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Experimentally-friendly methods of generation and detection of quantum correlations

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A Dissertation
submitted in partial fulfillment of the requirements
for the Degree of Doctor of The Physical Sciences
at the Adam Mickiewicz University in Poznań

Poznań, 2012

Acknowledgements

I would like to express my gratitude towards Prof. UAM dr. hab. Adam Miranowicz for providing me with valuable advice and help in the preparation of it, but also for the patience in guiding me towards the field of quantum optics and quantum information. Both the publications and the Rector of Adam Mickiewicz University award, which we were granted last year, I owe to my cooperation with him.

I would also like to say thank you to Prof. Lianao Wu, under whose supervision I completed my internship in the Basque Country University (EHU/UPV) in Bilbao (Spain). I appreciate his providing me during the work on the schemes for amplifying the nonlinear Kerr effect.

Last but not least, I would like to express my appreciation towards all of the Division of Nonlinear Physics employees for creating an amiable atmosphere during my four years at the department.

A part of the research included in this thesis was created with the aid of the Adam Mickiewicz University, which funded my internship in Bilbao as a part of „UAM: Unikatowy Absolwent = Możliwości. Wzrost potencjału dydaktycznego Uniwersytetu im. Adama Mickiewicza poprzez proinnowacyjne kształcenie w języku angielskim, interdyscyplinarność, e-learning, inwestycje w kadry" (POKL.04.01.01.00-019/10). The research included in the following thesis was supported from the Polish Ministry of Science and Higher Education under Grants No. 2619/B/H03/2010/38.

Oświadczenie

Ja, niżej podpisana Monika Bartkowiak, doktorantka Wydziału Fizyki Uniwersytetu im. Adama Mickiewicza w Poznaniu oświadczam, że przedkładaną rozprawę doktorską pt. *Experimentally-friendly methods of generation and detection of quantum correlations* napisałam samodzielnie. Oznacza to, że przy pisaniu pracy, poza niezbędnymi konsultacjami, nie korzystałam z pomocy innych osób, a w szczególności nie zlecałam opracowania rozprawy lub jej części innym osobom, ani nie odpisywałam tej rozprawy lub jej części od innych osób. Jednocześnie przyjmuję do wiadomości, że gdyby powyższe oświadczenie okazało się nieprawdziwe, decyzja o nadaniu mi stopnia doktora zostanie cofnięta.

....

Abstract

One of the most relevant problems in the quantum theory is the question whether the appropriate state of the system can be described within a classical theory. The famous examples of nonclassical states are Schrödinger-cats states or entanglement states. In spite of many practical applications of entanglement as a special kind of nonclassicality (like quantum teleportation [1], dense coding [2], or implementation of super-fast [3] and fast algorithms [4]), we still have difficulties with describing and specifying it. One cannot find such an operational optimal method, which would allow us to maximally use the technological and scientific potential arising from the adaptation properties of quantum correlations. Thus, my thesis is focused on giving a proposal of a theoretical recipe for constructing experimentally achievable procedures for the study of quantum correlations.

In my thesis I have focused on quantum correlations described in terms of nonclassicality (quantumness) and specific kind of it,i.e., entanglement. The nonclassicality definition, used by me in the thesis, is based on the Glauber-Sudarshan function (P-function). One can assume, that a state is nonclassical, if its P-function is negative or more singular than Dirac's delta [5]. As follows, the state is nonclassical, when its P-function cannot be treated as a "real" probabilistic distribution. Based on specific properties of the P-function, Agarwal and Tara [6], as well as, Shchukin, Richter and Vogel [7, 8] proposed nonclassicality and entanglement [9] criteria based on matrices of moments of annihilation and creation operators. The operational procedures for analyzing nonclassicality and an efficient method for measuring such moments developed by Shchukin and Vogel [10] creates basis for my thesis. It can be seen that the nonclassicality criteria based on matrices of moments offer an effective way of deriving specific inequalities, which might be useful in the verification of nonclassicality of particular states generated in experiments. Therefore, criteria constructed based on the above definition of nonclassicality can be used to find practical and effective methods of generating and testing nonclassicality and, therefore, also quantum entanglement of optical fields. The results presented in my thesis can be divided into three main groups:

1. Finding operational and practical criteria to classify states in terms of nonclassicality and entanglement based on fundamental classical inequalities like Cauchy-Schwarz inequality. It was also shown how some known entanglement inequalities can be derived as nonclassicality inequalities or as sums of more than one inequality [Bartkowiak2010a].

2. Describing properties and behaviour of quantum correlations for different optical fields (e.g. for multi- and single-mode systems, for interacting and noninteracting modes). In particular, general occurrence of sudden vanishing of nonclassicality, which can be observed not only for two- or multimode but also for single-mode fields, was proven [Bartkowiak2011].

3. Finding the methods of generating and testing of quantum correlations, which would be practical and easy to implement with available resources in both linear and nonlinear optics.

Two setups for implementing linear optical universal quantum gates (the controlled-NOT and controlled-sign gates), and two setups for improving the usage of cross-Kerr effect as the controlled-phase gate were proposed. The experimental aspects of those implementations were stressed and imperfections and noise connected with available resources were taken into account [Bartkowiak2010b,Bartkowiak2012].

Summarizing, the main aim of this thesis was to stress the experimental aspects of theoretical criteria of nonclassicality, which are based on fundamental classical inequalities. More precisely, my goal was to achieve the experimentally available implementations by linking nonclassicality criteria based on moments of annihilation and creation operators with technological simplicity of linear optical or nonlinear schemes.

Contents

Chapter I

Introduction

From the very beginning of quantum theory the problem of whether a given state can be described in a classical manner was of most interest among researchers. This issue appears in almost all branches of quantum theory e.g., quantum optics [5, 11, 12, 13, 14], condensed matter (see, e.g., [13, 15]), nanomechanics [16], or quantum biology (see, e.g., [17]). This question seems even more interesting in the context of famous examples of nonclassical states. Especially, that macroscopic quantum superpositions (being at the heart of the Schrödinger-cat paradox) and related entangled states (which are at the core of the Einstein-Podolsky-Rosen paradox and Bell's theorem), previously known mainly as physical curiosities, are now fundamental resources for quantum-information processing [18].

The threshold between *nonclassical* and *classical* can be set by taking into account different properties of states appearing in quantum physics. In the literature one can find a variety of definitions of nonclassicality criteria or entanglement measures. For instance, it is possible to define nonclassicality (called also quantumness) based on an ability to create a state using only classical operations on classical bits [19]. It can also be linked with noncommutative properties of operators representing the states [20], in the sense that the higher degree of noncommutation properties, the higher is also quantumness.

Different proposals of operational criteria of nonclassicality of single-mode (see, e.g., [5, 13] and references therein) and multimode fields were developed (see, e.g., textbooks [5, 13, 14]), and tested experimentally (see, e.g., Refs. [21, 22, 23, 24, 25, 26, 27]). For nonclassicality one can also construct witnesses based on the previously evoked criteria, e.g. witnesses based on noncommutativity [19, 20] or possibility of measurement-induced disturbance of states [28].

Moreover, for entanglement only, beside the most common one like concurrence or negativity, there also exists a lot of other operational definitions of measures based on entropy like: the entanglement of formation, the entanglement cost, the distillable entanglement [29, 30, 31], the relative entropy of entanglement [32, 33, 34, 35]. There were also some proposals of creating entanglement criteria based on separability of states and partial transposition like the Peres-Horodecki criterion [36, 37] and its improvement (more information can be found in Ref. [38]).

One needs to realize that according to this general definitions nonclassicality/quantumness is a wider term than entanglement (which formal definition in terms of nonseparability will be presented in Section II.3). Nonclassicality also contains other possible quantum correlations which cannot be reduced to entanglement. Thus, there is a need to stress that every entanglement state is nonclassical but one can find quantum correlations which are not connected with separability of states.

Recently, the idea of quantum discord [39, 40, 41] as a measure of a difference between total correlations (defined as mutual information) and classical ones has appeared (under some measurement). The subject of quantum discord is still under strong investigations and new definitions have been developed e.g. geometric discord (more information can be found in Ref. [42]). The analyses of the differences between all quantum correlations and entanglement can be found in the article of Modi *et al.* [19]. They presented how to find a common dominator between all measures of quantum correlations as the measures of distance from a given state to the one without a considered property (like e.g. nonseparability). For that they studied not only entanglement, discord, classical correlations, but they also introduced dissonance as a manifestation of quantum correlations with the exception of entanglement.

To obtain a general definition of nonclassicality which will detect the boundary between a classical and a nonclassical state, one is also able to use an analogy in a description of quantum and classical state by a probability distribution. Thus, there is a need to emphasize that the term "classical" is being used by me in an arbitrary way, in a sense that some quantum states are closer to the classical ones (like e.g. the coherent states as the most classical pure states of harmonic oscillator). Nevertheless, all states considered in this thesis are quantum states. On the pages of this thesis nonclassicality will be understood in the following way [11, 12]:

Criterion 1 *A quantum state is nonclassical if its Glauber-Sudarshan P-function cannot be interpreted as a true probability density.*

The above definition can be applied not only to a pure but also to a mixed states and, therefore, to quantum correlations which are not connected purely with entanglement. Using the above definition it is possible to formulate some criteria to detect nonclassicality of an arbitrary state. However, using the P-function as a criterion for detecting nonclassicality can be hindered, as this function can be very irregular and singular. Thus, still based on the Glauber-Sudarshan function, Agarwal and Tara [6], Shchukin, Richter and Vogel [7, 8] presented criteria for nonclassicality relying on matrices of moments of annihilation and creation operators for single-mode fields.

In this manner, analogously to the Shchukin-Richter-Vogel approach, entanglement criteria were proposed by Shchukin and Vogel [9] using additional partial transposition to detect nonseparability of states. The choice of the Shchukin-Richter-Vogel criteria as the basis for the analysis and the search of experimentally-friendly tests of nonclassicality presented further in this thesis is justified. Especially that there is a proposal, given by Shchukin and Vogel [10], of an effective method for measuring arbitrary moments of creation and annihilation operators.

1 Goals and methods of the thesis

The aim of my thesis is to obtain effective criteria for testing whether a given state of a system can be described within a classical theory, and providing technologically available implementations to realize them. The main problem is to find operational criteria which can be implemented using common experimental resources and then to create schemes which would enable measurement of appropriate quantities necessary for testing nonclassicality. In this thesis relations between different kinds of criteria of quantum correlations and obtained experimentally-friendly schemes to generate quantum correlations and to operate on them by testing their nonclassicality will be presented.

As far as nonclassicality (and by this also entanglement) is considered as a manifestation of quantum correlations, an optimal theoretical description of the boundary between classical and nonclassical states and analyzing the behaviour of not only entanglement [43] but also nonclassicality of dissipative systems, would deepen and complete knowledge about quantumness. On the other hand, one of the method to encode qubits is using light, and the easiest and most available way to perform operations on photons are linear-optical schemes. They can be implemented using the available devices such as half-wave or quarter-wave plates. The knowledge about technical parameters of those devices can be easily included in theoretical models and allows one to faithfully predict the influence of imperfections of resources on fidelity of implementations. However, as far as linear optics is considered, the Bell no-go theorem prevents designing of a deterministic implementations of two-qubit universal gates which are crucial for quantum computing. The other idea is to use inner nonlinearity of media to perform an interaction between photons. However, nowadays the phase noise in the available media prevent obtaining any significant results.

By linking criteria which are constructed using fundamental classical inequalities with technological simplicity of linear and nonlinear optics implementations one can obtain experimentally-friendly methods of characterizing nonclassicality of states. The proposals of implementations presented in this thesis give a theoretical recipe for constructing simple schemes to create and measure quantum correlations.

Based on assumptions and definitions of nonclassicality I focus on three main approaches to the matter of quantum correlations:

- finding basic and fundamental inequalities rooted in common properties of states, which will be broken for nonclassical, in particular- entanglement states [Bartkowiak2010a];
- analyzing the behaviour of nonclassicality witnesses obtained from the above inequalities for evaluating systems [Bartkowiak2011];
- finding experimentally-friendly implementations to create and test quantum correlations (using linear- and nonlinear- optical implementations) [Bartkowiak2010b,Bartkowiak2012].

The basis of my scientific methods are operational definitions, which were proposed by Richter, Shchukin, Vogel (for nonclassicality) [7, 8] and Shchukin, Vogel (entanglement) [9]. To obtain the results presented in this thesis I have used numerical and analytical methods of quantum optics and quantum information like e.g.

- including properties of detectors in linear-optical systems [Bartkowiak2010b];
- descriptions of states in term of quasiprobability distribution [Bartkowiak2010a, Bartkowiak2011];
- methods of solving master equation [Bartkowiak2011];
- presented methods of derivation of nonclassicality/entanglement witnesses [Bartkowiak2010a, Bartkowiak2011];
- group theory applied to quantum optics [Bartkowiak2012];
- analyzing spectral effects in nonlinear media (Subsection II.3).

2 Structure of the thesis

The structure of my thesis can be seen in the Fig. I.1. The thesis is divided into two main parts. The first (Chapter II) refers to introducing nonclassicality and entanglement criteria based on the Glauber-Sudarshan P-function [7, 8, 9], constructing witnesses and giving examples of practical applications of these witnesses to analyze the properties of optical systems.

The Section II.1 contains basic information and definitions of a statistical description of a state in terms of quasidistributions. Those definitions are further used in II.2 to formulate operational criteria of nonclassicality based on matrices of moments of creation and annihilation operators. In this section it is also shown how the defined criteria can be linked with the known inequalities for multimode effects and how to construct witnesses of nonclassicality.

Section II.3 refers to entanglement as a particular kind of nonclassicality defined based on nonseparability of a state. A criterion for entanglement and examples of entanglement witnesses are presented. In further part of this section more attention is given to entanglement inequalities (e.g. of Duan *et al.* [44], and Hillery and Zubairy [45]) which can be also constructed as nonclassicality criteria. One can find here the general recipe for how to find an entanglement inequality as a sum of the nonclassical conditions, in particular entanglement criterion of Simon [46] is being analyzed.

The last Section II.4 of this part of the thesis corresponds to the examples of applications of the criteria defined in the previous sections to analyze properties of optical systems. Using a constructed witnesses it is possible to reconstruct results of You and Eberly [43] and show that problem of the sudden vanishing of quantum correlations is a universal phenomenon.

The second part (Chapter III) presents the methods of quantum correlations generation using linear and nonlinear optics. Analysing nonclassicality, in particular entanglement, is even more interesting as quantum correlations enable one to achieve goals which cannot be realized by the means of the classical theory of information e.g. quantum teleportation [1], dense coding [2], or implementation of super-fast [3] and fast [4] algorithms. Quantum entanglement is nowadays commercially used in quantum cryptography [47]. Even thought there are many possibilities of adaptations of entanglement, we are still not able to characterize it precisely.

This Chapter focuses on a two-qubit quantum gates which can be used to generate entanglement between qubits and used in quantum computation protocols.

The first section of this part III.1 and the beginning of the Section III.2 contain a short review of proposals of optical implementations of two-qubit gates in particular a linear-optical ones.

Further in Section III.2 two proposals of experimentally-friendly implementations of two-qubit quantum gates are presented. They were designed taking into account an experimental accessibility and imperfection of the available optical devices.

The last Section III.3 of this part contains a setup to enhance nonlinearity of the cross-Kerr medium due to squeezing operation. The proposed scheme can be used to overcome difficulties connected with the implementations of two-qubit entangling gates using internal nonlinearity of medium. In this section one can also find a review of possible squeezing operation implementations. The last part of this section contains calculations of the impact of spectral effects in considered nonlinear media. It is shown that fidelity of the appropriate two-qubit gate can be improved even after performing one squeezing operation on the state.
The thesis finishes with concluding remarks and a list of the most important results.

In Fig. I.2 relations between the most important terms considered in the thesis are presented. Before each section there is a diagram with the currently investigated terms marked.

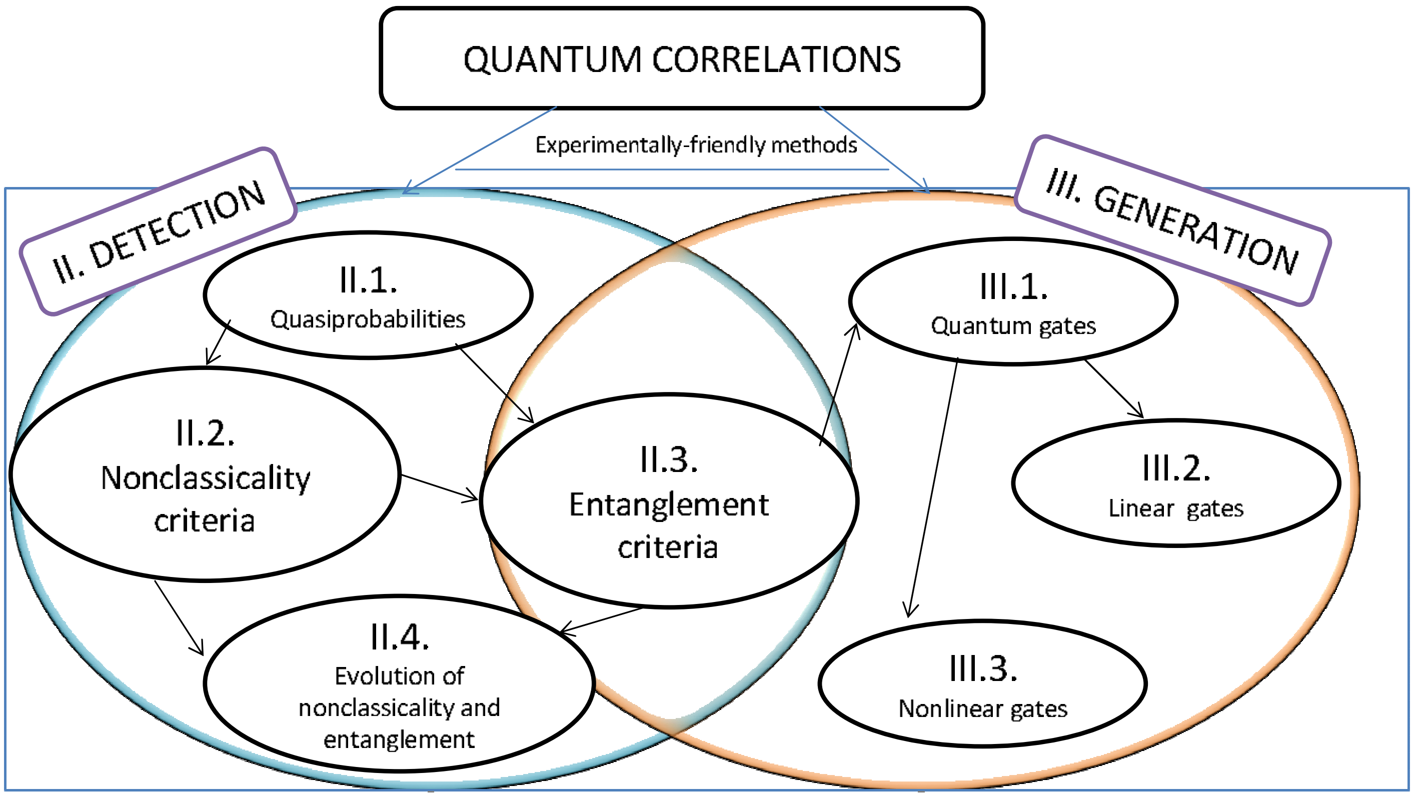


Figure I.1: A diagram presenting the structure of the thesis according to the interrelations between chapters and sections.

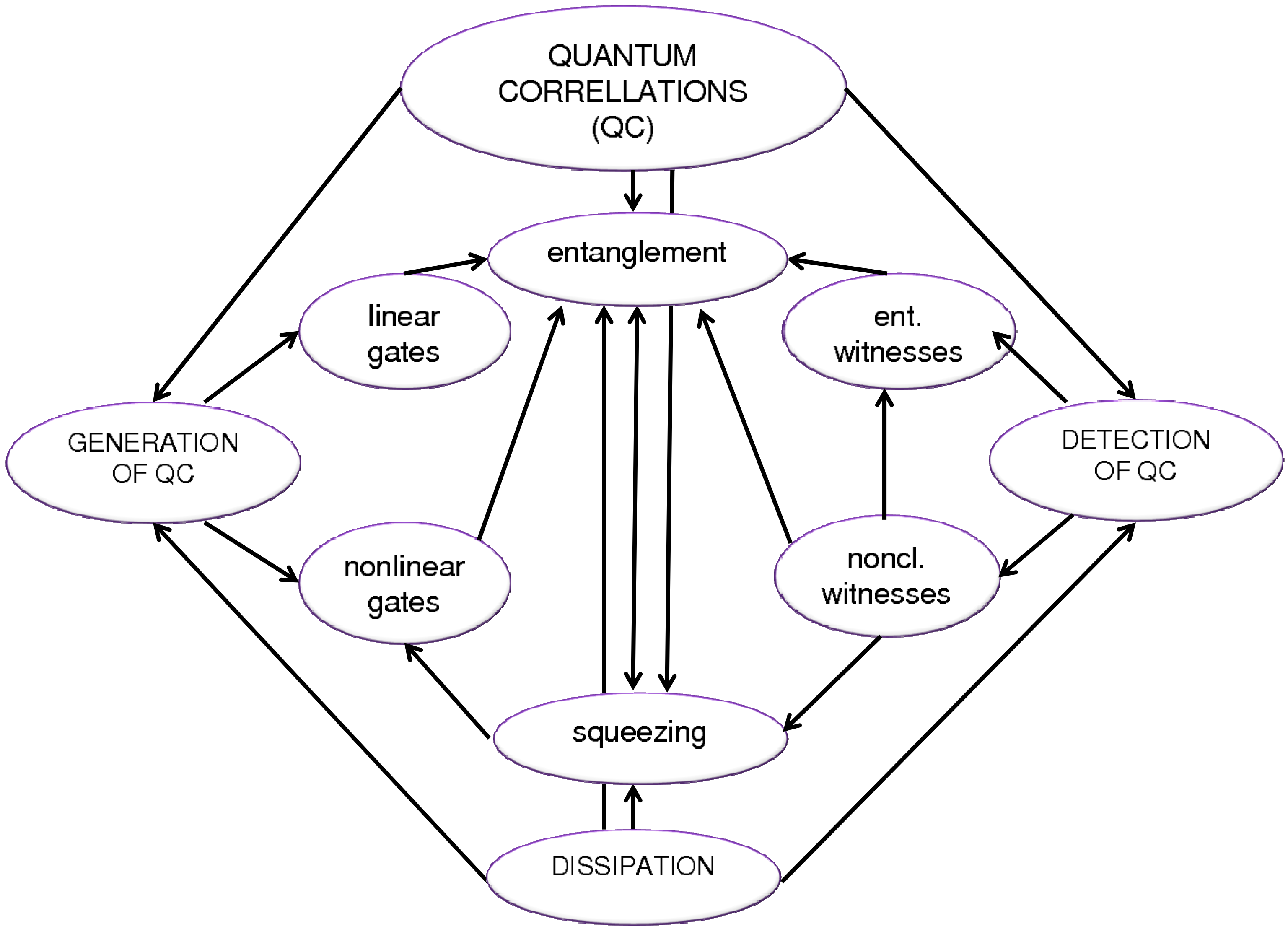
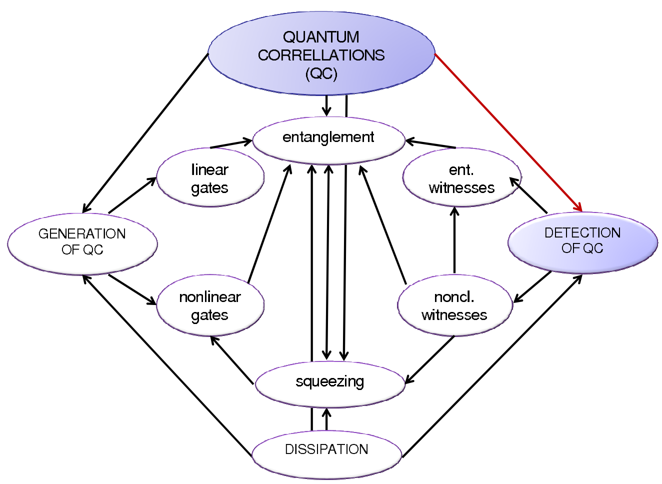


Figure I.2: A diagram presenting relations between the most important terms in the thesis.

Chapter II

Experimentally-friendly methods of the quantum correlations detection

1 Introduction- quasiprobability distributions



A search for an analogy between operators and classic statistical functions has become the basis for the definition of nonclassicality. It has seemed that the calculation of the averages in both manners: of operators as well as the classical phase-space ones, reduces to the integration of functions of classical phase-space variables against the quasidistributions. The analogy, however, should not be led too far. In contrast to the classical statistical physics in the quantum case we cannot define the variables in a phase-space with a complete certainty and match them with the standard probability distribution functions. In 1953 Glauber [11] and Sudarshan [12] proposed a representation of an electromagnetical field which is an explication of the idea of a correspondence between the classical and the quantum world. This representation was firstly formulated for a description of a statistical mixture of coherent states, the states which are the most classical among quantum states as far as an analogy to the classical states of oscillator is considered. The coherent states $|\alpha\rangle$ are eigenvectors of an annihilation operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{II.1}$$

and they are related to the Fock states as follows

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{II.2}$$

where α is an arbitrary complex number $\alpha = |\alpha|e^{i\phi}$. The coherent state has a few properties which enable it to be used in order to define appropriate analogies for classical averages. Mainly:

1. a nonorthogonality: $|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$;
2. a normalization property: $\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|d^2\alpha = 1$;
3. over-completeness (concludes from 1. and 2.), which allows one to find a diagonal representation of an arbitrary state in their basis.

It is possible to define a representation of an M-mode bosonic state $\hat{\rho}$ using the above presented properties of coherent states in the following manner [11, 12]:

$$\hat{\rho} \;=\; \int \mathrm{d}^2\boldsymbol{\alpha}\, P(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|, \tag{II.3}$$

where P is the Glauber-Sudarshan function,

$$\begin{aligned} |\boldsymbol{\alpha}\rangle &= \prod_{m=1}^{M}|\alpha_m\rangle, \\ \mathrm{d}^2\boldsymbol{\alpha} &= \prod_m \mathrm{d}^2\alpha_m, \end{aligned} \tag{II.4}$$

and $|\alpha_m\rangle$ is the mth-mode coherent state, i.e., the eigenstate of the mth-mode annihilation operator $\hat{a}_m$, $\boldsymbol{\alpha}$ denotes the complex multivariable $(\alpha_1, \alpha_2, ..., \alpha_M)$. The density matrix $\hat{\rho}$ can be presented on a tensor product of either infinite-dimensional or finite-dimensional Hilbert spaces. For simplicity, M is assumed to be finite, however one is able to generalize the results for an infinite number of modes. Thus it is possible to define the normally ordered moments of creation and annihilation operators as [48]:

$$\begin{aligned} \langle(\hat{a}^\dagger)^n\hat{a}^m\rangle &= Tr[\hat{\rho}(\hat{a}^\dagger)^n\hat{a}^m] = Tr[\int d^2\boldsymbol{\alpha}|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|P(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)(\hat{a}^\dagger)^n\hat{a}^m] \\ &= \int d^2\boldsymbol{\alpha}P(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)(\alpha^*)^n\alpha^m. \end{aligned} \tag{II.5}$$

From the Eq. (II.5) one can see that the normally ordered average is defined in analogy to the classical statistics with $P(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)$ as a probability function. For $n = 0,\; m = 0$ Eq. (II.5) is a normalization condition for probability. The analogy between the classical and the quantum case, however, needs to be treated with caution. Unlike the classical probability the P-function can be not only negative for some states but also more singular than Dirac's delta (e.g. derivative of Dirac's delta from the Fock states). Therefore, this function (and two other connected with a different kind of order) are called quasidistributions or quasiprobabilities.

A representation in terms of the P-function described above is defined for the normal order. There also exist other methods of finding a quantum-classical correspondence e.g. apart from using the normally ordered averages it is also possible to use an anti-normal or a symmetric order. These three representations can be related with each other through their characteristics functions, which are simply the Fourier transformations (if such exist) of the quasiprobabilities connected with an appropriate order. For the Glauber-Sudarshan distribution, so for a normal order, a characteristic function is as follows

$$\kappa^N(\xi^*,\xi) = Tr[\hat{\rho}e^{i\xi^*\hat{a}^\dagger}e^{i\xi\hat{a}}] = \int d^2\alpha P(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)e^{i\xi^*\alpha^*}e^{i\xi\alpha}. \tag{II.6}$$

Analogously, for the anti-normal order one can define the characteristic function as

$$\kappa^A(\xi^*,\xi) = Tr[\hat{\rho}e^{i\xi\hat{a}}e^{i\xi^*\hat{a}^\dagger}] = \int d^2\alpha Q(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)e^{i\xi^*\alpha^*}e^{i\xi\alpha}, \tag{II.7}$$

where $Q(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)$ is the Husimi function. For the Weyl order (a symmetric order) the characteristic function has the form of

$$\kappa^S(\xi^*,\xi) = Tr[\rho e^{i\xi^*\hat{a}^\dagger+i\xi\hat{a}}] = \int d^2\alpha W(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)e^{i\xi^*\alpha^*}e^{i\xi\alpha}, \tag{II.8}$$

where $W(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)$ is the Wigner function. The averages of moments for an appropriate order can be rewritten in terms of the characteristic functions as

$$\begin{aligned}\langle(\hat{a}^\dagger)^n\hat{a}^m\rangle &= \frac{\partial^{n+m}}{\partial(i\xi^*)^n\partial(i\xi)^m}\chi^N(\xi^*,\xi)\mid_{\xi=\xi^*=0},\\ \langle\hat{a}^m(\hat{a}^\dagger)^n\rangle &= \frac{\partial^{n+m}}{\partial(i\xi)^m\partial(i\xi^*)^n}\chi^A(\xi^*,\xi)\mid_{\xi=\xi^*=0}.\end{aligned} \tag{II.9}$$

The relations between quasidistributions can be derived for the corresponding characteristic functions as follows:

$$\begin{aligned}\chi^S(\xi^*,\xi) &= e^{-\frac{1}{2}|\xi|^2}\chi^N(\xi^*,\xi),\\ \chi^S(\xi^*,\xi) &= e^{\frac{1}{2}|\xi|^2}\chi^A(\xi^*,\xi),\\ \chi^A(\xi^*,\xi) &= e^{-|\xi|^2}\chi^N(\xi^*,\xi).\end{aligned} \tag{II.10}$$

To generalize, the idea of the usage of a different operator ordering enables one to introduce an s-parametrized displacement operator like [5]:

$$\hat{D}(\alpha;s') = \hat{D}(\alpha)e^{\frac{|\alpha|^2 s}{2}} = e^{\frac{1}{2}(s-s')|\alpha|^2}\hat{D}(\alpha;s'). \tag{II.11}$$

For $s=0$ which refers to a symmetric order, one obtains an original displacement operator. For $s=\pm1$ it is possible to describe the other ordering in the following way

$$\begin{aligned}\hat{D}(\alpha;1) &= e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}} = :\hat{D}(\alpha):,\\ \hat{D}(\alpha;-1) &= e^{-\alpha^*\hat{a}}e^{\alpha\hat{a}^\dagger} = +\hat{D}(\alpha)+,\end{aligned} \tag{II.12}$$

where $::$ denotes normal order and $++$ anti-normal order of operators. It is possible to construct a general s-parametrized quasiprobability distribution (QPD) function defined for $-1\le s\le 1$ by [49]:

$$\mathcal{W}^{(s)}(\boldsymbol{\alpha}) = \frac{1}{\pi}\,\mathrm{Tr}\left(\hat{\rho}\prod_{k=1}^{M}\hat{T}^{(s)}(\alpha_k)\right), \tag{II.13}$$

where

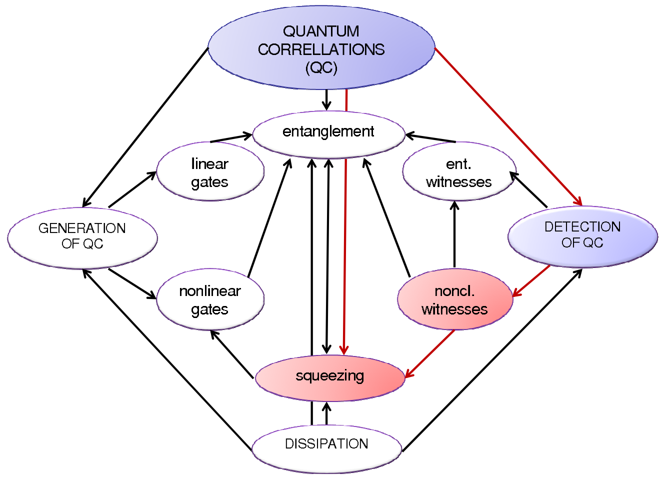
$$\hat{T}^{(s)}(\alpha_k) = \frac{1}{\pi}\int \exp\left(\alpha_k\xi^* - \alpha_k^*\xi + \frac{s}{2}|\xi|^2\right)\hat{D}(\xi)\,\mathrm{d}^2\xi, \tag{II.14}$$

and $\hat{D}(\xi)$ is a displacement operator, $\boldsymbol{\alpha}$ is a complex multivariable $(\alpha_1,\alpha_2,...,\alpha_M)$, and M is a number of modes. In special cases (for $s=1,0,-1$), the QPD reduces to the standard Glauber-Sudarshan P-function, Wigner W-function, and Husimi Q-function, respectively.

In contrary to the P-function and Wigner function, Q-function is non-negative for an arbitrary state. From the comparison of the properties of the P- and Q-functions for a given nonclassical state $\hat{\rho}$ one can see that it is possible to find such value of the parameter $s_0 \in (0, 1]$ for which $\mathcal{W}^{(s_0)}(\hat{\rho})$ can be treated as a classical probability distribution. Due to the critical behaviour of the parameter s_0, it is often considered to be a quantitative measure of nonclassicality of a given state $\hat{\rho}$ [50, 51]. Moreover, also the volume of the negative part of the Wigner function [52] can be treated as an indicator of nonclassicality.

However, there exist states for which the P-function fulfils a condition for nonclassicality and Wigner function is regular and positive (so it behaves like classical probability density) like e.g. a squeezed state. Therefore, in this thesis the P-function is considered to be the most fundamental of QPDs and is believed to justify the basis for construction of the nonclassicality criteria.

2 Nonclassicality criteria as a method to detect quantum correlations



Having defined quasiprobabilities it is possible to introduce a definition of nonclassicality which will be used on pages of this thesis. A construction of nonclassicality criterion is based on properties of the P-function, as it is the most fundamental one from the three showed in Section III.1. The Glauber-Sudarshan function is defined in such a way that it can be reconsidered in analogy to the classical probability distribution for coherent states. Coherent states, being connected with a harmonic oscillator, are the most classical among all the quantum states. The advantage over the Wigner function, which is easier to measure due to its regularity, is that the P-function can detect wider range of nonclassical states e.g. the squeezed states (which are defined by bellowing quantum noise threshold) for which Wigner function is Gaussian and positive. Due to singularity of the P-function, Criterion 1 is not operationally useful as it is extremely difficult (although sometimes possible [53]) to reconstruct the P-function directly from an experimental data. According to the properties and the definition of the P-function one can construct a very general criterion of nonclassicality.

2.1 Definition and criteria for testing nonclassicality

The most intuitive definition, which is also a necessary and a sufficient condition for nonclassicality, can be formulated as follows [54]:

Criterion 1 *A multimode bosonic state $\hat{\rho}$ is considered to be nonclassical if its Glauber-Sudarshan P-function cannot be interpreted as classical probability density, i.e., it is nonpositive or more singular than Dirac's delta function. Conversely, a state is called classical if it is described by the P-function being classical probability density.*

It is worth stressing that recently both conditions in the above Criterion 1 have been equivalent. Lately, Sperling [55] have shown that higher order singularity (in terms of Dirac's delta) is compatible to nonpositivity of the P-function [e.g., given by the nth derivative of $\delta(\alpha)$ for $n = 1, 2, ...$]. However, due to experimental difficulties connected with properties of the P-function it would be useful to reconstruct Criterion 1 in operational and easy to implement terms.

To fulfil this purpose let me construct a countable set $\hat{F} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_i, \ldots)$, that would be possibly infinite, $\hat{f}_i$ would be a function of M-mode operators dependant on creation and annihilation operators [$\hat{f}_i \equiv \hat{f}_i(\hat{\mathbf{a}}, \hat{\mathbf{a}}^\dagger)$, where $\hat{\mathbf{a}} \equiv (\hat{a}_1, \hat{a}_2, ..., \hat{a}_M)$]. In particular, one can construct such a function in a form of monomials

$$\hat{f}_i = \prod_{m=1}^{M} (\hat{a}_m^\dagger)^{i_{2m-1}} \hat{a}_m^{i_{2m}}, \tag{II.15}$$

where i stands for the multi-index $\mathbf{i} \equiv (i_1, i_2, ..., i_{2M})$ or polynomials of creation and annihilation operators. By introducing

$$\hat{f} = \sum_i c_i \hat{f}_i, \tag{II.16}$$

where c_i are arbitrary complex numbers, it is possible to define $\langle : \hat{f}^\dagger \hat{f} : \rangle$ using the P-function in the following manner [7, 56]:

$$\langle : \hat{f}^\dagger \hat{f} : \rangle \;\; = \;\; \int \mathrm{d}^2 \boldsymbol{\alpha} \, |f(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*)|^2 P(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*). \tag{II.17}$$

This average is normally ordered (denoted by ::) what corresponds also to the Shchukin, Richter and Vogel [7, 8] approach. The Shchukin-Richter-Vogel proposal showed hierarchy of operational criteria for detecting nonclassicality of single-mode bosonic states. An infinite set of these criteria (by inclusion of the correction analogous to that given in Ref. [57]) corresponds to a single-mode version of Criterion 1.
Criterion 1 can be reformulated in terms of moments from Eq. (II.17) as follows [7]:

Observation 1 *If the P-function for a given state is a classical probability density, then $\langle : \hat{f}^\dagger \hat{f} : \rangle \geq 0$ for any function $\hat{f}$. Conversely, if $\langle : \hat{f}^\dagger \hat{f} : \rangle < 0$ for some $\hat{f}$, then the P-function is not a classical probability density.*

If one restricts themselves to the two-mode case (which at mostly I am going to analyze in this thesis) and uses monomials of creation and annihilation operators [Eq. (II.16)] they are able to write Eq. (II.17) as

$$\langle : \hat{f}^\dagger \hat{f} : \rangle \;\; = \;\; \sum_{i,j} c_i^* c_j M_{ij}^{(\mathrm{n})}(\hat{\rho}), \tag{II.18}$$

where $M_{ij}^{(\mathrm{n})}$ is a matrix constructed through normally ordered correlation functions

$$M_{ij}^{(\mathrm{n})}(\hat{\rho}) \;\; = \;\; \mathrm{Tr}\,(: \hat{f}_i^\dagger \hat{f}_j : \hat{\rho}). \tag{II.19}$$

The superscript (n) denotes a normal order of field operators. To redefine the criterion and to make it even simpler, one is able to use fixed set of $\hat{F} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_i, \ldots)$ to obtain Hermitian matrix formed by the correlations from Eq. (II.19) in the form of

$$M^{(\mathrm{n})}(\hat{\rho}) = [M_{ij}^{(\mathrm{n})}(\hat{\rho})], \tag{II.20}$$

where

$$M_{ij}^{(\mathrm{n})}(\hat{\rho}) = \mathrm{Tr}\left[: (\hat{a}^{\dagger i_1} \hat{a}^{i_2} \hat{b}^{\dagger i_3} \hat{b}^{i_4})^\dagger (\hat{a}^{\dagger j_1} \hat{a}^{j_2} \hat{b}^{\dagger j_3} \hat{b}^{j_4}) : \hat{\rho} \right] \tag{II.21}$$

with $\hat{a} = \hat{a}_1$ and $\hat{b} = \hat{a}_2$. It is worth noting that there is an efficient optical scheme [10] for measuring correlation functions from Eq. (II.21).

Using the term of matrix $M_{\hat{F}}^{(\mathrm{n})}(\hat{\rho})$ (depending on the choice of $\hat{F}$) it would be possible to generalize single-mode criterion (analogously to the Vogel approach [58]) by applying Sylvester's criterion to the matrix from Eq. (II.20) [59, 57].

Criterion 2 *For any choice of $\hat{F} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_i, \ldots)$, a multimode state $\hat{\rho}$ is nonclassical if there exists a negative principal minor, i.e., $\det[M_{\hat{F}}^{(\mathrm{n})}(\hat{\rho})]_{\mathbf{r}} < 0$, for some $\mathbf{r} \equiv (r_1, \ldots, r_N)$, with $1 \leq r_1 < r_2 < \ldots < r_N$,*

where $[M^{(\mathrm{n})}(\hat{\rho})]_{\mathbf{r}}$ ($\mathbf{r} = (r_1, \ldots, r_N)$) denotes, received from $M_{\hat{F}}^{(\mathrm{n})}(\hat{\rho})$ matrix, $N \times N$ principal submatrix in such a way, that all rows and columns with the exception of the ones labelled by $r_1, \ldots, r_N$, are deleted.

To find a connection between $\langle : \hat{f}^\dagger \hat{f} : \rangle$ and Criterion 2 one can consider a subset $\hat{F}' \subset \hat{F}$ with $\hat{F}' = (\hat{f}_{r_1}, \hat{f}_{r_2}, ..., \hat{f}_{r_N})$, i.e., $[M_{\hat{F}}^{(\mathrm{n})}(\hat{\rho})]_{\mathbf{r}} = M_{\hat{F}'}^{(\mathrm{n})}(\hat{\rho})$. Thus, $[M_{\hat{F}}^{(\mathrm{n})}(\hat{\rho})]_{\mathbf{r}}$ is equivalent to $M_{\hat{F}'}^{\mathrm{n})}(\hat{\rho})$, and can be written as

$$M_{\hat{F}'}^{(\mathrm{n})}(\hat{\rho}) \equiv [M_{\hat{F}}^{(\mathrm{n})}(\hat{\rho})]_{\mathbf{r}} = \begin{pmatrix} \langle : \hat{f}_{r_1}^\dagger \hat{f}_{r_1} : \rangle & \langle : \hat{f}_{r_1}^\dagger \hat{f}_{r_2} : \rangle & \cdots & \langle : \hat{f}_{r_1}^\dagger \hat{f}_{r_N} : \rangle \\ \langle : \hat{f}_{r_2}^\dagger \hat{f}_{r_1} : \rangle & \langle : \hat{f}_{r_2}^\dagger \hat{f}_{r_2} : \rangle & \cdots & \langle : \hat{f}_{r_2}^\dagger \hat{f}_{r_N} : \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle : \hat{f}_{r_N}^\dagger \hat{f}_{r_1} : \rangle & \langle : \hat{f}_{r_N}^\dagger \hat{f}_{r_2} : \rangle & \cdots & \langle : \hat{f}_{r_N}^\dagger \hat{f}_{r_N} : \rangle \end{pmatrix}, \tag{II.22}$$

with the determinant

$$d_{\hat{F}'}^{(\mathrm{n})}(\hat{\rho}) \equiv \det\, M_{\hat{F}'}^{(\mathrm{n})}(\hat{\rho}). \tag{II.23}$$

Using this formulation of matrix from Eq. (II.22), Criterion 2 can be rewritten as [Bartkowiak2010a]:

Criterion 3 *A multimode bosonic state $\hat{\rho}$ is nonclassical if there exists $\hat{F}$, such that $d_{\hat{F}}^{(\mathrm{n})}(\hat{\rho})$ is negative.*

To clarify and emphasize the operational condition for nonclassicality, above Criterion 3 can be formulated in compact shape as

$$\begin{aligned} \hat{\rho} \text{ is classical} \quad &\Rightarrow \quad \forall \hat{F}: \quad d_{\hat{F}}^{(\mathrm{n})}(\hat{\rho}) \geq 0, \\ \hat{\rho} \text{ is nonclassical} \quad &\Leftarrow \quad \exists \hat{F}: \quad d_{\hat{F}}^{(\mathrm{n})}(\hat{\rho}) < 0. \end{aligned} \tag{II.24}$$

In this place I would like to introduce a symbol $\overset{\mathrm{ncl}}{<}$ and $\overset{\mathrm{cl}}{\geq}$, which denote that a given inequality *can* be satisfied only for *nonclassical* states and inequality *must* be satisfied for all *classical* states, respectively.
The procedural recipe for describing Criterion 3 introduced above is

1. choose set of $\hat{F} = (\hat{f}_1, \hat{f}_2, \ldots)$;
2. compute a corresponding matrix $M_{\hat{F}}^{(n)}$;
3. check positivity of its determinant (for $\hat{F}$: $\hat{f} = \sum_i c_i \hat{f}_i$ this point would be equivalent to checking positivity of all $\langle : \hat{f}^\dagger \hat{f} : \rangle$).

Obviously one can notice that adding the operators to set $\hat{F}$ increases a dimension of $M_{\hat{F}'}^{(n)}$ and introduces a hierarchy of criteria. To be more specific, it can be done by choosing $\hat{f}_i$'s which would be more general than monomials, e.g. polynomials. However, it can be easily seen that the criteria based on matrix with $\hat{F}$ with the polynomial expansion are not stronger than those with the monomial ones. Though, the price one needs to pay is an increase in the dimension of matrix $M_{\hat{F}}^{(n)}$. However, considering a more general set of $\hat{F}$ it would be possible to obtain some interesting and physically relevant inequalities straightforwardly (which is shown also in contexts of entanglement criteria in Ref. [57]). In this thesis I am not focusing on hierarchy of criteria. I am mainly interested in studying the possibility of using matrices of expectation values to obtain criteria of nonclassicality. While considering such a hierarchy of criteria one would need to face with a possible singularity of matrices (when one moves to scalar inequalities considering determinants).

