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SOLUTION OF THE INVARIANT SUBSPACE
PROBLEM FOR NON-ARCHIMEDEAN
KÖTHE SPACES



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ROZWIĄZANIE PROBLEMU ISTNIENIA
PODPRZESTRZENI NIEZMIENNICZYCH DLA
OPERATORÓW LINIOWYCH CIĄGLYCH NA
NIEARCHIMEDESOWYCH
PRZESTRZENIACH KÖTHEGO



ROZPRAWA DOKTORSKA Z NAUK MATEMATYCZNYCH
W ZAKRESIE MATEMATYKI POD KIERUNKIEM
PROF. DRA HAB. WIESŁAWA ŚLIWY

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Abstract

This thesis generalizes theorems from [8] about the existence of linear and continuous operators without nontrivial closed invariant subspaces on some non-Archimedean Köthe spaces. The methods from the paper are extended to essentially more non-Archimedean Köthe spaces that have matrices with coefficients not necessarily growing.

In particular the following theorems are true:

Theorem 3.11. *Let $\Lambda(A)$, where $A = (a_n^k)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space.*

Assume that:

(1) *For every $m \in \mathbb{N}$ there exists a sequence $(b_n^m)_{n \in \mathbb{N}} \subset \mathbb{R}$ of positive numbers such that:*

(a) *For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that*

$$a_n^{k+1} b_n^m \geq a_{n+i}^k b_{n+i}^m$$

for $n \geq j$.

(b) *For every $l \in \mathbb{N}$ there exists $n \in \mathbb{N}$, $n \geq l$ such that*

$$a_i^m b_i^m \leq a_n^m b_n^m \leq a_j^m b_j^m$$

for $1 \leq i \leq n \leq j$.

(2) *There exists a sequence $(c_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ of positive numbers such that:*

(a) *For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that*

$$a_n^{k+1} c_n \geq a_{n+i}^k c_{n+i}$$

for $n \geq j$.

(b) For every $l, k, i \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq l$, $m > i$ such that

$$a_n^j c_n \leq a_m^{j+1} c_m$$

for $1 \leq j \leq k$, $m - i \leq n < m$.

(c)

$$\inf_{n \in \mathbb{N}} a_n^1 c_n > 0.$$

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Theorem 3.15. Let $\Lambda(A)$, where $A = (a_n^k)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space.

Assume that:

(1) For every $k, m, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\frac{a_n^{k+1}}{a_{n+i}^k} \geq \frac{a_n^m}{a_{n+i}^m}$$

for $n \geq j$.

(2) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$, $j > i$ such that

$$\frac{a_n^{k+1}}{a_{n-i}^k} \geq \frac{a_n^1}{a_{n-i}^1}$$

for $n \geq j$.

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

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Preface

This dissertation is written to further investigate the existence of linear and continuous operators on non-Archimedean Köthe spaces that have no nontrivial closed invariant subspaces. In the first chapter, we formulate and prove the General Theorem. The General Theorem is not one theorem, but a scheme of many theorems. In the second chapter, we define and prove two specific schemes, which are consequences of the General Theorem. These schemes are easily proved for the function $f \equiv 0$, and then are easily generalized for any function f . And in the third chapter, we show that non-Archimedean Köthe spaces fulfilling certain properties are isomorphic with the schemes from the second part.

In comparison with [8], we use the following facilities: using the function f with real values, duality and algebraization. The function $f: \mathbb{N} \rightarrow \mathbb{R}$ plays a key role. By means of this function, we define a linear operator $T_0: \Lambda_0(A) \rightarrow \Lambda_0(A)$ on a dense subspace $\Lambda_0(A) = \text{Lin}\{e_1, e_2, \dots\}$ of a non-Archimedean Köthe space $\Lambda(A)$. In [8], the function f is defined with integer values. Now the function f is used with real values. To directly define the operator T_0 we use the function $\hat{f}: \mathbb{N} \rightarrow \mathbb{Z}$, $n \mapsto [f(n)]$. This gives large advantages. Among other things, it allows to create a duality. The duality means that the function f can be replaced with any other function, but the matrix coefficients of a given non-Archimedean Köthe space must change accordingly. Using the function $f \equiv 0$ we get a simple form of the operator T_0 , and using the function f other than zero it can be easier to use geometric properties of a considered non-Archimedean Köthe space. Algebraization is based on the fact that in one aspect the operator T_0 can be defined completely arbitrarily and the correctness of Lemma 1.6 is obtained automatically.

In the first paragraph of the first chapter, we formulate the assumptions of the General Theorem. This theorem says that if for a given function $f: \mathbb{N} \rightarrow \mathbb{R}$ a linear operator $T_0: \Lambda_0(A) \rightarrow \Lambda_0(A)$, $a^{\hat{f}(n)}e_n \mapsto \sum_{i=1}^n \varepsilon_{n,i} a^{\hat{f}(i)}e_i +$

$a^{f(n+1)}e_{n+1}$ is defined that fulfills these assumptions, then T_0 is continuous and extends to a linear and continuous operator $T: \Lambda(A) \rightarrow \Lambda(A)$ that has no nontrivial closed invariant subspaces. These assumptions are defined so that the duality can be used.

In the second paragraph we assume that $f \equiv 0$ and prove a few lemmas needed to prove Theorem 1.9 in the third paragraph, which is a special case of the General Theorem for $f \equiv 0$.

By using Theorem 1.11, we create the duality. This theorem says how an operator $T_0: \Lambda_0(A) \rightarrow \Lambda_0(A)$ defined by a function f can be replaced with an operator $R_0: \Lambda_0(B) \rightarrow \Lambda_0(B)$ defined by any other function g , with the spaces $\Lambda(A)$ and $\Lambda(B)$ being isomorphic. From Theorem 1.9 and Theorem 1.11 we derive the General Theorem, i.e. Theorem 1.12. With the assertion proved for $f \equiv 0$ and the fact that f can be replaced with any other function, we get the General Theorem.

In the first paragraph of the second chapter, we create and prove the first specific scheme for $f \equiv 0$, which then generalize for any f in the second paragraph. In the third paragraph, we introduce the second specific scheme and its generalization.

In the first paragraph of the third chapter, we prove the basic theorem proved in [8], using the first specific scheme for $f \equiv 0$. In the second paragraph, we use the generalized first specific scheme and we do not assume that the matrixes coefficients of non-Archimedean Köthe spaces are bounded from below.

In the further part of the chapter, we assume that the coefficients of matrixes of non-Archimedean Köthe spaces are limited from below, so we can use the second specific scheme. The division of non-Archimedean Köthe spaces due to the type of the coefficients is not strict and depends to a large extent on the function f .

From Theorem 3.6, we get successively theorems 3.9, 3.10 and 3.11. Theorem 3.11 generalizes Theorem 3.8 from [8]. And from Theorem 3.11 we obtain successively theorems 3.14, 3.15, 3.16 and 3.17. Theorem 3.15 generalizes Theorem 4.1 from [8] for nuclear non-Archimedean Köthe spaces [7]. And Theorem 3.17 generalizes the results for non-Archimedean analytic functions proved in [8].

When Diara [3] in 1979 studied p-adic representations, the problem arose of existence continuous operators on infinitely dimensional non-Archimedean Banach spaces without nontrivial closed invariant subspaces. If there were such operators on infinitely dimensional non-Archimedean K -Banach spaces, where $\mathbb{Q}_p \subset K$, that would mean that there exist p-adic irreducible repre-

sentations in an infinitely dimensional non-Archimedean Banach spaces.

Let K be a nontrivial complete non-Archimedean field, C an algebraic closure of K and Γ a spherical completion of C .

Let $a \in \Gamma$, $a \neq 0$ and let $b \in C$ be such that

$$|b - a| < |a|.$$

Let $b = b_1, b_2, \dots, b_n$ be all conjugates of b over K and let

$$W(x) = \left(1 - \frac{x}{b_1}\right) \left(1 - \frac{x}{b_2}\right) \dots \left(1 - \frac{x}{b_n}\right).$$

Then $W(x) \in K[x]$, $|W(a)| < 1$ and

$$1 = aQ(a) \frac{1}{1 - W(a)} = aQ(a) (1 + W(a) + W^2(a) + \dots),$$

where $Q(a) = \frac{1 - W(a)}{a}$.

Thus a is invertible in the K -Banach algebra generated by a .

Hence the K -Banach algebra $K(a)$ is a field.

If $a \in \Gamma \setminus C$, then a is transcendental over K , and the operator

$$\tau: K(a) \rightarrow K(a), \tau: x \mapsto ax$$

has no nontrivial closed invariant subspaces.

Therefore p -adic irreducible representations in an infinitely dimensional non-Archimedean Banach spaces exist since \mathbb{C}_p is not spherically complete.

In classical analysis the problem of invariant subspaces was first solved by Enflo [4] and Read [12, 13, 14] for some Banach spaces.

In 1992 van Rooij and Schikhof [16] stated the problem for infinitely dimensional algebraically closed and spherically complete non-Archimedean Banach spaces. The problem was solved by Śliwa [18] in 2008 for all infinitely dimensional non-Archimedean Banach spaces. Śliwa [19] also gave a simple explicit example of an operator without invariant subspaces in Banach space ℓ_1 .

Atzmon [1] was the first who proved the existence of operators without invariant subspaces on Fréchet space that is not a Banach space.

Goliński [5, 6] constructed operators without invariant subspace on some Köthe spaces whose norms are the ℓ_1 norms.

Introduction

By K , we denote a nontrivial complete non-Archimedean field, i.e. a field K with a function

$$|\cdot| : K \rightarrow [0, \infty)$$

called a valuation that satisfies the conditions:

- (1) $|\lambda| = 0$ if and only if $\lambda = 0$,
- (2) $|\lambda\mu| = |\lambda||\mu|$,
- (3) $|\lambda + \mu| \leq \max(|\lambda|, |\mu|)$

for $\lambda, \mu \in K$. And we assume that the topology generated by the valuation is complete and nontrivial, i.e. there exists $\lambda \in K$, $\lambda \neq 0$ and $|\lambda| \neq 1$.

A norm on a K -vector space E is a map

$$\|\cdot\| : E \rightarrow [0, \infty)$$

such that:

- (1) $\|x\| = 0$ if and only if $x = 0$,
- (2) $\|\lambda x\| = |\lambda|\|x\|$,
- (3) $\|x + y\| \leq \max(\|x\|, \|y\|)$

for $\lambda \in K$, $x, y \in E$.

A K -vector space E with a norm is called a K -normed space. If the topology generated by the norm is complete, it is called a K -Banach space. If a K -normed space E contains a countable set that forms a linear hull dense in E , we say that it is of countable type. In some cases, we drop the prefix " K ".

If $\|\cdot\|$ is a norm on a K -vector space E , $x, y \in E$ and $\|x\| > \|y\|$, then

$$\|x + y\| = \|x\|.$$

The space

$$\Lambda(b) = \left\{ (\lambda_1, \lambda_2, \dots) \in K^{\mathbb{N}} : \lim_{n \rightarrow \infty} |\lambda_n| b_n = 0 \right\},$$

where $b = (b_n)_{n \in \mathbb{N}}$ is a sequence of positive numbers, with the norm

$$\|\lambda\| = \max_{n \in \mathbb{N}} |\lambda_n| b_n$$

for $\lambda \in \Lambda(b)$, $\lambda = (\lambda_1, \lambda_2, \dots)$ is a Banach space. In particular,

$$c_0 = \{(\lambda_1, \lambda_2, \dots) \in K^{\mathbb{N}} : \lim_{n \rightarrow \infty} |\lambda_n| = 0\},$$

where $\|\lambda\| = \max_{n \in \mathbb{N}} |\lambda_n|$, $\lambda = (\lambda_1, \lambda_2, \dots) \in c_0$, is a Banach space.

Each infinitely dimensional Banach space of countable type is isomorphic to c_0 .

A non-Archimedean Köthe space [11] is a K -vector locally convex space $\Lambda(B)$, where $B = (b_n^k)_{k, n \in \mathbb{N}}$, i.e.

$$\Lambda(B) = \left\{ (\lambda_1, \lambda_2, \dots) \in K^{\mathbb{N}} : \lim_{n \rightarrow \infty} |\lambda_n| b_n^k = 0 \text{ for } k \in \mathbb{N} \right\},$$

where $b_n^k \in \mathbb{R}$, $b_n^k > 0$ for $k, n \in \mathbb{N}$, and the topology is generated by the collection of norms

$$\mathcal{P} = \{p_k : k \in \mathbb{N}\},$$

where

$$p_k(\lambda) = \max_{n \in \mathbb{N}} |\lambda_n| b_n^k$$

for $k \in \mathbb{N}$, $\lambda \in \Lambda(A)$, $\lambda = (\lambda_1, \lambda_2, \dots)$.

We assume for non-Archimedean Köthe spaces $\Lambda(B)$, where $B = (b_n^k)_{k, n \in \mathbb{N}}$, that $b_n^k \leq b_n^{k+1}$ for $k, n \in \mathbb{N}$, unless otherwise stated.

Non-Archimedean Köthe spaces are complete.

We can treat the Banach spaces $\Lambda((b_n)_{n \in \mathbb{N}})$ as non-Archimedean Köthe spaces $\Lambda((b_n^k)_{k, n \in \mathbb{N}})$, where $b_n^k = b_n$ for $k, n \in \mathbb{N}$.

Each non-Archimedean Köthe space $\Lambda(B)$ is metrizable by an invariant ultrametric, such as by

$$d(\lambda, \mu) = \max_{k \in \mathbb{N}} \min(p_k(\lambda - \mu), 2^{-k})$$

and

$$\lim_{n \rightarrow \infty} x_n = x$$

for $x_n, x \in \Lambda(B)$, $n \in \mathbb{N}$ if and only if

$$\lim_{n \rightarrow \infty} p_k(x_n - x) = 0$$

for $k \in \mathbb{N}$.

Every linear and continuous operator $T_0 : \Lambda_0 \rightarrow \Lambda(B)$, where Λ_0 is a dense linear subspace of the non-Archimedean Köthe space $\Lambda(B)$, extends uniquely to a linear and continuous operator $T : \Lambda(B) \rightarrow \Lambda(B)$.

If $f : \mathbb{N} \rightarrow \mathbb{R}$, then by \hat{f} we denote the function such that $\hat{f} : \mathbb{N} \rightarrow \mathbb{Z}$ and $\hat{f}(n) = [f(n)]$ for $n \in \mathbb{N}$.

The symbols e_n always mean $(\delta_{n,j})_{j \in \mathbb{N}}$ and if $\lambda = (\lambda_n)_{n \in \mathbb{N}}$, then we also write $\lambda(n)$ instead of λ_n , $n \in \mathbb{N}$.

By $[n, m], [n, m), (n, m)$ for $n, m \in \mathbb{N}$, $n \leq m$, we denote the sets $\{i \in \mathbb{N} : n \leq i \leq m\}$, $\{i \in \mathbb{N} : n \leq i < m\}$, $\{i \in \mathbb{N} : n < i < m\}$ respectively.

We say that an operator $\varphi : \Phi \rightarrow \Phi$ of a linear topological space Φ has an invariant subspace $\Psi \subset \Phi$ if $\varphi(\Psi) \subset \Psi$ and Ψ is closed, and it is nontrivial if $\Psi \neq \{0\}$ and $\Psi \neq \Phi$.

For more details from non-Archimedean functional analysis, the reader is referred to [2], [10], [11], [15], and [17].

Chapter 1

The General Scheme

Let K be a non-Archimedean field and let $a \in K$, $|a| > 1$ be a fixed element. By \hat{K} we denote the smallest ring contained in K to which 1 belongs. All non-Archimedean Köthe spaces $\Lambda(A)$ are over K . By $\Lambda_0(A)$ we denote $\text{Lin}\{e_1, e_2, \dots\} \subset \Lambda(A)$.

1.1 The Assumptions

From this point until Theorem 1.9, the following assumptions are applied.

Assumption 1.1.

(1) Let $\Lambda(A)$, where $A = \left(|a|^{\alpha_n^k}\right)_{k,n \in \mathbb{N}}$, $\alpha_n^k \in \mathbb{R}$ for $k, n \in \mathbb{N}$, be a non-Archimedean Köthe space.

(2) Assume that

$$\alpha_n^k + 1 \leq \alpha_n^{k+1}$$

for $k, n \in \mathbb{N}$.

(3) Let

$$\mu: \mathbb{N} \rightarrow \mathbb{N}$$

be a function that takes any natural number infinitely many times.

(4) Let the functions

$$\begin{aligned} x &: \mathbb{N} \rightarrow \mathbb{N}, \\ y &: \mathbb{N} \rightarrow \mathbb{N}, \\ z &: \mathbb{N} \rightarrow \mathbb{N}, \\ \chi &: \mathbb{N} \rightarrow \mathbb{N}, \\ \psi &: \mathbb{N} \rightarrow \mathbb{N}, \\ \varphi &: \mathbb{N} \rightarrow \mathbb{N} \end{aligned}$$

be defined such that

$$\begin{aligned} x(k) &\leq \chi(k) < \chi(k) + x(k) \leq y(k) < \psi(k) < z(k) \leq x(k+1), \\ \varphi(k) &= x(k) + y(k) - 1 \end{aligned}$$

for $k \in \mathbb{N}$.

(5) Let

$$f: \mathbb{N} \rightarrow \mathbb{R}$$

and let

$$T_0: \Lambda_0(A) \rightarrow \Lambda_0(A)$$

be a linear operator such that

$$T_0 a^{\hat{f}(n)} e_n = \sum_{i=1}^n \varepsilon_{n,i} a^{\hat{f}(i)} e_i + a^{\hat{f}(n+1)} e_{n+1}.$$

for $n \in \mathbb{N}$, where $\varepsilon_{n,i} \in \hat{K}$ for $n \in \mathbb{N}$, $1 \leq i \leq n$.

Lemma 1.2. Let $\Lambda(C)$ be a non-Archimedean Köthe space and let

$$\bar{T}: \Lambda_0(C) \rightarrow \Lambda_0(C)$$

be a linear operator such that

$$\bar{T} a^{\bar{f}(n)} e_n = \sum_{l=1}^n \bar{\varepsilon}_{n,l} a^{\bar{f}(l)} e_l + a^{\bar{f}(n+1)} e_{n+1} \quad (1.1)$$

for $n \in \mathbb{N}$, where $\bar{f}: \mathbb{N} \rightarrow \mathbb{Z}$ and $\bar{\varepsilon}_{n,l} \in \hat{K}$ for $n \in \mathbb{N}$, $1 \leq l \leq n$.

Then there exist $\bar{\varepsilon}_{i,j,l} \in \hat{K}$ for $i, j \in \mathbb{N}$, $1 \leq l \leq i+j$ such that

$$\bar{T}^i a^{\bar{f}(j)} e_j = \sum_{l=1}^{i+j} \bar{\varepsilon}_{i,j,l} a^{\bar{f}(l)} e_l, \quad (1.2)$$

where $\bar{\varepsilon}_{i,j,i+j} = 1$ for $i, j \in \mathbb{N}$, and the $\bar{\varepsilon}_{i,j,l}$ are independent of \bar{f} .

Proof. By (1.1), equality (1.2) is true when $i = 1$ and $j \in \mathbb{N}$.

Suppose that (1.2) is true for some $i \in \mathbb{N}$ and every $j \in \mathbb{N}$. Let $k \in \mathbb{N}$, then

$$\begin{aligned}
\bar{T}^{i+1} a^{\bar{f}(k)} e_k &= \bar{T} \bar{T}^i a^{\bar{f}(k)} e_k = \bar{T} \sum_{n=1}^{i+k} \bar{\varepsilon}_{i,k,n} a^{\bar{f}(n)} e_n \\
&= \sum_{n=1}^{i+k} \left(\bar{\varepsilon}_{i,k,n} \sum_{l=1}^n \bar{\varepsilon}_{n,l} a^{\bar{f}(l)} e_l + \bar{\varepsilon}_{i,k,n} a^{\bar{f}(n+1)} e_{n+1} \right) \\
&= \sum_{l=1}^{i+k} \sum_{n=l}^{i+k} \bar{\varepsilon}_{i,k,n} \bar{\varepsilon}_{n,l} a^{\bar{f}(l)} e_l + \sum_{l=2}^{i+k} \bar{\varepsilon}_{i,k,l-1} a^{\bar{f}(l)} e_l \\
&\quad + \bar{\varepsilon}_{i,k,i+k} a^{\bar{f}(i+k+1)} e_{i+k+1} \\
&= \left(\sum_{n=1}^{i+k} \bar{\varepsilon}_{i,k,n} \bar{\varepsilon}_{n,1} \right) a^{\bar{f}(1)} e_1 + \sum_{l=2}^{i+k} \left(\sum_{n=l}^{i+k} \bar{\varepsilon}_{i,k,n} \bar{\varepsilon}_{n,l} + \bar{\varepsilon}_{i,k,l-1} \right) a^{\bar{f}(l)} e_l \\
&\quad + a^{\bar{f}(i+k+1)} e_{i+k+1}.
\end{aligned}$$

Therefore, by induction, (1.2) is true for $i, j \in \mathbb{N}$ and the $\bar{\varepsilon}_{i,j,l}$ are independent of \bar{f} . \square

Lemma 1.3. *Let $\Lambda(C)$ be a non-Archimedean Köthe space and let*

$$\bar{T}: \Lambda_0(C) \rightarrow \Lambda_0(C)$$

be a linear operator such that

$$\bar{T} a^{\bar{f}(n)} e_n = \sum_{l=1}^n \bar{\varepsilon}_{n,l} a^{\bar{f}(l)} e_l + a^{\bar{f}(n+1)} e_{n+1}$$

for $n \in \mathbb{N}$, where $\bar{f}: \mathbb{N} \rightarrow \mathbb{Z}$ and $\bar{\varepsilon}_{n,l} \in \hat{K}$ for $n \in \mathbb{N}$, $1 \leq l \leq n$.

Then for every $m \in \mathbb{N}$

$$\bar{T} a^{\bar{f}(m)} e_m = -\bar{T} \left(\bar{T}^{m-1} a^{\bar{f}(1)} e_1 - a^{\bar{f}(m)} e_m \right) + a^{\bar{f}(m+1)} e_{m+1} \quad (1.3)$$

if and only if

$$\bar{T}^m a^{\bar{f}(1)} e_1 = a^{\bar{f}(m+1)} e_{m+1}. \quad (1.4)$$

Proof. The equalities (1.3) and (1.4) are equivalent since

$$-\bar{T} \left(\bar{T}^{m-1} a^{\bar{f}(1)} e_1 - a^{\bar{f}(m)} e_m \right) = -\bar{T}^m a^{\bar{f}(1)} e_1 + \bar{T} a^{\bar{f}(m)} e_m$$

for $m \in \mathbb{N}$. \square

Assumption 1.4.

(1) Now, by induction, we redefine the operator T_0 , i.e. we define some of the $\varepsilon_{n,i}$.

Let

$$T_0 a^{\hat{f}(n)} e_n = a^{\hat{f}(n+1)} e_{n+1}$$

for $1 \leq n < \psi(1)$, $n \neq y(1)$;

$$T_0 a^{\hat{f}(y(1))} e_{y(1)} = a^{\hat{f}(y(1)+1)} e_{y(1)+1} + a^{\hat{f}(1)} e_1;$$

$$T_0 a^{\hat{f}(n)} e_n = \sum_{i=1}^n \varepsilon_{n,i} a^{\hat{f}(i)} e_i + a^{\hat{f}(n+1)} e_{n+1}$$

for $\psi(1) \leq n < z(1)$. If we have defined $T_0 e_n$ for $1 \leq n < z(1)$, then we define $T_0 e_{z(1)}$ as follows

$$\begin{aligned} T_0 a^{\hat{f}(z(1))} e_{z(1)} &= -T_0 \left(T_0^{z(1)-1} a^{\hat{f}(1)} e_1 - a^{\hat{f}(z(1))} e_{z(1)} \right) \\ &\quad + a^{\hat{f}(z(1)+1)} e_{z(1)+1}. \end{aligned}$$

Let $k \in \mathbb{N}$, $k \geq 2$. Suppose that we have defined $T_0 e_n$ for $1 \leq n \leq z(k-1)$, then we define $T_0 e_n$ for $z(k-1) < n \leq z(k)$.

