

ADAM MICKIEWICZ UNIVERSITY

DOCTORAL THESIS

Secure Sets in Graphs

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*A thesis submitted in fulfilment of the requirements
for the degree of Doctor of Philosophy*

to the

Faculty of Mathematics and Computer Science

January 13, 2014

UNIwersytet im. Adama Mickiewicza

ROZPRAWA DOKTORSKA

Zbiory bezpieczne w grafach

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*Praca złożona w procesie ubiegania się o stopień doktora nauk
matematycznych w zakresie informatyki*

na

Wydziale Matematyki i Informatyki

13 stycznia 2014

Acknowledgements

I would like to thank Professor Maciej M. Sysło for being my supervisor. His constructive criticism allowed me to improve this thesis.

Furthermore, I would like to thank all the members of the Discrete Mathematics and Theoretical Computer Science Group at University of Zielona Góra for their support and everyday discussions.

Finally, I must express my gratitude to my family. Their love and faith in me have helped to make all this possible. I dedicate this thesis to them.

I would also like to acknowledge that the research presented in this thesis was partially supported by the National Science Centre on the basis of the decision DEC-2011/01/N/ST6/00922.

Zbiory bezpieczne w grafach

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Streszczenie

Praca dotyczy zbiorów bezpiecznych w grafach. Rozpocznijmy zatem od podstawowych definicji z teorii grafów. Niech $G = (V, E)$ będzie grafem o zbiorze wierzchołków V i zbiorze krawędzi E . Mówimy, że dwa wierzchołki grafu są sąsiednie, gdy istnieje pomiędzy nimi krawędź. Otwartym sąsiedztwem wierzchołka v nazywamy zbiór wszystkich wierzchołków sąsiednich z v , który oznaczamy $N(v)$. Natomiast przez zamknięte sąsiedztwo wierzchołka v rozumiemy zbiór wierzchołków $\{v\} \cup N(v)$ i oznaczamy $N[v]$. Analogicznie definiujemy otwarte i zamknięte sąsiedztwo zbioru wierzchołków. Mianowicie, niech X będzie dowolnym podzbiorem V . Wówczas $N(X) = \bigcup_{v \in X} N(v)$ a $N[X] = N(X) \cup X$. Te oraz inne podstawowe definicje z teorii grafów znajdują się w rozdziale 1.

W rozdziale 2 wprowadzone zostają dwa kluczowe pojęcia: koalicji i zbioru bezpiecznego. Koalicja została zdefiniowana w pracy [32] jako zbiór wierzchołków $S \subseteq V$ taki, że dla każdego $v \in S$ zachodzi $|N[v] \cap S| \geq |N(v) \cap (V - S)|$. Koalicja w naturalny sposób opisuje grupę obiektów, które mają wspólne cechy, cele lub łączy je pewna więź. Aby opisać jeszcze silniejsze związki pomiędzy elementami, Brigham, Dutton i Hedetniemi wprowadzili w pracy [6] pojęcie zbiorów bezpiecznych. Ich formalna definicja opiera się na idei ataku i obrony. Dla zbioru wierzchołków $S = \{s_1, s_2, \dots, s_k\} \subseteq V$, $1 \leq k \leq |V|$, atakiem na S nazywamy k -elementową rodzinę parami rozłącznych zbiorów $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$, dla których zachodzi $A_i \subseteq N[s_i] - S$, dla $1 \leq i \leq k$. Obroną zbioru S nazywamy k -elementową rodzinę parami rozłącznych zbiorów $\mathcal{D} = \{D_1, D_2, \dots, D_k\}$, dla których zachodzi $D_i \subseteq N[s_i] \cap S$, dla $1 \leq i \leq k$. Atak \mathcal{A} można odeprzeć, jeżeli istnieje obrona \mathcal{D} taka, że $|D_i| \geq |A_i|$, dla $1 \leq i \leq k$. Zbiór S jest bezpieczny wtedy i tylko wtedy, gdy każdy atak na S można odeprzeć. W tej samej pracy udowodniono, że zbiór $S \subseteq V$ jest bezpieczny wtedy i tylko wtedy, gdy dla każdego $X \subseteq S$ zachodzi $|N[X] \cap S| \geq |N[X] - S|$. W pracy rozważamy również silne zbiory bezpieczne zdefiniowane jako zbiory bezpieczne, które są przygotowane na atak

jednego dodatkowego, zewnętrznego wroga. Innymi słowy zbiór $S' \subseteq V$ jest silnie bezpieczny wtedy i tylko wtedy, gdy dla każdego niepustego $X \subseteq S'$ zachodzi $|N[X] \cap S'| > |N[X] - S'|$. W rozdziale 2 podajemy podstawowe własności oraz zastosowania zbiorów (silnie) bezpiecznych. Zarówno koalicje jak i zbiory bezpieczne mogą opisywać relacje pomiędzy ludźmi, partiami politycznymi, podmiotami gospodarczymi oraz członkami sojuszy militarnych [32]. Oddają one także charakter skupienia w analizie skupień [33, 41]. Przez analizę skupień rozumiemy taki podział pewnego zbioru obiektów na podgrupy zwane skupieniami, że elementy jednego skupienia są bardziej podobne do siebie (w pewnym sensie), niż do obiektów z innych skupień.

Kolejnym interesującym wariantem koalicji i zbiorów bezpiecznych jest ich wersja, w której dodany jest warunek dominowania, tzn. koalicja (zbiór bezpieczny) musi być także zbiorem dominującym, czyli każdy z wierzchołków spoza zbioru ma w nim sąsiada. Takie koalicje i zbiory bezpieczne nazywamy globalnymi. Tym zbiorom jest poświęcony rozdział 3. Ważną jego część stanowią rozważania dotyczące minimalnych globalnych zbiorów bezpiecznych w drzewach i kaktusach. Drzewami nazywamy grafy, które nie zawierają cykli, natomiast kaktusy to grafy, w których każda krawędź należy do co najwyżej jednego cyklu. Dowodzimy, że każde drzewo i kaktus o n wierzchołkach zawiera globalny zbiór bezpieczny o mocy nie przekraczającej $\frac{2}{3}n$. Wymienione wyniki zostały opublikowane w pracy [28].

W przypadku zbiorów dominujących wiemy, że jeżeli graf $G = (V, E)$ zawiera zbiór dominujący o mocy k , gdzie $k < |V|$, to graf G zawiera zbiór dominujący o mocy $k + 1$. Analogiczne stwierdzenie nie jest jednak prawdziwe w przypadku globalnych zbiorów bezpiecznych. W rozdziale 4 wprowadzamy pojęcie grafu γ_s -monotonicznego. Mówimy, że graf jest γ_s -monotoniczny, jeżeli zawiera globalne zbiory bezpieczne o każdej dopuszczalnej mocy k (dla $\gamma_s(G) \leq k \leq |V|$, gdzie $\gamma_s(G)$ oznacza moc najmniejszego globalnego zbioru bezpiecznego w grafie G). Pokazujemy, że wszystkie kografy nieparzystego rzędu są γ_s -monotoniczne. Dowodzimy także, że istnieje nieskończona klasa kografów parzystego rzędu, oznaczmy ją przez \mathcal{P}' , zawierająca kografy, które są prawie γ_s -monotoniczne. W każdym grafie $G \in \mathcal{P}'$ brakuje globalnego zbioru bezpiecznego o mocy równej $\gamma_s(G) + 1$. Ponadto wykazujemy, że jeżeli graf G jest kografem, to $\gamma_s(G) \leq \left\lceil \frac{|V(G)|+1}{2} \right\rceil$. Wykorzystując modułarną dekompozycję kografów, wskazujemy również nietrywialne podklasy kografów, dla których można w czasie liniowym znaleźć globalny zbiór bezpieczny o dowolnej, dopuszczalnej mocy. Wyniki przedstawione w rozdziale 4 zostały opublikowane w pracach [29, 30].