Let us now focus on a relation between both Criteria 2 and 3 and the Shchukin, Richter and Vogel criterion in its amended version that takes into account the issue of singular matrices. It is worth emphasizing that, if one denotes by $M_N^{(\mathrm{n})}(\hat{\rho})$ submatrix corresponding to the first N rows and columns of $M^{(\mathrm{n})}(\hat{\rho})$, one can show that Criterion 2 does not reduce to the original Shchukin, Richter and Vogel criterion (Theorem 3 in Ref. [8]), even for single-mode fields and $\hat{f}_i$ given by Eq. (II.15). The Shchukin-Richter-Vogel criterion fails for singular (i.e., $\det M_N^{(\mathrm{n})}(\hat{\rho}) = 0$) matrices of moments (more explanation will be given in case of entanglement). In the original Shchukin-Richter-Vogel criterion a single-mode state is nonclassical if it is possible to find N corresponding to the number of rows and columns of $M^{(\mathrm{n})}(\hat{\rho})$, for which a leading principal minor of $\det M_N^{(\mathrm{n})}(\hat{\rho})$ is negative. As can be seen in a definition of Criterion 3 it can be effectively understood as checking positivity of an infinite matrix $M_{ij}^{(n)}$ (defined in Eq. (II.19)) Thus, it is simply the matrix of $\langle : \hat{f}^\dagger \hat{f} : \rangle$ with $\hat{f}_i$'s chosen as monomials given by Eq. (II.15). Eventually, one can see that Criterion 3 is defined in an operational way and can be written in terms of annihilation and creation operators. Although, to preserve this condition, $M^{(n)}$ needs to be constructed in such a way, that the normal ordering matters. Obviously, this condition depends on an appropriate choice of f_is. However, the dependency of functions f_i on both types of operators (creation and annihilation ones) seems crucial for validity of Criterion 3. Without this dependency it is impossible to obtain nonpositive determinants for some states .

It is important to mention that the above criteria were criticised by e.g., Wünsche [60], who pointed out that:

1. In vicinity of an arbitrary classical state there always exists a nonclassical one. No measurement can distinguish, to arbitrary precision, between the outcomes of such two states (the same situation appears for separable and entanglement states [61] [1]).
2. There exists a state which is quasiclassical, but recognizable using the criteria (in particular Criterion 1) formulated above. For instance squeezing of thermal states cannot result in obtaining straightforwardly nonclassical states.

2.2 Nonclassicality and the Cauchy-Schwarz inequality

Using the criteria formulated above it is possible to define a condition, which is based on classical inequalities and simultaneously gives condition for nonclassicality in agreement with the previously introduced definition. The Cauchy-Schwarz inequality, which can also be used by derivation of uncertainty rule, is an example of such condition and can be written as (see, e.g., Ref. [14]):

$$\langle \hat{A}^\dagger \hat{A} \rangle \langle \hat{B}^\dagger \hat{B} \rangle \quad \geq \quad |\langle \hat{A}^\dagger \hat{B} \rangle|^2, \tag{II.25}$$

where $\hat{A}$ and $\hat{B}$ are arbitrary operators for which the above expectations exist. In analogy to $\langle \hat{A}^\dagger \hat{B} \rangle \equiv \mathrm{Tr}\,(\rho \hat{A}^\dagger \hat{B})$, which is a valid inner product due to positivity of ρ, it is possible to define

[1] It is worth stressing that this is the case only for continuous-variable systems: in the finite dimensional case, the set of separable states has finite volume.

the analogue to inner product for the P-function. It can be done by assigning $\hat{A} = f_1(\mathbf{a}, \mathbf{a}^\dagger)$ and $\hat{B} = f_2(\mathbf{a}, \mathbf{a}^\dagger)$. This way $\langle : \hat{f}_i^\dagger \hat{f}_j : \rangle$ has the following form

$$\langle : \hat{f}_i^\dagger \hat{f}_j : \rangle = \int \mathrm{d}^2\boldsymbol{\alpha}\ f_i^*(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) f_j(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) P(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*). \tag{II.26}$$

Thus, the Cauchy-Schwarz inequality can be written as

$$\langle : \hat{f}_1^\dagger \hat{f}_1 : \rangle \langle : \hat{f}_2^\dagger \hat{f}_2 : \rangle \overset{\mathrm{cl}}{\geq} |\langle : \hat{f}_1^\dagger \hat{f}_2 : \rangle|^2. \tag{II.27}$$

For a given choice of f_1 and f_2 the Cauchy-Schwarz inequality would be violated for a nonclassical field with the nonpositive P-function. This results from the fact that Eq. (II.26) is not actually the scalar product.

In terms of Criterion 3 we can write a violation of the Cauchy-Schwarz inequality for $\hat{F} = (\hat{f}_1, \hat{f}_2)$ with given choice of operators $\hat{f}_1$ and $\hat{f}_2$ as

$$d_{\hat{F}}^{(\mathrm{n})} = \begin{vmatrix} \langle : \hat{f}_1^\dagger \hat{f}_1 : \rangle & \langle : \hat{f}_1^\dagger \hat{f}_2 : \rangle \\ \langle : \hat{f}_1 \hat{f}_2^\dagger : \rangle & \langle : \hat{f}_2^\dagger \hat{f}_2 : \rangle \end{vmatrix} \overset{\mathrm{ncl}}{<} 0. \tag{II.28}$$

2.3 Examples of nonclassicality criteria based on quadrature squeezing conditions

One of the most known states, which is representative for nonclassicality, is a squeezed state. I would like to recall the well known definition of the squeezed states as a group of the states with the minimum uncertainty. It is possible to decrease the noise in one of the two quadratures by obeying the uncertainty relation. By manipulating the squeeze parameter one is able to decrease the minimum variance and increase the maximum one. The squeezed states are characterized by an asymmetric Wigner distribution function which is in agreement with lowering of the noise below quantum limit in one variance (while obeying the uncertainty principle).

Below I analyze a few examples of construction nonclassicality criteria for different kinds of quadrature squeezing. To obtain a general definition of quadrature squeezing with multimode quadrature operators defined as

$$\hat{X}_{\boldsymbol{\phi}} = \sum_{m=1}^{M} c_m\ \hat{x}_m(\phi_m), \tag{II.29}$$

given in terms of single-mode phase-rotated quadratures

$$\hat{x}_m(\phi_m) = \hat{a}_m \exp(i\phi_m) + \hat{a}_m^\dagger \exp(-i\phi_m), \tag{II.30}$$

it is convenient to use a normally ordered variance [5, 62, 63]. Thus, quadrature squeezing of multimode field is present if [64, 65]

$$\langle : (\Delta \hat{X}_{\boldsymbol{\phi}})^2 : \rangle < 0 \tag{II.31}$$

with $\Delta \hat{X}_{\boldsymbol{\phi}} = \hat{X}_{\boldsymbol{\phi}} - \langle \hat{X}_{\boldsymbol{\phi}} \rangle$. In Eq. (II.29), $\boldsymbol{\phi} = (\phi_1, ..., \phi_M)$ and c_m are real parameters. This formulation of quadrature squeezing is valid for all quadratures and orthogonal phases $\hat{x}_m(\phi_m)$ and $\hat{x}_m(\phi_m + \pi/2)$. Usually they are linked to physical systems by identifying $\hat{x}_m(0)$ with the canonical position operator and $\hat{x}_m(\pi/2)$ with the momentum one. It is also common to consider the annihilation ($\hat{a}_m$) and creation ($\hat{a}_m^\dagger$) operators corresponding to slowly-varying operators.

To link quadrature squeezing condition with Criterion 3 one needs to express normally ordered variance in terms of the P-function. It can be done as follows

$$\langle : (\Delta \hat{X}_{\boldsymbol{\phi}})^2 : \rangle = \int \mathrm{d}^2\boldsymbol{\alpha}\ P(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*)[X_{\boldsymbol{\phi}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) - \langle \hat{X}_{\boldsymbol{\phi}} \rangle]^2, \tag{II.32}$$

where

$$X_{\boldsymbol{\phi}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*) = \sum_{m=1}^{M} c_m(\alpha_m e^{i\phi_m} + \alpha_m^* e^{-i\phi_m}) \tag{II.33}$$

and $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_M)$. From the definition in Eq. (II.32) it can be seen that negativity of the P-function in some regions of phase space is implied by the presence of squeezing, so by negative value of $\langle : (\Delta\hat{X}_{\boldsymbol{\phi}})^2 : \rangle$. Thus, the multimode quadrature squeezing is a nonclassical effect. By applying Criterion 3 and choosing $\hat{F} = (1, \hat{X}_{\boldsymbol{\phi}})$ one can come to the same conclusion

$$d_{\hat{F}}^{(\mathrm{n})} = \begin{vmatrix} 1 & \langle \hat{X}_{\boldsymbol{\phi}} \rangle \\ \langle \hat{X}_{\boldsymbol{\phi}} \rangle & \langle : \hat{X}_{\boldsymbol{\phi}}^2 : \rangle \end{vmatrix} = \langle : (\Delta\hat{X}_{\boldsymbol{\phi}})^2 : \rangle \overset{\mathrm{ncl}}{<} 0, \tag{II.34}$$

which is the squeezing condition from Eq. (II.31).

To illustrate the above result one can base their considerations on analysing the two-mode ($M = 2$) case for $c_1 = c_2 = 1$ for which squeezing can be defined as

$$\min_{\boldsymbol{\phi}} \langle : (\Delta\hat{X}_{\boldsymbol{\phi}})^2 : \rangle < 0. \tag{II.35}$$

The optimization over ϕ of the Eq. (II.31) is, in fact, a definition of two-mode *principal* (quadrature) squeezing. The condition of two-mode principal squeezing is already known. Lukš *et al.* [66] showed (by applying the Schrödinger-Robertson indeterminacy relation [67]) that

$$\langle \Delta\hat{a}_{12}^\dagger \Delta\hat{a}_{12} \rangle < |\langle (\Delta\hat{a}_{12})^2 \rangle|, \tag{II.36}$$

where

$$\hat{a}_{12} = \hat{a}_1 + \hat{a}_2, \quad \Delta\hat{a}_{12} = \hat{a}_{12} - \langle \hat{a}_{12} \rangle.$$

This can be easily linked with Criterion 3 via selecting $\hat{F} = (\Delta\hat{a}_{12}^\dagger, \Delta\hat{a}_{12})$:

$$d_{\hat{F}}^{(\mathrm{n})} = \begin{vmatrix} \langle \Delta\hat{a}_{12}^\dagger \Delta\hat{a}_{12} \rangle & \langle (\Delta\hat{a}_{12})^2 \rangle \\ \langle (\Delta\hat{a}_{12}^\dagger)^2 \rangle & \langle \Delta\hat{a}_{12}^\dagger \Delta\hat{a}_{12} \rangle \end{vmatrix} \overset{\mathrm{ncl}}{<} 0 \tag{II.37}$$

or equivalently by choosing $\hat{F} = (1, \hat{a}_{12}^\dagger, \hat{a}_{12})$:

$$d_{\hat{F}}^{(\mathrm{n})} = \begin{vmatrix} 1 & \langle \hat{a}_{12}^\dagger \rangle & \langle \hat{a}_{12} \rangle \\ \langle \hat{a}_{12} \rangle & \langle \hat{n}_{12} \rangle & \langle (\hat{a}_{12})^2 \rangle \\ \langle \hat{a}_{12}^\dagger \rangle & \langle (\hat{a}_{12}^\dagger)^2 \rangle & \langle \hat{n}_{12} \rangle \end{vmatrix}, \tag{II.38}$$

where

$$\hat{n}_{12} = \hat{a}_{12}^\dagger \hat{a}_{12} = \hat{n}_1 + \hat{n}_2 + 2\mathrm{Re}(\hat{a}_1^\dagger \hat{a}_2).$$

The above two determinants, given by Eqs. (II.37) and (II.38), point out an advantage of choosing $\hat{f}_i$s as polynomials over monomials. From this example it can be seen, that the usage of polynomial expansion leads to criteria with matrices of lower dimensions. Both expansions are equivalent but polynomial one is simpler and more intuitive.

Other example that can implement Criterion 3 as criterion for detecting nonclassicality is a two-mode *sum squeezing*. According to Hillery [68] it occurs in the direction ϕ if variance defined as

$$\hat{V}_\phi = \frac{1}{2}(\hat{a}_1 \hat{a}_2 e^{-i\phi} + \hat{a}_1^\dagger \hat{a}_2^\dagger e^{i\phi}) \tag{II.39}$$

fulfils such inequality

$$\langle(\Delta\hat{V}_\phi)^2\rangle \quad < \quad \frac{1}{2}\langle\hat{V}_z\rangle, \tag{II.40}$$

where

$$\hat{V}_z = \frac{1}{2}(\hat{n}_1 + \hat{n}_2 + 1)$$

and $\hat{n}_m = \hat{a}_m^\dagger \hat{a}_m$ for $m = 1, 2$. For $\hat{V}_x = \hat{V}(\phi = 0)$ and $\hat{V}_y = \hat{V}(\phi = \pi/2)$ set of operators $\hat{V}_x$, $(-\hat{V}_y)$ and $\hat{V}_z$ satisfies the commutation relation for generators of SU(1,1) Lie group. Equation (II.40) is justified by the uncertainty relation

$$\langle(\Delta\hat{V}_x)^2\rangle\langle(\Delta\hat{V}_y)^2\rangle \geq \frac{1}{4}\langle\hat{V}_z\rangle^2,$$

which is a straightforward result of properties of the group (to be more specific-commutation relations). In terms of previous examples one can write a condition for sum squeezing as

$$\min\{\langle(\Delta\hat{V}_x)^2\rangle, \langle(\Delta\hat{V}_y)^2\rangle\} < \frac{1}{2}\langle\hat{V}_z\rangle,$$

or more generally as Eq. (II.40).

To connect it with Criterion 3, first, it is worth to notice that by minimizing $\langle(\Delta\hat{V}_\phi)^2\rangle$ over ϕ one can define the principal sum squeezing

$$\min_\phi \langle(\Delta\hat{V}_\phi)^2\rangle < \frac{1}{2}\langle\hat{V}_z\rangle. \tag{II.41}$$

Thus, because it can be seen that

$$\langle(\Delta\hat{V}_\phi)^2\rangle \quad = \quad \langle:(\Delta\hat{V}_\phi)^2:\rangle + \frac{1}{2}\langle\hat{V}_z\rangle, \tag{II.42}$$

the negative value of $\langle:(\Delta\hat{V}_\phi)^2:\rangle$ implies the sum squeezing. Hence, in analogy to previous example, Eqs. (II.40) and (II.41) can be written in terms of Criterion 3 with applying $\hat{F} = (1, \hat{V}_\phi)$ as

$$d_{\hat{F}}^{(\mathrm{n})} = \begin{vmatrix} 1 & \langle\hat{V}_\phi\rangle \\ \langle\hat{V}_\phi\rangle & \langle:\hat{V}_\phi^2:\rangle \end{vmatrix} = \langle:(\Delta\hat{V}_\phi)^2:\rangle \overset{\mathrm{ncl}}{<} 0. \tag{II.43}$$

As a conclusion one can say that in the sense of Criterion 1 the sum squeezing is a nonclassical effect.

A generalization of the above case can be done straightforwardly for any number of modes and leads to subsequent application of presented criteria. Analogously to previous examples, the multimode sum squeezing along the direction ϕ occurs if

$$\langle(\Delta\hat{\mathcal{V}}_\phi)^2\rangle < \frac{|\langle\hat{C}\rangle|}{4}. \tag{II.44}$$

The variance

$$\hat{\mathcal{V}}_\phi = \frac{1}{2}\left(\mathrm{e}^{-i\phi}\prod_j \hat{a}_j + \mathrm{e}^{i\phi}\prod_j \hat{a}_j^\dagger\right) \tag{II.45}$$

is M-mode phase-dependent operator [69], which satisfies the commutation relations

$$[\hat{\mathcal{V}}_\phi, \hat{\mathcal{V}}_{\phi+\pi/2}] = \frac{i}{2}\hat{C}, \quad \hat{C} = \prod_j(1 + \hat{n}_j) - \prod_j \hat{n}_j. \tag{II.46}$$

Hereafter $j = 1, ..., M$, and $|\langle\hat{C}\rangle| = \langle\hat{C}\rangle$. Since

$$\langle(\Delta\hat{\mathcal{V}}_\phi)^2\rangle = \langle:(\Delta\hat{\mathcal{V}}_\phi)^2:\rangle + \frac{|\langle\hat{C}\rangle|}{4}, \tag{II.47}$$

it is practicable to apply Criterion 3 with $\hat{F}=(1,\hat{\mathcal{V}}_\phi)$. In terms of the nonclassicality criteria, the sum squeezing condition can be written, equivalently to Eq. (II.44), as follows

$$\langle:(\Delta\hat{\mathcal{V}}_\phi)^2:\rangle = d^{(\mathrm{n})}_{\hat{F}} \overset{\mathrm{ncl}}{<} 0. \tag{II.48}$$

The next example has been previously defined by Hillery [68], as: the two-mode *difference squeezing* in the direction ϕ occurs if

$$\langle(\Delta\hat{W}_\phi)^2\rangle \quad < \quad \frac{1}{2}|\langle\hat{W}_z\rangle|, \tag{II.49}$$

where

$$\hat{W}_\phi \quad = \quad \frac{1}{2}(\hat{a}_1\hat{a}_2^\dagger e^{i\phi}+\hat{a}_1^\dagger\hat{a}_2 e^{-i\phi}) \tag{II.50}$$

and $\hat{W}_z=\frac{1}{2}(\hat{n}_1-\hat{n}_2)$. Introducing ϕ-optimization, in analogy to the principal quadrature squeezing and the principal sum squeezing, one can define the principal difference squeezing as

$$\min_\phi\langle(\Delta\hat{W}_\phi)^2\rangle < \frac{1}{2}|\langle\hat{W}_z\rangle|. \tag{II.51}$$

The uncertainty relation for $\hat{W}_x=\hat{W}(\phi=0)$, $\hat{W}_y=\hat{W}(\phi=\pi/2)$ and $\hat{W}_z$ is following

$$\langle(\Delta\hat{W}_x)^2\rangle\langle(\Delta\hat{W}_y)^2\rangle \geq (1/4)|\langle\hat{W}_z\rangle|^2, \tag{II.52}$$

as generators $\hat{W}_i$ satisfy commutation relation of SU(2) Lie group (in contradiction to $\hat{V}_i$ operators for sum squeezing). The uncertainty relation for these operators, which justifies defining difference squeezing by Eq. (II.49), has the form of

$$\langle(\Delta\hat{W}_x)^2\rangle\langle(\Delta\hat{W}_y)^2\rangle \geq (1/4)|\langle\hat{W}_z\rangle|^2. \tag{II.53}$$

After defying

$$\langle(\Delta\hat{W}_\phi)^2\rangle = \langle:(\Delta\hat{W}_\phi)^2:\rangle + \frac{1}{4}(\langle\hat{n}_1\rangle+\langle\hat{n}_2\rangle), \tag{II.54}$$

like before in Eq. (II.43), one can obtain Criterion 3 by choosing $\hat{F}=(1,\hat{W}_\phi)$ as

$$d^{(\mathrm{n})}_{\hat{F}} = \langle:(\Delta\hat{W}_\phi)^2:\rangle \overset{\mathrm{ncl}}{<} 0. \tag{II.55}$$

This way the condition for sum squeezing, given by Eq. (II.49), can be reformulated to

$$d^{(\mathrm{n})}_{\hat{F}} < -\frac{1}{2}\min_{i=1,2}\langle\hat{n}_i\rangle. \tag{II.56}$$

It is worth to emphasize that not all states for which difference squeezing occurs are nonclassical. Though, also states which fulfil

$$\frac{1}{4}|\langle\hat{n}_1\rangle-\langle\hat{n}_2\rangle| \leq \langle(\Delta\hat{W}_\phi)^2\rangle < \frac{1}{4}(\langle\hat{n}_1\rangle+\langle\hat{n}_2\rangle) \tag{II.57}$$

are nonclassical even *not* exhibiting difference squeezing. The Equation (II.57) is contrary to squeezing condition given by Eq. (II.49).

Applying the generalization for multimode fields one can write the multimode difference squeezing, which can be defined using the operator [70]:

$$\hat{\mathcal{W}}_\phi = \frac{1}{2}\mathrm{e}^{-i\phi}\prod_{k=1}^{K}\hat{a}_k\prod_{m=K+1}^{M}\hat{a}_m^\dagger + \mathrm{H.c.} \tag{II.58}$$

for any $K < M$ (for simplicity limits of multiplication in $\prod_k$ and $\prod_m$ are omitted) with commutation relations as

$$[\hat{\mathcal{W}}_\phi, \hat{\mathcal{W}}_{\phi+\pi/2}] = \frac{i}{2}\hat{C}, \tag{II.59}$$

where

$$\hat{C} = \prod_k (1+\hat{n}_k) \prod_m \hat{n}_m - \prod_k \hat{n}_k \prod_m (1+\hat{n}_m). \tag{II.60}$$

The above commutation relations imply the following condition for multimode difference squeezing along the direction ϕ [70]:

$$\langle (\Delta\hat{\mathcal{W}}_\phi)^2 \rangle < \frac{|\langle \hat{C} \rangle|}{4}, \tag{II.61}$$

with

$$\langle (\Delta\hat{\mathcal{W}}_\phi)^2 \rangle = \langle : (\Delta\hat{\mathcal{W}}_\phi)^2 : \rangle + \frac{|\langle \hat{D} \rangle|}{4}, \tag{II.62}$$

where

$$\hat{D} = \prod_k (1+\hat{n}_k) \prod_m \hat{n}_m + \prod_k \hat{n}_k \prod_m (1+\hat{n}_m) - 2\prod_{j=1}^{M} \hat{n}_j. \tag{II.63}$$

Criterion 3 can be applied for $\hat{F} = (1, \hat{\mathcal{W}}_\phi)$ as (with agreement to original condition from Eq. (II.61))

$$d_{\hat{F}}^{(\mathrm{n})} = \langle : (\Delta\hat{\mathcal{W}}_\phi)^2 : \rangle < \frac{1}{4}\left(|\langle \hat{C} \rangle| - \langle \hat{D} \rangle \right). \tag{II.64}$$

If $\langle \hat{C} \rangle > 0$ then

$$\hat{C} - \hat{D} = -2\prod_k \hat{n}_k \left(\prod_m (1+\hat{n}_m) - \prod_m \hat{n}_m \right) < 0, \tag{II.65}$$

otherwise

$$\hat{C} - \hat{D} = -2\left(\prod_k (1+\hat{n}_k) - \prod_k \hat{n}_k \right) \prod_m \hat{n}_m < 0. \tag{II.66}$$

Therefore, it can be seen that for states exhibiting difference squeezing, the right-hand side of Eq. (II.64) is negative.

There is a need to enhance the fact that the difference squeezing condition is stronger than the nonclassicality condition $d_{\hat{F}}^{(\mathrm{n})} \overset{\mathrm{ncl}}{<} 0$. Summarizing, one can conclude that the states satisfying inequalities

$$\frac{1}{4}\left(|\langle \hat{C} \rangle| - \langle \hat{D} \rangle \right) \le \langle : (\Delta\hat{\mathcal{W}}_\phi)^2 : \rangle < 0 \tag{II.67}$$

are nonclassical but *not* exhibiting difference squeezing.

2.4 Criteria for some known nonclassical effects

Apart from the squeezed states there also exist other effects in quantum optics, which can be interpreted as nonclassical and for which one can not find the analogy in a classical world, like e.g. photon bunching. The previously defined criteria make it possible to verify nonclassicality also for such cases and compare them with the criteria known for the presence of nonclassical photon-number intermode phenomena in two-mode radiation fields (see, e.g., Refs. [5, 13, **?**, 14, 71, 72, 73, 74]).

In this subsection I have presented a few examples of optical nonclassical effects manifested by single-time and two-time moments. To use Criterion 3 for such examples it is necessary to describe nonclassicality in terms of space-time correlations and the dynamic of radiation sources.

According to Vogel [58] it can be done for Criterion 2 (and the following Criterion 3) by adopting the generalized definition of the P-function

$$P(\boldsymbol{\alpha},\boldsymbol{\alpha}^*) \;=\; \left\langle {}^{\circ}_{\circ}\prod_{i=1}^{M}\delta(\hat{a}_i-\alpha_i){}^{\circ}_{\circ}\right\rangle, \tag{II.68}$$

where $\boldsymbol{\alpha}=(\alpha_1,...,\alpha_M)$, and $\alpha_i=\alpha_i(\mathbf{r}_i,t_i)$ depends on the space-time arguments $(\mathbf{r}_i,t_i)$. Symbol ${}^{\circ}_{\circ}\,{}^{\circ}_{\circ}$ denotes time and normal ordering of field operators i.e., time arguments increase to the right (left) in products of creation (annihilation) operators [5]. Introducing such a definition of P it is possible to formulate nonclassicality of photon antibunching and hyperbunching which are presented below.

The *sub-Poisson statistics* of photons is one of the most known nonclassical effects of quantum light. The photon-number sum/difference sub-Poisson statistics can be achieved by squeezing of the sum ($\hat{n}_+=\hat{n}_1+\hat{n}_2$) or difference ($\hat{n}_-=\hat{n}_1-\hat{n}_2$) of photon numbers respectively [74]. From previous examples it is known that squeezing can lead to nonclassicality criteria. The condition for squeezing of the sum/difference can be formulated as

$$\langle :(\Delta\hat{n}_\pm)^2:\rangle \;<\; 0. \tag{II.69}$$

Applying a definition of mean value in terms of P one can write

$$\langle :(\Delta\hat{n}_\pm)^2:\rangle=\int \mathrm{d}^2\boldsymbol{\alpha}\; P(\boldsymbol{\alpha},\boldsymbol{\alpha}^*)[(|\alpha_1|^2\pm|\alpha_2|^2)-\langle\hat{n}_\pm\rangle]^2, \tag{II.70}$$

where $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$. It can be seen that these phenomena are nonclassical as long as sup-Poisson statistics implies nonpositivity of P. Using Criterion 3 for $\hat{F}_\pm=(1,\hat{n}_\pm)$ it is possible to derive the same result as follows

$$d^{(\mathrm{n})}_{\hat{F}_\pm}=\begin{vmatrix} 1 & \langle\hat{n}_\pm\rangle \\ \langle\hat{n}_\pm\rangle & \langle:\hat{n}_\pm^2:\rangle \end{vmatrix}=\langle :(\Delta\hat{n}_\pm)^2:\rangle \overset{\mathrm{ncl}}{<} 0. \tag{II.71}$$

Analyzing sup-Poisson statistics leads directly to the *photon antibunching* [5, 14, 22, 65, 75] of a stationary or nonstationary single-mode field as a nonclassical phenomenon. In order to define it, one is able to introduce two kinds of quantities:

i) the two-time second-order intensity correlation functions given by

$$G^{(2)}(t,t+\tau)=\langle{}^{\circ}_{\circ}\hat{n}(t)\hat{n}(t+\tau){}^{\circ}_{\circ}\rangle=\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle$$

or

ii) its normalized intensity correlation functions defined as

$$g^{(2)}(t,t+\tau)=\frac{G^{(2)}(t,t+\tau)}{\sqrt{G^{(2)}(t,t)G^{(2)}(t+\tau,t+\tau)}}, \tag{II.72}$$

where ${}^{\circ}_{\circ}\,{}^{\circ}_{\circ}$ denotes the time order and normal order of field operators. One can, therefore, formulate a definition of the antibunching of photons in two manners:

1. It appears if $g^{(2)}(t,t)$ is a strict local minimum at $\tau=0$ for $g^{(2)}(t,t+\tau)$ considered as a function of τ (see, e.g., Refs. [14, 76]):

   $$g^{(2)}(t,t+\tau)>g^{(2)}(t,t). \tag{II.73}$$

   It is worth stressing that this definition reduces to the standard one [5, 14]:

   $$g^{(2)}(\tau)>g^{(2)}(0). \tag{II.74}$$

   if the considered fields are stationary [i.e., those satisfying $G^{(2)}(t,t+\tau)=G^{(2)}(\tau)$ so $g^{(2)}(t,t+\tau)=g^{(2)}(\tau)$].

2. Defined in Eq. (II.73) photon antibunching can be formulated as a violation of the Cauchy-Schwarz inequality

$$G^{(2)}(t,t)G^{(2)}(t+\tau,t+\tau) \overset{\rm cl}{\geq} \left[G^{(2)}(t,t+\tau)\right]^2. \tag{II.75}$$

It is worth to emphasizing that in accordance with a definition 1) *photon bunching* appears for decreasing of $g^{(2)}(t,t+\tau)$ and, in contrary, *photon unbunching* appears when $g^{(2)}(t,t+\tau)$ is locally constant.

Criterion 3 can be formulated for these phenomena via the usage of the generalized P function from Eq. (II.68) with $\hat{F} = (\hat{n}(t), \hat{n}(t+\tau))$ as [58]:

$$d_{\hat{F}}^{(\rm n)} = \begin{vmatrix} \langle {}^{\circ}_{\circ}\hat{n}^2(t){}^{\circ}_{\circ}\rangle & \langle {}^{\circ}_{\circ}\hat{n}(t)\hat{n}(t+\tau){}^{\circ}_{\circ}\rangle \\ \langle {}^{\circ}_{\circ}\hat{n}(t)\hat{n}(t+\tau){}^{\circ}_{\circ}\rangle & \langle {}^{\circ}_{\circ}\hat{n}^2(t+\tau){}^{\circ}_{\circ}\rangle \end{vmatrix} = \begin{vmatrix} G^{(2)}(t,t) & G^{(2)}(t,t+\tau) \\ G^{(2)}(t,t+\tau) & G^{(2)}(t+\tau,t+\tau) \end{vmatrix} \overset{\rm ncl}{<} 0.$$

For nonstationary fields one can define (also referred to a photon antibunching effect [77]) *photon hyperbunching* [78] as:

$$\overline{g}^{(2)}(t,t+\tau) > \overline{g}^{(2)}(t,t). \tag{II.76}$$

Inequality from Eq. (II.76) is written in terms of the correlation coefficient [79]:

$$\overline{g}^{(2)}(t,t+\tau) = \frac{\overline{G}^{(2)}(t,t+\tau)}{\sqrt{\overline{G}^{(2)}(t,t)\overline{G}^{(2)}(t+\tau,t+\tau)}}, \tag{II.77}$$

and covariance $\overline{G}^{(2)}(t,t+\tau)$ is defined as

$$\overline{G}^{(2)}(t,t+\tau) = G^{(2)}(t,t+\tau) - G^{(1)}(t)G^{(1)}(t+\tau), \tag{II.78}$$

where

$$G^{(1)}(t) = \langle \hat{n}(t) \rangle = \langle \hat{a}^{\dagger}(t)\hat{a}(t) \rangle \tag{II.79}$$

refers to intensity of light. For *stationary* fields, the inequalities definitions given by Eqs. (II.73) and (II.76) are equivalent. They are also equivalent to the formulation of the photon antibunching in terms of other normalized correlation functions, e.g.,

$$\tilde{g}^{(2)}(t,t+\tau) = \frac{G^{(2)}(t,t+\tau)}{[G^{(1)}(t)]^2}. \tag{II.80}$$

Nevertheless, there is still a need to emphasize the fact that for *nonstationary* fields these two definitions are interpreted as two different photon antibunching effects [76, 77, 78].

By evoking the Cauchy-Schwarz inequality it is possible to write inequality which would be violated for the fields manifesting photon hyperbunching (defined in Eq. (II.76)) as

$$\overline{G}^{(2)}(t,t)\overline{G}^{(2)}(t+\tau,t+\tau) \overset{\rm cl}{\geq} \left[\overline{G}^{(2)}(t,t+\tau)\right]^2. \tag{II.81}$$

On the other hand Criterion 3 can be applied by defying $\hat{F} = (\Delta\hat{n}(t), \Delta\hat{n}(t+\tau))$, where $\Delta\hat{n}(t) = \hat{n}(t) - \langle\hat{n}(t)\rangle$ as

$$d_{\hat{F}}^{(\rm n)} = \begin{vmatrix} \overline{G}^{(2)}(t,t) & \overline{G}^{(2)}(t,t+\tau) \\ \overline{G}^{(2)}(t,t+\tau) & \overline{G}^{(2)}(t+\tau,t+\tau) \end{vmatrix} \overset{\rm ncl}{<} 0. \tag{II.82}$$

This leads to a condition equivalent to Eq. (II.76). Alternatively, it can be done using $\hat{F} = (1, \hat{n}(t), \hat{n}(t+\tau))$ (in agreement with determinant given by Eq. (II.82)):

$$d_{\hat{F}}^{(\mathrm{n})} = \begin{vmatrix} 1 & \langle \hat{n}(t) \rangle & \langle \hat{n}(t+\tau) \rangle \\ \langle \hat{n}(t) \rangle & \langle {}^{\circ}_{\circ} \hat{n}^2(t) {}^{\circ}_{\circ} \rangle & \langle {}^{\circ}_{\circ} \hat{n}(t)\hat{n}(t+\tau) {}^{\circ}_{\circ} \rangle \\ \langle \hat{n}(t+\tau) \rangle & \langle {}^{\circ}_{\circ} \hat{n}(t)\hat{n}(t+\tau) {}^{\circ}_{\circ} \rangle & \langle {}^{\circ}_{\circ} \hat{n}^2(t+\tau) {}^{\circ}_{\circ} \rangle \end{vmatrix}. \tag{II.83}$$

Hyperbunching (beside of Eqs. (II.37) and (II.38)) is another example of the advantage of using polynomials over monomial functions of moments in $\hat{F}$ (it can be easily seen by comparing Eqs. (II.82) and (II.83)). However, there is a need to emphasize the fact that antibunching defined by $\langle : (\Delta \hat{n})^2 : \rangle < 0$ refers to *single-mode sub-Poisson* photon-number statistics. This phenomenon has different effects from those presented above (Eqs. (II.73) and (II.76)), as shown by examples in Ref. [80]).

Profiting from the usage of the Cauchy-Schwarz inequality as a basic inequality for testing nonclassicality one can find examples of applying Criterion 3 for two modes of the same evolution time or single-mode for different evolution times (in relation to photon antibunching and hyperbunching). For the same evolution time but two-modes it is possible to write the following inequality (based on the violation of the Cauchy-Schwarz inequality)

$$\langle : \hat{n}_1^2 : \rangle \langle : \hat{n}_2^2 : \rangle \quad \overset{\mathrm{cl}}{\geq} \quad \langle \hat{n}_1 \hat{n}_2 \rangle^2. \tag{II.84}$$

Such effect can be described by a parameter introduced by Agarwal [72] defined as

$$I_{12} = \frac{\sqrt{\langle : \hat{n}_1^2 : \rangle \langle : \hat{n}_2^2 : \rangle}}{\langle \hat{n}_1 \hat{n}_2 \rangle} - 1. \tag{II.85}$$

It can be seen that nonclassicality occurs for the negative value of I_{12}. The Equation (II.85) corresponds to Criterion 3 with chosen $\hat{F}$ as $\hat{F} = (\hat{n}_1, \hat{n}_2)$ in the following way

$$d_{\hat{F}}^{(\mathrm{n})} \quad = \quad \begin{vmatrix} \langle : \hat{n}_1^2 : \rangle & \langle \hat{n}_1 \hat{n}_2 \rangle \\ \langle \hat{n}_1 \hat{n}_2 \rangle & \langle : \hat{n}_2^2 : \rangle \end{vmatrix} \overset{\mathrm{ncl}}{<} 0. \tag{II.86}$$

For the single-mode but different times case one can formulate the following condition

$$D_{12} = \langle : \hat{n}_1^2 : \rangle + \langle : \hat{n}_2^2 : \rangle - 2\langle \hat{n}_1 \hat{n}_2 \rangle \overset{\mathrm{cl}}{\geq} 0. \tag{II.87}$$

This inequality (Eq.(II.87)) was formulated by Muirhead [81] as a generalization of the arithmetic-geometric mean inequality and reformulated to the shape presented in Eq. (II.87) by Lee [73]. By applying Criterion 3 with $\hat{F} = (\hat{n}_1 - \hat{n}_2) \equiv (\hat{n}_-)$ one obtains

$$D_{12} = \langle : \hat{n}_-^2 : \rangle \overset{\mathrm{ncl}}{<} 0. \tag{II.88}$$

It is also interesting to analyze the condition arising from the choice of $\hat{F} = (1, \hat{n}_-)$, as it yields to

$$d_{\hat{F}}^{(\mathrm{n})} = \langle : \hat{n}_-^2 : \rangle - \langle \hat{n}_- \rangle^2 \overset{\mathrm{cl}}{\geq} 0. \tag{II.89}$$

To clarify, one can write a simple condition

$$D_{12} < 0 \Rightarrow d_{\hat{F}}^{(\mathrm{n})} \overset{\mathrm{ncl}}{<} 0. \tag{II.90}$$

It is important to stress the fact that the condition given by Eq. (II.89) make it possible to distinguish more nonclassical states than the one with D_{12} parameter. .

2.5 Other examples of usage of the nonclassicality criteria

Criterion 3 allows one to construct a huge amount of variety nonclassicality criteria linked with physical phenomena, as can be seen in previous Subsections II.2.3 and II.2.4. Table II.1 contains different examples of multimode nonclassicality conditions derived from Criterion 3. Below, some simple examples of the application of Criterion 3 will be presented which, for my knowledge, have not been previously introduced in literature. To be more specific, inequalities are limited to the ones based on particularly defined determinants

$$D(x,y,z) \quad = \quad \begin{vmatrix} 1 & x & x^* \\ x^* & z & y^* \\ x & y & z \end{vmatrix}. \tag{II.91}$$

In such a case, Criterion 3 can be applied for a different choice of $\hat{F}$
(i) for $\hat{F} = (1, \hat{a}_1\hat{a}_2, \hat{a}_1^\dagger\hat{a}_2^\dagger)$, it leads to

$$d_{\hat{F}}^{(\mathrm{n})} = D(\langle\hat{a}_1\hat{a}_2\rangle, \langle\hat{a}_1^2\hat{a}_2^2\rangle, \langle\hat{n}_1\hat{n}_2\rangle) \overset{\mathrm{ncl}}{<} 0, \tag{II.92}$$

where $\hat{n}_1 = \hat{a}_1^\dagger\hat{a}_1$ and $\hat{n}_2 = \hat{a}_2^\dagger\hat{a}_2$.
(ii) For $\hat{F} = (1, \hat{a}_1\hat{a}_2^\dagger, \hat{a}_1^\dagger\hat{a}_2)$ a new condition can be written as

$$d_{\hat{F}}^{(\mathrm{n})} = D(\langle\hat{a}_1\hat{a}_2^\dagger\rangle, \langle\hat{a}_1^2(\hat{a}_2^\dagger)^2\rangle, \langle\hat{n}_1\hat{n}_2\rangle) \overset{\mathrm{ncl}}{<} 0. \tag{II.93}$$

(iii) For $\hat{F} = (1, \hat{a}_1 + \hat{a}_2^\dagger, \hat{a}_1^\dagger + \hat{a}_2)$, one obtains

$$d_{\hat{F}}^{(\mathrm{n})} = D(\langle\hat{a}_1 + \hat{a}_2^\dagger\rangle, \langle(\hat{a}_1 + \hat{a}_2^\dagger)^2\rangle, z) \overset{\mathrm{ncl}}{<} 0, \tag{II.94}$$

where $z = \langle\hat{n}_1\rangle + \langle\hat{n}_2\rangle + 2\mathrm{Re}\langle\hat{a}_1\hat{a}_2\rangle$.
(iv) For $\hat{F} = (1, \hat{a}_1 + \hat{a}_2, \hat{a}_1^\dagger + \hat{a}_2^\dagger)$ Criterion 3 gives

$$d_{\hat{F}}^{(\mathrm{n})} = D(\langle\hat{a}_1 + \hat{a}_2\rangle, \langle(\hat{a}_1 + \hat{a}_2)^2\rangle, z) \overset{\mathrm{ncl}}{<} 0, \tag{II.95}$$

where $z = \langle\hat{n}_1\rangle + \langle\hat{n}_2\rangle + 2\mathrm{Re}\langle\hat{a}_1\hat{a}_2^\dagger\rangle$.
What makes these nonclassicality criteria (Eqs. (II.92)–(II.95)) even more interesting is the fact that, as can be seen in the Subsection II.3.3, one can link them with the entanglement criteria. Using Criterion 3 it is also possible to connect appropriate nonclassicality condition choosing $\hat{F} = (1, \hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2^\dagger, \hat{a}_2)$:

$$d_{\hat{F}}^{(\mathrm{n})} = \begin{vmatrix} 1 & \langle\hat{a}_1\rangle & \langle\hat{a}_1^\dagger\rangle & \langle\hat{a}_2^\dagger\rangle & \langle\hat{a}_2\rangle \\ \langle\hat{a}_1^\dagger\rangle & \langle\hat{a}_1^\dagger\hat{a}_1\rangle & \langle(\hat{a}_1^\dagger)^2\rangle & \langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle & \langle\hat{a}_1^\dagger\hat{a}_2\rangle \\ \langle\hat{a}_1\rangle & \langle\hat{a}_1^2\rangle & \langle\hat{a}_1^\dagger\hat{a}_1\rangle & \langle\hat{a}_1\hat{a}_2^\dagger\rangle & \langle\hat{a}_1\hat{a}_2\rangle \\ \langle\hat{a}_2\rangle & \langle\hat{a}_1\hat{a}_2\rangle & \langle\hat{a}_1^\dagger\hat{a}_2\rangle & \langle\hat{a}_2^\dagger\hat{a}_2\rangle & \langle\hat{a}_2^2\rangle \\ \langle\hat{a}_2^\dagger\rangle & \langle\hat{a}_1\hat{a}_2^\dagger\rangle & \langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle & \langle(\hat{a}_2^\dagger)^2\rangle & \langle\hat{a}_2^\dagger\hat{a}_2\rangle \end{vmatrix} \overset{\mathrm{ncl}}{<} 0 \tag{II.96}$$

with the Simon entanglement criterion [46] (this case is discussed in Subsection II.3.3) .

Table II.1: Nonclassicality criteria for single-time effects in two-mode (TM) and multimode (MM) fields, and two-time effects in single-mode (SM) fields [Bartkowiak2010a] .

Nonclassical effect	**Criterion**	Equations
MM quadrature squeezing	$d^{(\mathrm{n})}(1, \hat{X}_{\boldsymbol{\phi}}) < 0$	(II.31), (II.34)
TM principal squeezing of Lukš *et al.* [66]	$d^{(\mathrm{n})}(\Delta\hat{a}_{12}^{\dagger}, \Delta\hat{a}_{12}) = d^{(\mathrm{n})}(1, \hat{a}_{12}^{\dagger}, \hat{a}_{12}) < 0$	(II.35)–(II.38)
TM sum squeezing of Hillery [68]	$d^{(\mathrm{n})}(1, \hat{V}_{\phi}) < 0$	(II.40), (II.43)
MM sum squeezing of An-Tinh [69]	$d^{(\mathrm{n})}(1, \hat{\mathcal{V}}_{\phi}) < 0$	(II.44), (II.48)
TM difference squeezing of Hillery [68]	$d^{(\mathrm{n})}(1, \hat{W}_{\phi}) < -\frac{1}{2}\min\left(\langle\hat{n}_1\rangle, \langle\hat{n}_2\rangle\right)$	(II.49), (II.55), (II.56)
MM difference squeezing of An-Tinh [70]	$d^{(\mathrm{n})}(1, \hat{\mathcal{W}}_{\phi}) < -\frac{1}{4}\left\| \|\langle\hat{C}\rangle\| - \langle\hat{D}\rangle \right\|$	(II.61), (II.64)
TM sub-Poisson photon-number correlations	$d^{(\mathrm{n})}(1, \hat{n}_1 \pm \hat{n}_2) < 0$	(II.69), (II.71)
Cauchy-Schwarz inequality violation	$d^{(\mathrm{n})}(\hat{f}_1, \hat{f}_2) < 0$	(II.27), (II.28)
TM Cauchy-Schwarz inequality violation via Agarwal's test [72]	$d^{(\mathrm{n})}(\hat{n}_1, \hat{n}_2) < 0$	(II.84), (II.86)
TM Muirhead inequality violation via Lee's test [73]	$d^{(\mathrm{n})}(\hat{n}_1 - \hat{n}_2) < 0$	(II.87), (II.88)
SM photon antibunching	$d^{(\mathrm{n})}[\hat{n}(t), \hat{n}(t+\tau)] < 0$	(II.73), (II.76)
SM photon hyperbunching	$d^{(\mathrm{n})}[\Delta\hat{n}(t), \Delta\hat{n}(t+\tau)]$ $= d^{(\mathrm{n})}[1, \hat{n}(t), \hat{n}(t+\tau)] < 0$	(II.76), (II.82), (II.83)
Other TM nonclassical effects	$d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2, \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}) < 0$	(II.92)
	$d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2^{\dagger}, \hat{a}_1^{\dagger}\hat{a}_2) < 0$	(II.93)
	$d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^{\dagger}, \hat{a}_1^{\dagger} + \hat{a}_2) < 0$	(II.94)
	$d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2, \hat{a}_1^{\dagger} + \hat{a}_2^{\dagger}) < 0$	(II.95)
	$d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_1^{\dagger}, \hat{a}_2^{\dagger}, \hat{a}_2) < 0$	(II.96)

2.6 How to construct nonclassicality witness

To use the criteria effectively in order to analyze the behaviour of nonclassicality in various physical systems one can construct quantities, which would be sensitive for breaking classical inequalities- witnesses of nonclassicality. However, for this purpose it would be useful to describe the method of construction of such witnesses in terms of previously defined criteria. Nonclassicality witness can be defined in the following way [82]:
Let $\hat{O}$ be an operator, where expectation value is nonnegative for all classical states ρ_{cl}

$$\langle\hat{O}\rangle_{cl} = Tr[\rho_{cl}\hat{O}] \geq 0. \tag{II.97}$$

If $\langle\hat{O}\rangle$ for some arbitrary state ρ is negative

$$\langle\hat{O}\rangle = Tr[\rho\hat{O}] < 0, \tag{II.98}$$

the state ρ is nonclassical and the operator $\hat{O}$ can be called nonclassicality witness.

