{-ta}). \tag{3.1}$$

Hence for every $a \in \mathbb{R}$, $t \geq 0$ we have $f_t \in H(\mathbb{C}_\infty \setminus \frac{1}{\mathbb{R}})$ and $(T_t,(e^{tan}))$ is a multiplier. On the other hand, if $a \notin \mathbb{R}$ then for every t such that $ta \neq k\pi i$, $k \in \mathbb{Z}$, we have $f_t \notin H(\mathbb{C}_\infty \setminus \frac{1}{\mathbb{R}})$ and E does not generate a C_0-semigroup.

To finish the proof we need to show that under the assumption $a \in \mathbb{R}$ the semigroup $(T_t)_{t\geq 0}$ is strongly continuous, i.e., we need to prove the continuity of the map

$$Tf : \mathbb{R}_+ \to \mathscr{A}(\mathbb{R}), \quad Tf(t) = T_t f$$

for arbitrary $f \in \mathscr{A}(\mathbb{R})$. We can extend the map $Tf : \mathbb{R}_+ \to \mathscr{A}(\mathbb{R})$ to the map $Tf : \mathbb{R} \to \mathscr{A}(\mathbb{R})$. Indeed, by (3.1) for every $t < 0$ functions f_t belong to $H_0(\mathbb{C} \setminus \mathbb{R})$ and so there exists a multiplier T_t with the multiplier sequence (e^{tan}), $t < 0$.

To prove the continuity we will use the explicit formula of the multipliers T_t with $(m_n^t) = (e^{tan})$. We claim that $T_t f(x) = f(e^{ta}x)$. Indeed, for a monomial x^n we have

$$T_t x^n(y) = e^{tan} x^n(y) = e^{tan} y^n = x^n(e^{ta} y).$$

Moreover, observe that the dilation map $f \mapsto g$, $g(x) = f(e^{ta}x)$ is linear and continuous on $\mathscr{A}(\mathbb{R})$ for any $a, t \in \mathbb{R}$. Thus the claim follows from the density of polynomials in $\mathscr{A}(\mathbb{R})$.

As $T_t f - T_{t+s} f = T_t(f - T_s f)$ and $s \in \mathbb{R}$ it is enough to show the continuity at $t = 0$. Recall that $T_{t_n} f \to f$ in $\mathscr{A}(\mathbb{R})$ as $t_n \to 0$ if and only if there exists an open complex neighbourhood $U \supset \mathbb{R}$ such that $T_{t_n} f \in H(U)$ for every $n \in \mathbb{N}$ and $T_{t_n} f \to f$ in $H(U)$.

Let U be a complex open neighbourhood of $\mathbb{R}$ such that $f \in H(U)$. Let U' be a starlike subset of U containing $\mathbb{R}$ and put $V := \frac{1}{2}U'$. We choose $\varepsilon > 0$ such that $e^{|a|\varepsilon} < 2$. Then for $|t| < \varepsilon$ we have $e^{ta} V \subset U' \subset U$ and $T_t f \in H(V)$.

Now we will show that $T_{t_n} f \to f$ in $H(V)$. Let K be a compact set in V. For any compact set K_2 such that $K \subset K_2 \subset V$, $K \subset \operatorname{Int} K_2$, and for t_n small enough we have $e^{t_n a} K \subset K_2 \subset V$ and

$$\lim_{t_n \to 0} \| T_{t_n} f - f \|_K = \lim_{t_n \to 0} \sup_{z \in K} |f(e^{t_n a} z) - f(z)| = 0,$$

since f is uniformly continuous on compact sets.

We have proved that $(T_t)_{t\geq 0}$ is strongly continuous. Finally, by Theorem 3.2.1 the operator E is the generator of the C_0-semigroup $(T_t)_{t\geq 0}$.

□

Now we consider the Euler differential operators of higher orders.

Theorem 3.2.5. *Let $P(\theta) = \sum_{k=0}^{K} a_k\theta^k$, $\theta f(x) = xf'(x)$, be a finite order Euler differential operator of degree at least 2. The operator $P(\theta)$ does not generate a C_0-semigroup in the following cases:*

(1) $\operatorname{Re} a_K = \ldots = \operatorname{Re} a_{l+1} = 0$ *and* $\operatorname{Re} a_l > 0$ *for some* $l \geq 2$.

(2) $a_K, \ldots, a_2 \in i\mathbb{Q}$.

Proof. (1): The multiplier sequence of $P(\theta)$ is given by $(m_n) = (P(n))$. Assume that $P(\theta)$ generates a C_0-semigroup $(T_t)_{t\geq 0}$. Then, by Corollary 3.2.2, for all $t \geq 0$ the operator $(T_t, e^{tP(n)})$ is a multiplier and the function f_t, $f_t(z) = \sum_{n=0}^{\infty} e^t P(n) z^n$ around 0, extends to a holomorphic function in $H(\mathbb{C} \setminus \frac{1}{\mathbb{R}})$. In particular, f_t is analytic in some neighborhood of zero. But, for every $R > 0$ we have

$$\sup_{n\in\mathbb{N}} \left| e^{tP(n)} \right| R^n = \sup_{n\in\mathbb{N}} e^{t \operatorname{Re} P(n)} R^n > \sup_{n\in\mathbb{N}} e^{t(a_l - \varepsilon)n^l} R^n = \infty$$

for some $\varepsilon > 0$.

(2): We start with the case $P(\theta) = \sum_{k=1}^{K} a_k\theta^k$ such that $a_k \in i\mathbb{Q}$ for every $1 \leq k \leq K$ and $a_0 = 0$. We will show, that for every such polynomial P there exists $t_0 \in \mathbb{R}_+$ such that $(m_n^{t_0})_{n\in\mathbb{N}} = (\exp(t_0 P(n)))_{n\in\mathbb{N}}$ is not a multiplier sequence.

Let $\widetilde{P}(x) = \sum_{k=1}^{K} \widetilde{a}_k x^k$ be a polynomial such that $\widetilde{a}_k \in \mathbb{Z}$ for all $k \leq K$ and $m_n = \frac{i}{S}\widetilde{P}(n)$, where S is the common denominator of all the coefficients $\frac{a_k}{i}$. As $\widetilde{a}_0 = 0$ we have that $\widetilde{P}(0) = 0$. Let $n_0 \in \mathbb{N}$ be such that

1. $\left|\widetilde{P}(n_0 + 2)\right| = q$, $q > 2$,
2. $\widetilde{P}(n_0) \not\equiv \widetilde{P}(n_0 + 2) \pmod{2q}$.

It is clear that such n_0 exists. Indeed, take n_0 such that $P(n)$ is monotonous for $n \geq n_0$. Then $\left|\widetilde{P}(n_0)\right| < \left|\widetilde{P}(n_0 + 2)\right| < 2q$.

Take $t_0 = \frac{S\pi}{q}$ and consider the function

$$f_{t_0}(z) = \sum_{n=0}^{\infty} m_n^{t_0} z^n = \sum_{n=0}^{\infty} \exp\left(\frac{\widetilde{P}(n)}{q}\pi i\right) z^n \qquad \text{around } 0.$$

The expression $\exp\left(\frac{\widetilde{P}(n)}{q}\pi i\right)$ takes at most $2q$ different values and

$$\exp\left(\frac{\widetilde{P}(n)}{q}\pi i\right) = \exp\left(\frac{\widetilde{P}(2q + n)}{q}\pi i\right).$$

Denote $\xi_n = \exp\left(\frac{\widetilde{P}(n)}{q}\pi i\right)$. Hence we have

$$\begin{aligned} f_{t_0}(z) &= \sum_{n=0}^{\infty} \xi_n z^n = z^0 + \xi_1 z^1 + \xi_2 z^2 + \ldots + z^{2q} + \xi_1 z^{2q+1} + \xi_2 z^{2q+2} + \ldots \\ &= \frac{\sum_{n=0}^{2q-1} \xi_n z^n}{1 - z^{2q}}. \end{aligned}$$

This implies that f_{t_0} is defined on $\mathbb{C}$ except it can have poles of order 1 at $2q$-roots of unity. Now we will show that $f_{t_0} \notin H(\mathbb{C} \setminus \frac{1}{\mathbb{R}})$. Assume that $f_{t_0} \in H(\mathbb{C} \setminus \frac{1}{\mathbb{R}})$, so f_{t_0} would have only poles of order 1 in points ± 1. Then $g(z) = (1-z^2)f_{t_0}(z) \in H(\mathbb{C})$. But

$$g(z) = (1-z^2)f_{t_0}(z) = (1-z^2)\sum_{n=0}^{\infty} \xi_n z^n = \sum_{n=0}^{\infty} (\xi_n z^n - \xi_n z^{n+2})$$
$$= 1 + \xi_1 z + \sum_{n=2}^{\infty} (\xi_n - \xi_{n-2}) z^n.$$

For every $k \in \mathbb{N}$ we have

$$\xi_{2kq+n_0+2} = \xi_{n_0+2} \neq \xi_{n_0} = \xi_{2kq+n_0}$$

and

$$\left|\xi_{2kq+n_0+2} - \xi_{2kq+n_0}\right| = \delta$$

for some $\delta > 0$. Hence

$$\limsup_{n\to\infty} \sqrt[n]{|\xi_n - \xi_{n-2}|} = 1$$

and we get a contradiction. Hence $f_{t_0} \notin H(\mathbb{C} \setminus \frac{1}{\mathbb{R}})$ and $(P(\theta), (P(n)))$ does not generate a semigroup.

Now consider $P(\theta) = \sum_{k=1}^{K} a_k \theta^k$, with $a_K, \ldots, a_2 \in i\mathbb{Q}$, $a_1 = ir$, $r \in \mathbb{R} \setminus \mathbb{Q}$. Taking $t_0 = 2S\pi$, where S denotes the common denominator of $\frac{a_K}{i}, \ldots \frac{a_2}{i}$, we get that

$$e^{t_0 P(n)} = e^{2Sr\pi in}.$$

By Theorem 3.1.2 the operator $(T_{t_0}, (e^{2r\pi in}))$ is not a multiplier since $e^{2Sr\pi i} \notin \mathbb{R}$ for $r \notin \mathbb{Q}$ and

$$f_{t_0}(z) = \sum_{n=0}^{\infty} e^{2Sr\pi in} z^n = \frac{1}{1 - e^{2Sr\pi i} z} \notin H(\mathbb{C} \setminus \frac{1}{\mathbb{R}}).$$

By Theorem 3.2.1, $(P(\theta), P(n))$ cannot generate a semigroup.

Summarizing, we have proved that a multiplier $(P(\theta), (P(n)))$ with $P(\theta) = \sum_{k=1}^{K} a_k \theta^k$, $a_K, \ldots a_2 \in i\mathbb{Q}$, $a_1 \in i\mathbb{R}$ does not generate a semigroup. Now take a multiplier $Q(\theta) = P(\theta) + b_1\theta + c$ with $b_1 \in \mathbb{R}$. As the operators $(M_{-b}, (-b_1 n - c))$, $(M_b, (b_1 n + c))$ generate C_0-semigroups (Theorem 3.2.4) and the sum of multipliers being generators is a generator (Lemma 3.2.3) we conlude that $(Q(\theta), (Q(n)))$ generates the semigroup if and only if $(P(\theta), (P(n)))$ does, which finishes the proof. □

3.3 The C_0-semigroup generated by the Hardy operator

Now we will give another example of a multiplier that generates a strongly continuous semigroup on $\mathscr{A}(\mathbb{R})$, i.e., we will show that the Hardy operator, $Hf(x) = \frac{1}{x}\int_0^x f(t)dt$, is a generator of a C_0-semigroup. The proof is based on the so called Mellin functions, which we define as in [15].

Definition 3.3.1. Let $(\kappa_n)_{n\in\mathbb{N}}$, $(K_n)_{n\in\mathbb{N}}$ be increasing sequences of real numbers such that $\kappa_1 < 0$ and $0 < K_n \to \infty$. We define an *asymptotic halfplane* ω by

$$\omega = \bigcup_{n=1}^{\infty}(\kappa_n + \omega_{K_n}) \text{ for } \omega_{K_n} := \{z \in \mathbb{C} : |\operatorname{Im} z| < K_n \operatorname{Re} z\}.$$

Roughly speaking an asymptotic halfplane is the union of shifted cones. We call a holomorphic function $f \in H(\omega)$ a *Mellin function* for the sequence $(m_n)_{n\in\mathbb{N}}$ if there exists a constant $C > 0$ such that

$$|f(z)| \leq Ce^{C|\operatorname{Re} z|} \text{ for } z \in \omega$$

and

$$f(n) = m_n.$$

We will denote the space of Mellin functions by $\mathscr{M}(\omega)$.

Definition 3.3.2. For $a \in \mathbb{R}$ we define

$$\mathscr{M}_a(\omega) = \{f \in \mathscr{M}(\omega) : \forall j \sup_{z\in\Gamma_j} |f(z)|\, e^{-(a+\frac{1}{j})\operatorname{Re} z} < \infty\}$$

where $\Gamma_j = \overline{\bigcup_{n\leq j}(\kappa_n + 1/j + \omega_{K_n})}$.

The space $\mathscr{M}_a(\omega)$ is a Fréchet space with the fundamental system of seminorms $(\|\cdot\|_j)_{j\in\mathbb{N}}$ given by

$$\|f\|_j = \sup_{z\in\Gamma_j} |f(z)|\, e^{-\left(a+\frac{1}{j}\right)\operatorname{Re} z}.$$

Theorem 3.3.3 ([16, 4.1])**.** *There exists a continuous, linear and surjective mapping*

$$H_a^+ : \mathscr{M}_a(\omega) \to H([0, e^a])'$$

satisfying

$$\langle H_a^+(f), x^n\rangle = f(n) \textit{ for every } n \in \mathbb{N}.$$

Now we can prove our main theorem of this section

Theorem 3.3.4. *Let $H \in \mathcal{L}(\mathscr{A}(\mathbb{R}))$ be the Hardy operator, $Hf(x) = \frac{1}{x}\int_0^x f(y)dy$. The operator $A = \sum_{k=0}^K a_k H^k$, $a_1, \ldots, a_K \in \mathbb{C}$ generates a C_0-semigroup on $\mathscr{A}(\mathbb{R})$.*

Proof. The multiplier sequence of the Hardy operator H equals $(\frac{1}{n+1})_{n\in\mathbb{N}}$. Hence the multiplier sequence of $(A, (m_n))$ equals $m_n = \sum_{k=0}^K \frac{a_k}{(n+1)^k}$. By Theorem 3.2.1 it is enough to show that the sequences $\left(\exp\left(\sum_{k=0}^K \frac{ta_k}{(n+1)^k}\right)\right)_{n\in\mathbb{N}}$ are multiplier sequences for the multipliers T_t and that the mapping $Tf : \mathbb{R} \to \mathscr{A}(\mathbb{R})$, $Tf(t) = T_t f$ is continuous for every $f \in \mathscr{A}(\mathbb{R})$. By Theorem 3.1.1 the sequences $\left(\exp\left(\sum_{k=0}^K \frac{ta_k}{(n+1)^k}\right)\right)_{n\in\mathbb{N}}$ are multiplier sequences if and only if there exist functionals

$F_t \in \mathscr{A}(\mathbb{R})'$ satisfying $\langle F_t, x^n\rangle = \exp\left(\sum_{k=0}^K \frac{ta_k}{(n+1)^k}\right)$, which by Theorem 3.3.3 is equivalent to the existence of Mellin functions $\mu_t \in \mathscr{M}_{a(t)}(\omega_t)$ for $\left(\exp\left(\sum_{k=0}^K \frac{ta_k}{(n+1)^k}\right)\right)_{n\in\mathbb{N}}$.

For the proof it is enough to find $a \in \mathbb{R}$, an asymptotic halfplane ω and Mellin functions $\mu_t \in \mathscr{M}_a(\omega)$ such that the mapping $\varphi: \mathbb{R} \to \mathscr{M}_a(\omega)$, $t \mapsto \mu_t$ is continuous. Indeed, consider the following diagram

$$\mathbb{R}_+ \xrightarrow{\varphi} \mathscr{M}_a(\omega) \xrightarrow{H_a^+} H([0,e^a])' \xrightarrow{\mathscr{B}} M(\mathbb{R}).$$

Recall that H_a^+, $\mathscr{B}$ are continuous (Theorems 3.3.3, 3.1.1) with $\mathscr{B}\circ H^+ \circ \varphi(t) = T_t$. Hence, if the function φ is continuous then the map $t \mapsto T_t f$ is continuous.

Let ω be an asymptotic halfplane such that $\kappa_1 = -\frac{1}{2}$, $\kappa_n = 0$ for all $n \geq 2$ and consider the functions $\mu_t(z) = \exp\left(\sum_{k=0}^K \frac{ta_k}{(z+1)^k}\right)$, $t \geq 0$.

The function μ_t is clearly holomorphic on ω and for $z \in \omega \subset \{\operatorname{Re} z > -\frac{1}{2}\}$ it satisfies

$$\begin{aligned} |\mu_t(z)| &= \left|\exp\left(\sum_{k=0}^K \frac{ta_k}{(z+1)^k}\right)\right| \leq \exp\left(\sum_{k=0}^K \left|\frac{ta_k}{(z+1)^k}\right|\right) \leq \exp\left(\sum_{k=0}^K 2^k t\,|a_k|\right) \\ &< \exp\left(\sum_{k=0}^K t2^k\,|a_k| + \frac{1}{2}\right)\exp(\operatorname{Re} z). \end{aligned}$$

Hence $\{\mu_t\}_{t\geq 0} \subset \mathscr{M}(\omega)$. Since $\mu_t(n) = \exp\left(\sum_{k=0}^K \frac{ta_k}{(n+1)^k}\right)$, we get that functions μ_t are Mellin functions for the sequence $\left(\exp\left(\sum_{k=0}^K \frac{ta_k}{(n+1)^k}\right)\right)_{n\in\mathbb{N}}$.

Now we will show that $\mu_t \in \mathscr{M}_a(\omega)$ for any $a > 0$ and all $t \in \mathbb{R}_+$. We compute

$$\begin{aligned} \sup_{z\in\Gamma_j} |\mu_t(z)|\, e^{-(a+\frac{1}{j})\operatorname{Re} z} &\leq \exp\left(\sum_{k=0}^K 2^k t\,|a_k|\right)\sup_{z\in\Gamma_j}\exp\left(-\left(a+\frac{1}{j}\right)\operatorname{Re} z\right) \\ &< \exp\left(\sum_{k=0}^K 2^k t\,|a_k|\right)\exp\left(\left(a+\frac{1}{j}\right)\frac{1}{2}\right) < \infty. \end{aligned}$$

To finish the proof we need to prove the continuity of the map

$$\varphi: \mathbb{R}_+ \to \mathscr{M}_a(\omega), \quad \varphi(t) = \mu_t.$$

Fix $t \geq 0$, $j \geq 1$. Then

$$\begin{aligned} \|\mu_t - \mu_{t+h}\|_j &= \sup_{z\in\Gamma_j} |\mu_t(z) - \mu_{t+h}(z)| \exp\left(-\left(a+\frac{1}{j}\right)\operatorname{Re} z\right) \\ &= \sup_{z\in\Gamma_j} |\mu_t(z)|\,|1-\mu_h(z)| \exp\left(-\left(a+\frac{1}{j}\right)\operatorname{Re} z\right) \\ &< \exp\left(\sum_{k=0}^K 2^k\,|ta_k| + \frac{1}{2}(a+\frac{1}{j})\right)\sup_{z\in\Gamma_j}|1-\mu_h(z)|. \end{aligned}$$

For the last component we have that

$$\left|\sum_{k=0}^{K} \frac{h a_k}{(z+1)^k}\right| \leq h \sum_{k=0}^{K} \left|\frac{a_k}{(z+1)^k}\right| < h \sum_{k=0}^{K} 2^k \left|a_k\right|$$

for all $z \in \Gamma_j$. Hence $\mu_h(z) \xrightarrow{h\to 0} 1$ uniformly on Γ_j and

$$\left\|\mu_t - \mu_{t+h}\right\|_j \xrightarrow{h\to 0} 0.$$

□

Hankel operators

4

An infinite matrix is called a Hankel matrix if it is of the form

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_1 & a_2 & a_3 & a_4 & \dots \\ a_2 & a_3 & a_4 & a_5 & \dots \\ a_3 & a_4 & a_5 & a_6 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

where $(a_n)_{n\geq 0}$ is a sequence of complex numbers. A Hankel operator is a continuous operator for which the associated matrix is a Hankel matrix.

In this chapter, we study Hankel operators on $\mathscr{A}(\mathbb{R})$. First we give a representation theorem for Hankel operators and study their properties. In the second part of the chapter, we show the relation between Hankel operators on $\mathscr{A}(\mathbb{R})$ and Hankel operators on the Hardy space $H^2(\mathbb{D})$ and investigate the spectrum of Hankel operators on $\mathscr{A}(\mathbb{R})$.

4.1 Characterization of Hankel operators on $\mathscr{A}(\mathbb{R})$

In this section we give a representation theorem for Hankel operators on $\mathscr{A}(\mathbb{R})$. We start with the more precise definition of a Hankel operator.

Definition 4.1.1. We say that a continuous operator $\Gamma\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is a *Hankel operator* if there exist a sequence of complex numbers $(a_n)_{n\geq 0}$ such that for all monomials x^n and for ξ near zero

$$\Gamma x^n(\xi) = \sum_{k=0}^{\infty} a_{n+k}\xi^k.$$

We call the function $\Gamma x^0 \in \mathscr{A}(\mathbb{R})$, $\Gamma x^0 = \sum_{k=0}^{\infty} a_k \xi^k$ around zero, the *symbol* of the operator Γ.

We denote by B the *backward shift operator*, i.e. $Bf(x) = \frac{f(x)-f(0)}{x}$ for $f \in \mathscr{A}(\mathbb{R})$. Since for every function $\varphi \in \mathscr{A}(\mathbb{R})$, $\varphi(z) = \sum_{n=0}^{\infty} \varphi_n z^n$ for z around zero, and for every $k \in \mathbb{N}$ we have $B^k\varphi(z) = \sum_{n=0}^{\infty} \varphi_{n+k} z^n$ for small z, we can also describe the Hankel operator via its relation with the backward shift.

Fact 4.1.2. *An operator $\Gamma \in L_b(\mathscr{A}(\mathbb{R}))$ is a Hankel operator with the symbol $\varphi \in \mathscr{A}(\mathbb{R})$ if and only if $\Gamma x^n = B^n\varphi$ for all n.*

We denote by $\mathscr{H}$ the set of all Hankel operators on $\mathscr{A}(\mathbb{R})$. From the definition of Hankel operators it is clear that the space $\mathscr{H}$ is a linear subspace of the space of all linear continuous operators $L_b(\mathscr{A}(\mathbb{R}))$ and we equip $\mathscr{H}$ with the topology induced from $L_b(\mathscr{A}(\mathbb{R}))$, i.e. the topology of uniform convergence on bounded subsets of $\mathscr{A}(\mathbb{R})$.

Proposition 4.1.3. *The space of Hankel operators $\mathscr{H}$ is a closed subspace of $L_b(\mathscr{A}(\mathbb{R}))$.*

Proof. Let $(\Gamma_\alpha)_\alpha$ be a net converging to Γ in $L_b(\mathscr{A}(\mathbb{R}))$. Let $f_\alpha := \Gamma_\alpha x^0$ and $f := \Gamma x^0 = \lim \Gamma_\alpha x^0 = \lim f_\alpha$. By Fact 4.1.2 we have that $\Gamma_\alpha x^n = B^n f_\alpha$. For any $n > 0$ we have $\Gamma x^n = \lim \Gamma_\alpha x^n = \lim B^n f_\alpha = B^n(\lim f_\alpha) = B^n f$. This shows that Γ is a Hankel operator. □

We will now show the integral representation of Hankel operators acting on $\mathscr{A}(\mathbb{R})$. If γ is a C^∞ smooth Jordan curve then by the Jordan's theorem it divides the plane into interior region bounded by γ and an unbounded exterior region. We will denote them by $\mathrm{Int}(\gamma)$ and $\mathrm{Ext}(\gamma)$ respectively.