Ze względu na to, że koalicje i zbiory bezpieczne mogą w szczególności odzwierciedlać strukturę sojuszy militarnych, wprowadzamy w rozdziale 5 pojęcie rozszerzalności zbioru bezpiecznego. I tak mówimy, że (globalny) zbiór bezpieczny S jest rozszerzalny w grafie $G = (V, E)$, jeżeli jego moc jest mniejsza od $|V|$ oraz istnieje wierzchołek $v \in V - S$ taki, że $S \cup \{v\}$ jest (globalnym) zbiorem bezpiecznym w grafie G . Analogicznie definiujemy pojęcie rozszerzalności dla koalicji. Dowodzimy, że każda koalicja i zbiór bezpieczny w grafie o maksymalnym stopniu wierzchołka równym 3, jest rozszerzalna. Badamy również klasę k -drzew, gdzie k -drzewem nazywamy klikę o k wierzchołkach lub graf T , który po usunięciu wierzchołka, którego sąsiedztwo w T indukują k -klikę, jest k -drzewem. Pokazujemy, że w przypadku 2-drzew każda ich globalna koalicja i globalny zbiór bezpieczny jest rozszerzalny. Ponadto, dla każdego $k \geq 3$ wskazujemy k -drzewo, które zawiera nierozszerzalną globalną koalicję i nierozszerzalny globalny zbiór bezpieczny. Powracamy również do klasy kografów. Nie wszystkie globalne zbiory bezpieczne kografów są rozszerzalne. Jednakże istnieje ich podklasa (co-qt grafy), w której każdy graf ma minimalny globalny zbiór bezpieczny, który możemy rozszerzać, dodając w każdym kroku tylko jeden wierzchołek, dopóki zbiór bezpieczny nie obejmie całego grafu. Wymienione wyniki zostały opublikowane w [30].

W rozdziale 6 rozważamy powiązania omawianych problemów z problemami dekompozycji grafów. Przedstawiamy wyniki dotyczące dekomponowalności wybranych klas grafów. Przez dekomponowalność grafu rozumiemy istnienie dwukolorowania jego wierzchołków, które spełnia następujące warunki:

- oba kolory zostały użyte co najmniej raz,
- wszystkie wierzchołki grafu zostały pokolorowane,
- każdy wierzchołek ma co najwyżej jednego sąsiada w drugim kolorze.

Prezentujemy twierdzenia dotyczące grafów o średnicy 2, które zostały opublikowane w [4] oraz dowodzimy NP-zupełności problemu dekomponowalności w klasie grafów 5-regularnych. Pokazujemy także silny związek dekomponowalności grafu z problemem rozszerzania koalicji i zbiorów bezpiecznych.

Contents

Acknowledgements	i
Abstract (in polish language)	ii
Symbols	vii
1 Introduction	1
1.1 Dissertation Outline	1
1.2 Basic graph theory definitions and notations	2
2 Preliminaries on the secure sets	5
2.1 Alliances and secure sets in graphs	5
2.2 Basic properties	9
2.3 Secure sets in cubic graphs	13
3 Global secure sets	20
3.1 Properties of global secure sets	20
3.2 Global secure sets in trees	23
3.3 Global secure sets in cactus trees	31
4 Possible cardinalities of alliances and secure sets in graphs	38
4.1 Introduction	38
4.2 Properties of γ_s -monotone graphs	42
4.3 Cographs are not γ_s -monotone	44
5 Expansion of alliances and secure sets	59
5.1 Introduction	59
5.2 Basic properties	60
5.3 Expansion of global (strong) secure sets and global defensive alliances	62
5.3.1 k-trees	63
5.3.2 Co-quasi-threshold graphs	65
6 Graph partitioning and the alliances	69
6.1 Introduction	69
6.2 Matching cutset	70
6.2.1 Introduction	70

6.2.2	Matching cutset in 5-regular graphs	74
6.3	Role of the graph partitioning in the expansion	76
	List of Figures	80

Symbols

<i>Symbol</i>	<i>Meaning</i>	<i>Page</i>
$V(G)$	vertex set of a graph G	2
$E(G)$	edge set of a graph G	2
$N_G(v)$	open neighbourhood of a vertex v in a graph G	3
$N_G[v]$	closed neighbourhood of a vertex v in a graph G	3
$N_G(X)$	open neighbourhood of a set $X \subseteq V(G)$	3
$N_G[X]$	closed neighbourhood of a set $X \subseteq V(G)$	3
$\deg_G v$	degree of a vertex v in a graph G	3
$\mathcal{U}(G)$	set of all universal vertices of a graph G	3
$\delta(G)$	minimum degree of a graph G	3
$\Delta(G)$	maximum degree of a graph G	3
$d_G(u, v)$	distance between $u, v \in V(G)$	3
$\text{diam } G$	diameter of a graph G	3
$\langle S \rangle$	subgraph induced by a set $S \subseteq V(G)$	3
P_n	path of order n	3
C_n	cycle of order n	3
K_n	complete graph of order n	3
$K_{n,m}$	complete bipartite graph with one partite set of order n and the second one of order m	3
S_k	star with $k + 1$ vertices	24
$S_{r,k}$	double star with $r + k + 2$ vertices	24
$G_1 + G_2$	join of graphs G_1 and G_2	3
$G_1 \cup G_2$	disjoint union of graphs G_1 and G_2	3
$G_1 \times G_2$	Cartesian product of graphs G_1 and G_2	3
$\gamma(G)$	domination number of a graph G	4
$a(G)$	alliance number of a graph G	7

$\hat{a}(G)$	strong alliance number of a graph G	8
$\gamma_a(G)$	global alliance number of a graph G	8
$\gamma_{\hat{a}}(G)$	global strong alliance number of a graph G	8
$s(G)$	security number of a graph G	8
$\hat{s}(G)$	strong security number of a graph G	8
$\gamma_s(G)$	global security number of a graph G	8
$\gamma_{s_A}(G)$	minimum cardinality of a global secure set of G that contains a set $A \subseteq V(G)$	30
$\gamma_{\hat{s}}(G)$	global strong security number of a graph G	8
$\gamma_s(G)$ -set	minimum global secure set of a graph G	22
$\gamma_{s_A}(G)$ -set	minimum global secure set of a graph G that contains a set $A \subseteq V(G)$	30
$A(G)_k$	defensive alliance of cardinality k in a graph G	8
$AD(G)_k$	global defensive alliance of cardinality k in a graph G	8
$S(G)_k$	secure set of cardinality k in a graph G	8
$\hat{S}(G)_k$	strong secure set of cardinality k in a graph G	8
$SD(G)_k$	global secure set of cardinality k in a graph G	8
$\hat{S}D(G)_k$	global strong secure set of cardinality k in a graph G	8
$L(v)$	set of leaves adjacent to a vertex v	24
$l(v)$	number of leaves adjacent to a vertex v	24
\mathcal{MSD}_k	class of graphs that are nearly γ_s -monotone and have no global secure set of cardinality k	42
$R(G)$	set of vertices of G coloured in red	70
$B(G)$	set of vertices of G coloured in blue	70
$\mathcal{UC}(G)$	the set of uncoloured vertices of G	71
$E_2(G)$	set of edges whose end-vertices are coloured in different colours	70

Chapter 1

Introduction

1.1 Dissertation Outline

The dissertation is organized as follows. Chapter 1 introduces the basic definitions of the graph theory including the notion of the graph, the neighbourhood of the vertices, and a dominating set.