To construct nonclassicality witnesses I make use of the method proposed in Refs. [7, 56] and developed in Refs. [58, 83]. One can also benefit from the proposal of Alicki *et al.* [84, 85, 86]. As it can be directly seen from the formulation of Criterion 1 and 2 the normally-ordered operator $:\hat{f}^\dagger\hat{f}:$ can be interpreted as nonclassicality witness [56]. To simplify, in both cases (nonclassicality and later entanglement witnesses) witness will refer also to expectation value of operator. It should be stressed that the term *witness of nonclassicality* is not limited only to operators (see, e.g., Refs. [86, 87]). By evoking Criterion 3 it is possible to define nonclassicality witness as matrices of normally-ordered moments $M_{\hat{f}}^{(\mathrm{n})}(\hat{\rho})$ and their functions (e.g., determinants). However, to unify the form of nonclassicality, entanglement witnesses and entanglement measures their definition is reformulated and the following recipe for their construction is given:

1. Find an appropriate nonclassicality witness $\hat{O}$ based on Criterion 3;
2. Do the truncation of $\hat{O}$ in the following way

$$O \rightarrow \tilde{O} = \max(0, O_0 - O), \tag{II.99}$$

   where O_0 is some threshold value.

In this thesis O and $\tilde{O}$ denotes the untruncated and truncated *nonclassicality witnesses*, respectively.

Motivation behind such a redefinition was finding similarity between the form of a such witness an entanglement measures e.g. concurrence or negativity, to be able to compare them (what can be seen in the following Section II.4). Concurrence for two-qubit system described by $\hat{\rho}$ is defined as [88]:

$$C(\hat{\rho}) = \max\Big(0, 2\max_i \lambda_i - \sum_i \lambda_i\Big), \tag{II.100}$$

where the λ_i's are the square roots of the eigenvalues of $\hat{\rho}(\hat{\sigma}_2\otimes\hat{\sigma}_2)\hat{\rho}^*(\hat{\sigma}_2\otimes\hat{\sigma}_2)$ and $\hat{\sigma}_2$ is the Pauli spin matrix. In turn, negativity can be formulated as [36, 37]:

$$N(\hat{\rho}) = \max\Big(0, -2\min_j \mu_j\Big), \tag{II.101}$$

where μ_j's are the eigenvalues of the partial transpose $\hat{\rho}^\Gamma$ and factor 2 is chosen for proper scaling, i.e., to obtain $N(\hat{\rho}) = 1$ for Bell's states. Despite similarity of the formal definition, it is important to emphasize that nonclassicality witnesses are, in general, equivalent neither to entanglement witnesses (which will be shown in next Subsection II.3.4) nor the more to entanglement measures. In general entanglement witnesses are also nonclassical ones but not the other way around.

One of the best nontrivial examples of nonclassicality witnesses, which is not necessary entanglement witness and which have already appeared in context of nonclassicality, is squeezing. In this case is analyzed the squeezing (or sub-Poisson statistics) of the photon-number difference $(\hat{n}_1 - \hat{n}_2)$ in two systems. This takes place when the normally-ordered variance

$$S = \langle : [\Delta(\hat{n}_1 - \hat{n}_2)]^2 : \rangle \tag{II.102}$$

is negative. Any field for which $S \overset{\text{cl}}{\geq} 0$ is obviously the classical one as far as squeezing as the nonclassical effect is considered. It is worth stressing that for the arbitrarily chosen $S_0 \geq 0$ also $S + S_0 \overset{\text{cl}}{\geq} 0$ is true for classical fields. Using this quantity one can apply the recipe formulated above and construct truncated nonclassicality witness as

$$\tilde{S} = \max\left(0, -\langle : [\Delta(\hat{n}_1 - \hat{n}_2)]^2 : \rangle - S_0\right) \overset{\text{ncl}}{>} 0. \tag{II.103}$$

Another example can be obtained simply by replacing $\Delta(\hat{n}_1 - \hat{n}_2)$ with $(\hat{n}_1 - \hat{n}_2)$ in Eq. (II.102).This way another normally-ordered witness $\tilde{D}'$ resulting from the classical inequality is derived

$$D' = \langle : (c_1\hat{n}_1 + c_2\hat{n}_2 + c_3)^2 : \rangle + |c_4|^2 \overset{\text{cl}}{\geq} 0, \tag{II.104}$$

where c_k ($k = 1, 2, 3, 4$) are real parameters. After truncation the witness has the following form

$$\tilde{D} = \max(0, -\langle : (\hat{n}_1 - \hat{n}_2 + D_0)^2 : \rangle) \overset{\text{ncl}}{>} 0, \tag{II.105}$$

which is a special case of $\tilde{D}'$ for $(c_1, c_2, c_3, c_4) = (1, -1, D_0, 0)$.

Obviously, Criterion 3 allows one to construct not only two-mode witnesses but also single-mode or multi-modes ones. To be consistent, also examples based on two kinds of squeezing have been shown: the photon-number and the quadrature one. For single- mode case one can use the Mandel's Q-parameter,

$$Q = \frac{\langle : (\Delta\hat{n})^2 : \rangle}{\langle : \hat{n} : \rangle}.$$

Its negativity is the manifestation of single-mode photon-number squeezing (also called sub-Poisson photon-number statistics). It can be easily obtained from Criterion 3. Truncated witness has the form of

$$\tilde{Q} = \max\left(0, -\frac{\langle : (\Delta\hat{n})^2 : \rangle}{\langle : \hat{n} : \rangle}\right) \overset{\text{ncl}}{>} 0. \tag{II.106}$$

For the M-mode case the quadrature squeezing is formulated as follows (the standard, $S_0 = 0$, and the strong one, $S_0 > 0$)

$$S_{x_\phi} = \langle : (\Delta\hat{x}_{\boldsymbol{\phi}})^2 : \rangle \overset{\text{ncl}}{<} (-S_0). \tag{II.107}$$

After truncation one obtains

$$\tilde{S}_{X_\phi} = \max(0, -\langle : (\Delta\hat{X}_{\boldsymbol{\phi}})^2 : \rangle - S_0) \overset{\text{ncl}}{>} 0, \tag{II.108}$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2, ..., \phi_M)$. The multimode quadrature operator is defined by Eq.(II.29). Such formulated witness $\tilde{S}_{x_\phi}$, can also be applied to a single-mode case. The principal squeezing (ϕ-optimized quadrature squeezing) defined by [66, 83]:

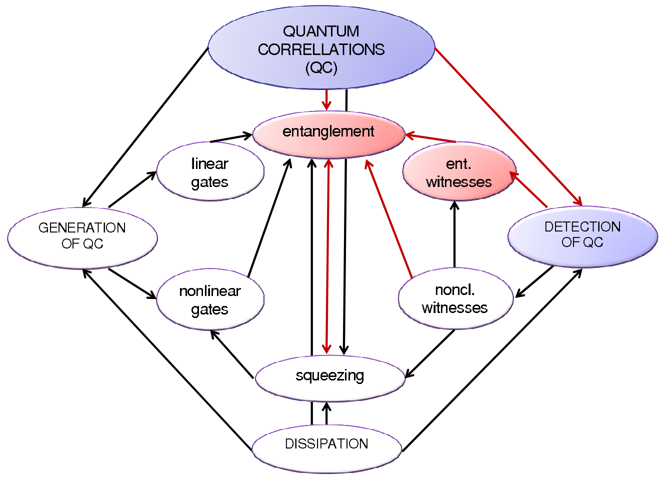
$$S_{\text{opt}} = \min_{\boldsymbol{\phi}} S_{x_\phi} \overset{\text{ncl}}{<} 0, \tag{II.109}$$

can lead to the following truncated witness

$$\tilde{S}_{\rm opt} = \max(0, -S_{\rm opt} - S_0) = \max_{\boldsymbol{\phi}} \tilde{S}_{x_\phi} \overset{\rm ncl}{>} 0. \tag{II.110}$$

In terms of violation of classical inequalities, one is able to construct many other two- and multi-mode nonclassicality witnesses. Explicit examples can be found in, e.g., Refs. [5, 56, 71, 76, 77, 89, 83, 90].

3 Entanglement as a quantum correlation



3.1 A definition of entanglement and the Shchukin-Vogel entanglement criterion

One of the most known quantum correlation is entanglement. From the previous Section II.2 one can see that nonclassicality is a wider term than entanglement. However, due to the significance of entanglement in quantum information protocols, I have decided to put the analyses concerning entanglement into a separate section. First, a formal definition of entanglement in terms of nonseparability will be given. It allows one to formulate criteria of entanglement in a similar shape as Criterion 3 and also makes it possible to construct entanglement witnesses which can be analyzed with nonclassicality and entanglement measures. The connection between nonseparability and entanglement was suggested by Werner in Ref. [91]:
If $\hat{\rho}_1$ and $\hat{\rho}_2$ are density operators for two modes, bipartite state described by $\hat{\rho}$ is *separable* if we can factorized it as

$$\hat{\rho} = \hat{\rho}_1 \otimes \hat{\rho}_2.$$

For a mixed state this can be rewritten to the following form:
The state described by the $\hat{\rho}_{\rm mix}$ is *separable* if it can be factorized as:

$$\hat{\rho}_{\rm mix} = \sum_k p_k \hat{\rho}_1^{(k)} \otimes \hat{\rho}_2^{(k)}.$$

The state, pure or mixed, which cannot be factorized in this way is *inseparable* or, in other words, *entangled*.

It is also important that entanglement states are also nonclassical according to a definition from Criterion 1. Although, this relation is not both-sided. Nonclassicality is a wider term in sense that one can find states which are separable but still nonclassical.

In order to detect separability of states it is common to use partial transposition, which for inseparable states introduces negative eigenvalues in density matrix. Before introducing a definition of partial transposition it is worth to recall a definition of the standard one. If T denotes transposition of an arbitrary operator, e.g, defined in Fock basis $|n\rangle$ for $n = 0, 1, ...$, $\hat{A} = |m\rangle\langle n|$, the action of T on $\hat{A}$ can be written as

$$T(|m\rangle\langle n|) = |n\rangle\langle m|.$$

Due to the properties of transposition (preserving the trace, hermicity and positivity of transposed operator) the trace of transposed density matrix should give 1 also for the “new" quantum state. For a bipartite state one can now define partial transposition (PT) as (with respect to the first mode)

$$\hat{A}^{\Gamma} = (T \otimes \mathrm{id})(\hat{A}),$$

with T denoting transposition acting on the first mode and id the identity operation effectively doing nothing on the remaining modes, respectively. What is important in this case is that partial transposition preserves separability of a state. For a separable bipartite state after PT one obtains a physical density matrix. This feature is useful in the context of detecting entanglement defined as inseparability. As long as after PT in density matrix there appear negative eigenvalues, one has to do with entanglement of bipartite state. Thus, distinguishing between entangled and not entangled state is shifted to the problem of separability and inseparability of states respectively. This solution of the problem was used by Shchukin and Vogel in a formulation of entanglement criterion (SV) [9, 57, 61] in terms of states with positive partial transposition (PPT) and nonpositive partial transposition (NPT). This criterion has a similar form to Criterion 3 of nonclassicality presented in Section II.2. Analogously to Eqs.(II.20) and (II.21) one can construct matrix of moments $M(\hat{\rho}) = [M_{ij}(\hat{\rho})]$ such as

$$M_{ij}(\hat{\rho}) = \mathrm{Tr}\left[(\hat{a}^{\dagger i_1}\hat{a}^{i_2}\hat{b}^{\dagger i_3}\hat{b}^{i_4})^{\dagger}(\hat{a}^{\dagger j_1}\hat{a}^{j_2}\hat{b}^{\dagger j_3}\hat{b}^{j_4})\hat{\rho}\right], \tag{II.111}$$

where the subscripts i and j correspond to multi-indices (i_1, i_2, i_3, i_4) and (j_1, j_2, j_3, j_4) respectively. Such a formulation leads also to a criterion in terms of moments of creation and annihilation operators, which can be easily measured [10] (as it was mentioned in the case of nonclassicality). It also crucial to emphasize the fact that in the definition from Eq. (II.111) the creation and annihilation operators are *not* normally ordered (in contrast to Eq. (II.21)). For a separable state $\hat{\rho}$ the matrix of moments $M(\hat{\rho})$ is also separable, as it was shown in Ref. [61]. For example this condition can be written in the following form

$$\hat{\rho} = \sum_i p_i \hat{\rho}_i^A \otimes \hat{\rho}_i^B \Rightarrow M(\hat{\rho}) = \sum_i p_i M^A(\hat{\rho}_i^A) \otimes M^B(\hat{\rho}_i^B), \tag{II.112}$$

where $p_i \geq 0$, $\sum_i p_i = 1$, and

$$M^A(\hat{\rho}^A) = \sum_{i'j'} M_{i'j'}(\hat{\rho}^A)|i'\rangle\langle j'| \tag{II.113}$$

is given in a formal basis $\{|i'\rangle\}$ for $i' = (i_1, i_2, 0, 0)$ and $j' = (j_1, j_2, 0, 0)$; $M^B(\hat{\rho}^B)$ is defined analogously. It is possible to write a first criterion for entanglement states using such defined $M(\hat{\rho})$ [9]:

Criterion 4 *A bipartite quantum state $\hat{\rho}$ is NPT if and only if $M(\hat{\rho}^{\Gamma})$ is NPT .*

The recipe for calculating elements of matrix of moments, $M(\hat{\rho}^{\Gamma}) = [M_{ij}(\hat{\rho}^{\Gamma})]$, where Γ denotes partial transposition in some fixed basis, is the following

$$\begin{aligned} M_{ij}(\hat{\rho}^{\Gamma}) &= \mathrm{Tr}\left[(\hat{a}^{\dagger i_1}\hat{a}^{i_2}\hat{b}^{\dagger i_3}\hat{b}^{i_4})^{\dagger}(\hat{a}^{\dagger j_1}\hat{a}^{j_2}\hat{b}^{\dagger j_3}\hat{b}^{j_4})\hat{\rho}^{\Gamma}\right] \\ &= \mathrm{Tr}\left[(\hat{a}^{\dagger i_1}\hat{a}^{i_2}\hat{b}^{\dagger j_3}\hat{b}^{j_4})^{\dagger}(\hat{a}^{\dagger j_1}\hat{a}^{j_2}\hat{b}^{\dagger i_3}\hat{b}^{i_4})\hat{\rho}\right]. \end{aligned} \tag{II.114}$$

To obtain Criterion similar to Criterion 3 for nonclassicality one can reformulate Criterion 4 in such a way [Bartkowiak2010a][61]:

Criterion 5 *A bipartite state $\hat{\rho}$ is NPT if and only if there exists $\hat{F}$, such that $d_{\hat{F}}^{\Gamma}(\hat{\rho})$ is negative,*

where

$$d_{\hat F}^{\Gamma}(\hat\rho) = \begin{vmatrix} \langle \hat f_{r_1}^\dagger \hat f_{r_1}\rangle^\Gamma & \langle \hat f_{r_1}^\dagger \hat f_{r_2}\rangle^\Gamma & \cdots & \langle \hat f_{r_1}^\dagger \hat f_{r_N}\rangle^\Gamma \\ \langle \hat f_{r_2}^\dagger \hat f_{r_1}\rangle^\Gamma & \langle \hat f_{r_2}^\dagger \hat f_{r_2}\rangle^\Gamma & \cdots & \langle \hat f_{r_2}^\dagger \hat f_{r_N}\rangle^\Gamma \\ \vdots & \vdots & \ddots & \vdots \\ \langle \hat f_{r_N}^\dagger \hat f_{r_1}\rangle^\Gamma & \langle \hat f_{r_N}^\dagger \hat f_{r_2}\rangle^\Gamma & \cdots & \langle \hat f_{r_N}^\dagger \hat f_{r_N}\rangle^\Gamma \end{vmatrix} \tag{II.115}$$

is given in terms of $\langle \hat f_{r_i}^\dagger \hat f_{r_j}\rangle^\Gamma \equiv \langle (\hat f_{r_i}^\dagger \hat f_{r_j})^\Gamma\rangle$ ($i, j = 1, ..., N$).
As in the case of Criterion 3 one can write Criterion 5 in more clear and compact form as

$$\begin{aligned} \hat\rho \text{ is PPT} \quad &\Leftrightarrow \quad \forall \hat F: \quad d_{\hat F}^{\Gamma}(\hat\rho) \geq 0, \\ \hat\rho \text{ is NPT} \quad &\Leftrightarrow \quad \exists \hat F: \quad d_{\hat F}^{\Gamma}(\hat\rho) < 0. \end{aligned} \tag{II.116}$$

A set of monomial functions of creation and annihilation operators is denoted by $\hat F$. Obviously, a such formulation of this criterion can be used to detect not only two-mode but also multimode fields [61, 92]. It is worth stressing that the disadvantage of Criterion 5 is that it can not recognize PPT entanglement (e.g. so-called bound entangled states [38]). Here $\overset{\text{ent}}{<}$ means that inequality is true only for entangled states. The above definition of entanglement allows to derive some relations between nonclassicality and inseparability of states. To find them I am going to use the already known criteria of entanglement such as: Hillery and Zubairy [45], Duan *et al.* [44], Simon [46], or Mancini *et al.* [93].

3.2 Entanglement and the Cauchy-Schwarz inequality

Like in case of nonclassicality one can also analyze a relation between Criterion 5 and the Cauchy-Schwarz inequality. Because of linearity of the matrix $M_{\hat F}^{(n)}(\hat\rho)$ (in it's state $\hat\rho = \sum_i p_i \hat\rho_i$) one can write the following inequality

$$M_{\hat F}^{(n)}(\hat\rho) = \sum_i p_i M_{\hat F}^{(n)}(\hat\rho_i) \geq 0, \tag{II.117}$$

if $M_{\hat F}^{(n)}(\hat\rho_i) \geq 0$ for all $\hat\rho_i$. Therefore, positivity of factorized separable state implies positivity of $M_{\hat F}^{(n)}$. For

$$\hat F = (\hat f_1, \ldots, \hat f_N) \tag{II.118}$$

with functions $\hat f_i = \hat f_{i1} \hat f_{i2} \cdots \hat f_{iM}$, where

$$\hat f_{ij} = \begin{cases} 1 & \text{if } i \neq k_j \\ \text{either } g_j(\hat a_j) \text{ or } g_j(\hat a_j^\dagger) & \text{if } i = k_j, \end{cases} \tag{II.119}$$

it is possible to present $M_{\hat F}^{(n)}$ in a formal basis $\{|k\rangle\}$, as

$$\begin{aligned} M_{\hat F}^{(n)} &= \sum_{kl} \langle : \hat f_k^\dagger \hat f_l : \rangle |k\rangle\langle l| \\ &= \sum_{kl} \langle : \hat f_{k1}^\dagger \hat f_{l1} \ldots \hat f_{kM}^\dagger \hat f_{lM} : \rangle |k\rangle\langle l|. \end{aligned} \tag{II.120}$$

In the above equations i is the index of the element $\hat f_i$ in $\hat F$, and index j refers to the mode. For one unique $i = k_j$, $\hat f_{ij}$ is equal to a function g_j of creation or (not and) annihilation operators of mode j and it should be possibly different from the identity.

For a factorized state one can write $M_{\hat{F}}^{(n)}$ as

$$\begin{aligned}
M_{\hat{F}}^{(n)} &= \sum_{kl}\langle: \hat{f}_{k1}^{\dagger}\hat{f}_{l1} :\rangle\cdots\langle: \hat{f}_{kM}^{\dagger}\hat{f}_{lM} :\rangle|k\rangle\langle l| \\
&= \sum_{k}\langle: \hat{f}_{k1}^{\dagger}\hat{f}_{k1} :\rangle\cdots\langle: \hat{f}_{kM}^{\dagger}\hat{f}_{kM} :\rangle|k\rangle\langle l| + \sum_{k\neq l}\langle: \hat{f}_{k1}^{\dagger}\hat{f}_{l1} :\rangle\cdots\langle: \hat{f}_{kM}^{\dagger}\hat{f}_{lM} :\rangle|k\rangle\langle l| \\
&= \sum_{k}\langle: \hat{f}_{k1}^{\dagger}\hat{f}_{k1} :\rangle\cdots\langle: \hat{f}_{kM}^{\dagger}\hat{f}_{kM} :\rangle|k\rangle\langle l| + \sum_{k\neq l}\langle\hat{f}_{k1}^{\dagger}\rangle\langle\hat{f}_{l1}\rangle\cdots\langle\hat{f}_{kM}^{\dagger}\rangle\langle\hat{f}_{lM}\rangle|k\rangle\langle l| \\
&\geq \sum_{k}\langle\hat{f}_{k1}^{\dagger}\rangle\langle\hat{f}_{k1}\rangle\cdots\langle\hat{f}_{kM}^{\dagger}\rangle\langle\hat{f}_{kM}\rangle|k\rangle\langle l| + \sum_{k\neq l}\langle\hat{f}_{k1}^{\dagger}\rangle\langle\hat{f}_{l1}\rangle\cdots\langle\hat{f}_{kM}^{\dagger}\rangle\langle\hat{f}_{lM}\rangle|k\rangle\langle l| \\
&= \Big(\sum_{k}\langle\hat{f}_{k1}^{\dagger}\rangle\cdots\langle\hat{f}_{kM}^{\dagger}\rangle|k\rangle\Big)\Big(\sum_{l}\langle\hat{f}_{l1}\rangle\cdots\langle\hat{f}_{lM}\rangle\langle l|\Big) \geq 0.
\end{aligned} \tag{II.121}$$

To clarify Eq. (II.121) one is able to list the most important steps:

1. First equality is implied by the factorization of the state;
2. Third equality is connected with the definition of $\hat{f}_{ij}$s, which are functions of either annihilation or creation operators (not and), so $\langle: \hat{f}_{k1}^{\dagger}\hat{f}_{l1} :\rangle = \langle\hat{f}_{k1}^{\dagger}\hat{f}_{l1}\rangle$ or $\langle: \hat{f}_{k1}^{\dagger}\hat{f}_{l1} :\rangle = \langle\hat{f}_{l1}\hat{f}_{k1}^{\dagger}\rangle$, and that for $k\neq l$ at least one among $\hat{f}_{k1}^{\dagger}$ and $\hat{f}_{l1}$, let us say, e.g., $\hat{f}_{l1}$, is equal to the identity (in particular this implies that its expectation value is equal to $\langle\hat{f}_{l1}\rangle = 1$);
3. The first inequality is due to the fact that $\langle: \hat{f}_{k1}^{\dagger}\hat{f}_{k1} :\rangle = \langle\hat{f}_{k1}^{\dagger}\hat{f}_{k1}\rangle$ or $\langle: \hat{f}_{k1}^{\dagger}\hat{f}_{k1} :\rangle = \langle\hat{f}_{k1}\hat{f}_{k1}^{\dagger}\rangle$, and because of the Cauchy-Schwarz inequality.

3.3 A zoo of entanglement criteria and their connection with nonclassicality

Using Criterion 5 one can derive entanglement criteria which are already known in literature but which can be also obtained via Criterion 3 of nonclassicality.
For two-mode fields Hillery and Zubairy [45] formulated a few entanglement inequalities

$$\langle\hat{n}_1\hat{n}_2\rangle \overset{\text{ent}}{<} |\langle\hat{a}_1\hat{a}_2^{\dagger}\rangle|^2, \tag{II.122}$$
$$\langle\hat{n}_1\rangle\langle\hat{n}_2\rangle \overset{\text{ent}}{<} |\langle\hat{a}_1\hat{a}_2\rangle|^2, \tag{II.123}$$

and for the three-mode fields

$$\langle\hat{n}_1\hat{n}_2\hat{n}_3\rangle \overset{\text{ent}}{<} |\langle\hat{a}_1^{\dagger}\hat{a}_2\hat{a}_3\rangle|^2. \tag{II.124}$$

Criterion 5 [9, 61] can be also applied in order to obtain the above inequalities by choosing $\hat{F} = (1, \hat{a}_1\hat{a}_2)$ to obtain Eq. (II.122), $\hat{F} = (\hat{a}_1, \hat{a}_2)$ for Eq. (II.123), and $\hat{F} = (1, \hat{a}_1\hat{a}_2\hat{a}_3)$ for Eq. (II.124). The Hillery-Zubairy conditions are the more interesting that it is possible to obtain them also using Criterion 3 for nonclassicality. Therefore, to obtain Eq. (II.122) one needs to use $\hat{F} = (1, \hat{a}_1\hat{a}_2^{\dagger})$, which gives

$$d_{\hat{F}}^{(\mathrm{n})} = \begin{vmatrix} 1 & \langle\hat{a}_1\hat{a}_2^{\dagger}\rangle \\ \langle\hat{a}_1^{\dagger}\hat{a}_2\rangle & \langle\hat{n}_1\hat{n}_2\rangle \end{vmatrix} \overset{\text{ncl}}{<} 0. \tag{II.125}$$

Choosing $\hat{F} = (\hat{a}_1, \hat{a}_2^{\dagger})$ leads to an inequality equivalent to Eq. (II.123):

$$d_{\hat{F}}^{(\mathrm{n})} = \begin{vmatrix} \langle\hat{n}_1\rangle & \langle\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\rangle \\ \langle\hat{a}_1\hat{a}_2\rangle & \langle\hat{n}_2\rangle \end{vmatrix} \overset{\text{ncl}}{<} 0. \tag{II.126}$$

Assuming three-mode operators $\hat{F} = (1, \hat{a}_1^\dagger \hat{a}_2 \hat{a}_3)$ one can derive Eq. (II.124):

$$d_{\hat{F}}^{(\mathrm{n})} = \begin{vmatrix} 1 & \langle \hat{a}_1^\dagger \hat{a}_2 \hat{a}_3 \rangle \\ \langle \hat{a}_1 \hat{a}_2^\dagger \hat{a}_3^\dagger \rangle & \langle \hat{n}_1 \hat{n}_2 \hat{n}_3 \rangle \end{vmatrix} \stackrel{\mathrm{ncl}}{<} 0. \tag{II.127}$$

Criterion 3 also gives an possibility to derive other entanglement criteria, like, e.g the one shown in Ref. [61] and obtained from the entanglement Criterion 5. It can be achieved by choosing $\hat{F} = (\hat{a}_1^\dagger, \hat{a}_2 \hat{a}_3)$:

$$d_{\hat{F}}^{(\mathrm{n})} = \begin{vmatrix} \langle \hat{n}_1 \rangle & \langle \hat{a}_1 \hat{a}_2 \hat{a}_3 \rangle \\ \langle \hat{a}_1 \hat{a}_2 \hat{a}_3 \rangle^* & \langle \hat{n}_2 \hat{n}_3 \rangle \end{vmatrix} \stackrel{\mathrm{ncl}}{<} 0. \tag{II.128}$$

To generalize the Hillery-Zubairy condition Eq. (II.122) it is convenient to use the Cauchy-Schwarz inequality, as it was done in Ref. [45] in the following manner

$$\langle (\hat{a}_1^\dagger)^m \hat{a}_1^m (\hat{a}_2^\dagger)^n \hat{a}_2^n \rangle \stackrel{\mathrm{ent}}{<} |\langle \hat{a}_1^m (\hat{a}_2^\dagger)^n \rangle|^2. \tag{II.129}$$

The above inequality can be obtained via Criterion 3 assuming $\hat{F} = (1, \hat{a}_1^m (\hat{a}_2^\dagger)^n)$ as

$$d_{\hat{F}}^{(\mathrm{n})} = \begin{vmatrix} 1 & \langle \hat{a}_1^m (\hat{a}_2^\dagger)^n \rangle \\ \langle (\hat{a}_1^\dagger)^m \hat{a}_2^n \rangle & \langle (\hat{a}_1^\dagger)^m \hat{a}_1^m (\hat{a}_2^\dagger)^n \hat{a}_2^n \rangle \end{vmatrix} \stackrel{\mathrm{ncl}}{<} 0 \tag{II.130}$$

or, equivalently, via Criterion 5 with $\hat{F} = (1, \hat{a}_1^m \hat{a}_2^n)$. Summarizing one can write that

$$d^{(\mathrm{n})}(1, \hat{a}_1^m (\hat{a}_2^\dagger)^n) = d^\Gamma(1, \hat{a}_1^m \hat{a}_2^n) \stackrel{\mathrm{ent}}{<} 0, \tag{II.131}$$

where, for clarity, I have used the notation $d^k(\hat{F})$ instead of $d_{\hat{F}}^k$ for $k = (n)$, and Γ denotes partial transposition. Analogously a condition from Eq. (II.124) can be generalized in as follows

$$\langle \hat{n}_1^k \hat{n}_2^l \hat{n}_3^m \rangle \stackrel{\mathrm{ent}}{<} |\langle (\hat{a}_1^\dagger)^k \hat{a}_2^l \hat{a}_3^m \rangle|^2 \tag{II.132}$$

for arbitrary integers $k, l, m > 0$. This inequality can also be derived via the usage of both criteria, the one for detecting nonclassicality (Criterion 3) and the one for entanglement (Criterion 5)

$$d^{(\mathrm{n})}(1, (\hat{a}_1^\dagger)^k \hat{a}_2^l \hat{a}_3^m) = d^\Gamma(1, \hat{a}_1^k \hat{a}_2^l \hat{a}_3^m) = \begin{vmatrix} 1 & \langle (\hat{a}_1^\dagger)^k \hat{a}_2^l \hat{a}_3^m \rangle \\ \langle (\hat{a}_1^\dagger)^k \hat{a}_2^l \hat{a}_3^m \rangle^* & \langle \hat{n}_1^k \hat{n}_2^l \hat{n}_3^m \rangle \end{vmatrix} \stackrel{\mathrm{ncl}}{<} 0, \tag{II.133}$$

where the first mode is partially-transposed. A generalization of Eq. (II.128) can be done as

$$\langle \hat{n}_1^k \rangle \langle \hat{n}_2^l \hat{n}_3^m \rangle \stackrel{\mathrm{ent}}{<} |\langle \hat{a}_1^k \hat{a}_2^l \hat{a}_3^m \rangle|^2. \tag{II.134}$$

Such entanglement condition can also be obtained using Criteria 3 and 5:

$$d^{(\mathrm{n})}((\hat{a}_1^\dagger)^k, \hat{a}_2^l \hat{a}_3^m) = d^\Gamma(\hat{a}_1^k, \hat{a}_2^l \hat{a}_3^m) = \begin{vmatrix} \langle \hat{n}_1^k \rangle & \langle \hat{a}_1^k \hat{a}_2^l \hat{a}_3^m \rangle \\ \langle \hat{a}_1^k \hat{a}_2^l \hat{a}_3^m \rangle^* & \langle \hat{n}_2^l \hat{n}_3^m \rangle \end{vmatrix} \stackrel{\mathrm{ncl}}{<} 0. \tag{II.135}$$

The above inequalities are, in fact, $\stackrel{\mathrm{ent}}{<}$ inequalities. They are also derived from nonclassical Criterion 3, that is why $\stackrel{\mathrm{ncl}}{<}$ is marked in them. One needs to be aware that they can be satisfied only by entangled states, which is easy to be shown. The only nontrivial determinant condition is the one for establishing the positivity of the involved 2×2 matrices. It can be proven that matrices stay positive under factorization of the state. It implies positivity of their determinants for separable states. This can be shown on the example of Eq. (II.126) for a factorized state. For simplicity, I have analyzed positivity of the 2×2 matrix. In other cases the proof can be built analogously. Thus, for a factorized state, as a special case of inequalities given in Eq. (II.121), one can write

$$\begin{aligned}\begin{pmatrix} \langle\hat{n}_1\rangle & \langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle \\ \langle\hat{a}_1\hat{a}_2\rangle & \langle\hat{n}_2\rangle \end{pmatrix} &= \begin{pmatrix} \langle\hat{a}_1^\dagger\hat{a}_1\rangle & \langle\hat{a}_1^\dagger\rangle\langle\hat{a}_2^\dagger\rangle \\ \langle\hat{a}_1\rangle\langle\hat{a}_2\rangle & \langle\hat{a}_2^\dagger\hat{a}_2\rangle \end{pmatrix} \\ &\geq \begin{pmatrix} \langle\hat{a}_1^\dagger\rangle\langle\hat{a}_1\rangle & \langle\hat{a}_1^\dagger\rangle\langle\hat{a}_2^\dagger\rangle \\ \langle\hat{a}_1\rangle\langle\hat{a}_2\rangle & \langle\hat{a}_2^\dagger\rangle\langle\hat{a}_2\rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle\hat{a}_1^\dagger\rangle \\ \langle\hat{a}_2\rangle \end{pmatrix} \begin{pmatrix} \langle\hat{a}_1\rangle & \langle\hat{a}_2^\dagger\rangle \end{pmatrix} \geq 0.\end{aligned}$$

The Cauchy-Schwarz inequality $\langle\hat{X}^\dagger\hat{X}\rangle \geq |\langle\hat{X}\rangle|^2$ was applied to obtain the first inequality.

Another entanglement criteria were introduced by Duan *et al.* [44]. It can be defined in the form of the following condition [9]:

$$\langle\Delta\hat{a}_1^\dagger\Delta\hat{a}_1\rangle\langle\Delta\hat{a}_2^\dagger\Delta\hat{a}_2\rangle \overset{\text{ent}}{<} |\langle\Delta\hat{a}_1\Delta\hat{a}_2\rangle|^2, \tag{II.136}$$

where $\Delta\hat{a}_i = \hat{a}_i - \langle\hat{a}_i\rangle$ for $i = 1, 2$. It can be derived from Criterion 5 by applying $\hat{F} = (1, \hat{a}_1, \hat{a}_2)$ [9] or, equivalently, by $\hat{F} = (\Delta\hat{a}_1, \Delta\hat{a}_2)$. To obtain Eq. (II.136) from Criterion 3 one can apply a choice of $\hat{F} = (\Delta\hat{a}_1, \Delta\hat{a}_2^\dagger)$ and obtain

$$d_{\hat{F}}^{(\text{n})} = \begin{vmatrix} \langle\Delta\hat{a}_1^\dagger\Delta\hat{a}_1\rangle & \langle\Delta\hat{a}_1^\dagger\Delta\hat{a}_2^\dagger\rangle \\ \langle\Delta\hat{a}_1\Delta\hat{a}_2\rangle & \langle\Delta\hat{a}_2^\dagger\Delta\hat{a}_2\rangle \end{vmatrix} \overset{\text{ncl}}{<} 0. \tag{II.137}$$

Also a different choice of $\hat{F}$ leads to Eq. (II.137). Thus, for $\hat{F} = (1, \hat{a}_1, \hat{a}_2^\dagger)$, one obtains

$$d_{\hat{F}}^{(\text{n})} = \begin{vmatrix} 1 & \langle\hat{a}_1\rangle & \langle\hat{a}_2^\dagger\rangle \\ \langle\hat{a}_1^\dagger\rangle & \langle\hat{n}_1\rangle & \langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle \\ \langle\hat{a}_2\rangle & \langle\hat{a}_1\hat{a}_2\rangle & \langle\hat{n}_2\rangle \end{vmatrix}. \tag{II.138}$$

It can be seen that the Duan criterion is also equal to the nonclassicality criterion. As in the case of comparison of Eqs. (II.37) and (II.38) or Eqs. (II.82) and (II.83) one can see the advantage of the usage of polynomials functions over monomial ones in the $\hat{F}$ definition.

However, the equivalence of nonclassical and entanglement criteria is not a general relation. Initially, there were presented the examples of entanglement conditions derived from Criterion 5 and their relations with Criterion 3, which is simply equality and can be derived by application $\hat{F}_2 = \hat{F}_1^\Gamma$ (where $\hat{F}_1$ is a set of functions for Criterion 3, and $\hat{F}_2$ for Criterion 5; $^\Gamma$ corresponds to partial transposition). Here I am interested in cases for which these two criteria cannot be simply related with the usage of $\hat{F}_2 = \hat{F}_1^\Gamma$. Obviously, the states fulfilling Criterion 5 for inseparability are also nonclassical in the sense of Criterion 1 (as any entangled state is necessarily nonclassical). I have shown the particular examples of states which satisfy entanglement conditions and simultaneously more than one nonclassical inequality. Due to this feature it is possible to analyze inseparability for a given nonclassicality. To find a relation between entanglement condition and nonclassicality inequality one can look for a linear combination of some $d^{(\text{n})}(\hat{F}^{(k)})$ needed to express in the term $d_{\hat{F}}^\Gamma \equiv d^\Gamma(\hat{F})$ as , i.e.:

$$d_{\hat{F}}^\Gamma = \sum_k c_k d^{(\text{n})}(\hat{F}^{(k)}), \tag{II.139}$$

where $c_k > 0$. As shown in Ref. [Bartkowiak2010a] three properties of determinants can help for this purpose:

(i) The Laplace expansion formula along any row (or column):

$$\det M = \sum_j (-1)^{i+j} M_{ij}\mu_{ij},$$

where μ_{ij} is a minor of a matrix $M = (M_{ij})$.
(ii) Swapping rule: By exchanging any two rows (columns) of a determinant, the value of the determinant is the same as the one of the original determinant but with opposite sign.
(iii) A summation rule: If some (or all) elements of a column (row) are sum of two terms, then the determinant can be given as the sum of two determinants, e.g.,

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

This nontrivial relation can be obtained both by using the already known entanglement criteria and by a construction of a new one based on Criterion 5. At first it will be analyzed the known Simon's entanglement criterion [46]. Obviously, as the entanglement inequality it can be derived from Criterion 5 for $d_{\hat{F}}^{\Gamma} \overset{\text{ent}}{<} 0$ with $\hat{F} = (1, \hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2, \hat{a}_2^\dagger)$. Simon's criterion can be also obtained as the sum of nonclassicality criteria in the following way

$$d_{\hat{F}}^{\Gamma} = d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2^\dagger, \hat{a}_2) + d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_2^\dagger) + d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2^\dagger) + d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_2^\dagger, \hat{a}_2), \quad \text{(II.140)}$$

where $d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2^\dagger, \hat{a}_2)$ is given by Eq. (II.96). Furthermore, $d_{\hat{F}}^{(\mathrm{n})}$, by analyzing its principal minors, for choosing $\hat{F} = (1, \hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2^\dagger)$, $\hat{F} = (1, \hat{a}_1, \hat{a}_2^\dagger, \hat{a}_2)$ and $\hat{F} = (1, \hat{a}_1, \hat{a}_2^\dagger)$ were derived from Eq. (II.96). Therefore, checking Simon's entanglement condition is equivalent to testing the violation of specific classical inequalities derived from the nonclassicality Criterion 3. It is possible to find other, simpler examples of entanglement conditions corresponding to the sum of nonclassical inequalities for a particularly defined determinant

$$D(x, y, z, z') = \begin{vmatrix} 1 & x & x^* \\ x^* & z & y^* \\ x & y & z' \end{vmatrix}. \quad \text{(II.141)}$$

(i) Criterion 5 for $\hat{F} = (1, \hat{a}_1\hat{a}_2, \hat{a}_1^\dagger\hat{a}_2^\dagger)$ leads to

$$d_{\hat{F}}^{\Gamma} = D\left(\langle \hat{a}_1\hat{a}_2^\dagger\rangle, \langle \hat{a}_1^2(\hat{a}_2^\dagger)^2\rangle, \langle \hat{n}_1\hat{n}_2\rangle, z'\right) \overset{\text{ent}}{<} 0, \quad \text{(II.142)}$$

where $z' = \langle(\hat{n}_1 + 1)(\hat{n}_2 + 1)\rangle$. The properties of determinants mentioned above allow one to find a relation between Eq. (II.142) and the following nonclassicality conditions (obtained from Criterion 3):

$$d_{\hat{F}}^{\Gamma} = d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2^\dagger, \hat{a}_1^\dagger\hat{a}_2) + (\langle \hat{n}_1\rangle + \langle \hat{n}_2\rangle + 1)\, d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2^\dagger). \quad \text{(II.143)}$$

(ii) For $\hat{F} = (1, \hat{a}_1\hat{a}_2^\dagger, \hat{a}_1^\dagger\hat{a}_2)$ one obtains

$$d_{\hat{F}}^{\Gamma} = D(\langle \hat{a}_1\hat{a}_2\rangle, \langle \hat{a}_1^2\hat{a}_2^2\rangle, z, z') \overset{\text{ent}}{<} 0, \quad \text{(II.144)}$$

where $z = \langle \hat{n}_1\hat{n}_2\rangle + \langle \hat{n}_1\rangle$ and $z' = \langle \hat{n}_1\hat{n}_2\rangle + \langle \hat{n}_2\rangle$. The entanglement criterion defined in Eq. (II.144) can be written in terms of nonclassicality criteria as

$$d_{\hat{F}}^{\Gamma} = d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2, \hat{a}_1^\dagger\hat{a}_2^\dagger) + \langle \hat{n}_1\rangle\langle \hat{n}_2\rangle + (\langle \hat{n}_1\rangle + \langle \hat{n}_2\rangle)\, d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2). \quad \text{(II.145)}$$

(iii) For $\hat{F} = (1, \hat{a}_1 + \hat{a}_2^\dagger, \hat{a}_1^\dagger + \hat{a}_2)$ one can obtain the following condition

$$d_{\hat{F}}^{\Gamma} = D(\langle \hat{a}_1 + \hat{a}_2 \rangle, \langle (\hat{a}_1 + \hat{a}_2)^2 \rangle, z, z) \overset{\text{ent}}{<} 0, \tag{II.146}$$

where $z = \langle \hat{n}_1 \rangle + \langle \hat{n}_2 \rangle + 2\mathrm{Re}\langle \hat{a}_1 \hat{a}_2^\dagger \rangle + 1$. This entanglement inequality Eq. (II.146) can be written as a sum of nonclassicality inequalities as

$$d_{\hat{F}}^{\Gamma} = d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2, \hat{a}_1^\dagger + \hat{a}_2^\dagger) + 2d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2) + 1.$$

(iv) Finally by choosing $\hat{F} = (1, \hat{a}_1 + \hat{a}_2, \hat{a}_1^\dagger + \hat{a}_2^\dagger)$ to Criterion 5 one can obtain

$$d_{\hat{F}}^{\Gamma} = D(\langle \hat{a}_1 + \hat{a}_2^\dagger \rangle, \langle (\hat{a}_1 + \hat{a}_2^\dagger)^2 \rangle, z, z') \overset{\text{ent}}{<} 0, \tag{II.147}$$

where $z = \langle \hat{n}_1 \rangle + \langle \hat{n}_2 \rangle + 2\mathrm{Re}\langle \hat{a}_1 \hat{a}_2 \rangle$ and $z' = z + 2$. The relation between Eq. (II.147) (which corresponds to the entanglement criterion of Mancini *et al.* [93], [9]) and nonclassicality criteria is following

$$d_{\hat{F}}^{\Gamma} = d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^\dagger, \hat{a}_1^\dagger + \hat{a}_2) + 2d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^\dagger), \tag{II.148}$$

where $d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^\dagger, \hat{a}_1^\dagger + \hat{a}_2)$ is given by Eq. (II.94), and $d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^\dagger)$ is given by its principal minor. In the Table II.2 are collected different examples of entanglement criteria which can be expressed directly in terms of nonclassicality criteria or as sum of various nonclassicality conditions.

Table II.2: Entanglement criteria via nonclassicality criteria [Bartkowiak2010a].