Lemma 4.1.4. *Let $\varphi \in H(\mathbb{C})$ and let g be an analytic function at zero. Define*

$$\Gamma_\varphi g(z) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\xi) g(\frac{1}{\xi})}{\xi - z} d\xi, \quad z \in \mathbb{C}, \tag{4.1}$$

where γ is a positively oriented C^∞ smooth Jordan curve such that $z, 0 \in \mathrm{Int}(\gamma)$ and $g(\frac{1}{\xi})$ is holomorphic on γ and in $\mathrm{Ext}(\gamma)$. Then the definition does not depend on the choice of γ and

(1) *$\Gamma_\varphi : H(\{0\}) \to H(\mathbb{C})$ is a continuous operator,*

(2) *$\Gamma_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is a bounded Hankel operator.*

Proof. (1): The space of germs at zero, $H(\{0\})$, is the inductive limit of the spaces of bounded, holomorphic functions $H^\infty(\frac{1}{n}\mathbb{D})$ [3, pp. 64]. Hence to prove that the operator $\Gamma_\varphi : H(\{0\}) = \mathrm{ind}_n H^\infty(\frac{1}{n}\mathbb{D}) \to H(\mathbb{C})$ is continuous, it sufficies to show that for every n the operator $\Gamma_\varphi : H^\infty(\frac{1}{n}\mathbb{D}) \to H(\mathbb{C})$ is well-defined and continuous. For $g \in H^\infty(\frac{1}{n}\mathbb{D})$, and $z \in \mathbb{C}$ we have

$$\Gamma_\varphi g(z) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\xi) g(\frac{1}{\xi})}{\xi - z} d\xi,$$

where γ is a positively oriented C^∞ smooth Jordan curve such that $z, 0 \in \mathrm{Int}(\gamma)$ and $g(\frac{1}{\xi})$ is holomorphic on γ and in $\mathrm{Ext}(\gamma)$. By the Cauchy's Theorem $\Gamma_\varphi g(z)$ does not depend on γ as long as z and zero are in the interior of γ. Clearly $\Gamma_\varphi g$ is differentiable in a neighborhood of z and, as z is arbitrary, $\Gamma_\varphi g$ is an entire function.

To show the continuity of Γ_φ we take an arbitrary compact set $K \subset \mathbb{C}$. For γ we take a circle of radius $\rho > n$ such that $K \subset \mathrm{Int}(\gamma)$ and compute

$$\|\Gamma_\varphi g\|_K = \sup_{z \in K}\left| \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\xi) g(\frac{1}{\xi})}{\xi - z} d\xi \right| \le \|\varphi\|_\gamma \, \rho \, \mathrm{dist}(\gamma, K)^{-1} \|g\|_\infty ,$$

where $\|\varphi\|_\gamma = \sup_{z \in \gamma} |\varphi(z)|$, $\|g\|_\infty = \sup_{z \in \frac{1}{n}\mathbb{D}} |g(z)|$.

(2): Since the space $\mathscr{A}(\mathbb{R})$ carries the projective and injective limit topology, the inclusions $i : \mathscr{A}(\mathbb{R}) \to H(\{0\})$ and $j : H(\mathbb{C}) \to \mathscr{A}(\mathbb{R})$ are continuous. It follows that the operator $\Gamma_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ factorizes through a continuous linear operator from an LB-space $H(\{0\}) = \mathrm{ind}_n H^\infty(\frac{1}{n}\mathbb{D})$ to a Fréchet space $H(\mathbb{C})$. Hence Γ_φ is bounded by Proposition 2.1.13.

It remains to show that Γ_φ is a Hankel operator. Let $\varphi(z) = \sum_{k=0}^\infty \varphi_k z^k$. For every monomial x^n and $|z| < R$ we have

$$\Gamma_\varphi x^n(z) = \frac{1}{2\pi i}\int_{|\xi|=R} \frac{\varphi(\xi)\xi^{-n}}{\xi - z} d\xi = \frac{1}{2\pi i}\int_{|\xi|=R} \sum_{k=0}^\infty \varphi_k \xi^{k-n} \sum_{j=0}^\infty \frac{z^j}{\xi^{j+1}} d\xi$$
$$= \sum_{j=0}^\infty z^j \sum_{k=0}^\infty \frac{1}{2\pi i}\int_{|\xi|=R} \varphi_k \xi^{k-n-j-1} d\xi = \sum_{j=0}^\infty \varphi_{j+n} z^j.$$

□

The next theorem shows that there are no other Hankel operators.

Theorem 4.1.5. *If $\Gamma\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is a Hankel operator then there exists an entire function φ such that $\Gamma = \Gamma_\varphi$.*

Proof. Let Γ be a Hankel operator. For every monomial we have $\Gamma x^n(\xi) = \sum_{k=0}^\infty a_{n+k}\xi^k$ for ξ near zero. The function $\varphi := \Gamma x^0$, $\varphi(z) = \sum_{k=0}^\infty a_k z^k$ is real analytic and there exists an open, complex neighborhood $U \supset \mathbb{R}$ such that $\varphi \in H(U)$. On the other hand, we consider the functional $\delta_0 \circ \Gamma \in \mathscr{A}(\mathbb{R})'$, where δ_0 denotes the evaluation at 0. We compute its moments

$$\langle x^n, \delta_0 \circ \Gamma \rangle = \Gamma x^n(0) = a_n$$

for $n \geq 0$. By the theory of multipliers (Corollary 3.1.3) there exists a function f, $f(z) = \sum_{k=0}^\infty a_k z^k$ around zero, which is holomorphic at zero and extends to a holomorphic function on $\mathbb{C} \setminus ((\infty, -\varepsilon) \cup (\varepsilon, \infty))$ for some $\varepsilon > 0$. As $f = \varphi$ on the neighborhood of zero, we have that $f = \varphi$. As φ extends to a function holomorphic on $U \supset \mathbb{R}$ and holomorphic on $\mathbb{C} \setminus ((\infty, -\varepsilon) \cup (\varepsilon, \infty))$ we get that φ is an entire function.

□

We proved that there is a $1-1$ correspondence between Hankel operators on $\mathscr{A}(\mathbb{R})$ and entire functions. Our next theorem shows that the spaces $\mathscr{H}$ and $H(\mathbb{C})$ are even topologically isomorphic.

Theorem 4.1.6. *The space of Hankel operators $\mathscr{H}$ is topologically isomorphic to the space of entire functions $H(\mathbb{C})$. The isomorphism is given by*

$$\mathcal{Q}\colon H(\mathbb{C}) \to \mathscr{H}, \quad \mathcal{Q}(\varphi) = \Gamma_\varphi$$

$$\mathcal{R}\colon \mathscr{H} \to H(\mathbb{C}), \quad \mathcal{R}(\Gamma) = \Gamma x^0.$$

and $\mathcal{QR} = \mathcal{RQ} = I$.

Proof. Lemma 4.1.4 shows that the map $\mathcal{Q}\colon H(\mathbb{C}) \to \mathscr{H}$ is well defined. The map $\mathcal{Q}$ is surjective by Theorem 4.1.5 and because $\varphi = \Gamma_\varphi x^0$ it is clearly injective.

Now, we prove that the map $\mathcal{Q}\colon H(\mathbb{C}) \to \mathscr{H}$, $\mathcal{Q}(\varphi) = \Gamma_\varphi$ is continuous. Since the space $L_b(\mathscr{A}(\mathbb{R}))$ is webbed (Lemma 2.1.11) and $H(\mathbb{C})$ is ultrabornological as a Fréchet space ([40, 24.15]), we can use the closed graph theorem. Take $(\varphi_\alpha, \Gamma_{\varphi_\alpha}) \to (\varphi, \Gamma)$ in $H(\mathbb{C}) \times L_b(\mathscr{A}(\mathbb{R}))$.

Then Γ is a Hankel operator by Proposition 4.1.3 and, by Theorem 4.1.5, there exists an entire function ψ such that $\Gamma = \Gamma_\psi$.

From $\Gamma_{\varphi_\alpha} \to \Gamma_\psi$ in $L_b(\mathscr{A}(\mathbb{R}))$ it follows that

$$\varphi_\alpha = \Gamma_{\varphi_\alpha} x^0 \to \Gamma_\psi x^0 = \psi \text{ in } \mathscr{A}(\mathbb{R}).$$

and from $\varphi_\alpha \to \varphi$ in $H(\mathbb{C})$ we get that

$$\varphi_\alpha \to \varphi \text{ in } \mathscr{A}(\mathbb{R}).$$

Hence $\varphi = \psi$, $\Gamma_\varphi = \Gamma_\psi = \Gamma$ and the map $\mathcal{Q}$ is indeed continuous.

In order to prove that $\mathcal{R}$ is a continuous inverse of $\mathcal{Q}$ we consider the following diagram

$$\mathscr{H} \xrightarrow{\mathcal{S}} H(\{0\})' \xrightarrow{\mathscr{C}} H_0(\mathbb{C}_\infty \setminus \{0\}) \xrightarrow{\mathcal{F}} H(\mathbb{C}).$$

The map $\mathcal{S}\colon \mathscr{H} \to H(\{0\})'$, $\mathcal{S}(\Gamma_\varphi) = \delta_0 \circ \Gamma_\varphi$, is continuous by Lemma 4.1.4(1). By the Köthe-Grothendieck-da Silva duality ([36, pp. 372-378]) the functional $\mathcal{S}(\Gamma_\varphi) \in H(\{0\})'$ corresponds to a function $\widetilde{\varphi} \in H_0(\mathbb{C}_\infty \setminus \{0\})$. We denote by $\mathscr{C}$ the Cauchy transform, $\mathscr{C}\colon H(\{0\})' \to H_0(\mathbb{C}_\infty \setminus \{0\})$, defined by

$$\mathscr{C}(T)(z) = \left\langle T, \frac{1}{z - \bullet} \right\rangle \quad \text{for } T \in H(\{0\})'.$$

Since $H_0(\mathbb{C}_\infty \setminus \{0\})$ is isomorphic to the space $H(\mathbb{C})$ (the isomorphism is given by the map $\mathcal{F}(f)(z) = \frac{1}{z} f(\frac{1}{z})$) we get, that the map $\mathcal{R}\colon \mathscr{H} \to H(\mathbb{C})$, $\mathcal{R} = \mathcal{F} \circ \mathscr{C} \circ \mathcal{S}$ is continuous.

It remains to show that $\mathcal{R}(\Gamma_\varphi) = \varphi = \Gamma_\varphi x^0$ and it is indeed the inverse of $\mathcal{Q}$. We compute

$$(\mathscr{C} \circ \mathcal{S})(\Gamma_\varphi)(z) = \mathscr{C}(\delta_0 \circ \Gamma_\varphi)(z) = \left\langle \delta_0 \circ \Gamma_\varphi, \frac{1}{z - \bullet} \right\rangle = \frac{1}{2\pi i} \int_{\partial D(0,R)} \frac{\varphi(\xi)}{\xi} \frac{1}{z - \frac{1}{\xi}} d\xi,$$

where $D(0,R)$ denotes a disc around zero with radius $R > \frac{1}{|z|}$. Using the residue theorem we get

$$\mathscr{C}(\delta_0 \circ \Gamma_\varphi)(z) = \operatorname*{Res}_{\xi = \frac{1}{z}} \left(\frac{\varphi(\xi)}{\xi z - 1} \right) = \frac{\varphi(\frac{1}{z})}{z}.$$

Since $\mathcal{F}(\frac{\varphi(\frac{1}{\bullet})}{\bullet})(z) = \varphi(z)$ we get

$$\mathcal{R}(\Gamma_\varphi)(z) = (\mathcal{F} \circ \mathscr{C} \circ \mathcal{S})(\Gamma_\varphi)(z) = \varphi(z). \qquad \square$$

Corollary 4.1.7. *The space of Hankel operators $\mathscr{H}$ is isomorphic to the space $H(\{0\})'$. Moreover if T is a continuous linear functional on $H(\{0\})$ then the symbol of the corresponding Hankel operator is defined by:*

$$\varphi(z) = \left\langle T, \frac{1}{1 - z\bullet} \right\rangle$$

Proof. The proof of Theorem 4.1.6 shows that $H(\mathbb{C}) \simeq H_0(\mathbb{C}_\infty \setminus \{0\}) \simeq H(\{0\})'$ and that the isomorphism $\mathcal{U}: H(\{0\})' \to H(\mathbb{C})$ is given by $\mathcal{U} = \mathcal{F} \circ \mathscr{C}$. Hence, for $T \in H(\{0\})'$ we have

$$\mathcal{U}(T)(z) = \frac{1}{z}\left\langle T, \frac{1}{\frac{1}{z} - \bullet}\right\rangle = \left\langle T, \frac{1}{1 - z\bullet}\right\rangle. \qquad \square$$

Our next theorem shows the properties of the Hankel operator Γ_φ.

Theorem 4.1.8. *Let φ be an entire function. The following assertions are equivalent*

(1) *φ is a polynomial,*

(2) *Γ_φ has finite rank,*

(3) *Γ_φ is not injective,*

(4) *the image of Γ_φ contains a polynomial.*

Proof. (1) $\Leftrightarrow$ (2): The equivalence is proved in the same way as the corresponding theorem for Hankel operators on $H^2(\mathbb{T})$ of Kronecker (see for instance [48, Theorem I.3.1]).

Let $\Gamma_\varphi: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ be a finite rank Hankel operator with the symbol $\varphi(z) = \sum_{j=0}^\infty \varphi_j z^j \in H(\mathbb{C})$ and rank $\Gamma_\varphi = n$. We denote the monomials by η_k,

$$\eta_k(z) = z^k.$$

By the assumption $\Gamma_\varphi\eta_0, \Gamma_\varphi\eta_1, \dots, \Gamma_\varphi\eta_n$ are linearly dependent and there exist complex numbers $c_0, \dots, c_n$, not all 0, such that $c_0\Gamma_\varphi\eta_0 + c_1\Gamma_\varphi\eta_1 + \dots + c_n\Gamma_\varphi\eta_n = 0$.

For every monomial we have $\Gamma_\varphi\eta_k(z) = \sum_{j=0}^\infty \varphi_{j+k}z^j$. So, denoting by B the backward shift operator we have that $\Gamma_\varphi\eta_k = B^k\Gamma_\varphi\eta_0 = B^k\varphi$ and

$$c_0\varphi + c_1B\varphi + \dots + c_nB^n\varphi = 0.$$

Denote by S the forward shift operator, $Sf(z) := zf(z)$. For $k \le n$ we can calculate

$$S^nB^k\varphi(z) = \sum_{j=0}^\infty \varphi_{j+k}z^{j+n} = \sum_{j=k}^\infty \varphi_j z^{j+n-k} = S^{n-k}\varphi(z) - \sum_{j=0}^{k-1}\varphi_j z^{j+n-k} = S^{n-k}\varphi(z) - S^{n-k}\sum_{j=0}^{k-1}\varphi_j z^j.$$

It follows that

$$0 = \sum_{k=0}^n c_kB^k\varphi = S^n\sum_{k=0}^n c_kB^k\varphi = \sum_{k=0}^n c_kS^nB^k\varphi = \sum_{k=0}^n c_kS^{n-k}\varphi - p,$$

where p is a polynomial of degree at most $n-1$.

Put

$$q(z) = \sum_{j=0}^n c_{n-j}z^j.$$

Then

$$q(z)\varphi(z) = \sum_{j=0}^n c_{n-j}S^j\varphi(z) = \sum_{k=0}^n c_kS^{n-k}\varphi(z) = p(z).$$

Hence $\varphi = \frac{p}{q}$. Since $\varphi \in H(\mathbb{C})$, we conclude that φ is a polynomial

Conversely, let φ be a polynomial of degree n. Then for every $k > n$ we have $\Gamma_\varphi x^k = B^k\varphi = 0$ and $\Gamma_\varphi(\operatorname{span}\{z^k : z \leq n\}) \subset \overline{\operatorname{span}\{z^k : z \leq n\}}$. Since polynomials are dense in $\mathscr{A}(\mathbb{R})$ we conlude that Γ_φ has finite rank.

(1) $\Leftrightarrow$ (3):

If φ is a polynomial of degree n then the monomial z^{n+1} belongs to the kernel of Γ_φ since $\Gamma_\varphi z^n = B^{n+1}\varphi = 0$.

For any $\varphi \in H(\mathbb{C})$ which is not a polynomial, $g \in \mathscr{A}(\mathbb{R})$ we put $\varphi(z)g(\frac{1}{z}) = \sum_{n=-\infty}^{\infty} c_n z^n$ and compute

$$\Gamma_\varphi g(z) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\xi)g(\frac{1}{\xi})}{\xi - z}d\xi = \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty}\frac{z^n}{\xi^{n+1}}\sum_{k=-\infty}^{\infty} c_k\xi^k d\xi = \sum_{n=0}^{\infty} c_n z^n$$

Hence $\Gamma_\varphi g = 0$ if and only if all coefficients with positive indices in the Laurent expansion of the function $\varphi(z)g(\frac{1}{z})$ are zero. Equivalently, $\Gamma_\varphi g = 0$ if and only if all coefficients with negative indices in the Laurent expansion of the function $\varphi(\frac{1}{z})g(z)$ are zero. Since φ is not a polynomial the function $\varphi(\frac{1}{z})$ has an essential singularity at zero and for any $g \in \mathscr{A}(\mathbb{R})$ not identically equal zero, $\varphi(\frac{1}{z})g(z)$ also has an essential singularity at zero. Hence $g \notin \ker\Gamma_\varphi$ for all $g \in \mathscr{A}(\mathbb{R})$, $g \not\equiv 0$.

(1) $\Leftrightarrow$ (4): Notice first that we have the following relation between Hankel operators and the backward shift B:

$$B \circ \Gamma_\varphi = \Gamma_{B\varphi}. \tag{4.2}$$

Indeed, for every monomial we have $B\Gamma_\varphi x^n = B(B^n\varphi) = B^n(B\varphi) = \Gamma_{B\varphi}x^n$. Hence (4.2) holds for polynomials. The general case follows by density of polynomials.

Assume that there exists a function $g \in \mathscr{A}(\mathbb{R})$ and a polynomial p of degree n such that $\Gamma_\varphi g = p$. Since $B^{n+1}p = 0$, we have that $B^{n+1}\Gamma_\varphi g = \Gamma_{B^{n+1}\varphi}g = 0$. Hence $\Gamma_{B^{n+1}\varphi}$ is not an injective operator and by ((1) $\Leftrightarrow$ (4)) the function $B^{n+1}\varphi$ is a polynomial. Hence φ is a polynomial.

The other direction is clear, since $\varphi \in \operatorname{Im}\Gamma_\varphi$. □

4.2 Spectra of Hankel operators

In this section we study the spectrum of a Hankel operator $\Gamma_\varphi\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$.

First we show that the spectrum of a Hankel operator $\Gamma_\varphi\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is the point spectrum. We will use the following lemma

Lemma 4.2.1. *[57, Satz 1] Let E be a Mackey complete locally convex space and let T be a continuous, bounded, strictly singular operator on E. Then*

1. *for all $\lambda \in \mathbb{C} \setminus \{0\}$ the operator $\lambda I - T$ is Fredholm with index 0,*
2. *The resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$ is an $L_b(E)$-valued meromorphic function on $\mathbb{C} \setminus \{0\}$ for which the coefficients of all the principal parts in all its poles are finite-dimensional operators.*

Proposition 4.2.2. *For every function $\varphi \in H(\mathbb{C})$ and every $\lambda \in \mathbb{C} \setminus \{0\}$ the operator $\Gamma_\varphi - \lambda I$ is a Fredholm operator with index* 0*. Moreover the spectrum of the Hankel operator Γ_φ is the sum of point spectrum and* 0*.*

Proof. We proved that the operator Γ_φ is bounded (Lemma 4.1.4) and strictly singular (Proposition 2.1.14). Hence the first part follows from the preceding lemma. Theorem 4.1.8 shows that for any function $\varphi \in H(\mathbb{C})$ the operator Γ_φ is not invertible, hence zero always belongs to the spectrum. □

The class of Hankel operators have been intensively studied on the classical Banach spaces of analytic functions. We will now show the relation between Hankel operators on $\mathscr{A}(\mathbb{R})$ and the small Hankel operators acting on the Hardy space $H^2(\mathbb{D})$. For the Hardy space case we use the following notation. Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle with normalized arc length $d\sigma$. Let $H^2 = H^2(\mathbb{D})$ be the Hardy space, i.e.

$$H^2 = \left\{ f \in H(\mathbb{D}) : \sup_{0\le r<1} \int_0^{2\pi} \left|f(re^{it})\right|^2 dt \right\}$$

and let $P\colon L^2(\mathbb{T}, d\sigma) \to H^2$ denote the Szegö projection,

$$Pf(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{f(w)}{1 - z\overline{w}} d\sigma(w).$$

For a function $f \in L^\infty(\mathbb{T})$ the small Hankel operator $H_f\colon H^2 \to H^2$ is defined by

$$H_f g = P(f\overline{g}), \qquad g \in H^2.$$

Let the space $X \neq H(\mathbb{C})$ be such that the inclusions $X \hookrightarrow H(\{0\}), H(\mathbb{C}) \hookrightarrow X$ are continuous. The space $L(H(\{0\}), H(\mathbb{C}))$ consists of all continuous operators on X which can be extended to operators mapping $H(\{0\})$ into $H(\mathbb{C})$. We add I_X, the identity operator on X, to this algebra and denote the new algebra by $L(H(\{0\}), H(\mathbb{C}))^*$. Then $L(H(\{0\}), H(\mathbb{C}))^* \subset L(X)$ and we have the following theorem

Theorem 4.2.3. *([10, Th. 2.3]) An element $A \in L(H(\{0\}), H(\mathbb{C}))^*$ is invertible in $L(X)$ if and only if it is invertible in $L(H(\{0\}), H(\mathbb{C}))^*$.*

The proof is the analogue of the proof in [10, Th. 2.3]. For the completness we present it here.

Proof. It is clear that if an element is invertible in $L(H(\{0\}), H(\mathbb{C}))^*$ then it is invertible in $L(X)$.

For the other direction let $A + \lambda \in L(H(\{0\}), H(\mathbb{C}))^*$ be invertible in $L(X)$. Since $\operatorname{Im} A \subset H(\mathbb{C}) \neq X$ the operator A is not invertible, hence $\lambda \neq 0$. Let $B + \frac{1}{\lambda} \in L(X)$ be an inverse of $A + \lambda$, i.e.

$$(A + \lambda)\left(B + \frac{1}{\lambda}\right) = I_X \qquad \left(B + \frac{1}{\lambda}\right)(A + \lambda) = I_X.$$

By the first equation we have

$$B = -\frac{1}{\lambda^2}A - \frac{1}{\lambda}AB.$$

Since $\operatorname{Im} A \subset H(\mathbb{C})$ we have $B: X \to H(\mathbb{C})$. By the second equation we have

$$B = -\frac{1}{\lambda^2}A - \frac{1}{\lambda}BA.$$

Since $BA: H(\{0\}) \to H(\mathbb{C})$ then $B: H(\{0\}) \to H(\mathbb{C})$. Hence $B \in L(H(\{0\}), H(\mathbb{C}))$ and $(B + \frac{1}{\lambda}) \in L(H(\{0\}), H(\mathbb{C}))^*$ is the inverse of $A + \lambda$.

□

Theorem 4.2.4. *Let $\varphi \in H(\mathbb{C})$. Then $\sigma(\Gamma_\varphi) = \sigma(H_\varphi)$.*

Proof. For any function $\varphi \in H(\mathbb{C})$ the operator $\Gamma_\varphi: H(\{0\}) \to H(\mathbb{C})$ is continuous by Lemma 4.1.4. From the projective and injective description of the topology of $\mathscr{A}(\mathbb{R})$ we have continuous inclusions $\mathscr{A}(\mathbb{R}) \subset H(\{0\})$, $H(\mathbb{C}) \subset \mathscr{A}(\mathbb{R})$. Hence, due to Theorem 4.2.3 the spectrum of the operator $\Gamma_\varphi \in L(\mathscr{A}(\mathbb{R}))$ is the same as the spectrum of Γ_φ in $L(H(\{0\}), H(\mathbb{C}))^*$. But we also have continuous inclusions $H^2(\mathbb{D}) \subset H(\{0\})$, $H(\mathbb{C}) \subset H^2(\mathbb{D})$ and we can apply Theorem 4.2.3 to the space $X = H^2(\mathbb{D})$. Hence the spectrum of $H_\varphi \in L(H(\{0\}), H(\mathbb{C}))^*$ is the same as the spectrum of H_φ in $L(H^2(\mathbb{D}))$. Because on $H^2(\mathbb{D})$ we have $\Gamma_\varphi = H_\varphi$ we get that $\sigma(\Gamma_\varphi) = \sigma(H_\varphi)$.