Let

$$T_0 a^{\hat{f}(n)} e_n = a^{\hat{f}(n+1)} e_{n+1}$$

for $z(k-1) < n < \psi(k)$, $n \neq y(k)$;

$$T_0 a^{\hat{f}(y(k))} e_{y(k)} = a^{\hat{f}(y(k)+1)} e_{y(k)+1} + a^{\hat{f}(1)} e_1;$$

$$T_0 a^{\hat{f}(n)} e_n = \sum_{i=1}^n \varepsilon_{n,i} a^{\hat{f}(i)} e_i + a^{\hat{f}(n+1)} e_{n+1}$$

for $\psi(k) \leq n < z(k)$. Assume that we have defined $T_0 e_n$ for $z(k-1) < n < z(k)$, then

$$\begin{aligned} T_0 a^{\hat{f}(z(k))} e_{z(k)} &= -T_0 \left(T_0^{z(k)-1} a^{\hat{f}(1)} e_1 - a^{\hat{f}(z(k))} e_{z(k)} \right) \\ &\quad + a^{\hat{f}(z(k)+1)} e_{z(k)+1}. \end{aligned}$$

(2) Let

(a) for every $k \in \mathbb{N}$ and $x(k) < n \leq \chi(k) + x(k)$

$$\alpha_n^{\mu(k)} + f(n) \leq 0;$$

(b) for every $k \in \mathbb{N}$ and $\chi(k) < n \leq y(k)$

$$\alpha_n^{\mu(k)+1} + f(n) \geq 0;$$

(c) for every $k \in \mathbb{N}$

$$\alpha_{y(k)+1}^{\mu(k)+2} + f(y(k) + 1) \leq 0$$

and

$$\alpha_n^{\mu(k)+2} + f(n) \geq \alpha_{n+1}^{\mu(k)+2} + f(n+1)$$

if $y(k) < n < \psi(k)$.

(3) Let $\varepsilon_{i,j,l} \in \hat{K}$ for $i, j \in \mathbb{N}$, $1 \leq l \leq i+j$ be such that

$$T_0^i a^{\hat{f}(j)} e_j = \sum_{l=1}^{i+j} \varepsilon_{i,j,l} a^{\hat{f}(l)} e_l$$

for $i, j \in \mathbb{N}$. Assume that

$$\max_{\substack{1 \leq l \leq i+j \\ \varepsilon_{i,j,l} \neq 0}} \left(-\alpha_j^{\mu(k)+2} - f(j) + \alpha_l^{\mu(k)} + f(l) \right) \leq 0$$

for $k \in \mathbb{N}$, $j \geq \psi(k)$ and $1 \leq i \leq \varphi(k)$.

1.2 Some more Lemmas

From now on to the Theorem 1.9, we assume that $f \equiv 0$.

Lemma 1.5. For every $n \in \mathbb{N}$ there exist $\hat{\varepsilon}_{n,i} \in \hat{K}$ for $0 \leq i < n$ such that

$$e_n = \sum_{i=0}^{n-1} \hat{\varepsilon}_{n,i} T_0^i e_1. \quad (1.5)$$

Proof. We have $e_1 = T_0^0 e_1$, $e_2 = -\varepsilon_{1,1} T_0^0 e_1 + T_0 e_1$. Let $m \in \mathbb{N}$, $m \geq 2$. Suppose that (1.5) is true for $1 \leq n \leq m$. Then

$$e_{m+1} = T_0 e_m - \sum_{l=1}^m \varepsilon_{m,l} e_l = T_0 \sum_{i=0}^{m-1} \hat{\varepsilon}_{m,i} T_0^i e_1 - \sum_{l=1}^m \varepsilon_{m,l} \sum_{i=0}^{l-1} \hat{\varepsilon}_{l,i} T_0^i e_1$$

$$\begin{aligned}
&= \sum_{i=1}^m \hat{\varepsilon}_{m,i-1} T_0^i e_1 - \sum_{i=0}^{m-1} \sum_{l=i+1}^m \varepsilon_{m,l} \hat{\varepsilon}_{l,i} T_0^i e_1 \\
&= \left(- \sum_{l=1}^m \varepsilon_{m,l} \hat{\varepsilon}_{l,0} \right) T_0^0 e_1 + \sum_{i=1}^{m-1} \left(\hat{\varepsilon}_{m,i-1} - \sum_{l=i+1}^m \varepsilon_{m,l} \hat{\varepsilon}_{l,i} \right) T_0^i e_1 + \hat{\varepsilon}_{m,m-1} T_0^m e_1
\end{aligned}$$

Therefore, by induction, (1.5) is true for $n \in \mathbb{N}$. \square

Lemma 1.6. *The equalities*

$$T_0^{x(k)} e_n = \sum_{l=x(k)+1}^{x(k)+n} \varepsilon_{x(k),n,l} e_l \quad (1.6)$$

are true for $k \in \mathbb{N}$, $1 \leq n \leq x(k)$.

Proof. If $k = 1$, $n = 1$, then

$$T_0^{x(1)} e_1 = e_{x(1)+1}.$$

Let $k \in \mathbb{N}$, $k \geq 2$, $n = 1$. Then, by Assumption 1.4 (1) and Lemma 1.3,

$$T_0^{x(k)} e_1 = T_0^{x(k)-z(k-1)} T_0^{z(k-1)} e_1 = T_0^{x(k)-z(k-1)} e_{z(k-1)+1} = e_{x(k)+1}.$$

Assume that $k \in \mathbb{N}$, $2 \leq n \leq x(k)$. Then, by Lemma 1.5, there exist $\hat{\varepsilon}_{n,i} \in \hat{K}$ for $0 \leq i \leq n-1$ such that

$$e_n = \sum_{i=0}^{n-1} \hat{\varepsilon}_{n,i} T_0^i e_1.$$

Hence

$$\begin{aligned}
T_0^{x(k)} e_n &= T_0^{x(k)} \sum_{i=0}^{n-1} \hat{\varepsilon}_{n,i} T_0^i e_1 = \sum_{i=0}^{n-1} \hat{\varepsilon}_{n,i} T_0^i T_0^{x(k)} e_1 \\
&= \sum_{i=0}^{n-1} \hat{\varepsilon}_{n,i} T_0^i e_{x(k)+1} = \sum_{i=1}^n \hat{\varepsilon}_{n,i-1} e_{x(k)+i}.
\end{aligned}$$

Therefore (1.6) is true. \square

Lemma 1.7. *The inequalities*

$$|a|^{-\alpha_j^{\mu(k)+2}} p_{\mu(k)}(T_0^i e_j) \leq 1$$

are true for $k \in \mathbb{N}$, $j > y(k)$ and $0 \leq i \leq \varphi(k)$.

Proof. Let $k \in \mathbb{N}$, $j > y(k)$ and $0 \leq i \leq \varphi(k)$.

If $i = 0$, then

$$|a|^{-\alpha_j^{\mu(k)+2}} p_{\mu(k)}(T_0^0 e_j) = |a|^{-\alpha_j^{\mu(k)+2}} |a|^{\alpha_j^{\mu(k)}} < 1.$$

Assume that $i \geq 1$ and $j < \psi(k)$. Then, by Assumption 1.4 (2)(c),

$$\begin{aligned} |a|^{-\alpha_j^{\mu(k)+2}} p_{\mu(k)}(T_0^i e_j) &= |a|^{-\alpha_j^{\mu(k)+2}} p_{\mu(k)}(T_0^{i-1} e_{j+1}) \\ &\leq |a|^{-\alpha_{j+1}^{\mu(k)+2}} p_{\mu(k)}(T_0^{i-1} e_{j+1}). \end{aligned}$$

Hence, by induction, we may assume that $j \geq \psi(k)$ and $1 \leq i \leq \varphi(k)$.

By Assumption 1.4 (3), we have

$$T_0^i e_j = \sum_{l=1}^{i+j} \varepsilon_{i,j,l} e_l$$

and

$$\max_{\substack{1 \leq l \leq i+j \\ \varepsilon_{i,j,l} \neq 0}} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) \leq 0.$$

Therefore

$$\begin{aligned} |a|^{-\alpha_j^{\mu(k)+2}} p_{\mu(k)}(T_0^i e_j) &= |a|^{-\alpha_j^{\mu(k)+2}} p_{\mu(k)} \left(\sum_{l=1}^{i+j} \varepsilon_{i,j,l} e_l \right) \\ &\leq \max_{\substack{1 \leq l \leq i+j \\ \varepsilon_{i,j,l} \neq 0}} |a|^{-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)}} \leq 1. \end{aligned}$$

□

Lemma 1.8. *The operator T_0 is continuous and extends uniquely to a continuous operator*

$$T: \Lambda(A) \rightarrow \Lambda(A).$$

Proof. Let $n \in \mathbb{N}$. By Lemma 1.7, there exists $m \in \mathbb{N}$ such that

$$p_n(T_0 e_i) \leq |a|^{\alpha_i^{n+2}} = p_{n+2}(e_i)$$

for $i \geq m$. Hence there exists $M_n > 0$ such that

$$p_n(T_0 e_i) \leq M_n p_{n+2}(e_i)$$

for $i \in \mathbb{N}$.

Let $\lambda = \sum_{i=1}^j \lambda_i e_i$, $\lambda_i \in K$, $i = 1, 2, \dots, j$, $j \in \mathbb{N}$. We have

$$\begin{aligned} p_n(T_0\lambda) &= p_n\left(T_0 \sum_{i=1}^j \lambda_i e_i\right) \leq \max_{1 \leq i \leq j} |\lambda_i| p_n(T_0 e_i) \\ &\leq M_n \max_{1 \leq i \leq j} |\lambda_i| p_{n+2}(e_i) = M_n p_{n+2}(\lambda). \end{aligned}$$

Therefore T_0 is continuous. Since $\Lambda_0(A)$ is dense in $\Lambda(A)$ and $\Lambda(A)$ is a non-Archimedean Köthe space, there exists a unique linear and continuous extension T of T_0 to $\Lambda(A)$. \square

1.3 The Fundamental Theorems

Theorem 1.9. *The operator T_0 extends to a linear and continuous operator*

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. By Lemma 1.8, the operator T_0 extends uniquely to a linear and continuous operator $T: \Lambda(A) \rightarrow \Lambda(A)$. Now we show that the operator T has no nontrivial invariant subspaces.

Let $\xi \in \Lambda(A)$, $\xi \neq 0$ and let X be the smallest invariant subspace of T such that $\xi \in X$.

It is sufficient to show that there exists a sequence $(\bar{\xi}_n)_{n \in \mathbb{N}} \subset X$ such that

$$\lim_{n \rightarrow \infty} p_k(\bar{\xi}_n - e_1) = 0$$

for every $k \in \mathbb{N}$, i.e. $\lim_{n \rightarrow \infty} \bar{\xi}_n = e_1$. Since then $e_1 \in X$, and if $\text{Lin}\{e_1, e_2, \dots, e_n\} \subset X$, then $\text{Lin}\{e_1, e_2, \dots, e_{n+1}\} \subset X$ for $e_{n+1} = T e_n + \rho$, where $\rho \in \text{Lin}\{e_1, e_2, \dots, e_n\}$. Thus it follows, by induction, that $\text{Lin}\{e_1, e_2, \dots\} \subset X$. Hence $\Lambda(A) = \text{cl } \text{Lin}\{e_1, e_2, \dots\} \subset X$, so $X = \Lambda(A)$. And this means that every non-null invariant subspace of T is equal to $\Lambda(A)$. Therefore we obtain that T does not have nontrivial invariant subspaces.

In order to prove this, we show that there exists a sequence $(\bar{\xi}_n)_{n \in \mathbb{N}} \subset X$ such that

$$p_n(\bar{\xi}_n - e_1) \leq 1 \tag{1.7}$$

for $n \in \mathbb{N}$. Then

$$p_k(\bar{\xi}_n - e_1) = \max_{m \in \mathbb{N}} |(\bar{\xi}_n - e_1)(m)| |a|^{\alpha_m^k}$$

$$\begin{aligned}
&= \max_{m \in \mathbb{N}} |(\bar{\xi}_n - e_1)(m)| |a|^{\alpha_m^n} |a|^{-\alpha_m^n} |a|^{\alpha_m^k} \\
&\leq p_n (\bar{\xi}_n - e_1) \sup_{m \in \mathbb{N}} |a|^{-\alpha_m^n} |a|^{\alpha_m^k} \leq \sup_{m \in \mathbb{N}} |a|^{-\alpha_m^n + \alpha_m^k} \leq |a|^{-n+k}
\end{aligned}$$

for $n \geq k$. Hence

$$\lim_{n \rightarrow \infty} p_k (\bar{\xi}_n - e_1) = 0$$

for $k \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and let $k \in \mathbb{N}$ be such that: $\mu(k) = n$; $\xi(l) \neq 0$ for some $l \in \mathbb{N}$, $l \leq \chi(k)$; $|\xi_1(m)| < 1$ for $\chi(k) < m \leq y(k)$, where $\xi_1 = \frac{1}{\xi(l)}\xi$; and

$$|\xi_1(m)| \leq |a|^{-\alpha_m^{\mu(k)+2}}$$

for $m > y(k)$.

Such $k \in \mathbb{N}$ exists by the facts that the set $\{k \in \mathbb{N} : \mu(k) = n\}$ is infinite, $\xi \neq 0$, $\lim_{m \rightarrow \infty} |a|^{\alpha_m^{\mu(k)+1}} \xi_1(m) = \lim_{m \rightarrow \infty} |a|^{\alpha_m^{\mu(k)+2}} \xi_1(m) = 0$ and by Assumption 1.4 (2)(b).

Let $v \in K$ be such that

$$|v| = \max_{1 \leq m \leq \chi(k)} |\xi_1(m)|$$

and let $\xi_2 = v^{-1}\xi_1$. Then

$$\max_{1 \leq m \leq \chi(k)} |\xi_2(m)| = 1.$$

Since $\max_{1 \leq m \leq \chi(k)} |\xi_1(m)| \geq |\xi_1(l)| = 1$, $|v^{-1}| \leq 1$.

Thus

$$|\xi_2(m)| \leq |\xi_1(m)| < 1$$

for $\chi(k) < m \leq y(k)$ and

$$|\xi_2(m)| \leq |a|^{-\alpha_m^{\mu(k)+2}} \tag{1.8}$$

for $m > y(k)$.

Let r be the largest integer such that

$$1 \leq r \leq \chi(k)$$

and

$$|\xi_2(r)| = 1.$$

Hence

$$\begin{cases} |\xi_2(m)| \leq 1 & \text{if } 1 \leq m < r, \\ |\xi_2(r)| = 1, \\ |\xi_2(m)| < 1 & \text{if } r < m \leq y(k). \end{cases} \quad (1.9)$$

Let

$$T_1: \Lambda(A) \rightarrow \Lambda(A)$$

be a linear operator such that

$$T_1 \left(\sum_{m=1}^{\infty} \lambda_m e_m \right) = T \left(\sum_{m=1}^{y(k)-1} \lambda_m e_m \right) + \lambda_{y(k)} e_1$$

for $\lambda \in \Lambda(A)$, $\lambda = \sum_{m=1}^{\infty} \lambda_m e_m$ and let

$$T_2 = T - T_1.$$

Then

$$T_2 \left(\sum_{m=1}^{\infty} \lambda_m e_m \right) = \lambda_{y(k)} e_{y(k)+1} + T \left(\sum_{m=y(k)+1}^{\infty} \lambda_m e_m \right)$$

for $\lambda \in \Lambda(A)$, $\lambda = \sum_{m=1}^{\infty} \lambda_m e_m$.

We show that

$$\begin{cases} |(T_1^i \xi_2)(m)| \leq 1 & \text{if } 1 \leq m \leq y(k), i \in \mathbb{N} \cup \{0\}, \\ |(T_1^i \xi_2)(r+i)| = 1 & \text{if } r \leq r+i \leq y(k), \\ |(T_1^i \xi_2)(m)| < 1 & \text{if } r \leq r+i < m \leq y(k). \end{cases} \quad (1.10)$$

If $i = 0$, then (1.10) is true by (1.9). Let $i \in \mathbb{N} \cup \{0\}$. By the definition of T_1 , we have

$$\begin{aligned} T_1^{i+1} \xi_2 &= T_1 \sum_{m=1}^{y(k)} (T_1^i \xi_2)(m) e_m \\ &= \sum_{m=1}^{y(k)-1} \left(\sum_{j=1}^m \varepsilon_{m,j} (T_1^i \xi_2)(m) e_j + (T_1^i \xi_2)(m) e_{m+1} \right) + (T_1^i \xi_2)(y(k)) e_1. \end{aligned}$$

Hence

$$(T_1^{i+1} \xi_2)(1) = \sum_{m=1}^{y(k)-1} \varepsilon_{m,1} (T_1^i \xi_2)(m) + (T_1^i \xi_2)(y(k)),$$

$$(T_1^{i+1}\xi_2)(j) = (T_1^i\xi_2)(j-1) + \sum_{m=j}^{y(k)-1} \varepsilon_{m,j}(T_1^i\xi_2)(m)$$

if $1 < j < y(k)$ and

$$(T_1^{i+1}\xi_2)(y(k)) = (T_1^i\xi_2)(y(k)-1).$$

Suppose that

$$|(T_1^i\xi_2)(m)| \leq 1$$

for $1 \leq m \leq y(k)$.

Then

$$\begin{aligned} |(T_1^{i+1}\xi_2)(1)| &= \left| \sum_{m=1}^{y(k)-1} \varepsilon_{m,1}(T_1^i\xi_2)(m) + (T_1^i\xi_2)(y(k)) \right| \\ &\leq \max \left(\max_{1 \leq m < y(k)} |\varepsilon_{m,1}| |(T_1^i\xi_2)(m)|, |(T_1^i\xi_2)(y(k))| \right) \leq 1, \\ |(T_1^{i+1}\xi_2)(j)| &= \left| (T_1^i\xi_2)(j-1) + \sum_{m=j}^{y(k)-1} \varepsilon_{m,j}(T_1^i\xi_2)(m) \right| \\ &\leq \max \left(|(T_1^i\xi_2)(j-1)|, \max_{j \leq m < y(k)} |\varepsilon_{m,j}| |(T_1^i\xi_2)(m)| \right) \leq 1 \end{aligned}$$

if $1 < j < y(k)$,

$$|(T_1^{i+1}\xi_2)(y(k))| = |(T_1^i\xi_2)(y(k)-1)| \leq 1.$$

Thus

$$|(T_1^{i+1}\xi_2)(m)| \leq 1$$

for $1 \leq m \leq y(k)$.

Now assume that $r \leq r+i < y(k)$,

$$|(T_1^i\xi_2)(r+i)| = 1$$

and

$$|(T_1^i\xi_2)(m)| < 1$$

if $r+i < m \leq y(k)$.

If $r + i + 1 < y(k)$, then

$$|(T_1^{i+1}\xi_2)(r + i + 1)| = \left| (T_1^i\xi_2)(r + i) + \sum_{m=r+i+1}^{y(k)-1} \varepsilon_{m,r+i+1}(T_1^i\xi_2)(m) \right| = 1$$

since

$$|(T_1^i\xi_2)(r + i)| = 1$$

and

$$\max_{r+i+1 \leq m < y(k)} |\varepsilon_{m,r+i+1}(T_1^i\xi_2)(m)| < 1.$$

If $r + i + 1 = y(k)$, then

$$|(T_1^{i+1}\xi_2)(y(k))| = |(T_1^i\xi_2)(y(k) - 1)| = |(T_1^i\xi_2)(r + i)| = 1.$$

Thus

$$|(T_1^{i+1}\xi_2)(r + i + 1)| = 1.$$

If $r + i + 1 < j < y(k)$, then

$$|(T_1^{i+1}\xi_2)(j)| = \left| (T_1^i\xi_2)(j - 1) + \sum_{m=j}^{y(k)-1} \varepsilon_{m,j}(T_1^i\xi_2)(m) \right| < 1$$

since

$$|(T_1^i\xi_2)(j - 1)| < 1$$

and

$$\max_{j \leq m < y(k)} |\varepsilon_{m,j}(T_1^i\xi_2)(m)| < 1.$$

If $r + i + 1 < j = y(k)$, then

$$|(T_1^{i+1}\xi_2)(y(k))| = |(T_1^i\xi_2)(y(k) - 1)| = |(T_1^i\xi_2)(j - 1)| < 1.$$

Thus

$$|(T_1^{i+1}\xi_2)(j)| < 1$$

if $r + i + 1 < j \leq y(k)$.

Therefore (1.10) is true by induction.

Let $s = \chi(k) - r$, $t = y(k) - \chi(k)$.

Let

$$M = (a_{ij})_{1 \leq i, j \leq t}$$

be the matrix with entries $a_{ij} = (T_1^{s+i}\xi_2)(\chi(k) + j)$, $i, j = 1, 2, \dots, t$,

$$u = y(k) - x(k) + 1,$$

and let

$$\rho = (\rho_1, \rho_2, \dots, \rho_t)$$

be such that $\rho_j = 0$ when $j \neq u - \chi(k)$ and $\rho_{u-\chi(k)} = 1$.

By (1.10) we get

$$\begin{aligned} |a_{ij}| &< 1 && \text{when } i < j, \\ |a_{ij}| &= 1 && \text{when } i = j, \\ |a_{ij}| &\leq 1 && \text{when } i > j. \end{aligned}$$

Hence

$$|\det M| = \left| \sum_{\sigma} \varepsilon(\sigma) \prod_{j=1}^t a_{\sigma(j)j} \right| = \left| \prod_{j=1}^t a_{jj} \right| = 1.$$

Therefore by the Cramer's rule

$$(a_1, a_2, \dots, a_t)M = \rho, \tag{1.11}$$

where $a_m = \frac{\det M_m}{\det M}$ and the matrix $M_m = (a_{ij}^m)_{1 \leq i, j \leq t}$ for $1 \leq m \leq t$ is created from M by replacing the m -th row by instead of entering ρ , while

$$|a_m| \leq 1 \tag{1.12}$$

for $1 \leq m \leq t$ since

$$|\det M_m| = \left| \sum_{\sigma} \varepsilon(\sigma) \prod_{j=1}^t a_{\sigma(j)j}^m \right| \leq \prod_{j=1}^t 1 = 1$$

for $1 \leq m \leq t$.

Let

$$\xi_3 = T^{x(k)} \sum_{i=1}^t a_i T^{s+i} \xi_2.$$

Then $\xi_3 \in X$.

We show that (1.7) is true with $\bar{\xi}_n = \xi_3$.

We use the equality

$$T^m = T_1^m + \sum_{j=0}^{m-1} T^{m-j-1} T_2 T_1^j, \quad m \in \mathbb{N}.$$

The equality is true for $m = 1$. Assume that this equality is true for some $m \in \mathbb{N}$.

Then

$$\begin{aligned} T^{m+1} &= T(T_1^m + \sum_{j=0}^{m-1} T^{m-j-1} T_2 T_1^j) = T_1^{m+1} + T_2 T_1^m + \sum_{j=0}^{m-1} T^{m-j} T_2 T_1^j \\ &= T_1^{m+1} + \sum_{j=0}^m T^{m+1-j-1} T_2 T_1^j. \end{aligned}$$

Thus, by induction, the equality is true for every $m \in \mathbb{N}$.

We have

$$\begin{aligned} \xi_3 &= T^{x(k)} \sum_{i=1}^t a_i T^{s+i} \xi_2 \\ &= \sum_{i=1}^t a_i T^{x(k)+s+i} \left(\sum_{m=1}^{y(k)} \xi_2(m) e_m + \sum_{m=y(k)+1}^{\infty} \xi_2(m) e_m \right) \\ &= \sum_{i=1}^t a_i T^{x(k)+s+i} \sum_{m=1}^{y(k)} \xi_2(m) e_m + \sum_{i=1}^t a_i T^{x(k)+s+i} \sum_{m=y(k)+1}^{\infty} \xi_2(m) e_m \\ &= \sum_{i=1}^t a_i \left(T_1^{x(k)+s+i} + \sum_{j=0}^{x(k)+s+i-1} T^{x(k)+s+i-j-1} T_2 T_1^j \right) \sum_{m=1}^{y(k)} \xi_2(m) e_m \\ &\quad + \sum_{i=1}^t a_i T^{x(k)+s+i} \sum_{m=y(k)+1}^{\infty} \xi_2(m) e_m \\ &= T_1^{x(k)} \sum_{i=1}^t \sum_{m=1}^{y(k)} a_i (T_1^{s+i} \xi_2)(m) e_m \\ &\quad + \sum_{i=1}^t \sum_{j=0}^{x(k)+s+i-1} a_i T^{x(k)+s+i-j-1} T_2 (T_1^j \xi_2)(y(k)) e_{y(k)} \\ &\quad + \sum_{i=1}^t a_i T^{x(k)+s+i} \sum_{m=y(k)+1}^{\infty} \xi_2(m) e_m \\ &= T_1^{x(k)} \sum_{m=1}^{y(k)} \left(\sum_{i=1}^t a_i T_1^{s+i} \xi_2 \right) (m) e_m \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^t \sum_{j=0}^{x(k)+s+i-1} a_i T^{x(k)+s+i-j-1} (T_1^j \xi_2)(y(k)) e_{y(k)+1} \\
& + \sum_{i=1}^t a_i T^{x(k)+s+i} \sum_{m=y(k)+1}^{\infty} \xi_2(m) e_m.
\end{aligned}$$

By (1.11), we obtain

$$\begin{aligned}
& T_1^{x(k)} \sum_{m=1}^{y(k)} \left(\sum_{i=1}^t a_i T_1^{s+i} \xi_2 \right) (m) e_m \\
& = T_1^{x(k)} \left(\sum_{m=1}^{\chi(k)} \left(\sum_{i=1}^t a_i T_1^{s+i} \xi_2 \right) (m) e_m + e_u \right) \\
& = T_1^{x(k)} \sum_{m=1}^{\chi(k)} \left(\sum_{i=1}^t a_i T_1^{s+i} \xi_2 \right) (m) e_m + T_1^{x(k)} e_{y(k)-x(k)+1} \\
& = T^{x(k)} \sum_{m=1}^{\chi(k)} \left(\sum_{i=1}^t a_i T_1^{s+i} \xi_2 \right) (m) e_m + e_1.
\end{aligned}$$

Thus

$$\xi_3 - e_1 = \xi_4 + \xi_5 + \xi_6,$$

where

$$\begin{aligned}
\xi_4 & = T^{x(k)} \sum_{m=1}^{\chi(k)} \left(\sum_{i=1}^t a_i T_1^{s+i} \xi_2 \right) (m) e_m, \\
\xi_5 & = \sum_{i=1}^t \sum_{j=0}^{x(k)+s+i-1} a_i T^{x(k)+s+i-j-1} (T_1^j \xi_2)(y(k)) e_{y(k)+1}, \\
\xi_6 & = \sum_{i=1}^t a_i T^{x(k)+s+i} \sum_{m=y(k)+1}^{\infty} \xi_2(m) e_m.
\end{aligned}$$

We still need to prove that $p_{\mu(k)}(\xi_m) \leq 1$ for $m = 4, 5, 6$.