Reference	**Entanglement criterion**	**Equivalent nonclassicality criterion**	Equations
Duan *et al.* [44]	$d^{\Gamma}(\Delta\hat{a}_1, \Delta\hat{a}_2) = d^{\Gamma}(1, \hat{a}_1, \hat{a}_2) < 0$	$d^{(\mathrm{n})}(\Delta\hat{a}_1, \Delta\hat{a}_2^{\dagger}) = d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_2^{\dagger}) < 0$	(II.136)–(II.138)
Simon [46]	$d^{\Gamma}(1, \hat{a}_1, \hat{a}_1^{\dagger}, \hat{a}_2, \hat{a}_2^{\dagger}) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_1^{\dagger}, \hat{a}_2^{\dagger}, \hat{a}_2) + d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_2^{\dagger})$	(II.140)
		$+d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_1^{\dagger}, \hat{a}_2^{\dagger}) + d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_2^{\dagger}, \hat{a}_2) < 0$	
Mancini *et al.* [93]	$d^{\Gamma}(1, \hat{a}_1 + \hat{a}_2, \hat{a}_1^{\dagger} + \hat{a}_2^{\dagger}) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^{\dagger}, \hat{a}_1^{\dagger} + \hat{a}_2) + 2d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^{\dagger}) + 1 < 0$	(II.146), (II.147)
Hillery & Zubairy [45]	$d^{\Gamma}(1, \hat{a}_1\hat{a}_2) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2^{\dagger}) < 0$	(II.122), (II.125)
ditto	$d^{\Gamma}(1, \hat{a}_1^{m}\hat{a}_2^{n}) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1^{m}(\hat{a}_2^{\dagger})^{n}) < 0$	(II.129)–(II.131)
ditto	$d^{\Gamma}(\hat{a}_1, \hat{a}_2) < 0$	$d^{(\mathrm{n})}(\hat{a}_1, \hat{a}_2^{\dagger}) < 0$	(II.123), (II.126)
ditto	$d^{\Gamma}(1, \hat{a}_1\hat{a}_2\hat{a}_3) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1^{\dagger}\hat{a}_2\hat{a}_3) < 0$	(II.124), (II.127)
Miranowicz *et al.* [61]	$d^{\Gamma}(\hat{a}_1, \hat{a}_2\hat{a}_3) < 0$	$d^{(\mathrm{n})}(\hat{a}_1^{\dagger}, \hat{a}_2\hat{a}_3) < 0$	(II.128)
Other entanglement tests	$d^{\Gamma}(1, \hat{a}_1^{k}\hat{a}_2^{l}\hat{a}_3^{m}) < 0$	$d^{(\mathrm{n})}(1, (\hat{a}_1^{\dagger})^{k}\hat{a}_2^{l}\hat{a}_3^{m}) < 0$	(II.132), (II.133)
	$d^{\Gamma}(\hat{a}_1^{k}, \hat{a}_2^{l}\hat{a}_3^{m}) < 0$	$d^{(\mathrm{n})}((\hat{a}_1^{\dagger})^{k}, \hat{a}_2^{l}\hat{a}_3^{m}) < 0$	(II.134), (II.135)
	$d^{\Gamma}(1, \hat{a}_1\hat{a}_2, \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2^{\dagger}, \hat{a}_1^{\dagger}\hat{a}_2) + (\langle\hat{n}_1 + \hat{n}_2\rangle + 1)\, d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2^{\dagger}) < 0$	(II.142), (II.143)
	$d^{\Gamma}(1, \hat{a}_1\hat{a}_2^{\dagger}, \hat{a}_1^{\dagger}\hat{a}_2) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2, \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}) + \langle\hat{n}_1\rangle\langle\hat{n}_2\rangle + \langle\hat{n}_1 + \hat{n}_2\rangle\, d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2) < 0$	(II.144), (II.145)
	$d^{\Gamma}(1, \hat{a}_1 + \hat{a}_2, \hat{a}_1^{\dagger} + \hat{a}_2^{\dagger}) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^{\dagger}, \hat{a}_1^{\dagger} + \hat{a}_2) + 2d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^{\dagger}) < 0$	(II.147), (II.148)

3.4 Examples of entanglement witnesses

To use entanglement inequality effectively, analogously to nonclassicality condition, one can construct an entanglement witnesses. As in the next section I have analyzed a behaviour of the nonclassicality and entanglement witnesses and entanglement measures, I have used the unified recipe for construction witnesses (also in this subsection the idea of the truncated witness introduced in II.2.6 is maintained). Before providing some examples of entanglement witness, for clarity, the formal definition will be recalled [37]:
An entanglement witness is a Hermitian operator $\hat{O}_{\rm ent}$ such that $\mathrm{tr}(\hat{O}_{\rm ent}\hat{\rho}_{\rm sep}) \geq 0$ for all separable states $\hat{\rho}_{\rm sep}$, while $\mathrm{tr}(\hat{O}_{\rm ent}\hat{\rho}_{\rm ent}) < 0$ for some entangled states $\hat{\rho}_{\rm ent}$.

From the formulation of the above definition it can be seen that the entanglement witness corresponds to observables rather than to expectation values. The concept presented in the above definition was later generalized to nonlinear entanglement witnesses [94, 95]. The term of entanglement witness used in this thesis differs slightly from the original definition. However, the idea of entanglement witness is kept unchanged and the slightly different usage of this term can clarify some part of this thesis. Analogously, as in the recipe for the construction of nonclassicality witnesses from Subsection II.2.6, the first step would be to apply Criterion 5 to obtain appropriate entanglement inequality. Here only two examples of such constructed witnesses are presented. The first has been already shown while introducing application of Criterion 5. The Hillery-Zubairy classical inequalities has the form of [45]:

$$\langle \hat{n}_1\hat{n}_2\rangle \overset{\rm cl}{\geq} |\langle \hat{a}_1\hat{a}_2^\dagger\rangle|^2, \quad \langle \hat{n}_1\rangle\langle \hat{n}_2\rangle \overset{\rm cl}{\geq} |\langle \hat{a}_1\hat{a}_2\rangle|^2, \tag{II.149}$$

where $\hat{n}_i = \hat{a}_i^\dagger\hat{a}_i$ is the photon number operator, and $\hat{a}_i$ ($\hat{a}_i^\dagger$) is the annihilation (creation) operator for mode $i = 1, 2$. Using the earlier introduced recipe one can construct the following truncated witnesses

$$\tilde{H} = \max(0, |\langle \hat{a}_1\hat{a}_2^\dagger\rangle|^2 - \langle \hat{n}_1\hat{n}_2\rangle) \overset{\rm ent}{<} 0, \tag{II.150}$$
$$\tilde{H}' = \max(0, |\langle \hat{a}_1\hat{a}_2\rangle|^2 - \langle \hat{n}_1\rangle\langle \hat{n}_2\rangle) \overset{\rm ent}{<} 0. \tag{II.151}$$

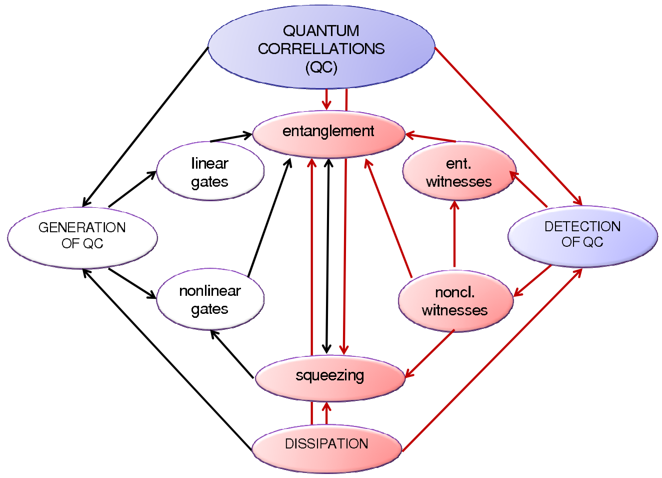
The two above witnesses are positive only for *entangled* states (because of the reformulation of witness made in the second part of the recipe presented in Section II.2.6). $\tilde{H}$ and $\tilde{H}'$ can be derived both by using partial transposition [9, 57, 61] and via the usage of the Cauchy-Schwarz inequality [45]. As Eq. (II.149) can be obtained from Criterion 5 based on the P-function [83], it is valid not only for separable states but also for the classical ones (it is marked by $\overset{\rm cl}{\geq}$).

The second example I want to analyze in this subsection is joined with violation of Bell's inequality. Such defined witness seems to be the natural choice of entanglement witness, as it is related with one of the first and qualitative definitions of entanglement. The witness for a two-qubit state can be identified with the degree of violation of Bell's inequality. In its version due to Clauser, Horne, Shimony, and Holt (CHSH) [96] the entanglement witness has the form [97, 98]:

$$B^2(\hat{\rho}) \equiv \max\left[0, \max_{j<k}(u_j + u_k) - 1\right], \tag{II.152}$$

where u_j $(j = 1, 2, 3)$ are the eigenvalues of $U_{\hat{\rho}} = T_{\hat{\rho}}^T T_{\hat{\rho}}$, $T_{\hat{\rho}}$ is a real matrix with elements $t_{ij} = \mathrm{Tr}\left[\hat{\rho}\left(\hat{\sigma}_i \otimes \hat{\sigma}_j\right)\right]$, and $\hat{\sigma}_j$ are the Pauli's spin matrices. B is often called *nonlocality* (measure), but it is crucial to emphasize that this term is not precise. There is a need to notice that B refers to entanglement witness, not measure. Thus, one can find mixed states $\hat{\rho}$ (e.g., Werner's states discussed in next section) for which $C(\hat{\rho}) > 0$ and $B(\hat{\rho}) = 0$. However, if a two-qubit state violates Bell inequality, it is entangled. For two-qubit pure states valid is equality $B(\hat{\rho}) = C(\hat{\rho})$. In this case entangled witness is also an entangled measure. It can be seen that due to the reformulation of the witnesses, given in Subsection II.2.6, B and $\tilde{H}$/ $\tilde{H}'$ have a definitions similar to the maximum of zero and another quantity.

4 Time evolution of nonclassicality and entanglement witnesses



In the previous sections the quantities called nonclassicality and entanglement witnesses have been introduced. Their construction is based on a general criteria connected with the P-function and allows one to find a great number of various conditions for nonclassicality and entanglement and, therefore, also witnesses of those quantities. In this section I have treated the nonclassicality (and a specific kind of it–entanglement) as property of particular optical systems.

It is well-known that decoherence is a main obstacle in effective implementation of quantum information processing and quantum state engineering. Entanglement, as manifestation of a quantum correlation can be especially sensitive when decoherence is taken into account. It is, therefore, possible to use the criteria defined previously to describe the influence of decoherence on not only entanglement but also more general nonclassicality in a system.

The first time the unlike behaviour of entanglement (in comparison to other quantities) in a dissipative system was presented by Życzkowski and the Horodecki family [99], as well as Yu and Eberly [43] (see also earlier studies in Refs. [100, 101, 102, 103]). They have shown that this kind of correlations decays in a finite time. These days this phenomenon acquired a dramatic name of a "sudden death" of entanglement. In this thesis I would like to use the term entanglement sudden vanishing (SV). It is worth noting that SV of entanglement (in general nonclassicality) can be followed by its sudden reappearance (sudden rebirth—SR) [99, 100, 101, 102, 103, 104, 105, 106]. After the article of You and Eberly being published analyzing entanglement losses in various systems (for reviews see Ref. [107]) has become much of the interest. This effect has been also observed experimentally [108, 109, 110].

In contrary to the time evolution of other physical correlations in dissipative system, SV of entanglement was new and unusual form of decay. Therefore, I would like to stress *general occurrence of sudden finite-time decays and periodic vanishings of nonclassical correlations* [Bartkowiak2011].

The main goal of this section is to show that SV and SR can also appear during the analysis of previously constructed nonclassicality (also called quantumness witnesses) [7, 56, 84, 85, 87, 86, 83] and entanglement witnesses [37, 94, 95] (for a review see Ref. [38]). The first one corresponds also to violation of classical inequalities. The final recipe for constructing witnesses was based on a similarity to the definitions of entanglement measures (which has been already

mentioned in Subsection II.2.6). The standard approach to analyze SV and SR refers to the study of the time evolution of entanglement measures e.g., concurrence or, equivalently, negativity or relative entropy of entanglement [38]. The definitions have been already given in the Subsection II.2.6. However, it should be emphasized that not all SRs and SVs of witnesses can be considered as standard. The SR needs to appear after a finite time of evolution and it should be preceded with the earliest SV. It can be easily explained using the following example:
For $|\cos t|$ and $\max(0, \cos t)$ one can observe vanishing of function at $\pi/2$. According to the assumption given above only vanishing of the latter one can be joined with the proper SV and SR.
It can be seen that both of the previously recalled entanglement measures (concurrence and negativity) have such a form. This clearly explains the occurrence of SV if $\hat{\rho}$, for which they were calculated, changes in time. Contradictory, if one took

$$C'(\hat{\rho}) = 2\max_i \lambda_i - \sum_i \lambda_i,$$

and

$$N'(\hat{\rho}) = -2\min_j \mu_j,$$

(for λ_i and μ_j with continuous derivatives in time) the SV would not appear. My deduction was that the SV and SR can be seen every time one has to deal with witnesses described as maximum and zero of some function. Thus, the SV and SR effects can be observed for any arbitrary time-dependent parameter $O(t)$, in comparison to some threshold value O_0. With a view to analyzing nonclassicality the most interesting parameters O are the ones related with breaking of some classical inequalities $O \overset{\text{cl}}{\geq} O_0$ (violated by some *nonclassical* fields, i.e., $O \overset{\text{ncl}}{<} O_0$). In contrary, by $\overset{\text{cl}}{\geq}$ it is stressed that the analyzed inequality *must* be fulfilled for all *classical* states. The concept that for truncated witnesses, constructed in such a way (see Subsection II.2.6 $O \to \tilde{O} = \max(0, O_0 - O)$), one can see SV and SR is illustrated in Fig. II.1.

4.1 Sudden decays of nonclassicality witnesses for noninteracting modes

Firstly there will be analyzed the environment-induced sudden vanishing of entanglement and nonclassicality, which as a concept, is similar to the original idea of finite-time sudden death of entanglement shown by Yu and Eberly [43]. It has been also used the more general nonclassicality witnesses to check my assumption of common occurrence of the SV effects. As it will be seen, SV of nonclassicality witnesses appear in different times than for entanglement witnesses and measures. In this subsection it is studied a case of two modes (qubits), which are not directly interacting with each other but via independent reservoirs. To the need of exposition of the generality of SV effect it will be analyzed a time evolution of initial entanglement states. The SV appears via interaction with reservoirs under Markov's approximation. The standard master equation for the reduced density operator $\hat{\rho}$ can be written in a form

$$\begin{aligned}\frac{\partial}{\partial t}\hat{\rho} \quad &= \quad \sum_{k=1,2} \frac{\gamma_k}{2}[\bar{n}_k(2\hat{a}_k^\dagger \hat{\rho}_k \hat{a}_k - \hat{a}_k \hat{a}_k^\dagger \hat{\rho} - \hat{\rho} \hat{a}_k \hat{a}_k^\dagger) \\ &\quad + (\bar{n}_k + 1)(2\hat{a}_k \hat{\rho} \hat{a}_k^\dagger - \hat{a}_k^\dagger \hat{a}_k \hat{\rho} - \hat{\rho} \hat{a}_k^\dagger \hat{a}_k)] - \frac{i}{\hbar}[\hat{\mathcal{H}}_S, \hat{\rho}], \qquad \text{(II.153)}\end{aligned}$$

where γ_k are the damping rates, $\bar{n}_k$ are the mean thermal photon numbers, $\bar{n}_k = \{\exp[\hbar\omega_k/(k_B T)] - 1\}^{-1}$, T is the reservoirs temperature at thermal equilibrium, and k_B is a Boltzmann's constant. Having made the assumption that the reservoirs has zero temperature it is justified to

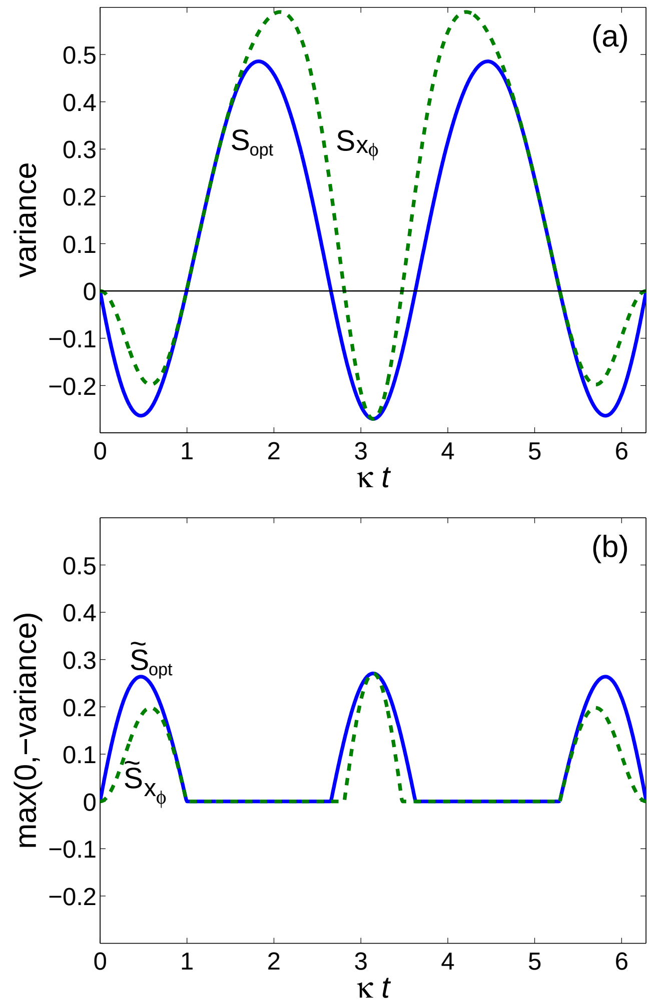


Figure II.1: An exemplification of a recipe for obtaining the SV and SR of nonclassicality witnesses for a unitary evolution of single-mode squeezing in the anharmonic oscillator model given by the Hamiltonian from Eq. (II.193). Figure (a) presents normally-ordered variances S_{x_ϕ} (a dashed curve) and $S_{\rm opt}$ (a solid curve), given by Eqs. (II.195) and (II.196), and Figure (b) shows truncation of normally-ordered variances $\tilde{S}_{x_\phi}$ (a dashed curve) and $\tilde{S}_{\rm opt}$ (a solid curve), given by Eqs. (II.108) and (II.110), respectively. Thus, for $S_{x_\phi} < 0$ or, equivalently, for the truncated witness $\tilde{S}_{x_\phi} > 0$ quadrature squeezing takes place. Analogously, a principal squeezing is present for $S_{\rm opt} < 0$ or if the truncated witness is considered $\tilde{S}_{\rm opt} > 0$. Here $|\alpha_0|^2 = 1/2, \phi_0 = \phi = 0$, and $S_0 = 0$. Introducing damping leads to results being analogous to the standard sudden death of entanglement [Bartkowiak2011].

put $\bar{n}_1 = \bar{n}_2 = 0$. The Hamiltonian $\hat{\mathcal{H}}_S$ can be understood as a sum of free Hamiltonians for the two noninteracting system modes. One of the methods of solving a master equation is to use Monte Carlo wave function simulation. To do this I assume the following collapse operators [111]

$$\hat{c}_{1k} = \sqrt{\gamma(1+\bar{n}_k)}\hat{a}_k,$$

$$\hat{c}_{2k} = \sqrt{\gamma\bar{n}_k}\hat{a}_k^\dagger.$$

In this context, it is very important to emphasize that in a case of the quantum entanglement between two systems, and the related violation of Bell's inequalities, the systems should be spatially separated and physically uncoupled [112]. Accordingly, my example is a system with two independent reservoirs (contrary to the models studied in the following subsections). However, in some cases of the coupled reservoirs entanglement for two qubits and two modes can be enhanced. Anyway, as has been shown [113, 114] it is possible rather due to the mixing mechanism than induced via interaction among them.

It has been analyzed time behaviour of nonclassicality and entanglement witnesses for this system for two initial states. The first is Werner-like state with the form [98]:

$$\hat{\rho}_m(0) = p|\Psi_m\rangle\langle\Psi_m| + \frac{1-p}{4}\hat{I}, \tag{II.154}$$

for $0 \leq p \leq 1$, $m = 1$, $|\Psi_1\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and $\hat{I}$ is the identity operator. Assuming such initial state one can solve the master equation analytically [98]. The time-dependent reduced density matrix in computational basis can be written as

$$\hat{\rho}_1(t) = \frac{1}{4}\begin{bmatrix} h^{(+)} & 0 & 0 & 2p\sqrt{g_1g_2} \\ 0 & h_1^{(+)} & 0 & 0 \\ 0 & 0 & h_2^{(+)} & 0 \\ 2p\sqrt{g_1g_2} & 0 & 0 & (1+p)g_1g_2 \end{bmatrix}, \tag{II.155}$$

where

$$h^{(+)} = (2-g_1)(2-g_2) + pg_1g_2,$$

$$h_k^{(+)} = g_{3-k}[2-(1+p)g_k],$$

and $g_k = \exp(-\gamma_k t)$ for $k = 1, 2$.
Now it is possible to analyze time evolution of concurrence [98]:

$$C(t) = \max\left\{0, \frac{1}{2}\sqrt{g_1g_2}\Big(2p - \sqrt{[2-(1+p)g_1][2-(1+p)g_2]}\Big)\right\}, \tag{II.156}$$

entanglement witness

$$B^2(t) = \max\left(0, 2p^2g_1g_2 - 1\right), \tag{II.157}$$

and nonclassicality witnesses

$$\tilde{S}(t) = \max\left[0, \frac{1}{4}(g_1^2 + g_2^2 + 2pg_1g_2) - S_0\right], \tag{II.158}$$

$$\tilde{D}(t) = \max\left[0, \frac{1}{2}g_1g_2(1+p) - D_0^2 - D_0(g_1 - g_2)\right]. \tag{II.159}$$

The formulas for times of SV for appropriate entanglement/nonclassicality witnesses and concurrence are given by (with assumption of the same reservoir damping rate γ, so $g_1 = g_2 \equiv g$)

$$t_{\mathrm{SV}}^{(C)} = \frac{1}{\gamma}\ln\left(\frac{1+p}{2(1-p)}\right), \tag{II.160}$$

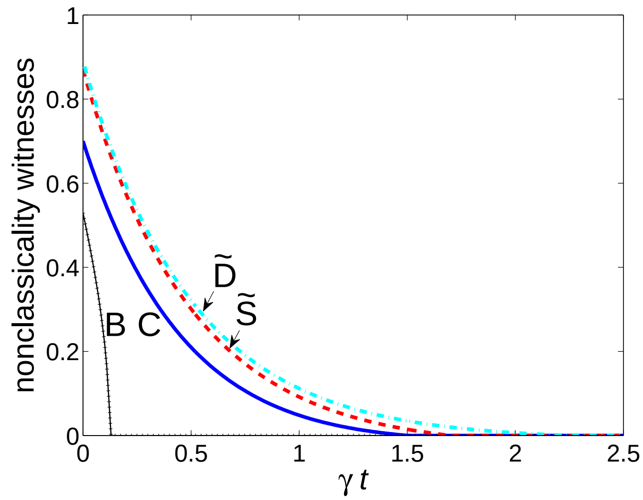


Figure II.2: The time behaviour of nonclassicality witnesses for two *noninteracting* modes described by the damping model from Subsection II.4.1 with environment-induced sudden vanishings of witnesses. The initial state is the Werner-like state ρ_1 with $p = 0.8$. Key: C– the concurrence C (a solid curve), B– nonlocality B (a dotted curve), $\tilde{S}$ (a dashed curve) for $S_0 = 0.03$ and $\tilde{D}$ (a dot-dashed curve) for $D_0 = 0.1$– two witnesses describing the photon-number-difference correlations [Bartkowiak2011].

$$t_{\rm SV}^{(B)} = \frac{1}{\gamma}\ln\left(\sqrt{2}p\right), \tag{II.161}$$

$$t_{\rm SV}^{(\tilde{S})} = \frac{1}{2\gamma}\ln\left(\frac{1+p}{2S_0}\right), \tag{II.162}$$

$$t_{\rm SV}^{(\tilde{D})} = \frac{1}{2\gamma}\ln\left(\frac{1+p}{2D_0^2}\right). \tag{II.163}$$

In Fig. II.2 it can be seen that they differ from each other. In the picture specific values of the damping constant γ and the initial Werner state $\hat{\rho}_1(0)$ with parameter p are assumed.

As another example of the SV effect, it has been studied the same system as in the previous example, but with a different initial state. This time it is the standard Werner state $\hat{\rho}_2(0)$, given by Eq. (II.154) for $m = 2$ and $|\Psi_2\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. The solution of a master equation has the form [98]:

$$\hat{\rho}_2(t) = \frac{1}{4}\begin{bmatrix} h^{(-)} & 0 & 0 & 0 \\ 0 & h_1^{(-)} & -2p\sqrt{g_1g_2} & 0 \\ 0 & -2p\sqrt{g_1g_2} & h_2^{(-)} & 0 \\ 0 & 0 & 0 & (1-p)g_1g_2 \end{bmatrix}, \tag{II.164}$$

where

$$h^{(-)} = (2-g_1)(2-g_2) - pg_1g_2$$

and

$$h_k^{(-)} = g_{3-k}[2-(1-p)g_k]$$

for $k = 1, 2$. The time evolution of $B(t)$ in this case is the same as in Eq. (II.157). The concurrence is given by [98]:

$$C(t) = \max\left[0, \frac{1}{2}\sqrt{g_1g_2}(2p - \sqrt{1-p}\sqrt{(2-g_1)(2-g_2)-pg_1g_2})\right], \tag{II.165}$$

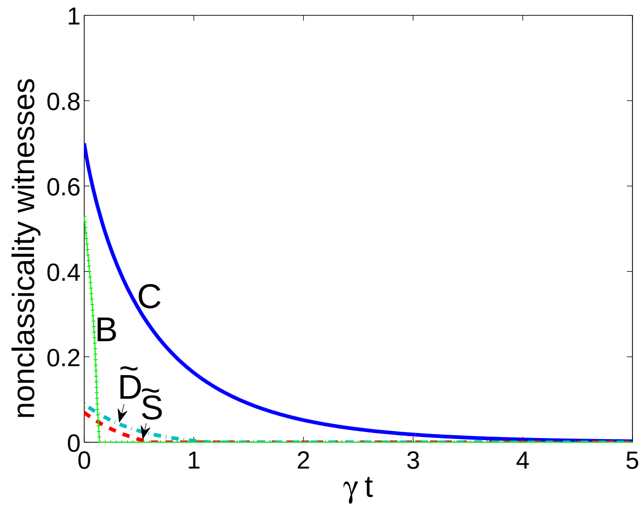


Figure II.3: Same as in Fig. II.2 but for the initial Werner-like state ρ_2 [Bartkowiak2011].

which is different than in Eq. (II.156). The behaviour of nonclassicality witnesses can be analyzed based on the following formulas

$$\tilde{S}(t) = \max\left[0, \frac{1}{4}(g_1^2 + g_2^2 - 2pg_1g_2) - S_0\right], \tag{II.166}$$

$$\tilde{D}(t) = \max\left[0, \frac{1}{2}g_1g_2(1-p) - D_0^2\left(1 - p\sqrt{g_1g_2}\right) - D_0(g_1 - g_2)\right], \tag{II.167}$$

which also differ from Eqs. (II.158) and (II.159). The times of SVs for $\hat{\rho}_2(t)$ are following

$$t_{\rm SV}^{(C)} = -\frac{1}{\gamma}\log\left(\frac{2\sqrt{p(1+p)}+2}{1-p}\right), \tag{II.168}$$

$$t_{\rm SV}^{(\tilde{S})} = \frac{1}{\gamma}\log\sqrt{\frac{1-p}{2S_0}}, \tag{II.169}$$

$$t_{\rm SV}^{(\tilde{D})} = \frac{1}{\gamma}\log\left(\frac{\sqrt{2+p\left(D_0^2p-2\right)}+D_0p}{2D_0}\right), \tag{II.170}$$

and $t_{\rm SV}^{(B)}$ is given by Eq. (II.161). One can see that analogously to Eqs. (II.160)–(II.162), times of SVs in considered example are different for various witnesses. SV of the witnesses and measures described above can be seen in Fig. II.3 for particular choices of the damping constant γ and parameter p of the initial Werner state $\hat{\rho}_2(0)$.

4.2 Periodic sudden vanishing of nonclassicality witnesses for interacting modes

A second type of time evolution that is related to the truncation of witnesses is *periodic* sudden vanishing of nonclassicality witnesses. In contrast to evolution of dissipative systems this effect appears under unitary evolution of two interacting modes. It should be stressed that dissipation can be easily introduced to the system, which results in the behaviour of quantum correlations analogous to the one analyzed in previous subsection.

To present the effect of periodic SV and SR it is analyzed the parametric frequency conversion, which can be described by the interaction Hamiltonian

$$\hat{\mathcal{H}} = \hbar\kappa[\hat{a}_1^\dagger \hat{a}_2 \exp(-i\Delta\omega t) + \hat{a}_1 \hat{a}_2^\dagger \exp(i\Delta\omega t)]. \quad \text{(II.171)}$$

This form of Hamiltonian is prototype of two linearly harmonic oscillators which are coupled. This can model e.g. the process of exchanging photons between two optical fields of different frequencies: a signal mode with frequency ω_1 and an idler mode with frequency ω_2. Then $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators for the signal and idler modes, respectively, and κ is the real coupling constant. For simplicity, it is assumed a resonant case $\Delta\omega = \omega + \omega_2 - \omega_1$.
The solution for the signal, $\hat{b}_1(t)$, and idler, $\hat{b}_2(t)$, modes of motion Heisenberg equation are the following [115]:

$$\begin{aligned} \hat{b}_1(t) &= \hat{a}_1 \cos(\kappa t) - i\,\hat{a}_2 \sin(\kappa t), \\ \hat{b}_2(t) &= \hat{a}_2 \cos(\kappa t) - i\hat{a}_1 \sin(\kappa t). \end{aligned} \quad \text{(II.172)}$$

The Schrödinger equation has the form of

$$|\psi(t)\rangle = \sum_{n_1,n_2} c_{n_1,n_2} \frac{[\hat{b}_1^\dagger(-t)]^{n_1}}{\sqrt{n_1!}} \frac{[\hat{b}_2^\dagger(-t)]^{n_2}}{\sqrt{n_2!}} |00\rangle, \quad \text{(II.173)}$$

where initial state is a superposition of the Fock states

$$|\psi(0)\rangle = \sum_{n_1,n_2} c_{n_1,n_2} |n_1, n_2\rangle.$$

A total number of photons is considered to be a constant of motion $\hat{n}_1(t) + \hat{n}_2(t) =$const. The time evolution of the QPD for the frequency-converter model (applying the results of Refs. [116, 117, 118]) can be written as follows (with arbitrary initial fields)

$$\mathcal{W}^{(s)}(\alpha_1, \alpha_2, t) = \mathcal{W}^{(s)}\left[\beta_1(\alpha_1, \alpha_2, -t), \beta_2(\alpha_1, \alpha_2, -t), 0\right]. \quad \text{(II.174)}$$

In the above Eq. (II.174) $\beta_{1,2}(\alpha_1, \alpha_2, t)$ refers to solutions of the corresponding *classical* equations of motion for the frequency conversion model

$$\begin{aligned} \beta_1(\alpha_1, \alpha_2, t) &= \alpha_1 \cos(\kappa t) - i\alpha_2 \sin(\kappa t), \\ \beta_2(\alpha_1, \alpha_2, t) &= \alpha_2 \cos(\kappa t) - i\alpha_1 \sin(\kappa t). \end{aligned} \quad \text{(II.175)}$$

The two-mode QPD can be understood as a constant along the classical trajectories as it can be interpreted from Eq. (II.174). There also exist an important property of the (undamped) parametric frequency model that degree of nonclassicality of the system (as defined, e.g., in Refs. [50, 51, 52]) remains unchanged at any evolution time of the system. However, for this case one can also observer SV and SR of entanglement and nonclassicality witnesses for both the pure initial state and the mixed one.

At first let me assume the initial state as a pure one in the form of $|\psi(0)\rangle = |01\rangle$. Evolution of the state of the system is ruled by Eq. (II.173) and has the form of

$$|\psi(t)\rangle = \cos(\kappa t)|01\rangle - i\sin(\kappa t)|10\rangle. \quad \text{(II.176)}$$

In terms of the P-function it can be described as a function more singular than Dirac's δ as

$$P(\alpha_1, \alpha_2, t) = \delta[\beta_1(\alpha_1, \alpha_2, t)] \left(1 + \frac{\partial}{\partial\beta_2(\alpha_1, \alpha_2, t)} \frac{\partial}{\partial\beta_2^*(\alpha_1, \alpha_2, t)}\right) \delta[\beta_2(\alpha_1, \alpha_2, t)]. \quad \text{(II.177)}$$

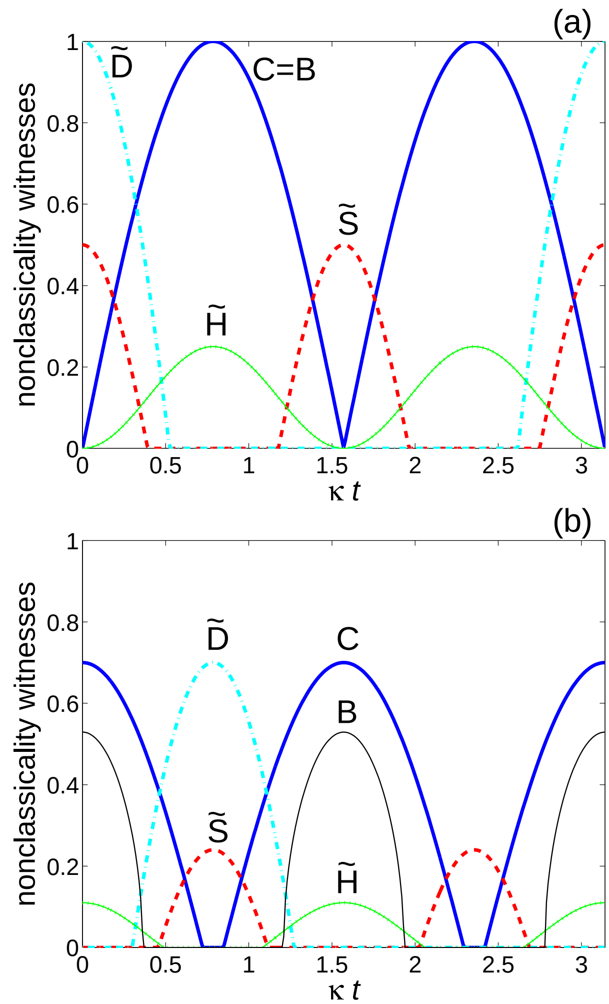


Figure II.4: The time behaviour of concurrence and other truncated nonclassicality witnesses for two interacting modes with noticeable sudden vanishings and reappearances. The time evolution is ruled by a unitary evolution of the frequency model with the following assumptions: (a) the initial pure state $|01\rangle$ and (b) the initial mixed state, given by Eq. (II.182) with $p = 0.8$, both analyzed in Section II.4 . Key: C (a thick solid curve)– the concurrence, B (a thin solid curve)– the nonlocality; $\tilde{H}$ (a dotted curve)– the entanglement witness, given by Eq. (II.150), linked with a violation of the first Hillery-Zubairy inequality; $\tilde{S}$ (a dashed curve) for $S_0 = 1/2$ and $\tilde{D}$ (a dot-dashed curve) for $D_0 = 1$– nonclassicality witnesses describing the photon-number-difference correlations (Eqs. (II.103) and (II.105) respectively). It is worth stressing that according to the standard approach, the SR should appear after some finite time after SV for an appropriate witness. This condition is fulfilled for all witnesses of the mixed-state evolution (b), but only for some witnesses of the pure-state evolution (a) [Bartkowiak2011].

It can be seen that the state is nonclassical according to a definition from Criterion 1 because it contains the derivative of Dirac's δ function. The β functions are solutions of classical equations of motion, given by Eq. (II.175). For such system concurrence and nonlocality can be expressed by simple formulas as

$$C(t) = B(t) = |\sin(2\kappa t)|, \tag{II.178}$$

the Hillery-Zubairy entanglement witness as

$$\tilde{H}(t) = \frac{1}{4}\sin^2(2\kappa t), \tag{II.179}$$

and nonclassicality witnesses as

$$\tilde{S}(t) = \max\left[0, \cos^2(2\kappa t) - S_0\right], \tag{II.180}$$

$$\tilde{D}(t) = \max\left\{0, D_0[2\cos(2\kappa t) - D_0]\right\}. \tag{II.181}$$

The behaviour of the witnesses and concurrence is depicted in Fig. II.4 (a). It is clear that all used witnesses (and concurrence) show the periodic SV and SR effects. It is worth noting that SV and SR of concurrence corresponds to the maximum value of $\tilde{S}$. One can also use Mandel's parameters to find the photon-number sub-Poisson statistics of the fields. The truncated Mandel's parameters are equal to $\tilde{Q}_1 = \sin^2(\kappa t)$ and $\tilde{Q}_2 = \cos^2(\kappa t)$. The behaviour of these parameters can be understood by recalling the classical-like interpretation of two linearly coupled oscillators when one of them is initially excited ($Q_2 > 0$) and the other one unexcited ($Q_1 = 0$). Then, the out-of-phase SVs and SRs take place. This effect can be interpreted as periodically transferred excitations between oscillators.

It should be also mentioned that from an orthodox point of view a SV (of some witness) should not be instantly followed by a SR. For $\tilde{D}$ with $D_0 > 0$ and for $\tilde{S}$ with $S_0 > 0$ (as it is shown in Fig. II.4(a)) this condition is fulfilled. However, one can raise an objection concerning the behaviour of $\tilde{D}$ for $D_0 = 0$, and $\tilde{S}$ for $S_0 = 0$ and Mandel's parameters. In these cases, and for concurrence, a SV is instantly followed by a SR, so they are not proper SV and SR effects.

Other examples of the SV and SR can be obtained by analyzing the system of two noninteracting modes with initially mixed states. It is possible to choose initial state as a Werner-like state $\hat{\rho}_0(0)$, given by Eq. (II.154) for $m = 0$ and $|\Psi_0\rangle = (|01\rangle - i|10\rangle)/\sqrt{2}$. Density matrix of the system evolves as follows

$$\hat{\rho}_0(t) = p|\Psi_0(t)\rangle\langle\Psi_0(t)| + \frac{1-p}{4}\hat{I}, \tag{II.182}$$

where

$$|\Psi_0(t)\rangle = \frac{1}{\sqrt{2}}\left[f_-(t)|01\rangle - if_+(t)|10\rangle\right] \tag{II.183}$$

with $f_\pm(t) = \cos(\kappa t) \pm \sin(\kappa t)$. Concurrence and entanglement witnesses with corresponding times of the first SV have the form of

$$C(t) = \max[0, p|c| - (1-p)/2] \;\Rightarrow\; t_{\rm SV}^{(C)} = f\left(\frac{1-p}{2p}\right), \tag{II.184}$$

$$B^2(t) = \max[0, p^2\left(1+c^2\right) - 1] \;\Rightarrow\; t_{\rm SV}^{(B)} = f\left(\frac{\sqrt{1-p^2}}{p}\right), \tag{II.185}$$

$$\tilde{H}(t) = \frac{1}{4}\max[0, (pc)^2 - (1-p)] \;\Rightarrow\; t_{\rm SV}^{(\tilde{H})} = f\left(\frac{\sqrt{1-p}}{p}\right), \tag{II.186}$$

where $f(x) = \arccos x/(2\kappa)$ and $c = \cos(2\kappa t)$. One can see that the first SR appears at the time equals

$$\kappa t_{\rm SR}^{(i)} = \pi/2 - \kappa t_{\rm SV}^{(i)} \tag{II.187}$$

for $i = C, B, \tilde{H}$. For $p = 0.8$ SVs and SRs occur in the following order (Fig. II.4(b)):

$$t_{\rm SV}^{(B)} < t_{\rm SV}^{(\tilde{H})} < t_{\rm SV}^{(C)} \;\Rightarrow\; t_{SR}^{(B)} > t_{\rm SR}^{(\tilde{H})} > t_{\rm SR}^{(C)}. \tag{II.188}$$

The nonclassicality witnesses $\tilde{D}$ and $\tilde{S}$, from Eqs. (II.103) and (II.105), respectively, evolve as

$$\tilde{S}(t) = \max[0, \frac{1}{2}(1-p) + p^2 \sin^2(2\kappa t) - S_0], \tag{II.189}$$

$$\tilde{D}(t) = \max[0, \frac{1}{2}(1-p) + 2D_0 p \sin(2\kappa t) - D_0^2]. \tag{II.190}$$

It needs to be stressed that for $S_0 = 0$ and $p < 1$, there is no complete vanishing of $\tilde{S}(t)$. For an initial Bell state ($S_0 = 0$ and $p = 1$), $\tilde{S}(t)$ periodically vanishes to zero and instantly increases. Thus, this case is not a good example of the SV and SR effects. However, it can be a proper one for $0 < p < 1$ (Fig. II.4(b)). The first SVs occur at the times

$$t_{\rm SV}^{(\tilde{S})} = \frac{\pi}{4\kappa} + f\left(\frac{\sqrt{2S_0 + p - 1}}{\sqrt{2}p}\right), \tag{II.191}$$

$$t_{\rm SV}^{(\tilde{D})} = \frac{\pi}{4\kappa} + f\left(\frac{2D_0^2 + p - 1}{4D_0 p}\right), \tag{II.192}$$

and the first SRs appear at $t_{\rm SR}^{(\tilde{S})} = \pi/\kappa - t_{\rm SV}^{(\tilde{S})}$ and $t_{\rm SR}^{(\tilde{D})} = 3\pi/(2\kappa) - t_{\rm SV}^{(\tilde{D})}$.
It is worth stressing that the first occurrence of these witnesses can be seen at earlier times, i.e., $t = \pi/(2\kappa) - t_{\rm SV}^{(i)}$ for $i = \tilde{S}, \tilde{D}$. I can conclude that one can choose the threshold values S_0 and D_0 for any $0 < p < 1$, that would it make possible to obtain the SVs and SRs of these witnesses for the photon-number-difference correlations at arbitrary evolution times (also for an initially disentangled system).

4.3 Periodic sudden vanishing of nonclassicality witnesses for a single mode

In this subsection it will be shown that the SV and SR of nonclassicality occur also for a single-mode case. In order to present this, single-mode anharmonic oscillator is analyzed, which can be modelled by the interaction Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2}\hbar\kappa(\hat{a}^\dagger)^2\hat{a}^2, \tag{II.193}$$

for describing e.g. the optical Kerr effect. The coherent state $|\alpha_0\rangle$ under evolution ruled by this interaction evolves periodically into a nonclassical state

$$|\psi(t)\rangle = e^{-|\alpha_0|^2/2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} \exp\left[\frac{i}{2}n(n-1)\tau\right] |n\rangle, \tag{II.194}$$

where τ is a rescaled time κt. The known Schrödinger cat and kitten states can be obtained from the state given by Eq. (II.194) after some evolution time as superposition of macroscopically distinguishable two [119] or more [120, 121] coherent states, respectively. The choice of this example is justified by a high-degree quadrature squeezing [66, 122, 123] of the Kerr state (beside some other intriguing nonclassical properties of this model–see, e.g., Ref. [124] and references therein). A single-mode normally-ordered variance S_{x_ϕ} can be compactly written as follows

$$S_{x_\phi} = 2|\alpha_0|^2[1 + f_{12}\cos(\tau_{12} + \tau) - f_{21}(\cos\tau_{21} + 1)] \tag{II.195}$$

in terms of auxiliary functions defined by

$$\tau_{kl} = k|\alpha_0|^2 \sin(l\tau) + 2(\phi - \phi_0)$$

and

$$f_{kl} = \exp\{k|\alpha_0|^2[\cos(l\tau) - 1]\}$$

with $\alpha_0 = |\alpha_0|\exp(i\phi_0)$. One can detect quadrature squeezing if $S_{x_\phi} \overset{\text{ncl}}{<} 0$ or, equivalently, if the truncated witness $\tilde{S}_{x_\phi} \overset{\text{ncl}}{>} 0$, defined by Eq. (II.108) with Eq. (II.195) and $\boldsymbol{\phi} = \phi$ (with a threshold value S_0 to be zero in this section and in Fig.II.1). The principal squeezing S_{opt} can be described as follows (by applying the results of Refs. [66, 122, 123]):

$$S_{\text{opt}}(t) = 2|\alpha_0|^2\Big(1 - f_{21} - \sqrt{f_{22} + f_{41} - 2f_{12}f_{21}\cos\tau'}\Big), \tag{II.196}$$

where $\tau' = \tau_{12} - \tau_{21} + \tau$. As before, the principal squeezing occurs if $S_{\text{opt}} \overset{\text{ncl}}{<} 0$ or for the truncated witness $\tilde{S}_{\text{opt}} \overset{\text{ncl}}{>} 0$, as given by Eqs. (II.110) and (II.196). The results following from these conditions can be seen in Fig. II.1 for some specific amplitude of the initial coherent state.

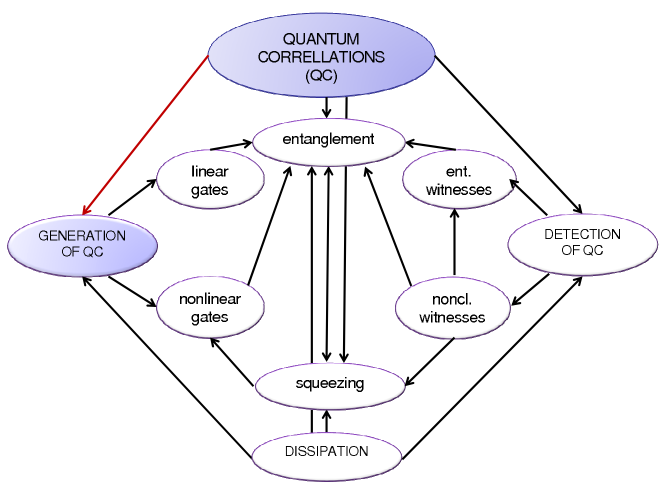
To obtain a sudden decay of entanglement or nonclassicality in the original form one can easily introduce dissipation to the system. In such a case, it is possible to solve a master equation in Markovian approximation as in Subsection II.4.1 (one can use Eq. (II.153) for a case of single mode– $k = 1$). Introducing dissipation would obviously break periodicity of SVs and SRs. Depending on dissipation, SV would appear after some evolution time. However, it should be stressed that the introducing dissipation is not a necessary condition to observe SV in this model. In Ref. [125, 126, 127] it is analyzed sudden vanishing of entanglement in two-mode dissipative coupled Kerr models more specifically .

It is important to emphasize the fact that oscillations of the entanglement measures in systems interacting with the non-Markovian reservoirs (see, e.g., Ref. [128]) cannot be compared to the periodic vanishing of the entanglement and nonclassicality witnesses analyzed in this thesis and obtained via unitary evolution of state.