□

In the last part of the chapter we will need the notions of the Besov class B_p^s, the sequence spaces ℓ_p and the space s of rapidly decreasing sequences. There are several equivalent definitions of Besov spaces with certain restrictions on the parameters. We recall here one of them, the one with the modulus of smoothness of the function. Please note that in case $s > \frac{1}{p} - 1$ this definition is equvalent to the one based on Fourier transform.

Definition 4.2.5. Let $0 < p < \infty$, $s > 0$, $n \in \mathbb{N}$, $n > s$. The Besov class B_p^s is defined by

$$B_p^s = \left\{ f \in L^p(\mathbb{T}) : \int_{-\pi}^{\pi} \frac{\|\Delta_t^n f\|_p^p}{t^{1+sp}} dt < \infty \right\},$$

where $\Delta_t f(e^{ix}) = f(e^{i(x+t)}) - f(e^{ix})$ and $\Delta_t^n = \Delta_t \Delta_t^{n-1}$.

Definition 4.2.6. We define the space of rapidly decreasing sequences by

$$s = \left\{ x \in \mathbb{C}^{\mathbb{N}} : \ \|x\|_k = \sum_{j=0}^{\infty} |x_j| \, j^k < \infty \quad \forall k \in \mathbb{N} \right\}.$$

Now, we recall the basic facts concerning the Schatten-von Neumann class of operators $\mathfrak{S}_p$ on the arbitrary Hilbert space H. We say that an operator T belongs to $\mathfrak{S}_p$ if the sequence $(s_n(T))$ of eigenvalues of $(T^*T)^{1/2}$ belongs to ℓ_p. From the Weyl theorem (see for instance [50, 3.5.5]) it follows that the operator ideal $\mathfrak{S}_p$ is of eigenvalue type ℓ_p, which means that for every operator $T \in \mathfrak{S}_p$ the sequence of its eigenvalues is in ℓ_p, i.e. $(\lambda_n(T)) \in \ell_p$.

The following theorem of Peller characterizes the Schatten-von Neumann classes of Hankel operators on H^2.

Theorem 4.2.7. *([47, Th. 1']) Suppose φ is analytic in the disc $\mathbb{D}$, and $0 < p < \infty$. Then $H_\varphi \in \mathfrak{S}_p$ if and only if $\varphi \in B_p^{\frac{1}{p}}$.*

Proposition 4.2.8. *Let $\varphi \in H(\mathbb{C})$ and $\sigma(\Gamma_\varphi) = \sigma(H_\varphi) = \{0\} \cup \{\lambda_n : n = 1, 2 \ldots\}$. Then $(\lambda_n)_{n=0}^{\infty} \in s$.*

Proof. Since φ is an entire function it belongs to each Besov space, i.e. $\varphi \in B_p^{\frac{1}{p}}$ for all $p > 0$ ([56, 3.5.1 Theorem 1]). Hence, by Theorem 4.2.7, $H_\varphi \in \mathfrak{S}_p$ for all $p > 0$. This implies, that the sequence of eigenvalues of H_φ belongs to the space ℓ_p for all $p > 0$. Because every decreasing sequence in $\bigcap_{p>0} \ell_p$ belongs to s ([51, 8.5.5 Lemma 2]) we get that $(\lambda_n)_{n=0}^{\infty} \in s$. □

Toeplitz operators

5

In this chapter we study the Toeplitz operators on $\mathscr{A}(\mathbb{R})$. The class of Toeplitz operators were thourogly studied on the classical Hardy space $H^2(\mathbb{D})$. In [13] Domański and Jasiczak developed the theory of Toeplitz operators on $\mathscr{A}(\mathbb{R})$. The theory was further extended by Jasiczak in [32],[33].

We start with a short introduction to the theory of Toeplitz operators on $\mathscr{A}(\mathbb{R})$. We introduce the symbol space, state the representation theorem and the theorem on characterization of Fredholm Topelitz operators. Finally we prove the analouge of Wiener-Hopf factorization theorem. In the next section we study the problem of one sided invertibility of Toeplitz operators. Finally, we investigate commutators of Teoplitz operators and show when such commutator has finite rank.

5.1 Toeplitz operators on $\mathscr{A}(\mathbb{R})$ and the symbol space $\mathcal{X}$

An infinite matrix is called a Toeplitz matrix if it is of the form

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \dots \\ a_1 & a_0 & a_{-1} & a_{-2} & \dots \\ a_2 & a_1 & a_0 & a_{-1} & \dots \\ a_3 & a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

where $(a_n)_{n\geq 0}$ is a sequence of complex numbers. A Toeplitz operator is a continuous linear operator for which the associated matrix is a Toeplitz matrix. More precisely

Definition 5.1.1. We say that a continuous linear operator $T\colon\mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is a *Toeplitz operator* if there exist complex numbers $\dots, a_{-1}, a_0, a_1, \dots \in \mathbb{C}$ such that for all $n \in \mathbb{N}$ locally near zero

$$Tx^n(\xi) = a_{-n} + a_{-n+1}\xi + a_{-n+2}\xi^2 + \dots.$$

Similarly to the classical case, a Toeplitz operator $T\colon\mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is the compression of a multiplication operator [13]. Roughly speaking T is of the form PM_F, where P is a projection onto $\mathscr{A}(\mathbb{R})$ and M_F is the multiplication by a function from some space $\mathcal{X}$, called the *symbol space*. We will now describe the symbol space $\mathcal{X}$ and the projection P.

Let U be an open neigbourhood of $\mathbb{R}$ and $K \subset U$ a compact subset of $\mathbb{R}$ and let $H(U \setminus K)$ denote the Fréchet space of functions holomorphic in $U \setminus K$. Define the symbol space as the inductive limit of Fréchet spaces

$$\mathcal{X} := \operatorname{ind}_{U,K} H(U \setminus K),$$

where U runs over all open neigbourhood of $\mathbb{R}$ and $K \subset U$ through all compact subset of $\mathbb{R}$. For the proof that the corresponding inductive topology exists, we refer to [13]. Notice that it is enough to take the inductive limit ind $H(U \setminus K)$, where U is simple connected and K is connected. Further on we will assume this.

Two functions $F_1 \in H(U_1 \setminus K_1)$, $F_2 \in H(U_2 \setminus K_2)$ are considered equivalent in $\mathcal{X}$ if and only if there exist an open set $U \subset U_1 \cap U_2$ and compact set $K \supset (K_1 \cup K_2)$ such that $F_1|_{U \setminus K} = F_2|_{U \setminus K}$. Elements of $\mathcal{X}$, which we will call *symbols* of Toeplitz operators, are equivalence classes with respect to the above defined equivalence relation. In this thesis we will usually identify function $F \in H(U \setminus K)$ with its equivalence class with respect to this relation.This should not lead to confusion.

Recall that we denoted by $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ the space $\mathrm{ind}_K\, H_0(\mathbb{C}_\infty \setminus K)$, where K runs over all compact subsets of $\mathbb{R}$.

Theorem 5.1.2. *[13, Thm. 3.3] The space $\mathcal{X}$ can be decomposed as*

$$\mathcal{X} = \mathscr{A}(\mathbb{R}) \oplus H_0(\mathbb{C}_\infty \setminus \mathbb{R}) \simeq \mathscr{A}(\mathbb{R}) \oplus \mathscr{A}(\mathbb{R})'.$$

For any function $F \in \mathcal{X}$ there exist an open and simply connected set $U \supset \mathbb{R}$ and a compact, connected subset $K \subset \mathbb{R}$, $0 \in K$ such that $F \in H(U \setminus K)$. Let $\mathcal{C}\colon \mathcal{X} \to \mathcal{X}$ be the Cauchy transform, i.e.,

$$\mathcal{C}F(z) := \mathcal{C}_\gamma F(z) = \frac{1}{2\pi} \int_\gamma \frac{F(\xi)}{\xi - z},$$

where γ denotes a C^∞ smooth Jordan curve in $U \setminus K$ seperating ∞ and K such that $\mathrm{ind}_\gamma(z) = 1$. The operator $\mathcal{C}$ is a continuous projection onto $\mathscr{A}(\mathbb{R})$([13, Thm. 3.2]).

For any function $F \in \mathcal{X}$ we denote by $M_F\colon \mathcal{X} \to \mathcal{X}$ the multiplication operator, $M_F f = Ff$ for $f \in \mathcal{X}$.

Theorem 5.1.3. *[13, Thm. 1] The following assertions are equivalent*

1. *$T\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is a Toeplitz operator, i.e. T is continuous operator such that locally near zero*

   $$T(x^n)(z) = a_{-n} + a_{-n+1}z + a_{-n+2}z^2, \tag{5.1}$$

   for some complex numbers a_n, $n \in \mathbb{N}$.

2. *there exists a function $F \in \mathcal{X}$ such that*

   $$T = \mathcal{C}M_F,$$

   where M_F is the multiplication operator and $\mathcal{C}$ is the Cauchy projection.

   Then (5.1) holds with

   $$a_n = \frac{1}{2\pi i} \int_\gamma F(\xi)\xi^{n-1} d\xi,$$

   where γ is a C^∞ smooth Jordan curve in $U \setminus K$ surrounding K and $F \in H(U \setminus K)$.

A continuous operator $T\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is called *Fredholm* if it has finite dimensional kernel and cokernel. If T is Fredholm we define its index by

$$\mathrm{index}\, T = \dim\ker T - \dim\mathrm{coker}\, T.$$

Theorem 5.1.4. *[13, Thm. 2] A Toeplitz operator $T_F: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ with $F \in \mathcal{X}$ is a Fredholm operator if and only if there exist an open complex set $U \supset \mathbb{R}$ and a compact set $K \in \mathbb{R}$ such that $F(z) \neq 0$ for $z \in U \setminus K$. If T_F is Fredholm then the index of T_F is equal to the winding number of F, i.e.,*

$$\operatorname{index} T_F = -\operatorname{winding} F.$$

Recall the definition of the winding number. Let $F \in H(U \setminus K)$ be not-vanishing on $U \setminus K$. Let $\gamma: \mathbb{T} \to U \setminus K$ be a diffeomorphism such that $\operatorname{Ind}_\gamma(0) = 1$. We define the winding number of F as $\operatorname{Ind}_{F \circ \gamma}(0)$ and denote it by winding F. Note that this definition does not depend on the choice of γ.

Theorem 5.1.5. *[32, Thm. 5.1] A Toeplitz operator $T_F: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is invertible if and only if it is a Fredholm operator of index zero.*

Theorem 5.1.6. *[32, Thm. 1.2] Let $T_F: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ be a Topelitz operator with non-zero $F \in \mathcal{X}$. Then either* $\ker T_F = \{0\}$ *or* $\ker T_F' = \{0\}$.

Now, we prove an analogue of the Simonenko theorem on Wiener-Hopf factorization of Toeplitz operators acting on the weighted Hardy spaces on curves (see, e.g., [6]). The proof is similar to the classical one. Analogously to the classical setting, we say that $f \in \mathcal{X}$ admits a *Wiener Hopf factorization* in $\mathcal{X}$, if f can be written in the form

$$f(z) = f_- z^k f_+,$$

where $f_-, f_-^{-1} \in \operatorname{ind}_K H(\mathbb{C}_\infty \setminus K)$, $f_+, f_+^{-1} \in \mathscr{A}(\mathbb{R})$ and $k \in \mathbb{Z}$. We use the notation $f^{-1} := {}^1/_f$.

We denote by $Q: \mathcal{X} \to \mathcal{X}$ the projection complementary to $\mathcal{C}$, i.e. $Q = I - \mathcal{C}$.

Lemma 5.1.7. *Assume that at least one of the condition holds : $F \in H(\mathbb{C}_\infty \setminus \mathbb{R})$, $G \in \mathscr{A}(\mathbb{R})$. Then*

$$T_{FG} = T_F T_G. \tag{5.2}$$

Proof. The Toeplitz operator $T_F: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ can be written in the form $T_F = \mathcal{C} M_F \mathcal{C}$. We can calculate

$$T_{FG} = \mathcal{C} M_{FG} \mathcal{C} = \mathcal{C} M_F (\mathcal{C} M_G + Q M_G) \mathcal{C} = \mathcal{C} M_F \mathcal{C} M_G \mathcal{C} + \mathcal{C} M_F Q M_G \mathcal{C} = T_F T_G + \mathcal{C} M_F Q M_G \mathcal{C}.$$

If $F \in H(\mathbb{C}_\infty \setminus \mathbb{R})$ then $\mathcal{C} M_F Q = 0$. If $G \in \mathscr{A}(\mathbb{R})$ then $Q M_G \mathcal{C} = 0$.

□

Lemma 5.1.8. *Let $F \in \mathcal{X}$ and consider three operators:*

- $T_F: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$,
- $(\mathcal{C} M_F \mathcal{C} + Q): \mathcal{X} \to \mathcal{X}$,
- $(M_F \mathcal{C} + Q): \mathcal{X} \to \mathcal{X}$.

If one of these operators is invertible then so are the other two.

Proof. By Theorem 5.1.2 the space $\mathcal{X}$ can be decomposed as a direct sum:

$$\mathcal{X} = \mathscr{A}(\mathbb{R}) \oplus H_0(\mathbb{C}_\infty \setminus \mathbb{R}).$$

The operator $\mathcal{C}M_F\mathcal{C} + Q$ can be written as the operator matrix

$$\begin{pmatrix} T_F & 0 \\ 0 & I \end{pmatrix} : \begin{pmatrix} \mathscr{A}(\mathbb{R}) \\ H_0(\mathbb{C}_\infty \setminus \mathbb{R}) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathscr{A}(\mathbb{R}) \\ H_0(\mathbb{C}_\infty \setminus \mathbb{R}) \end{pmatrix}$$

Hence,

$$\begin{aligned} \operatorname{Im}(\mathcal{C}M_F\mathcal{C} + Q) &= \operatorname{Im} T_F \oplus H_0(\mathbb{C}_\infty \setminus \mathbb{R}), \\ \ker(\mathcal{C}M_F\mathcal{C} + Q) &= \ker T_F, \end{aligned}$$

which prove that T_F is invertible if and only if $\mathcal{C}M_F\mathcal{C} + Q$ is invertible.

To get the assertion for the pair $M_F\mathcal{C} + Q$ and $\mathcal{C}M_F\mathcal{C} + Q$ notice that

$$M_F\mathcal{C} + Q = (\mathcal{C}M_F\mathcal{C} + Q)(I + QM_F\mathcal{C}) \quad \text{and} \quad (I + QM_F\mathcal{C})^{-1} = (I - QM_F\mathcal{C}).$$

Indeed

$$\begin{aligned} (\mathcal{C}M_F\mathcal{C} + Q)(I + QM_F\mathcal{C}) &= \mathcal{C}M_F\mathcal{C} + \mathcal{C}M_F\mathcal{C}QM_F\mathcal{C} + Q + QM_F\mathcal{C} \\ &= (\mathcal{C} + Q)M_F\mathcal{C} + Q = M_F\mathcal{C} + Q, \\ (I + QM_F\mathcal{C})(I - QM_F\mathcal{C}) &= I - QM_F\mathcal{C} + QM_F\mathcal{C} - QM_F\mathcal{C}QM_F\mathcal{C} = I. \end{aligned}$$

□

Now we state the factorization theorem.

Theorem 5.1.9. *A Toeplitz operator $T_F\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$, $F \in \mathcal{X}$, is Fredholm if and only if F admits a Wiener-Hopf factorization in $\mathcal{X}$, i.e.*

$$F = F_- z^k F_+,$$

where $F_+, F_+^{-1} \in \mathscr{A}(\mathbb{R})$, $F_-, F_-^{-1} \in H(\mathbb{C}_\infty \setminus \mathbb{R})$ and $k = -\operatorname{index} T_F$.

Proof. (⇒): Let T_F be a Fredholm operator with index k. We put $G = Fz^k$. By Lemma 5.1.7

$$T_G = T_F T_{z^k} \ \ (\text{if } k > 0) \quad or \quad T_G = T_{z^k} T_F \ \ (\text{if } k < 0). \tag{5.3}$$

By Theorem 5.1.4 the operator T_{z^k} is Fredholm with index $T_{z^k} = -k$. Hence (5.3) implies that ind $T_G = 0$. Using Theorem 5.1.4 again we get that the winding number of G and G^{-1} is zero. Hence, by Theorem 5.1.5, we get that both operators T_G and $T_{G^{-1}}$ are invertible.

By Lemma 5.1.8 we also get invertibility of the operators $M_G\mathcal{C} + Q$ and $M_{G^{-1}}\mathcal{C} + Q$. Let $\varphi, \psi \in \mathcal{X}$ be the solutions of the equations:

$$(M_G\mathcal{C} + Q)\varphi = 1, \qquad (M_{G^{-1}}\mathcal{C} + Q)\psi = 1.$$

We put

$$\varphi_+ := \mathcal{C}\varphi, \quad \psi_+ := \mathcal{C}\psi.$$

Since $\mathcal{C}$ is the projection onto $\mathscr{A}(\mathbb{R})$, there exists an open complex neigbourhood U of $\mathbb{R}$ such that $\varphi_+, \psi_+ \in H(U)$. Hence

$$G\varphi_+ = 1 - Q\varphi, \quad G^{-1}\psi_+ = 1 - Q\psi$$

with $Q\varphi, Q\psi \in H_0(C_\infty \setminus K)$ for some compact set $K \subset \mathbb{R}$. For an open set $U \supset \mathbb{R}$ and a compact set $K \subset \mathbb{R}$ by the Liouville theorem we have $H(U) \cap H(\mathbb{C}_\infty \setminus K) = \{\text{constant functions}\}$. Since $\varphi_+\psi_+ = \varphi_+ GG^{-1}\psi_+ = (1 - Q\varphi)(1 - Q\psi)$ and $(1 - Q\varphi)(1 - Q\psi)(\infty) = 1$ we get that $\varphi_+\psi_+ = \text{constant} = 1$.

Finally we put

$$F_+ = \psi_+, \quad F_- = 1 - Q\varphi$$

Then $F_+^{-1} = \varphi_+$, $F_-^{-1} = 1 - Q\psi$ and

$$F_- z^{-k} F_+ = (1 - Q\varphi)z^{-k}\psi_+ = (1 - Q\varphi)z^{-k}GG^{-1}\psi_+ = (1 - Q\varphi)z^{-k}G(1 - Q\psi) = z^{-k}G = F.$$

($\Leftarrow$): If F admits a factorization of this form then $F \neq 0$ on some $U \setminus K$, U - open complex neighborhood of $\mathbb{R}$, K - compact subset of $\mathbb{R}$, and T_F is Fredholm by Theorem 5.1.4. Since by Lemma 5.1.7 we have

$$T_{F_+} T_{F_+^{-1}} = I \quad \text{and} \quad T_{F_-} T_{F_-^{-1}} = I,$$

Theorem 5.1.5 asserts that $\operatorname{ind} T_{F_+} = 0$ and $\operatorname{ind} T_{F_-} = 0$. Finally, we use Lemma 5.1.7 once again, and get

$$\operatorname{ind} T_F = \operatorname{ind}\left(T_{F_-} T_{z^k} T_{F_+}\right) = -k.$$

□

5.2 INVERTIBILITY OF TOEPLITZ OPERATORS

Theorem 5.1.5 from the previous section states that a Topelitz operator $T_F: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is invertible if and only if it is a Fredholm operator of index zero. In this section, which is based on joint work with Jasiczak [27], we study one sided invertibility of the Toeplitz operators.

Our starting point is Theorem 5.2.1, but first we need to introduce some notation. Let $F \in \mathcal{X}$ and let $\widetilde{F} \in H(U \setminus K)$, $K \subset \mathbb{R}$-compact, $U \supset \mathbb{R}$ open, be its representative, i.e. $F = [\widetilde{F}]_\sim$. We say

(i) that F *has real zeros going to infinity* if there are $x_n \in \mathbb{R}$ such that $\lim |x_n| = \infty$ and $\widetilde{F}(x_n) = 0$.

(ii) that F *has non-real zeros accumulating at a real point* if the function $\widetilde{F}$ has zeros $z_n \notin \mathbb{R}$ whose limit $\lim z_n$ exists and belongs to $\mathbb{R}$.

Observe that these properties depend only on the germ $F \in \mathcal{X}$ and do not depend on the choice of U, K and the representative $\widetilde{F} \in H(U \setminus K)$ of F.

Theorem 5.2.1. *[33, Thm. 1.6] Assume that $F \in \mathcal{X}$ does not vanish identically. Consider the Toeplitz operator $T_F: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$.*

(i) *If F has no real zeros going to infinity and no non-real zeros accumulating at a real point, then the kernel of T_F is finite dimensional, the cokernel is finite dimensional and the image of T_F is closed;*

(ii) *If F has non-real zeros accumulating at a real point, but has no real zeros going to infinity, then the operator T_F is surjective and the kernel of T_F is infinite dimensional;*

(iii) *If F has real zeros going to infinity, but has no non-real zeros accumulating at a real point, then the operator T_F is injective and the range of T_F is a closed subspace of infinite codimension;*

(iv) *If F has both real zeros going to infinity and non-real zeros accumulating at a real point, then the operator T_F is injective and the range of T_F is a dense subspace of infinite codimension.*

For any $F \in \mathcal{X}$ on of the above four cases holds true.

It is clear that if the operator is left invertible then it is injective. Hence the operator T_F is not left invertible if the case (ii) holds. Moreover, if A is a left inverse of T_F then $T_F \circ A$ is a continuous projection onto the range of T_F. Hence the operator T_F is left invertible only if the range of T_F is a complemented subspace of $\mathscr{A}(\mathbb{R})$.

Theorem 5.2.2 (Jasiczak). *Assume that $F \in \mathcal{X}$ has real zeros going to infinity, but has no non-real zeros accumulating at a real point. Then there exists a sequence of continuous linear functionals $(\xi_n)_{n\geq 0} \subset \mathscr{A}(\mathbb{R})'_b$ such that*

$$\operatorname{Im} T_F = \bigcap_{n=1}^{\infty} \ker \xi_n.$$

The range of the operator T_F is not a complemented subspace of $\mathscr{A}(\mathbb{R})$.

It follows that a Toeplitz operator can be left-invertible only if (i) holds, i.e. it is a Fredholm operator. More precisely,

Theorem 5.2.3 (Jasiczak). *A Toeplitz operator $T_F\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$, $F \in \mathcal{X}$, is left-invertible if and only if T_F is an injective Fredholm operator.*

A right-invertible operator is surjective. Hence, a Toeplitz operator T_F can be right-invertible only in cases (i) and (ii). We will show that, in fact, T_F is right-invertible if and only if it is a surjective Fredholm operator. To exclude the (ii) case, we use the adjoints and show that the operator $T'_F\colon \mathscr{A}(\mathbb{R})'_b \to \mathscr{A}(\mathbb{R})'_b$ is not left-invertible. Similarly to Theorem 5.2.2 we will prove

Theorem 5.2.4. *Assume that $F \in \mathcal{X}$ has non-real zeros accumulating at a real point, but has no real zeros going to infinity. Then the range of T'_F is not complemented in $\mathscr{A}(\mathbb{R})'$ and the operator T'_F is not left-invertible.*

Throughout the proof we identify the space $\mathscr{A}(\mathbb{R})'_b$ with the function space $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$. For $F \in \mathcal{X}$ the operator $T'_F\colon H_0(\mathbb{C}_\infty \setminus \mathbb{R}) \to H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ is given by

$$T'_F g(z) = \frac{1}{2\pi i} \int_{-\gamma} \frac{F(\xi)g(\xi)}{\xi - z} d\xi \quad \text{for } z \in \operatorname{Ext}(\gamma),$$

where $F \in H(U \setminus K)$, $g \in H_0(\mathbb{C}_\infty \setminus L)$ and γ is a positively oriented C^∞ smooth Jordan curve in U such that $K \cup L \subset \operatorname{Int}(\gamma)$.