By (1.10) and (1.12), we get

$$\begin{aligned}
p_{\mu(k)}(\xi_4) & \leq \max_{1 \leq m \leq \chi(k)} \max_{1 \leq i \leq t} |a_i| |(T_1^{s+i} \xi_2)(m)| p_{\mu(k)} \left(T^{x(k)} e_m \right) \\
& \leq \max_{1 \leq m \leq \chi(k)} p_{\mu(k)} \left(T^{x(k)} e_m \right).
\end{aligned}$$

- If $1 \leq m \leq x(k)$, then by Lemma 1.6 and Assumption 1.4 (2)(a),

$$\begin{aligned} p_{\mu(k)} \left(T^{x(k)} e_m \right) &= p_{\mu(k)} \left(\sum_{i=x(k)+1}^{x(k)+m} \varepsilon_{x(k),m,i} e_i \right) \\ &\leq \max_{x(k) < i \leq 2x(k)} p_{\mu(k)}(e_i) \leq \max_{x(k) < i \leq \chi(k)+x(k)} p_{\mu(k)}(e_i) \\ &= \max_{x(k) < i \leq \chi(k)+x(k)} |a|^{\alpha_i^{\mu(k)}} \leq 1. \end{aligned}$$

- If $x(k) < m \leq \chi(k)$, then by Assumption 1.4 (2)(a),

$$p_{\mu(k)} \left(T^{x(k)} e_m \right) = p_{\mu(k)}(e_{x(k)+m}) = |a|^{\alpha_{x(k)+m}^{\mu(k)}} \leq 1.$$

Hence

$$p_{\mu(k)}(\xi_4) \leq 1.$$

By (1.10), (1.12), Assumption 1.4 (2)(c) and Lemma 1.7, we have

$$\begin{aligned} p_{\mu(k)}(\xi_5) &\leq \max_{1 \leq i \leq t} \max_{0 \leq j \leq x(k)+s+i-1} |a_i| \left| (T_1^j \xi_2)(y(k)) \right| \\ &\quad \cdot p_{\mu(k)} \left(T^{x(k)+s+i-j-1} e_{y(k)+1} \right) \\ &\leq \max_{0 \leq i \leq x(k)+s+t-1} p_{\mu(k)}(T^i e_{y(k)+1}) \leq \max_{0 \leq i < \varphi(k)} p_{\mu(k)}(T^i e_{y(k)+1}) \\ &\leq \max_{0 \leq i < \varphi(k)} |a|^{-\alpha_{y(k)+1}^{\mu(k)+2}} p_{\mu(k)}(T^i e_{y(k)+1}) \leq 1 \end{aligned}$$

since

$$x(k) + s + t - 1 = x(k) + \chi(k) - r + y(k) - \chi(k) - 1 < \varphi(k).$$

Finally, by (1.8), (1.12) and Lemma 1.7,

$$\begin{aligned} p_{\mu(k)}(\xi_6) &\leq \max_{1 \leq i \leq t} \max_{m > y(k)} |a_1| |\xi_2(m)| p_{\mu(k)} \left(T^{x(k)+s+i} e_m \right) \\ &\leq \max_{x(k) < i \leq \varphi(k)} \sup_{m > y(k)} |a|^{-\alpha_m^{\mu(k)+2}} p_{\mu(k)}(T^i e_m) \leq 1. \end{aligned}$$

Hence for every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $\mu(k) = n$ and there exists $\bar{\xi}_n \subset X$ such that

$$p_{\mu(k)}(\bar{\xi}_n - e_1) \leq 1.$$

Therefore the claim (1.7) is satisfied and

$$\lim_{n \rightarrow \infty} \bar{\xi}_n = e_1.$$

□

Lemma 1.10. *Let $\Lambda(B)$, where $B = (b_n^k)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space and it is not assumed that $b_n^k \leq b_n^{k+1}$ for $k, n \in \mathbb{N}$.*

(1) *Let $(c_n^k)_{n \in \mathbb{N}} \subset \mathbb{R}$ for $k \in \mathbb{N}$, $c_n^k > 0$ for $k, n \in \mathbb{N}$. If the inequalities*

$$0 < \inf_{n \in \mathbb{N}} c_n^k \leq \sup_{n \in \mathbb{N}} c_n^k < \infty \quad (1.13)$$

are satisfied for $k \in \mathbb{N}$, then the identity

$$I: \Lambda(B) \rightarrow \Lambda(C), \lambda \mapsto \lambda,$$

where $C = (c_n^k b_n^k)_{k,n \in \mathbb{N}}$, is an isomorphism.

(2) *Let $(\nu_n)_{n \in \mathbb{N}} \subset K^*$ and let $c_n = |\nu_n|$ for $n \in \mathbb{N}$. Then the map*

$$\tau: \Lambda(B) \rightarrow \Lambda(C), \sum_{n=1}^{\infty} \lambda_n e_n \mapsto \sum_{n=1}^{\infty} \frac{\lambda_n}{\nu_n} e_n,$$

where $C = (c_n b_n^k)_{k,n \in \mathbb{N}}$, is an isomorphism.

(3) *Let $(c_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, $c_n > 0$ for $n \in \mathbb{N}$. Then the spaces $\Lambda(B)$ and $\Lambda(C)$, where $C = (c_n b_n^k)_{k,n \in \mathbb{N}}$, are isomorphic.*

Proof. (1) Suppose that (1.13) is satisfied. We have

$$\begin{aligned} \Lambda(B) &= \left\{ (\lambda_1, \lambda_2, \dots) \in K^{\mathbb{N}} : \lim_{n \rightarrow \infty} |\lambda_n| b_n^k = 0 \text{ for } k \in \mathbb{N} \right\} \\ &= \left\{ (\lambda_1, \lambda_2, \dots) \in K^{\mathbb{N}} : \lim_{n \rightarrow \infty} |\lambda_n| c_n^k b_n^k = 0 \text{ for } k \in \mathbb{N} \right\} = \Lambda(C). \end{aligned}$$

The identity

$$I: \Lambda(B) \rightarrow \Lambda(C), \lambda \mapsto \lambda$$

is an isomorphism since for any sequence $(\lambda_m)_{m \in \mathbb{N}} = ((\lambda_n^m)_{n \in \mathbb{N}})_{m \in \mathbb{N}} \subset \Lambda(B)$

$$\lim_{m \rightarrow \infty} \max_{n \in \mathbb{N}} |\lambda_n^m| b_n^k = 0$$

if and only if

$$\lim_{m \rightarrow \infty} \max_{n \in \mathbb{N}} |\lambda_n^m| c_n^k b_n^k = 0$$

for $k \in \mathbb{N}$, i.e.

$$\lim_{m \rightarrow \infty} \lambda_m = 0$$

if and only if

$$\lim_{m \rightarrow \infty} I(\lambda_m) = 0.$$

(2) Since

$$\lim_{n \rightarrow \infty} |\lambda_n| b_n^k = 0$$

if and only if

$$\lim_{n \rightarrow \infty} \left| \frac{\lambda_n}{\nu_n} \right| c_n b_n^k = 0$$

and

$$\sup_{n \in \mathbb{N}} |\lambda_n| b_n^k = \sup_{n \in \mathbb{N}} \left| \frac{\lambda_n}{\nu_n} \right| c_n b_n^k$$

for $(\lambda_n)_{n \in \mathbb{N}} \in K^{\mathbb{N}}$, $k \in \mathbb{N}$, the map τ is a linear bijection such that

$$p_k(\lambda) = p_k(\tau(\lambda))$$

for $\lambda \in \Lambda(B)$, $k \in \mathbb{N}$.

Therefore τ is an isomorphism.

(3) Let $c_n = |a|^{\gamma_n}$, $\gamma_n \in \mathbb{R}$ for $n \in \mathbb{N}$. The spaces $\Lambda\left((b_n^k)_{k,n \in \mathbb{N}}\right)$ and

$\Lambda\left((|a|^{\gamma_n - [\gamma_n]} b_n^k)_{k,n \in \mathbb{N}}\right)$ are isomorphic by (1) and the spaces

$\Lambda\left((|a|^{\gamma_n - [\gamma_n]} b_n^k)_{k,n \in \mathbb{N}}\right)$ and $\Lambda\left((|a|^{\gamma_n} b_n^k)_{k,n \in \mathbb{N}}\right)$ are isomorphic by (2).

Therefore the spaces $\Lambda(B)$ and $\Lambda(C)$ are isomorphic. □

Theorem 1.11. *Suppose that a non-Archimedean Köthe space $\Lambda(A)$, where $A = \left(|a|^{\alpha_n^k}\right)_{k,n \in \mathbb{N}}$, functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and an operator*

$T_0: \Lambda_0(A) \rightarrow \Lambda_0(A)$ such that

$$T_0 a^{\hat{f}(n)} e_n = \sum_{i=1}^n \varepsilon_{n,i} a^{\hat{f}(i)} e_i + a^{\hat{f}(n+1)} e_{n+1}$$

for $n \in \mathbb{N}$, where $\varepsilon_{n,i} \in \hat{K}$ for $n \in \mathbb{N}$, $1 \leq i \leq n$, satisfy Assumptions 1.1 and 1.4 (1), (2).

Let $\Lambda(B)$, where $B = \left(|a|^{\beta_n^k}\right)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space and let $g: \mathbb{N} \rightarrow \mathbb{R}$.

Assume that

$$\alpha_n^k + f(n) = \beta_n^k + g(n) \quad (1.14)$$

for $k, n \in \mathbb{N}$.

Then the non-Archimedean Köthe space $\Lambda(B)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, g$ and the operator $R_0: \Lambda_0(B) \rightarrow \Lambda_0(B)$ such that

$$R_0 a^{\hat{g}(n)} e_n = \sum_{i=1}^n \varepsilon_{n,i} a^{\hat{g}(i)} e_i + a^{\hat{g}(n+1)} e_{n+1}$$

for $n \in \mathbb{N}$ satisfy Assumptions 1.1 and 1.4 (1), (2).

If the non-Archimedean Köthe space $\Lambda(A)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and the operator $T_0: \Lambda_0(A) \rightarrow \Lambda_0(A)$ satisfy Assumptions 1.1 and 1.4, then the non-Archimedean Köthe space $\Lambda(B)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, g$ and the operator R_0 satisfy Assumptions 1.1 and 1.4.

Proof. $\Lambda(B)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, g$ and the operator R_0 satisfy Assumption 1.1. Let

$$T_0^i a^{\hat{f}(j)} e_j = \sum_{l=1}^{i+j} \varepsilon_{i,j,l} a^{\hat{f}(l)} e_l \quad (1.15)$$

for $i, j \in \mathbb{N}$, where $\varepsilon_{i,j,l} \in \hat{K}$ for $i, j \in \mathbb{N}$, $1 \leq l \leq i+j$. Then, by Lemma 1.2,

$$R_0^i a^{\hat{g}(j)} e_j = \sum_{l=1}^{i+j} \varepsilon_{i,j,l} a^{\hat{g}(l)} e_l \quad (1.16)$$

for $i, j \in \mathbb{N}$.

By Lemma 1.3,

$$T_0^{z(k)} a^{\hat{f}(1)} e_1 = a^{\hat{f}(z(k)+1)} e_{z(k)+1}$$

for $k \in \mathbb{N}$ since

$$T_0 a^{\hat{f}(z(k))} e_{z(k)} = -T_0 \left(T_0^{z(k)-1} a^{\hat{f}(1)} e_1 - a^{\hat{f}(z(k))} e_{z(k)} \right) + a^{\hat{f}(z(k)+1)} e_{z(k)+1}$$

for $k \in \mathbb{N}$.

Thus

$$R_0^{z^{(k)}} a^{\hat{g}(1)} e_1 = a^{\hat{g}(z^{(k)}+1)} e_{z^{(k)}+1}$$

for $k \in \mathbb{N}$. And, by Lemma 1.3,

$$T_0 a^{\hat{g}(z^{(k)})} e_{z^{(k)}} = -T_0 \left(T_0^{z^{(k)}-1} a^{\hat{g}(1)} e_1 - a^{\hat{g}(z^{(k)})} e_{z^{(k)}} \right) + a^{\hat{g}(z^{(k)}+1)} e_{z^{(k)}+1}$$

for $k \in \mathbb{N}$.

Hence $\Lambda(B)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, g$ and the operator R_0 satisfy Assumption 1.4 (1).

Moreover, they satisfy Assumption 1.4 (2) by (1.14).

Therefore $\Lambda(B)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, g$ and the operator R_0 satisfy Assumptions 1.1 and 1.4 (1), (2).

If $\Lambda(A)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and the operator T_0 satisfy Assumption 1.1 and 1.4, then, by the above, $\Lambda(B)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, g$ and the operator R_0 satisfy Assumptions 1.1 and 1.4 (1), (2). And they satisfy Assumption 1.4 (3) by (1.14)-(1.16). Therefore $\Lambda(B)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, g$ and the operator R_0 satisfy Assumptions 1.1 and 1.4. \square

Theorem 1.12. *Let a non-Archimedean Köthe space $\Lambda(A)$, where $A = \left(|a|^{\alpha_n^k} \right)_{k,n \in \mathbb{N}}$, functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and an operator T_0 satisfy Assumptions 1.1 and 1.4.*

Then T_0 extends to a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. Let $B = \left(|a|^{\beta_n^k} \right)_{k,n \in \mathbb{N}}$, $C = \left(|a|^{\gamma_n^k} \right)_{k,n \in \mathbb{N}}$, where $\beta_n^k = \alpha_n^k + f(n) - \hat{f}(n)$, $\gamma_n^k = \alpha_n^k + f(n)$ for $k, n \in \mathbb{N}$, and let $T_0: \Lambda_0(A) \rightarrow \Lambda_0(A)$, $R_0: \Lambda_0(C) \rightarrow \Lambda_0(C)$ be such that

$$T_0 a^{\hat{f}(n)} e_n = \sum_{i=1}^n \varepsilon_{n,i} a^{\hat{f}(i)} e_i + a^{\hat{f}(n+1)} e_{n+1}$$

for $n \in \mathbb{N}$, where $\varepsilon_{n,i} \in \hat{K}$ for $n \in \mathbb{N}$, $1 \leq i \leq n$, and

$$R_0 e_n = \sum_{i=1}^n \varepsilon_{n,i} e_i + e_{n+1}$$

for $n \in \mathbb{N}$.

Since

$$\alpha_n^k + f(n) = \gamma_n^k + g(n)$$

for $k, n \in \mathbb{N}$, where $g: \mathbb{N} \rightarrow \mathbb{R}$, $g \equiv 0$, by Theorem 1.11, $\Lambda(C)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, g$ and the operator R_0 satisfy Assumptions 1.1 and 1.4.

Hence, by Theorem 1.9, R_0 extends to a linear and continuous operator

$$R: \Lambda(C) \rightarrow \Lambda(C)$$

that has no nontrivial invariant subspaces.

By Lemma 1.10,

$$\tau: \Lambda(A) \rightarrow \Lambda(C)$$

is an isomorphism, where $\tau = \hat{\tau}I$, $I: \Lambda(A) \rightarrow \Lambda(B)$, $\lambda \mapsto \lambda$, and $\hat{\tau}: \Lambda(B) \rightarrow \Lambda(C)$, $\sum_{n=1}^{\infty} \lambda_n e_n \mapsto \sum_{n=1}^{\infty} \lambda_n a^{-\hat{f}(n)} e_n$.

We have

$$\begin{aligned} T_0 a^{\hat{f}(n)} e_n &= \sum_{i=1}^n \varepsilon_{n,i} a^{\hat{f}(i)} e_i + a^{\hat{f}(n+1)} e_{n+1} = \tau^{-1} \left(\sum_{i=1}^n \varepsilon_{n,i} e_i + e_{n+1} \right) \\ &= \tau^{-1} R_0 e_n = \tau^{-1} R_0 \tau a^{\hat{f}(n)} e_n = \tau^{-1} R \tau a^{\hat{f}(n)} e_n \end{aligned}$$

for $n \in \mathbb{N}$.

Therefore T_0 extends to a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. □

Chapter 2

The Specific Schemes

Now we prove some specific schemes. We assume that the schemes satisfy Assumptions 1.1 and 1.4 (1), (2) and have some additional properties. Next we prove that from these assumptions and the additional properties follows Assumption 1.4 (3). In this way, we prove the truth of the schemes.

2.1 The first Specific Scheme

Theorem 2.1. *Suppose that a non-Archimedean Köthe space $\Lambda(A)$, where $A = \left(|a|^{\alpha_n^k}\right)_{k,n \in \mathbb{N}}$, functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and an operator T_0 satisfy Assumptions 1.1, 1.4 (1), (2) and the following additional conditions:*

$$(1) \quad f \equiv 0.$$

$$(2) \quad T_0 e_n = e_{n+1}$$

for $k \in \mathbb{N}$, $\psi(k) \leq n < z(k)$.

$$(3) \quad \alpha_n^{\mu(k)+2} \geq \alpha_{n+i}^{\mu(k)}$$

for $k \in \mathbb{N}$, $n \geq \psi(k)$, $1 \leq i \leq \varphi(k)$.

$$(4) \quad \alpha_n^{\mu(k)+2} \geq \alpha_{n+i}^{\mu(k)+1}$$

for $k \in \mathbb{N}$, $n \geq \psi(k)$, $1 \leq i < \varphi(k)$ if $n+i = y(r)$ for some $r > k$
or $n+i = z(s)$ for some $s \geq k$.

(5)

$$\alpha_i^{\mu(j)} \leq \alpha_{y(k)}^{\mu(j)+1}$$

for $k \in \mathbb{N}$, $k \geq 2$, $1 \leq j < k$, $1 \leq i \leq \varphi(j)$.

(6)

$$\alpha_i^{\mu(j)} \leq \alpha_{z(k)}^{\mu(j)+1}$$

for $k \in \mathbb{N}$, $1 \leq j \leq k$, $1 \leq i < z(k)$.

Then the non-Archimedean Köthe space $\Lambda(A)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and the operator T_0 satisfy Assumptions 1.1, 1.4 and T_0 extends to a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. Let $\varepsilon_{i,j,l} \in \hat{K}$ for $i, j \in \mathbb{N}$, $1 \leq l \leq i+j$ be such that

$$T_0^i e_j = \sum_{l=1}^{i+j} \varepsilon_{i,j,l} e_l$$

for $i, j \in \mathbb{N}$.

We shall prove that Assumption 1.4 (3) is true, i.e.

$$\max_{\substack{1 \leq l \leq i+j \\ \varepsilon_{i,j,l} \neq 0}} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) \leq 0$$

for $k \in \mathbb{N}$, $j \geq \psi(k)$ and $1 \leq i \leq \varphi(k)$.

Let $k \in \mathbb{N}$, $j \geq \psi(k)$ and $1 \leq i \leq \varphi(k)$.

- If $y(m) \notin [j, j+i)$ for $m > k$ and $z(m) \notin [j, j+i)$ for $m \geq k$, then

$$T_0^i e_j = e_{i+j}.$$

Thus

$$\max_{\substack{1 \leq l \leq i+j \\ \varepsilon_{i,j,l} \neq 0}} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) = -\alpha_j^{\mu(k)+2} + \alpha_{i+j}^{\mu(k)} \leq 0.$$

- If $y(m) \in [j, j+i)$ for some $m > k$ and $z(r) \notin [j, j+i)$ for $r \geq k$, then

$$T_0^i e_j = \sum_{l=1}^{\varphi(k)} \varepsilon_{i,j,l} e_l + e_{j+i}.$$

Hence

$$\begin{aligned}
& \max_{\substack{1 \leq l \leq i+j \\ \varepsilon_{i,j,l} \neq 0}} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) \\
& \leq \max \left(\max_{1 \leq l \leq \varphi(k)} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right), -\alpha_j^{\mu(k)+2} + \alpha_{j+i}^{\mu(k)} \right) \\
& \leq \max \left(-\alpha_j^{\mu(k)+2} + \alpha_{y(m)}^{\mu(k)+1}, -\alpha_j^{\mu(k)+2} + \alpha_{j+i}^{\mu(k)} \right) \leq 0.
\end{aligned}$$

- If $z(m) \in [j, j+i]$ for some $m \geq k$, then

$$\begin{aligned}
& \max_{\substack{1 \leq l \leq i+j \\ \varepsilon_{i,j,l} \neq 0}} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) \\
& \leq \max \left(\max_{1 \leq l < z(m)} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right), \max_{z(m) \leq l \leq j+i} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) \right) \\
& \leq \max \left(-\alpha_j^{\mu(k)+2} + \alpha_{z(m)}^{\mu(k)+1}, \max_{z(m) \leq l \leq j+i} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) \right) \leq 0.
\end{aligned}$$

Thus Assumption 1.4 (3) is satisfied.

Hence the non-Archimedean Köthe space $\Lambda(A)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and the operator T_0 satisfy Assumptions 1.1, 1.4.

And therefore, by Theorem 1.9, T_0 extends to a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. \square

2.2 The more general setting for the first Specific Scheme

Theorem 2.2. *Suppose that a non-Archimedean Köthe space $\Lambda(A)$, where $A = \left(|a|^{\alpha_n^k} \right)_{k,n \in \mathbb{N}}$, functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and an operator T_0 satisfy Assumptions 1.1, 1.4 (1), (2) and the following additional conditions:*

(1)

$$T_0 a^{\hat{f}(n)} e_n = a^{\hat{f}(n+1)} e_{n+1}$$

for $k \in \mathbb{N}$, $\psi(k) \leq n < z(k)$.

(2)

$$\alpha_n^{\mu(k)+2} + f(n) \geq \alpha_{n+i}^{\mu(k)} + f(n+i)$$

for $k \in \mathbb{N}$, $n \geq \psi(k)$, $1 \leq i \leq \varphi(k)$.

(3)

$$\alpha_n^{\mu(k)+2} + f(n) \geq \alpha_{n+i}^{\mu(k)+1} + f(n+i)$$

for $k \in \mathbb{N}$, $n \geq \psi(k)$, $1 \leq i < \varphi(k)$ if $n+i = y(r)$ for some $r > k$ or $n+i = z(s)$ for some $s \geq k$.

(4)

$$\alpha_i^{\mu(j)} + f(i) \leq \alpha_{y(k)}^{\mu(j)+1} + f(y(k))$$

for $k \in \mathbb{N}$, $k \geq 2$, $1 \leq j < k$, $1 \leq i \leq \varphi(j)$.

(5)

$$\alpha_i^{\mu(j)} + f(i) \leq \alpha_{z(k)}^{\mu(j)+1} + f(z(k))$$

for $k \in \mathbb{N}$, $1 \leq j \leq k$, $1 \leq i < z(k)$.

Then T_0 extends to a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. Let $\Lambda(B)$, where $B = \left(|a|^{\beta_n^k}\right)_{k,n \in \mathbb{N}}$, $\beta_n^k = \alpha_n^k + f(n)$ for $k, n \in \mathbb{N}$, be a non-Archimedean Köthe space, $T_0 a^{\hat{f}(n)} e_n = \sum_{i=1}^n \varepsilon_{n,i} a^{\hat{f}(i)} e_i + a^{\hat{f}(n+1)} e_{n+1}$ for $n \in \mathbb{N}$, and let $R_0: \Lambda_0(B) \rightarrow \Lambda_0(B)$ be such that $R_0 e_n = \sum_{i=1}^n \varepsilon_{n,i} e_i + e_{n+1}$ for $n \in \mathbb{N}$. We have

$$\alpha_n^k + f(n) = \beta_n^k + g(n)$$

for $k, n \in \mathbb{N}$, where $g: \mathbb{N} \rightarrow \mathbb{R}$, $g \equiv 0$.