Chapter III

Experimentally- friendly methods of generation of quantum correlations: Efficient implementation of quantum gates

1 A definition and types of quantum gates



From the point of view of quantum computation, quantum correlations are the most important features which are supposed to be used in protocols. They are a manifestation of a nontrivial interaction between two physical objects interpreted as qubits. As the analogy with classical computation is taken into account one can see that the most important are operations which combine two bits together in the form of gates like the NAND or NOR gates. In a quantum regime such gates introduce quantum correlations (in particular entanglement) between two quantum bits (the bits encoded in quantum objects). There exist many different ways of the implementation of both single-qubit and two- (or multi-) qubit quantum gates like atom and ion-traps, superconducting

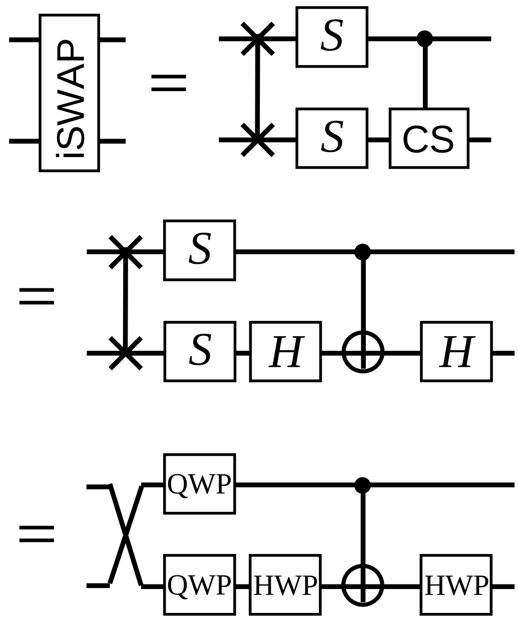


Figure III.1: The proposal of the decomposition of the iSWAP gate into the CS or equivalently the CNOT gate with the usage of the Hadamard gate, the S-phase gate and the SWAP gate (the more specific description can be found in text)[Bartkowiak2010b].

charge and flux qubits, nuclear magnetic resonance, spin- and charge-based quantum dots, nuclear spin quantum computing [131, 132].

The term of quantum gate was constructed in analogy to the definition of classical gate, so the quantum gate is a device which can perform operation on bits, in this case qubits. In other words, it is a kind of a black box which can manipulate quantum objects in the way described by the unitary operation representing action of quantum logic gate. The most fundamental single-qubit quantum gates are the ones represented by the Pauli matrices X,Y,Z and the Hadamard gate. However, to perform the more complicated operations on qubits it should be possible to perform operations more complicated than single-qubit ones. Extending the analogy to the classical computation the idea of controlled-NOT (CNOT) gate appeared. The prototype for this gate was the classical NAND gate, which can be used to build up any classical logic circuit. Obviously, one cannot drag analogy too far in a sense that one should remember about features of qubits. In case of quantum multiqubits gates, in fact, one has to deal with introducing interaction between two quantum objects. Moreover, the so called universal two-qubit gates (like e.g. the CNOT gate) can be used to construct arbitrary quantum circuit (with additional single-qubit gates) [18].

There exists a set of such universal gates like the iSWAP, CNOT and CS gates, which are formally equivalent. Thus, the choice of the universal gate depends on a particular case and needs to be connected with experimental feasibility or specific qubit interactions in studied systems. For instance, as far as solid-state system is taken into account it is more convenient to consider an implementation of the iSWAP gate (than the CNOT or CS) due to it can be described in terms of the Heisenberg or XY models. In comparison, the CNOT operations can be described by the less common Ising interactions. Therefore, the iSWAP gates for the solid state qubits were analyzed as the efficient quantum-information processing [133]. However, one can easily show that under unitary transformation the iSWAP and the CNOT gates are equivalent. The action of the iSWAP gate on an arbitrary pure state of two photon-polarization qubits can be described as the transformation of the state

$$|\psi_{\rm in}\rangle = \alpha_1|HH\rangle + \alpha_2|HV\rangle + \alpha_3|VH\rangle + \alpha_4|VV\rangle \tag{III.1}$$

into

$$|\psi_{\rm iswap}\rangle = \alpha_1|HH\rangle + i\alpha_2|VH\rangle + i\alpha_3|HV\rangle + \alpha_4|VV\rangle,$$

where, e.g., $|HV\rangle = |H\rangle|V\rangle = |H\rangle \otimes |V\rangle$ and $|H\rangle$ and $|V\rangle$ represent horizontal and vertical polarization states, respectively. According to Schuch and Siewert [134] the CNOT gate can be decomposed into two iSWAP gates or SWAP and iSWAP gates. One can see that the iSWAP gate can be obtained as (see the top circuit in Fig. III.1)

$$U_{\rm iSWAP} = U_{\rm CS}(S \otimes S)U_{\rm SWAP} \tag{III.2}$$

by inverting the Schuch-Siewert relation and replacing the CNOT by the CS gate; S is the phase gate (which can be implemented by a quarter-wave plate (QWP) with fast horizontal axis)

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

and the CS gate can be written as

$$U_{\rm CS} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The last part of the scheme is the SWAP gate, which is a classical gate and can be implemented deterministically, e.g., by brute-force exchanging qubits or waveguides carrying single qubits. Obviously, using the Hadamard gate it is possible to implement the iSWAP gate also using the CNOT gate instead of the CS, which can be seen in Fig. III.1 (center). The relation is as follows

$$U_{\rm CS} = (I \otimes H)U_{\rm CNOT}(I \otimes H).$$

The Hadamard gate is a deterministic gate which can be easily implemented by the half-wave plate (HWP) tilted at $\theta = \pi/8$ and is described by

$$U_{\rm HWP}(\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}. \tag{III.3}$$

In this part of my thesis I have focused on presenting optical methods of implementing such circuits. Photons seem to be the best potential carrier of quantum information in the sense that they are potentially free from decoherence. However, there appear different kinds of problems in the implementation of optical quantum circuits. The most available and the easiest to perform are linear-optical implementations. The single-qubit gate operations can be reconstructed by using simple optical devices like beams-splitters, half quarter wave-plates or phase shifters. However, one needs to remember that to build up a full quantum circuit one needs to introduce an interaction between photons. In case of linear-optical implementation it possible to do this but only by introducing projective measurements with photodetectors. On this account, such gates are no longer deterministic. Their action is successful only with a certain probability. Still, it is possible to distinguish between desired and unwanted events. The history of research under linear-optical implementations can be divided into two periods for which the boundary is the appearance of article of Knill, Laflamme and Milburn (KLM) [129]. Before publishing the article of KLM common was the opinion that to implement two-qubit gate the nonlinear element, which can introduce interaction between photons, is necessary. They proposed scheme to implement a linear -optical two-qubit gate including single-photon sources, quantum teleportation and error correction. Their article was crucial not only because it used photodetection to introduce nonlinearity between photons, but also as they showed how to construct the scalable quantum circuit. After this paper the interest of the linear-optical implementation increased, especially due to the improved scalability of the KLM's proposal. Although in the case of linear-optical implementation one needs to struggle with nondeterministic entangling gates, still they remain one of the most accessible experimentally due to available and easy to obtain resources. In a further part of thesis one can find an analysis of the most common types of linear-optical implementations of two-qubit entangled gates and two proposals designed with the view of experimental efficiency of experimental realizations.

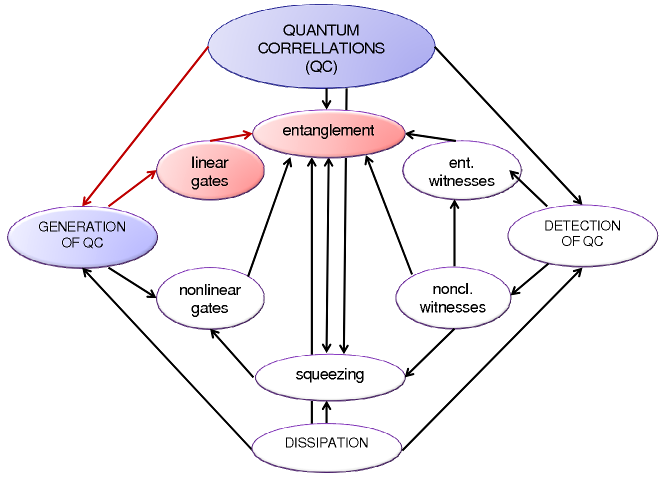
One of the ideas of improving scalability of KLM protocol was to introduce cluster-states, which are highly entangled states of multiple qubits. Raussendorf and Briegel [135] proposed a method of quantum computation called one-way computing or graph-state computing (cluster states are often represented by nodes with entanglement marked as lines), which is based on cluster states and local measurements in appropriate basis on cluster state [131]. Using this protocol it is possible to implement both single and two-qubits operations (in particular universal gates like the CS or CNOT gates) nearly deterministic with additional feedforward and even conventional detectors. It is worth stressing that such gates can be deterministic only under condition that a specific cluster-state is given. Moreover, a nonprobabilistic character of such gates is due to the fact that it is assumed to be something strictly easier than applying the true CNOT gate on *independently* prepared input photonic qubits. As far as linear optics is taken into account the no-go theorem for the Bell measurement is valid and it disables two-qubit gates to be truly deterministic. In a further part of my thesis implementations using this idea are deliberately omitted. Some exemplification of the implementations could be found in [135, 136, 137, 138, 139, 140, 141, 142].

The second type of trials of optical multi-qubit universal gates realization is usage of nonlinearity directly. For this purpose one can use internal nonlinearity of media characterized by refractive index n_{Kerr} [131]:

$$n_{Kerr} = n_0 + \chi^{(3)} E^2, \tag{III.4}$$

where n_0 is the ordinary refractive index, E^2- optical intensity of a probe beam with proportional constant $\chi^{(3)}$. This kind of media is called the Kerr medium and as one can see from Eq.(III.4) it affects beam passing through the medium by introducing an additional phase shift, proportional to its intensity. In case of the so called cross-Kerr medium phase shift of a signal depends on the second probe beam. The last effect can therefore be used to perform the CS gate. This type of the construction of a quantum gate seems to be more natural than using a set of linear-optical devices, especially as it can be performed without destructive measurement. However, now the accessible nonlinearity is too weak to perform the action of quantum gate in a significant way in case of single photons ($\chi^{(3)} \simeq 10^{-22} m^2 V^{-2}$ [131]). Not only the available nonlinearity of Kerr media is very small but also in such types of media there exist other effects which can prevent the gate operation. In a further Section III.3 it will be shown a setup to improve nonlinearity in the Kerr medium to perform the CS gate more effectively and analyzed the influence of spectral effects on fidelity of gate.

2 Linear quantum gates



In this section it will be shown via the analysis of examples known from literature (see Table III.1), different types of linear-optical implementations of quantum gates. Obviously, due to their simplicity and accessibility, linear-optical implementations using postselection and based on counts of photodetectors (see a review [131] references therein) stimulate the interest of researchers. Linear implementations became especially interesting after pioneering articles of Knill, Laflamme, and Milburn (KLM) [129] and Koashi, Yamamoto, and Imoto (KYI) [130]. These works had a significant impact on development and simplicity of new implementations of entangling two-qubit gates i.e. the controlled-NOT (CNOT) and controlled-sign (CS) gates as listed in Table III.1. However, as far as most proposals of implementations are theoretical draft, they are based on selective detectors (i.e., single-photon or photon-number resolving). Thus, using such detectors one is able to obtain gates with higher probability of success than with the more available nonselective ones. Better probability of success takes place therefore at the expense of lack of simplicity and accessibility of resources. Further it will be shown two examples of implementations which are based on conventional detectors (also referred to as bucket detectors) which indicate the presence or absence of photons only.

However, a kind of detectors is not the only obstacle which appears during an analysis of linear-optical implementations of two-qubit quantum gates. The second problem is nondestructiveness of gates (ability to operate on qubits without destroying control and target qubits– in case that any of them would be destroyed the gate will be called in this thesis as a destructive gate). From an experimental point of view implementations which combine those two features (nonselective detectors and being nondestructive) would be most desired. However, according to my knowledge, there are only few proposals of such schemes. As one can see from the list in Table III.1 only the proposal of Zou *et al.* [143] really fulfilled both conditions. This implementation is based on a quantum encoder (described in Subsection III.2.1), which allows one to use conventional detectors and preserve both needed outcomes (a control and a target qubit). The other gates which likewise use the idea of quantum encoder device are schemes by Gasparoni *et al.* [144] (scheme #14) and Zhao *et al.* [145] (scheme #15). In fact, both are the instances of experimental realizations of the modified Pittman *et al.* gate [146] (scheme #12) without feedforward. The gate of Pittman was designed for photon-number resolving detectors. However, in the realization performed by Gasparoni due to the lack of detectors for appropriate wavelength conventional detectors were used during the experiment. To be able to realize this implementation with conventional detectors unfortunately two additional ones were needed. Thus, in the end they realized a destructive version of the nondestructive CNOT gate of Pittman *et al.* [146], due to necessity of measurement of each of the outcomes.

Below other kinds of linear quantum gates have been analyzed in more detail. The implementations can be gathered in several groups depending on specific characteristic features (they are listed in Table III.1). My goal was to classify them based on resources needed to implement them. I have divided them as follows [Bartkowiak2010b]:

1. with unentangled ancillae,
2. with entangled ancillae (not only the EPR states, but also the Gottesman-Chuang four-entangled state and the GHZ states),
3. without ancillae at all.

The implementations were compared taking into account the total probability of success, destructiveness/nondestructivness of gate, type of detectors and the presence of feedforward mechanism. To clarify a definition of a classical feedforward will be recalled. In this thesis *feedforward* means that it is possible to use classical outcomes of measurement included in the scheme to preform unitary operations on the remaining modes.

As mentioned before, in I group ancillae were prepared in unentangled states. The highest probability of success for this kind of implementation is equal to $2/27$ [149] (for scheme #3 in Table III.1). It is important to emphasize that there exists only numerical proof [160] (not analytical one in contrast to the nonlinear sign shift gate [161]) that in case of a gate with two unentangled ancillae $2/27$ is the rigorous tight upper bound on the success probability. Furthermore, adding more ancillae does not increase the value of this probability. Obviously implementations with usage of feedforward can achieve higher probability of success. In Table III.1 one can see example of the gate with probability of success accounts for $1/8$ for gates with two ancillae [151, 152] (schemes #6 and #7) or even to $1/4$ with one ancilla [150] (schemes #4 and #5) at the expense of destructing the output states. One should realize that destruction (measuring) of any of outcomes always increases probability of success (this concerns all groups analyzed here).

In group II, the highest probability of success is equal to $1/16$ without feedforward [130, 146] (schemes #9 and #12) and $1/4$ with feedforward [129, 146, 151] (schemes #8, #13, and #16).

Group III contains schemes of the CS/CNOT inspired by ideas of Hofmann and Takeuchi [154] (scheme #25), and Ralph *et al.* [155] (scheme #26). Beside, the three gates mentioned above, remaining implementations are experimental realizations of #25 and #26 . They use a beam splitter with the reflection coefficient equal to $1/3$, which is characteristic for those implementations. All of them are destructive (perform the measurement of both the control and target bits for postselection) and gain probability of success equal to $1/9$.

It should be stressed that although cluster type gates were deliberately neglected in Table III.1, one can find in considered implementations two examples of the usage of cluster states . The implementations of Gottesman and Chuang [147] (scheme #18) and closely related proposal of Pittman *et al.* [146] (scheme #19) are the schemes which in nondestructive and nondeterministic way perform the CNOT gate operation via the usage of a four-photon entangled state $|\chi\rangle$. This state is equivalent to a four-qubit cluster state, under a local unitary transformation. It is included in Table III.1 as it uses the state $|\chi\rangle$ as an ancillae (far from the type of usage of cluster states in the Raussendorf-Briegel protocol).

2.1 Schemes with conventional detectors and ancillae in the GHZ states

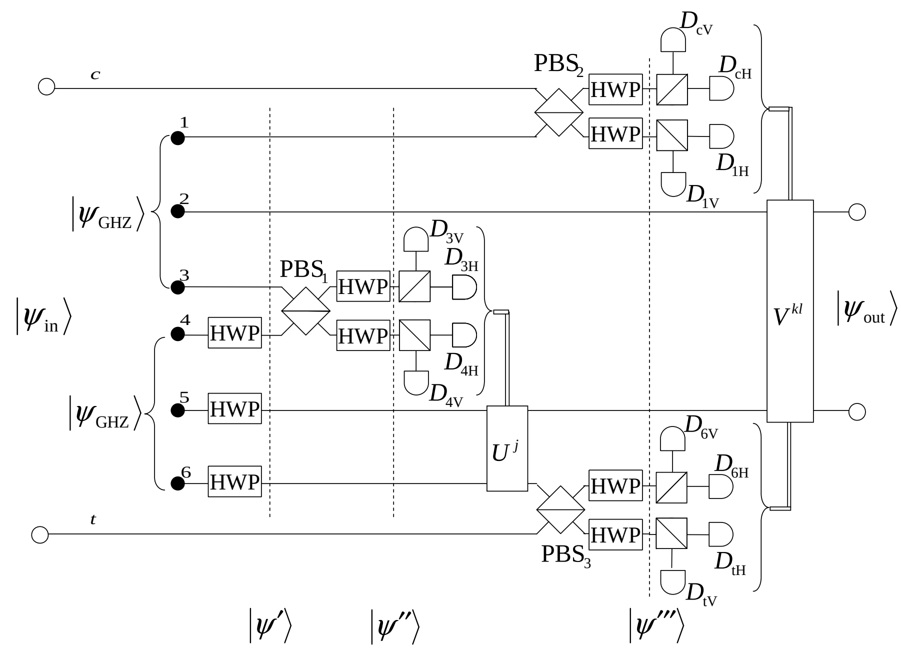


Figure III.2: Scheme I– a proposal of the implementation of the CNOT gate with the usage of conventional detectors and the GHZ states, given as $|\psi_{\rm GHZ}\rangle$, as ancillae. Key: $\mathrm{HWP} = U_{\rm HWP}(\pi/8)$ implements the Hadamard gate H; U^j and V^{kl} are conditional unitary operations given in Table III.2, where σ_z is implemented by $U_{\rm HWP}(0)$; D_k are photodetectors; PBS_i are polarizing beam-splitters in the *HV*-basis [Bartkowiak2010b].

In the following Subsections (III.2.1 and III.2.2) presented two schemes for implementations of the CNOT and CS gates in a nondestructive way using conventional detectors have been presented. Firstly I have focused on the scheme of the CNOT gate assuming ancillae in the GHZ state presented in Fig. III.2 (referred as Scheme I). The implementation presented in Scheme I is designed based on the schemes of Gottesman and Chuang [147] (scheme #18 in Table III.1) and Pittman *et al.* [146] (scheme #19). However, scheme #19 was created for selective detectors. My main goal in this subsection is to present the possibility of modification of #19 to adjust it for conventional detectors. In both schemes (#18 and #19) ancilla was the cluster-type state, equivalent (under local unitary transformations) to the standard four-qubit cluster states [135], of the form

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|HH\rangle|\Phi^+\rangle + |VV\rangle|\Psi^+\rangle), \tag{III.5}$$

where

$$\begin{aligned} |\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle), \\ |\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle), \end{aligned} \tag{III.6}$$

are the Bell's states (the EPR states). The state $|\chi\rangle$ can be obtained in different ways, i.e. using the linear-optical nondestructive scheme proposed by Wang *et al.* [162] with success probability accounts for $\eta^3/8$. However, Gottesman-Chuang [147] designed a scheme to create such a state with probability $\eta^2/2$. This proposal is further applied to Scheme I. The more detailed proposal of Scheme I, with a scheme to generation $|\chi\rangle$ included, can be seen in the picture Fig. III.2. As an input state an arbitrary state $|\psi_{\rm in}\rangle$ was assumed, which can be given by

$$|\psi_{\rm in}\rangle = \alpha_1|HH\rangle + \alpha_2|HV\rangle + \alpha_3|VH\rangle + \alpha_4|VV\rangle, \qquad \text{(III.7)}$$

which is in modes c (control) and t (target). Other $1-6$ modes correspond to two ancillae in the GHZ states of the form

$$|\psi_{\rm GHZ}\rangle = \frac{1}{\sqrt{2}}(|HHH\rangle + |VVV\rangle). \qquad \text{(III.8)}$$

the GHZ states are treated in Scheme I as a resource.

The particular operations which are done on the input states ($|\psi_{\rm in}\rangle$ and two ancilla) are following. Firstly, the photons in modes 4, 5, and 6 go through the Hadamard gates. This is a deterministic gate which can be easily implemented by the HWP tilted at $\theta = \pi/8$ (according to Eq. (III.3)) and which is described by transformations

$$|H\rangle \to \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$$

and

$$|V\rangle \to \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$$

and for two photons with different polarizations,

$$|HV\rangle \equiv |1_H 1_V\rangle \to \frac{1}{\sqrt{2}}(|2_H, 0_V\rangle - |0_H, 2_V\rangle). \qquad \text{(III.9)}$$

The Hadamard gate can also be equivalently implemented by a 50/50 beam splitter when one of the input modes is H-polarized and the other is V-polarized, together with two $(-\pi/2)$ phase shifters [131]. The latter implementation is particularly useful in understanding the Hadamard transformation applied to more than one photon. After those operations the total input state has the form of

$$\begin{aligned} |\psi'\rangle &= \frac{1}{2\sqrt{2}}(|H\rangle_1|H\rangle_2|H\rangle_3 + |V\rangle_1|V\rangle_2|V\rangle_3) \\ &\quad \otimes (|H\rangle_4|H\rangle_5|H\rangle_6 + |V\rangle_4|V\rangle_5|H\rangle_6 + |V\rangle_4|H\rangle_5|V\rangle_6 + |H\rangle_4|V\rangle_5|V\rangle_6). \end{aligned} \qquad \text{(III.10)}$$

Then the above state is changed by polarizing beam-splitter PBS$_1$ performed in the HV-basis (i.e., which transmits H-polarized states and reflects V-polarized states) and the two Hadamard gates into

$$\begin{aligned} |\psi''\rangle &= \frac{1}{2}\left(|\Phi^+\rangle_{34}U^0 + |\Psi^+\rangle_{34}U^1\right)|\chi\rangle_{1256} \\ &\quad + \frac{1}{2}\left(|V\rangle_1|V\rangle_2|\xi\rangle_3|0\rangle_4|\Phi^+\rangle_{56} + |H\rangle_1|H\rangle_2|0\rangle_3|\xi\rangle_4|\Psi^+\rangle_{56}\right), \end{aligned}$$

where

$$U^j = (\sigma_z^{(5)} \otimes \sigma_z^{(6)})^j$$

(j=0,1) are given in terms of the Pauli's matrices σ_z, $|\xi\rangle = \frac{1}{\sqrt{2}}(|2_H\rangle - |2_V\rangle)$, and $|0\rangle \equiv |0_H\rangle|0_V\rangle$ denotes no photon in H and V modes. The detection measurement using detectors D_{3H}, D_{4H},

D_{3V} and D_{4V} closes the part for generating $|\chi\rangle$ state, which is further used as ancilla to the second part of the Scheme I. The desired state is obtained with probability $\eta^2/2$ whenever two photons separately reach detectors D_{3H} and D_{4H} or D_{3V} and D_{4V} (see Table III.2). The success events are also the ones connected with two separate clicks in detectors D_{3H} and D_{4V} or D_{3V} and D_{4H}. However, in this case to obtain the state $|\chi\rangle$ one needs to apply two additional Pauli's gates σ_z on photons in modes 5 and 6. As in the case of the Hadamard gate, the Pauli σ_z gate can be implemented by the HWP at $\theta = 0$ according to Eq. (III.3).

Such generated state is used as ancilla in the further part of Scheme I, which is the CNOT gate performed on modes c and t with the input state $|\psi_{\rm in}\rangle$, given by Eq. (III.7). The state $|\psi''\rangle$ is the state after measuring modes 3 and 4. This state is sent through PBS$_2$ and PBS$_3$ in the HV-basis and four HWPs which transform it into

$$\begin{aligned} |\psi'''\rangle \quad = \quad & \frac{1}{4}[|\Phi^+\rangle_{c1}(|\Phi^+\rangle_{6t}V^{00} + |\Psi^+\rangle_{6t}V^{11}) \\ & +|\Psi^+\rangle_{c1}(|\Phi^+\rangle_{6t}V^{10} + |\Psi^+\rangle_{6t}V^{01})]|\psi_{\rm out}\rangle_{25} + \frac{\sqrt{3}}{2}|\psi_{\rm err}\rangle, \end{aligned} \tag{III.11}$$

where

$$V^{kl} = (\sigma_z^{(2)})^k \otimes (\sigma_z^{(5)})^l$$

for $k, l = 0, 1$. In the Eq. (III.11) part called $|\psi_{\rm err}\rangle$ refers to all events in which two photons enter one detector. Such situation is a source of errors caused by the fact that conventional detectors can only detect the presence of photons and not their number. The successful events are those in which four separate detectors clicks, D_{iH} or D_{iV} for some i ($i = c, 1, 6, t$) (two detectors from part to generate state $|\chi\rangle$ and two from the CNOT part). This scheme is suitable for using conventional detectors because there are only four photons present in the setup (without counting output photons). The Hadamard operations performed before polarizing beam-splitters allow one to recognize individual cases of successful events and use feedforward to correct the output via unitary operations, when it is needed. Finally, one obtains $|\psi_{\rm out}\rangle_{25} = |\psi_{\rm cnot}\rangle$, where

$$|\psi_{\rm cnot}\rangle = \alpha_1|HH\rangle + \alpha_2|HV\rangle + \alpha_3|VV\rangle + \alpha_4|VH\rangle, \tag{III.12}$$

as required by the CNOT operation for the input state given by Eq. (III.7).

The probability of a success for this setup of the CNOT operation implementation accounts for $\eta^4/4$ if the state $|\chi\rangle$ is treated as resource and $\eta^6/8$ for the whole Scheme I shown in Fig. III.2, including the generation of the state $|\chi\rangle$ (with the GHZ states as resource). The GHZ states can be obtained from e.g., EPR-state pairs. Zeilinger *et al.* [163] presented a proposal of the EPR states generation by the usage of a nondestructive optical method. As for the first time the GHZ state was experimentally obtained by Bouwmeester *et al.* [164], there appeared various optical proposals for creating the GHZ state (see, e.g., Refs. [165, 166, 167, 168]). In principle this methods can be used to obtain the GHZ state needed to the proposed Scheme I.

2.2 Scheme with conventional detectors and ancillae in the EPR states

As a second example of the nondestructive implementations based on conventional detectors is the scheme of the CS gate (equivalent to the CNOT under two Hadamard operations), with ancillae in the EPR or EPR-like states, shown in Fig. III.3. To analyze experimentally-oriented effectiveness of this scheme it has been also considered a few kinds of detector imperfections (dark counts, finite efficiency, and no photon-number resolution) and realistic sources of the ancilla and input states.

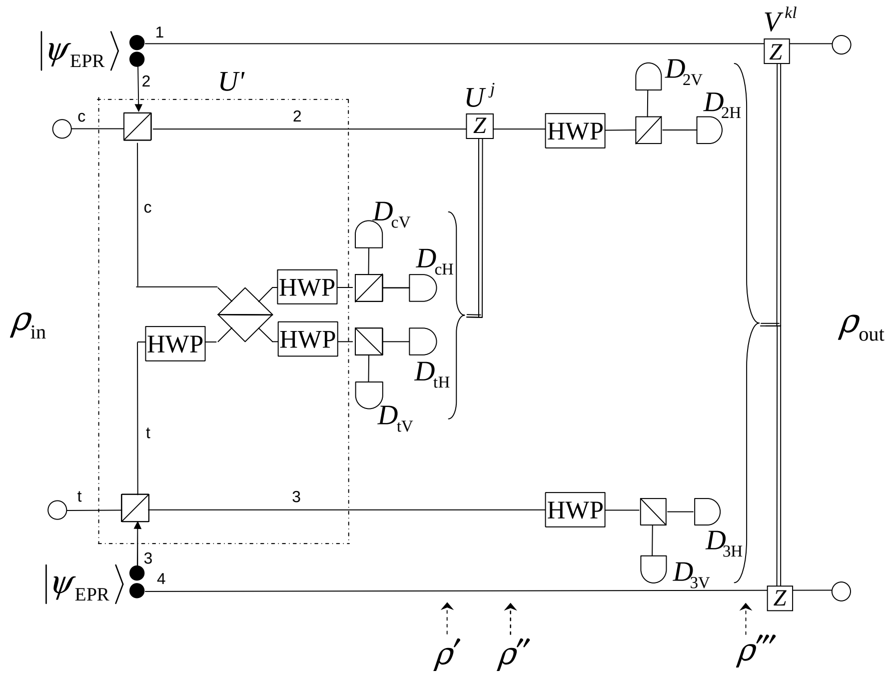


Figure III.3: Scheme II- a proposal of the CS gate implementation with the usage of conventional detectors and two perfect or the non-perfect EPR states $|\psi_{\rm EPR}\rangle$ as ancillae. Notation is in agreement with the one in Fig. III.2. In this case $\rm HWP = U_{HWP}(\pi/8)$ corresponds to the Hadamard gate H. In the text one can find definitions of states and unitary operations U', U'', U^j, and V^{kl} (and in Table III.3) [Bartkowiak2010b].

This setup (Scheme II) is based on gates of Pittman *et al.* [146] (scheme #12) and Zou *et al.* [143] (scheme #17). It is worth emphasizing that it had been previously exploited by Wang *et al.* [169] as a part of their iSWAP scheme. The main idea of Zou *et al.* behind obtaining nondestructive gates with conventional detectors was to use the quantum encoder mentioned previously. The quantum encoder refers to a device which changes, with the probability of success equal to $1/2$ (to compare with $1/4$ without feedforward), an input state $\alpha|H\rangle + \beta|V\rangle$ into $\alpha|HH\rangle + \beta|VV\rangle$. This device is convenient to use to avoid the problem of destructiveness of the gate. Thus, both Refs. [143] and [169] exploited it (to be more specific two encoders with feedforward) in their implementations to finally perform a nondestructive gate. Similarly to the scheme #17 Scheme II relies on a double usage of quantum encoder and a triple usage of feedforward. It needs to be stressed that the main concept is slightly different. The idea is to measure outcomes of quantum encoders, which were previously combined. This is in contrary to the previous usage of quantum encoder in scheme #17 where output states of the encoders are measured separately. Due to combining two encoders it is possible to obtain a cluster-like state (two single-qubit quantum encoders in scheme #17 can give two separate EPR pairs). The perfect CS gate changes the arbitrary pure input state, given by Eq. (III.7) as follows

$$|\psi_{\rm cs}\rangle = \alpha_1|HH\rangle + \alpha_2|HV\rangle + \alpha_3|VH\rangle - \alpha_4|VV\rangle. \tag{III.13}$$

The fidelity can be expressed as

$$F = \langle\psi_{\rm cs}|\rho_{\rm out}|\psi_{\rm cs}\rangle, \tag{III.14}$$

where F refers to fidelity and is interpreted as a deviation of the output state $\rho_{\rm out}$ of a realistic CS gate from the state $|\psi_{\rm cs}\rangle$ of an ideal CS gate. A part of a scheme marked by U' consists of six gates (inside of dot-dashed box in Fig. III.3) and it can be written as

$$U' = U^{(c)}_{\rm HWP} U^{(t)}_{\rm HWP} U^{(ct)}_{\rm PBS} U^{(t)}_{\rm HWP} U^{(2c)}_{\rm PBS} U^{(t3)}_{\rm PBS}, \tag{III.15}$$

where $U^{(kl)}_{\rm PBS}$ denotes the PBS unitary transformation of k and l lines. Note that the PBS operation in the dual-line (dual-rail) notation (and assuming labelling of lines as shown Fig. III.3) can be understood as swapping of H-polarized modes and no action on V-polarized modes. As in the previous case (Scheme I from Subsection III.2.1) $U_{\rm HWP} = U_{\rm HWP}(\pi/8)$ is the Hadamard gate. Assuming that ancillae are the perfect EPR states, $|\psi_{\rm EPR}\rangle = |\Phi^+\rangle$, one can write the total initial state as

$$|\Psi_{\rm in}\rangle = |\psi_{\rm in}\rangle_{ct} |\psi_{\rm EPR}\rangle_{12} |\psi_{\rm EPR}\rangle_{34}, \tag{III.16}$$

where $|\psi_{\rm in}\rangle_{ct}$ is given by Eq. (III.7). After the action of the multigate U' on the initial state $|\Psi_{\rm in}\rangle$ one gets

$$U'|\Psi_{\rm in}\rangle = N_{\rm ok}|\psi_{\rm ok}\rangle + N_{\rm err1}|\psi_{\rm err1}\rangle + N_{\rm err2}|\psi_{\rm err2}\rangle, \tag{III.17}$$

where

$$|\psi_{\rm ok}\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle_{ct} U^0 + |\Psi^+\rangle_{ct} U^1)|\tilde{C}_4\rangle_{1234} \tag{III.18}$$

with $U^j = (\sigma_z^{(2)})^j$ (j=0,1), and

$$\begin{aligned} N^2_{\rm ok} &= 1/8, \\ N^2_{\rm err1} &= (8|\alpha_1|^2 + 7|\alpha_2|^2 + 6)/16, \\ N^2_{\rm err2} &= |\alpha_2|^2/16 + (|\alpha_3|^2 + |\alpha_4|^2)/2. \end{aligned}$$

In general, $|\tilde{C}_4\rangle_{1234}$ is of the form

$$|\tilde{C}_4\rangle_{1234} = \alpha_1|HHHH\rangle + \alpha_2|HHVV\rangle + \alpha_3|VVHH\rangle - \alpha_4|VVVV\rangle. \tag{III.19}$$

In the special case of all equal coefficients state from Eq. (III.19) reduces to a four-entangled cluster state $|C_4\rangle$. As before, states $|\psi_{\rm err1}\rangle$ and $|\psi_{\rm err2}\rangle$ correspond to unsuccessful events. However, the first one can be excluded by measuring only modes c and t (the first postselection). The second one refers to all events in which more than one photon reaches a detector. The usage of conventional detectors does not allow one to distinguish this cases from single-photon states. However, even though $|\psi_{\rm err2}\rangle$ corresponds to the undesired cases, which *cannot* be uniquely excluded via the first postselection, they *can* be later excluded after measuring modes 2 and 3 (the second postselection). In an ideal case, by assuming conventional detectors without dark counts and the ancillae to be in the perfect EPR states, the probability of success accounts for $P = \eta^4/8$ and fidelity is equal to one as in the original scheme of Zou *et al.* [143](η^4 refers to clicks of four out of eight detectors– see Table III.3). Moreover, the factor 1/8 is just equal to $N^2_{\rm ok}$ in Eq. (III.17).

So far, it has been presented the transformations of states by assuming perfect sources of the ancilla states and no dark counts of detectors, both for Schemes I and II. Here, in contrast, I have used a numerical method assuming non-perfect sources of ancillae and input states, and dark counts. Let me analyze an imperfection introduced to the scheme by efficiency of detectors, mean dark count rate ν. The positive-operator-valued measure (POVM) elements associated with distinguishing vacuum (Π_0) and the presence of at least one photon (Π_1) can be written as

$$\begin{aligned} \Pi_0 &= \sum_{m=0}^{\infty} e^{-\nu}(1-\eta)^m |m\rangle\langle m|, \\ \Pi_1 &= 1 - \Pi_0\,, \end{aligned} \tag{III.20}$$

where $\nu = \tau_{\rm res} R_{\rm dark}$ is given in terms of the dark count rate, $R_{\rm dark}$, and the detector resolution time, $\tau_{\rm res}$ [170]. The EPR states as resource can be generated via spontaneous parametric down-conversion (SPDC). Approximated EPR-like states can be obtained as the output state of a type-II SPDC crystal or two type-I SPDC crystals sandwiched together and can be written in the form (see, e.g., Refs. [171, 172]):

$$|\psi_{\rm EPR}\rangle = (1-\gamma^2)^{-1/2}[|0\rangle|0\rangle + \gamma(|HH\rangle + |VV\rangle)] + \mathcal{O}(\gamma^2). \tag{III.21}$$

Parameter γ is given by the product of interaction time of the pump field and the crystal, their coupling constant, and a complex amplitude of the pump field. It differs from the EPR state $|\Phi^+\rangle$ as it contains both additional vacuum and higher order states. Parameter γ^2 is usually of the order 10^{-4}/pulse [170] and it describes the rate of single-photon pair generation per pulse of the pump field. Therefore, vacuum appears in superposition with high probability and it cannot be neglected when action of the gate is considered.

Each line in Schemes I and II can carry an arbitrary number of photons in H and V polarizations. In a dual-line notation it is possible to express qubits as $|H\rangle = |1\rangle_H|0\rangle_V \equiv |1_H, 0_V\rangle$, $|V\rangle = |0\rangle_H|1\rangle_V$, and $|0\rangle = |0\rangle_H|0\rangle_V$. After the action of U', performing measurement of photons by the detectors D_{cH}, D_{cV}, D_{tH}, and D_{tV}, ρ' has the form of

$$\rho' = \mathcal{N}\,{\rm Tr}_{ct}\left[\Pi_m^{(cH)}\Pi_{m'}^{(cV)}\Pi_n^{(tH)}\Pi_{n'}^{(tV)}U'\rho_{\rm in}(U')^\dagger\right], \tag{III.22}$$

where ${\rm Tr}_{ct} \equiv {\rm Tr}_{cH,cV,tH,tV}$, $\rho_{\rm in} = |\Psi_{\rm in}\rangle\langle\Psi_{\rm in}|$, $\mathcal{N}$ is a renormalization constant, and the POVM elements are given by Eq. (III.20); $m, m', n,$ and n' are equal to 1 or 0, and refer to clicks or no clicks of the detectors according to Table III.3.A. The state $\rho'' = U^j\rho'(U^j)^\dagger$ is obtained by performing conditional operations $U^j = (\sigma_z^{(2)})^j$ with $j = 0, 1$, defined in Table III.3.A. The state ρ'' is sent through the Hadamard gates at lines 2 and 3 ($U'' = U_{\rm HWP}^{(2)}U_{\rm HWP}^{(3)}$) and then by the detectors D_{2H}, D_{2V}, D_{3H}, and D_{3V}. After that one obtains

$$\rho''' = \mathcal{N}\,{\rm Tr}_{23}\left[\Pi_m^{(2H)}\Pi_{m'}^{(2V)}\Pi_n^{(3H)}\Pi_{n'}^{(3V)}U''\rho''(U'')^\dagger\right], \tag{III.23}$$

where ${\rm Tr}_{23} \equiv {\rm Tr}_{2H,2V,3H,3V}$, while m, n, m', and n' correspond to clicks or no clicks of the detectors according to Table III.3.B.

It is worth noting that the PBSs operations applied before measurement just convert polarization qubits into dual-line qubits. One can see that they can be omitted in a dual-line notation consistently in our numerical approach. To obtain the final output state one needs to apply the conditional gates

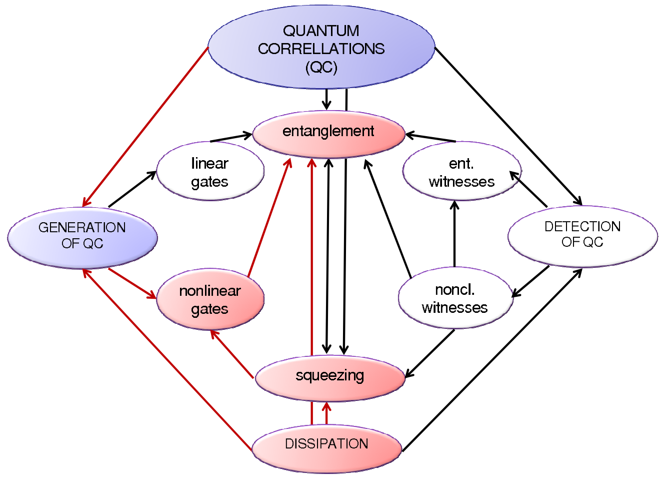
$$V^{kl} = (\sigma_z^{(1)})^k \otimes (\sigma_z^{(4)})^l$$

($k, l = 0, 1$) according to Table III.3.B. After this ρ'' is transformed into $\rho_{\rm out} = V^{kl}\rho''(V^{kl})^\dagger$. For simplicity, in our numerical calculations it has been reserved three-dimensional Hilbert space for each mode. Thus it has been set $|0\rangle_H = [1; 0; 0]$, $|1\rangle_H = [0; 1; 0]$, and $|2\rangle_H = [0; 0; 1]$, and analogously for V polarization. This is valid by assuming dark count rates and γ parameter to be relatively low. Otherwise, the higher-dimensional Hilbert spaces should be set.

To estimate the numerically realistic fidelity of a setup one needs to assume realistic values of conventional detectors [173] (see also Refs. [170, 174]): the detector efficiency to be $\eta = 0.7$, the dark count rate $R_{\rm dark} = 100\ {\rm s}^{-1}$, the detector resolution time $\tau_{\rm res}$=10 ns. For convenience, it has been assumed that all detectors are the same. The rate of single-photon pair generation per pulse of the pump field is set to be $\gamma^2 = 10^{-4}$/pulse [170]. Only the first cases in Table III.3 have been considered, to omit the necessity of performing the conditional operations. For simplicity and experimental verification of Scheme II, it is convenient to assume that SPDC was also used to generate the input state $|\psi_{\rm in}\rangle$ given by Eq. (III.21). The fidelity which has been obtained under these assumptions is relatively high and equal to $F \approx 0.97$.

As it has been shown in Section III.1 the universal gate CNOT, CS and iSWAP are equivalent under unitary transformations. Thus, the iSWAP gate can be expressed in terms of the CNOT or CS gates as it can be seen in Fig. III.1. Recently, Wang *et al.* [169] has proposed the linear-optical scheme for performing the iSWAP operation. Wang used the two EPR states as ancillae, classical feedforward and conventional detectors. The proposal of Wang is based also on the idea of the quantum encoder firstly proposed by Zou *et al.* [143]. However, the probability of success of this iSWAP implementation is surprisingly low and accounts for $\eta^4/32$. According to decomposition of the iSWAP gate presented in Fig. III.1 one can see that probability of success, in fact, can be reduced to probability of the CNOT or CS gates (as the other gates in decomposition can be implemented deterministically using the half-wave plates or quarter-wave plates). Evoking both the list of gates presented in Table III.1 and the Schemes I and II proposed by me in the last two subsections it is possible, to implement the iSWAP gate with probability of success even eight times higher than in the proposal of Wang (when the GHZ states are assumed as a resource) or four times higher assuming the same resources.

3 Possibility of usage an nonlinear medium



After a proposal of Knill, Laflamme and Milburn [129], which showed that photodetection can introduce nonlinearity to an optical computation, there was a big concern regarding the usage of a linear optics to provide the linear-optical quantum gates (as has been presented in Section III.2). Single photons proved to be good carriers for quantum information and linear optical schemes are experimentally available and easy to build with simple devices (like half-wave plates or beam-splitters). Nevertheless, one can not accomplish an entangling operation on qubits (e.g. the use linear optics to design arbitrary quantum circuits having one of the entangling operations and some other deterministic gates (e.g. Hadamard gate, X,Y,Z gates), it can only occur with a success probability lower than 100%, with an agreement with a no-go Bell theorem.

After realizing that the CNOT/CPHASE gate requires a nonlinear interaction between single photons, the attention of researchers has moved to nonlinear materials. Chuang and Yamamoto [175] have presented that a cross-Kerr effect can be used to perform the CNOT gate. They relied on a strong nonlinearity interaction between two single photons in a cross-Kerr medium. The usage of the nonlinear media gives hope for an implementation of entangling gates in a deterministic way. It would allow one to overcome problems connected with the probabilistic operation based on linear optics. A few proposals of using a cross-Kerr medium to implement entangling gates have appeared [176, 177]. Experimentally, the Kerr nonlinearity was achieved by Matsuda *et al* [178] who reported a small conditional phase shift (10^{-7}rad) for single photons in an optical fiber. Also another proposal has been put forward [179] according to which obtaining entanglement between the photon number in one mode and the optical phase in another one using a spontaneous parametric down conversion and quantum interferometry is achievable. Also Resh *et al* [179] reported an enhancement of the introduced nonlinearity.

In the subsections below I would like to present a schemes which can be helpful for enhancing nonlinearity in the cross-Kerr media. The results are based on theoretical analyses using properties arising from a group theory. It can be experimentally approximated by a unitary operation, which can be implemented in a deterministic way in opposition to the linear-optical ones. Due to the general theoretical background of a group theory this proposal is valid for every kind of implementation of quadrature squeezing and cross-Kerr modulation. Two setups have been analyzed which can be implemented for both: single-mode and two-mode squeezing, relativity of the availability of the resources, and reviewing experimental features for realizing them.

The Hamiltonian for a single frequency cross-Kerr medium with modes a and b can be written as

$$\hat{H} = \chi \hat{n}_a \hat{n}_b, \tag{III.24}$$

where χ denotes the interaction strength, $\hat{n}_a = \hat{a}^\dagger \hat{a},\ \hat{n}_b = \hat{b}^\dagger \hat{b}$ are photon numbers operators for modes a and b . Using an appropriate strong cross-Kerr interaction it is possible to perform the CPHASE operation on two single qubits in such a way, that states $|00\rangle, |01\rangle, |10\rangle$ will stay untouched and the two single-photon state will gain an additional phase $|11\rangle \rightarrow e^{i\delta}|11\rangle$. In particular, for $\delta = \pi$ one obtains the CS gate, which under unitary transformation is equivalent to the CNOT gate.