In Chapter 2, we introduce the reader to (strong) secure sets and their basic properties. We also define alliance in a graph with particular emphasis on a defensive alliance, which is a progenitor of a secure set that is the main subject of research of this thesis. The possible applications of the presented graph models are also given. In the third part of Chapter 2 we concentrate on (strong) secure sets in cubic graphs.

Chapter 3 begins with general properties of global secure sets. We investigate the global security number $\gamma_s(G)$ of a graph G , which is the minimum cardinality of a global secure set in a given graph. The second part of the chapter concerns the upper bound on the global security number in trees and cactus trees (graphs in which any two cycles have at most one vertex in common). In particular, we show that for any cactus tree T of order n , $\gamma_s(T) \leq \frac{2}{3}n$.

Any graph has a secure set. However, from the fact that a graph has a secure set of cardinality k , it does not follow that the graph has a secure set of greater cardinality. In Chapter 4, we search for graph classes that are γ_s -monotone, i.e., graphs that have global secure sets of any cardinality k , where k is at most the order and at least the global security number of the considered graph. In particular, we show that any cograph of odd order is γ_s -monotone, whether there exist

cographs of even order that have no global secure set of cardinality $\gamma_s(G) + 1$. Moreover, we present two subclasses of cographs that contain only γ_s -monotone graphs. For any graph from these subclasses, a global secure set of any cardinality that is greater than or equal to $\lceil \frac{n}{2} \rceil$ can be found in linear time, where n is the order of the input graph.

Chapter 5 is devoted to the problem of an expansion of alliances and secure sets. By the expansion we mean the possibility of adding new members so that the obtained set is still an alliance or a secure set. We consider a situation in which in each step of the expansion we add only one new member. We show that any defensive alliance and secure set of a graph with maximum degree less than or equal to 3, can be expanded. We also consider the possibility of the expansion of global defensive alliances and global secure sets in k -trees. We prove that in any k -tree, where $k \in \{1, 2\}$, every global defensive alliance and global secure set of cardinality less than the order of the graph, can be expanded. Furthermore, we show that, for any $k \geq 3$, there is a k -tree with a global defensive alliance and a global secure set that cannot be expanded. In the last part of the chapter we revisit the class of cographs. We know that not every global secure set of a γ_s -monotone cograph can be expanded. However, we show that there are cographs, called co-qt graphs, with the following property: a co-qt graph contains a minimum global secure set that can be sequentially expanded, until it covers the whole set of vertices.

In Chapter 6 we discuss the relation between secure sets and well-known problems of graph partitioning. We especially focus on the matching cutset problem, which, as we show, is sometimes a key to determining whether any (strong) secure set or defensive alliance in a graph is expandable. In the first part of the chapter we present results concerning matching cutsets in graphs with diameter two, and 5-regular graphs. We also show how other known graph decompositions relate to the problem of the expansion of alliances and secure sets.

1.2 Basic graph theory definitions and notations

First we introduce basic terminology and notations, generally we follow that in [8]. A *graph* $G = (V(G), E(G))$ is defined as a pair of two sets, where $V(G)$ is a nonempty and finite set of *vertices*, and $E(G)$ is a set of unordered pairs of distinct vertices of G called *edges*. The set of edges of the graph can be empty. We write V instead $V(G)$, and E instead $E(G)$, whenever the graph is clear from the context.

Let $e = \{u, v\}$ be an edge of the graph. Then, we say that u and v are *adjacent*, and e is *incident* to u and v . For simplicity, the edge $e = \{u, v\}$ we denote by uv . For a vertex $v \in V(G)$ its *open neighbourhood* is the set $N_G(v) = \{x \in V(G) : vx \in E(G)\}$, whereas the *closed neighbourhood* of v is the set $N_G[x] = N_G(v) \cup \{v\}$. Similarly we define the open and closed neighbourhood of a set $X \subseteq V(G)$, i.e. $N_G(X) = \bigcup_{v \in X} N_G(v)$ and $N_G[X] = N_G(X) \cup X$. The *degree* of a vertex v in a graph G , denoted by $\deg_G v$, is the number of vertices adjacent to v . The *minimum (maximum) degree* of a graph is the minimum (maximum) degree among the vertices of the graph and is denoted by $\delta(G)$ ($\Delta(G)$). A vertex is *universal* in a graph G if it is adjacent to all other vertices in G . The set of all universal vertices of a graph G is denoted by $\mathcal{U}(G)$. We also distinguish *pendant* vertices, i.e., vertices that are adjacent to exactly one vertex. Let u, v be two vertices of a connected graph G . The number of edges in a shortest path between u and v is the *distance* between u and v , denoted by $d_G(u, v)$. Furthermore, by $\text{diam } G$ we denote the greatest distance between any two vertices of G . In all the above notations we omit the subscript G when the graph under consideration is clear.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ (in this case we say that G *contains* H). Moreover, by $\langle S \rangle$, we denote a subgraph *induced* by a set $S \subseteq V(G)$, i.e., subgraph whose vertex set is S , and the edge set consists of all edges incident to two elements of S . A *connected component* or simply a *component* of a graph G is a maximal connected subgraph of G .

We use standard notations for basic graph classes:

- P_n – denotes a path of order n and length $n - 1$;
- C_n – denotes a cycle of order n ;
- K_n – denotes a complete graph of order n ;
- $K_{n,m}$ – denotes a complete bipartite graph with one partite set of order n and the second of order m .

A graph G is a *join* of graphs G_1 and G_2 , denoted by $G = G_1 + G_2$, if $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{e = uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. The *disjoint union* of graphs G_1 and G_2 is the graph $G = G_1 \cup G_2$ such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The *Cartesian product* $G = G_1 \times G_2$ has $V(G) = V(G_1) \times V(G_2)$ and $E(G) = \{(u_1, u_2)(v_1, v_2) : u_1 =$

v_1 and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$ }.

A set of vertices $D \subseteq V(G)$ is called a *dominating set*, if $N[D] = V(G)$, and the minimum cardinality of a dominating set in a graph G is a *domination number*, denoted $\gamma(G)$.

Other terminology and notation will be introduced when needed.

Chapter 2

Preliminaries on the secure sets

In this chapter we introduce secure sets and discuss their applications.

2.1 Alliances and secure sets in graphs

The notion of an alliance in graphs was introduced in [32] by Kristiansen, Hedetniemi and Hedetniemi. Mostly motivated by military applications, the authors defined several types of alliances, e.g., defensive, offensive and powerful alliances. Let $G = (V, E)$ be a graph. The nonempty set $A \subseteq V$ is a *defensive alliance* if for every $x \in A$, $|N[x] \cap A| \geq |N[x] - A|$. A defensive alliance is *strong* if for every vertex $v \in A$, $|N[v] \cap A| > |N[v] - A|$. According to the authors of [32], we may think of the members of A as defenders and of the rest of the vertices as attackers. Only one member of an alliance may be attacked at a time by its neighbours that are not in the alliance.