$\Lambda(A)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and the operator T_0 satisfy Assumptions 1.1 and 1.4 (1), (2). Then, by Theorem 1.11, $\Lambda(B)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, g$ and the operator R_0 satisfy Assumptions 1.1 and 1.4 (1), (2). Since $\Lambda(B)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, g$ and the operator R_0 satisfy conditions (1)-(6) of Theorem 2.1, they satisfy Assumptions 1.1 and 1.4.

Hence, by Theorem 1.11, $\Lambda(A)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and the operator T_0 satisfy Assumptions 1.1 and 1.4.

Therefore, by Theorem 1.12, T_0 extends to a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. \square

Remark 2.3. The function f can be expressed literally, i.e. the Assumptions 1.4 (2), (3) can be written such that:

- Let

– for every $k \in \mathbb{N}$ and $x(k) < n \leq \chi(k) + x(k)$

$$f(n) \leq -\alpha_n^{\mu(k)};$$

– for every $k \in \mathbb{N}$ and $\chi(k) < n \leq y(k)$

$$f(n) \geq -\alpha_n^{\mu(k)+1};$$

– for every $k \in \mathbb{N}$

$$f(y(k) + 1) \leq -\alpha_{y(k)+1}^{\mu(k)+2}$$

and

$$f(n + 1) - f(n) \leq -\alpha_{n+1}^{\mu(k)+2} - \left(-\alpha_n^{\mu(k)+2}\right)$$

if $y(k) < n < \psi(k)$.

- If $\varepsilon_{i,j,l} \neq 0$, then

$$f(l) - f(j) \leq -\alpha_l^{\mu(k)} - \left(-\alpha_j^{\mu(k)+2}\right)$$

for $k \in \mathbb{N}$, $j \geq \psi(k)$, $1 \leq i \leq \varphi(k)$, $1 \leq l \leq j + i$.

And the conditions (2)-(5) of Theorem 2.2 can be written such that:

-

$$f(n + i) - f(n) \leq -\alpha_{n+i}^{\mu(k)} - \left(-\alpha_n^{\mu(k)+2}\right)$$

for $k \in \mathbb{N}$, $n \geq \psi(k)$, $1 \leq i \leq \varphi(k)$.

-

$$f(n + i) - f(n) \leq -\alpha_{n+i}^{\mu(k)+1} - \left(-\alpha_n^{\mu(k)+2}\right)$$

for $k \in \mathbb{N}$, $n \geq \psi(k)$, $1 \leq i < \varphi(k)$ if $n + i = y(r)$ for some $r > k$ or $n + i = z(s)$ for some $s \geq k$.

-

$$f(i) - f(y(k)) \leq -\alpha_i^{\mu(j)} - \left(-\alpha_{y(k)}^{\mu(j)+1}\right)$$

for $k \in \mathbb{N}$, $k \geq 2$, $1 \leq j < k$, $1 \leq i \leq \varphi(j)$.

-

$$f(i) - f(z(k)) \leq -\alpha_i^{\mu(j)} - \left(-\alpha_{z(k)}^{\mu(j)+1}\right)$$

for $k \in \mathbb{N}$, $1 \leq j \leq k$, $1 \leq i < z(k)$.

2.3 The second Specific Scheme and the generalization

Theorem 2.4. *Suppose that a non-Archimedean Köthe space $\Lambda(A)$, where $A = \left(|a|^{\alpha_n^k}\right)_{k,n \in \mathbb{N}}$, functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and an operator T_0 satisfy Assumptions 1.1, 1.4 (1), (2) and the following additional conditions:*

- (1) *There exist function ν , sets X_k for $k \in \mathbb{N} \setminus \{1\}$ and functions v_k for $k \in \mathbb{N} \setminus \{1\}$ such that*

$$\nu: \mathbb{N} \setminus \{1\} \rightarrow \mathbb{N} \setminus \{1\};$$

$$\psi(k) \leq \nu(k)y(k) < (\nu(k) + 1)y(k) < z(k),$$

$$X_k = \{m \geq \nu(k): my(k) < z(k), m = 0 \pmod{\nu(k)} \text{ or} \\ m = 1 \pmod{\nu(k)}\},$$

$$v_k: X_k \rightarrow \mathbb{N}, m \mapsto my(k)$$

for $k \in \mathbb{N} \setminus \{1\}$.

- (2)

$$f \equiv 0.$$

- (3)

$$T_0 e_n = e_{n+1}$$

if $k \in \mathbb{N}$, $\psi(k) \leq n < z(k)$ and $n \neq v_k(m)$ for $m \in X_k$ if $k \geq 2$;

$$T_0 e_n = -e_1 + e_{n+1}$$

if $n = v_k(m)$ for some $m \in X_k$, $k \geq 2$ and $m = 0 \pmod{\nu(k)}$;

$$T_0 e_n = e_1 + e_{n+1}$$

if $n = v_k(m)$ for some $m \in X_k$, $k \geq 2$ and $m = 1 \pmod{\nu(k)}$.

- (4)

$$\alpha_n^{\mu(k)+2} \geq \alpha_{n+i}^{\mu(k)}$$

for $k \in \mathbb{N}$, $n \geq \psi(k)$, $1 \leq i \leq \varphi(k)$.

(5)

$$\alpha_n^{\mu(k)+2} \geq \alpha_{n+i}^{\mu(k)+1}$$

for $k \in \mathbb{N}$, $n \geq \psi(k)$, $1 \leq i < \varphi(k)$ if $n+i = y(r)$ for some $r > k$ or $n+i = z(s)$ for some $s \geq k$ or $n+i = v_t(m)$ for some $t \geq k$, $t \geq 2$, $m \in X_t$.

(6)

$$\alpha_i^{\mu(1)} \leq \alpha_{z(1)}^{\mu(1)+1}$$

for $1 \leq i < z(1)$.

(7)

$$\alpha_i^{\mu(j)} \leq \alpha_{y(k)}^{\mu(j)+1}$$

for $k \in \mathbb{N}$, $k \geq 2$, $1 \leq j < k$, $1 \leq i \leq \varphi(j)$.

(8)

$$\alpha_i^{\mu(j)} \leq \alpha_{v_k(m)}^{\mu(j)+1}$$

for $k \in \mathbb{N}$, $k \geq 2$, $m \in X_k$, $1 \leq j \leq k$, $1 \leq i \leq \varphi(j)$.

(9)

$$\alpha_i^{\mu(j)} \leq \alpha_{z(k)}^{\mu(j)+1}$$

for $k \in \mathbb{N}$, $k \geq 2$, $1 \leq j \leq k$, $1 \leq i \leq \varphi(j)$.

(10)

$$\alpha_i^{\mu(j)} \leq \alpha_{z(k)}^{\mu(j)+1}$$

for $k \in \mathbb{N}$, $k \geq 2$, $1 \leq j \leq k$, $z(k) - (\nu(k) - 1)y(k) < i < z(k)$.

Then the non-Archimedean Köthe space $\Lambda(A)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and the operator T_0 satisfy Assumptions 1.1, 1.4 and T_0 extends to a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. Let $\varepsilon_{i,j,l} \in \hat{K}$ for $i, j \in \mathbb{N}$, $1 \leq l \leq i+j$ be such that

$$T_0^i e_j = \sum_{l=1}^{i+j} \varepsilon_{i,j,l} e_l$$

for $i, j \in \mathbb{N}$.

By Theorem 1.9, it is enough to prove that Assumption 1.4 (3) is satisfied, i.e.

$$\max_{\substack{1 \leq l \leq i+j \\ \varepsilon_{i,j,l} \neq 0}} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) \leq 0$$

for $k \in \mathbb{N}$, $j \geq \psi(k)$ and $1 \leq i \leq \varphi(k)$.

Let $k \in \mathbb{N}$, $k \geq 2$. We have

$$T_0^{y(k)-1} e_1 = e_{y(k)}$$

since $T_0^{z(k-1)} e_1 = e_{z(k-1)+1}$ and $T_0 e_n = e_{n+1}$ for $z(k-1) < n < y(k)$. Thus

$$T_0^{y(k)} e_1 = e_{y(k)+1} + e_1.$$

Since $T_0 e_n = e_{n+1}$ for $y(k) < n < \nu(k)y(k)$, by induction,

$$T_0^{\nu(k)y(k)-1} e_1 = \sum_{m=1}^{\nu(k)} e_{my(k)}.$$

Hence, using induction again, we have

$$T_0^{\nu(k)y(k)-1+n} e_1 = \sum_{m=1}^{\nu(k)} e_{my(k)+n}$$

for $1 \leq n \leq z(k) - \nu(k)y(k)$ since

$$T_0 e_{y(k)} = e_{y(k)+1} + e_1,$$

$$T_0 e_{\nu(k)y(k)} = e_{\nu(k)y(k)+1} - e_1,$$

$$T_0 e_{v_k(m)} = e_{v_k(m)+1} - e_1$$

if $m \in X_k$ and $m = 0 \pmod{\nu(k)}$,

$$T_0 e_{v_k(m)} = e_{v_k(m)+1} + e_1$$

if $m \in X_k$ and $m = 1 \pmod{\nu(k)}$ and

$$T_0 e_n = e_{n+1}$$

if $y(k) < n < z(k)$ and $n \neq v_k(m)$ for $m \in X_k$.

In particular,

$$\begin{aligned} T_0^{z(k)-1} e_1 &= \sum_{m=1}^{\nu(k)} e_{z(k)-\nu(k)y(k)+my(k)} \\ &= \left(\sum_{m=1}^{\nu(k)-1} e_{z(k)-\nu(k)y(k)+my(k)} \right) + e_{z(k)}. \end{aligned}$$

The operator T_0 is defined such that

$$T_0^{z(k)} e_1 = e_{z(k)+1}.$$

Therefore if $z(k) = 0 \pmod{\nu(k)y(k)}$, then

$$T_0 e_{z(k)} = -e_1 - \sum_{m=1}^{\nu(k)-1} e_{z(k)-\nu(k)y(k)+my(k)+1} + e_{z(k)+1};$$

if $z(k) = y(k) \pmod{\nu(k)y(k)}$, then

$$T_0 e_{z(k)} = e_1 - \sum_{m=1}^{\nu(k)-1} e_{z(k)-\nu(k)y(k)+my(k)+1} + e_{z(k)+1};$$

and if $z(k) \neq 0 \pmod{\nu(k)y(k)}$ and $z(k) \neq y(k) \pmod{\nu(k)y(k)}$, then

$$T_0 e_{z(k)} = - \sum_{m=1}^{\nu(k)-1} e_{z(k)-\nu(k)y(k)+my(k)+1} + e_{z(k)+1}.$$

Now we show that Assumption 1.4 (3) is satisfied.

Let $k \in \mathbb{N}$, $j \geq \psi(k)$, $1 \leq i \leq \varphi(k)$.

- Suppose that $z(s) \notin [j, j+i)$ for $s \geq k$.

- If $y(r) \notin [j, j+i)$ for $r > k$ and $v_t(m) \notin [j, j+i)$ for $t \geq k$, $t \geq 2$, $m \in X_t$, then

$$T_0^i e_j = e_{i+j}.$$

Thus

$$\max_{\substack{1 \leq l \leq i+j \\ \varepsilon_{i,j,l} \neq 0}} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) = -\alpha_j^{\mu(k)+2} + \alpha_{i+j}^{\mu(k)} \leq 0.$$

– If $y(r) \in [j, j + i)$ for some $r > k$, then

$$T_0^i e_j = \sum_{l=1}^{\varphi(k)} \varepsilon_{i,j,l} e_l + e_{i+j}.$$

Hence

$$\begin{aligned} & \max_{\substack{1 \leq l \leq i+j \\ \varepsilon_{i,j,l} \neq 0}} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) \\ & \leq \max \left(\max_{1 \leq l \leq \varphi(k)} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right), -\alpha_j^{\mu(k)+2} + \alpha_{i+j}^{\mu(k)} \right) \\ & \leq \max \left(-\alpha_j^{\mu(k)+2} + \alpha_{y(r)}^{\mu(k)+1}, -\alpha_j^{\mu(k)+2} + \alpha_{i+j}^{\mu(k)} \right) \leq 0. \end{aligned}$$

– If $v_s(m) \in [j, j + i)$ for some $s \geq k$, $s \geq 2$, $m \in X_t$, then

$$T_0^i e_j = \sum_{l=1}^{\varphi(k)} \varepsilon_{i,j,l} e_l + e_{i+j}.$$

Hence

$$\begin{aligned} & \max_{\substack{1 \leq l \leq i+j \\ \varepsilon_{i,j,l} \neq 0}} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) \\ & \leq \max \left(\max_{1 \leq l \leq \varphi(k)} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right), -\alpha_j^{\mu(k)+2} + \alpha_{i+j}^{\mu(k)} \right) \\ & \leq \max \left(-\alpha_j^{\mu(k)+2} + \alpha_{v_s(m)}^{\mu(k)+1}, -\alpha_j^{\mu(k)+2} + \alpha_{i+j}^{\mu(k)} \right) \leq 0. \end{aligned}$$

• Now suppose that $z(s) \in [j, j + i)$ for some $s \geq k$.

– Let $s = 1$. Then $k = 1$ and we have

$$\begin{aligned} & \max_{\substack{1 \leq l \leq i+j \\ \varepsilon_{i,j,l} \neq 0}} \left(-\alpha_j^{\mu(1)+2} + \alpha_l^{\mu(1)} \right) \\ & \leq \max \left(\max_{1 \leq l < z(1)} \left(-\alpha_j^{\mu(1)+2} + \alpha_l^{\mu(1)} \right), \max_{z(1) \leq l \leq i+j} \left(-\alpha_j^{\mu(1)+2} + \alpha_l^{\mu(1)} \right) \right) \\ & \leq \max \left(-\alpha_j^{\mu(1)+2} + \alpha_{z(1)}^{\mu(1)+1}, \max_{z(1) \leq l \leq i+j} \left(-\alpha_j^{\mu(1)+2} + \alpha_l^{\mu(1)} \right) \right) \leq 0. \end{aligned}$$

– Assume that $s \geq 2$.

Then

$$\varphi(k) < z(s) - (\nu(s) - 1)y(s) + 1$$

since

$$\begin{aligned} z(s) - (\nu(s) - 1)y(s) + 1 - \varphi(k) &> (\nu(s) + 1)y(s) \\ - (\nu(s) - 1)y(s) - 2y(k) &= 2y(s) - 2y(k) \geq 0. \end{aligned}$$

Thus

$$T_0^i e_j = \sum_{l=1}^{\varphi(k)} \varepsilon_{i,j,l} e_l + \sum_{l=m}^{i+j} \varepsilon_{i,j,l} e_l,$$

where $m = z(s) - (\nu(s) - 1)y(s) + 1$.

And we get

$$\begin{aligned} &\max_{\substack{1 \leq l \leq i+j \\ \varepsilon_{i,j,l} \neq 0}} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) \\ \leq &\max \left(\max_{1 \leq l \leq \varphi(k)} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right), \max_{m \leq l < z(s)} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right), \right. \\ &\quad \left. \max_{z(s) \leq l \leq i+j} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) \right) \\ \leq &\max \left(-\alpha_j^{\mu(k)+2} + \alpha_{z(s)}^{\mu(k)+1}, -\alpha_j^{\mu(k)+2} + \alpha_{z(s)}^{\mu(k)+1}, \right. \\ &\quad \left. \max_{z(s) \leq l \leq i+j} \left(-\alpha_j^{\mu(k)+2} + \alpha_l^{\mu(k)} \right) \right) \leq 0. \end{aligned}$$

Thus Assumption 1.4 (3) is satisfied.

Hence the non-Archimedean Köthe space $\Lambda(A)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and the operator T_0 satisfy Assumptions 1.1, 1.4.

And therefore, by Theorem 1.9, T_0 extends to a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. \square

Theorem 2.5. *Suppose that a non-Archimedean Köthe space $\Lambda(A)$, where $A = \left(|a|^{\alpha_n^k} \right)_{k,n \in \mathbb{N}}$, functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and an operator T_0 satisfy Assumptions 1.1, 1.4 (1), (2) and the following additional conditions:*

(1) There exist function ν , sets X_k for $k \in \mathbb{N} \setminus \{1\}$ and functions v_k for $k \in \mathbb{N} \setminus \{1\}$ such that

$$\nu: \mathbb{N} \setminus \{1\} \rightarrow \mathbb{N} \setminus \{1\};$$

$$\psi(k) \leq \nu(k)y(k) < (\nu(k) + 1)y(k) < z(k),$$

$$X_k = \{m \geq \nu(k): my(k) < z(k), m = 0 \pmod{\nu(k)} \text{ or} \\ m = 1 \pmod{\nu(k)}\},$$

$$v_k: X_k \rightarrow \mathbb{N}, m \mapsto my(k)$$

for $k \in \mathbb{N} \setminus \{1\}$.

(2)

$$T_0 a^{\hat{f}(n)} e_n = a^{\hat{f}(n+1)} e_{n+1}$$

if $k \in \mathbb{N}$, $\psi(k) \leq n < z(k)$ and $n \neq v_k(m)$ for $m \in X_k$ if $k \geq 2$;

$$T_0 a^{\hat{f}(n)} e_n = -a^{\hat{f}(1)} e_1 + a^{\hat{f}(n+1)} e_{n+1}$$

if $n = v_k(m)$ for some $m \in X_k$, $k \geq 2$ and $m = 0 \pmod{\nu(k)}$;

$$T_0 a^{\hat{f}(n)} e_n = a^{\hat{f}(1)} e_1 + a^{\hat{f}(n+1)} e_{n+1}$$

if $n = v_k(m)$ for some $m \in X_k$, $k \geq 2$ and $m = 1 \pmod{\nu(k)}$.

(3)

$$\alpha_n^{\mu(k)+2} + f(n) \geq \alpha_{n+i}^{\mu(k)} + f(n+i)$$

for $k \in \mathbb{N}$, $n \geq \psi(k)$, $1 \leq i \leq \varphi(k)$.

(4)

$$\alpha_n^{\mu(k)+2} + f(n) \geq \alpha_{n+i}^{\mu(k)+1} + f(n+i)$$

for $k \in \mathbb{N}$, $n \geq \psi(k)$, $1 \leq i < \varphi(k)$ if $n+i = y(r)$ for some $r > k$ or $n+i = z(s)$ for some $s \geq k$ or $n+i = v_t(m)$ for some $t \geq k$, $t \geq 2$, $m \in X_t$.

(5)

$$\alpha_i^{\mu(1)} + f(i) \leq \alpha_{z(1)}^{\mu(1)+1} + f(z(1))$$

for $1 \leq i < z(1)$.

$$(6) \quad \alpha_i^{\mu(j)} + f(i) \leq \alpha_{y(k)}^{\mu(j)+1} + f(y(k))$$

for $k \in \mathbb{N}$, $k \geq 2$, $1 \leq j < k$, $1 \leq i \leq \varphi(j)$.

$$(7) \quad \alpha_i^{\mu(j)} + f(i) \leq \alpha_{v_k(m)}^{\mu(j)+1} + f(v_k(m))$$

for $k \in \mathbb{N}$, $k \geq 2$, $m \in X_k$, $1 \leq j \leq k$, $1 \leq i \leq \varphi(j)$.

$$(8) \quad \alpha_i^{\mu(j)} + f(i) \leq \alpha_{z(k)}^{\mu(j)+1} + f(z(k))$$

for $k \in \mathbb{N}$, $k \geq 2$, $1 \leq j \leq k$, $1 \leq i \leq \varphi(j)$.

$$(9) \quad \alpha_i^{\mu(j)} + f(i) \leq \alpha_{z(k)}^{\mu(j)+1} + f(z(k))$$

for $k \in \mathbb{N}$, $k \geq 2$, $1 \leq j \leq k$, $z(k) - (\nu(k) - 1)y(k) < i < z(k)$.

Then T_0 extends to a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. Let $\Lambda(B)$, where $B = \left(|a|^{\beta_n^k}\right)_{k,n \in \mathbb{N}}$, $\beta_n^k = \alpha_n^k + f(n)$ for $k, n \in \mathbb{N}$, be a non-Archimedean Köthe space, $T_0 a^{\hat{f}(n)} e_n = \sum_{i=1}^n \varepsilon_{n,i} a^{\hat{f}(i)} e_i + a^{\hat{f}(n+1)} e_{n+1}$ for $n \in \mathbb{N}$, and let $R_0: \Lambda_0(B) \rightarrow \Lambda_0(B)$ be such that $R_0 e_n = \sum_{i=1}^n \varepsilon_{n,i} e_i + e_{n+1}$ for $n \in \mathbb{N}$. We have

$$\alpha_n^k + f(n) = \beta_n^k + g(n)$$

for $k, n \in \mathbb{N}$, where $g: \mathbb{N} \rightarrow \mathbb{R}$, $g \equiv 0$.

$\Lambda(A)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and the operator T_0 satisfy Assumptions 1.1 and 1.4 (1), (2), then, by Theorem 1.11, $\Lambda(B)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, g$ and the operator R_0 satisfy Assumptions 1.1 and 1.4 (1), (2). Since $\Lambda(B)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, g$ and the operator R_0 satisfy conditions (1)-(10) of Theorem 2.4, they satisfy Assumptions 1.1 and 1.4.

Hence, by Theorem 1.11, $\Lambda(A)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and the operator T_0 satisfy Assumptions 1.1 and 1.4.

Therefore, by Theorem 1.12, T_0 extends to a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. □

Chapter 3

Operators on non-Archimedean Köthe spaces without invariant subspaces

3.1 Non-decreasing coefficients

Theorem 3.1 ([8, 9]). *Let $\Lambda(A)$, where $A = (|a|^{\alpha_n^k})_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space.*

Assume that:

(1) *For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that*

$$\alpha_n^{k+1} \geq \alpha_{n+i}^k$$

for $n \geq j$.

(2)

$$\alpha_n^k \leq \alpha_{n+1}^k$$

for $k, n \in \mathbb{N}$.

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. Let $h: \mathbb{N} \rightarrow \mathbb{N}$, $h(n) = m$ for $n, m \in \mathbb{N}$, $2^{m-1} \leq n < 2^m$.

The function h has the properties: $h(n) \leq h(n+1)$ for $n \in \mathbb{N}$; for every $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $h(n) + 1 \geq h(n+i)$ for $n \geq j$; and $\lim_{n \rightarrow \infty} h(n) = \infty$.

Let $\Lambda(B)$ be a non-Archimedean Köthe space such that $B = \left(|a|^{\beta_n^k} \right)_{k, n \in \mathbb{N}}$, where $\beta_n^k = \alpha_n^k + k + h(n)$ for $k, n \in \mathbb{N}$.

We have

$$\beta_n^k + 1 = \alpha_n^k + k + 1 + h(n) \leq \alpha_n^{k+1} + k + 1 + h(n) = \beta_n^{k+1} \quad (3.1)$$

for $k, n \in \mathbb{N}$; by (1) and the properties of h , for every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\beta_n^{k+1} = \alpha_n^{k+1} + k + 1 + h(n) \geq \alpha_{n+i}^k + k + h(n+i) = \beta_{n+i}^k$$

for $n \geq j$; for every $k, n \in \mathbb{N}$

$$\begin{aligned} \beta_n^k &= \alpha_n^k + k + h(n) \leq \alpha_{n+1}^k + k + h(n+1) \leq \beta_{n+1}^k; \\ \lim_{n \rightarrow \infty} \beta_n^1 &= \lim_{n \rightarrow \infty} (\alpha_n^1 + 1 + h(n)) = \infty. \end{aligned} \quad (3.2)$$

Hence $(\beta_n^k)_{k, n \in \mathbb{N}}$ satisfies condition (3.1), assumptions (1), (2) and the additional condition (3.2).

Since the non-Archimedean Köthe space $\Lambda(B)$ is isomorphic to $\Lambda(A)$, we can assume that

$$\alpha_n^k + 1 \leq \alpha_n^{k+1}$$

for $k, n \in \mathbb{N}$; conditions (1) and (2) are satisfied; and that the condition

$$(3) \quad \lim_{n \rightarrow \infty} \alpha_n^1 = \infty.$$

is satisfied.

Now we define a non-Archimedean Köthe space $\Lambda(C)$, where $C = \left(|a|^{\gamma_n^k} \right)_{k, n \in \mathbb{N}}$, $\gamma_n^k = \alpha_n^k + \xi_n$ for $k, n \in \mathbb{N}$, $(\xi_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, isomorphic to $\Lambda(A)$, functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and an operator T_0 that satisfy assumptions of Theorem 2.1.