In the proof of Theorem 5.2.4 we use the dual space of $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$, which is the space $\mathscr{A}(\mathbb{R})$ itself, i.e., for any continuous functional $\varphi\colon H_0(\mathbb{C}_\infty \setminus \mathbb{R}) \to \mathbb{C}$ there exist an open set $U \supset \mathbb{R}$ and a function $g \in H(U)$ such that

$$\varphi(f) = \langle f, g\rangle = \int_\gamma f(z)g(z)dz, \quad f \in H_0(\mathbb{C}_\infty \setminus \mathbb{R}), \tag{5.4}$$

where $f \in H_0(\mathbb{C}_\infty \setminus L)$ for some compact set $L \subset \mathbb{R}$ and γ is a C^∞ smooth Jordan curve in $U \setminus L$ such that $L \in \operatorname{Int}(\gamma)$. Since for every functional $\varphi \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})'$ there exist the unique function g satisfiying (5.4), to simplify notation we shall identify the functional φ with the corresponding function $g \in \mathscr{A}(\mathbb{R})$.

For any open sets $U, W \supset \mathbb{R}$, compact sets $K, L \subset \mathbb{R}$ and functions $F \in H(U \setminus K)$, $f \in H_0(\mathbb{C}_\infty \setminus L)$, $g \in H(W)$ we have

$$\langle T'_F f, g\rangle = \langle f, T_F g\rangle = \int_\gamma F(z)f(z)g(z)dz \tag{5.5}$$

where γ is a positively oriented C^∞ smooth Jordan curve in $(U \cap W)$ such that $(K \cup L) \in \operatorname{Int}(\gamma)$. Indeed, by the Fubini theorem we have

$$\begin{aligned}\langle T'_F f, g\rangle &= \int_{\gamma_1} T'_F f(\xi)g(\xi)d\xi = \int_{\gamma_1} g(\xi)\int_{-\gamma_2} \frac{F(z)f(z)}{z-\xi}dz\\ &= \int_{\gamma_1} g(\xi)\int_{\gamma_2}\frac{F(z)f(z)}{\xi - z}dz = \int_{\gamma_2} F(z)f(z)\int_{\gamma_1}\frac{g(\xi)}{\xi - z}d\xi dz\\ &= \int_{\gamma_2} F(z)f(z)g(z)dz,\end{aligned}$$

where γ_1, γ_2 are positively oriented C^∞ smooth Jordan curves in $(U \cap W) \setminus (K \cup L)$ such that $(K \cup L) \in \operatorname{Int}(\gamma_1)$ and $(K \cup L) \in \operatorname{Int}(\gamma_2)$.

We will need the following auxiliary lemmas and facts.

Lemma 5.2.5. *If an operator $A\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is surjective then the image of the adjoint operator $A'\colon \mathscr{A}(\mathbb{R})'_b \to \mathscr{A}(\mathbb{R})'_b$ is closed.*

Proof. Since $\ker A = (\operatorname{Im} A')^\circ$ we have

$$(\ker A)^\circ = (\operatorname{Im} A')^{\circ\circ} \supset \operatorname{Im} A'.$$

To show that $(\ker A)^\circ \subset \operatorname{Im} A'$ we define the map $A_0\colon \mathscr{A}(\mathbb{R})/\ker A \to \mathscr{A}(\mathbb{R})$, $A_0(x + \ker A) = Ax$. The map A_0 is a continuous linear bijection. The space $\mathscr{A}(\mathbb{R})$ is ultrabornological. Since $\mathscr{A}(\mathbb{R})$ has a web, so does the space $\mathscr{A}(\mathbb{R})/\ker A$ [40, Lemma 24.28]. Hence we can use the open mapping theorem and get the continuous inverse map

$$A_0^{-1}\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})/\ker A.$$

Take $y \in (\ker A)^\circ$ and define the functional $\overline{y}: \mathscr{A}(\mathbb{R})/\ker A \to \mathbb{C}$ by

$$\overline{y} : (x + \ker A) \mapsto y(x).$$

Let

$$\eta := \overline{y} \circ A_0^{-1}.$$

Obviously $\eta \in \mathscr{A}(\mathbb{R})'$. For $x \in \mathscr{A}(\mathbb{R})$ we have

$$\langle A'\eta, x\rangle = \langle \eta, Ax\rangle = \langle \eta, A_0(x + \ker A)\rangle = \overline{y}(x + \ker A) = y(x).$$

Hence $(\ker A)^\circ \subset \operatorname{Im} A'$. Finally, we have $(\ker A)^\circ = \operatorname{Im} A'$ and because $(\ker A)^\circ$ is a closed set we get that $\operatorname{Im} A'$ is closed.

□

Lemma 5.2.6. *Let $F \in H(U \setminus K)$ vanish on $z_n \in \mathbb{C} \setminus \mathbb{R}$ with accumulating points in compact set $K \subset \mathbb{R}$. The image of the Toeplitz operator $T_F': \mathscr{A}(\mathbb{R})_b' \to \mathscr{A}(\mathbb{R})_b'$ is closed.*

Proof. Under this assumption, by Theorem 5.2.1, the operator $T_F: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is surjective and we can apply Lemma 5.2.5. □

Proposition 5.2.7. *If an operator $A: \mathscr{A}(\mathbb{R})_b' \to \mathscr{A}(\mathbb{R})_b'$ is Fredholm then its image is closed.*

Proof. Similarly to the classical Banach case (see e.g. [34, Lemma 332]), the proof is based on the open mapping theorem. Since the space $\mathscr{A}(\mathbb{R})$ is Schwartz its dual $\mathscr{A}(\mathbb{R})_b'$ is ultrabornological ([40, Prop. 24.23]). For a compact set $K \subset \mathbb{R}$ the space $H_0(\mathbb{C}_\infty \setminus K)$ admits a web as a Fréchet space ([40, 24.29]), and the space $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ admits a web as an inductive limit of webbed spaces ([40, Lemma 24.28]). Let Y be a closed finite dimensional subspace of $\mathscr{A}(\mathbb{R})_b'$ such that $\operatorname{Im} A \oplus Y = \mathscr{A}(\mathbb{R})'$. By Lemma 2.1.7 the space $\mathscr{A}(\mathbb{R})_b'/\ker A + Y$ also admits a web. Now consider the operator

$$\widetilde{A}: \mathscr{A}(\mathbb{R})_b'/\ker A \oplus Y \to \mathscr{A}(\mathbb{R})', \quad \widetilde{A}(f + \ker A, y) = Af + y.$$

The operator $\widetilde{A}$ is clearly continuous and surjective. By the open mapping theorem it is open. Note that $\operatorname{Im} A = \operatorname{Im} \widetilde{A}(\mathscr{A}(\mathbb{R})_b'/\ker A \oplus \{0\})$, which completes the proof. □

Theorem 5.2.8. *If an operator $T_F: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$, $F \in \mathcal{X}$, is a Fredholm operator, then the adjoint operator $T_F': \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is also a Fredholm operator. Moreover*

$$\operatorname{index} T_F' = -\operatorname{index} T_F.$$

Proof. By the assumption $\operatorname{Im} T_F$ is a subspace of finite codimension of the bornological space $\mathscr{A}(\mathbb{R})$. Hence, by [31, Thm 13.5.2] it is bornological. Since it is closed ([13, Prop. 5.1]) in a complete space, it is ultrabornological ([40, 24.15(b)]). Hence we can use the open mapping theorem for the map $T_F: \mathscr{A}(\mathbb{R}) \to \operatorname{Im} T_F$ to conclude that T_F is open. According to [52, Satz 1.1] the operator T_F is a ρ-transformation ([52, Definition 1.1]). By [52, Satz 7.1] the adjoint operator $T_F: \mathscr{A}(\mathbb{R})_b' \to \mathscr{A}(\mathbb{R})_b'$ is a ρ-transformation and the following holds

$$\dim\ker T_F = \operatorname{codim} \operatorname{Im} T_F', \quad \dim \operatorname{codim} T_F = \dim\ker T_F'.$$ □

Lemma 5.2.9. *Let $F \in H(U \setminus K)$ and let γ be a C^∞ smooth Jordan curve in U and $K \subset \mathrm{Int}(\gamma)$. If*

$$\int_\gamma F(z)z^{-k}dz = 0 \quad k = 1, 2, \dots,$$

then F can be holomorphically extended to $\mathbb{C}_\infty \setminus K$.

Proof. By Theorem 5.1.2 we can decompose the function F into $F = F_+ + F_-$, where $F_+ \in H(U)$, $F_- \in H_0(\mathbb{C}_\infty \setminus K)$. Define the curve $\widetilde{\gamma}$ by $\widetilde{\gamma}(t) = \frac{1}{\gamma(t)}$. We have

$$\int_\gamma F(z)z^{-k}dz = -\int_{\widetilde{\gamma}} F\left(\frac{1}{z}\right) z^{k-2}dz = -\int_{\widetilde{\gamma}} F_+\left(\frac{1}{z}\right) z^{k-2}dz - \int_{\widetilde{\gamma}} \frac{F_-\left(\frac{1}{z}\right)}{z} z^{k-1}dz$$

Since the function $\frac{F_-(\frac{1}{z})}{z}$ is holomorphic in $\mathrm{Int}(\widetilde{\gamma})$ the second term equals zero. We are left with

$$-\int_{\widetilde{\gamma}} F_+\left(\frac{1}{z}\right) z^{k-2}dz = \int_\gamma F_+(z)z^{-k}dz = \langle z^{-k}, F_+\rangle.$$

If the functional F_+ vanishes on all the z^{-k} then it has to be zero, since $\mathrm{span}\{z^{-k} : k = 1, 2, \dots\}$ is dense in $H_0(\mathbb{C}_\infty \setminus K)$. □

Recall that for any $f, g \in H_0(\mathbb{C}_\infty \setminus K)$, $K \in \mathbb{R}$ compact, we have

$$\int_\gamma fg = 0$$

for every γ such that $K \subset \mathrm{Int}(\gamma)$. Indeed, we change the order of integration and get

$$\int_\gamma f(z)g(z)dz = \int_{-\frac{1}{\gamma}} f\left(\frac{1}{z}\right) g\left(\frac{1}{z}\right) z^{-2}dz = 0,$$

since both functions $f(\frac{1}{z})/z$ and $g(\frac{1}{z})/z$ are holomorphic in $\mathrm{Int}(1/\gamma)$.

Now we set the stage for the proof of Theorem 5.2.4.

Let $F \in \mathcal{X}$ satisfy condition (ii) in Theorem 5.2.1. Let (z_n) be a sequence of non-real zeros of F and let $(c_n)_{n\in\mathbb{N}}$ be a sequence in K such that

$$\mathrm{dist}(z_n, K) = |z_n - c_n|.$$

We assume that

$$\mathrm{dist}(z_1, K) \geq \mathrm{dist}(z_2, K) \geq \dots.$$

We factorize the symbol function F as

$$F(z) = \prod_{n=1}^\infty E_{p_n}^{m_n}\left(\frac{z_n - c_n}{z - c_n}\right) F_0(z),$$

where $E_{p_n}^{m_n}$ denotes the Weierstrass elementary factor, i.e.,

$$E_{p_n}^{m_n}(z) = (1 - z)^{m_n} \exp\left(m_n\left(z + \frac{z^2}{2} + \dots + \frac{z^{p_n}}{p_n}\right)\right),$$

the function F_0 does not vanish on $U \setminus K$, m_n denotes the multiplicity of z_n and $(p_n)_{n\in\mathbb{N}}$ is a sequence of natural numbers chosen in such a way that

$$\sum_{n=1}^{\infty} m_n(r\,|z_n - c_n|)^{p_n+1} < \infty \quad \text{for all } r > 0.$$

Notice that we can always find a function $\widetilde{F}$ from the equivalence class of F in $\mathcal{X}$ such that $\widetilde{F}$ factorizes as

$$\widetilde{F}(z) = \mathcal{E}(z)\widetilde{F}_0(z),$$

where $\mathcal{E} \in H(\mathbb{C}_\infty \setminus \overline{\{c_1, c_2, \ldots\}})$ is a Weierstrass product, $\widetilde{F}_0$ does not vanish on some $V \setminus K$ and index $T'_{\widetilde{F}_0} = -1$. Indeed, let k be the index of T'_{F_0} and choose the natural numbers N, l_N, $l_N \le m_N$ such that

$$m_1 + \ldots m_{N-1} + l_N = |k+1|\,.$$

Then, if $k \ge 0$ we put $G_0 := \prod_{n=1}^{N-1} E_{p_n}^{m_n}\left(\frac{z_n-c_n}{z-c_n}\right) E_{p_N}^{l_N}\left(\frac{z_n-c_n}{z-c_n}\right) F_0(z)$. By Theorem 5.2.8 we have $\operatorname{ind} T'_{G_0} = -\operatorname{ind} T_{G_0}$, hence we get

$$\begin{aligned}\text{index } T'_{G_0} &= \text{winding}\, G_0 = -\sum_{n=1}^{N-1} m_n - l_N + \text{winding}\, F_0 = -\sum_{n=1}^{N-1} m_n - l_N + \text{index } T'_{F_0}\\ &= -\sum_{n=1}^{N-1} m_n - l_N + k = -1\end{aligned}$$

In the case $k < 0$ we put $G_0 := \frac{F_0}{\prod_{n=1}^{N-1} E_{p_n}^{m_n}\left(\frac{z_n-c_n}{z-c_n}\right) E_{p_N}^{l_N}\left(\frac{z_n-c_n}{z-c_n}\right)}$ and get

$$\begin{aligned}\text{index } T'_{G_0} &= \text{winding}\, G_0 = \sum_{n=1}^{N-1} m_n + l_N + \text{winding}\, F_0 = \sum_{n=1}^{N-1} m_n + l_N + \text{index } T'_{F_0}\\ &= \sum_{n=1}^{N-1} m_n + l_N - |k| = -1.\end{aligned}$$

Naturally, in both cases there exist an open neigbourhood V of $\mathbb{R}$ such that $\{z_1, \ldots, z_{N-1}\} \notin V$, which means that G_0 does not vanish on $V \setminus K$. Now, we write

$$F(z) := \mathcal{E}(z)G_0(z),$$

where $\mathcal{E}(z) = E_{p_N}^{m_N-l_N}\left(\frac{z_n-c_n}{z-c_n}\right) \prod_{n=N+1}^{\infty} E_{p_n}^{m_n}\left(\frac{z_n-c_n}{z-c_n}\right)$ or
$\mathcal{E}(z) = \prod_{n=1}^{N-1} E_{p_n}^{2m_n}\left(\frac{z_n-c_n}{z-c_n}\right) E_{p_N}^{m_N+l_N}\left(\frac{z_n-c_n}{z-c_n}\right) \prod_{n=N+1}^{\infty} E_{p_n}^{m_n}\left(\frac{z_n-c_n}{z-c_n}\right)$.

The next lemma is the key factor in the proof. Recall that we identify the functionals on $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ with the corresponding functions from $\mathscr{A}(\mathbb{R})$ satisfiying (5.4).

Lemma 5.2.10. *Assume that $G_0 \in H(U \setminus K)$ does not vanish in $U \setminus K$ for some open neighbourhood U of $\mathbb{R}$ and compact set $K \subset \mathbb{R}$, and let T'_{G_0} be a Fredholm operator with* index $T'_{G_0} = -1$. *There exists a function $\varphi \in \mathscr{A}(\mathbb{R})$ such that*

$$\operatorname{Im}(T'_{G_0}) = \ker \varphi.$$

Proof. Since T'_{G_0} is a Fredholm operator, by Proposition 5.2.7, it has closed image. From index $T'_{G_0} = -1$ we have that $\dim \operatorname{coker} T'_{G_0} > 0$. Consider the space $\mathscr{A}(\mathbb{R})'/\operatorname{Im} T'_{G_0}$ and let $\widetilde{\xi}$ be a non-zero functional on it. Denote by π the quotient map

$$\pi \colon \mathscr{A}(\mathbb{R})' \to \mathscr{A}(\mathbb{R})'/\operatorname{Im} T'_{G_0}.$$

The functional $\xi := \widetilde{\xi} \circ \pi$ is a non-zero functional on $\mathscr{A}(\mathbb{R})'$ such that $\langle f, \xi \rangle = 0$ for all functions $f \in \operatorname{Im} T'_{G_0}$. Hence $\operatorname{Im} T_{G_0} \subset \ker \xi$. Since index $T_{G_0} = 1$, by Theorem 5.1.6, we get that $\ker T'_{G_0} = \{0\}$ and $\dim \operatorname{coker} T'_{G_0} = 1$. Hence $\operatorname{Im} T'_{G_0} = \ker \xi$. □

Lemma 5.2.11. *Assume that $G_0 \in H(U \setminus K)$ does not vanish in $U \setminus K$ for some open neighbourhood U of $\mathbb{R}$ and a compact set $K \subset \mathbb{R}$ and let T'_{G_0} be a Fredholm operator with* index $T'_{G_0} = -1$. *Let $\varphi \in \mathscr{A}(\mathbb{R})$ satisfy* $\operatorname{Im} T'_{G_0} = \ker \varphi$.

(*i*) *Put*

$$G(z) := \prod_{n=1}^{N} E_{p_n}^{i_n}\left(\frac{z_n - c_n}{z - c_n}\right) G_0(z) := \mathcal{E}(z) G_0(z)$$

for some sequences $(c_n) \subset K$, $(z_n) \subset U \setminus K$ such that $\operatorname{dist}(z_n, K) = |z_n - c_n|$. *Then*

$$\operatorname{Im} T'_G = \ker \varphi \cap \bigcap_{n=1}^{N} \bigcap_{k=1}^{i_n} \ker \frac{\varphi}{(z - z_n)^k}.$$

In other words, if ψ vanishes on $\operatorname{Im} T'_G$ *then*

$$\psi \in \operatorname{span}\left\{\varphi, \frac{\varphi}{z - z_1}, \frac{\varphi}{(z - z_1)^2}, \ldots, \frac{\varphi}{(z - z_1)^{i_1}}, \ldots, \frac{\varphi}{z - z_N}, \ldots, \frac{\varphi}{(z - z_N)^{i_N}}\right\}.$$

(*ii*) *Put*

$$G(z) := \prod_{n=1}^{\infty} E_{p_n}^{i_n}\left(\frac{z_n - c_n}{z - c_n}\right) G_0(z) := \mathcal{E}(z) G_0(z)$$

for some sequences $(c_n) \subset K$, $(z_n) \subset U \setminus K$ such that $\operatorname{dist}(z_n, K) = |z_n - c_n|$ *and sequences $(i_n), (p_n) \subset \mathbb{N}$ such that the Weierstrass product is convergent. Then*

$$\operatorname{Im} T'_G = \ker \varphi \cap \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{i_n} \ker \frac{\varphi}{(z - z_n)^k}.$$

(*iii*) *Assume that $\varphi \in H(V)$ for some open set $V \supset \mathbb{R}$. There exists a function $a \in H_0(\mathbb{C}_\infty \setminus K)$ such that*

$$\varphi(z) = \frac{a(z)}{G_0(z)},$$

for $z \in (U \cap V) \setminus K$.

Proof. (i): The function G has finitely many zeros in $U \setminus K$, all of them at the points z_n, $n \in \mathbb{N}$. Hence there exists an open set $V \subset U$, containing $\mathbb{R}$, such that G does not vanish on $V \setminus K$. It follows from Theorem 5.2.1 that T_G is a Fredholm operator and we can compute

$$\operatorname{index} T'_G = -\operatorname{index} T_G = \operatorname{winding} G = -\sum_{n=1}^{N} i_n + \operatorname{winding} G_0 = -\sum_{n=1}^{N} i_n - 1.$$

Hence index $T_G > 0$ and $\dim\ker T_G > 0$. By Theorem 5.1.6 we get $\ker T'_G = \{0\}$ and $\dim\operatorname{coker} T'_G = \sum_{n=1}^{N} i_n + 1$

Consider the family of functions

$$\mathcal{G} = \left\{\varphi(z), \frac{\varphi(z)}{z-z_1}, \frac{\varphi(z)}{(z-z_1)^2}, \dots, \frac{\varphi(z)}{(z-z_1)^{i_1}}, \dots, \frac{\varphi(z)}{z-z_N}, \dots, \frac{\varphi(z)}{(z-z_N)^{i_N}}\right\}.$$

We treat the family $\mathcal{G} \subset \mathscr{A}(\mathbb{R})$ as functionals on $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$. We prove that the functionals from $\mathcal{G}$ vanish on $\operatorname{Im} T'_G$. Indeed, take $\psi \in \mathcal{G}$. For $f \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$, $n \le N$ and $j \le i_n$ we have

$$\langle T'_G f, \psi\rangle = \frac{1}{2\pi i}\int_\gamma G(z)f(z)\psi(z)dz = \frac{1}{2\pi i}\int_\gamma \frac{G_0(z)\mathcal{E}(z)\varphi(z)f(z)}{(z-z_n)^j}dz = \left\langle T'_{G_0}g, \varphi\right\rangle$$

where $g(z) = \frac{\mathcal{E}(z)f(z)}{(z-z_n)^j} \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$. Since the functional φ vanishes on $\operatorname{Im} T'_{G_0}$ we get that $\langle T'_G f, \psi\rangle = 0$.

We proved that $\operatorname{Im} T'_G \subset \bigcap_{\psi\in\mathcal{G}} \ker\psi$. We will show that $\dim\operatorname{span}\mathcal{G} = \sum_{n=1}^{N} i_n + 1$. This ends the proof since it means that both spaces $\operatorname{Im} T'_G$ and $\bigcap_{\psi\in\mathcal{G}} \ker\psi$ have the same codimension and have to be equal.

To prove that functions from $\mathcal{G}$ are linearly independent assume that

$$0 = \alpha_0\varphi + \alpha_{11}\frac{\varphi}{z-z_1} + \alpha_{12}\frac{\varphi}{(z-z_1)^2} + \dots + \alpha_{1m_1}\frac{\varphi}{(z-z_1)^{m_1}} + \alpha_{21}\frac{\varphi}{z-z_2} + \dots + \alpha_{N,l}\frac{\varphi}{(z-z_N)^l}.$$

Thus

$$0 = \alpha_0 + \alpha_{11}\frac{1}{z-z_1} + \alpha_{12}\frac{1}{(z-z_1)^2} + \dots + \alpha_{1m_1}\frac{1}{(z-z_1)^{m_1}} + \alpha_{21}\frac{1}{z-z_2} + \dots + \alpha_{N,l}\frac{1}{(z-z_N)^l}.$$

Since on the right we have a sum of meromorphic functions with poles of different order it follows that all the α_{ij} have to be zero.