A different point of view on coalitions is captured in the notion of an offensive alliance. We say that a nonempty set of vertices $Z \subseteq V$ is an *offensive alliance* if for every vertex $u \in \bigcup_{v \in Z} N(v) - Z$, $|N[u] \cap Z| \geq |N[u] \cap (V - Z)|$. The set of vertices is a *powerful alliance* if it is both defensive and offensive [5]. Another variant of alliances is a *global alliance*, where global means that the alliance is also a dominating set.

Defensive alliances are mathematical models of web communities [18], i.e., pages having more hyperlinks to the members of the set (including themselves) than to non-members. Secondly, they can describe the relations between: people, political parties, groups in botany, companies and countries in military entente

[32]. Recently global alliances were applied in bioinformatics in analysis of RNA structure [24] and in fault tolerant computing [44, 45]. In [41], Shafique showed usefulness of defensive alliances in data clustering, i.e., dividing given data into groups of objects that are in some sense more similar to each other than to the other objects of the data set.

Many problems related to alliances have been investigated. We recall some of them. First of all, the problem of deciding for a given graph G and integer k whether G has a defensive or powerful alliance of size at most k is NP-complete. This result, together with an extensive study of the complexity of problems defined for various types of alliances, can be found in [27]. The similar problem for global defensive (powerful, offensive) alliances is also NP-complete, see [7]. Many authors also investigated the possibility of partitioning a graph into alliances or alliance-free sets, see for instance [41], [42], [25], [16] and [43].

Secure sets were introduced by Brigham, Dutton and Hedetniemi in [6]. Their definition includes the notion of an attack and a defence.

Definition 2.1. [6] Let $G = (V, E)$ be a graph. For any $S = \{s_1, s_2, \dots, s_k\} \subseteq V$, an *attack* on S is any k mutually disjoint sets $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ for which $A_i \subseteq N[s_i] - S$, $1 \leq i \leq k$. A *defence* of S is any k mutually disjoint sets $\mathcal{D} = \{D_1, D_2, \dots, D_k\}$ for which $D_i \subseteq N[s_i] \cap S$, $1 \leq i \leq k$. Attack \mathcal{A} is *defendable* if there exists a defence \mathcal{D} such that $|D_i| \geq |A_i|$ for $1 \leq i \leq k$. Set S is *secure* if and only if every attack on S is defendable.

Let $S = \{s_1, s_2, \dots, s_k\}$ be a secure set of $G = (V, E)$, $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ be an attack on S , and $\mathcal{D} = \{D_1, D_2, \dots, D_k\}$ be a defence against \mathcal{A} . Following the authors of [6], we say that the members of S are the *defenders* and the vertices of $N[S] - S$ are the *attackers*. Moreover, we say that a vertex $v \in N[s_i] - S$ *attacks* $s_i \in S$ if $v \in A_i$. Similarly, a vertex $s_j \in S$ *defends* $s_i \in S$ if $s_j \in D_i$. In such a case, s_j *fights against* some vertex of A_i . An attack is *maximal* if it involves all the attackers. Moreover, for a given secure set S , there is exactly one maximal attack if and only if every attacker has exactly one neighbour in S . Clearly, to determine whether a set of vertices is secure, it is enough to find a defence for every maximal attack. In [6] the authors proved the following theorem.

Theorem 2.2. [6] *Set $S \subseteq V$ is secure if and only if $\forall X \subseteq S$,*

$$|N[X] \cap S| \geq |N[X] - S|. \quad (2.1)$$

This theorem is now often used as an alternative definition of the secure sets. Now, when we look at Condition (2.1), we see that the secure sets are a stronger version of defensive alliances. If we consider an attack on a defensive alliance then we assume that, at a given time, only one member of the alliance, say v , is being attacked, and a defence is built only by neighbours of v . In the case of secure sets, by the definition, any subset of vertices of the secure set can be attacked at a given time. Hence, the concept of secure sets applies better, e.g., if we want to model a military unions.

Similarly as for the alliances, we define strong secure sets and global secure sets. We say that a set $S \subseteq V$ is *strong secure* if and only if for any nonempty $X \subseteq S$,

$$|N[X] \cap S| > |N[X] - S|. \quad (2.2)$$

The strong secure sets can be seen as secure sets that are prepared for a raid of one additional foe that can attack any member of the set. Moreover, if a (strong) secure set is a dominating set, then we say that it is a *global (strong) secure set*.

The applications of secure sets can be found in the areas in which the defensive alliances are applied. As we have already noticed, in certain cases, they will even better fit the real world situations. Now we present another area of applications of global defensive alliances that was given in [23]. Let us consider a computer network with a selected set of nodes S . Moreover, let S be a dominating set such that each node of S has a desired resource. Furthermore, any vertex without that resource can gain access to this resource by accessing a node at distance at most one. Suppose that all nodes in $V - S$ that are adjacent to a vertex $v \in S$ simultaneously demand the resource of v . In most cases v alone would not provide such access. However, if S is a global defensive alliance, then such demand can be satisfied (within distance two) by v and its neighbours in S . In this model we assumed that, at a given time, there are simultaneous demands in only one node of the network. In many networks this condition cannot be fulfilled. We can solve such problem by selecting S in such a way that it is a global secure set. Now, for any node of $V - S$, there is a resource within distance two, and the nodes can send a request to one of their neighbours in S at any time.

With the introduced types of secure sets and defensive alliances we associate the following invariants:

- $a(G)$ is the *alliance number*, i.e., the minimum cardinality of a defensive alliance in a graph G ,

- $\hat{a}(G)$ is the *strong alliance number*, i.e., the minimum cardinality of a strong defensive alliance in a graph G ,
- $\gamma_a(G)$ is the *global alliance number*, i.e., the minimum cardinality of a global defensive alliance in a graph G ,
- $\gamma_{\hat{a}}(G)$ is the *global strong alliance number*, i.e., the minimum cardinality of a global strong defensive alliance in a graph G .
- $s(G)$ is the *security number*, i.e., the minimum cardinality of a secure set in a graph G ,
- $\hat{s}(G)$ is the *strong security number*, i.e., the minimum cardinality of a strong secure set in a graph G ,
- $\gamma_s(G)$ is the *global security number*, i.e., the minimum cardinality of a global secure set in a graph G ,
- $\gamma_{\hat{s}}(G)$ is the *global strong security number*, i.e., the minimum cardinality of a global strong secure set in a graph G .

In the next chapters we will also search for secure sets of specified cardinality. For this reason we introduce these notations:

- $A(G)_k$ denotes a defensive alliance of cardinality k in a graph G ,
- $AD(G)_k$ denotes a global defensive alliance of cardinality k in a graph G ,
- $S(G)_k$ denotes a secure set of cardinality k in a graph G ,
- $\hat{S}(G)_k$ denotes a strong secure set of cardinality k in a graph G ,
- $SD(G)_k$ denotes a global secure set of cardinality k in a graph G ,
- $\hat{S}D(G)_k$ denotes a global strong secure set of cardinality k in a graph G .

We write simply A_k , AD_k , S_k , \hat{S}_k , SD_k , $\hat{S}D_k$ if the graph under consideration is clear from the context.

2.2 Basic properties

Let $G = (V, E)$ be a connected graph. Clearly, V is both a secure set and a strong secure set. Moreover, if $|V| \geq 2$ then any subset of V that contains $|V| - 1$ vertices is a secure set. However, a similar statement is not true for strong secure sets. Take for instance a path on three vertices, and let X be the set of its two pendant vertices. Clearly, X is not a strong secure set. On the other hand, it is easy to see that the following remark is true.