The quantum gate, which entangles photons (qubits) using its internal nonlinearity emerged from the features of the medium, as an idea, is much simpler then building of the complicated schemes even with using very simple devices. It provides to a possibility of getting deterministic implementations of gates, in opposite to a linear optical one, not limited by a no-go theorem. Unfortunately, Shapiro *et al* [180, 181] proved, that the phase-noise in the cross-Kerr modulation would be significant and preclude the effective implementation of the CPHASE gate. Nevertheless, as mentioned before, the small conditional phase shift induced by Kerr nonlinearity was successfully measured by Matsuda*et al* [178].

3.1 Scheme for the Kerr nonlinearity amplification with one-mode squeezing

It has been presented the scheme for amplifying conditional phase-shift induced by the Kerr effect using one-mode squeezing operation. Describing the quantum computation process as the CPHASE gate one can restrict ourselves to qubits and define the states with photon numbers 0 and 1 as $|0\rangle$ and $a^\dagger|0\rangle$, respectively. It means that we have to restrict ourselves only to a subspace of the total photon-number space in terms of a certain experimental method. Therefore, in the subspace used for quantum computation one can introduce an operator $Z_a = 2n_a - 1$, which has eigenvalues only 1 and -1 so that $Z_a^2 = 1$. Such defined operator is later used to construct one of the generators of the $SU(1,1)$ group Γ_3, which would approximate (with some additional phase shift operation on qubits in both modes) the quantum Kerr effect described by Eq. (III.24):

$$\hat{\Gamma}_3 = \frac{1}{2}(2\hat{n}_b + 1)\hat{Z}_a = \frac{1}{2}(4 \underbrace{\hat{n}_a \hat{n}_b}_{\text{Kerr effect}} + 2\hat{n}_a - 2\hat{n}_b - 1). \tag{III.25}$$

To preserve the bosonic commutation rules for the generators of $SU(1,1)$

$$\begin{aligned} [\hat{\Gamma}_1, \hat{\Gamma}_2] &= -i2\hat{\Gamma}_3, \\ [\hat{\Gamma}_2, \hat{\Gamma}_3] &= i2\hat{\Gamma}_1, \\ [\hat{\Gamma}_3, \hat{\Gamma}_1] &= i2\hat{\Gamma}_2, \end{aligned} \tag{III.26}$$

it is possible to compose the remaining generators in the following manner

$$\begin{aligned} \hat{\Gamma}_1 &= \frac{1}{2}(\hat{b}\hat{b} + \hat{b}^\dagger \hat{b}^\dagger)\hat{Z}_a, \\ \hat{\Gamma}_2 &= -\frac{i}{2}(\hat{b}\hat{b} - \hat{b}^\dagger \hat{b}^\dagger), \end{aligned} \tag{III.27}$$

where $\hat{b}$ and $\hat{b}^\dagger$ fulfil the bosonic commutation relations.

Using the vector coherent state theory we find a configuration of the operations which needs to be performed on qubits to amplify a conditional phase-shift induced by the cross-Kerr modulation. The vector coherent state theory is based on the fact that the structural constants are dependent only on the commutation relations of generators, but independent of the dimensions of the representations of those generators, for example in the group $SU(1,1)$ true is below equality

$$e^{i\alpha\hat{\Gamma}_i}e^{i\beta\hat{\Gamma}_j} \quad = e^{(i\theta\hat{\Gamma}_1+i\phi\hat{\Gamma}_2+i\psi\hat{\Gamma}_3)}. \tag{III.28}$$

The structural constants θ,ϕ,ψ are independent of the dimension of generators Γ_i. The generators in the group $SU(1,1)$, which is noncompact and does not have any finite unitary representation, can, however, be written in simple two-dimensional non-Hermitian representation as

$$\begin{aligned}\hat{\Gamma}_1 &= i\hat{\sigma}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\\ \hat{\Gamma}_2 &= -i\hat{\sigma}_1 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix},\\ \hat{\Gamma}_3 &= \hat{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.\end{aligned} \tag{III.29}$$

According to Eq. (III.29) it is possible to design a setup for enhancing the Kerr nonlinearity (the coefficients are introduced to the angle definitions)

$$e^{i\theta\hat{\Gamma}_2}e^{i\frac{\delta}{2}\hat{\Gamma}_3}e^{i\phi\hat{\Gamma}_2}e^{i\frac{\delta}{2}\hat{\Gamma}_3}e^{i\theta\hat{\Gamma}_2} = e^{i\gamma\hat{\Gamma}_3}, \tag{III.30}$$

where

$$\begin{aligned}\phi &= \operatorname{arctanh}(-\cos\delta\tanh 2\theta),\\ \gamma &= \arctan(\tan\delta\cosh 2\theta).\end{aligned} \tag{III.31}$$

The proof of Eq. (III.30) can be written in the following method

$$\hat{V}\begin{pmatrix} e^{\frac{i\delta}{2}} & 0 \\ 0 & e^{-\frac{i\delta}{2}} \end{pmatrix} w \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} \times \begin{pmatrix} e^{\frac{i\delta}{2}} & 0 \\ 0 & e^{-\frac{i\delta}{2}} \end{pmatrix}\hat{V} = \begin{pmatrix} y & 0 \\ 0 & y^* \end{pmatrix}, \tag{III.32}$$

where

$$\begin{aligned}\hat{V} &= \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix},\\ w &= \frac{1}{\sqrt{1-\cos^2\delta\tanh^2 2\theta}},\\ x &= -\cos\delta\tanh 2\theta,\\ y &= \frac{\cos\delta + i\cosh 2\theta\sin\delta}{\cosh(2\theta)\sqrt{1-\cos^2\delta\tanh^2 2\theta}}.\end{aligned} \tag{III.33}$$

Despite the lack of a finite unitary representation of the group, results have been checked numerically for spaces with big dimensions.

An enhancement of a phase shift versus a squeeze parameter is plotted in Fig. III.5 for (a) achievable experimental parameter reported by Matsuda *et al* [178]($\delta = 10^{-7}$) and arbitrarily chosen $\delta = \pi/512$. b) in experimental regime around $10\mathrm{dB}$ of squeezing for $\delta = 10^{-7}$. As can be seen in Figs. III.5, it is possible to obtain a significant and strong improvement of nonlinear effect. As it turns out, when an appropriate squeezed light goes through two Kerr crystals (and phase shifters PS), the Kerr nonlinearity can be amplified (e.g. for $\theta = 5$ the Kerr nonlinearity can be enhanced about 74 times).

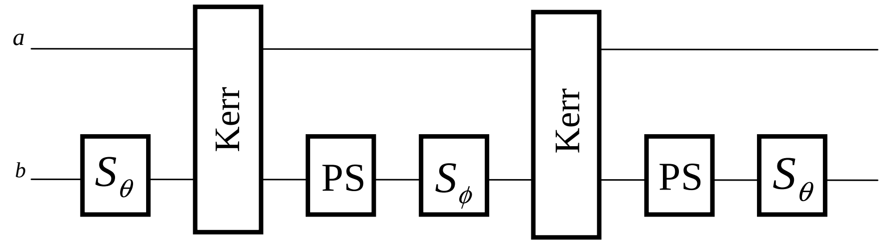


Figure III.4: Scheme III for an phase shift amplification induced in a cross-Kerr medium for two modes. Key:S$_i$- a one-mode squeezing operation ($i = \theta, \phi$) with the squeeze parameters from eq. (III.30), CP-controlled phase gate implemented via the Kerr nonlinearity, PS-phase shifter (implemented by a quarter-wave plate)[Bartkowiak2012].

According to Eq. (III.30) it is possible to design a scheme for an amplification of the Kerr nonlinearity as follows

$$\overbrace{\hat{S}_b(\theta)}^{\text{squeezing}}\ \overbrace{e^{i\frac{\delta}{4}(2\hat{n}_a\hat{n}_b-\hat{n}_b)}}^{\text{Kerr+PS}}\ \overbrace{\hat{S}_b(\phi)}^{\text{squeezing}}\ \overbrace{e^{i\frac{\delta}{4}(2\hat{n}_a\hat{n}_b-\hat{n}_b)}}^{\text{Kerr+PS}}\ \overbrace{\hat{S}_b(\theta)}^{\text{squeezing}} = e^{\frac{i}{2}(\gamma-\delta)(2\hat{n}_a-1)}\ \underbrace{e^{i\gamma(2\hat{n}_a\hat{n}_b-\hat{n}_b)}}_{\text{amplified Kerr+PS}}\ . \tag{III.34}$$

Scheme III is depicted in Fig. III.4. The unitary operation, which in the Scheme III is labelled by $\mathrm{S_i}$ (connected with exponent of Γ_2 from Eq. (III.27) with $i = \theta, \phi$) can be written as (for $\theta \in R$) :

$$\hat{S}_b(\theta) \quad = \quad \exp\left[\frac{\theta}{2}(\hat{b}\hat{b} - \hat{b}^\dagger\hat{b}^\dagger)\right] \tag{III.35}$$

and can transform the bosonic operator in the following manner

$$\hat{S}_b(\theta)\hat{b}\hat{S}_b(-\theta) = \cosh\left(\frac{\theta}{2}\right)\hat{b} - \sinh\left(\frac{\theta}{2}\right)\hat{b}^\dagger. \tag{III.36}$$

The above introduced operator is called squeezing operator and can be approximated by a real physical process. The process of squeezing the states is in fact lowering one of the variances of the quadrature (for single-mode phase-rotated quadratures are defined as $\hat{x}(\phi) = \hat{a}\exp(i\phi) + \hat{a}^\dagger\exp(-i\phi)$). In order to clarify the mathematical description of squeezing one can recall the well known definition of the squeeze states as a group of the states with the minimum uncertainty. It is possible to decrease the noise in one of the two quadratures by obeying an uncertainty relation. In general, the squeeze parameter (θ in Eq. (III.35)) is a complex number which can be written as $\theta = r_s\exp(i2\phi)$, where r_s is a degree of squeezing and ϕ an orientation of the squeezing axis. By manipulating the squeeze parameter one is able to decrease the minimum variance and increase the maximum one. Squeezed states are characterized by asymmetric distribution function, which is in agreement with lowering of the noise below the quantum limit in the one of variances (while obeying the uncertainty principle).

However, the process which is governed by operation from Eq. (III.35) cannot be found directly. Although, the form of the operator S_b can suggest what kind of phenomena can be used to approximate squeezing operation in experimental reality. Because the squeezing operator is expected to decrease the noise of the optical field and, therefore, squeeze the state of light, the phenomena of nonlinear interaction between light and medium are needed. Nonlinear process is

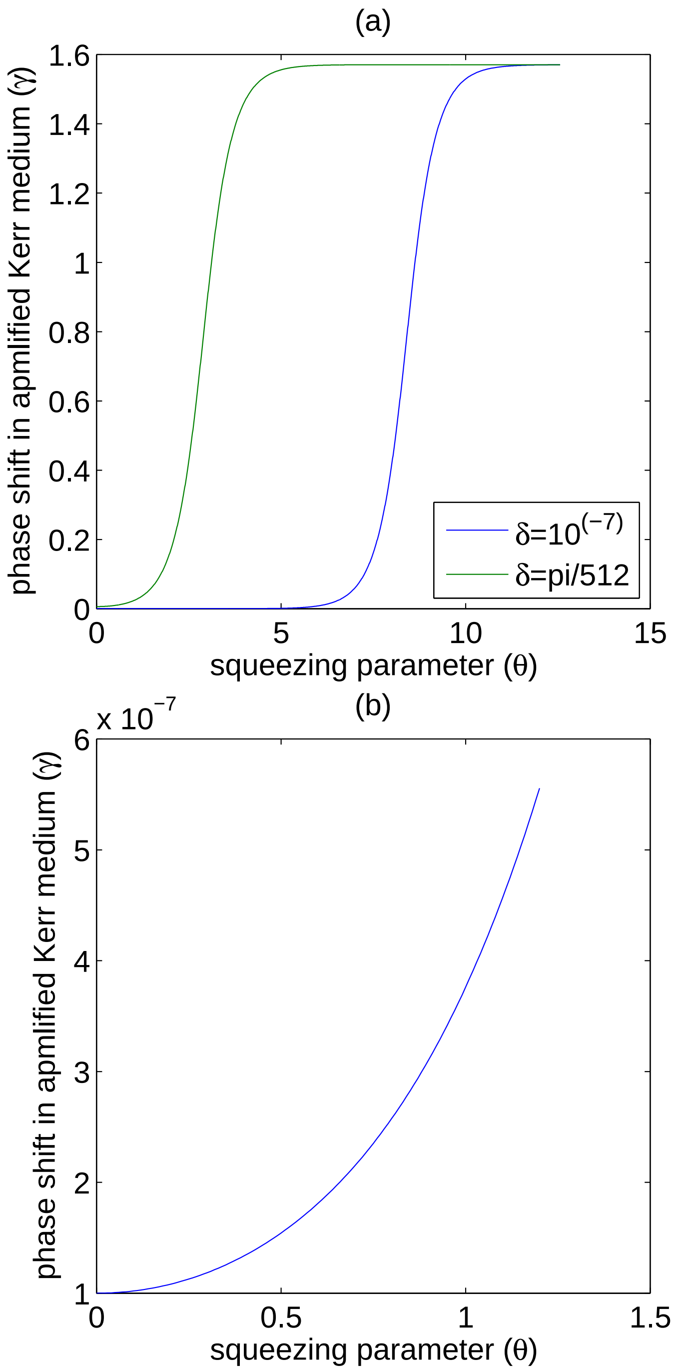


Figure III.5: An induced phase shift in the Kerr medium vs. squeezing parameter for (a) $\delta = 10^{-7} rad$ (in an agreement with Matsuda's *et al* results [178]) and $\delta = \frac{\pi}{512}$–arbitrarily chosen angle; (b) the induced phase shift for $\delta = 10^{-7} rad$ in experimental regime of the squeeze parameter (till $10\mathrm{dB}$) [Bartkowiak2012].

necessary to achieve correlation between quadratures. It is possible to distinguish two classes of interactions which are governed by the following Hamiltonians

$$\hat{H} = \quad i\hbar(\alpha\chi^{(2)}\hat{a}^2 - \alpha\chi^{(2)}\hat{a}^{\dagger 2}), \tag{III.37}$$

$$\hat{H} = \quad i\hbar(\alpha^2\chi^{(3)}\hat{a}^2 - \alpha^2\chi^{(3)}\hat{a}^{\dagger 2}), \tag{III.38}$$

where the $\chi^{(2)}$ and $\chi^{(3)}$ are the second and the third susceptibilities of the medium, and α is the amplitude of the pump field (with the assumption it is a strong classical light). Another type of interaction which can cause the squeezing effect is the one described by the Hamiltonian

$$\hat{H} = i\hbar\chi^{(2)}(b^\dagger a_1^\dagger a_2 - ba_1 a_2^\dagger), \tag{III.39}$$

which converts photons in the mode b into photons in the modes a_1 and a_2 introducing a correlation between quadratures for different modes (e.g. the second harmonic generation process). A list of the most popular methods of squeezing can be found in Table III.4.

Table III.1: A list of the main types of linear-optical implementations of the CS/CNOT gates. Key: P—the total probability of success, E (T)—experimental (theoretical) implementation, $|\chi\rangle$—the Gottesman-Chuang state equivalent to a four-qubit cluster state [147], ^{a}—measurement of both the control and target bits used for postselection, ^{b}—assuming perfect efficiency ($\eta = 1$) of detectors [Bartkowiak2010b].

#	Authors	E/T	Comments	P	Feedforward	Entangled ancillae	Destructive	Conventional detectors
				I. UNENTANGLED ANCILLAE				
1	KLM [129]	T		$\frac{1}{16}$	no	0	no	no
2	Ralph *et al.* [148]	T	simplified #1	$\frac{1}{16}$	no	0	no	no
3	Knill [149]	T	improved #1	$\frac{2}{27}$	no	0	no	no
4	Pittman *et al.* [150]	E		$\frac{1}{8}$	no	0	yes^a	no
5	*ditto*	T	modified #4	$\frac{1}{4}$	yes	0	yes^a	no
6	Giorgi *et al.* [151]	T	modified #16	$\frac{1}{8}$	yes	0	no	no
7	Bao *et al.* [152]	E	modified #13	$\frac{1}{8}$	yes	0	no	no
				II. ENTANGLED ANCILLAE				
8	KLM [129]	T		$\frac{1}{4}$	yes	EPR	no	no
9	KYI [130]	T		$\frac{1}{16}$	yes	EPR	no	no
10	*ditto*	T	modified #9	$\frac{1}{4}^b$	yes	3×EPR	no	no
11	*ditto*	T	modified #9	$\frac{1}{4}$	yes	5×EPR	no	no
12	Pittman *et al.* [146]	T		$\frac{1}{16}$	no	EPR	no	no
13	*ditto*	T	modified #12	$\frac{1}{4}$	yes	EPR	no	no
14	Gasparoni *et al.* [144]	E	realization of #12	$\frac{1}{16}$	no	EPR	yes^a	yes
15	Zhao *et al.* [145]	E	realization of #12	$\frac{1}{16}$	no	EPR	yes^a	yes
16	Giorgi *et al.* [151]	T	related to #12	$\frac{1}{4}$	yes	EPR	no	no
17	Zou *et al.* [143]	T	related to #12	$\frac{1}{8}$	yes	2×EPR	no	yes
18	Gottesman, Chuang [147]	T		—	yes	$\|\chi\rangle$	no	—
19	Pittman *et al.* [146]	T	based on #18	$\frac{1}{4}$	yes	$\|\chi\rangle$	no	no
				III. WITHOUT ANCILLAE				
20	Pittman *et al.* [146]	T		$\frac{1}{4}$	no	0	yes	no
21	*ditto*	T	modified #20	$\frac{1}{2}$	yes	0	yes	no
22	Pittman *et al.* [153]	E	realization of #20	$\frac{1}{4}$	no	0	yes	no
23	Giorgi *et al.* [151]	T	related to #20	$\frac{1}{4}$	no	0	yes	no
24	*ditto*	T	modified #23	$\frac{1}{2}$	yes	0	yes	no
25	Hofmann, Takeuchi [154]	T		$\frac{1}{9}$	no	0	yes^a	no
26	Ralph *et al.* [155]	T	equivalent to #25	$\frac{1}{9}$	no	0	yes^a	no
27	O'Brien [156]	E	realization of #25, #26	$\frac{1}{9}$	no	0	yes^a	no
28	Okamoto *et al.* [157]	E	realization of #25, #26	$\frac{1}{9}$	no	0	yes^a	no
29	Kiesel *et al.* [158]	E	simplified #25, #26	$\frac{1}{9}$	no	0	yes^a	no
30	Langford *et al.* [159]	E	equivalent to #29	$\frac{1}{9}$	no	0	yes^a	no

Table III.2: Required conditional operations $\hat{U}^j$ and $\hat{V}^{kl}$ for Scheme I for an appropriate clicks configuration of ideal detectors D_i [Bartkowiak2010b].

D_{3H}	D_{3V}	D_{4H}	D_{4V}	$\hat{U}^j$
1	0	1	0	$\hat{I}$
0	1	0	1	$\hat{I}$
1	0	0	1	$\hat{\sigma}_z^{(5)} \otimes \hat{\sigma}_z^{(6)}$
0	1	1	0	$\hat{\sigma}_z^{(5)} \otimes \hat{\sigma}_z^{(6)}$

D_{cH}	D_{cV}	D_{1H}	D_{1V}	D_{6H}	D_{6V}	D_{tH}	D_{tV}	$\hat{V}^{kl}$
1	0	1	0	1	0	1	0	$\hat{I}$
1	0	1	0	0	1	0	1	$\hat{I}$
0	1	0	1	1	0	1	0	$\hat{I}$
0	1	0	1	0	1	0	1	$\hat{I}$
1	0	0	1	1	0	1	0	$\hat{\sigma}_z^{(2)}$
1	0	0	1	0	1	0	1	$\hat{\sigma}_z^{(2)}$
0	1	1	0	1	0	1	0	$\hat{\sigma}_z^{(2)}$
0	1	1	0	0	1	0	1	$\hat{\sigma}_z^{(2)}$
1	0	0	1	1	0	0	1	$\hat{\sigma}_z^{(5)}$
1	0	0	1	0	1	1	0	$\hat{\sigma}_z^{(5)}$
0	1	1	0	1	0	0	1	$\hat{\sigma}_z^{(5)}$
0	1	1	0	0	1	1	0	$\hat{\sigma}_z^{(5)}$
1	0	1	0	1	0	0	1	$\hat{\sigma}_z^{(2)} \otimes \hat{\sigma}_z^{(5)}$
1	0	1	0	0	1	1	0	$\hat{\sigma}_z^{(2)} \otimes \hat{\sigma}_z^{(5)}$
0	1	0	1	1	0	0	1	$\hat{\sigma}_z^{(2)} \otimes \hat{\sigma}_z^{(5)}$
0	1	0	1	0	1	1	0	$\hat{\sigma}_z^{(2)} \otimes \hat{\sigma}_z^{(5)}$

Table III.3: Same as Table III.2.A. but for Scheme II [Bartkowiak2010b].

D_{2H}	D_{2V}	D_{3H}	D_{3V}	$\hat{U}^j$
1	0	1	0	$\hat{I}$
0	1	0	1	$\hat{I}$
1	0	0	1	$\hat{\sigma}_z^{(2)}$
0	1	1	0	$\hat{\sigma}_z^{(2)}$

D_{cH}	D_{cV}	D_{tH}	D_{tV}	$\hat{V}^{kl}$
1	0	1	0	$\hat{I}$
0	1	1	0	$\hat{\sigma}_z^{(1)}$
1	0	0	1	$\hat{\sigma}_z^{(4)}$
0	1	0	1	$\hat{\sigma}_z^{(1)} \otimes \hat{\sigma}_z^{(4)}$

Table III.4: A list of the most popular methods of the light squeezing, based on [203]; Key: H-type of Hamiltonian from Eqs. (III.37,III.38, III.39); PS- possibility of the pulse squeezing usage to enhance nonlinearity.

TYPE OF MEDIUM	PHYSICAL PROCESS	H	PS	DESCRIPTION
$\chi^{(2)}$	OPA [182, 183, 184, 185, 186]	(3)	yes [187, 188]	· one of the most successful results in the early stage of research; · the most popular medium–$LiNbO_3$; · the most characteristic feature of the squeezed light has been shown, like noise suppression, preservation of uncertainty rule, enhancement in the opposite quadrature
	SHG [189, 190, 191, 192]	(3)	yes [187]	· the most reliable and effective for amplitude squeezing;
$\chi^{(3)}$	4WM [193]	(2)		· the first experiment;
	Kerr media			· a problem with extra sources of noise; · a need of high $\chi^{(3)}$, interaction length and high pump intensity;
	1. Optical fiber · the pioneering experiment- [196] · usage of solitons [197, 198, 199]		yes [194, 195]	· $\chi^{(3)}$ possibility to use long (several hundred of meters) media, and strong pulses instead of high $\chi^{(3)}$; · popular for photon-number squeezing; ·problem: Brillouin scattering (can be avoided using solitons or pulse squeezing);
	2. Resonator media			· $\chi^{(3)}$ enhanced several orders of magnitude by operating close to atomic resonance;
	3. Cascade of $\chi^{(2)}$ [200, 201, 202]			· simultaneous OPA or SGH;

Following the Ref. [203] we can distinguish a few most important types of experimental realizations of the squeezing process (Table III.4). The first squeezed light was experimentally realized by nondegenerate four-wave mixing (4WM) via the usage of sodium atoms in a cavity as a nonlinear medium [193]. Although, with a view of obtaining large squeezing this method has been reported to be not stable enough to achieve $10\mathrm{dB}$. The second group consist of optical parametric processes (OPA), the usage of which was reported for the first time by Wu *et al* [182]. Nonlinearity was enhanced by placing the $\chi^{(2)}$ medium in a cavity. The most popular medium used in such experiments is $\mathrm{LiNb0_3}$. One of the best results was achieved by Pereira *et al* [183] ($-7.1\mathrm{dB}$).

The most reliable and effective method to decrease the noise is the second harmonic generation (SHG). Using this method it is possible to obtain a consistent noise reduction for many hours. Large squeezing was obtained by Kürz *et al* [192]. Referring to the Eq. (III.38) it can be seen that approximation of a squeezing operation can also be obtained by the Kerr effect. There are a few main methods of obtaining squeezing via $\chi^{(3)}$ media, like: i) via the usage of optical fibres, ii) via nonresonant media inside a cavity, iii) via resonant media with enhanced $\chi^{(3)}$, or iv) using a cascade of $\chi^{(2)}$.

A big advantage of an optical fiber to perform squeezing is a possibility of the usage of the length of the medium to increase nonlinearity. Nevertheless, because of Brillouin scattering driven by a thermal mode, it is very hard to obtain large squeezing [196]. There appeared attempts to avoid this problem via cooling of the medium [204, 205].

Some of the best results of squeezing were achieved in the experiments which used pulsed squeezing with high intensities of the pump and nonlinearity [206]. For instantaneous response this technique can improve the experiments using SHG, OPA or Kerr effect. Another idea is to use solitons, which are the robust against a variety of perturbations after an interaction (they preserve the temporal shape of pulse, energy and momentum) [197, 198, 199]. The usage of these two techniques (pulse squeezing and solitons) is expected to make it possible to achieve large squeezing. For more details go to [203].

The CP in Scheme III is the CPHASE gate which was deterministically implemented by the Kerr nonlinearity via a cross-phase modulation. As mentioned before a small phase shift for a two single-photon state is achievable. Matsuda *et al* [178] used the Sagnac interferometry to measure the phase shift in a photonic crystal fiber. The last part of our scheme is a phase shifter, which can be implemented by a quarter -wave plane.

3.2 Scheme for the Kerr nonlinearity amplification with two-mode squeezing

In analogy to the case described earlier it will be shown Scheme IV for a three-modes case to adapt our setup also for the usage of two-mode squeezing operation (e.g. using the process approximated by Hamiltonian similar to Eq. (III.39) which is defined as follows

$$\hat{S}_{bc}(\theta) = \exp\left[\theta(\hat{b}\hat{c} - \hat{b}^{\dagger}\hat{c}^{\dagger})\right], \tag{III.40}$$

and satisfying the Bogolubov transformations

$$\begin{aligned} \hat{S}_{bc}(\theta)\hat{b}\hat{S}_{bc}(-\theta) &= \hat{b}\cosh\theta - \hat{c}^{\dagger}\sinh\theta, \\ \hat{S}_{bc}(\theta)\hat{c}\hat{S}_{bc}(-\theta) &= \hat{c}\cosh\theta - \hat{b}^{\dagger}\sinh\theta, \end{aligned} \tag{III.41}$$

analogous to Eq. (III.36) for the single-mode case. In this case one is able to improve the double Kerr effect on two modes separately. The generators of the $SU(1,1)$ group can be written in analogy to Eqs. (III.25)-(III.27) as

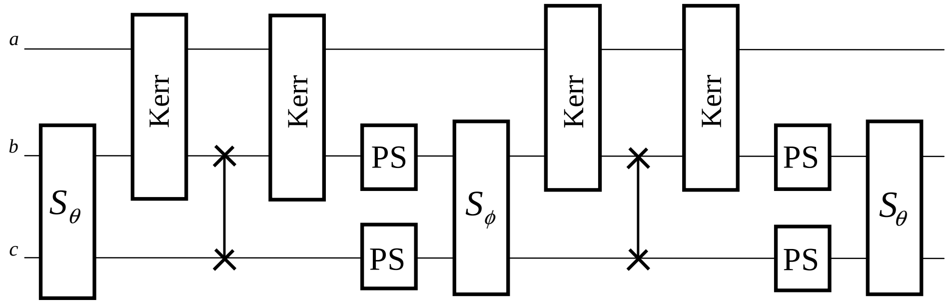


Figure III.6: Scheme IV for an phase shift amplification induced in a cross-Kerr medium for three modes (with a two-mode squeezing operation). A notation is similar to that in Fig. III.4 [Bartkowiak2012].

$$\begin{aligned}\hat{\Gamma}'_1 &= (\hat{b}\hat{c}+\hat{b}^\dagger\hat{c}^\dagger)\hat{Z}_a,\\ \hat{\Gamma}'_2 &= -i(\hat{b}\hat{c}-\hat{b}^\dagger\hat{c}^\dagger),\\ \hat{\Gamma}'_3 &= (\hat{n}_b+\hat{n}_c+\hat{1})\hat{Z}_a,\end{aligned} \tag{III.42}$$

with the same commutation relations for SU(1,1) group from Eq. (III.26). Relying on Eq. (III.30) we can derive the following equality

$$\begin{aligned}&\overbrace{\hat{S}_{bc}(\theta)}^{\text{squeezing}}\ \overbrace{e^{i\frac{\delta}{2}(2\hat{n}_a\hat{n}_b-\hat{n}_b)}}^{\text{Kerr}_{\text{ab}}+\text{PS}}\overbrace{e^{i\frac{\delta}{2}(2\hat{n}_a\hat{n}_c-\hat{n}_c)}}^{\text{Kerr}_{\text{ac}}+\text{PS}}\ \overbrace{S_{bc}(\phi)}^{\text{squeezing}}\ \cdot\\ &\overbrace{e^{i\frac{\delta}{2}(2\hat{n}_a\hat{n}_b-\hat{n}_b)}}^{\text{Kerr}_{\text{ab}}+\text{PS}}\overbrace{e^{i\frac{\delta}{2}(2\hat{n}_a\hat{n}_c-\hat{n}_c)}}^{\text{Kerr}_{\text{ac}}+\text{PS}}\ \overbrace{S_{bc}(\theta)}^{\text{squeezing}}\\ &=\underbrace{e^{i(\gamma-\delta)(2\hat{n}_a-1)}}_{\text{PS}}\underbrace{e^{\gamma(2\hat{n}_a\hat{n}_b-\hat{n}_b)}e^{i\gamma(2\hat{n}_a\hat{n}_c-\hat{n}_c)}}_{\text{amplified Kerr+PS}}.\end{aligned} \tag{III.43}$$

As can be seen in the Fig. III.6 Scheme IV consists of the Kerr effect applied simultaneously in two modes on the squeezed light. The enhancement is analogical with the two-mode case.

3.3 Experimental feasibilities of the schemes

The most experimentally challenging part of our schemes is an implementation of the squeezing operation which would be as strong as possible. In the recent years, many proposals connected with different kind of squeezing have appeared (as one can see in Table III.4). It is possible to implement a squeezing operation via microstructure fibers [207] as well as organic polymers or semiconducting materials [208, 209, 210]. To my knowledge the strongest squeezing reaches the value of about 10dB ($[dB] = \frac{20\theta}{\ln(10)}$, where θ is squeezing parameter), what corresponds to $\theta \simeq 1.15$. In the region that is experimentally available dependency is shown in Fig. III.5(b). As far as squeezing is taken into account the biggest losses are present during detection of the squeezed light. According to Valbruch *et al* [211], it is possible to obtain the quadrature squeezing of about 10dB via the I degenerate parametric optical oscillation with probability of success about 97%. Corney *et al* [212] achieved the polarization squeezing in an optical fiber with probability of about 98%.

The small Kerr nonlinearity in the ultrafast response regime was obtained and shown by Matsuda *et al* Ref. [178] for single photons. The losses they obtained equal approximately 50%.

Using the experimental data given above we can roughly estimate effectiveness of our scheme. In proposal presented in Subsections III.3.1 and III.3.2 three squeezing devices and other two that should produce the Kerr nonlinearity are used. Taking into account experimental results it is possible to obtain approximately 20% of success in Schemes III and IV.

3.4 A spectral effect in the non-instantaneous response in the cross-Kerr medium

In this subsection it have been described how spectral effects can affect fidelity of the cross-Kerr effect for a general input state. Further, it will be shown that even when spectral effects are taken into account applying squeezing operation on an input state before sending it through the Kerr effect presented before, corrects fidelity of the gate. At first, I will shortly review the results obtained by Leung [213] to further generalize them. It is important to emphasize that consideration presented below does not include a phase-shift noise. Obviously, taking into account also this kind of noise one needs to expect a deterioration of results.

A response of the Kerr medium depends on the frequency of an input states. Thus, it is crucial to understand what kind of influence the spectral response effect has on spectral profile of the photonic input state, and further, fidelity of quantum gate. Hamiltonian for the cross-Kerr medium (for instantaneous response and no self-Kerr effect) can be expressed as [213]:

$$\begin{aligned}\hat{H}(t) =&\chi^{(3)}\epsilon_0 \int_V d\mathbf{r}^3 \Big(\hat{E_p}^\dagger(\mathbf{r},t)\hat{E_p}(\mathbf{r},t)\Big(\int_0^\infty d\tau g(\tau)\hat{E_s}^\dagger(\mathbf{r},t-\tau)\hat{E_s}(\mathbf{r},t-\tau)+\kappa\hat{m}_p(\tau,r)\Big)\\ &+\Big(\int_0^\infty d\tau g(\tau')\hat{E_p}^\dagger(\mathbf{r},t-\tau')\hat{E_p}(\mathbf{r},t-\tau')+\kappa\hat{m}_s(\tau',r)\Big)\hat{E_s}^\dagger(\mathbf{r},t)\hat{E_s}(\mathbf{r},t)\Big).\end{aligned} \tag{III.44}$$

Evolution of the states will be considered in an interaction picture. Thus, following Leung [213], as the time-commutable interaction Hamiltonian is considered, one can write unitary evolution of state as

$$\hat{U}(t_1,t_0)|\psi\rangle = \exp\left(\mathcal{T}\Big\{-\frac{i}{\hbar}\int_{t_0}^{t_1}\hat{H}(t)dt\Big\}\right)|\psi\rangle \tag{III.45}$$

and expand it using the Taylor series (and neglect $\mathcal{T}$). The electric field operator of mode j for one spatial mode is defined as

$$\hat{E}_j^\dagger(z,t) = A_j\int_{-\infty}^{\infty} d\omega_j \hat{a_j}^\dagger(\omega_j)e^{i(k_j(\omega_j)z-\omega_j t)}, \tag{III.46}$$

where A_j as slowly varying amplitude can be factored outside of integral and written in the form of

$$A_j = A_j(\omega_j) = i\sqrt{\frac{\hbar\omega_j}{4\pi c\epsilon_0 n_j^2(\omega_0)S}},$$

where S – the cross section area of the beam and $n_j(\omega_j)$– the refractive index for mode j. It is assumed that the effects connected with a fast non-instantaneous regime dominate the dispersion effects (in such a case k_j becomes a constant). In an interaction Hamiltonian given by Eq. (III.44) $\hat{m}(\tau,z)$ refers to a noise operator. Its choice is ruled by a condition that it has to preserve the commutation of the filed operators in non-instantaneous response regime [213]. In the case considered in this thesis, that is the cross-phase modulation, the noise operator can be defined as

$$\hat{m}(\tau,z) = \int_0^\infty d\omega\sqrt{\frac{G(\omega)}{\pi}}\hat{d}_\omega^\dagger(z)e^{i\omega\tau} + h.c., \tag{III.47}$$

and can be interpreted as a set of localized and independent harmonic oscillators ($G(\omega) = \int d\tau \sin(\omega\tau)g(\tau)$). For simplicity it is assumed (following Ref. [213]) that for fast response regime I can approximate $g(\tau)$ in the following manner

$$g(\tau) \approx \frac{M}{\sqrt{\pi}} \exp(-M^2\tau^2).$$

In such a case Hamiltonian from Eq. (III.44) can be rewritten to form

$$\begin{aligned}\hat{H}(t) \simeq \chi^{(3)} L \Big(\int\int_{-\infty}^{\infty} d\omega_{p_+} d\omega_{p_-} \hat{a}^{\dagger}_{p_+} \hat{a}_{p_-} e^{i(-\omega_{p_+}+\omega_{p_-})t} \int\int_{-\infty}^{\infty} d\omega_{s_+} d\omega_{s_-} \\ \times \int_0^{\infty} d\tau g(\tau) \hat{a}^{\dagger}_{s_+} \hat{a}_{s_-} e^{i(-\omega_{s_+}+\omega_{s_-})t} e^{-i(-\omega_{s_+}+\omega_{s_-})\tau} \\ + \int\int_{-\infty}^{\infty} d\omega_{p_+} d\omega_{p_-} \int_0^{\infty} d\tau' g(\tau') \hat{a}^{\dagger}_{p_+} \hat{a}_{p_-} e^{i(-\omega_{p_+}+\omega_{p_-})t} \\ \times e^{-i(-\omega_{s_+}+\omega_{s_-})\tau'} \int\int_{-\infty}^{\infty} d\omega_{s_+} d\omega_{s_-} \hat{a}^{\dagger}_{s_+} \hat{a}_{s_-} e^{i(-\omega_{s_+}+\omega_{s_-})t} \Big). \qquad \text{(III.48)}\end{aligned}$$

After integration over τ and with assumed large M parameter the Hamiltonian above can be transformed into

$$\begin{aligned}\hat{H}(t) \approx & \frac{\chi L}{2} \iiiint_{-\infty}^{\infty} d\omega_{p+} d\omega_{p-} d\omega_{s+} d\omega_{s-} \\ & \hat{a}^{\dagger}_p(\omega_{p+}) \hat{a}_p(\omega_{p-}) \hat{a}^{\dagger}_s(\omega_{s+}) \hat{a}_s(\omega_{s-}) \exp(-i\Delta\omega t)) \left(e^{-\frac{(\omega_{p_+}-\omega_{p_-})^2}{4M^2}} + e^{-\frac{(\omega_{s_+}-\omega_{s_-})^2}{4M^2}} \right), \qquad \text{(III.49)}\end{aligned}$$

$\Delta\omega = \omega_{p+} - \omega_{p-} + \omega_{s+} - \omega_{s-}$ is the frequency detuning, L is the length of the medium and χ– the interaction strength (including some constants from an electric field). Dispersion can be omitted as the sum of the wavenumbers is zero.

Unfortunately this Hamiltonian does not commute for different times. However, Leung *et al.* [213] showed that similar spectral effects can appear while using both the Taylor and the Dyson series. Thus, for simplicity the Dyson series can be replaced with the Taylor series in the presented consideration. After integrating Hamiltonian from Eq. (III.50) over time (with bounds from $-\infty$ to ∞ for far fields) one can write

$$\hat{H} = \int_{-\infty}^{\infty} \hat{H}(t)dt = \chi L \iiint d\omega_{p_+} d\omega_{p_-} d\omega_{s+} \hat{a}^{\dagger}_p(\omega_{p_+}) \hat{a}_p(\omega_{p_-}) \hat{a}^{\dagger}_s(\omega_{s_+}) \hat{a}_s(\omega_{p_+} - \omega_{p_-} + \omega_{s_+}) \phi(\omega_{p_+}, \omega_{p_-}), \qquad \text{(III.50)}$$

where

$$\phi(\omega_{p_+}, \omega_{p_-}) = \exp\left(-\frac{(\omega_{p_+} - \omega_{p_-})^2}{4M^2} \right).$$

A single-photon state with a Gaussian spectral profile can be expressed as

$$|1\rangle = \int_{-\infty}^{\infty} d\omega \; \hat{a}^{\dagger}(\omega) f(\omega) |0\rangle, \qquad \text{(III.51)}$$

where

$$f(\omega) = \sqrt{\frac{1}{\sigma\sqrt{2\pi}}} \exp\left(-\frac{\nu^2}{4\sigma^2} \right),$$

and $\nu_j = \omega_j - \mu$ and μ is the mean frequency. Thus, an arbitrary spectrally separable input state with modes p and s can be written as

$$|x, y\rangle = \int_{-\infty}^{\infty} d^x\omega_p d^y\omega_s (\hat{a}^{\dagger}_p(\omega_p))^x (\hat{a}^{\dagger}_s(\omega_s))^y (f(\omega_p))^x (f(\omega_s))^y |0\rangle. \qquad \text{(III.52)}$$

A unitary action on an arbitrary -number photon state $|x, y\rangle$ has the form of

$$\hat{U}|x,y\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\hat{H}}{i\hbar}\right)^n |x,y\rangle = \sum_{n=0}^{\infty} \frac{\sigma^2}{n!} \int_{-\infty}^{\infty} d^x\omega_p d^y\omega_s f_n^{x,y}(\omega_p,\omega_s)(\hat{a}_p^\dagger(\omega_p))^x(\hat{a}_s^\dagger(\omega_s))^y|0\rangle. \quad \text{(III.53)}$$

In general case $f_n^{x,y}(\omega_p, \omega_s)$ can be expressed by

$$f_n^{x,y}(\omega_p,\omega_s) = Y^n V^{-(x+y)/4} Z^{-\frac{1}{2}} \exp\left(-\frac{h_1(\nu_p+\nu_s)^2 + h_2(x\nu_p^2 + y\nu_s^2) + 4\sigma^2 M^2 \nu_p \nu_s}{4Z\sigma^2}\right), \quad \text{(III.54)}$$

where

$$Y = 2\sqrt{\pi} Mxy, \quad V = 2\pi\sigma^2,$$

$$Z = 2M^4(n-1)(x+y-2) + M^2\sigma^2((n-1)(x+y)+2) + \sigma^4,$$

$$h_1 = M^2\Big((n-1)[xy\left(2M^2+\sigma^2\right) - M^2(x+y)] - \sigma^2\Big),$$

and

$$h_2 = 2M^2\sigma^2 + \sigma^4.$$

Leung *et al.* [213] and Shapiro [214] showed that the cross-Kerr medium induces a little phase shift onto a two single-photon state. Using Eqs. (III.53) and (III.54) it is possible to check the existence of a phase shift onto multi-photon states. It can be found for particular states computing argument from $\langle x, y|\hat{U}|x, y\rangle$.

As can be seen in the Figs. III.7 it is possible to obtain a little phase shift for an arbitrary input state (when phase-noise is not considered). Figs. III.7 present dependency of a phase shift for five different input states introduced to Kerr medium from the length of medium in which spectral response appears. In X variable is hidden dependence on susceptibility and length of the medium.