(ii): Recall, that by the assumption the function $\varphi \in \mathscr{A}(\mathbb{R})$ satisfies $\operatorname{Im}(T'_{G_0}) = \ker\varphi$. Take $f \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$. For any $n \in \mathbb{N}$ and $0 \le j \le i_n$ we have

$$\left\langle T'_G f, \frac{\varphi}{(z-z_n)^j}\right\rangle = \int_\gamma G_0(z)\frac{\mathcal{E}(z)}{(z-z_n)^j}f(z)\varphi(z)dz = \langle T_{G_0}g, \varphi\rangle = 0,$$

where $g(z) = \frac{\mathcal{E}(z)f(z)}{(z-z_n)^j}$, $g \in H_0(\mathbb{C}_\infty \setminus K)$. Hence

$$\operatorname{Im}(T'_G) \subset \ker\varphi \cap \bigcap_{n=1}^{\infty}\bigcap_{j=1}^{m_n} \ker\frac{\varphi}{(z-z_n)^j}.$$

To prove the other inclusion we assume that the functional $\psi \in H(V) \subset \mathscr{A}(\mathbb{R})$ vanishes on $\operatorname{Im}(T'_G)$, i.e., for any compact set $K \subset \mathbb{R}$ and $f \in H_0(\mathbb{C}_\infty \setminus K)$ we have $\langle T'_G f, \psi \rangle = 0$. Then

$$0 = \langle T'_G f, \psi \rangle = \int_\gamma G(z) f(z) \psi(z) dz = \int_\gamma (G\psi)(z) f(z) dz,$$

where γ is a C^∞ smooth Jordan curve in $(V \cap U) \setminus K$ such that $K \in \operatorname{Int}(\gamma)$. Since $G\psi \in H(V \cap U \setminus K)$, by Lemma 5.2.9, we get that $G\psi$ holomorphically extends to $\mathbb{C}_\infty \setminus K$. Put $a := G\psi$, $a \in H(\mathbb{C}_\infty \setminus K)$. We have

$$a(z) = \prod_{n=1}^{\infty} E_{p_n}^{i_n} \left(\frac{z_n - c_n}{z - c_n} \right) G_0(z) \psi(z) \quad \text{for } z \in (V \cap U) \setminus K.$$

Since a vanishes at points $z_n \in U \cap V$, we can find a function $b_N \in H_0(\mathbb{C}_\infty \setminus K)$ such that

$$b_N(z) = \prod_{n=1}^{N} E_{p_n}^{i_n} \left(\frac{z_n - c_n}{z - c_n} \right) G_0(z) \psi(z) \quad \text{for } z \in (V \cap U) \setminus K.$$

The number N should be chosen is such way that all $z_1, \dots, z_N$ lie in $\mathbb{C}_\infty \setminus (V \cap U)$. We can write

$$\psi(z) = \frac{1}{\prod_{n=1}^{N} E_{p_n}^{i_n} \left(\frac{z_n - c_n}{z - c_n} \right)} \frac{b_N(z)}{G_0(z)} \quad \text{for } z \in (V \cap U) \setminus K.$$

For $G_N(z) := \prod_{n=1}^{N} E_{p_n}^{i_n} \left(\frac{z_n - c_n}{z - c_n} \right) G_0(z)$ and $f \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ we have

$$\langle T'_{G_N} f, \psi \rangle = \int_\gamma G_N(z) f(z) \psi(z) dz = \int_\gamma f(z) b_N(z) dz = 0$$

since $f, b_N \in H_0(C_\infty \setminus L)$ for some compact set $L \subset \mathbb{R}$. This means that ψ vanishes on the $\operatorname{Im} T'_{G_N}$. From the point (i) of the Lemma we get that

$$\psi \in \operatorname{span} \left\{ \varphi, \frac{\varphi}{z - z_1}, \dots \frac{\varphi}{(z - z_1)^{i_1}}, \dots, \frac{\varphi}{(z - z_N)}, \dots \frac{\varphi}{(z - z_N)^{i_N}} \right\}.$$

At the begining of the proof we showed that $\operatorname{Im}(T'_G) \subset \ker \varphi \cap \bigcap_{n=1}^{\infty} \bigcap_{j=1}^{m_n} \ker \frac{\varphi}{(z - z_n)^j}$. Assume that $\operatorname{Im}(T'_G) \neq \ker \varphi \cap \bigcap_{n=1}^{\infty} \bigcap_{j=1}^{m_n} \ker \frac{\varphi}{(z - z_n)^j}$. Since $\operatorname{Im}(T'_G)$ is closed (by Lemma 5.2.6) there exists a functional ξ such that ξ vanishes on $\operatorname{Im}(T'_G)$ and $\xi(f) = 1$ for some $f \in \ker \varphi \cap \bigcap_{n=1}^{\infty} \bigcap_{j=1}^{m_n} \ker \frac{\varphi}{(z - z_n)^j}$. But we just proved that if ξ vanishes on $\operatorname{Im}(T'_G)$ then it belongs to

$$\operatorname{span} \left\{ \varphi, \frac{\varphi}{z - z_1}, \dots \frac{\varphi}{(z - z_1)^{i_1}}, \dots, \frac{\varphi}{(z - z_N)}, \dots \frac{\varphi}{(z - z_N)^{i_N}} \right\}$$

for some N. Hence ξ vanishes on f. We get a contradiction and we can finally conclude that

$$\operatorname{Im}(T'_G) = \ker \varphi \cap \bigcap_{n=1}^{\infty} \bigcap_{j=1}^{m_n} \ker \frac{\varphi}{(z - z_n)^j}.$$

(iii): We repeat a part of the proof of (ii). Since $\operatorname{Im} T'_{G_0} = \ker \varphi$, we have

$$0 = \langle T'_{G_0} f, \varphi \rangle = \int_\gamma G_0(z) f(z) \varphi(z) dz.$$

for every $f \in H_0(\mathbb{C}_\infty \setminus K)$ and any smooth Jordan curve in $U \cap V \setminus K$. Since $G_0 \psi \in H(V \cap U \setminus K)$, by Lemma 5.2.9, we can extend $G_0 \psi$ holomorphically to $\mathbb{C}_\infty \setminus K$. Finally, we put $a := G_0 \varphi$. □

We proved that for a Toeplitz operator $T'_F \colon H_0(\mathbb{C}_\infty \setminus \mathbb{R}) \to H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ with symbol function $F \in H(U \setminus K)$, where U is some open neigbourhood of $\mathbb{R}$ and K is a compact set in $\mathbb{R}$, satisfiying

$(*)$ there exists a sequence $z_n \in \mathbb{C} \setminus \mathbb{R}$ accumulating in a compact set $K \subset \mathbb{R}$ such that $F(z_n) = 0$ and these are the only zeroes of F in $U \setminus K$

we can find a sequence of continuous linear functionals ξ_n such that $\operatorname{Im}(T'_F) = \bigcap_n \ker \xi_n$. Now, we construct a sequence of functions (f_i) in $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ such that $\xi_n(f_i) = \delta_{i,n}$.

We can assume that function F factorizes as

$$F(z) = \prod_{n=1}^{\infty} E_{p_n}^{m_n} \left(\frac{z_n - c_n}{z - c_n} \right) F_0(z),$$

where F_0 does not vanish on $U \setminus K$, a sequence (c_n) satisfy $\operatorname{dist}(z_n, K) = |z_n - c_n|$ and index $T_{F_0} = -1$. We can also assume that the sequence of zeros (z_n) satisfies $\operatorname{dist}(z_n, K) \geq \operatorname{dist}(z_{n+1}, K)$ for all $n \in \mathbb{N}$.

Now we can be more precise and write

$$\operatorname{Im}(T'_F) = \ker \varphi \cap \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{m_n} \ker \frac{\varphi}{(z - z_n)^k}$$

for some non zero function $\varphi \in \mathscr{A}(\mathbb{R})$.

Notice that there are two posibilities: either

$$\ker \varphi \cap \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{m_n} \ker \frac{\varphi}{(z - z_n)^k} = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{m_n} \ker \frac{\varphi}{(z - z_n)^k} \tag{5.6}$$

or the equality does not hold. In the latter case there exists a non-zero function $f \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ such that

$$f \in \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{m_n} \ker \frac{\varphi}{(z - z_n)^k} \setminus \left(\ker \varphi \cap \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{m_n} \ker \frac{\varphi}{(z - z_n)^k} \right).$$

The function $f/\varphi(f)$ is the first element of the construction in this case. If (5.6) holds then we skip the first step. Then we have to construct functions $f \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ dual to the functionals of the form $\frac{\varphi}{(z-z_n)^k}$, $n \in \mathbb{N}$, $k \leq i_n$ and to φ.

Denote $\xi_{n,k} = \frac{\varphi}{(z-z_n)^k}$ for $n \in \mathbb{N}$ and $1 \le k \le m_n$. We will construct functions $f_{n,k}$ such that

$$\left\langle f_{n,k}, \frac{\varphi}{(z-z_i)^j} \right\rangle = 0, \quad \text{for } i \neq n \text{ and } 1 \le j \le m_i \tag{5.7}$$

$$\left\langle f_{n,k}, \frac{\varphi}{(z-z_n)^j} \right\rangle = 0, \quad \text{for } j \neq k \tag{5.8}$$

$$\left\langle f_{n,k}, \frac{\varphi}{(z-z_n)^k} \right\rangle = 1. \tag{5.9}$$

Recall that by Lemma 5.2.11(iii) we can find a function $a \in H_0(\mathbb{C}_\infty \setminus K)$ such that

$$\varphi = \frac{a}{F_0}. \tag{5.10}$$

Denote

$$A := \{n \in \mathbb{N} : a(z_n) = 0\}.$$

Notice that A is finite. This follows from (5.10) since $a(z) = 0$ implies $\varphi(z) = 0$ and φ is holomorphic in $V \supset K$ for some open neigbourhood of the real line V. For $n \in A$ let α_n be the multiplicity of z_n. Write

$$a(z) = \prod_{n \in A} (z - z_n)^{\alpha_n} b(z)$$

for some function $b \in H_0(\mathbb{C}_\infty \setminus K)$.

Consider the symbol function

$$F_n(z) := \frac{\prod_{i \neq n} E_{p_i}^{m_i}\left(\frac{z_i - c_n}{z - c_n}\right)}{\prod_{i \in A}(z - z_i)^{\alpha_i}} F_0(z).$$

Let $g \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ be arbitrary. For $l \neq n$ and $1 \le \kappa \le m_l$ it holds that

$$\begin{aligned}\left\langle T'_{F_n} g, \frac{\varphi}{(z-z_l)^\kappa} \right\rangle &= \int_\gamma F_n g(z) \frac{\varphi}{(z-z_l)^\kappa} dz \\ &= \int_\gamma \frac{\prod_{i \neq n} E_{p_i}^{m_i}\left(\frac{z_i - c_n}{z - c_n}\right)}{\prod_{i \in A}(z - z_i)^{\alpha_i}} F_0(z) \frac{1}{(z-z_l)^\kappa} g(z) \frac{\prod_{n \in A}(z-z_n)^{\alpha_n} b(z)}{F_0(z)} dz \\ &= \int_\gamma \frac{\prod_{i \neq n} E_{p_i}^{m_i}\left(\frac{z_i - c_n}{z - c_n}\right)}{(z-z_l)^\kappa} g(z) b(z) dz = 0.\end{aligned}$$

The integral equals zero because we have

$$\lim_{z \to \infty} \prod_{i \neq n} E_{p_i}^{m_i}\left(\frac{z_i - c_n}{z - c_n}\right) = 1$$

and the integrand is a product of functions from $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$.

A similar calculations shows that $\langle T'_{F_n} g, \varphi \rangle = 0$.

It looks like functions of the form $T'_{F_n}g$ are good candidates for the dual functions $(f_{n,k})$ satisfiying (5.7). We need to choose a function $g_{n,k} \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ such that

$$\left\langle T'_{F_n} g_{n,k}, \frac{\varphi}{(z-z_n)^j} \right\rangle = \delta_{kj}, \quad j = 1, \dots, m_n.$$

By the Cauchy integral formula we have

$$\begin{aligned}\left\langle T'_{F_n} g_{n,k}, \frac{\varphi}{(z-z_n)^j} \right\rangle &= \int_\gamma \frac{\prod_{i\neq n} E_{p_i}^{m_i}\left(\frac{z_i-c_n}{z-c_n}\right)}{(z-z_n)^j} g_{n,k}(z) b(z)\, dz \\ &= -\frac{2\pi i}{(j-1)!} \frac{d^{j-1}}{dz^{j-1}} \left(\prod_{i\neq n} E_{p_i}^{m_i}\left(\frac{z_i-c_n}{z-c_n}\right) g_{n,k}(z) b(z) \right)(z_n).\end{aligned}$$

Hence the function $g_{n,k}$ must be chosen in such way that

$$-\frac{2\pi i}{(j-1)!} \frac{d^{j-1}}{dz^{j-1}} \left(\prod_{i\neq n} E_{p_i}^{m_i}\left(\frac{z_i-c_n}{z-c_n}\right) g_{n,k}(z) b(z) \right)(z_n) = \delta_{kj}, \ j = 1, \dots, m_n. \tag{5.11}$$

The system (5.11) can be solved inductively for $g_{n,k}^{(j)}(z_n)$, $j = 0, 1, \dots, m_n - 1$. With given values of derivatives at points z_n, by Mittag-Leffler's theorem we can find the appropriate function $g_{n,k}$.

We are ready to prove Theorem 5.2.4.

Proof of Theorem 5.2.4. Assume that there exists a continuous projection $P: H_0(\mathbb{C}_\infty \setminus \mathbb{R}) \to H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ onto the range of the operator T'_F. Denote

$$X_N := \{ f \in H_0(\mathbb{C}_\infty \setminus \mathbb{R}) : \xi_{N+1}(f) = \xi_{N+2}(f) = \dots = 0 \},$$

where functionals ξ_n were constructed in Lemma 5.2.11 and satisfy $\operatorname{Im}(T'_F) = \bigcap_{n=1}^\infty \ker \xi_n$. Let

$$X = \bigcup_{N=1}^\infty X_N.$$

For any $f \in X$ the expression

$$f - \sum_{n=1}^\infty \xi_n(f) f_n,$$

where the functions f_n satisfy $\xi_m(f_n) = \delta_{mn}$ is well defined, since the sum is finite. For any m we have

$$\xi_m\left(f - \sum_{n=1}^\infty \xi_n(f) f_n \right) = \xi_m(f) - \xi_m(f) = 0.$$

Hence for any $f \in X$

$$f - \sum_{n=1}^\infty \xi_n(f) f_n \in \operatorname{Im}(T'_F)$$

and

$$P\left(f - \sum_{n=1}^\infty \xi_n(f) f_n \right) = f - \sum_{n=1}^\infty \xi_n(f) f_n.$$

In other words,

$$(I-P)f = \sum_{n=1}^{\infty} \xi_n(f)\psi_n,$$

where $\psi_n = (I-P)(f_n)$. Since $\xi_n(f_n) = 1$ it follows that $f_n \notin \operatorname{Im} T'_F$ and $\psi_n \neq 0$ for all n. Therefore, there exists $\widetilde{z} \in \mathbb{C}_\infty \smallsetminus \mathbb{R}$ such that $\psi_n(\widetilde{z}) \neq 0$ for every $n \in \mathbb{N}$.

Our aim is to construct a sequence of functions (g_n) convergent in $H_0(\mathbb{C}_\infty \smallsetminus \mathbb{R})$ such that $(I-P)g_n$ does not converge.

Step 1: There exists a function $f \in H(\mathbb{C}_\infty \smallsetminus K)$ such that

$$f(z_n) = f'(z_n) = \ldots = f^{m_n-1}(z_n) = 0 \quad \text{if } a(z_n) = 0$$

and

$$\frac{2\pi i}{j!}(af)^{(j)}(z_n) = -\frac{1}{\psi_M(\widetilde{z})}, \; j = 0, \ldots, m_n - 1 \quad \text{if } a(z_n) \neq 0 \tag{5.12}$$

where $\psi_M = (I-P)f_M$ and f_M is dual to $\xi_M = \frac{\varphi}{(z-z_n)^{j+1}}$. Indeed, if $a(z_n) \neq 0$, then the system (5.12) can be solved inductively. Let us recall that $a(z_n) = 0$ only for finitely many $n \in \mathbb{N}$.

Step 2: Let $(q_n)_{n\geq 0}$ be a sequence of natural numbers such that the product $\prod_{n=1}^{\infty} E_{q_n}^{m_n}\left(\frac{z_n-c_n}{z-c_n}\right)$ converges and consider the symbol function

$$F_N(z) = \prod_{n=N+1}^{\infty} E_{q_n}^{m_n}\left(\frac{z_n - c_n}{z - c_n}\right) F_0(z).$$

We claim that the sequences $(T'_{F_N} f)_{N\geq 0}$, which depend on the choice of $(q_n)_{n\geq 0}$, are convergent in $H_0(\mathbb{C}_\infty \smallsetminus \mathbb{R})$.

If $n > N$ and $1 \leq k \leq m_n$ then

$$\left\langle T'_{F_N} f, \frac{\varphi}{(z-z_n)^k} \right\rangle = \int_\gamma \frac{\prod_{n=N+1}^{\infty} E_{q_n}^{m_n}\left(\frac{z_n-c_n}{z-c_n}\right) F_0(z)}{(z-z_n)^k} f(z) \frac{a(z)}{F_0(z)} dz = 0, \tag{5.13}$$

since the integrand is the product of functions form $H_0(\mathbb{C}_\infty \smallsetminus \mathbb{R})$. This implies that $T'_{F_N} f \in X$ for any $N \in \mathbb{N}$.

The function

$$\prod_{n=N+1}^{\infty} E_{q_n}^{m_n}\left(\frac{z_n - c_n}{z - c_n}\right) f(z)$$

converges uniformly on compact sets of $\mathbb{C}_\infty \smallsetminus K$ to f. Hence it converges to f in $H_0(\mathbb{C}_\infty \smallsetminus \mathbb{R})$ and

$$T'_{F_N} f = T'_{F_0}\left(\prod_{n=N+1}^{\infty} E_{q_n}^{m_n}\left(\frac{z_n - c_n}{z - c_n}\right) f(z)\right) \to T'_{F_0} f.$$

Thus if P is continuous we should have

$$(I-P)T'_{F_N} f \to (I-P)T'_{F_0} f.$$

Step 3: We choose a sequence $(q_n)_{n\geq 0}$ such that $(I-P)(T'_{F_N}f)$ diverges.

For $1 \leq i \leq N, 1 \leq j \leq m_i$ we have

$$\begin{aligned}
\left\langle T_{F'_N} f, \frac{\varphi}{(z-z_i)^j} \right\rangle &= \int_\gamma \frac{\prod_{n=N+1}^\infty E_{q_n}^{m_n}\left(\frac{z_n-c}{z-c}\right) f(z)a(z)}{(z-z_i)^j} dz \\
&= -\frac{2\pi i}{(j-1)!}\frac{d^{j-1}}{dz^{j-1}}\left(\prod_{n=N+1} E_{q_n}^{m_n}\left(\frac{z_n-c_n}{z-c_n}\right) f(z)a(z)\right)(z_i) \\
&= -\frac{2\pi i}{(j-1)!}\sum_{k=0}^{j-1}\frac{(j-1)!}{k!(j-1-k)!}\left(\prod_{n=N+1} E_{q_n}^{m_n}\left(\frac{z_n-c_n}{z-c_n}\right)\right)^{(k)}(z_i)(fa)^{(j-1-k)}(z_i) \qquad (5.14)\\
&= -\frac{2\pi i}{(j-1)!}\left(\prod_{n=N+1} E_{q_n}^{m_n}\left(\frac{z_n-c_n}{z-c_n}\right)\right)(z_i)(fa)^{(j-1)}(z_i) \\
&\quad - \frac{2\pi i}{(j-1)!}\sum_{k=1}^{j-1}\frac{(j-1)!}{k!(j-1-k)!}\left(\prod_{n=N+1} E_{q_n}^{m_n}\left(\frac{z_n-c_n}{z-c_n}\right)\right)^{(k)}(z_i)(fa)^{(j-1-k)}(z_i)
\end{aligned}$$

Since for all $N \in \mathbb{N}$ the product $\prod_{n=N+1}^\infty E_{p_n}^{m_n}\left(\frac{z_n-c_n}{z-c_n}\right)$ converges to 1, its derivatives tend to 0 and we get

$$\lim_{N\to\infty}\left\langle T_{F'_N} f, \frac{\varphi}{(z-z_i)^j}\right\rangle = -\frac{2\pi i}{(j-1)!}(fa)^{(j-1)}(z_i) = \frac{1}{\psi_M(\widetilde{z})},$$

where, as before, $\psi_M = (I-P)f_M$ with f_M dual to $\xi_M = \frac{\psi}{(z-z_i)^j}$.

Without loss of generality we can assume that

$$\operatorname{dist}(z_1, K) > \operatorname{dist}(z_2, K) > \dots.$$

Otherwise, we group the points z_n into the subgroups of the same distance and then choose ε_N, $(p_n^N)_{n\geq 0}$ for these subgroups, i.e. if $\operatorname{dist}(z_i, K) = \operatorname{dist}(z_j, K)$ then we choose $\varepsilon_i = \varepsilon_j$ and $(p_n^i)_{n\geq 0} = (p_n^j)_{n\geq 0}$.

First, we choose the numbers δ_N to be such that

$$\operatorname{dist}(z_N, K) - \delta_N > \operatorname{dist}(z_{N+1}, K).$$

Next, we choose a sequence (ε_N), $\varepsilon_N < 1$ such that

$$\frac{2\pi}{(j-1)!}\sum_{k=1}^{j-1}\frac{(j-1)!}{k!(j-1-k)!}\frac{k!(\operatorname{dist}(z_N, K)-\delta_N)\varepsilon_N}{3\delta_N^{k+1}}\left|(fa)^{(j-1-k)}(z_i)\right||\psi_M(\widetilde{z})| \leq \frac{1}{3} \qquad (5.15)$$

for $j = 1, \dots N$. Finally, we choose a sequence (q_n) satisfying

$$\left|\prod_{n=N+1}^\infty E_{q_n}^{m_n}\left(\frac{z_n-c_n}{z-c_n}\right) - 1\right| \leq \frac{1}{3}\varepsilon_N, \qquad (5.16)$$

for every $N = 1, 2, \dots$ and z such that $\operatorname{dist}(z, K) > \operatorname{dist}(z, z_N) - \delta_N$. This is possible, because the product $\prod_{n=0}^\infty E_{q_n}^{m_n}\left(\frac{z_n-c_n}{z-c_n}\right)$ converges, so its tails converge to 1 and all its factors get closer to 1 with larger p_n.

We now show, that for such choice of $(q_n)_{n\geq 0}$, the sequence $(I-P)(T'_{F_N}f)$ diverges. Denote

$$S_N(z) := \prod_{n=N+1}^{\infty} E_{p_n^N}^{m_n}\left(\frac{z_n - c_n}{z - c_n}\right).$$

Recall that we assume that K is connected, hence

$$\gamma := \{\zeta \in \mathbb{C} : \operatorname{dist}(\zeta, K) = \operatorname{dist}(z_N, K) - \delta_N\}$$

is a piecewise smooth curve. By Cauchy's integral formula for $j > 0$ we have

$$S_N^{(j)}(z) = -\frac{j!}{2\pi i}\int_\gamma \frac{S_N(\zeta) - 1}{(\zeta - z)^{j+1}} d\zeta.$$

for $\operatorname{dist}(z, K) > \operatorname{dist}(z_N, K) - \delta_N$. For z such that $\operatorname{dist}(z, K) > \operatorname{dist}(z, z_N) - \delta_N$ we have

$$\left|S_N^j(z)\right| \leq j!(\operatorname{dist}(z_N, K) - \delta_N)\frac{\max_{\zeta\in\gamma}|S_N(\zeta) - 1|}{\delta_N^{j+1}} \leq j!(\operatorname{dist}(z_N, K) - \delta_N)\frac{\varepsilon_N}{3\delta_N^{j+1}}. \tag{5.17}$$

Let $\xi_M = \frac{\varphi}{(z - z_i)^j}$ with $i \leq N$ and $\psi_M = (I - P)f_M$, where f_M is dual to ξ_M. We use (5.14) and compute

$$\begin{aligned}
\left|\langle T'_{F_N}f, \xi_M\rangle\psi_M(\widetilde{z}) - 1\right| \leq & \left|\frac{2\pi i}{(j-1)!}\prod_{n=N+1}^{\infty} E_{q_n}^{m_n}\left(\frac{z_n - c_n}{z - c_n}\right)(af)^{(j-1)}(z_i)\psi_M(\widetilde{z}) - 1\right| \\
& + \frac{2\pi}{(j-1)!}\sum_{k=1}^{j-1}\frac{(j-1)!}{k!(j-1-k)!}\times \\
& \times \left|\left(\prod_{n=N+1} E_{q_n}^{m_n}\left(\frac{z_n - c_n}{z - c_n}\right)\right)^{(k)}(z_i)(fa)^{(j-1-k)}(z_i)\psi_M(\widetilde{z})\right| \\
\leq & \frac{2}{3}.
\end{aligned}$$

The first summand is $\leq \frac{1}{3}$ by (5.12) and (5.16). The second summand $\leq \frac{1}{3}$ by (5.15) and (5.17).