Remark 2.3. Let G be a graph of order $n \geq 3$ and H be its connected subgraph of order $n - 1$. Then, $V(H)$ is a strong secure set of G .

In [6] the authors presented the following properties of secure sets.

Proposition 2.4. [6] *If S_1 and S_2 are vertex disjoint secure sets in the same graph, then $S_1 \cup S_2$ is a secure set.*

Remark 2.5. [6] Let S be a minimum secure set. Then, $\langle S \rangle$ is connected.

The above proposition and remark remain true for strong secure sets. Clearly, having a (strong) secure set of cardinality k is not an induced hereditary property. However, if a graph G has a (strong) secure set S , then we can delete any edge that has at most one end-vertex in S or we can add a new edge such that both of its end-vertices are either in or outside S , and the set S is still (strong) secure in the so-obtained graph. In [15] one can find the following upper and lower bound on the security number.

Theorem 2.6. [15] *For any graph G on n vertices with minimum degree δ*

1. $s(G) \geq \lceil \frac{\delta+1}{2} \rceil$, and
2. $s(G) \leq n - \lceil \frac{\delta}{2} \rceil$.

We can formulate similar results for the strong secure sets.

Theorem 2.7. *For any graph G on n vertices with minimum degree δ*

1. $\hat{s}(G) \geq \lceil \frac{\delta}{2} \rceil + 1$, and
2. $\hat{s}(G) \leq n - \lfloor \frac{\delta}{2} \rfloor$.

Proof.

1. Let S be any strong secure set of G and let v be an arbitrary vertex that belongs to S . Since S is a strong secure set, $|N[v] \cap S| > |N[v] - S|$. Hence, S contains at least $\lceil \frac{\deg v}{2} \rceil$ neighbours of v . Thus, $|S| \geq \lceil \frac{\delta}{2} \rceil + 1$.
2. Let W be a subset of $V(G)$ such that $|W| = n - \lfloor \frac{\delta}{2} \rfloor$, and X be a nonempty subset of W . Then, $|N[X] \cap W| \geq 1 + \lceil \frac{\delta}{2} \rceil$ and $|N[X] - W| \leq \lfloor \frac{\delta}{2} \rfloor$. Hence, by the definition, the set W is a strong secure set.

□

The next proposition gives us the conditions that must be satisfied when the security number is small, i.e, less than or equal to 3. Moreover, the exact values of the security number of complete graphs, cycles, subcubic graphs, and some of grid-like graphs are given.

Proposition 2.8. [6] *Let G be a graph.*

1. (a) $s(G) = 1$ if and only if $\delta(G) \leq 1$.
 (b) $s(G) = 2$ if and only if $\delta(G) \geq 2$ and G has a set $S = \{u, v\}$ where u and v are adjacent and $|N[S] - S| \leq 2$.
 (c) $s(G) = 3$ if and only if $s(G) \notin \{1, 2\}$ and G has a set $S = \{u, v, w\}$ where $|N[S] - S| \leq 3$ and $\langle S \rangle$ is either K_3 or $P_3 = \langle u, v, w \rangle$ and, in the latter case, $|N(u) \cap (N[S] - S)| \leq 2$ and $|N(w) \cap (N[S] - S)| \leq 2$.
2. $s(K_n) = \lceil n/2 \rceil$.
3. $s(C_n) = 2$.
4. (a) $s(P_m \times P_n) = \min\{m, n, 3\}$.
 (b) $s(C_m \times P_n) \leq \{m, 2n, 6\}$.
 (c) $s(C_3 \times C_3) = 4$ and $s(C_m \times C_n) \leq \{2m, 2n, 12\}$.
5. Let G be a graph with maximum degree three. If G is not a forest, let g be its girth. Define k to be g if G has at most one vertex of degree 2, and to be the number of vertices in a shortest path between degree 2 vertices otherwise.

Then

$$s(G) = \begin{cases} 1 & \text{if } \delta(G) \leq 1, \\ 2 & \text{if } k = 2 \text{ or } G \text{ contains either} \\ & K_4 - e \text{ or a } K_3 \text{ with a degree 2} \\ & \text{vertex,} \\ \max\{3, \min\{k, g - 1\}\} & \text{if } G \text{ contains an induced } K_{2,3} \\ & \text{or a } C_g \text{ with a degree 2 vertex,} \\ \min\{k, g\} & \text{otherwise.} \end{cases}$$

In [15] Dutton et al. gave the exact value of security number for complete k -partite graphs.

Theorem 2.9. [15] *Let K_{n_1, n_2, \dots, n_k} be a complete k -partite graph of order $n = n_1 + \dots + n_k$. If K_{n_1, n_2, \dots, n_k} is not a star or an edgeless graph, then $s(K_{n_1, n_2, \dots, n_k}) = \lceil \frac{n}{2} \rceil$.*

In the forthcoming proposition we characterize graphs for which the strong security number equals 1, 2 or 3.

Proposition 2.10. *Let $G = (V, E)$ be a connected graph. Then,*

1. $\hat{s}(G) = 1$ if and only if $|V| = 1$,
2. $\hat{s}(G) = 2$ if and only if there exists a set $S = \{v_1, v_2\}$ of vertices such that v_1 and v_2 are adjacent and $|N[S] - S| \leq 1$,
3. $\hat{s}(G) = 3$ if and only if $\hat{s}(G) \notin \{1, 2\}$ and there exists a set $S = \{v_1, v_2, v_3\}$ such that $|N[S] - S| \leq 2$ and $\langle S \rangle$ is either K_3 or $P_3 = \langle v_1, v_2, v_3 \rangle$ and in the latter case $|N(v_1) \cap (N[S] - S)| \leq 1$ and $|N(v_3) \cap (N[S] - S)| \leq 1$.

Proof. Let S be a minimum strong secure set of G . The sufficiency of 1, 2 and 3 is straightforward. Hence, we concentrate on the necessity part of the proof. By Condition (2.2), $|N[S] - S| < \hat{s}(G)$. This establishes 1. Furthermore, the subgraph induced by S is connected. Hence, if $\hat{s}(G) = 2$, then the vertices of the minimum strong secure set must be adjacent, which establishes 2. If $\hat{s}(G) = 3$, then $\langle S \rangle$ is K_3 or P_3 and $|N[S] - S| \leq 2$. If $\langle S \rangle$ is a path $\langle v_1, v_2, v_3 \rangle$ and one of its pendant vertices, say v_1 , has two neighbours in $N[S] - S$, then $|N[v_1] \cap S| = 2$ and $|N[v_1] - S| = 2$, which contradicts the strong security of S . Thus, $|N(v_1) \cap (N[S] - S)| \leq 1$ and $|N(v_3) \cap (N[S] - S)| \leq 1$. \square

Now, we provide the values of the strong security number for complete graphs, complete bipartite graphs and cycles.

Proposition 2.11.

1. $\hat{s}(C_n) = 3$ if $n \geq 4$; otherwise $\hat{s}(C_n) = 2$.
2. $\hat{s}(K_n) = \lceil \frac{n+1}{2} \rceil$.
3. $\hat{s}(K_{n,m}) = \lceil \frac{n+m+1}{2} \rceil$.

Proof.