The sequence $(\xi_n)_{n \in \mathbb{N}}$ is defined such that

$$\xi_n \geq \xi_{n+1}$$

for $n \in \mathbb{N}$.

Let

$$f \equiv 0,$$

and let

$$\mu: \mathbb{N} \rightarrow \mathbb{N}$$

be a function that takes any natural number infinitely many times.

The sequence $(\xi_n)_{n \in \mathbb{N}}$, the functions $x, y, z, \chi, \psi, \varphi$ and the operator T_0 are defined by induction.

By (3), there exist $x(1)$ and $\chi(1)$ such that

$$x(1) \leq \chi(1),$$

$$\alpha_n^{\mu(1)} \leq 0$$

if $x(1) < n \leq \chi(1)$ and

$$\alpha_{\chi(1)+1}^{\mu(1)} \geq 0.$$

Let

$$y(1) = \chi(1) + x(1).$$

Let

$$\xi(n) = 0$$

for $1 \leq n \leq \chi(1)$ and

$$\xi(n) = -\alpha_n^{\mu(1)}$$

for $\chi(1) < n \leq \chi(1) + x(1)$.

Then

$$\gamma_n^{\mu(1)} \leq 0$$

for $x(1) < n \leq \chi(1) + x(1)$ and

$$\gamma_n^{\mu(1)+1} \geq 0$$

for $\chi(1) < n \leq y(1)$.

Let

$$\varphi(1) = x(1) + y(1) - 1.$$

By (1), there exists $\psi(1)$, $\psi(1) > y(1)$ such that

$$\alpha_n^{\mu(1)+2} \geq \alpha_{n+i}^{\mu(1)+1}$$

for $n \geq \psi(1)$, $1 \leq i \leq \varphi(1)$.

Let

$$\xi(n) = -\alpha_n^{\mu(1)+2}$$

for $y(1) < n \leq \psi(1)$.

Then

$$\gamma_{y(1)+1}^{\mu(1)+2} \leq 0$$

and

$$\gamma_n^{\mu(1)+2} \geq \gamma_{n+1}^{\mu(1)+2}$$

if $y(1) < n < \psi(1)$.

By (3), there exists $z(1)$, $z(1) > \psi(1)$ such that

$$\gamma_n^{\mu(1)} \leq \alpha_{z(1)}^{\mu(1)+1} + \xi_{\psi(1)}$$

for $1 \leq n \leq \psi(1)$.

Let

$$\xi_n = \xi_{\psi(1)}$$

for $\psi(1) < n \leq z(1)$.

Then

$$\gamma_n^{\mu(1)} \leq \gamma_{z(1)}^{\mu(1)+1}$$

for $n < z(1)$.

Let $k \in \mathbb{N}$, $k \geq 2$. Suppose that we have defined $x(m), y(m), z(m), \chi(m), \psi(m), \varphi(m)$ for $1 \leq m < k$ and ξ_n for $1 \leq n \leq z(k-1)$. Then we define $x(k), y(k), z(k), \chi(k), \psi(k), \varphi(k)$ and ξ_i for $z(k-1) < i \leq z(k)$.

By (3), there exist $x(k)$ and $\chi(k)$ such that

$$z(k-1) \leq x(k) \leq \chi(k),$$

$$\alpha_n^{\mu(k)} + \xi_{z(k-1)} \leq 0$$

if $x(k) < n \leq \chi(k)$ and

$$\alpha_{\chi(k)+1}^{\mu(k)} + \xi_{z(k-1)} \geq 0.$$

Let

$$\xi_n = \xi_{z(k-1)}$$

if $z(k-1) < n \leq \chi(k)$ and

$$\xi_n = -\alpha_n^{\mu(k)}$$

for $\chi(k) < n \leq \chi(k) + x(k)$.

By (3), there exists $y(k)$, $y(k) \geq \chi(k) + x(k)$ such that

$$\gamma_i^{\mu(j)} \leq \alpha_{y(k)}^{\mu(j)+1} + \xi_{\chi(k)+x(k)}$$

for $1 \leq j < k$, $1 \leq i \leq \varphi(j)$.

Let

$$\xi_n = \xi_{\chi(k)+x(k)}$$

if $\chi(k) + x(k) < n \leq y(k)$.

Then

$$\gamma_n^{\mu(k)} \leq 0$$

for $x(k) < n \leq \chi(k) + x(k)$,

$$\gamma_n^{\mu(k)+1} \geq 0$$

for $\chi(k) < n \leq y(k)$ and

$$\gamma_i^{\mu(j)} \leq \gamma_{y(k)}^{\mu(j)+1}$$

for $1 \leq j < k$, $1 \leq i \leq \varphi(j)$.

Let

$$\varphi(k) = x(k) + y(k) - 1.$$

By (1), there exists $\psi(k)$, $\psi(k) > y(k)$ such that

$$\alpha_n^{\mu(k)+2} \geq \alpha_{n+i}^{\mu(k)+1}$$

for $n \geq \psi(k)$, $1 \leq i \leq \varphi(k)$. Let

$$\xi_n = -\alpha_n^{\mu(k)+2}$$

for $y(k) < n \leq \psi(k)$.

Then

$$\gamma_{y(k)+1}^{\mu(k)+2} \leq 0$$

and

$$\gamma_n^{\mu(k)+2} \geq \gamma_{n+1}^{\mu(k)+2}$$

if $y(k) < n < \psi(k)$.

By (3), there exists $z(k)$, $z(k) > \psi(k)$ such that

$$\gamma_i^{\mu(j)} \leq \alpha_{z(k)}^{\mu(j)+1} + \xi_{\psi(k)}$$

for $1 \leq j \leq k$, $1 \leq i \leq \psi(k)$.

Let

$$\xi_n = \xi_{\psi(k)}$$

for $\psi(k) < n \leq z(k)$.

Then

$$\gamma_i^{\mu(j)} \leq \gamma_{z(k)}^{\mu(j)+1}$$

for $1 \leq j \leq k$, $1 \leq i < z(k)$.

Now we define the operator T_0 .

Let

$$T_0 e_n = e_{n+1}$$

for $1 \leq n < z(1)$, $n \neq y(1)$;

$$T_0 e_{y(1)} = e_{y(1)+1} + e_1.$$

If we have defined $T_0 e_n$ for $1 \leq n < z(1)$, then let

$$T_0 e_{z(1)} = -T_0 \left(T_0^{z(1)-1} e_1 - e_{z(1)} \right) + e_{z(1)+1}.$$

Let $k \in \mathbb{N}$, $k \geq 2$. Suppose that we have defined $T_0 e_n$ for $1 \leq n \leq z(k-1)$. Then we define $T_0 e_n$ for $z(k-1) < n \leq z(k)$.

Let

$$T_0 e_n = e_{n+1}$$

for $z(k-1) < n < z(k)$, $n \neq y(k)$;

$$T_0 e_{y(k)} = e_{y(k)+1} + e_1.$$

Suppose that we have defined $T_0 e_n$ for $1 \leq n < z(k)$, then let

$$T_0 e_{z(k)} = -T_0 \left(T_0^{z(k)-1} e_1 - e_{z(k)} \right) + e_{z(k)+1}.$$

The non-Archimedean Köthe space $\Lambda(C)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and the operator T_0 are defined such that they satisfy Assumptions 1.1, 1.4 (1), (2) and conditions (1), (2), (5), (6) of Theorem 2.1.

So it remains to check conditions (3), (4) of Theorem 2.1.

Let $k \in \mathbb{N}$, $n \geq \psi(k)$, $1 \leq i \leq \varphi(k)$.

Then

$$\gamma_n^{\mu(k)+2} \geq \gamma_{n+i}^{\mu(k)+1}$$

since

$$\alpha_n^{\mu(k)+2} \geq \alpha_{n+i}^{\mu(k)+1}$$

and

$$\xi_n \geq \xi_{n+i}.$$

Hence conditions (3), (4) of Theorem 2.1 are satisfied.

Therefore the operator T_0 extends to a linear and continuous operator

$$\hat{T}: \Lambda(C) \rightarrow \Lambda(C)$$

that has no nontrivial invariant subspaces.

Since $\Lambda(A)$ is isomorphic to $\Lambda(C)$, there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. □

Theorem 3.2. *The non-Archimedean Banach space c_0 has a linear and continuous operator*

$$T: c_0 \rightarrow c_0$$

that has no nontrivial invariant subspaces.

Proof. The space c_0 is isomorphic to the non-Archimedean Köthe space $\Lambda(A)$, where $A = (a_n^k)_{k,n \in \mathbb{N}}$, $a_n^k = 1$ for $k, n \in \mathbb{N}$.

Since $\Lambda(A)$ satisfies assumptions of Theorem 3.1, there exists a linear and continuous operator

$$T: c_0 \rightarrow c_0$$

that has no nontrivial invariant subspaces. □

Theorem 3.3 (W. Śliwa [18]). *Let X be an infinitely dimensional non-Archimedean Banach space of countable type.*

Then there exists a linear and continuous operator

$$T: X \rightarrow X$$

that has no nontrivial invariant subspaces.

Proof. Every infinitely dimensional non-Archimedean Banach space of countable type is isomorphic to c_0 .

Therefore, by Theorem 3.2, there exists a linear and continuous operator

$$T: X \rightarrow X$$

that has no nontrivial invariant subspaces. □

3.2 Unlimited coefficients from below

Theorem 3.4. *Let $\Lambda(A)$, where $A = \left(|a|^{|\alpha_n^k|}\right)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space.*

Assume that:

(1) *There exists a sequence $(\zeta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that:*

(a) *For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that*

$$\alpha_n^{k+1} + \zeta_n \geq \alpha_{n+i}^k + \zeta_{n+i}$$

for $n \geq j$.

(b) *For every $l, m \in \mathbb{N}$ and $M \in \mathbb{R}$, $M \geq 0$ there exist $r, s, t \in \mathbb{N}$ and a function $\alpha: [s+1, s+r] \rightarrow \mathbb{R}$ such that:*

(i)

$$l \leq r \leq s < s+r \leq t,$$

(ii)

$$\alpha_n^m + \zeta_n \leq \alpha(s+1)$$

if $r < n \leq s$,

(iii)

$$\alpha(s+1) \geq 0,$$

(iv)

$$\alpha(n) \leq \alpha(n+1)$$

if $s < n < s+r$,

(v)

$$\alpha_n^m + \zeta_n \leq \alpha(n) \leq \alpha_n^{m+1} + \zeta_n$$

for $s < n \leq s+r$,

(vi)

$$\alpha(s+r) \leq \alpha_n^{m+1} + \zeta_n$$

if $s+r < n \leq t$,

(vii)

$$\alpha_t^1 + \zeta_t - \alpha(s+r) \geq M.$$

(2) *There exists a sequence $(\vartheta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that:*

(a) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\alpha_n^{k+1} + \vartheta_n \geq \alpha_{n+i}^k + \vartheta_{n+i}$$

for $n \geq j$.

(b) For every $k, l \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq l$ such that

$$\alpha_n^i + \vartheta_n \leq \alpha_m^{i+1} + \vartheta_m$$

for $1 \leq i \leq k$, $1 \leq n \leq m$.

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. Let $h: \mathbb{N} \rightarrow \mathbb{N}$, $h(n) = m$ for $n, m \in \mathbb{N}$, $2^{m-1} \leq n < 2^m$, and let $\Lambda(B)$ be a non-Archimedean Köthe space such that $B = \left(|a|^{\beta_n^k} \right)_{k, n \in \mathbb{N}}$, $\beta_n^k = \alpha_n^k + k + h(n)$ for $k, n \in \mathbb{N}$.

We have

$$\beta_n^k + 1 = \alpha_n^k + k + 1 + h(n) \leq \alpha_n^{k+1} + k + 1 + h(n) = \beta_n^{k+1} \quad (3.3)$$

for $k, n \in \mathbb{N}$.

Now we show that $(\beta_n^k)_{k, n \in \mathbb{N}}$ and the sequences $(\zeta_n)_{n \in \mathbb{N}}$, $(\vartheta_n)_{n \in \mathbb{N}}$ satisfy conditions of assumptions (1) and (2) and some additional conditions.

By (1)(a) and the properties of h , for every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\beta_n^{k+1} + \zeta_n = \alpha_n^{k+1} + \zeta_n + k + 1 + h(n) \geq \alpha_{n+i}^k + \zeta_{n+i} + k + h(n+i) = \beta_{n+i}^k + \zeta_{n+i}$$

for $n \geq j$.

Let $l, m \in \mathbb{N}$, $M \in \mathbb{R}$, $M \geq 0$ and let $r, s, t \in \mathbb{N}$ and $\alpha: [s+1, s+r] \rightarrow \mathbb{R}$ be such that conditions of assumption (1)(b) are satisfied.

Let

$$\beta: [s+1, s+r] \rightarrow \mathbb{R}, \quad n \mapsto \alpha(n) + m + h(n).$$

Then

$$\beta_n^m + \zeta_n = \alpha_n^m + \zeta_n + m + h(n) \leq \alpha(s+1) + m + h(s+1) = \beta(s+1)$$

if $r < n \leq s$,

$$\beta(s+1) = \alpha(s+1) + m + h(s+1) \geq h(s+1), \quad (3.4)$$

$$\beta(n) = \alpha(n) + m + h(n) \leq \alpha(n+1) + m + h(n+1) = \beta(n+1)$$

if $s < n < s+r$,

$$\begin{aligned} \beta_n^m + \zeta_n &= \alpha_n^m + \zeta_n + m + h(n) \leq \alpha(n) + m + h(n) = \beta(n) \\ &\leq \alpha_n^{m+1} + \zeta_n + m + 1 + h(n) = \beta_n^{m+1} + \zeta_n \end{aligned}$$

for $s < n \leq s+r$,

$$\beta(s+r) = \alpha(s+r) + m + h(s+r) \leq \alpha_n^{m+1} + \zeta_n + m + 1 + h(n) = \beta_n^{m+1} + \zeta_n$$

if $s+r < n \leq t$,

$$\beta_t^1 + \zeta_t - \beta(s+r) = \alpha_t^1 + \zeta_t + 1 + h(t) - \alpha(s+r) - m - h(s+r) \geq M - m + 1$$

since $h(t) - h(s+r) \geq 0$. If we take $M + m - 1$ instead of M , then

$$\beta_t^1 + \zeta_t - \beta(s+r) \geq M.$$

By (2)(a) and the properties of h , for every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\beta_n^{k+1} + \vartheta_n = \alpha_n^{k+1} + \vartheta_n + k + 1 + h(n) \geq \alpha_{n+i}^k + \vartheta_{n+i} + k + h(n+i) = \beta_{n+i}^k + \vartheta_{n+i}$$

for $n \geq j$.

Let $k, l \in \mathbb{N}$ and let $m \in \mathbb{N}$, $m \geq l$ be such that condition (2)(b) is satisfied. Then

$$\beta_n^i + \vartheta_n = \alpha_n^i + \vartheta_n + i + h(n) \leq \alpha_m^{i+1} + \vartheta_m + i + 1 + h(m) = \beta_m^{i+1} + \vartheta_m$$

for $1 \leq i \leq k$, $1 \leq n \leq m$ and

$$\alpha_1^1 + \vartheta_1 + h(m) \leq \alpha_m^2 + \vartheta_m + 2 + h(m) = \beta_m^2 + \vartheta_m. \quad (3.5)$$

Hence $(\beta_n^k)_{k, n \in \mathbb{N}}$ satisfies condition (3.3), and $(\beta_n^k)_{k, n \in \mathbb{N}}$ and the sequences $(\zeta_n)_{n \in \mathbb{N}}$, $(\vartheta_n)_{n \in \mathbb{N}}$ satisfy conditions of assumptions (1), (2) and the additional conditions (3.4), (3.5).

Since the non-Archimedean Köthe spaces $\Lambda(A)$ is isomorphic to $\Lambda(B)$ and $\lim_{n \rightarrow \infty} h(n) = \infty$, we can assume that

$$\alpha_n^k + 1 \leq \alpha_n^{k+1}$$

for $k, n \in \mathbb{N}$ and that conditions (1), (2) are satisfied with the changes of (1)(b)(iii) and (2)(b) respectively on:

(1) (b) (iii')

$$\alpha(s+1) \geq M.$$

(2) (b') For every $k, l \in \mathbb{N}$ and $M \in \mathbb{R}$ there exists $m \in \mathbb{N}$, $m \geq l$ such that

$$\alpha_n^i + \vartheta_n \leq \alpha_m^{i+1} + \vartheta_m$$

for $1 \leq i \leq k$, $1 \leq n \leq m$ and $\alpha_m^2 + \vartheta_m \geq M$.

Now we define a non-Archimedean Köthe space $\Lambda(C)$, where $C = \left(|a|\gamma_n^k\right)_{k,n \in \mathbb{N}}$, $\gamma_n^k = \alpha_n^k + \xi_n$ for $k, n \in \mathbb{N}$, $(\xi_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, isomorphic to $\Lambda(A)$, functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and an operator T_0 that satisfy assumptions of Theorem 2.2. And additionally, we define functions $\alpha_k: [\chi(k) + 1, \chi(k) + x(k)] \rightarrow \mathbb{R}$ for $k \in \mathbb{N}$ and a function $\lambda: \mathbb{N} \rightarrow \mathbb{N}$.

The function f is defined such that

$$f(n) \geq f(n+1)$$

for $n \in \mathbb{N}$.

Let

$$\mu: \mathbb{N} \rightarrow \mathbb{N}$$

be a function that takes any natural number infinitely many times.

The sequence $(\xi_n)_{n \in \mathbb{N}}$, the functions $x, y, z, \chi, \psi, \varphi, f, \lambda, \alpha_k$ for $k \in \mathbb{N}$ and the operator T_0 are defined by induction.

By (1)(b), there exist $x(1), \chi(1), y(1)$ and α_1 such that

$$x(1) \leq \chi(1) < \chi(1) + x(1) = y(1),$$

$$\alpha_n^{\mu(1)} + \zeta_n \leq \alpha_1(\chi(1) + 1)$$

if $x(1) < n \leq \chi(1)$,

$$\alpha_1(\chi(1) + 1) \geq 0,$$

$$\alpha_1(n) \leq \alpha_1(n+1)$$

if $\chi(1) < n < \chi(1) + x(1)$ and

$$\alpha_n^{\mu(1)} + \zeta_n \leq \alpha_1(n) \leq \alpha_n^{\mu(1)+1} + \zeta_n$$

for $\chi(1) < n \leq \chi(1) + x(1)$.

Let

$$\xi_n = \zeta_n$$

for $1 \leq n \leq y(1)$,

$$f(n) = 0$$

for $1 \leq n \leq x(1)$,

$$f(n) = -\alpha_1(\chi(1) + 1)$$

if $x(1) < n \leq \chi(1)$ and

$$f(n) = -\alpha_1(n)$$

for $\chi(1) < n \leq \chi(1) + x(1)$.

Then

$$\gamma_n^{\mu(1)} + f(n) \leq 0$$

for $x(1) < n \leq \chi(1) + x(1)$ and

$$\gamma_n^{\mu(1)+1} + f(n) \geq 0$$

for $\chi(1) < n \leq y(1)$.

Let

$$\varphi(1) = x(1) + y(1) - 1.$$

By (2)(a), there exists $\psi(1)$ such that $\psi(1) > y(1)$,

$$\alpha_n^{\mu(1)+2} + \vartheta_n \geq \alpha_{n+i}^{\mu(1)+1} + \vartheta_{n+i}$$

and

$$\alpha_n^{\mu(1)+1} + \vartheta_n \geq \alpha_{n+i}^{\mu(1)} + \vartheta_{n+i}$$

for $n \geq \psi(1)$, $1 \leq i \leq \varphi(1)$.

Let

$$\xi_n = \vartheta_n$$

for $y(1) < n \leq \psi(1)$.

Let

$$f(y(1) + 1) = \min \left(f(y(1)), -\gamma_{y(1)+1}^{\mu(1)+2} \right)$$

and

$$f(n + 1) = f(n) + \min \left(0, \gamma_n^{\mu(1)+2} - \gamma_{n+1}^{\mu(1)+2} \right)$$

if $y(1) < n < \psi(1)$.

Then

$$\gamma_{y(1)+1}^{\mu(1)+2} + f(y(1) + 1) \leq 0$$

and

$$\gamma_n^{\mu(1)+2} + f(n) \geq \gamma_{n+1}^{\mu(1)+2} + f(n + 1)$$

if $y(1) < n < \psi(1)$.

Let $z(1)$ be such that $z(1) > \psi(1)$,

$$\gamma_n^{\mu(1)} + f(n) \leq \alpha_{z(1)}^{\mu(1)+1} + \vartheta_{z(1)} + f(\psi(1))$$

for $1 \leq n \leq \psi(1)$ and

$$\alpha_n^{\mu(1)} + \vartheta_n \leq \alpha_{z(1)}^{\mu(1)+1} + \vartheta_{z(1)}$$

for $\psi(1) < n \leq z(1)$. Such $z(1)$ exists by (2)(b').

Let

$$\xi_n = \vartheta_n$$

for $\psi(1) < n \leq z(1)$,

$$f(n) = f(\psi(1))$$

for $\psi(1) < n \leq z(1)$.

Then

$$\gamma_n^{\mu(1)} + f(n) \leq \gamma_{z(1)}^{\mu(1)+1} + f(z(1))$$

for $1 \leq n \leq z(1)$.

Let $\lambda(1)$ be such that $\lambda(1) \geq z(1)$,

$$\alpha_n^{\mu(1)+2} + \zeta_n \geq \alpha_{n+i}^{\mu(1)+1} + \zeta_{n+i}$$

and

$$\alpha_n^{\mu(1)+1} + \zeta_n \geq \alpha_{n+i}^{\mu(1)} + \zeta_{n+i}$$

for $n \geq \lambda(1)$, $1 \leq i \leq \varphi(1)$.

Let

$$\xi_n = \vartheta_n$$

if $z(1) < n \leq \lambda(1)$.

Let $k \in \mathbb{N}$, $k \geq 2$. Suppose that we have defined $x(m), y(m), z(m), \chi(m), \psi(m), \varphi(m), \alpha_m, \lambda(m)$ for $1 \leq m \leq k-1$, ξ_i for $1 \leq i \leq \lambda(k-1)$ and $f(i)$ for $1 \leq i \leq z(k-1)$. Then we define $x(k), y(k), z(k), \chi(k), \psi(k), \varphi(k), \alpha_k, \lambda(k)$, ξ_i for $\lambda(k-1) < i \leq \lambda(k)$ and $f(i)$ for $z(k-1) < i \leq z(k)$.

There exist $x(k), \chi(k), y(k)$, $\alpha_k: [\chi(k) + 1, \chi(k) + x(k)] \rightarrow \mathbb{R}$, ξ_i for $\lambda(k-1) < i \leq y(k)$ and $f(i)$ for $z(k-1) < i \leq y(k)$ such that

$$\lambda(k-1) \leq x(k) \leq \chi(k) < \chi(k) + x(k) \leq y(k);$$

$$\varphi(k-1) \leq y(k) - \lambda(k-1);$$

$$\xi_n = \xi_{\lambda(k-1)} - \zeta_{\lambda(k-1)} + \zeta_n$$

for $\lambda(k-1) < n \leq y(k)$;

$$\gamma_n^{\mu(k)} \leq \alpha_k(\chi(k) + 1)$$

if $x(k) < n \leq \chi(k)$;

$$\alpha_k(\chi(k) + 1) + f(z(k-1)) \geq 0;$$

$$\alpha_k(n) \leq \alpha_k(n+1)$$

for $\chi(k) < n < \chi(k) + x(k)$;

$$\gamma_n^{\mu(k)} \leq \alpha_k(n) \leq \gamma_n^{\mu(k)+1}$$

for $\chi(k) < n \leq \chi(k) + x(k)$;

$$\alpha_k(\chi(k) + x(k)) \leq \gamma_n^{\mu(k)+1}$$

if $\chi(k) + x(k) < n \leq y(k)$;

$$f(n) = -\alpha_k(\chi(k) + 1)$$

if $z(k-1) < n \leq \chi(k)$;

$$f(n) = -\alpha_k(n)$$

for $\chi(k) < n \leq \chi(k) + x(k)$;

$$f(n) = -\alpha_k(\chi(k) + x(k))$$

if $\chi(k) + x(k) < n \leq y(k)$;

$$\gamma_i^{\mu(j)} + f(i) \leq \gamma_{y(k)}^{\mu(j)+1} - \alpha_k(\chi(k) + x(k))$$

for $1 \leq j < k$, $1 \leq i \leq \varphi(j)$.