3.5 An analysis of spectral effects in a setup combining squeezing operation and cross-Kerr effect

As has been shown in Subsections III.3.1 and III.3.2 it is possible to enhance the nonlinear Kerr effect via the usage of a squeezed light using e.g. parametric down conversion in $\chi^{(2)}$ crystal. One can describe spectral properties of this process analogically to the Kerr effect. The interaction Hamiltonian for $\chi^{(2)}$ crystal can be written as [215]:

$$\hat{H}(t) = \chi L \iiint_{-\infty}^{\infty} d\omega_p d\omega_s d\omega_i \hat{a}_p^\dagger(\omega_p)\hat{a}_s(\omega_s)\hat{a}_i(\omega_i)\text{sinc}\left(\frac{L\Delta k}{2}\right) e^{\frac{iL\Delta k}{2}} e^{-i\Delta\omega t} + h.c., \quad \text{(III.55)}$$

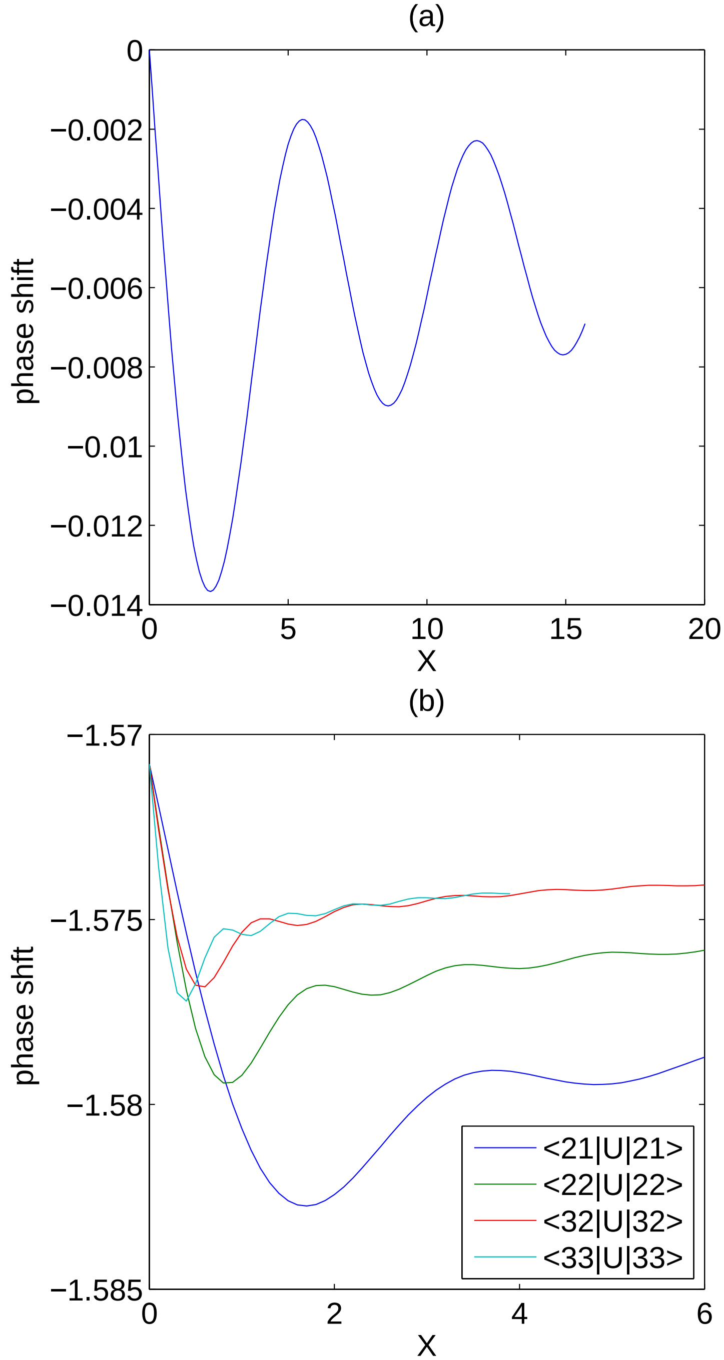
where

$$\Delta k = k_p(\omega_p) - k_s(\omega_s) - k_i(\omega_i)$$

is the phase mismatch and $\Delta\omega = \omega_p - \omega_s - \omega_i$ is the frequency detuning; χ like previously refers to the interaction strength (incorporated with some constants from the electric field expressions). The phase mismatch is expanded in Taylor series the way Grice and Walmsley [216] and Leung *et al.* [215] did it. Following Leung *et al.* [215] the assumption, that the higher orders can be omitted and crucial become the terms up to first order, will be kept. In this case

$$\Delta k \approx \Delta k^{(0)} + k_p'\nu_p - k_s'\nu_s - k_i'\nu_i,$$

Figure III.7: An induced phase-shift in Kerr medium in a case in which spectral effects are taken into account with increasing length of the medium. Key: $X = \frac{2\sqrt{\pi}\chi LM}{\hbar}$ for $\sigma = 10^9$ and $M = 10^{11}$. Figures present a phase shift gained for the different input states (a) $|11\rangle$ and (b)$|21\rangle$, $|22\rangle$, $|32\rangle$, and $|33\rangle$.



where $\nu_j = \omega_j - \mu_j$ and μ_j is the centre frequency of the photon in the mode j, and $\mu_s = \mu_i = \mu$ and $\mu_p = 2\mu$. Parameter k'_j links to the derivative of wavenumber k_j with respect to ω_j and evaluated at μ_j. Due to the conservation of momentum at the zeroth order term

$$\Delta k^{(0)} = k_p(\mu_p) - k_s(\mu_s) - k_i(\mu_i) = 0$$

and thus

$$\Delta k \approx k'_p \nu_p - k'_s \nu_s - k'_i \nu_i.$$

To perform a squeezing operation the type II parametric conversion is necessary, so one needs to assume $k'_s - k'_i \neq 0$. According to [215], strong nonlinearity can be obtained from many thin slices each with weak interaction. Therefore, one can impart a bulk $\chi^{(2)}$ with length $N \times L$ to N slices of length L [215]. In such a case a unitary operator has the following form

$$\hat{U}(t_1, t_0) = \lim_{N\to\infty} \left(1 + \frac{1}{i\hbar}\int_{t_1-\Delta T}^{t_1} \hat{H}(t)dt\right)\left(1 + \frac{1}{i\hbar}\int_{t_1-2\Delta T}^{t_1-\Delta T} \hat{H}(t)dt\right)\dots\left(1 + \frac{1}{i\hbar}\int_{t_0}^{t_0+\Delta T} \hat{H}(t)dt\right), \tag{III.56}$$

where $\Delta T = (t_1 - t_0)/N$. For the slices which are time separated (that the wave packet exits one slice before entering another) the unitary operator can be expressed as a Taylor series

$$\hat{U} = 1 + \frac{N}{i\hbar}\int_{-\infty}^{\infty} \hat{H}(t)dt + \frac{1}{2!}\left(\frac{N}{i\hbar}\int_{-\infty}^{\infty} \hat{H}(t)dt\right)^2 + \dots = \exp\left(\frac{N}{i\hbar}\int_{-\infty}^{\infty} \hat{H}(t)dt\right). \tag{III.57}$$

After integration over time one can write [215]:

$$\begin{aligned}\hat{H} &= \int_{-\infty}^{\infty} \hat{H}(t)dt \\ &= \chi L \iiint_{-\infty}^{\infty} d\omega_p d\omega_s d\omega_i \hat{a}_p^\dagger(\omega_p)\hat{a}_s(\omega_s)\hat{a}_i(\omega_i) \\ &\quad \times \operatorname{sinc}\left(\frac{L\Delta k}{2}\right)\exp\left(\frac{iL\Delta k}{2}\right)\delta(\Delta\omega) + h.c. = \hat{H}_+ + \hat{H}_-,\end{aligned} \tag{III.58}$$

where

$$\hat{H}_+ = \chi L \iiint_{-\infty}^{\infty} d\omega_p d\omega_s d\omega_i \hat{a}_p^\dagger(\omega_p)\hat{a}_s(\omega_s)\hat{a}_i(\omega_i)\vartheta(\omega_s,\omega_i)\delta(\Delta\omega), \tag{III.59}$$

$$\hat{H}_- = \chi^* L \iiint_{-\infty}^{\infty} d\omega_p d\omega_s d\omega_i \hat{a}_p(\omega_p)\hat{a}_s^\dagger(\omega_s)\hat{a}_i^\dagger(\omega_i)\vartheta^*(\omega_s,\omega_i)\delta(\Delta\omega), \tag{III.60}$$

and

$$\vartheta(\omega_s,\omega_i) = \operatorname{sinc}\left(\frac{L\Delta k}{2}\right)\exp\left(\frac{iL\Delta k}{2}\right). \tag{III.61}$$

For simplicity, it is possible to approximate Eq. (III.61) using $\operatorname{sinc} \sim \sqrt{\zeta\pi}\exp(-\zeta \mathrm{x}^2)$ where $\zeta = 0.193...$ [213]. Using Eqs. (III.57) to (III.61), and assuming $|\psi_{\rm out}\rangle = \hat{U}|\psi_0\rangle = |\psi_{\rm even}\rangle - i|\psi_{\rm odd}\rangle$, Leung *et al.* [215] obtained an expression for an even and an odd part of states after passing through $\chi^{(2)}$ medium. The odd and the even parts of the output states correspond to an up-conversion of two photons from signal and idler modes into a photon in the pump mode, and two photons from signal and idler modes remain in the two modes, respectively. For me the most interesting is the last case, which one can use as a new input state to the cross-Kerr medium. The even part of state has the form of

$$|\psi_{\rm even}\rangle = \cos\left(\frac{N}{\hbar}\hat{H}\right)|\psi_0\rangle = \left(1 - \frac{1}{2!}\left(\frac{|\chi|NL}{\hbar}\right)^2 \iint d\omega_s d\omega_i \hat{a}_s^\dagger(\omega_s)\hat{a}_i^\dagger(\omega_i)\vartheta^*(\omega_s,\omega_i)J_{s,i} + \cdots\right)|0\rangle, \tag{III.62}$$

where

$$J_{s,i} = \int d\omega f(\omega) f(\omega_s + \omega_i - \omega) \vartheta(\omega, \omega_s + \omega_i - \omega),$$

and $f(\omega)$ is defined in Subsection III.3.4.

The extended phase matching condition $k'_p = \frac{k'_s + k'_i}{2}$ and the condition $L^2 \gamma \sigma^2 \left(k'_s - k'_p\right) \left(k'_p - k'_i\right) = \frac{1}{2}$ are fulfilled for some choice of parameters, e.g. for a reasonable set $k'_s = 5.6 \times 10^{-9}(s/m)$, $k'_i = 5.2 \times 10^{-9}(s/m)$, and $\sigma = 10^9(\mathrm{Hz})$ [215]. Due to this choice of parameters the second term of the Taylor series is proportional to $|\psi_0\rangle$ and spectrally separable. Thus, it is possible to write

$$\vartheta^*(\omega_s, \omega_i) J_{s,i} = R f(\omega_s) f(\omega_i),$$

where

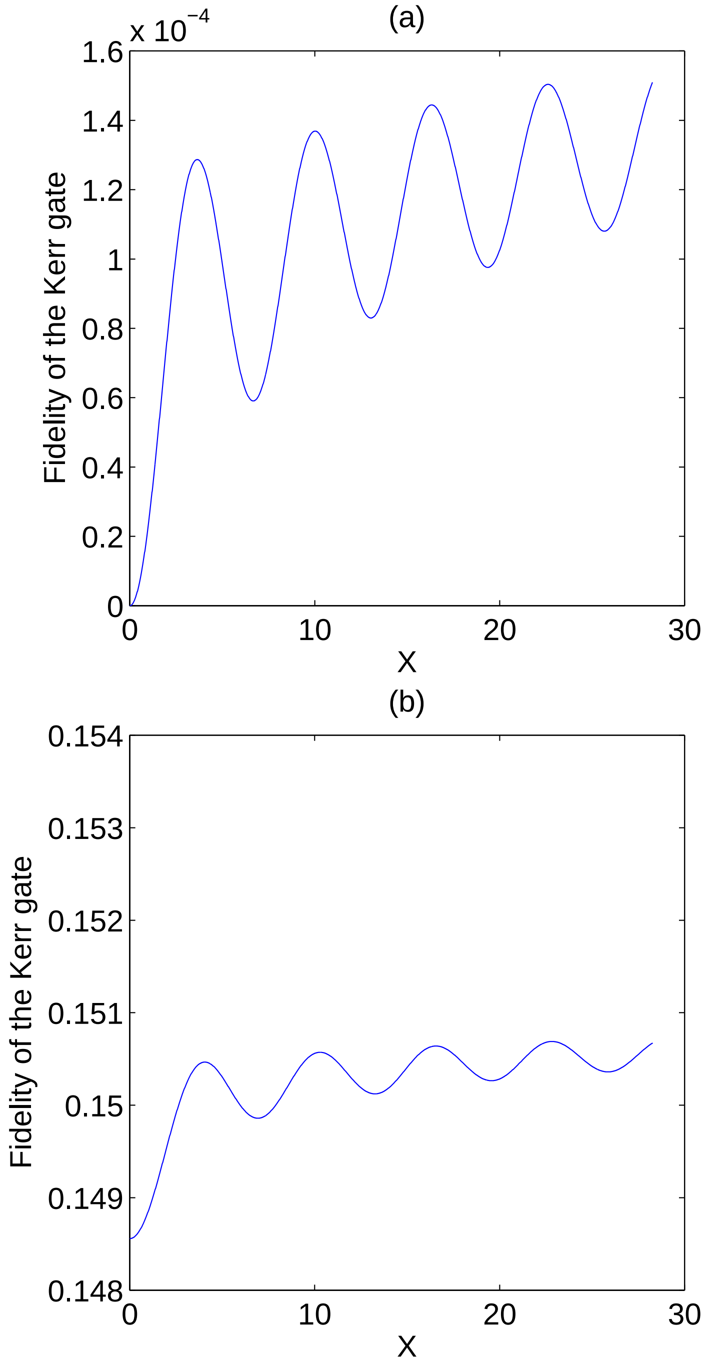
$$R_p = \int d\omega |\Phi(\omega, \omega_p - \omega)|^2 = \sqrt{\frac{2\zeta\pi^3}{L^2 \left(k'_s - k'_i\right)^2}}.$$

Thus, one can write

$$|\psi_{even}\rangle = \cos\left(\frac{|\chi| N L}{\hbar} \sqrt{R}\right) |\psi_0\rangle. \tag{III.63}$$

The state given by Eq. (III.63) after going through the cross-Kerr crystal will gain enhancement of a phase shift proportional to cosine. In the Figs.III.8 it is possible to see, how adding a squeezing of the input state can improve fidelity of the gate performed on the Kerr medium. Fig.III.8(a) reconstructed the results obtained also by Leung *et al.* [213]. As it can be seen in Fig.III.8(b) sending the input state previously through a bulk $\chi^{(2)}$ significantly improves fidelity of the gate (about three order of magnitude). Thus, Schemes III and IV presented in Subsections III.3.1 and III.3.2 can be helpful to perform the effective CPHASE even if the spectral effects are taken into account. Especially, that meaningful enhancement of the phase-shift can be seen already after adding one squeezing operation.

Figure III.8: A dependence of the quantum gate fidelity based on the Kerr medium on the length of the medium. Figure (a) refers to a case when the two single-photon input was introduced to the Kerr medium, Figure (b) when the squeezing operation accounting for 10dB was performed on the input state before sending it to the Kerr gate. The notation is in agreement with the one in Fig.III.7.



Chapter IV

Concluding remarks and main results

The results presented in this thesis are based on analyzing and generalizing an operational criterion of nonclassicality for multi-mode radiation fields of Vogel [58], which is a generalized version of the Shchukin-Richter-Vogel nonclassicality criterion [7, 8] for single-mode fields and the Shchukin-Vogel entanglement criterion [9, 57]. I was seeking an effective experimental method to test nonclassicality by linking the operational Shchukin-Richter-Vogel criteria and wide knowledge of optical implementations using the commonly available resources.

1 A detailed list of the obtained results

Concluding remarks are presented in the order of appearance in the thesis.
In Section II.2 classical inequalities for multimode bosonic fields which can be treated as nonclassicality criteria, as they are violated only by *nonclassical* fields, were derived. Criteria presented in the thesis are based on the generalized version of the Vogel criterion [58], which is a generalization of the analogous criteria for single-mode fields of Agarwal and Tara [6] and, more directly, of Shchukin, Richter, and Vogel [7, 8].

- In Subsection II.2.1 a generalization of the Vogel criterion was proposed for arbitrary multimode fields, which was based on polynomial functions of creation and operators, contrary to the original Vogel's criterion based on monomial functions.

- In Subsection II.2.2 it was shown how, in general, some nonclassicality criteria for one-mode fields based on moments can be reduced to the violation of the Cauchy-Schwarz inequality [14].

- Subsection II.2.3 contains examples of bringing nonclassicality criteria to multimode quadrature squeezing [5] and its generalizations.

- There were also presented (Subsection II.2.3) some criteria linked with the sum and difference squeezings defined by Hillery [68], An and Tinh [69, 70], as well the principal squeezing connected with the Schrödinger-Robertson indeterminacy relation [67] (defined by Lukš *et al.* [66]).

- Examples of the criteria based on the violation of single-time photon-number correlations of the two modes are shown in the Subsection II.2.4. In particular:

  - It was analyzed how it is possible to use the previously introduced nonclassicality criteria to obtain criteria connected with violations of conditions for squeezing of the sum and difference of photon numbers (which refers to the photon-number sum/difference sub-Poisson photon-number statistics) [74].
  - It was also presented how to link the nonclassicality criteria with violations of the Cauchy-Schwarz inequality [14] and violations of the Muirhead inequality [81, 73] (a generalization of the arithmetic-geometric mean inequality).
  - The usage of the introduced criteria in order to obtain the condition for the two-time photon-number correlations of single modes including photon antibunching [5, 14, 76] and photon hyperbunching [77, 78] for stationary and nonstationary fields was demonstrated.

- In the Subsection II.2.5 a few inequalities derived from the nonclassicality criteria were shown, which, as long as my knowledge is considered, have not yet been characterized in the literature.

- Subsection II.2.6 demonstrates a recipe for a construction of a infinite number of nonclassicality witnesses based on moments of the annihilation and creation operators.

In Section II.3 it was presented how entanglement criteria based on the Shchukin-Vogel entanglement criterion [9, 57] can be connected with the known condition for detecting entanglement and linked with nonclassicality criteria.

- In Subsection II.3.2 a connection between the entanglement criteria introduced previously and the Cauchy-Schwarz inequality was shown.

- The known entanglement conditions (e.g., of Duan *et al.* [44], and Hillery and Zubairy [45]) which can be also obtained from nonclassicality criteria are shown in Subsection II.3.3.

- In the thesis a general method for expressing inequalities derived from the Shchukin-Vogel entanglement criterion [9, 57] as a sum of nonclassicality conditions is demonstrated. In particular it was shown how the Simon entanglement inequality [46] can be expressed in terms of sums of nonclassicality inequalities (Subsection II.3.3).

- With the help of the entanglement criteria entanglement witnesses were constructed, which can be applied to analyze properties of different systems (Subsection II.3.4).

An application of the nonclassicality and entanglement witnesses constructed based on matrices of moments of annihilation and creation operators to analyze sudden vanishing and rebirth of quantum correlations in optical systems was shown in Section II.4. The concluding remarks from this part of the thesis are following:

- Both entanglement and nonclassicality (defined as violation of the classical inequalities) demonstrate sudden vanishing in a dissipative system [5, 83] (Subsection II.4.2).

- It was shown that the sudden vanishings can be observed also when dissipation is not considered in both bipartite or multipartite (multimode) interacting or noninteracting systems (Subsection II.4.2), in a single-qubit and single-mode systems (Subsection II.4.3). Those conclusions are given after analyzing single-mode squeezing of a photon number, squeezing of quadrature operators [5], and violations of other classical inequalities [83].

- Periodic sudden vanishings of quantum correlations can be observed even for non-dissipative systems which are initially in pure states. As an example, a quadrature squeezing of light in the Kerr medium is considered. It was shown that squeezing, treated as witness of nonclassicality, in such system exhibits periodic sudden vanishings for some finite periods of time. However, to obtain proper finite-time sudden decays (in terms of original sudden decays of entanglement [43]) the dissipation needs to be introduced to the system by coupling such systems to the environment. Taking into account damping, results in irregularity and the loss of periodicity of the evolution of the nonclassicality witnesses. Moreover, one can summarize that the introduced decoherence cannot be considered as a necessary condition for appearance of the sudden vanishings. However, there is a need to stress that decoherence can accelerate the occurrence of the first sudden vanishings (Subsection II.4.3).

In Section III.1 it was demonstrated how the iSWAP gate can be decomposed into the CS/CNOT gate and deterministic gates including the SWAP, phase or Hadamard gates. Thus, the probability of success of the ISWAP gate can be reduced to the probability of the CNOT or, equivalently, the CS gate. In such a manner it is possible to use both proposals of linear-optical implementations presented in Subsections III.2.1 and III.2.2. Therefore, using the presented Schemes I and II to implement the ISWAP gate one can obtain the probabilities eight (for the CNOT gate when the Gottesman-Chuang [147] state $|\chi\rangle$ is given) or four (for the CS gate when the EPR states are given) times higher than in the Wang *et al.* [169] proposal.
Section III.2 contains analyses of possibilities of implementation of universal linear-optical quantum gates. Two experimentally-friendly proposals of implementation of quantum gates with the usage of conventional detectors and feedforward operation, were presented.

- Also a possibility of the usage of conventional detectors in implementations of nondestructive gates originally designed for single-photon detectors were studied. In particular the scheme of Pittman *et al.* [146] was considered, that is a proposal of an implementation of the nondestructive CNOT gate. It was shown that conventional detectors can be used in this setup achieving the probability of success equal to $\eta^4/4$.The assumption that needs to be added is such that the Gottesman-Chuang four-qubit entangled state [147] is given. If the pair of ancillae in the GHZ state is considered as a resource, the probability of success accounts for $\eta^6/8$ (Subsection III.2.1).

- In Subsection III.2.2 one can find my second Scheme II described as a modified version of the scheme by Zou *et al.* [143], which is a proposal of an implementation of the CS gates with ancillae in the EPR or EPR-like states. The probability of success in an ideal case is equal to $\eta^4/8$. Then analyses of experimental feasibility of this scheme were performed. Under consideration was also the influence of an assumption of realistic sources of ancilla and input states, and detector imperfections to include dark counts, finite efficiency and no photon-number resolution on fidelity of a quantum gate. The obtained fidelity of proposed gate under realistic assumption is relatively high and accounts for 97% (Subsection III.2.2).

The results concerning improving nonlinear quantum gates are presented in the Section III.3.

- A vector coherent state theory has been applied to design the schemes for an amplification of the conditional phase-shift which can lead to a deterministic implementation of the CPHASE gate. Two types of the possible squeezing operations were taken into account: a one-mode (Subsection III.3.1) and two-mode (Subsection III.3.2) squeezings.

- The possible experimental implementations of the proposed setups were analyzed and their effectiveness, which has value of approximately 20% (as the cross-Kerr modulation is a process with a multiple amount of an additional noise), was estimated (Subsection III.3.3).
- Based on the results of Refs. [213, 215], spectral properties of cross-Kerr (Subsection III3.4) and $\chi^{(2)}$ media (Subsection III.3.5) were analyzed.
- It was shown that even when spectral properties of pump and media are considered, a squeezing operation can improve fidelity of quantum gates designed using the Kerr nonlinearity $\chi^{(3)}$(Subsection III.3.5).

2 The most important results of the thesis

The most important results of my thesis have been published or will be submitted for further publication [Bartkowiak2010a,Bartkowiak2010b,Bartkowiak2011,Bartkowiak2012]. As detailed list of results was presented in the previous subsections, here I would like to rather focus on meaning of obtained results.

2.1 Unifying derivation of classical inequalities [Bartkowiak2010a]

Some methods for creating operational criteria to test quantumness of multimode bosonic fields (or multiparty bosonic systems) have been analyzed. The procedure applied in the thesis allows one to unify derivation of many known inequalities and to propose new ones. My research is based on a criterion relying on analyses of the positivity of the multimode P-functions or, equivalently, the positivity of matrices of expectation values of, e.g., creation and annihilation operators. Under consideration there were not only monomials, but also polynomial functions of such moments. It is important to emphasize that the usage of polynomial functions sometimes enables simpler derivations of physically relevant inequalities, contrary to the Shchukin-Richter-Vogel nonclassicality criterion [7, 8].

It was demonstrated how nonclassicality criteria easily reduce to the well-known inequalities (see, e.g., textbooks [5, 13, 14, 74], reviews [63, 65, 75, 217], and Refs. [62, 64, 66, 68, 69, 70, 71, 72, 73, 76, 77, 78, 80, 218, 219, 220, 221]) describing various multimode nonclassical effects. Various examples are summarized in Tables II.1 and II.2.

Some general relations between the nonclassicality and entanglement criteria were obtained. In particular, I have analyzed relations resulting from the Cauchy-Schwarz inequality.

It was shown how some known entanglement inequalities can be derived as nonclassicality inequalities with the usage of introduced formalism. On the other hand, also some other known entanglement inequalities that can be seen as sums of more than one inequality derived from the nonclassicality criterion were studied.

It seems that the quantum-information community more or less ignores nonclassicality as something closely related to quantum entanglement. This thesis stresses a useful approach for analyzing both types of phenomena. It may be seen that by the usage of the introduced nonclassicality criteria based on matrices of moments one is able to find an effective way to derive specific inequalities. Such ability can be useful in verification of nonclassicality of particular states generated in experiments.

2.2 General occurrence of sudden vanishing and sudden reappearance of nonclassicality and entanglement witnesses [Bartkowiak2011]

The standard approach to study the sudden vanishing/sudden reappearance of quantum entanglement is based on the analysis of the time evolution of entanglement measures, e.g., the concurrence or, equivalently, the negativity or the relative entropy of entanglement [38]. A general operational recipe for construction of nonclassicality witnesses was presented, which emphasizes the fact that the most interesting parameters are the ones, which correspond to classical inequalities that can be violated for some nonclassical fields. Using introduced nonclassicality witnesses it was possible to reconstruct the results of Życzkowski and Horodecki [99], as well as Yu and Eberly [43] for dissipative systems and to analyze various finite-time decays (for dissipative systems) and analogous periodic vanishings (for unitary systems).

The analyses in this thesis dealt with various finite-time decays (for dissipative systems) and analogous periodic vanishings (for unitary systems) of nonclassical correlations as described by violations of classical inequalities and the corresponding nonclassicality witnesses, which are not necessarily entanglement witnesses. It has been shown that sudden vanishings are universal phenomena and can be observed for two- or multi-mode and for single-mode nonclassical fields.

There is a need to stress the fact that sudden vanishings of nonclassicality occur not solely for dissipative systems, and at evolution times which are usually different from those of sudden vanishings and reappearances of quantum entanglement.

These observations deepen an analysis of sudden vanishing and sudden reappearance of various nonclassicality witnesses in specific models and might serve as a motivation for further analyzes in experimental scenarios.

2.3 A linear-optical implementation of the CS and CNOT gates with the usage of conventional detectors [Bartkowiak2010b]

Linear-optical implementations of two-qubit universal gates including the iSWAP, CS and CNOT gates were studied. As shown in Table III.1, the majority of these realizations of nondestructive gates are based on single-photon detectors, even though a progress in constructing single-photon detectors (see Refs. [222, 223] and references therein) can be seen. One of the disadvantages of the usage of single-photon detectors is that they have dark count rates much higher than the conventional detectors [223]. Very attractive but experimentally underdeveloped proposals of multiple-photon resolving detectors including cascade arrays of conventional detectors (connected with beam splitters or with high-speed low-loss optical switches [224]) and fiber-loop detectors [225] are also available. Taking into account experimental accessibility two implementations, which are nondestructive (i.e., destroying only ancilla states) and work with conventional detectors (i.e., those which do not resolve number of photons), were proposed. I analysed schemes of the nondestructive universal gates using conventional detectors and entangled ancillae in a cluster state, Greenberger–Horne–Zeilinger states and Bell's states giving the success probability of $\eta^4/4$, $\eta^6/8$ (for the CNOT gate), and $\eta^4/8$ (for the CS gate), respectively. Deterioration of fidelity of the CS quantum gate in the case of detector imperfections (dark counts in addition to finite efficiency and no photon-number resolution) and imperfect sources of ancilla states were taken into account. It was demonstrated that the iSWAP gate can be implemented by the CNOT gate (or equivalently by the CS gate) with additional deterministic gates. Moreover, I have analyzed recently proposed linear-optical implementation of the iSWAP gate of Wang *et al.*[169] which probability of success accounts for $\eta^4/32$. I have presented, that using the same resources as

Wang *et al.* (the same ancillae, classical feedforward and even smaller number of conventional detectors) the iSWAP gate can be designed with success probability of $\eta^4/8$.

2.4 Schemes for the Kerr nonlinearity amplification [Bartkowiak2012]

Limitations of linear optics while building deterministic entangling quantum gates move the attention to nonlinear media as good resources for preforming necessary interaction between photons. Shapiro [180] showed that although there exists a small conditional phase-shift in cross-Kerr effect, the phase noise in $\chi^{(3)}$ media precludes from the effective usage of it as a conditional phase-shift gate. However, Matsuda *et al.* [178] showed experimentally that it is possible to induce and measure effectively conditional phase-shift induced by cross-Kerr modulation even for a few photons. Two schemes were proposed to enhance this phase-shift using i) one-mode squeezing and ii) two-mode squeezing operations. Vector coherent state theory has been applied to design the schemes for an amplification of the conditional phase-shift which can lead to a deterministic implementation of the CPHASE gate with efficiency that accounts for 20%. It has been demonstrated, that it is possible to improve the phase-shift obtained for two single-photon state in the cross-Kerr interaction. Such result gives the hope for using the Kerr medium as a deterministic, measurement- independent universal two-qubit entangling gate like the CPHASE or the CNOT. Presented schemes are based on properties of a group theory and as such can be adopted into arbitrary implementation, which operations can be described using $SU(1,1)$.

Streszczenie — Summary of the thesis in Polish

1 Wprowadzenie

Stany kwantowe, które wcześniej traktowano jako kurioza fizyki, takie jak makroskopowe superpozycje kwantowe (koty Schrödingera) czy stany splątane (np. stany Bella), obecnie odgrywają kluczową rolę w przetwarzaniu informacji kwantowej. Pytanie o możliwość opisu stanu kwantowego przy użyciu teorii zaczerpniętych z fizyki klasycznej stało się w tym kontekście jednym z fundamentalnych problemów wszystkich dziedzin teorii kwantowej (od kwantowej optyki, poprzez nanonauki, fizykę fazy skondensowanej, na kwantowej biologii kończąc [5,11-17]). Rozgraniczenie pomiędzy klasycznością a nieklasycznością można jednakże ustanowić biorąc pod uwagę różne aspekty świata kwantowego. Na stronach tej rozprawy nieklasyczność, jako przejaw występowania w układzie korelacji kwantowych, definiowana jest w oparciu o analogię pomiędzy opisem stanu w fizyce statystycznej i kwantowej. Należy zaznaczyć, że wszystkie stany, o których mowa, są stanami kwantowymi, jednakże wydaje się, że niektórym stanom bliżej jest do swoich klasycznych odpowiedników (np. stany koherentne nazywane są stanami klasycznymi mimo iż są kwantowe). Stany klasyczne w niniejszej rozprawie oznaczają zatem stany, które można opisać, podobnie jak stany w fizyce statystycznej, przy użyciu standardowo zdefiniowanych funkcji rozkładu prawdopodobieństwa. Przykładowo, takimi stanami będą stany koherentne, jako stany związane z oscylatorem harmonicznym. Wykorzystując własności stanów koherentnych dowolny M-modowy stan bozonowy można przedstawić w następujący sposób [11,12]:

$$\hat{\rho} \quad = \quad \int \mathrm{d}^2\boldsymbol{\alpha}\; P(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*)|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|, \tag{1}$$

gdzie P jest funkcją Glaubera-Sudarshana. Funkcja P, która w świecie klasycznym odgrywałaby rolę rozkładu prawdopodobieństwa, w reżimie kwantowym może okazać się ujemna lub bardzo osobliwa. Z uwagi na niestandardowe zachowanie, funkcja P (wraz z funkcją Wignera i Husimiego) określona została mianem funkcji quasiprawdopodobieństwa i ustanowiła podstawę do sformułowania kryterium nieklasyczności.

Stanem **nieklasycznym** nazywamy stan, dla którego funkcja P Glaubera-Sudarshan zachowuje się w niekonwencjonalny sposób (w stosunku do klasycznych rozkładów prawdopodobieństwa) tj. przyjmuje wartości ujemne lub jest bardziej osobliwa niż delta Diraca [5]. Warto zwrócić uwagę na zakres tak zdefiniowanej nieklasyczności. Mianowicie, zgodnie z tą definicją nie każdy stan nieklasyczny jest nieseparowalny, tj. splątany. Kryterium sformułowane w oparciu o funkcję P może wykryć kwantowe korelacje inne niż splątanie kwantowe.

Bazując na własnościach funkcji P, Agarwal i Tara [6] oraz Shchukin, Richter i Vogel (SRV) [7,8] zaproponowali kryteria nieklasyczności w oparciu o macierze skonstruowane z momentów operatorów kreacji i anihilacji. Analogicznie do kryteriów nieklasyczności, ale poprzez wprowadzenie dodatkowo częściowej transpozycji, zdefiniowane zostało również kryterium splątania [9]. Z uwagi na metodę badań momentów zaproponowaną przez Vogla i Shchukina [10], wybór kryterium Shchukina, Richtera i Vogla do badań przedstawionych w tej rozprawie wydaje się być uzasadniony, szczególnie, jeżeli celem jest powiązanie kryteriów nieklasyczności z przyjaznymi eksperymentalnie metodami ich rozstrzygania.

2 Cele i układ pracy

Rozważania przedstawione w rozprawie skupiają się na trzech aspektach analizy kwantowych korelacji:

- na odnajdywaniu fundamentalnych klasycznych nierówności, które byłyby łamane w obliczu kwantowych korelacji, w szczególności w obecności splątania kwantowego [Bartkowiak2010a];
- na badaniu czasowej ewolucji odpowiednio zdefiniowanych świadków nieklasyczności oraz splątania w układach optycznych (zarówno dysypatywnych, jak i unitarnych) [Bartkowiak2011];
- na szukaniu efektywnych i najprostszych doświadczalnie układów optycznych (liniowych i nieliniowych) do generacji i weryfikacji nieklasyczności [Bartkowiak2010b,Bartkowiak2012].

Celem mojej rozprawy jest powiązanie kryteriów nieklasyczności, które przy swoim wyprowadzeniu opierają się na teoretycznych klasycznych nierównościach, takich jak nierówność Cauchy'ego-Schwarza, z technologiczną eksperymentalną prostotą (w porównaniu z innymi metodami) oferowaną przez implementacje optyczne.

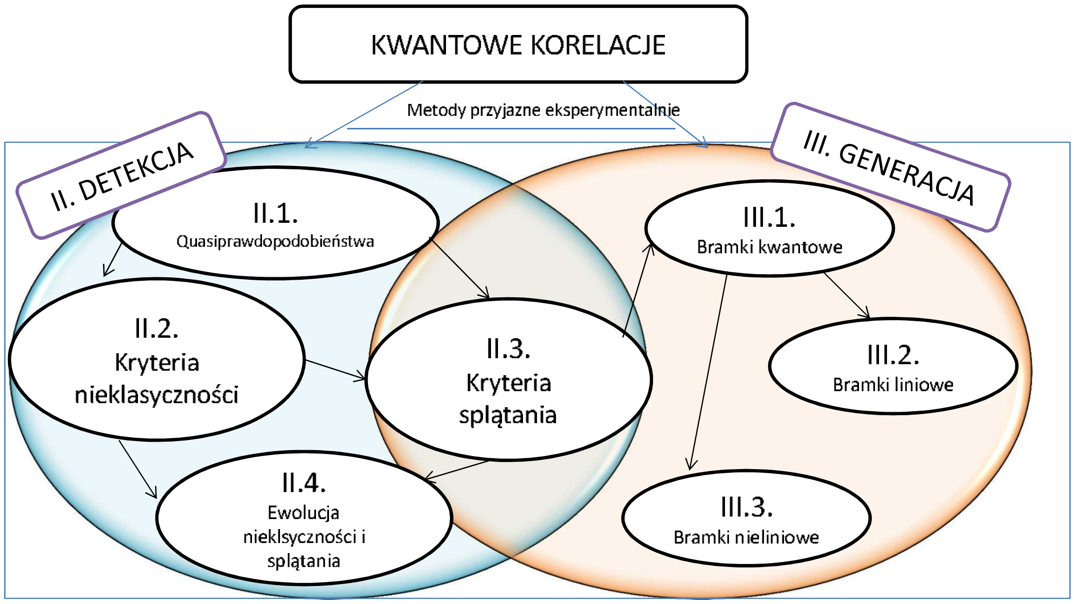
Rozprawa podzielona została na dwie główne części (poza wprowadzeniem, które można znaleźć w Rozdziale I). Pierwsza część, zawarta w Rozdziale II, wprowadza definicję nieklasyczności w oparciu o funkcję P Glaubera-Sudarshana, daje przepis na konstruowanie świadków zarówno nieklasyczności jak i splątania, rozumianego poprzez pojęcie separowalności stanów oraz przedstawia przykłady zastosowania ich później w celu analizy własności układów fizycznych.

Część druga (Rozdział III) skupia się na metodach generacji kwantowych korelacji przy wykorzystywaniu liniowej i nieliniowej optyki. Sposób powiązania ze sobą poszczególnych rozdziałów został zilustrowany na Rys. 1.

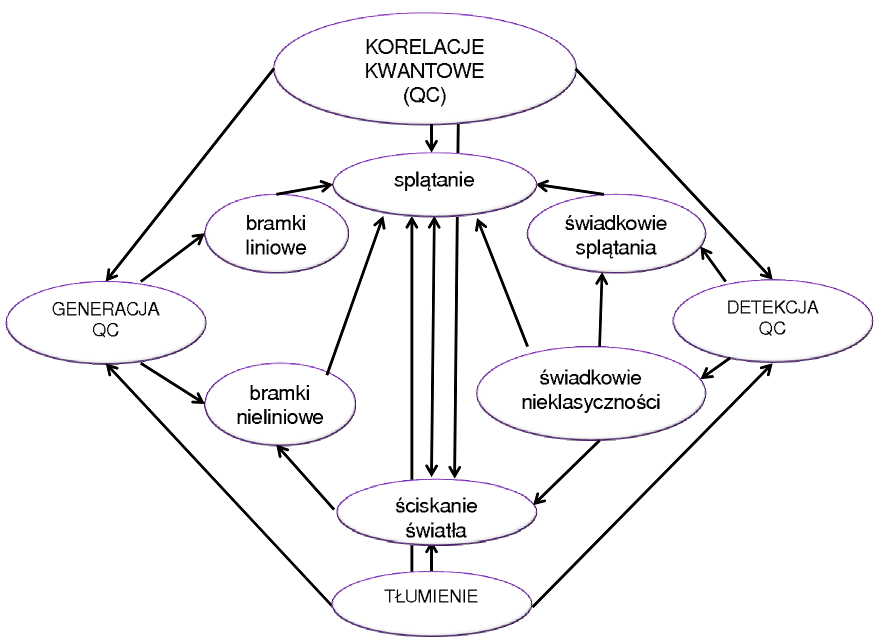
Rozprawa rozpoczyna się wprowadzeniem (Rozdział I), w którym można znaleźć uzasadnienie przeprowadzonych badań, założone cele oraz sposób ich realizacji. Sekcja II.1 ma na celu wprowadzenie formalizmu do opisu stanów w analogiczny sposób, jak ma to miejsce w fizyce statystycznej (przy użyciu rozkładów prawdopodobieństwa). W tej części zostaje wprowadzona definicja stanów koherentnych (Rów. (II.1), jako stanów własnych operatora anihilacji oraz przedstawienie ich własności, które umożliwiają opisać dowolny stan kwantowy przy użyciu funkcji quasiprawdopodobieństwa (Rów. (1)). Rozdział II.1 przedstawia uzasadnienie wyboru funkcji Glaubera-Sudarshana spośród pozostałych funkcji quasiprawdopodobieństwa, jako dobrej podstawy do budowania kryteriów nieklasyczności.
Najważniejsze pojęcia poruszone w pracy i relacje miedzy nimi są przedstawione na Rys. 2.

3 Kryteria nieklasyczności i splątania a łamanie klasycznych nierówności

Rozdział II.2 zawiera operacyjne kryteria nieklasyczności zdefiniowane w oparciu o macierze momentów operatorów kreacji i anihilacji. Definicję nieklasyczności można napisać formalnie w postaci [11,12]:
Kryterium 1: *Wielomodowy bozonowy stan $\hat{\rho}$ jest nieklasyczny, jeżeli jego funkcja Glaubera-Sudarshana P nie może być rozpatrywana jako klasyczna gęstość prawdopodobieństwa, tj. jest ujemna lub bardziej osobliwa niż delta Diraca. Zatem stan nazywamy klasycznym, jeżeli można go opisać za pomocą klasycznej gęstości prawdopodobieństwa.*



Rysunek 1: Diagram przedstawiający strukturę rozprawy w oparciu o relacje pomiędzy rozdziałami.



Rysunek 2: Diagram przedstawiający relacje pomiędzy najważniejszymi pojęciami, używanymi na stronach rozprawy.

Wykorzystując funkcję P, można zdefiniować następującą normalnie uporządkowaną średnią funkcji operatorów (Rów. (II.17)):

$$\langle : \hat{f}^\dagger \hat{f} : \rangle \quad = \quad \int \mathrm{d}^2\boldsymbol{\alpha} \, |f(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*)|^2 P(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*), \tag{2}$$

gdzie $\hat{f}$ są funkcjami M-modowych operatorów kreacji i anihilacji, które w szczególności mogą mieć postać jednomianów lub wielomianów (w przeciwieństwie do kryterium Shchukina, Richtera i Vogla [7,8], w którym zakłada się użycie wyłącznie jednomianów). Powyżej sformułowana średnia pozwala na przedefiniowanie Kryterium 1 w następujący sposób:

Obserwacja 1*: Jeżeli funkcja P danego stanu jest klasyczną gęstością prawdopodobieństwa, wówczas $\langle : \hat{f}^\dagger \hat{f} : \rangle \geq 0$ dla dowolnej funkcji $\hat{f}$. W przeciwnym przypadku, jeśli $\langle : \hat{f}^\dagger \hat{f} : \rangle < 0$ dla pewnych $\hat{f}$, wówczas funkcja P nie jest klasyczną gęstością prawdopodobieństwa.*

Normalnie uporządkowane momenty $\langle : \hat{f}^\dagger \hat{f} : \rangle$ mogą być zgrupowane w macierzy przedstawionej w Rów. (II.22):

$$M_{\hat{f}}^{(\mathrm{n})}(\hat{\rho}) = \begin{pmatrix} \langle : \hat{f}_1^\dagger \hat{f}_1 : \rangle & \langle : \hat{f}_1^\dagger \hat{f}_2 : \rangle & \cdots & \langle : \hat{f}_1^\dagger \hat{f}_N : \rangle \\ \langle : \hat{f}_2^\dagger \hat{f}_1 : \rangle & \langle : \hat{f}_2^\dagger \hat{f}_2 : \rangle & \cdots & \langle : \hat{f}_2^\dagger \hat{f}_N : \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle : \hat{f}_N^\dagger \hat{f}_1 : \rangle & \langle : \hat{f}_N^\dagger \hat{f}_2 : \rangle & \cdots & \langle : \hat{f}_N^\dagger \hat{f}_N : \rangle \end{pmatrix}, \tag{3}$$

gdzie $\hat{f} = \sum_i^N c_i \hat{f}_i$ dla dowolnych liczb zespolonych c_i. Kryterium nieklasyczności można zapisać ostatecznie jako [Bartkowiak2010a]:

Kryterium 3: *Wielomodowy stan bozonowy $\hat{\rho}$ jest nieklasyczny, jeżeli istnieje takie $\hat{f}$, że wyznacznik $\det[M_{\hat{f}}^{(\mathrm{n})}(\hat{\rho})]$ jest ujemny.*

Kolejne części Rozdziału II.2 mają na celu pokazanie zastosowań sformułowanych kryteriów do wyprowadzania fundamentalnych nierówności. W szczególności, w Rozdziale II.2.2 zostało pokazane, jak kryteria nieklasyczności dla jednomodowych pól mogą być zredukowane do nierówności Cauchy'ego-Schwarza [14]– Rów. (II.28).

Zdefiniowane przeze mnie kryteria zostały wykorzystane do wyprowadzenia warunków na występowanie znanych efektów kwantowych, jak np. ściskanie kwadraturowe. Rozdział II.2.3 jest egzemplifikacją tego, jak można powiązać nieklasyczność z warunkami na różnego typu ściskanie światła:

- na wielomodowe ściskanie kwadraturowe– Rów. (II.31), (II.34);
- dwumodowe fundamentalne ściskanie (ang. principal squeezing), związane z relacją nieokreśloności Schrödingera-Robertsona [67] (zdefiniowane przez Lukša *i in.* [66] i Loudona *i in.*[63])– Rów. (II.35)–(II.38);
- dwumodowe ściskanie sumy i różnicy liczby fotonów zdefiniowane przez Hillery'ego [68] i ich uogólnienia przedstawione przez Ana i Tinha [69,70]– Rów. (II.40),(II.43),(II.44),(II.48),(II.49),(II.55), (II.56),(II.61) i (II.64).

Przykłady zastosowań kryteriów nieklasyczności do pokazania łamania jednoczasowych korelacji liczby fotonów dwóch modów zostały pokazane w Rozdziale II.2.4. W tym celu została wprowadzona uogólniona definicja funkcji P, która z uwagi na zależność czasową jest nie tylko normalnie, ale również czasowo uporządkowana (Rów. (II.68)). W szczególności przeanalizowano:

- jak można uzyskać warunki na ściskanie sumy i różnicy liczby fotonów, co można powiązać z subpoissonowską statystyką fotonów [74]– Rów. (II.69), (II.71).

- jak można powiązać kryteria nieklasyczności z łamaniem nierówności Cauchy'ego-Schwarza [14] oraz nierównością Muirheada [73,81]– Rów.(II.84) i (II.86)–(II.88);

- w jaki sposób można uzyskać z kryteriów nieklasyczności warunki na dwuczasowe korelacje liczby fotonów dla pojedynczego modu, włączając antygrupowanie (ang. antibunching) [5,14,76]– Rów. (II.73), (II.76) i nadgrupowanie fotonów (ang. hyperbunching) [77,78]– Rów. (II.76),(II.82) i (II.83).

W kolejnym Rozdziale II.2.5 zostały przedstawione nierówności uzyskane z kryteriów nieklasyczności, które zgodnie z moją wiedzą nie były prezentowane do tej pory w literaturze– Rów. (II.92)–(II.95). Przykłady zastosowań kryteriów do wyprowadzania znanych warunków na efekty kwantowe oraz klasycznych nierówności są przedstawione również w Tabeli 1.

Następnie kryteria zostały zastosowane do konstrukcji świadków nieklasyczności, które dalej mogą być wykorzystywane do detekcji nieklasyczności układu. Rozdział II.2.6 zawiera przepis na konstruowanie świadków nieklasyczności w sposób, który sprowadza je do postaci analogicznej do miar splątania (Rów. (II.100) i (II.101)). W tej części przedstawione zostały również definicje kilku świadków, które zostały użyte do analizowania własności pól optycznych– Rów.(II.103), (II.105), (II.106), (II.108) i (II.110).

Z uwagi na znaczenie splątania, jako specyficznego rodzaju kwantowej korelacji, został poświęcony temu zagadnieniu osobny Rozdział II.3. Wprowadzono w nim kryterium Shchukina-Vogla [9,57] rozstrzygające o separowalności stanów, w oparciu o którą definiowane jest splątanie. Analogicznie jak w przypadku nieklasyczności, również w tym przypadku można zdefiniować macierz momentów operatorów kreacji i anihilacji, z tą różnicą, że aby wykryć separowalność stanu, wprowadza się częściową transpozycję

$$M_{\hat{f}}(\hat{\rho}^{\Gamma}) = \begin{pmatrix} \langle \hat{f}_1^\dagger \hat{f}_1 \rangle^\Gamma & \langle \hat{f}_1^\dagger \hat{f}_2 \rangle^\Gamma & \cdots & \langle \hat{f}_1^\dagger \hat{f}_N \rangle^\Gamma \\ \langle \hat{f}_2^\dagger \hat{f}_1 \rangle^\Gamma & \langle \hat{f}_2^\dagger \hat{f}_2 \rangle^\Gamma & \cdots & \langle \hat{f}_2^\dagger \hat{f}_N \rangle^\Gamma \\ \vdots & \vdots & \ddots & \vdots \\ \langle \hat{f}_N^\dagger \hat{f}_1 \rangle^\Gamma & \langle \hat{f}_N^\dagger \hat{f}_2 \rangle^\Gamma & \cdots & \langle \hat{f}_N^\dagger \hat{f}_N \rangle^\Gamma \end{pmatrix}, \tag{4}$$

gdzie $\langle \hat{f}_i^\dagger \hat{f}_j \rangle^\Gamma \equiv \mathrm{tr}(\hat{f}_i^\dagger \hat{f}_j \hat{\rho}^\Gamma)$ natomiast Γ oznacza częściową transpozycję. Kryterium splątania Shchukina-Vogla [9,57] można zapisać jako:

***Kryterium 4**: Dwucząstkowy stan $\hat{\rho}$ może zostać niedodatnio częściowo transponowany (NPT) wtedy i tylko wtedy, gdy istnieje takie $\hat{f}$, że $\det[M_{\hat{f}}(\hat{\rho}^\Gamma)]$ jest ujemne.*

W powyższym Kryterium 4 NPT jest równoważne z nieseparowalnością stanu. W dalszych częściach tego rozdziału zostały pokazane przykłady zastosowania Kryterium 4 do otrzymania znanych warunków na wykrywanie splątania i przeanalizowana ich relacji z kryteriami nieklasyczności.