1. The result follows from Proposition 2.10.
2. Let S be any subset of vertices in K_n such that $|S| = \lceil \frac{n+1}{2} \rceil$. Since every vertex of a complete graph is universal, for any $X \subseteq S$, $|N[X] \cap S| > |N[X] - S|$, and thus the set S is a strong secure set of K_n .
3. Let V_1 and V_2 be sets of independent vertices of $K_{n,m}$ of cardinality n and m , respectively. First, observe that any strong secure set S of G must contain vertices from V_1 and V_2 , which implies that it is also a dominating set. Hence, by Condition (2.2), S must contain more than half of the vertices of the graph. Now we show that there is a strong secure set of cardinality $\lceil \frac{n+m+1}{2} \rceil$. Let S be a union of $\lceil \frac{n}{2} \rceil$ vertices of V_1 and $\lceil \frac{m}{2} \rceil$ vertices of V_2 . If n and m are even we add one additional vertex that we can arbitrarily chose. Assume that in such a case we add a vertex of V_2 . Now, let X be any nonempty subset of S . If $X \cap V_1 \neq \emptyset$ and $X \cap V_2 \neq \emptyset$, then X is a dominating set and since $|S| > |V - S|$, $|N[X] \cap S| > |N[X] - S|$. Hence let us assume that $X \cap V_2 = \emptyset$. It follows that if n and m are even, then $|N[X] \cap S| = |X| + \frac{m}{2} + 1$ and $|N[X] - S| = \frac{m}{2} - 1$; otherwise $|N[X] \cap S| = |X| + \lceil \frac{m}{2} \rceil \geq 1 + \lceil \frac{m}{2} \rceil$ and $|N[X] - S| = \lceil \frac{m}{2} \rceil$. If $X \cap V_2 \neq \emptyset$, then $|N[X] \cap S| = |X| + \lceil \frac{n}{2} \rceil \geq 1 + \lceil \frac{n}{2} \rceil$ and $|N[X] - S| = \lceil \frac{n}{2} \rceil$. Hence, in every case $|N[X] \cap S| > |N[X] - S|$, which implies that the set S is strong secure.

□

From the definition of strong secure sets, it immediately follows that for any graph G , $\hat{s}(G) \geq s(G)$. The equality holds for, e.g., complete graphs of odd order.

However, there exists no constant k such that $\hat{s} \leq s(G) + k$. Let G be a graph obtained from a complete graph of even order $n \geq 4$ to which we add a pendant vertex (see Fig. 2.1). Clearly, $s(G) = 1$ (realized by a pendant vertex) and $\hat{s}(G) = \frac{n}{2} + 1$ (realized by any $\frac{n}{2} + 1$ vertices of degree $n - 1$).

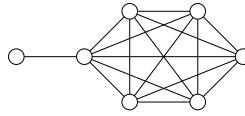


FIGURE 2.1: A graph G with $s(G) = 1$ and $\hat{s}(G) = 4$.

2.3 Secure sets in cubic graphs

In this section we concentrate on *cubic graphs*, i.e., graphs with all vertices of degree 3. In [6] the authors presented the following theorem.

Theorem 2.12. [6] *If G is a cubic graph with girth g , then*

$$s(G) = \begin{cases} 2 & \text{if } G \text{ contains a } K_4 - e, \\ 3 & \text{if } G \text{ contains a } K_{2,3}, \\ g & \text{otherwise.} \end{cases}$$

Before we present a similar result concerning strong security number, we give a necessary and sufficient conditions that must be satisfied by a subgraph of a cubic graph so that its vertices form a strong secure set. The forthcoming results are included in [31].

Lemma 2.13. *Let G be a cubic graph and H be its connected subgraph. $V(H)$ is a strong secure set of G if and only if $|N_G[V(H)] - V(H)| < |V(H)|$ and $\delta(H) \geq 2$.*

Proof. (\Rightarrow) First, let us assume that H is a connected subgraph of G such that $V(H)$ is a strong secure set of G . Thus $V(H)$ satisfies Condition (2.2). Moreover, every vertex of H also satisfies Condition (2.2), which implies that $\delta(H) \geq 2$, and the necessity (\Rightarrow) follows.

(\Leftarrow) Suppose that a connected subgraph H of G satisfies the following two conditions: $|N_G[V(H)] - V(H)| < |V(H)|$ and $\delta(H) \geq 2$. We show that $V(H)$ is a strong secure set of G . If $V(H) = V(G)$, then clearly the lemma is true. Hence suppose that $V(H) \subset V(G)$. Since $\delta(H) \geq 2$, every vertex of H has at most one neighbour outside H . If $V(H)$ is a strong secure set of G , then it must be a secure set in a graph obtained from G by adding a universal vertex. Let G' be such a graph with a universal vertex v . Now, any vertex of H is also adjacent to v . In any attack A on $V(H)$, there is at most one vertex that is being attacked by two vertices of $V(G') - V(H)$. Let us denote this vertex by x . Since $|N_{G'}[V(H)] - V(H)| \leq |V(H)|$, there is a vertex $y \in V(H)$ that is not attacked. If $xy \in E(H)$, then y can be a defender of x , and clearly the set is secure; otherwise let $P = \{v_1, \dots, v_k\}$ be a path connecting x and y such that $V(P) \subseteq V(H)$, $xv_1 \in E(G)$ and $v_ky \in E(G)$. Then the attack A can be repelled as follows: all vertices except $\{y, v_1, \dots, v_k\}$ defend themselves, y defends v_k , v_1 defends x , and v_j defends v_{j-1} , where $j \in \{2, \dots, k\}$. Thus, $V(H)$ is secure in G' , which implies that $V(H)$ is a strong secure set of G . \square

Now, we concentrate on the structure of strong secure sets of minimum cardinality. Let G be a cubic graph and H be an induced subgraph of G . We say that H is of

- *type 1* if H is a cycle such that at least two of its vertices have a common neighbour outside H ,
- *type 2* if the vertices of H induce two cycles C_1, C_2 such that $C_1 \cap C_2$ is a path with at least two vertices,
- *type 3* if the vertices of H induce two vertex disjoint cycles C_1, C_2 joined by a path.

Remark 2.14. Let H be an induced subgraph of a cubic graph G . If H is a graph of type 1, type 2 or type 3, then, by Lemma 2.13, $V(H)$ is a strong secure set.

Lemma 2.15. *Let G be a connected cubic graph. If a set S is a minimum strong secure set, then the vertices of S induce either a subgraph of type 1, type 2 or type 3.*

Proof. Let H be a subgraph of G induced by the vertices of S . Observe, that by condition (2.2), $\delta(H) \geq 2$. Thus, H cannot be acyclic. So let us suppose that H is a cycle C' . Since G is cubic, every vertex of S has exactly one neighbour outside

S . If no two vertices of S have a common neighbour, then $|N_G[S] - S| = |S|$, and we obtain a contradiction with the strong security of S . Hence, H is a subgraph of type 1. Observe that if H is not a cycle, then it must contain at least two cycles. Hence, let us assume that H has at least two cycles. Let us denote by C_1 and C_2 two shortest cycles in H . Thus, C_1 and C_2 are induced cycles. This fact implies that neither $V(C_1) \subset V(C_2)$ nor $V(C_2) \subset V(C_1)$.