Such $x(k)$, $\chi(k)$, $y(k)$, α_k , ξ_i for $\lambda(k-1) < i \leq y(k)$ and $f(i)$ for $z(k-1) < i \leq y(k)$ exist by (1)(b) and the fact that if

$$\alpha_{y(k)}^1 + \zeta_{y(k)} - \alpha_k(\chi(k) + x(k)) \geq \max_{\substack{1 \leq j < k \\ 1 \leq i \leq \varphi(j)}} \gamma_i^{\mu(j)},$$

then

$$\alpha_{y(k)}^{\mu(j)+1} + \zeta_{y(k)} - \alpha_k(\chi(k) + x(k)) \geq \alpha_{y(k)}^1 + \zeta_{y(k)} - \alpha_k(\chi(k) + x(k)) \geq \gamma_i^{\mu(j)} + f(i)$$

for $1 \leq j < k$, $1 \leq i \leq \varphi(j)$ since $f(n) \leq 0$ for $n \in \mathbb{N}$.

Hence

$$\gamma_n^{\mu(k)} + f(n) \leq 0$$

for $x(k) < n \leq \chi(k) + x(k)$,

$$\gamma_n^{\mu(k)+1} + f(n) \geq 0$$

for $\chi(k) < n \leq y(k)$ and

$$\gamma_i^{\mu(j)} + f(i) \leq \gamma_{y(k)}^{\mu(j)+1} + f(y(k))$$

for $1 \leq j < k$, $1 \leq i \leq \varphi(j)$.

Let

$$\varphi(k) = x(k) + y(k) - 1.$$

By (2)(a), there exists $\psi(k)$ such that $\psi(k) > y(k)$,

$$\alpha_n^{\mu(k)+2} + \vartheta_n \geq \alpha_{n+i}^{\mu(k)+1} + \vartheta_{n+i}$$

and

$$\alpha_n^{\mu(k)+1} + \vartheta_n \geq \alpha_{n+i}^{\mu(k)} + \vartheta_{n+i}$$

for $n \geq \psi(k)$, $1 \leq i \leq \varphi(k)$.

Let

$$\xi_n = \xi_{y(k)} - \vartheta_{y(k)} + \vartheta_n$$

for $y(k) < n \leq \psi(k)$.

Let

$$f(y(k) + 1) = \min \left(f(y(k)), -\gamma_{y(k)+1}^{\mu(k)+2} \right)$$

and

$$f(n + 1) = f(n) + \min \left(0, \gamma_n^{\mu(k)+2} - \gamma_{n+1}^{\mu(k)+2} \right)$$

if $y(k) < n < \psi(k)$.

Then

$$\gamma_{y(k)+1}^{\mu(k)+2} + f(y(k) + 1) \leq 0$$

and

$$\gamma_n^{\mu(k)+2} + f(n) \geq \gamma_{n+1}^{\mu(k)+2} + f(n + 1)$$

if $y(k) < n < \psi(k)$.

By (2)(b'), there exists $z(k)$, $z(k) > \psi(k)$ such that

$$\varphi(k - 1) \leq z(k) - y(k),$$

$$\gamma_i^{\mu(j)} + f(i) \leq \alpha_{z(k)}^{\mu(j)+1} + \xi_{\psi(k)} - \vartheta_{\psi(k)} + \vartheta_{z(k)} + f(\psi(k))$$

for $1 \leq j \leq k$, $1 \leq i \leq \psi(k)$ and

$$\alpha_i^{\mu(j)} + \vartheta_i \leq \alpha_{z(k)}^{\mu(j)+1} + \vartheta_{z(k)}$$

for $1 \leq j \leq k$, $\psi(k) < i \leq z(k)$.

Let

$$\xi_n = \xi_{\psi(k)} - \vartheta_{\psi(k)} + \vartheta_n$$

for $\psi(k) < n \leq z(k)$,

$$f(i) = f(\psi(k))$$

for $\psi(k) < i \leq z(k)$.

Then

$$\gamma_i^{\mu(j)} + f(i) \leq \gamma_{z(k)}^{\mu(j)+1} + f(z(k))$$

for $1 \leq j \leq k$, $1 \leq i < z(k)$.

Let $\lambda(k)$, $\lambda(k) \geq z(k)$ be such that

$$\alpha_n^{\mu(k)+2} + \zeta_n \geq \alpha_{n+i}^{\mu(k)+1} + \zeta_{n+i}$$

and

$$\alpha_n^{\mu(k)+1} + \zeta_n \geq \alpha_{n+i}^{\mu(k)} + \zeta_{n+i}$$

for $n \geq \lambda(k)$, $1 \leq i \leq \varphi(k)$.

Let

$$\xi_n = \xi_{z(k)} - \vartheta_{z(k)} + \vartheta_n$$

if $z(k) < n \leq \lambda(k)$.

Now we define the operator T_0 .

Let

$$T_0 a^{\hat{f}(n)} e_n = a^{\hat{f}(n+1)} e_{n+1}$$

for $1 \leq n < z(1)$, $n \neq y(1)$;

$$T_0 a^{\hat{f}(y(1))} e_{y(1)} = a^{\hat{f}(y(1)+1)} e_{y(1)+1} + a^{\hat{f}(1)} e_1.$$

If we have defined $T_0 e_n$ for $1 \leq n < z(1)$, then let

$$T_0 a^{\hat{f}(z(1))} e_{z(1)} = -T_0 \left(T_0^{z(1)-1} a^{\hat{f}(1)} e_1 - a^{\hat{f}(z(1))} e_{z(1)} \right) + a^{\hat{f}(z(1)+1)} e_{z(1)+1}.$$

Let $k \in \mathbb{N}$, $k \geq 2$. Suppose that we have defined $T_0 e_n$ for $1 \leq n \leq z(k-1)$. Then we define $T_0 e_n$ for $z(k-1) < n \leq z(k)$.

Let

$$T_0 a^{\hat{f}(n)} e_n = a^{\hat{f}(n+1)} e_{n+1}$$

for $z(k-1) < n < z(k)$, $n \neq y(k)$;

$$T_0 a^{\hat{f}(y(k))} e_{y(k)} = a^{\hat{f}(y(k)+1)} e_{y(k)+1} + a^{\hat{f}(1)} e_1.$$

Suppose that we have defined $T_0 e_n$ for $1 \leq n < z(k)$, then let

$$T_0 a^{\hat{f}(z(k))} e_{z(k)} = -T_0 \left(T_0^{z(k)-1} a^{\hat{f}(1)} e_1 - a^{\hat{f}(z(k))} e_{z(k)} \right) + a^{\hat{f}(z(k)+1)} e_{z(k)+1}.$$

The non-Archimedean Köthe space $\Lambda(C)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, f$ and the operator T_0 are defined such that they satisfy Assumptions 1.1, 1.4 (1), (2) and conditions (1), (4), (5) of Theorem 2.2, so it remains to check conditions (2), (3) of Theorem 2.2.

We use the fact that

$$f(n) \geq f(m)$$

for $n, m \in \mathbb{N}$, $n \leq m$.

Let $k \in \mathbb{N}$, $j \geq \psi(k)$, $1 \leq i \leq \varphi(k)$.

- If $[j, j+i] \subset [\psi(k), \lambda(k)]$, then

$$\begin{aligned} \gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)} &\geq \gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)+1} = \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)+1} + \xi_j - \xi_{j+i} \\ &= \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)+1} + \vartheta_j - \vartheta_{j+i} = \alpha_j^{\mu(k)+2} + \vartheta_j - \alpha_{j+i}^{\mu(k)+1} - \vartheta_{j+i} \geq 0, \end{aligned}$$

thus

$$\gamma_j^{\mu(k)+2} + f(j) \geq \gamma_{j+i}^{\mu(k)} + f(j+i).$$

- If $[j, j+i] \subset [y(m), \lambda(m)]$ for some $m > k$, then

$$\begin{aligned} \gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)} &\geq \gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)+1} = \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)+1} + \xi_j - \xi_{j+i} \\ &= \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)+1} + \vartheta_j - \vartheta_{j+i} = \alpha_j^{\mu(k)+2} + \vartheta_j - \alpha_{j+i}^{\mu(k)+1} - \vartheta_{j+i} \geq 0, \end{aligned}$$

thus

$$\gamma_j^{\mu(k)+2} + f(j) \geq \gamma_{j+i}^{\mu(k)} + f(j+i).$$

- If $[j, j+i] \subset [\lambda(m), y(m+1)]$ for some $m \geq k$, then

$$\begin{aligned} \gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)} &\geq \gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)+1} = \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)+1} + \xi_j - \xi_{j+i} \\ &= \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)+1} + \zeta_j - \zeta_{j+i} = \alpha_j^{\mu(k)+2} + \zeta_j - \alpha_{j+i}^{\mu(k)+1} - \zeta_{j+i} \geq 0, \end{aligned}$$

thus

$$\gamma_j^{\mu(k)+2} + f(j) \geq \gamma_{j+i}^{\mu(k)} + f(j+i).$$

- If $\lambda(k) \in (j, j+i)$, we have $j \geq \psi(k)$ and $j+i < y(k+1)$, thus

$$\begin{aligned}
\gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)} &= \gamma_j^{\mu(k)+2} - \gamma_{\lambda(k)}^{\mu(k)+1} + \gamma_{\lambda(k)}^{\mu(k)+1} - \gamma_{j+i}^{\mu(k)} \\
&= \alpha_j^{\mu(k)+2} - \alpha_{\lambda(k)}^{\mu(k)+1} + \xi_j - \xi_{\lambda(k)} + \alpha_{\lambda(k)}^{\mu(k)+1} - \alpha_{j+i}^{\mu(k)} + \xi_{\lambda(k)} - \xi_{j+i} \\
&= \alpha_j^{\mu(k)+2} - \alpha_{\lambda(k)}^{\mu(k)+1} + \vartheta_j - \vartheta_{\lambda(k)} + \alpha_{\lambda(k)}^{\mu(k)+1} - \alpha_{j+i}^{\mu(k)} + \zeta_{\lambda(k)} - \zeta_{j+i} \\
&= \alpha_j^{\mu(k)+2} + \vartheta_j - \alpha_{\lambda(k)}^{\mu(k)+1} - \vartheta_{\lambda(k)} + \alpha_{\lambda(k)}^{\mu(k)+1} + \zeta_{\lambda(k)} - \alpha_{j+i}^{\mu(k)} - \zeta_{j+i} \geq 0
\end{aligned}$$

and

$$\gamma_j^{\mu(k)+2} + f(j) \geq \gamma_{j+i}^{\mu(k)} + f(j+i).$$

- If $\lambda(m) \in (j, j+i)$ for some $m > k$, then $j > y(m)$ and $j+i < y(m+1)$, thus

$$\begin{aligned}
\gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)} &= \gamma_j^{\mu(k)+2} - \gamma_{\lambda(m)}^{\mu(k)+1} + \gamma_{\lambda(m)}^{\mu(k)+1} - \gamma_{j+i}^{\mu(k)} \\
&= \alpha_j^{\mu(k)+2} - \alpha_{\lambda(m)}^{\mu(k)+1} + \xi_j - \xi_{\lambda(m)} + \alpha_{\lambda(m)}^{\mu(k)+1} - \alpha_{j+i}^{\mu(k)} + \xi_{\lambda(m)} - \xi_{j+i} \\
&= \alpha_j^{\mu(k)+2} - \alpha_{\lambda(m)}^{\mu(k)+1} + \vartheta_j - \vartheta_{\lambda(m)} + \alpha_{\lambda(m)}^{\mu(k)+1} - \alpha_{j+i}^{\mu(k)} + \zeta_{\lambda(m)} - \zeta_{j+i} \\
&= \alpha_j^{\mu(k)+2} + \vartheta_j - \alpha_{\lambda(m)}^{\mu(k)+1} - \vartheta_{\lambda(m)} + \alpha_{\lambda(m)}^{\mu(k)+1} + \zeta_{\lambda(m)} - \alpha_{j+i}^{\mu(k)} - \zeta_{j+i} \geq 0
\end{aligned}$$

and

$$\gamma_j^{\mu(k)+2} + f(j) \geq \gamma_{j+i}^{\mu(k)} + f(j+i).$$

- If $y(m) \in (j, j+i)$ for some $m > k$, then $j > \lambda(m-1)$ and $j+i < \lambda(m)$, thus

$$\begin{aligned}
\gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)} &= \gamma_j^{\mu(k)+2} - \gamma_{y(m)}^{\mu(k)+1} + \gamma_{y(m)}^{\mu(k)+1} - \gamma_{j+i}^{\mu(k)} \\
&= \alpha_j^{\mu(k)+2} - \alpha_{y(m)}^{\mu(k)+1} + \xi_j - \xi_{y(m)} + \alpha_{y(m)}^{\mu(k)+1} - \alpha_{j+i}^{\mu(k)} + \xi_{y(m)} - \xi_{j+i} \\
&= \alpha_j^{\mu(k)+2} - \alpha_{y(m)}^{\mu(k)+1} + \zeta_j - \zeta_{y(m)} + \alpha_{y(m)}^{\mu(k)+1} - \alpha_{j+i}^{\mu(k)} + \vartheta_{y(m)} - \vartheta_{j+i} \\
&= \alpha_j^{\mu(k)+2} + \zeta_j - \alpha_{y(m)}^{\mu(k)+1} - \zeta_{y(m)} + \alpha_{y(m)}^{\mu(k)+1} + \vartheta_{y(m)} - \alpha_{j+i}^{\mu(k)} - \vartheta_{j+i} \geq 0
\end{aligned}$$

and

$$\gamma_j^{\mu(k)+2} + f(j) \geq \gamma_{j+i}^{\mu(k)} + f(j+i).$$

Hence condition (2) of Theorem 2.2 is satisfied.

Now let $k \in \mathbb{N}$, $j \geq \psi(k)$, $1 \leq i < \varphi(k)$.

- If $y(m) = j + i$ for some $m > k$, then $j > \lambda(m - 1)$, thus

$$\begin{aligned} \gamma_j^{\mu(k)+2} - \gamma_{y(m)}^{\mu(k)+1} &= \alpha_j^{\mu(k)+2} - \alpha_{y(m)}^{\mu(k)+1} + \xi_j - \xi_{y(m)} \\ &= \alpha_j^{\mu(k)+2} - \alpha_{y(m)}^{\mu(k)+1} + \zeta_j - \zeta_{y(m)} = \alpha_j^{\mu(k)+2} + \zeta_j - \alpha_{y(m)}^{\mu(k)+1} - \zeta_{y(m)} \geq 0, \end{aligned}$$

so

$$\gamma_j^{\mu(k)+2} + f(j) \geq \gamma_{y(m)}^{\mu(k)+1} + f(y(m)).$$

- If $z(k) = j + i$, we have $j \geq \psi(k)$, thus

$$\begin{aligned} \gamma_j^{\mu(k)+2} - \gamma_{z(k)}^{\mu(k)+1} &= \alpha_j^{\mu(k)+2} - \alpha_{z(k)}^{\mu(k)+1} + \xi_j - \xi_{z(k)} \\ &= \alpha_j^{\mu(k)+2} - \alpha_{z(k)}^{\mu(k)+1} + \vartheta_j - \vartheta_{z(k)} = \alpha_j^{\mu(k)+2} + \vartheta_j - \alpha_{z(k)}^{\mu(k)+1} - \vartheta_{z(k)} \geq 0, \end{aligned}$$

so

$$\gamma_j^{\mu(k)+2} + f(j) \geq \gamma_{z(k)}^{\mu(k)+1} + f(z(k)).$$

- If $z(m) = j + i$ for some $m > k$, then $j > y(m)$, thus

$$\begin{aligned} \gamma_j^{\mu(k)+2} - \gamma_{z(m)}^{\mu(k)+1} &= \alpha_j^{\mu(k)+2} - \alpha_{z(m)}^{\mu(k)+1} + \xi_j - \xi_{z(m)} \\ &= \alpha_j^{\mu(k)+2} - \alpha_{z(m)}^{\mu(k)+1} + \vartheta_j - \vartheta_{z(m)} = \alpha_j^{\mu(k)+2} + \vartheta_j - \alpha_{z(m)}^{\mu(k)+1} - \vartheta_{z(m)} \geq 0, \end{aligned}$$

so

$$\gamma_j^{\mu(k)+2} + f(j) \geq \gamma_{z(m)}^{\mu(k)+1} + f(z(m)).$$

Hence condition (3) of Theorem 2.2 is satisfied.

Therefore the operator T_0 extends to a linear and continuous operator

$$\hat{T}: \Lambda(C) \rightarrow \Lambda(C)$$

that has no nontrivial invariant subspaces.

Since $\Lambda(A)$ is isomorphic to $\Lambda(C)$, there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. \square

Theorem 3.5. *Let $\Lambda(A)$, where $A = \left(|a|^{\alpha_n^k}\right)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space.*

Assume that:

(1) There exists a sequence $(\zeta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that:

(a) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\alpha_n^{k+1} + \zeta_n \geq \alpha_{n+i}^k + \zeta_{n+i}$$

for $n \geq j$.

(b) For every $l, m \in \mathbb{N}$ and $M \in \mathbb{R}$, $M \geq 0$ there exist $u, v \in \mathbb{N}$ such that:

(i)

$$l \leq u \leq v,$$

(ii)

$$\alpha_n^m + \zeta_n \leq \alpha_u^m + \zeta_u$$

for $1 \leq n \leq u$,

(iii)

$$\alpha_u^m + \zeta_u \leq \alpha_n^{m+1} + \zeta_n$$

for $u \leq n \leq v$,

(iv)

$$\alpha_v^1 + \zeta_v - \alpha_u^m - \zeta_u \geq M.$$

(2) There exists a sequence $(\vartheta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that:

(a) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\alpha_n^{k+1} + \vartheta_n \geq \alpha_{n+i}^k + \vartheta_{n+i}$$

for $n \geq j$.

(b) For every $k, l \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq l$ such that

$$\alpha_n^i + \vartheta_n \leq \alpha_m^{i+1} + \vartheta_m$$

for $1 \leq i \leq k$, $1 \leq n \leq m$.

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. We show that conditions of Theorem 3.4 are satisfied.

Condition (1)(a) of Theorem 3.4 is satisfied by (1)(a).

Let $l, m \in \mathbb{N}$ and $M \in \mathbb{R}$, $M \geq 0$ and let $r \in \mathbb{N}$, $r \geq l$. By (1)(a), there exists $j \in \mathbb{N}$, $j \geq r$ such that

$$\alpha_n^{m+1} + \zeta_n \geq \alpha_{n+i}^m + \zeta_{n+i}$$

for $n > j$, $1 \leq i \leq r$.

Then, by (1)(b), there exist $u, v \in \mathbb{N}$ such that

$$j + r \leq u \leq v,$$

conditions (ii)-(iv) of (1)(b) are satisfied, and we can assume that $\alpha_u^m + \zeta_u \geq 0$. Let $s, t \in \mathbb{N}$ be such that

$$s + r = u$$

and

$$v = t.$$

Then conditions (i)-(vii) of (1)(b) of Theorem 3.4 are satisfied if we set $\alpha: [s + 1, s + r] \rightarrow \mathbb{R}$, $n \mapsto \alpha_{s+r}^m + \zeta_{s+r}$.

Hence condition (1) of Theorem 3.4 is satisfied by (1). And condition (2) of Theorem 3.4 is satisfied by (2).

Therefore there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. □

3.3 Limited coefficients from below

Theorem 3.6. *Let $\Lambda(A)$, where $A = \left(|a|^{\alpha_n^k}\right)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space.*

Assume that:

- (1) *For every $k, l, m \in \mathbb{N}$ and $M \in \mathbb{R}$, $M \geq 0$ there exist $r, s, t \in \mathbb{N}$, a finite sequence $(\zeta_n)_{r \leq n \leq t} \subset \mathbb{R}$ and a function $\alpha: [s + 1, s + r] \rightarrow \mathbb{R}$ such that:*

(a)

$$l \leq r \leq s < s + r \leq t.$$

(b) (i) If $[n, n+i] \subset [r, t]$ and $1 \leq i \leq l$, then

$$\alpha_n^{j+2} + \zeta_n \geq \alpha_{n+i}^j + \zeta_{n+i}$$

for $1 \leq j \leq k$;

(ii) if $1 \leq i \leq l$, then

$$\alpha_r^{j+1} + \zeta_r \geq \alpha_{r+i}^j + \zeta_{r+i}$$

for $1 \leq j \leq k$;

(iii) if $1 \leq t-n \leq l$, then

$$\alpha_n^{j+2} + \zeta_n \geq \alpha_t^{j+1} + \zeta_t$$

for $1 \leq j \leq k$.

(c) (i)

$$\alpha_n^m + \zeta_n \leq \alpha(s+1)$$

for $r \leq n \leq s$,

(ii)

$$\alpha(n) \leq \alpha(n+1)$$

if $s < n < s+r$,

(iii)

$$\alpha_n^m + \zeta_n \leq \alpha(n) \leq \alpha_n^{m+1} + \zeta_n$$

for $s < n \leq s+r$,

(iv)

$$\alpha(s+r) \leq \alpha_n^{m+1} + \zeta_n$$

if $s+r < n \leq t$,

(v)

$$\alpha_t^1 + \zeta_t - \alpha(s+r) \geq M.$$

(2) There exists a sequence $(\vartheta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that:

(a) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\alpha_n^{k+1} + \vartheta_n \geq \alpha_{n+i}^k + \vartheta_{n+i}$$

for $n \geq j$.

(b) For every $l, k, i \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq l$, $m > i$ such that

$$\alpha_n^j + \vartheta_n \leq \alpha_m^{j+1} + \vartheta_m$$

for $1 \leq j \leq k$, $m - i \leq n < m$.

(c)

$$\inf_{n \in \mathbb{N}} (\alpha_n^1 + \vartheta_n) > -\infty.$$

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. Let $\Lambda(B)$ be a non-Archimedean Köthe space such that $B = \left(|a|^{\beta_n^k}\right)_{k,n \in \mathbb{N}}$, $\beta_n^k = \alpha_n^k + k$ for $k, n \in \mathbb{N}$ and let $\left(\hat{\vartheta}_n\right)_{n \in \mathbb{N}}$ be a sequence such that $\hat{\vartheta}_n = \vartheta_n + h(n)$ for $n \in \mathbb{N}$, where $h: \mathbb{N} \rightarrow \mathbb{N}$ is a function such that $h(n) = m$ for $n, m \in \mathbb{N}$, $2^{m-1} \leq n < 2^m$.

We have

$$\beta_n^k + 1 = \alpha_n^k + k + 1 \leq \alpha_n^{k+1} + k + 1 = \beta_n^{k+1} \quad (3.6)$$

for $k, n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \left(\beta_n^1 + \hat{\vartheta}_n\right) = \lim_{n \rightarrow \infty} \left(\alpha_n^1 + 1 + \vartheta_n + h(n)\right) = \infty. \quad (3.7)$$

Hence $\left(\beta_n^k\right)_{k,n \in \mathbb{N}}$ satisfies condition (3.6), assumption (1), and $\left(\beta_n^k\right)_{k,n \in \mathbb{N}}$ and the sequence $\left(\hat{\vartheta}_n\right)_{n \in \mathbb{N}}$ satisfy conditions (2)(a), (2)(b) and the additional condition (3.7).

Since $\Lambda(B)$ is isomorphic to $\Lambda(A)$, we can assume that

$$\alpha_n^k + 1 \leq \alpha_n^{k+1}$$

for $k, n \in \mathbb{N}$ and that conditions (1) and (2) are satisfied with the change of (2)(c) on

(2) (c')

$$\lim_{n \rightarrow \infty} (\alpha_n^1 + \vartheta_n) = \infty.$$

Now we define a non-Archimedean Köthe space $\Lambda(C)$, where $C = \left(|a|^{\gamma_n^k}\right)_{k,n \in \mathbb{N}}$, $\gamma_n^k = \alpha_n^k + \xi_n$ for $k, n \in \mathbb{N}$, $(\xi_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, isomorphic to $\Lambda(A)$, functions $\mu, x, y, z, \chi, \psi, \varphi, f$, function ν , sets X_k for $k \in \mathbb{N} \setminus \{1\}$, functions ν_k for $k \in \mathbb{N} \setminus \{1\}$ and an operator T_0 that satisfy assumptions of Theorem 2.4. And additionally, we define functions $\alpha_k: [\chi(k) + 1, \chi(k) + x(k)] \rightarrow \mathbb{R}$ for $k \in \mathbb{N}$.

Let

$$f \equiv 0,$$

and let

$$\mu: \mathbb{N} \rightarrow \mathbb{N}$$

be a function that takes any natural number infinitely many times.

The sequence $(\xi_n)_{n \in \mathbb{N}}$, the functions $x, y, z, \chi, \psi, \varphi$, the operator T_0 , the function ν , the sets X_k for $k \in \mathbb{N} \setminus \{1\}$, the functions ν_k for $k \in \mathbb{N} \setminus \{1\}$ and the functions α_k for $k \in \mathbb{N}$ are defined by inductions.