Recall that, by (5.13) for $i > N$ and $1 \leq k \leq m_i$ we have $\langle T'_{F_N}f, \frac{\varphi}{(z - z_i)^k}\rangle = 0$. Hence

$$\left|(I - P)(T'_{F_N}f)(\widetilde{z}) - \sum_{n=1}^{N} m_n\right| = \left|\sum_{n=0}^{\infty}\langle T'_{F_N}f, \xi_M\rangle\psi_M(\widetilde{z}) - \sum_{n=1}^{N} m_n\right| \leq \frac{2}{3}\sum_{n=1}^{N} m_n$$

and

$$\left|(I - P)(T'_{F_N}f)(\widetilde{z})\right| \geq \sum_{n=1}^{N} m_n - \left|(I - P)(T'_{F_N}f)(\widetilde{z}) - \sum_{n=1}^{N} m_n\right| \geq \frac{1}{3}\sum_{n=1}^{N} m_n \xrightarrow{N\to\infty} \infty.$$

We proved that $\operatorname{Im} T'_F$ is not complemented in $\mathscr{A}(\mathbb{R})'$. Hence the operator T'_F is not left-invertible. □

Finally we can state the theorem

Theorem 5.2.12. *A Toeplitz operator $T_F: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ with symbol $F \in \mathcal{X}$ is right-invertible if and only if it is a surjective Fredholm operator. Moreover, the right-inverse is of the form $T_G T_H$, where $G \in \mathscr{A}(\mathbb{R})$.*

Proof. We proved that if T_F is not a Fredholm operator then it is not right-invertible. Let T_F be a surjective Fredholm operator. Then index $T_F = k$ with $k \geq 0$. By Theorem 5.1.9 the function F admits the following factorization

$$F = F_+ z^{-k} F_-,$$

where $F_+, F_+^{-1} \in H(U)$ and $F_-, F_-^{-1} \in H(C_\infty \setminus K)$ for some open complex neigbourhood U of $\mathbb{R}$, and a compact set $K \subset \mathbb{R}$ containing 0. We claim that the right-inverse of T_F is given by

$$T_{F_+^{-1}} T_{z^k F_-^{-1}}.$$

Indeed, using Lemma 5.1.7 we get

$$\begin{aligned} T_{F_+ z^{-k} F_-} \left(T_{F_+^{-1}} T_{z^k F_-^{-1}} \right) (g) &= T_{z^{-k} F_-^{-1}} T_{F_+} T_{F_+^{-1}} T_{z^k F_-^{-1}} g \\ &= T_{z^{-k} F_-^{-1}} T_{F_+ F_+^{-1}} T_{z^k F_-^{-1}} g \\ &= T_{z^{-k} F_-^{-1}} T_{z^k F_-^{-1}} g \\ &= T_{z^{-k} F_-^{-1} z^k F_-^{-1}} g \\ &= g. \end{aligned}$$

□

5.3 Commutators of Topelitz operators

A *commutator* of two operators A and B in $L_b(\mathscr{A}(\mathbb{R}))$ is given by

$$[A, B] := AB - BA.$$

In this section we characterize when the *commutator* $[T, S] = TS - ST$ of two Toeplitz operators has finite rank. The proof is similar to the proof of Ding and Zheng [9] for a corresponding result on $H^2(\mathbb{T})$.

Recall that the symbol space $\mathcal{X}$ can be decomposed as $\mathcal{X} = \mathscr{A}(\mathbb{R}) \oplus H_0(\mathbb{C}_\infty \setminus \mathbb{R})$)and $\mathcal{C}, Q$ denote the complementary projections onto $\mathscr{A}(\mathbb{R})$ and $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$, respectively. Up to this point we have only considered Toeplitz and Hankel operators acting on the space $\mathscr{A}(\mathbb{R})$. We will now need Toeplitz operators acting on $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$. We also need another definition of Hankel operators. For $f \in \mathcal{X}$ we define Toeplitz and Hankel operators with symbol f in the following way

$$\begin{aligned} &T_f: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R}), && T_f g = \mathcal{C}(fg) \\ &\widetilde{T}_f: H_0(\mathbb{C}_\infty \setminus \mathbb{R}) \to H_0(\mathbb{C}_\infty \setminus \mathbb{R}), && \widetilde{T}_f h = Q(fh) \\ &H_f: \mathscr{A}(\mathbb{R}) \to H_0(\mathbb{C}_\infty \setminus \mathbb{R}), && H_f(g) = Q(fg) \\ &\widetilde{H}_f: H_0(\mathbb{C}_\infty \setminus \mathbb{R}) \to \mathscr{A}(\mathbb{R}), && \widetilde{H}_f(h) = \mathcal{C}(fh) \end{aligned}$$

for $g \in \mathscr{A}(\mathbb{R})$ and $h \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$. The adjoint operators are the following

$$T'_f = \widetilde{T}_f, \quad \widetilde{T}'_f = T_f, \quad H'_f = H_f, \quad \widetilde{H}'_f = \widetilde{H}_f.$$

We have the following realtions between Toeplitz and Hankel operators

$$T_{fg} = T_f T_g + \widetilde{H}_f H_g, \tag{5.18}$$

$$\widetilde{T}_{fg} = \widetilde{T}_f \widetilde{T}_g + H_f \widetilde{H}_g \tag{5.19}$$

$$H_{fg} = H_f T_g + \widetilde{T}_f H_g \tag{5.20}$$

$$\widetilde{H}_{fg} = T_f \widetilde{H}_g + \widetilde{H}_f \widetilde{T}_g. \tag{5.21}$$

Indeed,

$$\begin{aligned}
T_{fg}(h) &= \mathcal{C}(fgh) = \mathcal{C}(f(\mathcal{C}+Q)(gh)) = \mathcal{C}(f\mathcal{C}(gh)) + \mathcal{C}(fQ(gh)) = T_f T_g(h) + \widetilde{H}_f H_g(h),\\
\widetilde{T}_{fg}(h) &= Q(fgh) = Q(f(\mathcal{C}+Q)(gh)) = Q(f\mathcal{C}(gh)) + Q(fQ(gh)) = H_f\widetilde{H}_g(h) + \widetilde{T}_f\widetilde{T}_g(h),\\
H_{fg}(h) &= Q(fgh) = Q(f(\mathcal{C}+Q)(gh)) = Q(f\mathcal{C}(gh)) + Q(fQ(gh)) = H_f T_g(h) + \widetilde{T}_f H_g(h),\\
\widetilde{H}_{fg}(h) &= \mathcal{C}(fgh) = \mathcal{C}(f(\mathcal{C}+Q)(gh)) = \mathcal{C}(f\mathcal{C}(gh)) + \mathcal{C}(fQ(gh)) = T_f\widetilde{H}_g(h) + \widetilde{H}_f\widetilde{T}_g(h).
\end{aligned}$$

Hence, for $f \in \mathscr{A}(\mathbb{R})$ we have $H_f = 0$ and

$$H_{fg} = \widetilde{T}_f H_g = H_g T_f. \tag{5.22}$$

For $f \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ we have $\widetilde{H}_f = 0$ and

$$\widetilde{H}_{fg} = T_f \widetilde{H}_g = \widetilde{H}_g \widetilde{T}_f. \tag{5.23}$$

Moreover, if $f \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ or $g \in \mathscr{A}(\mathbb{R})$ then we also have

$$T_{fg} = T_f T_g \tag{5.24}$$

and

$$\widetilde{T}_{fg} = \widetilde{T}_g \widetilde{T}_f \tag{5.25}$$

We investigate when the commutator of the Toeplitz operators, $[T_f, T_g] = T_f T_g - T_g T_f$ is of finite rank, i.e., its image is finite dimensional. Notice that from (5.18)

$$T_f T_g - T_g T_f = \widetilde{H}_f H_g - \widetilde{H}_g H_f$$

and so the rank of the commutator $[T_f, T_g]$ is related to the rank of the corresponding Hankel operators and their product.

We use Pol_+ to denote the set of all analytic polynomials (i.e., polynomials in z) and Pol_- to denote the set of polynomials in $\frac{1}{z}$. By the Kronecker theorem we get

Theorem 5.3.1 (Kronecker). *Let $f \in \mathcal{X}$. The Hankel operator H_f has finite rank if and only if there exists a polynomial $p \in \mathrm{Pol}_+$ such that $pf \in \mathscr{A}(\mathbb{R})$.*

Proof. The proof of implication $(\Rightarrow)$ is similar to the one given in Theorem 4.1.8, hence we present here just a sketch of it. Let $f_-(z) = Qf(z) = \sum_{n=1}^{\infty} a_n z^{-n}$ around infinity. Assume that $\operatorname{rank}(H_f) = N$. Then $\sum_{j=0}^{N} c_j H_f(z^j) = 0$ for some $c_1, \dots, c_N \in \mathbb{C}$. Let $p = \sum_{j=0}^{N} c_j z^n$, $q(z) = \sum_{j=0}^{N} \sum_{k=1}^{j} c_j a_k z^{j-k}$. Then $pf_- = q$.

For the other direction, $(\Leftarrow)$, let $f_- = Qf$, $f_-(z) = \sum_{n=1}^{\infty} a_n z^{-n}$ around infinity. Assume that $pf \in \mathscr{A}(\mathbb{R})$. Since $pf_- \in H(\mathbb{C} \setminus K)$, for some compact $K \subset \mathbb{R}$ and $pf_- \in \mathscr{A}(\mathbb{R})$, it follows that $pf_- \in H(\mathbb{C})$ and by Liouville theorem pf_- is a polynomial. Let

$$f_- = \frac{q}{p}, \qquad \deg(q) < \deg(p).$$

Assume that

$$p(z) = \sum_{j=0}^{N} c_j z^j.$$

Then

$$p(z) f_-(z) = \sum_{j=0}^{N} c_j z^j \sum_{k=1}^{\infty} a_k z^{-k} = \sum_{j=0}^{N} \sum_{k=1}^{\infty} c_j a_k z^{j-k} = q(z).$$

We have

$$\sum_{j=0}^{N} \sum_{k=1}^{\infty} c_j a_k z^{j-k} = \sum_{j=0}^{N} \sum_{l=-j+1}^{\infty} c_j a_{j+l} z^{-l} = \sum_{j=0}^{N} \sum_{l=-j+1}^{0} c_j a_{j+l} z^{-l} + \sum_{j=0}^{N} \sum_{l=1}^{\infty} c_j a_{j+l} z^{-l}.$$

Since $q \in \mathrm{Pol}_+$ we get

$$\sum_{j=0}^{N} \sum_{l=1}^{\infty} c_j a_{j+l} z^{-l} = \sum_{l=1}^{\infty} \left(\sum_{j=0}^{N} c_j a_{j+l} \right) z^{-l} = 0,$$

hence

$$\sum_{j=0}^{N} c_j a_{j+l} = 0 \quad \forall l \in \mathbb{N}.$$

For every $u \in \mathbb{N}$ we have

$$\sum_{j=0}^{N} c_J H_f(z^{j+u}) = \sum_{j=0}^{N} c_j \sum_{l=1}^{\infty} a_{j+u+l} z^{-l} = \sum_{l=1}^{\infty} \left(\sum_{j=0}^{N} c_j a_{j+l+u} \right) z^{-l} = 0.$$

Hence, for every N-tuple $z^k, \dots, z^{k+N}$ the vectors $H_f(z^k), \dots, H_f(z^{k+N})$ are linearly dependent and $\dim(H_f(\mathscr{A}(\mathbb{R}))) \le N$.

□

Analogously, we have

Theorem 5.3.2 (Kronecker). *Let $f \in \mathcal{X}$. The Hankel operator $\widetilde{H}_f$ has finite rank if and only if there exists a polynomial $\overline{q} \in \mathrm{Pol}_-$ such that $\overline{q} f \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$.*

Lemma 5.3.3. *Let $p, q \in \mathrm{Pol}_+$, $\overline{p}, \overline{q} \in \mathrm{Pol}_-$.*

(a) *Let* $A: \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ *be a continuous operator. If the operator* $T_{\overline{q}} A T_p$ *has finite rank, then* A *also has finite rank.*

(b) *Let* $A: H_0(\mathbb{C}_\infty \setminus \mathbb{R}) \to H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ *be a continuous operator. If the operator* $\widetilde{T}_p A \widetilde{T}_{\overline{q}}$ *has finite rank, then* A *also has finite rank.*

(c) *Let* $A: \mathscr{A}(\mathbb{R}) \to H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ *be a continuous operator. If the operator* $\widetilde{T}_q A T_p$ *has finite rank, then* A *also has finite rank.*

(d) *Let* $A: H_0(\mathbb{C}_\infty \setminus \mathbb{R}) \to \mathscr{A}(\mathbb{R})$ *be a continuous operator. If the operator* $T_{\overline{q}} A \widetilde{T}_{\overline{p}}$ *has finite rank, then* A *also has finite rank.*

Proof. First notice, that each polynomial $p \in \mathrm{Pol}_+$ can be decomposed into $p = \overline{p_-} z^k p_+$, with $\overline{p_-}, \overline{p_-}^{-1} \in H(\mathbb{C}_\infty \setminus \mathbb{R})$, $p_+, p_+^{-1} \in \mathscr{A}(\mathbb{R})$, $k = \deg p$. Indeed, let

$$p(z) = c \prod_{j=1}^{k} (z - a_j) = \prod_{a_j \in \mathbb{R}} \left(1 - \frac{a_j}{z}\right) z^k c \prod_{a_j \notin \mathbb{R}} \left(1 - \frac{a_j}{z}\right) = \overline{p_-} z^k p_+.$$

Similarly we can decompose $\overline{q}$ into $\overline{q} = q_+ z^{-l} \overline{q}_-$, with $\overline{q}_-, \overline{q}_-^{-1} \in H(\mathbb{C}_\infty \setminus \mathbb{R})$, $q_+, q_+^{-1} \in \mathscr{A}(\mathbb{R})$, $l \in \mathbb{N}$.

We prove (a). The other cases are similar. By (5.24) we have

$$T_p = T_{\overline{p}_- z^k p_+} = T_{\overline{p}_-} T_{z^k p_+} = T_{\overline{p}_-} T_{p_+} T_{z^k}.$$

Let $M = T_{\overline{q}} A T_p(\mathscr{A}(\mathbb{R}))$, $\dim M < \infty$. We claim that the operator $T_{\overline{q}} A T_{\overline{p}_-} T_{p_+}$ also has finite rank. Indeed,

$$\begin{aligned} N = T_{\overline{q}} A T_{\overline{p}_-} T_{p_+}(\mathscr{A}(\mathbb{R})) &= T_{\overline{q}} A T_{\overline{p}_-} T_{p_+}(T_{z^k}(\mathscr{A}(\mathbb{R})) \oplus X_k) \\ &= T_{\overline{q}} A T_p(\mathscr{A}(\mathbb{R})) \oplus T_{\overline{q}} A T_{\overline{p}_-} T_{p_+}(X_k), \end{aligned}$$

where $X_k = \{f \in \mathscr{A}(\mathbb{R}) : f^{(j)} = 0 \text{ for } j \geq k\}$. Hence $\dim N \leq \dim M + k < \infty$.

Notice that, by (5.24)

$$T_{\overline{q}} A T_{\overline{p}_-} = T_{\overline{q}} A T_{\overline{p}_-} T_{p_+ p_+^{-1}} = (T_{\overline{q}} A T_{\overline{p}_-} T_{p_+}) T_{p_+^{-1}}.$$

Hence $\operatorname{rank}(T_{\overline{q}} A T_{\overline{p}_-}) < \infty$. We repeat the last step

$$T_{\overline{q}} A = T_{\overline{q}} A T_{\overline{p}_- \overline{p}_-^{-1}} = T_{\overline{q}} A T_{\overline{p}_-} T_{\overline{p}_-^{-1}}$$

and get that $\operatorname{rank}(T_{\overline{q}} A) < \infty$.

We have $(T_{\overline{q}} A)' = A' \widetilde{T}_{\overline{q}}$. By the same argument we show that rank A' is finite. As before, by (5.25) we have

$$\widetilde{T}_{\overline{q}} = \widetilde{T}_{q_+ z^{-l} \overline{q}_-} = \widetilde{T}_{q_+} \widetilde{T}_{\overline{q_-}} \widetilde{T}_{z^{-l}}.$$

Let $\widetilde{M} = A' \widetilde{T}_{\overline{q}}(H_0(\mathbb{C}_\infty \setminus \mathbb{R}))$. We have

$$\begin{aligned} \widetilde{N} &= A' \widetilde{T}_{q_+} \widetilde{T}_{\overline{q}_-}(H_0(\mathbb{C}_\infty \setminus \mathbb{R})) = A' \widetilde{T}_{q_+} \widetilde{T}_{\overline{q}_-}(\widetilde{T}_{z^{-l}}(H_0(\mathbb{C}_\infty \setminus \mathbb{R})) \oplus Y_l) \\ &= A' \widetilde{T}_{\overline{q}}(H_0(\mathbb{C}_\infty \setminus \mathbb{R})) \oplus A' \widetilde{T}_{q_+} \widetilde{T}_{\overline{q}_-}(Y_l), \end{aligned}$$

where $Y_l = \{f \in H_0(\mathbb{C}_\infty \setminus \mathbb{R}), f(z) = \sum_{n=1}^{\infty} f_n z^{-n} : f_n = 0 \text{ for } n > l\}$. Hence $\dim \widetilde{N} \leq \dim \widetilde{M} + l < \infty$.

Finally, we have

$$A'\widetilde{T}_{q_+} = A'\widetilde{T}_{q_+}\widetilde{T}_{\overline{q}_-\overline{q}_-^{-1}} = A'\widetilde{T}_{q_+}\widetilde{T}_{\overline{q}_-}\widetilde{T}_{\overline{q}_-^{-1}}$$
$$A' = A'\widetilde{T}_{q_+q_+^{-1}} = A'\widetilde{T}_{q_+}\widetilde{T}_{q_+^{-1}}.$$

and so $\operatorname{rank}(A) = \operatorname{rank}(A') \leq \infty$. □

We will write

$$A = B \pmod{\mathcal{F}}$$

to denote that the operator $A - B$ has finite rank.

Recal that the dual space $\mathscr{A}(\mathbb{R})'$ is isomorphic to $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$. Hence, for $x \in \mathscr{A}(\mathbb{R})$, $y \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ we can define the rank one operator $(x \otimes y)$ by

$$(x \otimes y)(h) = \langle h, y \rangle x, \quad h \in \mathscr{A}(\mathbb{R})$$

Lemma 5.3.4. *Let $f_i, g_i \in \mathcal{X}$, $i = 1, \dots, n$. If $\operatorname{rank}(\sum_{i=1}^n \widetilde{H}_{f_i} H_{g_i}) = K$ then there exist polynomials $\overline{A}_i \in \mathrm{Pol}_-$ and $B_i \in \mathrm{Pol}_+$, such that*

$$\sum_{i=1}^{n} \overline{A}_i f_i \in H(\mathbb{C}_\infty \setminus \mathbb{R}) \quad \text{or} \quad \sum_{i=1}^{n} B_i g_i \in \mathscr{A}(\mathbb{R}).$$

Moreover,

$$\max_{1 \leq i \leq n} \left(\deg \overline{A}_i, \deg B_i\right) = K.$$

Proof. Let $n \in \mathbb{N}$ and $\operatorname{rank}(\sum_{i=1}^n \widetilde{H}_{f_i} H_{g_i}) = K$. We prove the result by induction on the rank K.

Let $K = 0$. Then

$$\sum_{i=1}^{n} \widetilde{H}_{f_i} H_{g_i} = 0.$$

If for some i one of the operators $\widetilde{H}_{f_i}, H_{g_i}$ is zero, then the result is clear – it is enough to take polynomials $\overline{A}_j = B_j = 0$ for $j \neq i$ and non zero constants $\overline{A}_i$ or B_i. Hence we assume that for all i, $\widetilde{H}_{f_i} \neq 0$ and $H_{g_i} \neq 0$.

Note that for $h \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$, $h = \sum_{n \geq 1}^{\infty} h_n z^{-n}$ around infinity, we have

$$(1 - \widetilde{T}_{z^{-1}}\widetilde{T}_z)(h) = h - Q(z^{-1}Q(zh)) = h - z^{-1}Q(zh) = h_1 z^{-1} = \langle h, 1 \rangle z^{-1}.$$

Hence for $h \in \mathscr{A}(\mathbb{R})$ and $i = 1, \dots n$ we have

$$\widetilde{H}_{f_i}(1 - \widetilde{T}_{z^{-1}}\widetilde{T}_z)H_{g_i}(h) = \widetilde{H}_{f_i}\left(\left\langle H_{g_i}(h), 1\right\rangle z^{-1}\right) = \left\langle h, H_{g_i}^*(1)\right\rangle \widetilde{H}_{f_i}(z^{-1})$$
$$= \left(\widetilde{H}_{f_i}(z^{-1}) \otimes H_{g_i}^*(1)\right)(h) = \left(\widetilde{H}_{f_i}(z^{-1}) \otimes H_{g_i}(1)\right)(h).$$

By (5.22) and (5.23) we get the identity

$$\begin{aligned}\sum_{i=1}^{n} \widetilde{H}_{f_i}(z^{-1}) \otimes H_{g_i}(1) &= \sum_{i=1}^{n} \widetilde{H}_{f_i}\left(1 - \widetilde{T}_{z^{-1}}\widetilde{T}_z\right) H_{g_i} \\ &= \sum_{i=1}^{n} \widetilde{H}_{f_i} H_{g_i} - T_{z^{-1}}\left(\sum_{i=1}^{n} \widetilde{H}_{f_i} H_{g_i}\right) T_z\end{aligned} \tag{5.26}$$

Taking adjoints we get

$$\begin{aligned}\sum_{i=1}^{n} H_{g_i}(1) \otimes \widetilde{H}_{f_i}(z^{-1}) &= \sum_{i=1}^{n} H_{g_i}\left(1 - T_z T_{z^{-1}}\right) \widetilde{H}_{f_i} \\ &= \sum_{i=1}^{n} H_{g_i} \widetilde{H}_{f_i} - \widetilde{T}_z\left(\sum_{i=1}^{n} H_{g_i}\widetilde{H}_{f_i}\right) \widetilde{T}_{z^{-1}}\end{aligned} \tag{5.27}$$

For $\lambda \in \mathbb{R}$ we put $h_\lambda(z) = \frac{1}{2\pi i}\frac{1}{z-\lambda}$. We have $h_\lambda \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ and $\langle f, h_\lambda \rangle = f(\lambda)$ for all $f \in \mathscr{A}(\mathbb{R})$. Since $\widetilde{H}_{f_i} \neq 0$ for all i, there exists λ such that $\widetilde{H}_{f_i}(z^{-1})(\lambda) \neq 0$ for all i. Since $\operatorname{rank}(\sum_{i=1}^{n} H_{g_i}\widetilde{H}_{f_i}) = 0$ we get

$$\sum_{i=1}^{n} H_{g_i}(1) \otimes \widetilde{H}_{f_i}(z^{-1})(h_\lambda) = \sum_{i=1}^{n} \left\langle h_\lambda, \widetilde{H}_{f_i}(z^{-1})\right\rangle H_{g_i}(1) = \sum_{i=1}^{n} \widetilde{H}_{f_i}(z^{-1})(\lambda) H_{g_i}(1) = 0.$$

Put $B_i := \widetilde{H}_{f_i}(z^{-1})(\lambda)$. Then $B_i \neq 0$ for all i and

$$0 = \sum_{i=1}^{n} B_i H_{g_i}(1) = H_{\sum_{i=1}^{n} B_i g_i}(1) = Q\left(\sum_{i=1}^{n} B_i g_i\right).$$

Hence $\sum_{i=1}^{n} B_i g_i \in \mathscr{A}(\mathbb{R})$. This concludes the case $K = 0$.