Suppose that C_1 and C_2 are vertex disjoint. Since S is a minimum strong secure set, H is connected. Thus, there is a path P in H with one end-vertex in C_1 and the second end-vertex in C_2 . Let $S' = V(P) \cup V(C_1) \cup V(C_2)$. If $S \neq S'$, then we obtain a contradiction with the minimality of S , since by Lemma 2.13, S' is a strong secure set. If $S = S'$ and H is not of type 3, then $E(H)$ must contain an edge e that joins a vertex of C_1 with a vertex of C_2 and $e \notin E(P)$. This implies that H contains a proper subgraph of type 2, which contradicts the minimality of H . So let us suppose that $M = V(C_1) \cap V(C_2) \neq \emptyset$. It follows, that $|V(C_1)| > 3$ and $|V(C_2)| > 3$ since otherwise H would have a proper induced subgraph of type 1. Since G is a cubic graph and C_1, C_2 are induced cycles of G , the vertices of M induce a linear forest F . If F is not a path, then H contains a proper induced subgraph of type 2, which contradicts the minimality of H . If $S \neq V(C_1) \cup V(C_2)$ or there is an edge between vertices that belong to $(V(C_1) \cup V(C_2)) - (V(C_1) \cap V(C_2))$, then as previously we obtain a contradiction with the minimality of S . Hence H is of type 2. \square

Remark 2.16. Let H be an induced subgraph of a cubic graph G . If H is a graph of type 1, type 2 or type 3 and we add to H a new edge e between the vertices of degree 2, then $H + e$ contains a proper induced subgraph of type 1, type 2 or type 3. This implies that if the vertices of a strong secure set $S \subseteq V(G)$ induce a graph that contains a proper subgraph (not necessarily induced) of type 1, type 2 or type 3, then S is not a minimum strong secure set.

On the basis of the Lemma 2.15, we can formulate an analogue of Theorem 2.12. Let us introduce two parameters for a cubic graph G . By $c(G)$ we denote the length of the shortest cycle C of G such that at least two of the vertices of C have a common neighbour outside the cycle, and by $t(G)$ we denote the cardinality of a minimum subgraph of G that has at least two cycles.

Theorem 2.17. *Let G be a cubic graph. Then, $\hat{s}(G) = \min \{c(G), t(G)\}$.*

In [31] Sidorowicz and Jesse-Józefczyk determined the following upper bound on the strong security number for cubic graphs.

Theorem 2.18. [31] *Let G be a connected cubic graph with n vertices, where $n \geq 12$. Then, $\hat{s}(G) \leq \frac{n}{2} + 1$.*

Proof. Assume the contrary that there is a connected cubic graph G such that $\hat{s}(G) \geq n/2 + 2 \geq 8$. Let us consider three cases.

Case 1. G has a minimum strong secure set that induces a graph H of type 3.

Thus, $|V(H)| \geq n/2 + 2$ and H contains two vertex disjoint cycles C_1, C_2 joined by a path P . Let $V(C_1) = \{v_1, \dots, v_k\}$, $V(C_2) = \{v'_1, \dots, v'_p\}$, $V(P) = \{v''_1, \dots, v''_t\}$. Furthermore, assume that $v_1 = v''_1$ and $v'_1 = v''_t$. Since G is cubic, each vertex of H except v_1 and v'_1 has a neighbour outside H . For $v \in V(H) - \{v_1, v'_1\}$ let $n(v)$ denote the neighbour of v outside H . We assumed that $|V(H)| \geq n/2 + 2$, it follows that $|V(G) - V(H)| \leq n/2 - 2$. This implies that there are two vertices of H that have the same neighbour in $V(G) - V(H)$. First, suppose that there are two vertices $v_i, v_j \in V(C_1)$ such that $n(v_i) = n(v_j)$. Thus, C_1 is a graph of type 1 and by Lemma 2.13, $V(C_1)$ is a strong secure set of G . This implies that G has a strong secure set with less vertices than H , a contradiction. Now, assume that there is a vertex $v_i \in V(C_1)$ and $v''_j \in V(P)$ such that $n(v_i) = n(v''_j) = u$. Thus, vertices $V(C_1) \cup \{u, v''_2, \dots, v''_t\}$ induce a graph containing a graph of type 2, which form a strong secure set of G with less vertices than H , a contradiction. Suppose that there are two vertices $v''_i, v''_j \in V(P)$ such that $n(v''_i) = n(v''_j) = u$. Thus, vertices $V(C_1) \cup \{v''_2, \dots, v''_t, u\}$ induce a graph containing a graph of type 3 with less vertices than H , a contradiction. The symmetry of H and the above arguments shows that there is no vertex in $V(G) - V(H)$ that is adjacent to

1. two vertices of C_1 , or
2. two vertices of C_2 , or
3. two vertices of P , or
4. a vertex of C_1 and a vertex of P , or
5. a vertex of C_2 and a vertex of P .

This implies that if $u \in V(G) - V(H)$ and $|N(u) \cap V(H)| > 1$, then u has one neighbour in C_1 and the other one in C_2 , and furthermore $|N(u) \cap V(H)| = 2$. Since

$|V(H)| \geq n/2 + 2$ and each vertex of $V(G) - V(H)$ has at most two neighbours in $V(H)$, it follows that in $V(G) - V(H)$ there are at least two vertices with two neighbours in $V(H)$. Thus, without loss of generality, we can assume there is a vertex u such that $u = n(v_i) = n(v'_j)$ and $j \leq \lceil p/2 \rceil$ and $i \leq \lceil k/2 \rceil$. Now, we can observe that if $p > 3$, then the vertices $V(C_1) \cup V(P) \cup \{v'_1, \dots, v'_j\} \cup \{u\}$ induce a subgraph containing a graph of type 2; otherwise the vertices $\{v_1, \dots, v_i\} \cup V(P) \cup \{v'_1, \dots, v'_j\} \cup \{u\}$ induce a subgraph containing a graph of type 1. In both cases the indicated sets have less vertices than H , and we obtain a contradiction with the assumption that $V(H)$ is a minimum strong secure set.

Case 2. G has a minimum strong secure set that induces a graph H of type 2.

Thus, H contains two cycles C_1, C_2 such that $C_1 \cap C_2$ is a path P . Let v_1 be the first vertex of P and v_t be the last vertex of P . Observe that we can see H as two vertices v_1, v_t joined by three paths. Without loss of generality we may assume that P is the shortest path joining v_1 and v_t . Let v'_{t+1}, \dots, v'_k be the vertices of the second path and v''_{t+1}, \dots, v''_p be the vertices of the third path such that $v_1, \dots, v_t, v'_{t+1}, \dots, v'_k, v_1$ is the cycle C_1 and $v_1, \dots, v_t, v''_{t+1}, \dots, v''_p, v_1$ is the cycle C_2 . Furthermore, assume that $p \geq k$. Since $n \geq 12$ and $|V(H)| \geq n/2 + 2$, we have that $|V(H)| \geq 8$. Since G is cubic, each vertex has a neighbour outside H except v_1 and v_t . For $v \in V(H) - \{v_1, v_t\}$, let $n(v)$ denote the neighbour of v outside H . Since $|V(H)| \geq n/2 + 2$, it follows that there are two vertices of H with the same neighbour outside H . First suppose that there are two vertices $e, f \in V(C_1)$ such that $n(e) = n(f)$. Thus, C_1 is a graph of type 1 and hence $V(C_1)$ is a strong secure set of G , which contradicts the minimality of $V(H)$. Similarly we can show that there are no two vertices $r, s \in V(C_2)$ such that $n(r) = n(s)$. This implies that if $u \in V(G) - V(H)$ and $|N(u) \cap V(H)| > 1$, then u has one neighbour in C_1 and the other one in C_2 and furthermore $|N(u) \cap V(H)| = 2$. Similarly as above, in $V(G) - V(H)$ there are at least two vertices with two neighbours in H . Let $u, w \in V(G) - V(H)$ be such vertices. Observe that if the path P has more than two vertices, then $(C_1 \cup C_2) - \{v_2, \dots, v_{t-1}\}$ is a graph of type 1 with less vertices than H . Hence $|V(P)| = 2$. Moreover, we have $|V(C_2)| - |V(P)| \geq 3$, since $|V(H)| \geq 8$. Let v''_j be a neighbour of u in C_2 and v''_s be a neighbour of w in C_2 . Without loss of generality we can assume that $j < s$. If v''_s is adjacent to v_1 , then the subgraph induced by $V(C_1) \cup \{v''_s, w\}$ induce a graph of type 2 with less vertices than H ; otherwise $V(C_1) \cup \{v''_{t+1}, \dots, v''_j\} \cup \{u\}$ induce a graph

of type 2 with less vertices than H . In both cases we obtain a contradiction with the minimality of the chosen strong secure set.