Rozdział II.3.2 przedstawia relację pomiędzy kryterium splatania i nierównością Cauchy'ego-Schwarza. Kolejny Rozdział II.3.3 prezentuje takie warunki na detekcję splątania, jak np. kryterium Duana *i in.* [44]– Rów. (II.136)–(II.138), albo Hillery'ego-Zubairy'ego [45]– Rów.(II.122),(II.125),(II.129)–(II.131),(II.123), (II.124),(II.126) i (II.127), które można również wyprowadzić z przedstawionych kryteriów nieklasyczności.

Tabela 1: Kryteria nieklasyczności dla efektów jednoczasowych dla dwumodowych (DM) i wielomodowych (WM) pól oraz dwuczasowe efekty dla jednomodowych (JM) pól [Bartkowiak2010a].

Efekt nieklasyczny	**Kryterium**	Równanie
WM ściskanie kwadraturowe	$d^{(\mathrm{n})}(1, \hat{X}_{\boldsymbol{\phi}}) < 0$	(II.31), (II.34)
DM ściskanie fundamentalne Lukša *i in.* [66]	$d^{(\mathrm{n})}(\Delta\hat{a}_{12}^{\dagger}, \Delta\hat{a}_{12}) = d^{(\mathrm{n})}(1, \hat{a}_{12}^{\dagger}, \hat{a}_{12}) < 0$	(II.35)–(II.38)
DM ściskanie sumy Hillery'ego [68]	$d^{(\mathrm{n})}(1, \hat{V}_{\phi}) < 0$	(II.40), (II.43)
WM ściskanie sumy Ana-Tinha [69]	$d^{(\mathrm{n})}(1, \hat{\mathcal{V}}_{\phi}) < 0$	(II.44), (II.48)
DM ściskanie różnicy Hillery'ego [68]	$d^{(\mathrm{n})}(1, \hat{W}_{\phi}) < -\frac{1}{2}\min(\langle\hat{n}_1\rangle, \langle\hat{n}_2\rangle)$	(II.49), (II.55), (II.56)
WM ściskanie różnicy Ana-Tinha [70]	$d^{(\mathrm{n})}(1, \hat{\mathcal{W}}_{\phi}) < -\frac{1}{4}\left\lvert \lvert\langle\hat{C}\rangle\rvert - \langle\hat{D}\rangle \right\rvert$	(II.61), (II.64)
WM subpoissonowskie korelacje liczby fotonów	$d^{(\mathrm{n})}(1, \hat{n}_1 \pm \hat{n}_2) < 0$	(II.69), (II.71)
naruszanie nierówności Cauchy'ego-Schwarza	$d^{(\mathrm{n})}(\hat{f}_1, \hat{f}_2) < 0$	(II.27), (II.28)
DM łamanie nierówności Cauchy'ego-Schwarza poprzez test Agarwala [72]	$d^{(\mathrm{n})}(\hat{n}_1, \hat{n}_2) < 0$	(II.84), (II.86)
DM łamanie nierówności Muirheada poprzez test Lee [73]	$d^{(\mathrm{n})}(\hat{n}_1 - \hat{n}_2) < 0$	(II.87), (II.88)
JM antygrupowanie fotonów	$d^{(\mathrm{n})}[\hat{n}(t), \hat{n}(t+\tau)] < 0$	(II.73), (II.76)
JM nadgrupowanie fotonów	$d^{(\mathrm{n})}[\Delta\hat{n}(t), \Delta\hat{n}(t+\tau)]$ $= d^{(\mathrm{n})}[1, \hat{n}(t), \hat{n}(t+\tau)] < 0$	(III.76), (II.82), (II.83)
Inne DM nieklasyczne efekty [Bartkowiak2010a]	$d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2, \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}) < 0$	(II.92)
	$d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2^{\dagger}, \hat{a}_1^{\dagger}\hat{a}_2) < 0$	(II.93)
	$d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^{\dagger}, \hat{a}_1^{\dagger} + \hat{a}_2) < 0$	(II.94)
	$d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2, \hat{a}_1^{\dagger} + \hat{a}_2^{\dagger}) < 0$	(II.95)
	$d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_1^{\dagger}, \hat{a}_2^{\dagger}, \hat{a}_2) < 0$	(II.96)

Tabela 2: Kryteria splątania versus kryteria nieklasyczności [Bartkowiak2010a].

Referencja	**Kryteria splątania**	**Równoważne kryteria nieklasyczności**	Równanie
Duan *i in.* [44]	$d^{\Gamma}(\Delta\hat{a}_1, \Delta\hat{a}_2) = d^{\Gamma}(1, \hat{a}_1, \hat{a}_2) < 0$	$d^{(\mathrm{n})}(\Delta\hat{a}_1, \Delta\hat{a}_2^{\dagger}) = d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_2^{\dagger}) < 0$	(II.136)–(II.138)
Simon [46]	$d^{\Gamma}(1, \hat{a}_1, \hat{a}_1^{\dagger}, \hat{a}_2, \hat{a}_2^{\dagger}) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_1^{\dagger}, \hat{a}_2^{\dagger}, \hat{a}_2) + d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_2^{\dagger})$	(II.140)
		$+d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_1^{\dagger}, \hat{a}_2^{\dagger}) + d^{(\mathrm{n})}(1, \hat{a}_1, \hat{a}_2^{\dagger}, \hat{a}_2) < 0$	
Mancini *i in.* [93]	$d^{\Gamma}(1, \hat{a}_1 + \hat{a}_2, \hat{a}_1^{\dagger} + \hat{a}_2^{\dagger}) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^{\dagger}, \hat{a}_1^{\dagger} + \hat{a}_2) + 2d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^{\dagger}) + 1 < 0$	(II.146), (II.147)
Hillery & Zubairy [45]	$d^{\Gamma}(1, \hat{a}_1\hat{a}_2) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2^{\dagger}) < 0$	(II.122), (II.125)
ditto	$d^{\Gamma}(1, \hat{a}_1^{m}\hat{a}_2^{n}) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1^{m}(\hat{a}_2^{\dagger})^{n}) < 0$	(II.129)–(II.131)
ditto	$d^{\Gamma}(\hat{a}_1, \hat{a}_2) < 0$	$d^{(\mathrm{n})}(\hat{a}_1, \hat{a}_2^{\dagger}) < 0$	(II.123), (II.126)
ditto	$d^{\Gamma}(1, \hat{a}_1\hat{a}_2\hat{a}_3) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1^{\dagger}\hat{a}_2\hat{a}_3) < 0$	(II.124), (II.127)
Miranowicz *i in.* [61]	$d^{\Gamma}(\hat{a}_1, \hat{a}_2\hat{a}_3) < 0$	$d^{(\mathrm{n})}(\hat{a}_1^{\dagger}, \hat{a}_2\hat{a}_3) < 0$	(II.128)
Inne testy na obecność splątania [Bartkowiak2010a]	$d^{\Gamma}(1, \hat{a}_1^{k}\hat{a}_2^{l}\hat{a}_3^{m}) < 0$	$d^{(\mathrm{n})}(1, (\hat{a}_1^{\dagger})^{k}\hat{a}_2^{l}\hat{a}_3^{m}) < 0$	(II.132), (II.133)
	$d^{\Gamma}(\hat{a}_1^{k}, \hat{a}_2^{l}\hat{a}_3^{m}) < 0$	$d^{(\mathrm{n})}((\hat{a}_1^{\dagger})^{k}, \hat{a}_2^{l}\hat{a}_3^{m}) < 0$	(II.134), (II.135)
	$d^{\Gamma}(1, \hat{a}_1\hat{a}_2, \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2^{\dagger}, \hat{a}_1^{\dagger}\hat{a}_2) + (\langle\hat{n}_1 + \hat{n}_2\rangle + 1)\, d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2^{\dagger}) < 0$	(II.142), (II.143)
	$d^{\Gamma}(1, \hat{a}_1\hat{a}_2^{\dagger}, \hat{a}_1^{\dagger}\hat{a}_2) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2, \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}) + \langle\hat{n}_1\rangle\langle\hat{n}_2\rangle + \langle\hat{n}_1 + \hat{n}_2\rangle\, d^{(\mathrm{n})}(1, \hat{a}_1\hat{a}_2) < 0$	(II.144), (II.145)
	$d^{\Gamma}(1, \hat{a}_1 + \hat{a}_2, \hat{a}_1^{\dagger} + \hat{a}_2^{\dagger}) < 0$	$d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^{\dagger}, \hat{a}_1^{\dagger} + \hat{a}_2) + 2d^{(\mathrm{n})}(1, \hat{a}_1 + \hat{a}_2^{\dagger}) < 0$	(II.147), (II.148)

Jednakże taka zależność nie jest ogólna, z tego względu została opracowana metoda na wyrażanie warunków splątania poprzez sumy kryteriów nieklasyczności. Przykładowo nierówność Simona [46]–

Rów. (II.140), lub Manciniego *i in.* [93]– Rów. (II.146) i (II.147), zostały wyprowadzone w oparciu o kryteria splątania i wyrażone za pomocą sumy kryteriów nieklasyczności. Ostatnia część tego rozdziału (Rozdział II.3.4) analogicznie, jak w przypadku tej dotyczącej nieklasyczności, podaje przykłady skonstruowanych świadków splątania– Rów. (II.150), (II.151) i (II.152).

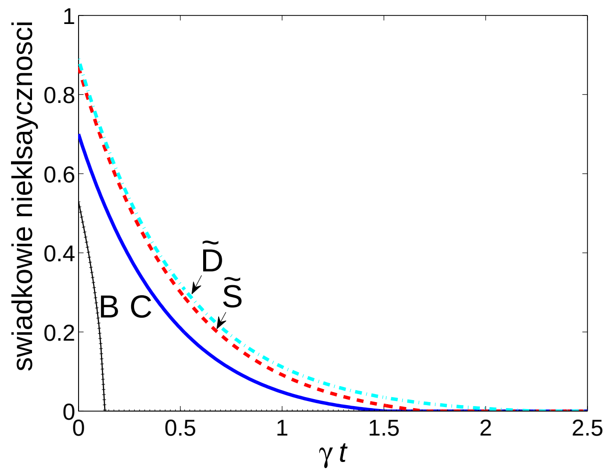
4 Nagłe zaniki i odrodzenia nieklasyczności w układach optycznych [Bartkowiak2011]

W każdym przypadku, zarówno kryteriów nieklasyczności jak i splątania, w ostatnich rozdziałach zdefiniowano świadków nieklasyczności i splątania wzorując się na znanych definicjach miar splątania. Logiczną kontynuacją zdefiniowania takowych świadków, wydaje się zastosowanie ich do konkretnych modeli fizycznych w celu testowania nieklasyczności oraz splątania. Rozważania dotyczące analizy zachowań zdefiniowanych świadków zostały przedstawione w Rozdziale II.4. Zostały one zainspirowane charakterystyczną wrażliwością kwantowych korelacji na wprowadzoną dekoherencje do układu i faktem, że jest ona jedną z głównych przeszkód zapobiegających manipulacji informacją kwantową. Pierwszy raz, nietypowe w porównaniu do innych zmiennych charakteryzujących układ, zachowanie splątania zostało przedstawione w pracach Życzkowskiego i Horodeckich [99] oraz Yu and Eberly'ego [43]. Pokazali oni, że korelacje tego typu zanikają w skończonym czasie. Do tego zjawiska przylgnęła bardzo dramatyczna nazwa śmierci splątania (na stronach tej rozprawy będzie używane pojęcie nagłego zaniku). Po przeanalizowaniu szeregu świadków splątania i nieklasyczności oraz porównaniu ich do miar splątania dla różnych modeli fizycznych pokazano, że nagły zanik nieklasyczności jest zjawiskiem powszechnym w układach dysypatywnych i nie dotyczy tylko splątania.

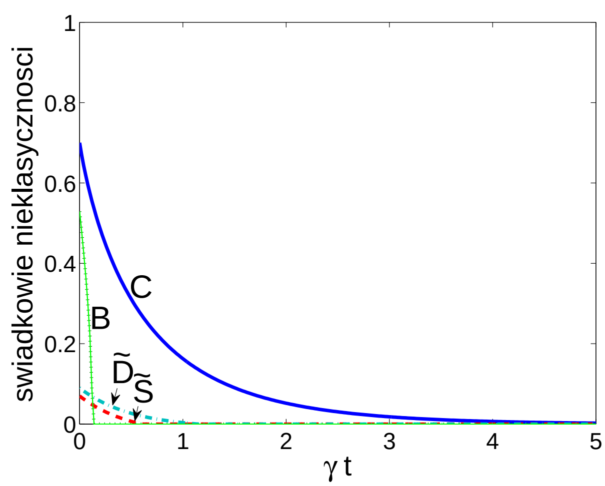
W Rozdziale II.4.2 rozpatrzony został przykład układu dwóch nieoddziałujących bezpośrednio modów. W założeniu mogą oddziaływać one tylko za pomocą swoich niezależnych rezerwuarów. Rozważania prowadzone były dla dwóch rożnych stanów wejściowych: stanu Wernera i jego modyfikacji. Używając równania master przedstawionego w Rów. (II.153) w przybliżeniu Markowa, wyprowadzone zostały analityczne formuły na poszczególnych świadków nieklasyczności, splątania i miar splątania– Rów. (II.156)– (II.159) i (II.165)–(II.167). Rysunki 3 i 4 przedstawiają oczekiwane zależności świadków splątania i nieklasyczności oraz miar splątania od czasu. Widać na nich wyraźnie, że po skończonych, aczkolwiek różnych czasach, każdy z nich spada do zera.

W dalszej części pracy pokazane zostały również modele, w których mimo braku dyssypacji można zaobserwować nagłe zaniki nieklasyczności i splątania. W Rozdziale II.4.2 rozważany był parametryczny konwerter częstości, którego hamiltonian jest opisany Rów. (II.171). Odpowiada on sytuacji dwóch liniowo sprzężonych wahadeł. Model ten przeanalizowany został dla przypadków o dwóch różnych stanach wejściowych: stanie czystym oraz mieszanym. Zostało pokazane, że nagłe zaniki i odrodzenia nieklasyczności mogą pojawiać się periodycznie zarówno, kiedy na początku układ jest w stanie czystym (który jest nieklasyczny, ale separowalny), jak i w stanie mieszanym (w stanie niemaksymalnie splątanym). Periodyczne zachowanie świadków nieklasyczności i splątania jest widoczne na Rys. 5.

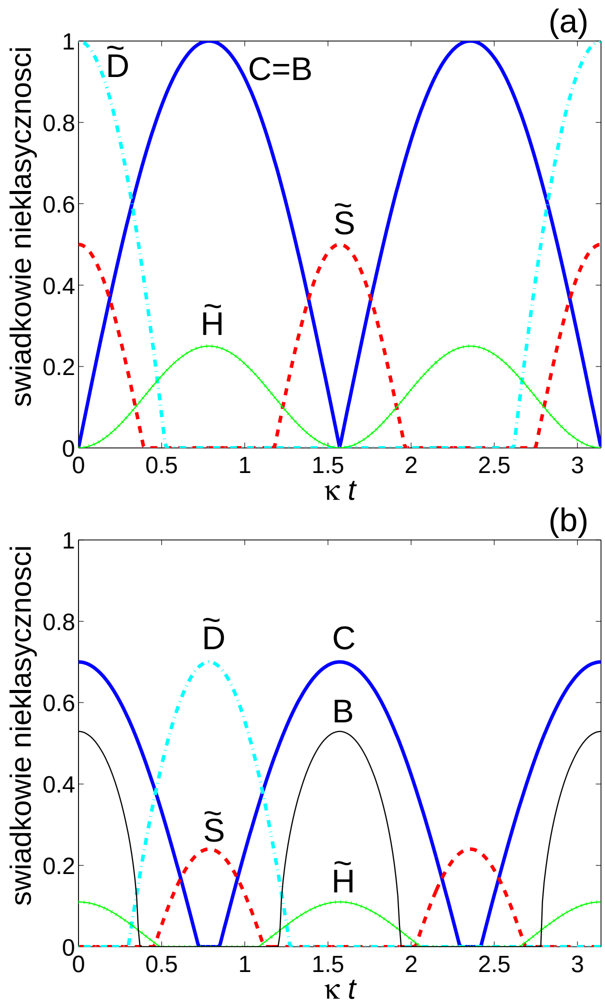
Okazuje się, że nagłe zaniki mogą być obserwowane w układach rządzonych unitarną ewolucją w zarówno dwucząstkowym jak i wielocząstkowych układach, dla modów oddziałujących jak i nieoddziałujących ze sobą (Rozdział II.4.2). Po przeanalizowaniu ściskania kwadraturowego w ośrodku kerrowskim zostało również pokazane, że nagłe zaniki i odrodzenia nieklasyczności mogą zostać także zaobserwowane w układzie jednomodowym (Rozdział II.4.3).



Rysunek 3: Ewolucja czasowa świadków nieklasyczności z widocznymi nagłymi zanikami dla dwóch nieoddziałujących modów, opisana modelem z Rozdziału II.4.1. Stanem początkowym jest stan podobny do stanu Wernera ρ_1 z $p = 0.8$ dany Rów. II.154. Klucz: C– zgodność (ang. concurrence), B– nielokalność B, $\tilde{S}$ dla $S_0 = 0.03$ oraz $\tilde{D}$ dla $D_0 = 0.1$– dwie ostatnie wielkości opisują korelacje różnicy liczby fotonów i są dane odpowiednio Rów. (II.103) oraz (II.105) [Bartkowiak2011].



Rysunek 4: Rysunek dotyczy tego samego modelu co Rys. 3, ale przy założeniu stanu początkowego ρ_2 danego Rów. II.154. [Bartkowiak2011].



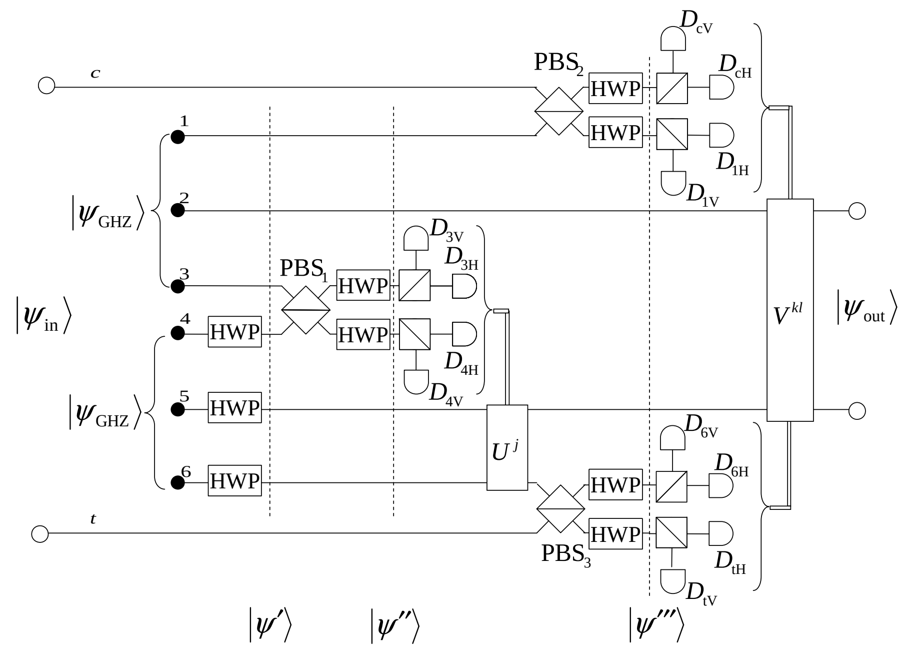
Rysunek 5: Przykłady nagłych zaników i odrodzeń świadków nieklasyczności dla dwóch oddziałujących modów. Ewolucja unitarna modów zakłada (a) stan początkowy jako $|01\rangle$ oraz (b) stan początkowy jako stan mieszany dany Rów. (II.182) z $p = 0.8$. Klucz: C – zgodność (ang. concurrence), B– nielokalność; $\tilde{H}$ – świadek splątania dany Rów. (II.150), związany z naruszaniem pierwszej nierówności Hillery'ego-Zubairy'ego; $\tilde{S}$ dla $S_0 = 1/2$ i $\tilde{D}$ dla $D_0 = 1$– świadkowie nieklasyczności opisujący korelacje różnicy liczby fotonów dane Rów. (II.103) oraz (II.105)[Bartkowiak2011].

5 Liniowo-optyczne implementacje bramek CS and CNOT z wykorzystaniem konwencjonalnych detektorów [Bartkowiak2010b]

Jak zostało wspomniane wcześniej, splątanie, jako emanacja kwantowości i źródło różnych interpretacji, stało się bardzo ważnym elementem wspomagającym protokoły inżynierii kwantowej. Część druga pracy została poświęcona implementacjom schematów do generacji kwantowych korelacji, w szczególności najbardziej popularnej z nich- splątania. Jednym ze sposób wprowadzania do układu kwantowych korelacji jest zastosowanie plączących bramek kwantowych. Posługując się analogią pomiędzy światem klasycznym i kwantowym, bramki kwantowe mogą być interpretowane jako urządzenia dokonujące operacji na kubitach. Zarówno w przypadku klasycznym jak i kwantowym mamy do czynienia z prostymi operacjami, które można implementować na pojedynczych bitach/ kubitach. Najbardziej fundamentalne okazują się jednak bramki, które potrafią dokonywać nietrywialnych operacji na bitach /kubitach. W przypadku świata klasycznego mamy do czynienia z np. bramką NAND, która jest nieodwracalną bramką uniwersalną. Mając dostępną tę bramkę i jednobitowe operacje, można zaimplementować każdą inną operację logiczną. Okazała się ona inspirująca do stworzenia analogicznej bramki w świecie kwantowym, zwanej bramką kontrolowanej negacji (CNOT). Należy jednak zaznaczyć, że z uwagi na własności fizyki kwantowej, bramki kwantowe posiadają znacząco różne właściwości od tych klasycznych, przykładowo są odwracalne (matematycznie są macierzami unitarnymi) oraz potrafią wprowadzać kwantowe korelacje, w szczególności splątanie pomiędzy kubitami. Bramka CNOT, podobnie jak np. iSWAP czy bramka kontrolowanej zmiany znaku (CS), są bramkami uniwersalnymi. Oznacza to, że używając jednej z nich i bramek jednokubitowych można zbudować dowolny układ. Bramki te są równoważne ze względu na unitarne transformacje. Sposób w jaki można wyrazić bramkę iSWAP za pomocą bramki CNOT lub CS i innych bramek jednokubitowych został przedstawiony na Rys.III.1.

W tej części pracy zostały zaprezentowane dwa podejścia do konstruowania bramek kwantowych. Ponieważ metody liniowo-optyczne są znane od bardzo dawna i część potrzebnych operacji np. bramki jednokubitowe można zaimplementować używając półfalówek, ćwierćfalówek, czy dzielników wiązki, zostały zaproponowane przeze mnie dwa schematy bramek liniowo-optycznych. W Rozdziale III.2 przedstawiona i omówiona została również lista najważniejszych implementacji liniowo-optycznych bramek kwantowych (Tabela III.1). Jak się okazuje większość implementacji, jako teoretyczne projekty, nie bierze pod uwagę eksperymentalnych możliwości realizacji bramek. Z wyjątkiem propozycji Zou *i in.* [143], wszystkie pozostałe przedstawione w Tabeli III.1 albo zakładają użycie trudno dostępnych detektorów, które rozpoznają ilość rejestrowanych fotonów albo/i niszczą stany wyjściowe, czyniąc tym samym bramkę bezużyteczną do dalszego użycia. Z tego względu zaprojektowano dwa schematy bramek kwantowych, które byłyby niedestrukcyjne (nieniszczące stanów wyjściowych) i zakładałyby użycie konwencjonalnych detektorów, które są powszechniejsze w użyciu w grupach eksperymentalnych.

W Rozdziale III.2 zostały przedstawione dwa schematy bramek kwantowych: bramka CNOT (Rozdział III.2.1)– Rys. 6, oraz bramka CS (Rozdział III.2.2)– Rys. 7, obie nieniszczące i zakładające użycie najprostszych detektorów. Pierwsza z nich, nazwana Schematem I, bazuje na propozycji Pittmana *i in.* [146] dostosowanej do użycia konwencjonalnych detektorów bez obniżania prawdopodobieństwa bramki, które wynosi $\eta^4/4$, gdzie η jest sprawnością detektorów. Prawdopodobieństwo $\eta^4/4$ oznacza, że w 1/4 przypadków schemat działa jak bramka CNOT (przy założeniu, ze czterosplątany stan Gottesmana-Chuanga [147] jest dany). Warto jednak zaznaczyć, że sytuacje, w których bramka działa w pożądany sposób, są jednoznacznie określone przez odpowiednie konfiguracje detektorów, przedstawione w Tabeli III.2. Na Rysunku 6 część wewnętrzna schematu generuje stan Gottesmana-Chuanga, zakładający bardziej popularne stany Greenbergera-Horne'a-Zeilingera jako stany pomocnicze. Dla obu połączonych schematów całkowite prawdopodobieństwo sukcesu wynosi $\eta^6/8$.



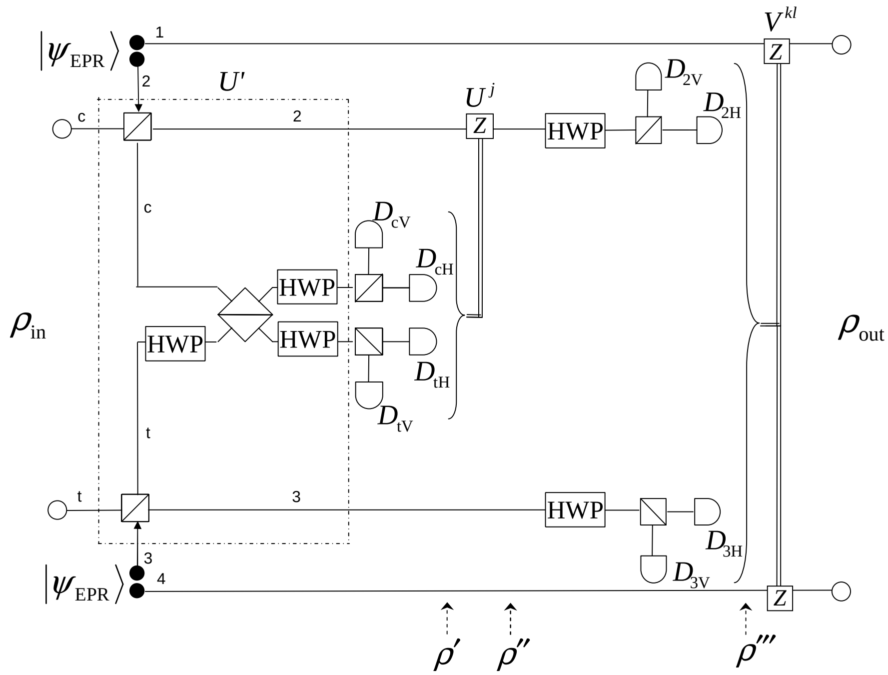
Rysunek 6: Schemat I– propozycja implementacji bramki CNOT wykorzystującej konwencjonalne detektory i kubity pomocnicze w stanie Greenbergera-Horne'a-Zeilingera (GHZ) danym $|\psi_{\mathrm{GHZ}}\rangle$. Klucz: $\mathrm{HWP} = U_{\mathrm{HWP}}(\pi/8)$–bramka Hadamarda H; U^j i V^{kl} są warunkowymi operacjami unitarnymi przedstawionymi w Tabeli III.2, gdzie σ_z może być zrealizowana jako $U_{\mathrm{HWP}}(0)$; D_k oznaczają fotodetektory; PBS_i są polaryzacyjnymi dzielnikami wiązki w bazie *HV* [Bartkowiak2010b].

Drugi Schemat II został przedstawiony na Rys.7 i opisany dokładnie w Rozdziale III.2.2. Jest to schemat do implementacji bramki CS przy założeniu, że stanami pomocniczymi są stany EPR. Inspiracje dla tego schematu stanowił układ zaproponowany przez Zou *i in.* [143]. Jest to również bramka zakładająca użycie konwencjonalnych detektorów i nieniszcząca stany wyjściowe. Prawdopodobieństwo sukcesu dla tego układu wynosi $\eta^4/8$ i może być rozumiane podobnie jak poprzednio. Sekwencje detektorów informujące o udanych zdarzeniach przedstawione są w Tabeli III.3. W przypadku tej bramki zostały również przeprowadzone rozważania biorące pod uwagę realne źródła generacji fotonów i stanów pomocniczych, niedoskonałości detektorów, włącznie z ciemnymi zliczeniami oraz skończoną efektywnością i ich wpływ na wierność bramki (wyniosła ona ok 97%).

Ostatnio Wang *i in.* [169] zaproponował liniowo-optyczną implementację bramki iSWAP, która jak wynika z dekompozycji przedstawionej na Rys.III.1, może zostać zastąpiona bramką CNOT lub CS. Prawdopodobieństwo które uzyskał Wang *i in.* wyniosło $\eta^4/32$. Wykorzystując dowolny z dwóch przedstawionych przez ze mnie schematów można uzyskać prawdopodobieństwo przynajmniej czterokrotnie większe, przy założeniu tych samych zasobów.

6 Schematy wzmacniające nieliniowość w ośrodkach kerrowskich [Bartkowiak2012]

Większość bramek kwantowych wykorzystujących liniową optykę jest probabilistyczna. Alternatywną metodą realizacji bramek dwufotonowych jest wykorzystanie ośrodka nieliniowego, w którym fotony oddziałują ze sobą. W szczególności, Chuang i Yamamoto [175] pokazali, że do tego celu może zostać użyty efekt Kerra. Zastosowanie ośrodków Kerra daje nadzieję na deterministyczne implementacje bramek uniwersalnych. W tym celu zaproponowałam, aby skorzystać z wewnętrznej nieliniowości niektórych ośrodków,



Rysunek 7: Schemat II- propozycja implementacji bramki CS wykorzystującej konwencjonalne detektory i kubity pomocnicze w dwóch idealnych lub nieidealnych stanach EPR $|\psi_{\rm EPR}\rangle$. Notacja jest w zgodzie z tą na Rys. 6. W tym przypadku HWP $= U_{\rm HWP}(\pi/8)$ reprezentuje bramkę Hadamarda H. W tekście można znaleźć definicje stanów operacji unitarnych U', U'', U^j, oraz V^{kl} (przedstawionych w Tabeli III.3) [Bartkowiak2010b].

charakteryzowanych współczynnikiem załamania

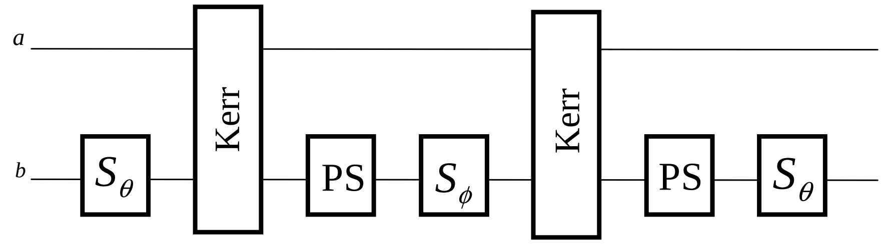
$$n_{\rm Kerr} = n_0 + \chi^{(3)} E^2, \tag{5}$$

gdzie n_0 jest współczynnikiem załamania światła, E^2– natężeniem pola elektrycznego padającego światła, $\chi^{(3)}$ jest stałą Kerra proporcjonalną do podatności magnetycznej ośrodka trzeciego rzędu. Ośrodki takiego typu nazywane są ośrodkami kerrowskimi i wpływają na wiązkę przechodzącą przez kryształ poprzez wprowadzanie dodatkowego przesunięcia fazowego, proporcjonalnego do intensywności wiązki. W przypadku ośrodka umożliwiającego sprzężenie kerrowskie dwu wiązek (tj. ośrodek typu cross-Kerr), przesunięcie fazowe zależy od drugiej wiązki i może być użyte do zaimplementowania bramki CS. Hamiltonian dla ośrodka typu cross-Kerr, dla modów p i s, można zapisać jako

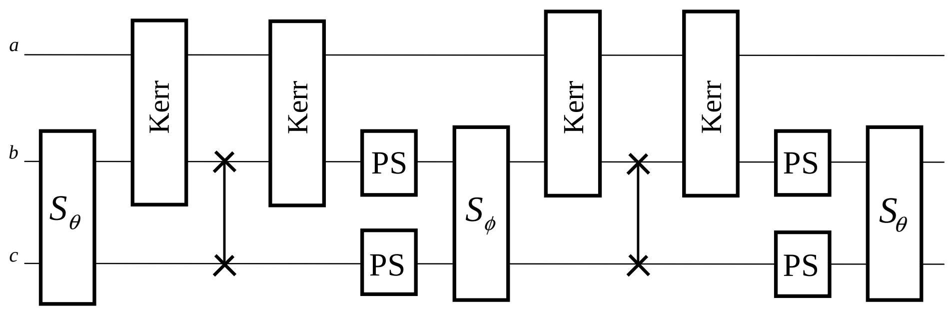
$$H = \kappa {a_p}^\dagger a_p {a_s}^\dagger a_s, \tag{6}$$

gdzie κ oznacza parametr oddziaływania, natomiast $a^\dagger(a)$ operatory kreacji (anihilacji). Jednak mimo iż ta metoda implementacji bramek wydaje się być bardziej naturalna, gdyż nie wymaga wprowadzania dodatkowych pomiarów do układu, okazuje się trudniejsza w realizacji. Eksperymentalnie dostępne są jedynie ośrodki o słabej nieliniowość ($\chi^{(3)} \simeq 10^{-22} m^2 V^{-2}$ [131]), na tyle małej, by inne efekty zdołały ją zdominować i zapobiec skutecznej implementacji bramki CS. Ostatnio udało się jednak pokazać, że można eksperymentalnie zmierzyć wyindukowane w ośrodku kerrowskim przesunięcie fazowe. Matsudo *i in.* [178] przedstawił wyniki eksperymentu, w którym udało się uzyskać małe, warunkowe przesunięcie fazowe (rzędu 10^{-7}rad) indukowane przez pojedyncze fotony w optycznym włóknie.

W Rozdziale III.3 przedstawione zostały ogólne schematy, które przy pomocy wprowadzonej dodatkowo operacji ściskania światła mogą polepszyć uzyskane przesunięcie fazowe: zawierające odpowied-



Rysunek 8: Schemat III– propozycja układu umożliwiającego wzmocnienie przesunięcia fazowego wywołanego ośrodkiem typu cross-Kerr dla dwóch modów. Klucz :S$_i$- jednomodowa operacja ściskania ($i = \theta, \phi$) z parametrami ściskania z Rów. (III.30), CP- bramka kontrolowanej fazy zaimplementowana poprzez ośrodek Kerra, PS- operacja przesuwania fazy [Bartkowiak2012].



Rysunek 9: Schemat IV–propozycja układu wzmacniającego przesunięcie fazowe wywołanego ośrodkiem typu cross-Kerr dla dwóch modów. Notacja podobna do tej przedstawionej na Rys. 8 [Bartkowiak2012].

nio operacje ściskania jednomodowego (Rozdział III.3.1)– Rys. 8, oraz dwumodowego (Rozdział III.3.2)– Rys. 9.

Do uzasadnienia tych schematów zostały użyte własności grupy SU(1,1) oraz teoria stanu koherentnego. Zatem schematy te są ogólne i mogą być zaimplementowane dowolnymi fizycznymi procesami, które można przybliżyć generatorami te grupy. Rozdział zawiera również analizę możliwych implementacji operacji ściskania, której najważniejsze rodzaje zostały przedstawione w Tabeli III.4. Została również oszacowana przybliżona efektywność bramki, w przypadku kiedy poszczególne procesy były brane pod uwagę (wynosi ona ok. 20%). Obniżenie efektywności schematu spowodowane jest bardzo dużym poziomem szumu w ośrodku kerrowskim. Z uwagi na spektralne efekty, które można zaobserwować zarówno w ośrodku Kerra, jak i w ośrodkach ściskających, przeanalizowany został wpływ tych efektów na wierność bramki CS. Na Rysunkach III.7 można zauważyć, że nawet w przypadku wprowadzenie do układu jednego ściśnięcia, wierność bramki zostaje znacznie poprawiona.

7 Podsumowanie

Oto najważniejsze wyniki przedstawione w rozprawie zostały opublikowane [Bartkowiak2010a, Bartkowiak2010b, Bartkowiak2011] lub wysłane do publikacji [Bartkowiak2012]:

- W pracy zostały wyprowadzone klasyczne nierówności dla wielomodowych bozonowych pól, które mogą być naruszone tylko przez pola nieklasyczne. Kryteria nieklasyczności sformułowano w oparciu o macierze momentów kreacji i anihilacji, które w istocie wiążą się z analizą dodatniości funkcji P Glaubera-Sudarshana. Do utworzenia odpowiednich macierzy momentów zostały użyte zarówno jednomianowe jak i wielomianowe funkcje momentów w przeciwieństwie do podejścia Shchukina-Vogla-Richtera [7,8], którzy wykorzystali jedynie jednomiany. Zastosowane przeze mnie podejście umożliwiło mi otrzymanie fizycznie istotnych nierówności w prostszy i bardziej intuicyjny sposób [Bartkowiak2010a].

- Rozprawa zawiera również przepis na to, jak można sprowadzić kryteria nieklasyczności do znanych warunków np. opisujących wielomodowe efekty nieklasyczne, czy powiązać je z nierównością Cauchy'ego-Schwarza. Uzyskana została również ogólna metoda na wyrażenie nierówności uzyskanych z kryterium splątania Shchukina-Vogla [59,60] poprzez sumę warunków na nieklasyczność. Z kryteriów nieklasyczności wyprowadzone zostały również nowe nierówności wykrywające nieklasyczność i splątanie [Bartkowiak2010a].

- Wyprowadzone przeze mnie nierówności wykrywające nieklasyczność i splątanie zostały wykorzystane do skonstruowania świadków splątania i nieklasyczności. Zastosowany został koncept nagłej śmierci miar splątania do analizy zachowania również świadków splątania i nieklasyczności. Zostało zademonstrowane, że nagłe zaniki pojawiają się w przypadku zarówno wielomodowych oddziałujących i nieoddziałujących układów, jak i w przypadkach jednomodowych (jednokubitowych) systemów. Nagłe zaniki można również zaobserwować w przypadku układów niedysypatywnych, które były początkowo także w stanie czystym [Bartkowiak2011].

- Przedstawiono dwie propozycje implementacji liniowo-optycznych bramek kwantowych CS i CNOT wykorzystujących postselekcję, mechanizm sprzężenia zwrotnego (ang. feedforward) oraz konwencjonalne detektory. Uwidocznione zostało, że schemat Pittmana *i in.* [146] zaprojektowany początkowo jako bramka CNOT z selektywnymi detektorami, może być zrealizowana również przy użyciu konwencjonalnych detektorów, zachowując przy tym prawdopodobieństwo sukcesu równe $\eta^4/4$ (zakładając, że stan Gottesmana-Chuanga jest dany). Zaproponowana została przeze mnie bramka CNOT opierająca się na idei bramki Pittmana jednak zawierająca dodatkowo schemat do kreacji stanu Gottesmana-Chuanga. Prawdopodobieństwo całego schematu wynosi $\eta^6/8$ [Bartkowiak2010b].

- Drugi schemat zaproponowany przez mnie w rozprawie, jest propozycją implementacji bramki CS i został zainspirowany układem Zou *i in.*[4]. Układ zakłada stany EPR jako stany pomocnicze. Bramka działa z prawdopodobieństwem $\eta^4/8$. Zweryfikowane zostały również eksperymentalne możliwości implementacji układu związane z obniżeniem wierności bramki poprzez wzięcie pod uwagę realnych źródeł, ciemnych zliczeń czy efektywności detektorów [Bartkowiak2010b].

- Przeanalizowany został układ Wanga *i in.* [169] implementujący bramkę iSWAP z prawdopodobieństwem $\eta^4/32$. Zostało zademonstrowane, że bramkę iSWAP można zrekonstruować przy pomocy dowolnej bramki CNOT lub CS i deterministycznych bramek jednokubitowych. Zgodnie z tym, można ją zaimplementować z prawdopodobieństwem równym temu odpowiadającemu bramce CNOT lub CS. Przy użyciu zaprezentowanych przeze mnie bramek można zatem zaprojektować bramkę iSWAP, która będzie działać z prawdopodobieństwem przynajmniej $\eta^4/8$, tj. czterokrotnie większym niż w przypadku bramki Wanga, przy założeniu tych samych stanów pomocniczych i mniejszej liczbie detektorów [Bartkowiak2010b].

- Zaproponowane zostały układy wzmacniające nieliniowość w ośrodkach kerrowskich, które umożliwiłyby użycie ich jako deterministycznych bramek CS (lub CNOT). Przy użyciu teorii grup zostało pokazane, że nawet w przypadku małego przesunięcia fazowego uzyskanego w ośrodku kerrowskim, dla pojedynczych fotonów, jest możliwe zwiększenie wierności takiej bramki, poprzez wprowadzenie dodatkowego ściskania w układzie. Zostały zaproponowane dwa układy wykorzystujące jednomodowe i dwumodowe ściskanie [Bartkowiak2012].

Wyniki przedstawione w pracy pozwalają zrozumieć relacje między wymienianymi rodzajami kwantowych korelacji oraz umożliwiają zbadanie fizycznych własności układów, w których mogą one występować. Przedstawione badania mają charakter fundamentalny, gdyż dotykają podstaw i odzwierciedlają esencję teorii kwantowej. Z uwagi na już w tej chwili bogaty potencjał wykorzystania zarówno kwantowych korelacji, jak i kwantowych pól optycznych, znalezienie efektywnych metod rozstrzygania i badania własności nieklasyczności i splątania staje się kluczowe.

List of Notation

Γ_i generators of group SU(1,1) 63

CPHASE Controlled-phase quantum gate 63

S_{bc} Two-mode squeezing operator acting on modes b and c 63

S_b One-mode squeezing operator acting on mode b 63

4WM Four-wave mixing mechanism 71

OPA Optical parametric amplifier 71

SHG Second harmonic generation 71

List of own publications

- [Bartkowiak2010a] "Testing nonclassicality in multimode fields: a unified derivation of classical inequalities", A. Miranowicz, M. Bartkowiak, X. Wang, Y.X. Liu, F. Nori, Phys. Rev. A **82**, 013824 (2010).

  Article was chosen to Virtual J. Quantum Information **10** (2010) Issue 8.

- [Bartkowiak2010b] "Linear-optical implementations of the iSWAP and controlled NOT gates based on conventional detectors", M. Bartkowiak, A. Miranowicz, J. Opt. Soc. Am. B **27**, 2369 (2010).

  Article was chosen to Virtual J. Nanoscale Sci. & Tech. 22 (2010) Issue 22.

- [Bartkowiak2011] "Sudden vanishing and reappearance of nonclassical effects: General occurrence of finite-time decays and periodic vanishings of nonclassicality and entanglement witnesses" , M. Bartkowiak, A. Miranowicz, X. Wang, Y.X. Liu, W. Leoński, F. Nori, Phys. Rev. A **83**, 053814, (2011).

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- A. Miranowicz, M. Bartkowiak, X. Wang, Y.X. Liu, and F. Nori: Experimentally-friendly tests of quantumness and entanglement in multiparty bosonic systems, a poster at the Asia-Europe Physics Summit (ASEPS 2011), Wroclaw, 26-29 Oct 2011.
- A. Miranowicz, M. Bartkowiak, X. Wang, Y.X. Liu, and F. Nori: A unified derivation of witnesses of quantumness and entanglement in multimode fields, a talk at the 18th Central European Workshop on Quantum Optics (CEWQO 2011) in Madrid (Spain), May 30-June 3, 2011.
- M. Bartkowiak, and A. Miranowicz: Effective linear-optical implementations of quantum controlled-sign gate, a poster at the 18th Central European Workshop on Quantum Optics (CEWQO 2011) in Madrid (Spain), May 30-June 3, 2011.
- M. Bartkowiak, and A. Miranowicz: Finite-time decays and periodic vanishings of nonclassicality and entanglement witnesses, a poster at the 18th Central European Workshop on Quantum Optics (CEWQO 2011) in Madrid (Spain), May 30-June 3, 2011.
- M. Bartkowiak and A. Miranowicz: Efficient universal optical gates based on conventional detectors, a poster at the 4th International Workshop on Solid State Quantum Computing (IWSSQC-4) at Fudan University, Shanghai, China, December 13-16, 2010.
- A. Miranowicz, M. Bartkowiak, Xiaoguang Wang, Yu-xi Liu, and Franco Nori: Experimentally friendly criteria of quantumness in multiparty bosonic systems, a poster at the 4th International Workshop on Solid State Quantum Computing (IWSSQC-4) at Fudan University, Shanghai, China, December 13-16, 2010.
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