By (1), there exist $x(1), \chi(1), y(1), (\zeta_n)_{x(1) \leq n \leq y(1)} \subset \mathbb{R}$ and α_1 such that

$$x(1) \leq \chi(1) < \chi(1) + x(1) = y(1),$$

$$\alpha_n^{\mu(1)} + \zeta_n \leq \alpha_1(\chi(1) + 1)$$

if $x(1) < n \leq \chi(1)$,

$$\alpha_1(n) \leq \alpha_1(n + 1)$$

if $\chi(1) < n < \chi(1) + x(1)$ and

$$\alpha_n^{\mu(1)} + \zeta_n \leq \alpha_1(n) \leq \alpha_n^{\mu(1)+1} + \zeta_n$$

for $\chi(1) < n \leq \chi(1) + x(1)$.

Let

$$\xi_n = \zeta_n$$

for $1 \leq n \leq x(1)$,

$$\xi_n = \zeta_n - \alpha_1(\chi(1) + 1)$$

if $x(1) < n \leq \chi(1)$,

$$\xi_n = \zeta_n - \alpha_1(n)$$

for $\chi(1) < n \leq \chi(1) + x(1)$.

Then

$$\gamma_n^{\mu(1)} \leq 0$$

for $x(1) < n \leq \chi(1) + x(1)$ and

$$\gamma_n^{\mu(1)+1} \geq 0$$

for $\chi(1) < n \leq y(1)$.

Let

$$\varphi(1) = x(1) + y(1) - 1.$$

By (2)(a), there exists $\psi(1)$, $\psi(1) > y(1)$ such that

$$\alpha_n^{\mu(1)+2} + \vartheta_n \geq \alpha_{n+i}^{\mu(1)+1} + \vartheta_{n+i}$$

and

$$\alpha_n^{\mu(1)+1} + \vartheta_n \geq \alpha_{n+i}^{\mu(1)} + \vartheta_{n+i}$$

for $n \geq \psi(1)$, $1 \leq i \leq \varphi(1)$.

Let

$$\xi_{y(1)+1} = -\alpha_{y(1)+1}^{\mu(1)+2},$$

$$\xi_{n+1} = \xi_n - \vartheta_n + \vartheta_{n+1} + \min\left(0, \alpha_n^{\mu(1)+2} + \vartheta_n - \alpha_{n+1}^{\mu(1)+2} - \vartheta_{n+1}\right)$$

if $y(1) < n < \psi(1)$.

Then

$$\gamma_{y(1)+1}^{\mu(1)+2} \leq 0$$

and

$$\gamma_n^{\mu(1)+2} \geq \gamma_{n+1}^{\mu(1)+2}$$

if $y(1) < n < \psi(1)$.

By (2)(c'), there exists $z(1)$, $z(1) > \psi(1)$ such that

$$z(1) \geq \varphi(1),$$

$$\gamma_n^{\mu(1)} \leq \alpha_{z(1)}^{\mu(1)+1} + \xi_{\psi(1)} - \vartheta_{\psi(1)} + \vartheta_{z(1)}$$

for $1 \leq n \leq \psi(1)$ and

$$\alpha_n^{\mu(1)} + \vartheta_n \leq \alpha_{z(1)}^{\mu(1)+1} + \vartheta_{z(1)}$$

for $\psi(1) < n \leq z(1)$.

Let

$$\xi_n = \xi_{\psi(1)} - \vartheta_{\psi(1)} + \vartheta_n$$

for $\psi(1) < n \leq z(1)$.

Then

$$\gamma_n^{\mu(1)} \leq \gamma_{z(1)}^{\mu(1)+1}$$

for $1 \leq n \leq z(1)$.

Now we define $T_0 e_n$ for $1 \leq n \leq z(1)$.

Let

$$T_0 e_n = e_{n+1}$$

for $1 \leq n < z(1)$, $n \neq y(1)$;

$$T_0 e_{y(1)} = e_{y(1)+1} + e_1.$$

If we have defined $T_0 e_n$ for $1 \leq n < z(1)$, then let

$$T_0 e_{z(1)} = -T_0 \left(T_0^{z(1)-1} e_1 - e_{z(1)} \right) + e_{z(1)+1}.$$

Let $k \in \mathbb{N}$, $k \geq 2$. Suppose that we have defined $x(m), y(m), z(m), \chi(m), \psi(m), \varphi(m), (\zeta_n)_{x(m) \leq n \leq y(m)}, \alpha_m$ for $1 \leq m < k$, $\nu(m), X_m, v_m$ for $2 \leq m < k$ if $k > 2$ and $\xi_i, T_0 e_i$ for $1 \leq i \leq z(k-1)$. Then we define $x(k), y(k), z(k), \chi(k), \psi(k), \varphi(k), (\zeta_n)_{x(k) \leq n \leq y(k)}, \alpha_k, \nu(k), X_k, v_k$ and $\xi_i, T_0 e_i$ for $z(k-1) < i \leq z(k)$.

By (1) and (2)(c'), there exist $x(k), \chi(k), y(k), (\zeta_n)_{x(k) \leq n \leq y(k)}, \alpha_k$ and ξ_i for $z(k-1) < i \leq y(k)$ such that

$$z(k-1) \leq x(k) \leq \chi(k) < \chi(k) + x(k) \leq y(k);$$

$$\alpha_{x(k)}^{\mu(k)} + \vartheta_{x(k)} \geq 0;$$

$$\alpha_n^{\mu(j)+2} + \zeta_n \geq \alpha_{n+i}^{\mu(j)} + \zeta_{n+i}$$

for $1 \leq j < k$, $[n, n+i] \subset [x(k), y(k)]$, $1 \leq i \leq \varphi(j)$;

$$\alpha_{x(k)}^{\mu(j)+1} + \zeta_{x(k)} \geq \alpha_{x(k)+i}^{\mu(j)} + \zeta_{x(k)+i}$$

for $1 \leq j < k$, $1 \leq i \leq \varphi(j)$;

$$\alpha_n^{\mu(j)+2} + \zeta_n \geq \alpha_{y(k)}^{\mu(j)+1} + \zeta_{y(k)}$$

for $1 \leq j < k$, $1 \leq y(k) - n \leq \varphi(j)$;

$$\alpha_n^{\mu(k)} + \zeta_n \leq \alpha_k(\chi(k) + 1)$$

for $x(k) \leq n \leq \chi(k)$;

$$\alpha_k(n) \leq \alpha_k(n+1)$$

for $\chi(k) < n < \chi(k) + x(k)$;

$$\alpha_n^{\mu(k)} + \zeta_n \leq \alpha_k(n) \leq \alpha_n^{\mu(k)+1} + \zeta_n$$

for $\chi(k) < n \leq \chi(k) + x(k)$;

$$\alpha_k(\chi(k) + x(k)) \leq \alpha_n^{\mu(k)+1} + \zeta_n$$

if $\chi(k) + x(k) < n \leq y(k)$;

$$\gamma_n^{\mu(j)} \leq \alpha_{y(k)}^{\mu(j)+1} + \zeta_{y(k)} - \alpha_k(\chi(k) + x(k))$$

for $1 \leq j < k$, $1 \leq n \leq \varphi(j)$;

$$\xi_n = \xi_{z(k-1)} - \vartheta_{z(k-1)} + \vartheta_n$$

if $z(k-1) < n \leq x(k)$;

$$\xi_n = \zeta_n - \alpha_k(\chi(k) + 1)$$

if $x(k) < n \leq \chi(k)$;

$$\xi_n = \zeta_n - \alpha_k(n)$$

for $\chi(k) < n \leq \chi(k) + x(k)$;

$$\xi_n = \zeta_n - \alpha_k(\chi(k) + x(k))$$

if $\chi(k) + x(k) < n \leq y(k)$.

Such $x(k), \chi(k), y(k), (\zeta_n)_{x(k) \leq n \leq y(k)}, \alpha_k$ and ξ_i for $z(k-1) < i \leq y(k)$ exist by (1) and the fact that if

$$\alpha_{y(k)}^1 + \zeta_{y(k)} - \alpha_k(\chi(k) + x(k)) \geq \max_{\substack{1 \leq j < k \\ 1 \leq i \leq \varphi(j)}} \gamma_i^{\mu(j)},$$

then

$$\alpha_{y(k)}^{\mu(j)+1} + \zeta_{y(k)} - \alpha_k(\chi(k) + x(k)) \geq \alpha_{y(k)}^1 + \zeta_{y(k)} - \alpha_k(\chi(k) + x(k)) \geq \gamma_i^{\mu(j)}$$

for $1 \leq j < k$, $1 \leq i \leq \varphi(j)$.

Hence

$$\gamma_n^{\mu(k)} \leq 0$$

for $x(k) < n \leq \chi(k) + x(k)$,

$$\gamma_n^{\mu(k)+1} \geq 0$$

for $\chi(k) < n \leq y(k)$ and

$$\gamma_i^{\mu(j)} \leq \gamma_{y(k)}^{\mu(j)+1}$$

for $1 \leq j < k$, $1 \leq i \leq \varphi(j)$.

Let

$$\varphi(k) = x(k) + y(k) - 1.$$

By (2)(a), there exists $\psi(k)$, $\psi(k) > y(k)$ such that

$$\alpha_n^{\mu(k)+2} + \vartheta_n \geq \alpha_n^{\mu(k)+1} + \vartheta_{n+i}$$

and

$$\alpha_n^{\mu(k)+1} + \vartheta_n \geq \alpha_n^{\mu(k)} + \vartheta_{n+i}$$

for $n \geq \psi(k)$, $1 \leq i \leq \varphi(k)$.

Let

$$\begin{aligned} \xi_{y(k)+1} &= \xi_{y(k)} - \vartheta_{y(k)} + \vartheta_{y(k)+1} + \min \left(0, -\alpha_{y(k)+1}^{\mu(k)+2} - \xi_{y(k)} \right. \\ &\quad \left. + \vartheta_{y(k)} - \vartheta_{y(k)+1} \right), \end{aligned}$$

$$\xi_{n+1} = \xi_n - \vartheta_n + \vartheta_{n+1} + \min \left(0, \alpha_n^{\mu(k)+2} + \vartheta_n - \alpha_{n+1}^{\mu(k)+2} - \vartheta_{n+1} \right)$$

if $y(k) < n < \psi(k)$.

Hence

$$\gamma_{y(k)+1}^{\mu(k)+2} \leq 0$$

and

$$\gamma_n^{\mu(k)+2} \geq \gamma_{n+1}^{\mu(k)+2}$$

if $y(k) < n < \psi(k)$.

Let $\nu(k)$, X_k , v_k , $z(k)$ and ξ_n for $\psi(k) < n \leq z(k)$ be such that

$$\psi(k) \leq \nu(k)y(k) < (\nu(k) + 1)y(k) < z(k);$$

$$\begin{aligned} X_k &= \{m \geq \nu(k) : my(k) < z(k), m = 0 \pmod{\nu(k)} \text{ or} \\ &\quad m = 1 \pmod{\nu(k)}\}; \end{aligned}$$

$$v_k : X_k \rightarrow \mathbb{N}, \quad v_k : m \mapsto my(k);$$

$$\xi_n = \xi_{\psi(k)} - \vartheta_{\psi(k)} + \vartheta_n$$

for $\psi(k) < n \leq z(k)$;

$$\gamma_i^{\mu(j)} \leq \gamma_{v_k(m)}^{\mu(j)+1}$$

for $m \in X_k$, $1 \leq j \leq k$, $1 \leq i \leq \varphi(j)$;

$$\gamma_i^{\mu(j)} \leq \gamma_{z(k)}^{\mu(j)+1}$$

for $1 \leq j \leq k$, $1 \leq i \leq \varphi(j)$;

$$\gamma_i^{\mu(j)} \leq \gamma_{z(k)}^{\mu(j)+1}$$

for $1 \leq j \leq k$, $z(k) - (\nu(k) - 1)y(k) < i < z(k)$.

Such $\nu(k)$, X_k , v_k , $z(k)$ and ξ_i for $\psi(k) < i \leq z(k)$ exist by (2)(b) and (2)(c').

And we define $T_0 e_n$ for $z(k-1) < n \leq z(k)$.

Let

$$T_0 e_n = e_{n+1}$$

if $z(k-1) < n < z(k)$, $n \neq y(k)$ and $n \neq v_k(m)$ for $m \in X_k$;

$$T_0 e_{y(k)} = e_{y(k)+1} + e_1;$$

$$T_0 e_n = -e_1 + e_{n+1}$$

if $n = v_k(m)$ for some $m \in X_k$ and $m = 0 \pmod{\nu(k)}$;

$$T_0 e_n = e_1 + e_{n+1}$$

if $n = v_k(m)$ for some $m \in X_k$ and $m = 1 \pmod{\nu(k)}$.

If we have defined $T_0 e_n$ for $1 \leq n < z(k)$, then let

$$T_0 e_{z(k)} = -T_0 \left(T_0^{z(k)-1} e_1 - e_{z(k)} \right) + e_{z(k)+1}.$$

The non-Archimedean Köthe space $\Lambda(C)$, the functions $\mu, x, y, z, \chi, \psi, \varphi, f$, the function ν , the sets X_k for $k \in \mathbb{N} \setminus \{1\}$, the functions v_k for $k \in \mathbb{N} \setminus \{1\}$ and the operator T_0 are defined so that they satisfy Assumptions 1.1, 1.4 (1), (2) and conditions (1)-(3), (6)-(10) of Theorem 2.4. Thus it remains to check conditions (4), (5) of Theorem 2.4.

Let $k \in \mathbb{N}$, $j \geq \psi(k)$, $1 \leq i \leq \varphi(k)$.

- If $[j, j+i] \subset [\psi(k), x(k+1)]$, then

$$\begin{aligned} \gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)} &\geq \gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)+1} = \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)+1} + \xi_j - \xi_{j+i} \\ &= \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)+1} + \vartheta_j - \vartheta_{j+i} = \alpha_j^{\mu(k)+2} + \vartheta_j - \alpha_{j+i}^{\mu(k)+1} - \vartheta_{j+i} \geq 0. \end{aligned}$$

- If $[j, j+i] \subset [y(m), x(m+1)]$ for some $m > k$, then

$$\gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)} \geq \gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)+1} = \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)+1} + \xi_j - \xi_{j+i}$$

$$\begin{aligned}
&= \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)+1} + \sum_{l=0}^{i-1} (\xi_{j+l} - \xi_{j+l+1}) \geq \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)+1} \\
&\quad + \sum_{l=0}^{i-1} (\vartheta_{j+l} - \vartheta_{j+l+1}) = \alpha_j^{\mu(k)+2} + \vartheta_j - \alpha_{j+i}^{\mu(k)+1} - \vartheta_{j+i} \geq 0.
\end{aligned}$$

- If $[j, j+i] \subset [x(m), y(m)]$ for some $m > k$, then

$$\begin{aligned}
&\gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)} = \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)} + \xi_j - \xi_{j+i} \\
&= \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)} + \sum_{l=0}^{i-1} (\xi_{j+l} - \xi_{j+l+1}) \geq \alpha_j^{\mu(k)+2} - \alpha_{j+i}^{\mu(k)} \\
&\quad + \sum_{l=0}^{i-1} (\zeta_{j+l} - \zeta_{j+l+1}) = \alpha_j^{\mu(k)+2} + \zeta_j - \alpha_{j+i}^{\mu(k)} - \zeta_{j+i} \geq 0.
\end{aligned}$$

- If $x(k+1) \in (j, j+i)$, we have $j \geq \psi(k)$ and $j+i < y(k+1)$, thus

$$\begin{aligned}
&\gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)} = \gamma_j^{\mu(k)+2} - \gamma_{x(k+1)}^{\mu(k)+1} + \gamma_{x(k+1)}^{\mu(k)+1} - \gamma_{j+i}^{\mu(k)} \\
&= \alpha_j^{\mu(k)+2} - \alpha_{x(k+1)}^{\mu(k)+1} + \xi_j - \xi_{x(k+1)} + \alpha_{x(k+1)}^{\mu(k)+1} - \alpha_{j+i}^{\mu(k)} + \xi_{x(k+1)} - \xi_{j+i} \\
&\geq \alpha_j^{\mu(k)+2} - \alpha_{x(k+1)}^{\mu(k)+1} + \vartheta_j - \vartheta_{x(k+1)} + \alpha_{x(k+1)}^{\mu(k)+1} - \alpha_{j+i}^{\mu(k)} + \zeta_{x(k+1)} - \zeta_{j+i} \\
&= \alpha_j^{\mu(k)+2} + \vartheta_j - \alpha_{x(k+1)}^{\mu(k)+1} - \vartheta_{x(k+1)} + \alpha_{x(k+1)}^{\mu(k)+1} + \zeta_{x(k+1)} - \alpha_{j+i}^{\mu(k)} - \zeta_{j+i} \geq 0.
\end{aligned}$$

- If $x(m) \in (j, j+i)$ for some $m > k+1$, then $j > y(m-1)$ and $j+i < y(m)$, thus

$$\begin{aligned}
&\gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)} = \gamma_j^{\mu(k)+2} - \gamma_{x(m)}^{\mu(k)+1} + \gamma_{x(m)}^{\mu(k)+1} - \gamma_{j+i}^{\mu(k)} \\
&= \alpha_j^{\mu(k)+2} - \alpha_{x(m)}^{\mu(k)+1} + \xi_j - \xi_{x(m)} + \alpha_{x(m)}^{\mu(k)+1} - \alpha_{j+i}^{\mu(k)} + \xi_{x(m)} - \xi_{j+i} \\
&\geq \alpha_j^{\mu(k)+2} - \alpha_{x(m)}^{\mu(k)+1} + \vartheta_j - \vartheta_{x(m)} + \alpha_{x(m)}^{\mu(k)+1} - \alpha_{j+i}^{\mu(k)} + \zeta_{x(m)} - \zeta_{j+i} \\
&= \alpha_j^{\mu(k)+2} + \vartheta_j - \alpha_{x(m)}^{\mu(k)+1} - \vartheta_{x(m)} + \alpha_{x(m)}^{\mu(k)+1} + \zeta_{x(m)} - \alpha_{j+i}^{\mu(k)} - \zeta_{j+i} \geq 0.
\end{aligned}$$

- If $y(m) \in (j, j+i)$ for some $m > k$, then $j > x(m)$ and $j+i \leq x(m+1)$, thus

$$\begin{aligned}
\gamma_j^{\mu(k)+2} - \gamma_{j+i}^{\mu(k)} &= \gamma_j^{\mu(k)+2} - \gamma_{y(m)}^{\mu(k)+1} + \gamma_{y(m)}^{\mu(k)+1} - \gamma_{j+i}^{\mu(k)} \\
&= \alpha_j^{\mu(k)+2} - \alpha_{y(m)}^{\mu(k)+1} + \xi_j - \xi_{y(m)} + \alpha_{y(m)}^{\mu(k)+1} - \alpha_{j+i}^{\mu(k)} + \xi_{y(m)} - \xi_{j+i} \\
&\geq \alpha_j^{\mu(k)+2} - \alpha_{y(m)}^{\mu(k)+1} + \zeta_j - \zeta_{y(m)} + \alpha_{y(m)}^{\mu(k)+1} - \alpha_{j+i}^{\mu(k)} + \vartheta_{y(m)} - \vartheta_{j+i} \\
&= \alpha_j^{\mu(k)+2} + \zeta_j - \alpha_{y(m)}^{\mu(k)+1} - \zeta_{y(m)} + \alpha_{y(m)}^{\mu(k)+1} + \vartheta_{y(m)} - \alpha_{j+i}^{\mu(k)} - \vartheta_{j+i} \geq 0.
\end{aligned}$$

Hence condition (4) of Theorem 2.4 is satisfied.

Now assume that $k \in \mathbb{N}$, $j \geq \psi(k)$, $1 \leq i < \varphi(k)$.

- If $y(m) = j + i$ for some $m > k$, then $j > x(m)$, thus

$$\begin{aligned}
\gamma_j^{\mu(k)+2} - \gamma_{y(m)}^{\mu(k)+1} &= \alpha_j^{\mu(k)+2} - \alpha_{y(m)}^{\mu(k)+1} + \xi_j - \xi_{y(m)} \\
&\geq \alpha_j^{\mu(k)+2} - \alpha_{y(m)}^{\mu(k)+1} + \zeta_j - \zeta_{y(m)} = \alpha_j^{\mu(k)+2} + \zeta_j - \alpha_{y(m)}^{\mu(k)+1} - \zeta_{y(m)} \geq 0.
\end{aligned}$$

- If $z(m) = j + i$ for some $m \geq k$, then $j \geq \psi(m)$, thus

$$\begin{aligned}
\gamma_j^{\mu(k)+2} - \gamma_{z(m)}^{\mu(k)+1} &= \alpha_j^{\mu(k)+2} - \alpha_{z(m)}^{\mu(k)+1} + \xi_j - \xi_{z(m)} \\
&= \alpha_j^{\mu(k)+2} - \alpha_{z(m)}^{\mu(k)+1} + \vartheta_j - \vartheta_{z(m)} = \alpha_j^{\mu(k)+2} + \vartheta_j - \alpha_{z(m)}^{\mu(k)+1} - \vartheta_{z(m)} \geq 0.
\end{aligned}$$

- If $v_k(r) = j + i$ for some $r \in X_k$, we have $j \geq \psi(k)$, thus

$$\begin{aligned}
\gamma_j^{\mu(k)+2} - \gamma_{v_k(r)}^{\mu(k)+1} &= \alpha_j^{\mu(k)+2} - \alpha_{v_k(r)}^{\mu(k)+1} + \xi_j - \xi_{v_k(r)} \\
&= \alpha_j^{\mu(k)+2} - \alpha_{v_k(r)}^{\mu(k)+1} + \vartheta_j - \vartheta_{v_k(r)} = \alpha_j^{\mu(k)+2} + \vartheta_j - \alpha_{v_k(r)}^{\mu(k)+1} - \vartheta_{v_k(r)} \geq 0.
\end{aligned}$$

- If $v_m(r) = j + i$ for some $r \in X_m$, $m > k$, then $j > y(m)$, thus

$$\begin{aligned}
\gamma_j^{\mu(k)+2} - \gamma_{v_m(r)}^{\mu(k)+1} &= \alpha_j^{\mu(k)+2} - \alpha_{v_m(r)}^{\mu(k)+1} + \xi_j - \xi_{v_m(r)} \\
&\geq \alpha_j^{\mu(k)+2} - \alpha_{v_m(r)}^{\mu(k)+1} + \vartheta_j - \vartheta_{v_m(r)} = \alpha_j^{\mu(k)+2} + \vartheta_j - \alpha_{v_m(r)}^{\mu(k)+1} - \vartheta_{v_m(r)} \geq 0.
\end{aligned}$$

Hence condition (5) of Theorem 2.4 is satisfied.

Therefore the operator T_0 extends to a linear and continuous operator

$$\hat{T}: \Lambda(C) \rightarrow \Lambda(C)$$

that has no nontrivial invariant subspaces.

Since $\Lambda(A)$ is isomorphic with $\Lambda(C)$, there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. \square

Theorem 3.7. *Let $\Lambda(A)$, where $A = \left(|a|^{\alpha_n^k}\right)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space.*

Assume that:

(1) *There exists a sequence $(\zeta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that:*

(a) *For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that*

$$\alpha_n^{k+1} + \zeta_n \geq \alpha_{n+i}^k + \zeta_{n+i}$$

for $n \geq j$.

(b) *For every $l, m \in \mathbb{N}$ and $M \in \mathbb{R}$, $M \geq 0$ there exist $r, s, t \in \mathbb{N}$ and a function $\alpha: [s+1, s+r] \rightarrow \mathbb{R}$ such that:*

(i)

$$l \leq r \leq s < s+r \leq t,$$

(ii)

$$\alpha_n^m + \zeta_n \leq \alpha(s+1)$$

for $r \leq n \leq s$,

(iii)

$$\alpha(n) \leq \alpha(n+1)$$

if $s < n < s+r$,

(iv)

$$\alpha_n^m + \zeta_n \leq \alpha(n) \leq \alpha_n^{m+1} + \zeta_n$$

for $s < n \leq s+r$,

(v)

$$\alpha(s+r) \leq \alpha_n^{m+1} + \zeta_n$$

if $s+r < n \leq t$,

(vi)

$$\alpha_t^1 + \zeta_t - \alpha(s+r) \geq M.$$

(2) There exists a sequence $(\vartheta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that:

(a) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\alpha_n^{k+1} + \vartheta_n \geq \alpha_{n+i}^k + \vartheta_{n+i}$$

for $n \geq j$.

(b) For every $l, k, i \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq l$, $m > i$ such that

$$\alpha_n^j + \vartheta_n \leq \alpha_m^{j+1} + \vartheta_m$$

for $1 \leq j \leq k$, $m - i \leq n < m$.

(c)

$$\inf_{n \in \mathbb{N}} (\alpha_n^1 + \vartheta_n) > -\infty.$$

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. We show that conditions of Theorem 3.6 are satisfied.