Assume that the result is true for $K < k$ and consider $K = k$. Let

$$\sum_{i=1}^{n} \widetilde{H}_{f_i} H_{g_i} = \sum_{j=1}^{k} x_j \otimes y_j,$$

for some $x_j \in \mathscr{A}(\mathbb{R})$, $y_j \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$, with

$$\dim \operatorname{span}\{x_1, \ldots, x_k\} = \dim \operatorname{span}\{y_1, \ldots, y_k\} = k.$$

We will consider three cases:

(1) the vectors $\widetilde{T}_z y_1, \ldots, \widetilde{T}_z y_k$ are linearly dependent,

(2) the vectors $\widetilde{T}_{z^{-1}} x_1, \ldots, \widetilde{T}_{z^{-1}} x_k$ are linearly dependent,

(3) both sets of vectors $\{\widetilde{T}_z y_1, \ldots, \widetilde{T}_z y_k\}$ are $\{\widetilde{T}_{z^{-1}} x_1, \ldots, T_{z^{-1}} x_k\}$ are linearly independent.

Case (1): Let

$$\widetilde{T}_z y_k = \sum_{j=1}^{k-1} c_j \widetilde{T}_z y_j.$$

For further reference let us observe that for $x \in \mathscr{A}(\mathbb{R})$ and $y \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ we have

$$\begin{aligned} T_{z^{-1}}(x \otimes y) T_z(h) &= T_{z^{-1}}(\langle T_z h, y\rangle x) = \langle h, T_z' y\rangle T_{z^{-1}} x \\ &= \langle h, \widetilde{T}_z y\rangle T_{z^{-1}} x = \left(T_{z^{-1}} x \otimes \widetilde{T}_z y\right)(h). \end{aligned}$$

Now, by (5.22) and (5.23) we get

$$\begin{aligned} \sum_{i=1}^{n} \widetilde{H}_{z^{-1} f_i} H_{z g_i} &= T_{z^{-1}} \left(\sum_{i=1}^{N} \widetilde{H}_{f_i} H_{g_i} \right) T_z = T_{z^{-1}} \left(\sum_{j=1}^{k} x_j \otimes y_j \right) T_z = \sum_{j=1}^{k} T_{z^{-1}} x_j \otimes \widetilde{T}_z y_j \\ &= \sum_{j=1}^{k-1} T_{z^{-1}} x_j \otimes \widetilde{T}_z y_j + T_{z^{-1}} x_k \otimes \widetilde{T}_z y_k \\ &= \sum_{j=1}^{k-1} T_{z^{-1}} x_j \otimes \widetilde{T}_z y_j + T_{z^{-1}} x_k \otimes \sum_{j=1}^{k-1} c_j \widetilde{T}_z y_j \\ &= \sum_{j=1}^{k-1} T_{z^{-1}} x_j \otimes \widetilde{T}_z y_j + \sum_{j=1}^{k-1} c_j T_{z^{-1}} x_k \otimes \widetilde{T}_z y_j \\ &= \sum_{j=1}^{k-1} T_{z^{-1}}(x_j + c_j x_k) \otimes \widetilde{T}_z y_j. \end{aligned}$$

We showed that $\operatorname{rank}(\sum_{i=1}^n \widetilde{H}_{z^{-1} f_i} H_{z g_i}) \leq k-1$. By the induction hypothesis there exist polynomials $\overline{a_i} \in \mathrm{Pol}_-$, $b_i \in \mathrm{Pol}_+$, such that $\max_i(\deg(a_i), \deg(b_i)) = k-1$ and

$$\sum_{i=1}^{n} \overline{a_i} z^{-1} f_i \in H(\mathbb{C}_\infty \setminus \mathbb{R}) \quad \text{or} \quad \sum_{i=1}^{n} z b_i g_i \in \mathscr{A}(\mathbb{R}).$$

The polynomials

$$\begin{aligned} \overline{A_i}(z) &:= z^{-1} \overline{a_i}(z) \\ B_i(z) &:= z b_i(z), \end{aligned}$$

have the required properties.

Case (2): The proof of is similar.

Case (3): We assume that $\widetilde{T}_z y_1, \ldots, \widetilde{T}_z y_k$ are linearly independent and $\widetilde{T}_{z^{-1}} x_1, \ldots, \widetilde{T}_{z^{-1}} x_k$ are also linearly independent. From (5.26) we have

$$\sum_{i=1}^{n} \widetilde{H}_{f_i}(z^{-1}) \otimes H_{g_i}(1) = \sum_{j=1}^{k} x_j \otimes y_j - \sum_{j=1}^{k} T_{z^{-1}} x_j \otimes \widetilde{T}_z y_j. \tag{5.28}$$

So, for any $w \in \mathscr{A}(\mathbb{R})$ we have

$$\sum_{i=1}^{n}\langle w, H_{g_i}(1)\rangle \widetilde{H}_{f_i}(z^{-1}) = \sum_{j=1}^{k}\langle w, y_j\rangle x_j - \sum_{j=1}^{k}\langle w, \widetilde{T}_z y_j\rangle T_{z^{-1}} x_j. \tag{5.29}$$

Since $\widetilde{T}_z y_1, \dots \widetilde{T}_z y_k$ are linearly independent we can find $w_1, \dots w_k \in \mathscr{A}(\mathbb{R})$ such that

$$\det\left(\langle w_l, \widetilde{T}_z y_j\rangle\right)_{lj} \neq 0.$$

Such choice is possible by the Hahn-Banach theorem. We treat the space $\mathscr{A}(\mathbb{R})$ and vectors $w_1, \dots, w_k$ as functionals on $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$. Hence for every $l = 1, \dots, k$ we find the vector w_l such that $\langle w_l, \widetilde{T}_z y_j\rangle = 1$ if $j = l$ and $\langle w_l, \widetilde{T}_z y_j\rangle = 0$ if $j \neq l$.

We define scalar matrices $A \in M_{k,k}$, $C \in M_{k,n}$ by

$$\begin{aligned} A &= (a_{lj}), \quad a_{lj} = \langle w_l, \widetilde{T}_z y_j\rangle, \\ C &= (c_{lj}), \quad c_{lj} = \langle w_l, H_{g_j}(1)\rangle \end{aligned}$$

and a $k \times k$ matrix B of polynomials in $\frac{1}{z}$ by

$$B = (b_{lj}), \quad b_{lj} = \langle w_l, y_j\rangle - a_{lj} z^{-1}.$$

Finally we denote

$$\begin{aligned} X &= (x_1, \dots, x_k)^T, \\ \underline{\widetilde{H}}_f &= \left(\widetilde{H}_{f_1}(z^{-1}), \dots, \widetilde{H}_{f_n}(z^{-1})\right)^T. \end{aligned}$$

Because

$$T_{z^{-1}} x_j = \frac{x_j - x_j(0)}{z} = z^{-1} x_j - z^{-1} x_j(0),$$

using (5.29) we have

$$C\underline{\widetilde{H}}_f = BX + AX(0) z^{-1}.$$

Indeed, we compute the l-th element

$$\begin{aligned} \left(C\underline{\widetilde{H}}_f\right)_l &= \sum_{j=1}^{n} c_{lj}\widetilde{H}_{f_j}(z^{-1}) = \sum_{j=1}^{n}\langle w_l, H_{g_j}(1)\rangle \widetilde{H}_{f_j}(z^{-1}) \\ &= \sum_{j=1}^{k}\langle w_l, y_j\rangle x_j - \sum_{j=1}^{k}\langle w_l, \widetilde{T}_z y_j\rangle T_{z^{-1}} x_j \\ &= \sum_{j=1}^{k}\langle w_l, y_j\rangle x_j - \sum_{j=1}^{k}\langle w_l, \widetilde{T}_z y_j\rangle z^{-1} x_j + \sum_{j=1}^{k}\langle w_l, \widetilde{T}_z y_j\rangle z^{-1} x_j(0) \\ &= (BX)_l + (AX(0))_l z^{-1}. \end{aligned}$$

Denote $\overline{B} = \det B$. From the definition of the matrix B it follows that

$$\overline{B}(z) = (-1)^k a z^{-k} + a_{k-1} z^{-k+1} + \dots + a_0,$$

where $a = \det A$ is non-zero by the choice of w_l, $l = 1, \dots, k$, and $a_0, \dots, a_{k-1}$ are constants. Hence $\deg \overline{B} = k$.

We denote by $\operatorname{adj}(B)$ the adjugate of the matrix B and have

$$\operatorname{adj}(B) C \underline{\widetilde{H}}_f = \overline{B} X + \operatorname{adj}(B) A X(0) z^{-1}.$$

Let $\overline{C} = \left(\overline{C}_{lj}\right)_{l \le k, j \le n} = \operatorname{adj}(B) C$. Each $\overline{C}_{lj}$ is a polynomial in $\frac{1}{z}$ of degree at most $k-1$. We apply the projection $\mathcal{C} \colon \mathcal{X} \to \mathscr{A}(\mathbb{R})$ to both sides and get

$$\mathcal{C}\left(\overline{C} \underline{\widetilde{H}}_f\right) = \mathcal{C}\left(\overline{B} X\right). \tag{5.30}$$

Notice, that for arbitrary $f \in \mathcal{X}$, $g, h \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ we have

$$\mathcal{C}(h \widetilde{H}_f g) = \mathcal{C}\left(h \mathcal{C}(fg)\right) = \mathcal{C}\left(h(Q + \mathcal{C})(fg) - hQ(fg)\right) = \mathcal{C}(hfg) = H_{hf} g.$$

Hence for each $i = 1, \dots k$ we have

$$\mathcal{C}\left(\left[\overline{C}_{i1}, \dots, \overline{C}_{in}\right] \underline{\widetilde{H}}_f\right) = \mathcal{C}\left(\sum_{j=1}^{n} \overline{C}_{ij} \widetilde{H}_{f_j}\left(z^{-1}\right)\right) = \mathcal{C}\left(\widetilde{H}_{\sum_{j=1}^n \overline{C}_{ij} f_j}\left(z^{-1}\right)\right) = \widetilde{H}_{\sum_{j=1}^n \overline{C}_{ij} f_j}\left(z^{-1}\right)$$

and by (5.30)

$$\begin{bmatrix} \widetilde{H}_{\sum_{j=1}^n \overline{C}_{1j} f_j}(z^{-1}) \\ \vdots \\ \widetilde{H}_{\sum_{j=1}^n \overline{C}_{kj} f_j}(z^{-1}) \end{bmatrix} = \begin{bmatrix} T_{\overline{B}} x_1 \\ \vdots \\ T_{\overline{B}} x_k \end{bmatrix}. \tag{5.31}$$

Analogously, using adjoints, from (5.27) we get

$$\sum_{i=1}^{n} H_{g_i}(1) \otimes \widetilde{H}_{f_i}(z^{-1}) = \sum_{j=1}^{k} y_j \otimes x_j - \sum_{j=1}^{k} \widetilde{T}_z y_j \otimes T_{z^{-1}} x_j. \tag{5.32}$$

and repeating the last procedure, we get a polynomial $B(z) \in \mathrm{Pol}_+$, $\deg B = k$ and polynomials $U_{l,j} \in \mathrm{Pol}_+$, $l = 1, \dots, n$, $j = 1, \dots k$, of the degree at most $k-1$ such that

$$\begin{bmatrix} H_{\sum_{j=1}^n U_{1,j} g_j}(1) \\ \vdots \\ H_{\sum_{j=1}^n U_{k,j} g_j}(1) \end{bmatrix} = \begin{bmatrix} \widetilde{T}_B x_1 \\ \vdots \\ \widetilde{T}_B x_k \end{bmatrix}. \tag{5.33}$$

We consider now three cases:

(i) the vectors $\widetilde{H}_{f_1}(z^{-1}), \dots, \widetilde{H}_{f_n}(z^{-1}), x_1, \dots, x_k$ are linearly dependent,

(ii) the vectors $H_{g_1}(1), \dots, H_{g_n}(1), y_1, \dots, y_k$ are linearly dependent,

(iii) sets of vectors $\{\widetilde{H}_{f_1}(z^{-1}), \dots, \widetilde{H}_{f_n}(z^{-1}), x_1, \dots, x_k\}$ and $\{H_{g_1}(1), \dots, H_{g_n}(1), y_1, \dots, y_k\}$ are linearly independent.

Case (i): In the first case we assume that $\widetilde{H}_{f_1}(z^{-1}), \dots, \widetilde{H}_{f_n}(z^{-1}), x_1, \dots, x_k$ are linearly dependent. Put

$$\widetilde{H}_{f_n}(z^{-1}) = \sum_{i=1}^{n-1} \alpha_i \widetilde{H}_{f_i}(z^{-1}) + \sum_{j=1}^{k} \beta_j x_j.$$

We apply the operator $T_{\overline{B}}$ to both sides and by (5.23) and (5.31) we get

$$\begin{aligned}\widetilde{H}_{\overline{B}f_n}(z^{-1}) = T_{\overline{B}}\widetilde{H}_{f_n}(z^{-1}) &= \sum_{i=1}^{n-1} \widetilde{H}_{\alpha_i \overline{B} f_i}(z^{-1}) + \sum_{j=1}^{k} \beta_j T_{\overline{B}} x_j \\ &= \sum_{i=1}^{n-1} \widetilde{H}_{\alpha_i \overline{B} f_i}(z^{-1}) + \sum_{j=1}^{k} \beta_j \sum_{l=1}^{n} \widetilde{H}_{\overline{C}_{jl} f_l}(z^{-1}) \\ &= \sum_{i=1}^{n-1} \widetilde{H}_{\alpha_i \overline{B} f_i}(z^{-1}) + \sum_{l=1}^{n} \widetilde{H}_{\sum_{j=1}^{k} \beta_j \overline{C}_{jl} f_l}(z^{-1}).\end{aligned}$$

Hence

$$\widetilde{H}_{f_n(\overline{B} - \sum_{j=1}^{k}\beta_j \overline{C}_{jn}) - \sum_{i=1}^{n-1} f_i(\alpha_i \overline{B} + \sum_{j=1}^{k}\beta_j \overline{C}_{ji})}(z^{-1}) = 0$$

and the symbol satisfies

$$f_n\left(\overline{B} - \sum_{j=1}^{k} \beta_j \overline{C}_{jn}\right) - \sum_{i=1}^{n-1} f_i \left(\alpha_i \overline{B} + \sum_{j=1}^{k} \beta_j \overline{C}_{ji}\right) \in H(\mathbb{C}_\infty \setminus \mathbb{R}).$$

Put

$$\begin{aligned}\overline{A}_n &:= \overline{B} - \sum_{j=1}^{k} \beta_j \overline{C}_{jn}, \\ \overline{A}_i &:= -\left(\alpha_i \overline{B} + \sum_{j=1}^{k} \beta_j \overline{C}_{ji}\right), \quad i = 1, \dots, n-1.\end{aligned}$$

Then,

$$\sum_{i=1}^{n} \overline{A}_i f_i \in H(\mathbb{C}_\infty \setminus \mathbb{R})$$

and $\deg \overline{A}_n = k$, $\deg \overline{A}_i \le k-1$ for $i = 1, \dots, n-1$.

Case (ii): In the second case we assume that $H_{g_1}(1), \dots, H_{g_n}(1), y_1, \dots, y_k$ are linearly dependent. We act similarly to the previuous case. We put

$$H_{g_n}(1) = \sum_{i=1}^{n-1} \alpha_i H_{g_i}(1) + \sum_{j=1}^{k} \beta_j y_j.$$

We apply the operator $\widetilde{T}_B$ to both sides and using (5.22) and (5.33) we get

$$H_{Bg_n}(1) = \widetilde{T}_B H_{g_n}(1) = \sum_{i=1}^{n-1} H_{\alpha_i B g_i}(1) + \sum_{l=1}^{n} H_{\sum_{j=1}^{k} \beta_j U_{jl} g_l}(1). \tag{5.34}$$

Hence

$$H_{Bg_n-\sum_{i=1}^{n-1}\alpha_i\overline{B}g_i-\sum_{l=1}^{n}g_l\sum_{j=1}^{k}\beta_jU_{jl}}(1)=0$$

and the symbol now satisfies

$$g_n\left(B-\sum_{j=1}^{k}\beta_jU_{jn}\right)-\sum_{i=1}^{n-1}g_i\left(\alpha_iB+\sum_{j=1}^{k}\beta_jU_{ji}\right)\in\mathscr{A}(\mathbb{R}).$$

We put

$$B_n:=B-\sum_{j=1}^{k}\beta_jU_{jn},$$
$$B_i:=\alpha_iB+\sum_{j=1}^{k}\beta_jU_{ji},\quad i=1,\dots,n-1$$

and get

$$\sum_{i=1}^{n}B_ig_i\in\mathscr{A}(\mathbb{R})$$

with polynomials B_i satisfying our conditions.

Case (iii): We are left with the situation in which $\widetilde{H}_{f_1}(z^{-1}),\dots,\widetilde{H}_{f_n}(z^{-1}),x_1,\dots,x_k$ are linearly independent and $H_{g_1}(1),\dots,H_{g_n}(1),y_1,\dots,y_k$ are linearly independent. We will show that such a situation cannot occur.

Since $H_{g_1}(1),\dots,H_{g_n}(1)$ are linearly indpendent, for each i there exists $\eta_i\in\mathscr{A}(\mathbb{R})$ such that $\langle H_{g_i}(1),\eta_i\rangle=1$ and $\langle H_{g_j}(1),\eta_i\rangle=0$ for $j\neq i$. Hence, from (5.29) we get

$$H_{f_i}(z^{-k})=\sum_{j=1}^{k}\langle\eta_i,y_j\rangle x_j-\sum_{j=1}^{k}\left\langle\eta_i,\widetilde{T}_zy_j\right\rangle T_{z^{-1}}x_j$$

and for $i=1,\dots,n$ we have

$$H_{f_i}(z^{-1})\in\operatorname{span}\{x_1,\dots,x_k,T_{z^{-1}}x_1,\dots,T_{z^{-1}}x_k\}.$$

Hence

$$\dim\operatorname{span}\{x_1,\dots,x_k,T_{z^{-1}}x_1,\dots,T_{z^{-1}}x_k\}\geq k+n.$$

Since by the assumptions of case (3) and case $(3)(iii)$ both sets of vectors $\{x_1,\dots,x_k\}$ and $\{T_{z^{-1}}x_1,\dots,T_{z^{-1}}x_k\}$ are linearly independent, by the Hahn-Banach theorem there exists $1\leq l\leq k$ and a functional $\psi\in\mathscr{A}(\mathbb{R})'$ such that

$$\langle T_{z^{-1}}x_i,\psi\rangle=0\quad i=1,\dots,k$$

and

$$\langle x_l,\psi\rangle=1.$$

By (5.32) we have

$$\sum_{i=1}^{n}\left\langle\psi,\widetilde{H}_{f_i}(z^{-1})\right\rangle H_{g_i}(1)=\sum_{j=1}^{k}\langle\psi,x_j\rangle y_j.$$

This is a contradiction since $H_{g_1}(1),\dots,H_{g_n}(1),y_1,\dots,y_k$ were assumed to be linearly independent. □

We can now prove the theorem on finite rank semicommutator of Toeplitz operators.

Theorem 5.3.5. *Semicommutator of the Toeplitz operators $[T_f,T_g)$ is finite dimensional if and only if one of the operators $\widetilde{H}_f$, H_g is of finite rank.*

Proof. By (5.18) we have

$$[T_f,T_g)=T_{fg}-T_fT_g=\widetilde{H}_fH_g.$$

Assume that $\operatorname{rank}(\left(T_f,T_g\right])<\infty$. By Lemma 5.3.4 there exists a polynomial $\overline{A}\in\mathrm{Pol}_-$ such that $\overline{A}f\in H(\mathbb{C}_\infty\smallsetminus\mathbb{R})$ or a polynomial $B\in\mathrm{Pol}_+$ such that $Bg\in\mathscr{A}(\mathbb{R})$. If $\overline{A}f\in H(\mathbb{C}_\infty\smallsetminus\mathbb{R})$ then

$$\widetilde{H}_{\overline{A}f}=T_{\overline{A}}\widetilde{H}_f=0.$$

By Lemma 5.3.3, $\widetilde{H}_f$ has finite rank. If $Bg\in\mathscr{A}(\mathbb{R})$ then

$$H_{Bg}=H_gT_B=0$$

and again by Lemma 5.3.3, H_g has finite rank. □

Theorem 5.3.6. *Let $f_1,f_2,g_1,g_2\in\mathcal{X}$. If none of the operators $\widetilde{H}_{f_1},\widetilde{H}_{f_2},H_{g_1},H_{g_2}$ has finite rank then*

$$\widetilde{H}_{f_1}H_{g_1}=\widetilde{H}_{f_2}H_{g_2}\pmod{\mathcal{F}}$$

if and only if there exist nonzero polynomials $\overline{A}_1,\overline{A}_2\in\mathrm{Pol}_-$, $B_1,B_2\in\mathrm{Pol}_+$ such that:

$$\overline{A}_1f_1+\overline{A}_2f_2\in H(\mathbb{C}_\infty\smallsetminus\mathbb{R}),\quad B_1g_1+B_2g_2\in\mathscr{A}(\mathbb{R})$$

and

$$\overline{A}_1B_1=\overline{A}_2B_2.$$

Proof. ⇒: Assume that $\widetilde{H}_{f_1}H_{g_1}=\widetilde{H}_{f_2}H_{g_2}\pmod{\mathcal{F}}$. By Lemma 5.3.4 there exist polynomials $\overline{A}_1,\overline{A}_2\in\mathrm{Pol}_-$ such that $\overline{A}_1f_1+\overline{A}_2f_2\in H(\mathbb{C}_\infty\smallsetminus\mathbb{R})$ or there exist polynomials $B_1,B_2\in\mathrm{Pol}_+$ such that $B_1g_1+B_2g_2\in\mathscr{A}(\mathbb{R})$.

First we assume that $\overline{A}_1f_1+\overline{A}_2f_2\in H(\mathbb{C}_\infty\smallsetminus\mathbb{R})$. It follows that $\widetilde{H}_{\overline{A}_1f_1+\overline{A}_2f_2}=0$ and so

$$\widetilde{H}_{\overline{A}_1f_1}=-\widetilde{H}_{\overline{A}_2f_2}.$$

Since, by the assumption, $\widetilde{H}_{f_1},\widetilde{H}_{f_2}$ are not finite rank operators and at least one of the polynomials is non-zero we get that both operators $\widetilde{H}_{\overline{A}_1f_1},\widetilde{H}_{\overline{A}_2f_2}$ are non-zero.

Let $n = \max\{\deg \overline{A}_1, \deg \overline{A}_2\}$ and write

$$\overline{A}_1(z) = a_n z^{-n} + \ldots + a_0$$
$$\overline{A}_2(z) = b_n z^{-n} + \ldots + b_0,$$

with $|a_n| + |b_n| > 0$. We have

$$\begin{aligned} T_{\overline{A}_1}\left[\widetilde{H}_{f_1}H_{g_1} - \widetilde{H}_{f_2}H_{g_2}\right] &= \widetilde{H}_{\overline{A}_1 f_1}H_{g_1} - \widetilde{H}_{\overline{A}_1 f_2}H_{g_2} \\ &= -\widetilde{H}_{\overline{A}_2 f_2}H_{g_1} - \widetilde{H}_{\overline{A}_1 f_2}H_{g_2} \\ &= -\widetilde{H}_{f_2}\widetilde{T}_{\overline{A}_2}H_{g_1} - \widetilde{H}_{f_2}\widetilde{T}_{\overline{A}_1}H_{g_2} \\ &= -\widetilde{H}_{f_2}\widetilde{T}_{z^{-n}}\widetilde{T}_{\sum_{j=0}^n b_j z^{n-j}}H_{g_1} - \widetilde{H}_{f_2}\widetilde{T}_{z^{-n}}\widetilde{T}_{\sum_{j=0}^n a_j z^{n-j}}H_{g_2} \\ &= -\widetilde{H}_{f_2 z^{-n}}H_{g_1 \sum_{j=0}^n b_j z^{n-j}} - \widetilde{H}_{f_2 z^{-n}}H_{g_2 \sum_{j=0}^n a_j z^{n-j}} \\ &= -\widetilde{H}_{f_2 z^{-n}}H_{g_1 \sum_{j=0}^n b_j z^{n-j} + g_2 \sum_{j=0}^n a_j z^{n-j}}. \end{aligned}$$

Since $\widetilde{H}_{f_2}$ is not of finite rank, then by Lemma 5.3.3 the operator $\widetilde{H}_{f_2 z^{-n}} = \widetilde{H}_{f_2}\widetilde{T}_{z^{-n}}$ is not of finite rank. Hence by Theorem 5.3.5

$$\operatorname{rank}\left(H_{g_1 \sum_{j=0}^n b_j z^{n-j} + g_2 \sum_{j=0}^n a_j z^{n-j}}\right) < \infty$$

and by Kronecker theorem there exists a polynomial $q \in \mathrm{Pol}_+$ such that

$$q(z)\left(g_1 \sum_{j=0}^n b_j z^{n-j} + g_2 \sum_{j=0}^n a_j z^{n-j}\right) \in \mathscr{A}(\mathbb{R}).$$

We put

$$B_1(z) = q(z)\sum_{j=0}^n b_j z^{n-j}, \quad B_2(z) = q(z)\sum_{j=0}^n a_j z^{n-j}.$$

Hence, $B_1 g_1 + B_2 g_2 \in \mathscr{A}(\mathbb{R})$ and

$$B_1(z) = q(z)\sum_{j=0}^n b_j z^{n-j} = q(z)z^n \sum_{j=0}^n b_j z^{-j} = q(z)z^n \overline{A}_2(z),$$
$$B_2(z) = q(z)\sum_{j=0}^n a_j z^{n-j} = q(z)z^n \sum_{j=0}^n a_j z^{-j} = q(z)z^n \overline{A}_1(z).$$

This proves that $\overline{A}_1 B_1 = \overline{A}_2 B_2$.

Now we consider the other case, i.e., $B_1 g_1 + B_2 g_2 \in \mathscr{A}(\mathbb{R})$. We act analogously. From $H_{B_1 g_1 + B_2 g_2} = 0$ it follows that

$$H_{B_1 g_1} = -H_{B_2 g_2} \neq 0.$$

We put

$$B_1(z) = a_n z^n + \ldots a_0$$
$$B_2(z) = b_n z^n + \ldots b_0.$$

with $|a_n| + |b_n| \geq 0$. We have

$$\begin{aligned}
\left(\widetilde{H}_{f_1}H_{g_1} - \widetilde{H}_{f_2}H_{g_2}\right)T_{B_1} &= \widetilde{H}_{f_1}H_{B_1g_1} - \widetilde{H}_{f_2}H_{B_1g_2} \\
&= -\widetilde{H}_{f_1}H_{B_2g_2} - \widetilde{H}_{f_2}H_{B_1g_2} \\
&= -\widetilde{H}_{f_1}\widetilde{T}_{B_2}H_{g_2} - \widetilde{H}_{f_2}\widetilde{T}_{B_1}H_{g_2} \\
&= -\widetilde{H}_{f_1}\widetilde{T}_{\sum_{j=0}^n b_j z^{-n+j}}\widetilde{T}_{z^n}H_{g_2} - \widetilde{H}_{f_2}\widetilde{T}_{\sum_{j=0}^n a_j z^{-n+j}}\widetilde{T}_{z^n}H_{g_2} \\
&= \left(\widetilde{H}_{f_1}\widetilde{T}_{\sum_{j=0}^n b_j z^{-n+j}} - \widetilde{H}_{f_2}\widetilde{T}_{\sum_{j=0}^n a_j z^{-n+j}}\right)\widetilde{T}_{z^n}H_{g_2} \\
&= \widetilde{H}_{f_1\sum_{j=0}^n b_j z^{-n+j} - f_2\sum_{j=0}^n a_j z^{-n+j}}H_{z^n g_2}.
\end{aligned}$$

Since H_{g_2} is not of finite rank, by Lemma 5.3.3(c) the operator $H_{z^n g_2} = \widetilde{T}_{z^n}H_{g_2}$ is not of finite rank. Hence, by Theorem 5.3.5,

$$\operatorname{rank}\left(\widetilde{H}_{f_1\sum_{j=0}^n b_j z^{-n+j} - f_2\sum_{j=0}^n a_j z^{-n+j}}\right) < \infty$$

and by Kronecker theorem there exists a polynomial $\overline{q} \in \mathrm{Pol}_-$ such that

$$\overline{q}\left(f_1\sum_{j=0}^n b_j z^{-n+j} - f_2\sum_{j=0}^n a_j z^{-n+j}\right) \in H(\mathbb{C}_\infty \setminus \mathbb{R}).$$

We put

$$\overline{A}_1(z^{-1}) = \overline{q}(z^{-1})\sum_{j=0}^n b_j z^{-n+j}, \quad \overline{A}_2(z^{-1}) = \overline{q}(z^{-1})\sum_{j=0}^n a_j z^{-n+j}.$$

Hence $\overline{A}_1 f_1 + \overline{A}_2 f_2 \in H(\mathbb{C}_\infty \setminus \mathbb{R})$ and

$$\begin{aligned}
\overline{A}_1(z^{-1}) &= \overline{q}(z^{-1})\sum_{j=0}^n b_j z^{-n+j} = \overline{q}(z^{-1})z^{-n}\sum_{j=0}^n b_j z^j = \overline{q}(z^{-1})z^{-n}B_2(z), \\
\overline{A}_2(z^{-1}) &= \overline{q}(z^{-1})\sum_{j=0}^n a_j z^{-n+j} = \overline{q}(z^{-1})z^{-n}\sum_{j=0}^n a_j z^j = \overline{q}(z^{-1})z^{-n}B_1(z)
\end{aligned}$$

gives $\overline{A}_1 B_1 = \overline{A}_2 B_2$.

$\Leftarrow$: Now we assume that there exist polynomials $\overline{A}_1, \overline{A}_2, B_1, B_2$ such that

$$\overline{A}_1 f_1 + \overline{A}_2 f_2 \in H(\mathbb{C}_\infty \setminus \mathbb{R}), \quad B_1 g_1 + B_2 g_2 \in \mathscr{A}(\mathbb{R})$$

and $\overline{A}_1 B_1 = \overline{A}_2 B_2$. Using (5.22), (5.23), (5.25) we compute

$$\begin{aligned}
T_{\overline{A}_1}(\widetilde{H}_{f_1}H_{g_1} - \widetilde{H}_{f_2}H_{g_2})T_{B_1} &= \widetilde{H}_{\overline{A}_1 f_1}H_{B_1g_1} - \widetilde{H}_{\overline{A}_1 f_2}H_{B_1g_2} \\
&= \widetilde{H}_{\overline{A}_2 f_2}H_{B_2g_2} - \widetilde{H}_{\overline{A}_1 f_2}H_{B_1g_2} \\
&= \widetilde{H}_{f_2}\left(\widetilde{T}_{\overline{A}_1}\widetilde{T}_{B_1} - \widetilde{T}_{\overline{A}_1}\widetilde{T}_{B_1}\right)H_{g_2} \\
&= \widetilde{H}_{f_2}(\widetilde{T}_{\overline{A}_2B_2} - \widetilde{T}_{\overline{A}_1B_1})H_{g_2} \pmod{\mathcal{F}} \\
&= \widetilde{H}_{f_2}(\widetilde{T}_{\overline{A}_2B_2 - \overline{A}_1B_1})H_{g_2} = 0.
\end{aligned}$$

Hence $\operatorname{rank}(T_{\overline{A}_1}(\widetilde{H}_{f_1}H_{g_1} - \widetilde{H}_{f_2}H_{g_2})T_{B_1}) < \infty$ and by Lemma 5.3.3 we get that $\operatorname{rank}(\widetilde{H}_{f_1}H_{g_1} - \widetilde{H}_{f_2}H_{g_2}) < \infty$. □

Finally we can answer the question when the commutator of Toeplitz operators has finite rank.

Theorem 5.3.7. *Let $f, g \in \mathcal{X}$. The commutator $[T_f, T_g]$ has finite rank if and only if one of the following holds:*

(1) *there exists a nonzero polynomial $p \in \mathrm{Pol}_+$ such that $pf \in \mathscr{A}(\mathbb{R})$ and $pg \in \mathscr{A}(\mathbb{R})$,*

(2) *there exists a nonzero polynomial $\overline{q} \in \mathrm{Pol}_-$ such that $\overline{q}f \in H(\mathbb{C}_\infty \setminus \mathbb{R})$ and $\overline{q}g \in H(\mathbb{C}_\infty \setminus \mathbb{R})$.*

(3) *there exist polynomials $\overline{A}_1, \overline{A}_2 \in \mathrm{Pol}_-$, $B_1, B_2 \in \mathrm{Pol}_+$ such that*

$$\overline{A}_1 f + \overline{A}_2 g \in H(\mathbb{C}_\infty \setminus \mathbb{R}), \quad B_1 g + B_2 f \in \mathscr{A}(\mathbb{R})$$

and

$$\overline{A}_1 B_1 = \overline{A}_2 B_2.$$

Proof. ($\Rightarrow$): Let $\operatorname{rank}([T_f, T_g]) = \operatorname{rank}(\widetilde{H}_f H_g - \widetilde{H}_g H_f) < \infty$. Assume first that H_f has finite rank. By Kronecker theorem there exists a polynomial $p_1 \in \mathrm{Pol}_+$ such that $p_1 f \in \mathscr{A}(\mathbb{R})$. Since

$$(\widetilde{H}_f H_g - \widetilde{H}_g H_f) T_{p_1} = \widetilde{H}_f H_{p_1 g} - \widetilde{H}_g H_{f p_1}$$

and $H_{f p_1} = 0$, we get that $\widetilde{H}_f H_{p_1 g}$ has finite rank. From Lemma 5.3.4 there exists $\overline{a_1} \in \mathrm{Pol}_-$ such that $\overline{a_1} f \in H(\mathbb{C}_\infty \setminus \mathbb{R})$ or there exists $b_1 \in \mathrm{Pol}_+$ such that $b_1 p_1 g \in \mathscr{A}(\mathbb{R})$.

If $\overline{a_1} f \in H(\mathbb{C}_\infty \setminus \mathbb{R})$ for some $\overline{a}_1 \in \mathrm{Pol}_-$ then (3) holds with $\overline{A}_1 = \overline{a_1}$, $\overline{A}_2 = 0$, $B_1 = 0$, $B_2 = p_1$.

If $b_1 p_1 g \in \mathscr{A}(\mathbb{R})$ for some $b_1 \in \mathrm{Pol}_+$ then (1) holds with $p = b_1 p_1$.

If one of the operators H_g, $\widetilde{H}_f$, $\widetilde{H}_g$ has finite rank we get a similar result.

If none of the operators $H_f, H_g, \widetilde{H}_f, \widetilde{H}_g$ has finite rank, then we use Theorem 5.3.6 and get that (3) holds.

($\Leftarrow$) : Now we prove the other direction, so we assume that one of the (1) – (3) holds. If (1) (or (2)) holds, then by Kronecker theorem $\widetilde{H}_f$ and $\widetilde{H}_g$ (or H_f and H_g) are finite dimensional and $[T_f, T_g] = \widetilde{H}_f H_g - \widetilde{H}_g H_f$ has finite rank. Assume then that (3) holds. Using (5.22), (5.23) we compute

$$\begin{aligned} T_{\overline{A}_1}(\widetilde{H}_f H_g - \widetilde{H}_g H_f) T_{B_2} &= \widetilde{H}_{\overline{A}_1 f} H_{B_2 g} - \widetilde{H}_{\overline{A}_1 g} H_{B_2 f} \\ &= \widetilde{H}_{-\overline{A}_2 g} H_{B_2 g} - \widetilde{H}_{\overline{A}_1 g} H_{-B_1 g} \\ &= \widetilde{H}_g(\widetilde{T}_{-\overline{A}_2} \widetilde{T}_{B_2}) H_g - \widetilde{H}_g(\widetilde{T}_{\overline{A}_1} \widetilde{T}_{-B_1}) H_g \\ &= \widetilde{H}_g(\widetilde{T}_{\overline{A}_1} \widetilde{T}_{B_1} - \widetilde{T}_{\overline{A}_2} \widetilde{T}_{B_2}) H_g. \end{aligned}$$

Now we prove that the operator $\widetilde{T}_{\overline{A}_1} \widetilde{T}_{B_1} - \widetilde{T}_{\overline{A}_2}$ has finite rank. By (5.19) we get the formula for a semicommutator

$$(\widetilde{T}_f, \widetilde{T}_g] = T_f T_g - T_{fg} = H_f \widetilde{H}_g.$$

Hence

$$\widetilde{T}_{\overline{A}_1} \widetilde{T}_{B_1} - \widetilde{T}_{\overline{A}_2} \widetilde{T}_{B_2} = \widetilde{T}_{\overline{A}_1} \widetilde{T}_{B_1} - \widetilde{T}_{\overline{A}_1 B_1} + \widetilde{T}_{\overline{A}_2 B_2} - \widetilde{T}_{\overline{A}_2} \widetilde{T}_{B_2} = H_{\overline{A}_1} \widetilde{H}_{B_1} - H_{\overline{A}_2} \widetilde{H}_g.$$

Since on the right hand side we have only finite rank operators, the operator $\widetilde{T}_{\overline{A}_1} \widetilde{T}_{B_1} - \widetilde{T}_{\overline{A}_2} \widetilde{T}_{B_2}$ has finite rank. Lemma 5.3.3 completes the proof. □

In Chapter 4 we have considered Hankel operators acting from $\mathscr{A}(\mathbb{R})$ into $\mathscr{A}(\mathbb{R})$. We will now derive the result concerning commutators of Hankel operators on $\mathscr{A}(\mathbb{R})$ from the just proved results on Hankel operators acting from $\mathscr{A}(\mathbb{R})$ into $H_0(\mathbb{C}_\infty \setminus \mathbb{R})$. Recall, that for an entire function $\varphi \in H(\mathbb{C})$ we denote by Γ_φ the Hankel operator acting on $\mathscr{A}(\mathbb{R})$, i.e., $\Gamma_\varphi\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$

$$\Gamma_\varphi g(z) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\xi)g(\frac{1}{\xi})}{\xi - z}d\xi, \quad g \in \mathscr{A}(\mathbb{R}),$$

where γ is a C^∞ smooth Jordan curve such that $z, 0 \in \operatorname{Int}(\gamma)$ and $g(\frac{1}{\xi})$ is holomorphic on γ and in $\operatorname{Ext}(\gamma)$.

We have the following relation between Γ_φ and H_φ

$$\Gamma_\varphi = RH_{R\varphi},$$

where R is the isomorphism between $H(\mathbb{C})$ and $H_0(\mathbb{C}_\infty \setminus \{0\})$, $Rf(z) = \frac{1}{z}f(\frac{1}{z})$. Indeed, take $\varphi(z) = \sum_{n=0}^\infty \varphi_n z^n \in H(\mathbb{C})$. Then for every monomial z^k we have

$$\Gamma_\varphi(z^k) = T^k_{z^{-1}}\varphi,$$

$$RH_{R\varphi}(z^k) = RQ(R\varphi \cdot z^k) = RQ(\sum_{n=0}^\infty \varphi_n z^{-n-1+k}) = R(\sum_{n=k}^\infty \varphi_n z^{-n-1+k}) = \sum_{n=k}^\infty \varphi_n z^{n-k} = T^k_{z^{-1}}\varphi.$$

Moreover, for every $f, g \in H(\mathbb{C})$ and $k \in \mathbb{N}$

$$\Gamma_f\Gamma_g(z^k) = \Gamma_f(T^k_{z^{-1}}g) = \mathcal{C}(fzRT^k_{z^{-1}}g),$$

$$\widetilde{H}_{zf}H_{Rg}(z^k) = \widetilde{H}_{zf}RT^k_{z^{-1}}g = \mathcal{C}(zfRT^k_{z^{-1}}g).$$

Hence

$$\Gamma_f\Gamma_g = \widetilde{H}_{zf}H_{Rg}.$$

Theorem 5.3.8. *Let $f, g, h \in H(\mathbb{C})$. The following are equivalent*

1. *$\Gamma_f\Gamma_g$ is a finite rank perturbation of Γ_h,*
2. *$\Gamma_f\Gamma_g$ and Γ_h are finite rank operators,*
3. *there are nonzero polynomials $\overline{A}, \overline{B}, \overline{C} \in \mathrm{Pol}_-$ such that*

$$\overline{A}f \in H(\mathbb{C}_\infty \setminus \mathbb{R}) \quad \textit{and} \quad \overline{C}h \in H(\mathbb{C}_\infty \setminus \mathbb{R})$$

or

$$\overline{B}g \in H(\mathbb{C}_\infty \setminus \mathbb{R}) \quad \textit{and} \quad \overline{C}h \in H(\mathbb{C}_\infty \setminus \mathbb{R}).$$

Proof. (2) ⇔ (3): Notice first that $B(Rg) \in \mathscr{A}(\mathbb{R})$ for some $B \in \mathrm{Pol}_+$ if and only if Rg is a rational function. Hence also g is rational. And this is equivalent to $\overline{B}g \in H_0(\mathbb{C}_\infty \setminus \mathbb{R})$ for some $\overline{B} \in \mathrm{Pol}_-$.

From the Kronecker theorem and the identity $\Gamma_h = RH_{Rh}$ it follows that Γ_h has finite rank if and only if $\overline{C}h \in H(\mathbb{C}_\infty \setminus \mathbb{R})$ for some $\overline{C} \in \mathrm{Pol}_-$.

Since $\Gamma_f\Gamma_g = \widetilde{H}_{zf}H_{Rg}$, the equivalence (2) ⇔ (3) follows from Lemma 5.3.4.

(1) ⇔ (2): We have to show (1) ⇒ (2). Suppose $\Gamma_f\Gamma_g - \Gamma_h$ is a finite rank operator. From (5.26) we have

$$\begin{aligned}\widetilde{H}_{zf}(z^{-1}) \otimes H_{Rg}(1) &= \widetilde{H}_{zf}H_{Rg} - T_{z^{-1}}\widetilde{H}_{zf}H_{Rg}T_z\\ &= \Gamma_f\Gamma_g - T_{z^{-1}}\Gamma_f\Gamma_g T_z\\ &= \Gamma_h - T_{z^{-1}}\Gamma_h T_z \pmod{\mathcal{F}}\\ &= \Gamma_h - \Gamma_{T^2_{z^{-1}}h}\\ &= \Gamma_{h-T^2_{z^{-1}}h}.\end{aligned}$$

Hence the operator $\Gamma_{h-T^2_{z^{-1}}h}$ has finite rank. By the Kronecker theorem for the Hankel operators acting from $\mathscr{A}(\mathbb{R})$ into $\mathscr{A}(\mathbb{R})$ (Theorem 4.1.8) the function $h - T^2_{z^{-1}}h$ is a polynomial. Hence h is a polynomial, the operator Γ_h is of finite rank and so is $\Gamma_f\Gamma_g$. □

Theorem 5.3.9. *Let $f, g, h \in H(\mathbb{C})$. The following are equivalent*

1. $[\Gamma_f, \Gamma_g] = \Gamma_h \pmod{\mathcal{F}}$,

2. Γ_h, $[\Gamma_f, \Gamma_g]$ *have finite rank,*

3. Γ_h, $\Gamma_f\Gamma_g$, $\Gamma_g\Gamma_f$ *have finite rank or there are nonzero polynomials* $\overline{A}_1, \overline{A}_2, \overline{C} \in \mathrm{Pol}_-$, $B_1, B_2 \in \mathscr{A}(\mathbb{R})$ *such that*

$$\overline{A}_1 f + \overline{A}_2 g \in H(\mathbb{C}_\infty \setminus \mathbb{R}), \quad B_1(Rg) + B_2(Rf) \in \mathscr{A}(\mathbb{R}), \quad \overline{C}h \in H(\mathbb{C}_\infty \setminus \mathbb{R})$$

and

$$\overline{A}_1 B_1 = \overline{A}_2 B_2.$$

Proof. (1) ⇒ (2): We act analogously to the proof of (1) ⇒ (2) in the previous lemma. By the assumption we have

$$[\Gamma_f, \Gamma_g] = \Gamma_h \pmod{\mathcal{F}},$$

which is equvalent to

$$\widetilde{H}_{zf}H_{Rg} - \widetilde{H}_{zg}H_{Rg} = RH_{Rh} \pmod{\mathcal{F}}.$$

From (5.26) it follows that

$$\begin{aligned}
\widetilde{H}_{zf}(z^{-1}) \otimes H_{Rg}(1) - \widetilde{H}_{zg}(z^{-1}) \otimes H_{Rf}(1) \\
&= \widetilde{H}_{zf}H_{Rg} - T_{z^{-1}}\widetilde{H}_{zf}H_{Rg}T_z - \widetilde{H}_{zg}H_{Rf} + T_{z^{-1}}\widetilde{H}_{zg}H_{Rf}T_z \\
&= \Gamma_f\Gamma_g - T_{z^{-1}}\Gamma_f\Gamma_g T_z - \Gamma_g\Gamma_f - T_{z^{-1}}\Gamma_g\Gamma_f T_z \\
&= [\Gamma_f, \Gamma_g] - T_{z^{-1}}[\Gamma_f, \Gamma_g]T_z \\
&= \Gamma_h - T_{z^{-1}}\Gamma_h T_z \pmod{\mathcal{F}} \\
&= \Gamma_h - \Gamma_{T^2_{z^{-1}}h} \\
&= \Gamma_{h - T^2_{z^{-1}}h}.
\end{aligned}$$

Hence the operator $\Gamma_{h-T^2_{z^{-1}}h}$ has finite rank. By the Kronecker theorem (Theorem 4.1.8) the function $h - T^2_{z^{-1}}h$ is a polynomial and it follows that h is a polynomial and the operators Γ_h , $\Gamma_f\Gamma_g$ have finite rank.

(2) $\Rightarrow$ (3): Assume that Γ_h and $[\Gamma_f, \Gamma_g] = \Gamma_f\Gamma_g - \Gamma_g\Gamma_f$ have finite rank. Since Γ_h has finite rank by Kronecker theorem (Theorem 4.1.8) h is a polynomial. Let k be a degree of h. We put $\overline{C} = z^{-k-1}$ and get that $\overline{C}h \in H(\mathbb{C}_\infty \setminus \mathbb{R})$. If $\Gamma_f\Gamma_g$ has finite rank then so does $\Gamma_g\Gamma_f$. If neither of $\Gamma_f\Gamma_g = \widetilde{H}_{zf}H_{Rg}$, $\Gamma_g\Gamma_f = \widetilde{H}_{zg}H_{Rf}$ has finite rank then we are in the following situation:

$$\widetilde{H}_{zf}H_{Rg} = \widetilde{H}_{zg}H_{Rf} \pmod{\mathcal{F}}$$

and neither of the operators $\widetilde{H}_{zf}, H_{Rg}, \widetilde{H}_{zg}, H_{Rf}$ has finite rank. Hence, by Theorem 5.3.6 there exists nonzero polynomials $\overline{A}_1, \overline{A}_2 \in \mathrm{Pol}_-$, $B_1, B_2 \in \mathrm{Pol}_+$ such that:

$$\overline{A}_1 zf + \overline{A}_2 zg \in H_0(\mathbb{C}_\infty \setminus \mathbb{R}), \quad B_1(Rg) + B_2(Rf) \in \mathscr{A}(\mathbb{R})$$

and

$$\overline{A_1}B_1 = \overline{A_2}B_2.$$

If needed, we multiply polynomials $\overline{A}_1, \overline{A}_2$ by z^{-1} and get the desired result.

(3) $\Rightarrow$ (1): The condition $\overline{C}h \in H(\mathbb{C}_\infty \setminus \mathbb{R})$ for some $\overline{C} \in \mathrm{Pol}_-$ means that Γ_h has finite rank. Hence we want to prove that from (3) it follows that

$$[\Gamma_f, \Gamma_g] = \widetilde{H}_{zf}H_{Rg} - \widetilde{H}_{zg}H_{Rf} = 0 \pmod{\mathcal{F}}.$$

Hence the result again follows from Theorem 5.3.6. □

Index of symbols

Bibliography

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