Let $k, l, m \in \mathbb{N}$ and $M \in \mathbb{R}$, $M \geq 0$. Then there exist $r, s, t \in \mathbb{N}$ and a function $\alpha: [s+1, s+r] \rightarrow \mathbb{R}$ such that

$$l \leq r \leq s < s+r \leq t;$$

$$\alpha_n^{j+1} + \zeta_n \geq \alpha_{n+i}^j + \zeta_{n+i}$$

for $n \geq r$, $1 \leq i \leq l$, $1 \leq j \leq k+1$; and conditions (ii)-(vi) of (1)(b) are satisfied. Then conditions (a)-(c) of (1) of Theorem 3.6 are satisfied.

Hence condition (1) of Theorem 3.6 is satisfied by (1). And condition (2) of Theorem 3.6 is satisfied by (2).

Therefore there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. □

Theorem 3.8. Let $\Lambda(A)$, where $A = \left(|a|^{\alpha_n^k}\right)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space.

Assume that:

(1) There exists a sequence $(\zeta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that:

(a) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\alpha_n^{k+1} + \zeta_n \geq \alpha_{n+i}^k + \zeta_{n+i}$$

for $n \geq j$.

(b) For every $l, m \in \mathbb{N}$ and $M \in \mathbb{R}$, $M \geq 0$ there exist $u, v \in \mathbb{N}$ such that:

(i)

$$l \leq u \leq v,$$

(ii)

$$\alpha_n^m + \zeta_n \leq \alpha_u^m + \zeta_u$$

for $1 \leq n \leq u$,

(iii)

$$\alpha_u^m + \zeta_u \leq \alpha_n^{m+1} + \zeta_n$$

for $u \leq n \leq v$,

(iv)

$$\alpha_v^1 + \zeta_v - \alpha_u^m - \zeta_u \geq M.$$

(2) There exists a sequence $(\vartheta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that:

(a) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\alpha_n^{k+1} + \vartheta_n \geq \alpha_{n+i}^k + \vartheta_{n+i}$$

for $n \geq j$.

(b) For every $l, k, i \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq l$, $m > i$ such that

$$\alpha_n^j + \vartheta_n \leq \alpha_m^{j+1} + \vartheta_m$$

for $1 \leq j \leq k$, $m - i \leq n < m$.

(c)

$$\inf_{n \in \mathbb{N}} (\alpha_n^1 + \vartheta_n) > -\infty.$$

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. We show that conditions of Theorem 3.7 are satisfied.

Condition (1)(a) of Theorem 3.7 is satisfied by (1)(a).

Let $l, m \in \mathbb{N}$ and $M \in \mathbb{R}$, $M \geq 0$ and let $r \in \mathbb{N}$, $r \geq l$. By (1)(a), there exists $j \in \mathbb{N}$, $j \geq r$ such that

$$\alpha_n^{m+1} + \zeta_n \geq \alpha_{n+i}^m + \zeta_{n+i}$$

for $n > j$, $1 \leq i \leq r$.

Then, by (1)(b), there exist $u, v \in \mathbb{N}$ such that

$$j + r \leq u \leq v$$

and conditions (ii)-(iv) of (1)(b) are satisfied. Let $s, t \in \mathbb{N}$ be such that

$$s + r = u$$

and

$$v = t.$$

Then conditions (i)-(vi) of (1)(b) of Theorem 3.7 are satisfied if we set $\alpha: [s+1, s+r] \rightarrow \mathbb{R}$, $n \mapsto \alpha_{s+r}^m + \zeta_{s+r}$.

Hence condition (1) of Theorem 3.7 is satisfied by (1). And condition (2) of Theorem 3.7 is satisfied by (2).

Therefore there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. □

3.4 Coefficients with the property of braking

Theorem 3.9. Let $\Lambda(A)$, where $A = \left(|a|^{\alpha_n^k} \right)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space.

Assume that:

- (1) For every $k, l, m \in \mathbb{N}$ there exist $r, s \in \mathbb{N}$, a sequence $(\zeta_n)_{n \geq r} \subset \mathbb{R}$ and a function $\alpha: [s+1, s+r] \rightarrow \mathbb{R}$ such that:

(a)

$$l \leq r \leq s.$$

(b) If $n \geq r$ and $1 \leq i \leq l$, then

$$\alpha_n^{j+1} + \zeta_n \geq \alpha_{n+i}^j + \zeta_{n+i}$$

for $1 \leq j \leq k+1$.

(c) (i)

$$\alpha_n^m + \zeta_n \leq \alpha(s+1)$$

if $r \leq n \leq s$,

(ii)

$$\alpha(n) \leq \alpha(n+1)$$

if $s < n < s+r$,

(iii)

$$\alpha_n^m + \zeta_n \leq \alpha(n) \leq \alpha_n^{m+1} + \zeta_n$$

for $s < n \leq s+r$,

(iv)

$$\alpha(s+r) \leq \alpha_n^{m+1} + \zeta_n$$

for $s+r < n$.

(2) There exists a sequence $(\vartheta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that:

(a) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\alpha_n^{k+1} + \vartheta_n \geq \alpha_{n+i}^k + \vartheta_{n+i}$$

for $n \geq j$.

(b) For every $l, k, i \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq l$, $m > i$ such that

$$\alpha_n^j + \vartheta_n \leq \alpha_m^{j+1} + \vartheta_m$$

for $1 \leq j \leq k$, $m-i \leq n < m$.

(c)

$$\inf_{n \in \mathbb{N}} (\alpha_n^1 + \vartheta_n) > -\infty.$$

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. Let $\Lambda(B)$ be a non-Archimedean Köthe space such that $B = \left(|a|^{\beta_n^k}\right)_{k,n \in \mathbb{N}}$, $\beta_n^k = \alpha_n^k + k$ for $k, n \in \mathbb{N}$ and let $\left(\hat{\vartheta}_n\right)_{n \in \mathbb{N}}$ be a sequence such that $\hat{\vartheta}_n = \vartheta_n + h(n)$ for $n \in \mathbb{N}$, where $h: \mathbb{N} \rightarrow \mathbb{N}$ is a function such that $h(n) = m$ for $n, m \in \mathbb{N}$, $2^{m-1} \leq n < 2^m$.

We have

$$\lim_{n \rightarrow \infty} \left(\beta_n^1 + \hat{\vartheta}_n\right) = \lim_{n \rightarrow \infty} \left(\alpha_n^1 + 1 + \vartheta_n + h(n)\right) = \infty. \quad (3.8)$$

Since $\left(\beta_n^k\right)_{k,n \in \mathbb{N}}$ satisfies condition (1), $\left(\beta_n^k\right)_{k,n \in \mathbb{N}}$ and the sequence $\left(\hat{\vartheta}_n\right)_{n \in \mathbb{N}}$ satisfy conditions (a)-(c) of (2), equations (3.8) is satisfied and $\Lambda(B)$ is isomorphic to $\Lambda(A)$, we may assume that conditions (1) and (2) are satisfied with the change of (2)(c) on

$$(2) \text{ (c')} \quad \lim_{n \rightarrow \infty} \left(\alpha_n^1 + \vartheta_n\right) = \infty.$$

We check that conditions of Theorem 3.6 are satisfied.

Let $k, l, m \in \mathbb{N}$ and $M \in \mathbb{R}$, $M \geq 0$.

By (1), there exist $r, s \in \mathbb{N}$, $(\zeta_n)_{n \geq r} \subset \mathbb{R}$ and $\alpha: [s+1, s+r] \rightarrow \mathbb{R}$ such that:

$$l \leq r \leq s,$$

$$\alpha_n^{j+1} + \zeta_n \geq \alpha_{n+i}^j + \zeta_{n+i}$$

for $n \geq r$, $1 \leq i \leq l$, $1 \leq j \leq k+1$ and condition (1)(c) is satisfied.

By (2), there exists $\lambda \in \mathbb{N}$, $\lambda \geq r+s$ such that:

$$\alpha_n^{j+1} + \vartheta_n \geq \alpha_{n+i}^j + \vartheta_{n+i}$$

for $1 \leq j \leq k+1$, $n \geq \lambda$, $1 \leq i \leq l$ and

$$\inf_{n \geq \lambda} \left(\alpha_n^{m+1} + \vartheta_n\right) = \alpha_\lambda^{m+1} + \vartheta_\lambda.$$

By (2)(c'), there exist $t \in \mathbb{N}$ such that

$$t - \lambda \geq l$$

and

$$\alpha_t^1 + \zeta_\lambda - \vartheta_\lambda + \vartheta_t - \alpha(s+r) \geq M.$$

Let

$$\xi_n = \zeta_n$$

for $r \leq n \leq \lambda$ and

$$\xi_n = \zeta_\lambda - \vartheta_\lambda + \vartheta_n$$

for $\lambda \leq n \leq t$.

Then

$$\alpha_t^1 + \xi_t - \alpha(s+r) \geq M.$$

To prove that condition (1) of Theorem 3.6 is satisfied it is enough to show that the finite sequence $(\xi_n)_{r \leq n \leq t}$ satisfies condition (1)(b) of Theorem 3.6.

Let $1 \leq j \leq k$, $[n, n+i] \subset [r, t]$ and $1 \leq i \leq l$.

- If $n+i \leq \lambda$, then

$$\alpha_n^{j+2} + \xi_n - \alpha_{n+i}^j - \xi_{n+i} \geq \alpha_n^{j+2} + \zeta_n - \alpha_{n+i}^{j+1} - \zeta_{n+i} \geq 0.$$

- If $n \geq \lambda$, then

$$\alpha_n^{j+2} + \xi_n - \alpha_{n+i}^j - \xi_{n+i} \geq \alpha_n^{j+2} + \vartheta_n - \alpha_{n+i}^{j+1} - \vartheta_{n+i} \geq 0.$$

- If $\lambda \in (n, n+i)$, then

$$\begin{aligned} \alpha_n^{j+2} + \xi_n - \alpha_{n+i}^j - \xi_{n+i} &= \alpha_n^{j+2} + \xi_n - \alpha_\lambda^{j+1} - \xi_\lambda + \alpha_\lambda^{j+1} + \xi_\lambda - \alpha_{n+i}^j - \xi_{n+i} \\ &= \alpha_n^{j+2} + \zeta_n - \alpha_\lambda^{j+1} - \zeta_\lambda + \alpha_\lambda^{j+1} + \vartheta_\lambda - \alpha_{n+i}^j - \vartheta_{n+i} \geq 0. \end{aligned}$$

- If $n = r$, then

$$\alpha_r^{j+1} + \xi_r - \alpha_{r+i}^j - \xi_{r+i} = \alpha_r^{j+1} + \zeta_n - \alpha_{r+i}^j - \zeta_{r+i} \geq 0.$$

If $n+i = t$, then

$$\alpha_n^{j+2} + \xi_n - \alpha_t^{j+1} - \xi_t = \alpha_n^{j+2} + \vartheta_n - \alpha_t^{j+1} - \vartheta_t \geq 0.$$

Hence condition (1) of Theorem 3.6 is satisfied by (1) and (2). And condition (2) of Theorem 3.6 is satisfied by (2).

Therefore there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. □

Theorem 3.10. Let $\Lambda(A)$, where $A = \left(|a|^{\alpha_n^k}\right)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space.

Assume that:

(1) For every $m \in \mathbb{N}$ there exists a sequence $(\zeta_n^m)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that:

(a) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\alpha_n^{k+1} + \zeta_n^m \geq \alpha_{n+i}^k + \zeta_{n+i}^m$$

for $n \geq j$.

(b) For every $l \in \mathbb{N}$ there exists $v \in \mathbb{N}$, $v \geq l$ such that:

(i)

$$\alpha_n^m + \zeta_n^m \leq \alpha_v^m + \zeta_v^m$$

for $1 \leq n \leq v$,

(ii)

$$\alpha_v^m + \zeta_v^m \leq \alpha_n^{m+1} + \zeta_n^m$$

for $v \leq n$.

(2) There exists a sequence $(\vartheta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that:

(a) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\alpha_n^{k+1} + \vartheta_n \geq \alpha_{n+i}^k + \vartheta_{n+i}$$

for $n \geq j$.

(b) For every $l, k, i \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq l$, $m > i$ such that

$$\alpha_n^j + \vartheta_n \leq \alpha_m^{j+1} + \vartheta_m$$

for $1 \leq j \leq k$, $m - i \leq n < m$.

(c)

$$\inf_{n \in \mathbb{N}} (\alpha_n^1 + \vartheta_n) > -\infty.$$

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. We check that conditions of Theorem 3.9 are satisfied.

Let $k, l, m \in \mathbb{N}$. By (1)(a), there exists $r \in \mathbb{N}$, $r \geq l$ such that $(\zeta_n^m)_{n \in \mathbb{N}}$ satisfies condition (1)(b) of Theorem 3.9 and there exists $j \in \mathbb{N}$, $j \geq r$, such that

$$\alpha_n^{m+1} + \zeta_n^m \geq \alpha_{n+i}^m + \zeta_{n+i}^m$$

for $n > j$, $1 \leq i \leq r$.

Then, by (1)(b), there exists $v \in \mathbb{N}$ such that

$$j + r \leq v$$

and conditions (i) and (ii) of (1)(b) are satisfied. Let $s \in \mathbb{N}$ be such that

$$s + r = v.$$

Let $(\zeta_n)_{n \geq r}$ be such that $\zeta_n = \zeta_n^m$ for $n \geq r$. Then conditions (a)-(c) of (1) of Theorem 3.9 are satisfied if we set $\alpha: [s+1, s+r] \rightarrow \mathbb{R}$, $n \mapsto \alpha_v^m + \zeta_v$.

Hence condition (1) of Theorem 3.9 is satisfied by (1). And condition (2) of Theorem 3.9 is satisfied by (2).

Therefore there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. □

Theorem 3.11. *Let $\Lambda(A)$, where $A = (a_n^k)_{k, n \in \mathbb{N}}$, be a non-Archimedean Köthe space.*

Assume that:

(1) *For every $m \in \mathbb{N}$ there exists a sequence $(b_n^m)_{n \in \mathbb{N}} \subset \mathbb{R}$ of positive numbers such that:*

(a) *For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that*

$$a_n^{k+1} b_n^m \geq a_{n+i}^k b_{n+i}^m$$

for $n \geq j$.

(b) *For every $l \in \mathbb{N}$ there exists $n \in \mathbb{N}$, $n \geq l$ such that*

$$a_i^m b_i^m \leq a_n^m b_n^m \leq a_j^m b_j^m$$

for $1 \leq i \leq n \leq j$.

(2) *There exists a sequence $(c_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ of positive numbers such that:*

(a) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$a_n^{k+1} c_n \geq a_{n+i}^k c_{n+i}$$

for $n \geq j$.

(b) For every $l, k, i \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq l$, $m > i$ such that

$$a_n^j c_n \leq a_m^{j+1} c_m$$

for $1 \leq j \leq k$, $m - i \leq n < m$.

(c)

$$\inf_{n \in \mathbb{N}} a_n^1 c_n > 0.$$

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. We show that conditions of Theorem 3.10 are satisfied.

Let $a_n^k = |a|^{\alpha_n^k}$ for $k, n \in \mathbb{N}$, $b_n^m = |a|^{\zeta_n^m}$ for $m, n \in \mathbb{N}$ and $c_n = |a|^{\vartheta_n}$ for $n \in \mathbb{N}$.

Let $m \in \mathbb{N}$. Then condition (1)(a) of Theorem 3.10 is satisfied by (1)(a) and since for every $l \in \mathbb{N}$ there exists $v \in \mathbb{N}$, $v \geq l$ such that

$$a_i^m b_i^m \leq a_v^m b_v^m \leq a_j^m b_j^m \leq a_j^{m+1} b_j^m$$

for $1 \leq i \leq v \leq j$, condition (1)(b) of Theorem 3.10 is satisfied by (1)(b).

Hence condition (1) of Theorem 3.10 is satisfied by (1). And condition (2) of Theorem 3.10 is satisfied by (2).

Therefore there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. □

Theorem 3.12. Let $\Lambda(A)$, where $A = (a_n^k)_{k, n \in \mathbb{N}}$, be a non-Archimedean Köthe space.

Assume that:

(1) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$a_n^{k+1} \geq a_{n+i}^k$$

for $n \geq j$.

(2) For every $l, m \in \mathbb{N}$ there exists $n \in \mathbb{N}$, $n \geq l$ such that

$$a_i^m \leq a_n^m \leq a_j^m$$

for $1 \leq i \leq n \leq j$.

(3) For every $l, k, i \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq l$, $m > i$ such that

$$a_n^j \leq a_m^{j+1}$$

for $1 \leq j \leq k$, $m - i \leq n < m$.

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. We check that conditions of Theorem 3.11 are satisfied.

Let $b_n^m = 1$ for $m, n \in \mathbb{N}$ and $c_n = 1$ for $n \in \mathbb{N}$.

Let $m \in \mathbb{N}$. Then conditions (a), (b) of (1) of Theorem 3.11 are satisfied by (1), (2) respectively. And conditions (a), (b), (c) of (2) of Theorem 3.11 are satisfied by (1), (3), (2) respectively.

Therefore there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. □

Theorem 3.13. Let $\Lambda(A)$, where $A = (a_n^k)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space.

Assume that:

(1) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$a_n^{k+1} \geq a_{n+i}^k$$

for $n \geq j$.

(2) For every $l, m \in \mathbb{N}$ there exists $n \in \mathbb{N}$, $n \geq l$ such that

$$a_i^m \leq a_n^m \leq a_j^m$$

for $1 \leq i \leq n \leq j$.

(3) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$, $j > i$ such that

$$a_{n-i}^k \leq a_n^{k+1}$$

for $n \geq j$.

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. Conditions (1)-(3) of Theorem 3.12 are satisfied by (1)-(3) respectively.

Therefore there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. □

3.5 Some global properties

Theorem 3.14. Let $\Lambda(A)$, where $A = (a_n^k)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space.

Assume that:

(1) For every $k, m, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\frac{a_n^{k+1}}{a_{n+i}^k} \geq \frac{a_n^m}{a_{n+i}^m}$$

for $n \geq j$.

(2) There exists a sequence $(c_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ of positive numbers such that:

(a) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$a_n^{k+1} c_n \geq a_{n+i}^k c_{n+i}$$

for $n \geq j$.

(b) For every $k, l, i \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq l$, $m > i$ such that

$$a_n^j c_n \leq a_m^{j+1} c_m$$

for $1 \leq j \leq k$, $m - i \leq n < m$.

(c)

$$\inf_{n \in \mathbb{N}} a_n^1 c_n > 0.$$

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. We show that conditions of Theorem 3.11 are satisfied.

Let $b_n^m = (a_n^m)^{-1}$ for $m, n \in \mathbb{N}$.

Let $m \in \mathbb{N}$. Then condition (1)(a) of Theorem 3.11 is satisfied by (1) and since

$$a_i^m b_i^m = a_i^m (a_i^m)^{-1} = 1$$

for $i \in \mathbb{N}$, condition (1)(b) of Theorem 3.11 is satisfied.

Hence condition (1) of Theorem 3.11 is satisfied by (1). And condition (2) of Theorem 3.11 is satisfied by (2).

Therefore there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. \square

Theorem 3.15. Let $\Lambda(A)$, where $A = (a_n^k)_{k,n \in \mathbb{N}}$, be a non-Archimedean Köthe space.

Assume that:

(1) For every $k, m, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\frac{a_n^{k+1}}{a_{n+i}^k} \geq \frac{a_n^m}{a_{n+i}^m}$$

for $n \geq j$.

(2) For every $k, i \in \mathbb{N}$ there exists $j \in \mathbb{N}$, $j > i$ such that

$$\frac{a_n^{k+1}}{a_{n-i}^k} \geq \frac{a_n^1}{a_{n-i}^1}$$

for $n \geq j$.

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. We show that conditions of Theorem 3.14 are satisfied.

Condition (1) of Theorem 3.14 is satisfied by (1).

Let $c_n = (a_n^1)^{-1}$ for $n \in \mathbb{N}$. Then, in particular, condition (2)(a) of Theorem 3.14 is satisfied by (1). Condition (2)(b) of Theorem 3.14 is satisfied by (2). And since

$$a_n^1 c_n = a_n^1 (a_n^1)^{-1} = 1$$

for $n \in \mathbb{N}$, condition (2)(c) of Theorem 3.14 is satisfied.

Hence condition (2) of Theorem 3.14 is satisfied by (1) and (2).

Therefore there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. \square

Theorem 3.16. *Let $(c_n^k)_{n \in \mathbb{N}} \subset \mathbb{R}$ for $k \in \mathbb{N}$ and $(\alpha_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ be sequences of positive numbers, and let $\Lambda(A)$, where $A = (a_n^k)_{k, n \in \mathbb{N}}$, $a_n^k = (c_n^k)^{\alpha_n}$ for $k, n \in \mathbb{N}$, be a non-Archimedean Köthe space.*

Assume that:

- (1) *For every $k, m, i \in \mathbb{N}$ there exist $j \in \mathbb{N}$ and $\varepsilon_{k, m, i} \in \mathbb{R}$, $\varepsilon_{k, m, i} > 1$ such that*

$$\frac{c_n^{k+1} c_{n+i}^m}{c_{n+i}^k c_n^m} \geq \varepsilon_{k, m, i}$$

for $n \geq j$.

- (2) *For every $k, i \in \mathbb{N}$ there exist $j \in \mathbb{N}$, $j > i$ and $\varepsilon_{k, i} \in \mathbb{R}$, $\varepsilon_{k, i} > 1$ such that*

$$\frac{c_n^{k+1} c_{n-i}^1}{c_{n-i}^k c_n^1} \geq \varepsilon_{k, i}$$

for $n \geq j$.

- (3) *For every $k, l \in \mathbb{N}$*

$$0 < \inf_{n \in \mathbb{N}} \frac{c_n^k}{c_n^l} \leq \sup_{n \in \mathbb{N}} \frac{c_n^k}{c_n^l} < \infty.$$

- (4)

$$\lim_{n \rightarrow \infty} \frac{\alpha_{n+1}}{\alpha_n} = 1.$$

Then there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. We show that conditions of Theorem 3.15 are satisfied.

Let $k, m, n, i \in \mathbb{N}$. We have

$$\frac{a_n^{k+1} a_{n+i}^m}{a_{n+i}^k a_n^m} = \frac{(c_n^{k+1})^{\alpha_n} (c_{n+i}^m)^{\alpha_{n+i}}}{(c_n^m)^{\alpha_n} (c_{n+i}^k)^{\alpha_{n+i}}} = \left(\frac{c_n^{k+1} c_{n+i}^m}{c_n^m c_{n+i}^k} \left(\frac{c_{n+i}^m}{c_{n+i}^k} \right)^{\frac{\alpha_{n+i}-1}{\alpha_n}} \right)^{\alpha_n}.$$

Thus, by (1), (3) and (4), condition (1) of Theorem 3.15 is satisfied since

$$\lim_{n \rightarrow \infty} \left(\frac{\alpha_{n+i}}{\alpha_n} - 1 \right) = 0$$

for $i \in \mathbb{N}$.

Let $k, n, i \in \mathbb{N}$, $i < n$. Then

$$\frac{a_n^{k+1} a_{n-i}^1}{a_{n-i}^k a_n^1} = \frac{(c_n^{k+1})^{\alpha_n} (c_{n-i}^1)^{\alpha_{n-i}}}{(c_n^1)^{\alpha_n} (c_{n-i}^k)^{\alpha_{n-i}}} = \left(\frac{c_n^{k+1} c_{n-i}^1}{c_n^1 c_{n-i}^k} \left(\frac{c_{n-i}^1}{c_{n-i}^k} \right)^{\frac{\alpha_{n-i}-1}{\alpha_n}} \right)^{\alpha_n}.$$

Hence, by (2), (3) and (4), condition (2) of Theorem 3.15 is satisfied since

$$\lim_{n \rightarrow \infty} \left(\frac{\alpha_{n-i}}{\alpha_n} - 1 \right) = 0$$

for $i \in \mathbb{N}$.

Therefore there exists a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. \square

Theorem 3.17. Let $(c_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ be a sequence of positive numbers such that $c_n < c_{n+1}$ for $n \in \mathbb{N}$, and let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that

$$\lim_{n \rightarrow \infty} \frac{\alpha_{n+1}}{\alpha_n} = 1.$$

Then the non-Archimedean Köthe space $\Lambda(A)$, where $A = (a_n^k)_{k, n \in \mathbb{N}}$, $a_n^k = (c_k)^{\alpha_n}$ for $k, n \in \mathbb{N}$, has a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces.

Proof. Let $c_n^k = c_k$ for $k, n \in \mathbb{N}$. Then conditions of Theorem 3.16 are satisfied.

Therefore the non-Archimedean Köthe space $\Lambda(A)$ has a linear and continuous operator

$$T: \Lambda(A) \rightarrow \Lambda(A)$$

that has no nontrivial invariant subspaces. □

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