Case 3. G has a minimum strong secure set that induces a graph H of type 1.

Thus, H is a cycle $v_1, v_2, \dots, v_k, v_1$. Suppose that there is a vertex $u \in V(G) - V(H)$ that has three neighbours v_j, v_s, v_t in H such that $1 \leq j < s < t \leq k$. Let P_1, P_2 and P_3 be three paths that join pairs $(v_j, v_s), (v_s, v_t)$ and (v_t, v_j) , respectively. Thus, $V(P_1) = \{v_j, \dots, v_s\}$, $V(P_2) = \{v_s, \dots, v_t\}$, and $V(P_3) = \{v_t, \dots, v_k, v_1, \dots, v_j\}$. Let V_i be the set of interior vertices of the path P_i . Assume that there is P_i that has at least two interior vertices. Then, the vertices $(V(H) - V_i) \cup \{u\}$ induce a graph of type 2 with less vertices than H , a contradiction. If there is P_i that has one interior vertex, then the vertices $(V(H) - V_i) \cup \{u\}$ induce a graph of type 2 with the same number of vertices as H and this case was considered in Case 2. Therefore, we have that $V_i = \emptyset$ for $i \in \{1, 2, 3\}$ which implies that $G = K_4$.

Thus, we may assume that each vertex of $V(G) - V(H)$ has at most two neighbours in H . Since $|V(G) - V(H)| \leq n/2 - 2$, at least two vertices of $V(G) - V(H)$ have two neighbours in H . Let c, d be such vertices and v_j, v_s, v_t, v_p be their neighbours in H . Assume that $1 \leq j < s < t < p \leq k$. Let P_1, P_2, P_3 and P_4 be paths that join pairs $(v_j, v_s), (v_s, v_t), (v_t, v_p)$ and v_p, v_j , respectively. Thus, $V(P_1) = \{v_j, \dots, v_s\}$, $V(P_2) = \{v_s, \dots, v_t\}$, $V(P_3) = \{v_t, \dots, v_p\}$ and $V(P_4) = \{v_p, \dots, v_k, v_1, \dots, v_j\}$. Let V_i be the set of interior vertices of the path P_i .

First assume that there is a path P_i having at least two interior vertices. Then, consider a subgraph H' induced by $(V(H) - V_i) \cup \{u, w\}$ where $|V_i| \geq 2$. H' contains a graph either of type 2 or 3. If $|V_i| \geq 3$, then H' has less vertices than H , a contradiction. If $|V_i| = 2$, then H' has the same number of vertices as H . If H' is not a subgraph of type 2 or 3, then it contains such a subgraph with less vertices than H , a contradiction; otherwise, this case was considered in Case 1 or Case 2.

Observe that if $n \geq 14$, H has at least 9 vertices and hence at least one path P_i with at least two interior vertices. Thus, to complete the proof we must consider the case when $n = 12$ (then H has exactly 8 vertices) and each path has exactly one interior vertex. First observe that if in $V(G) - V(H)$ there are more than two vertices having two neighbours in H , then there are two such vertices that at least one path between their neighbours has more than one interior vertex. Thus, without loss of generality, we may assume that in $V(G) - V(H)$ there are exactly

two vertices u, w having two neighbours in H and four having one neighbour in H . Assume that the vertices u and w are adjacent. Consider the subgraph H' induced by $V(H) - (V_1 \cup V_3)$. Observe that H' contains a graph either of type 2 or of type 3, regardless of existing edges between u, w and v_j, v_s, v_t, v_p . Thus, either we obtain a contradiction with the choice of H or we have a case that was already considered. Therefore, u has a neighbour $g \in V(G) - V(H)$ such that g is adjacent to $V(H) - \{v_j, v_s, v_t, v_p\}$. In this case we can always indicate a subgraph H' that contains a graph of type 2, contains u, g , and a subpath of H , and $V(H') \leq V(H)$. Thus, either we obtain a contradiction with the choice of H or we have a case that was already considered. \square

The bound presented in the above theorem is tight. There is a cubic graph of order 14 with a strong security number equal to 8 (see Fig. 2.2). All cubic graphs

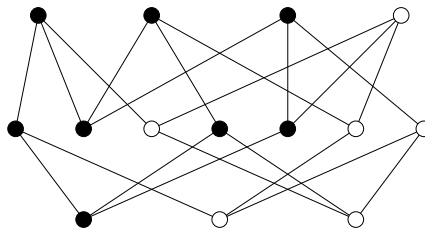


FIGURE 2.2: A cubic graph of order 14 with a strong security number equal to 8, the black vertices form a minimum strong secure set.

of order $n \leq 10$, except for those presented in the next figure, have a strong secure set of cardinality less or equal to $n/2 + 1$.



FIGURE 2.3: Graphs of order 8 with a strong security number equal to 6, black vertices form a minimum strong secure set.

Chapter 3

Global secure sets

In this chapter we deal with global secure sets. First we present some of their properties, and in the second part of the chapter we give an upper bound on the global security number for trees and cactus trees.

3.1 Properties of global secure sets

The global secure sets are both secure and dominating sets. It follows that for any global secure set S of a graph G , $|N[S] - S| = |V(G) - S|$. Hence, by Theorem 2.2,

$$\gamma_s(G) \geq \left\lceil \frac{|V(G)|}{2} \right\rceil. \quad (3.1)$$

By similar arguments it can be shown that

$$\gamma_s(G) \geq \left\lceil \frac{|V(G)| + 1}{2} \right\rceil. \quad (3.2)$$

From the definitions follows that for any graph G ,

$$\gamma(G) \leq \gamma_a(G) \leq \gamma_s(G) \leq \gamma_s(G). \quad (3.3)$$

There are many families of connected graphs for which $\gamma_a(G) = \gamma_s(G)$, e.g., complete graphs, cycles, paths. Since for any connected graph of order n the domination number is at most $\lfloor n/2 \rfloor$, there are no graphs with domination number

equal to global strong security number. The question is whether there exist graphs for which $\gamma(G) = \gamma_s(G)$. In [17] Fink et al. proved that for a graph of order $2k$, $\gamma(G) = k$ iff G is a cycle on four vertices or G can be obtained by adding a pendant vertex to each vertex of a connected graph H of order k . The structure of these graphs allows us to prove the following proposition.

Proposition 3.1. *Let G be a graph of order $2k$. If $\gamma(G) = k$, then $\gamma_s(G) = k$ and $\gamma_{\hat{s}}(G) = k + 1$.*

Proof. If G is C_4 , then clearly the proposition is true. Hence, assume that this is not the case. Let P be a set of pendant vertices of G . Since, $\gamma(G) = k$, $|P| = |V(G) - P| = k$ and every vertex of $V(G) - P$ has exactly one neighbour in P . Clearly, both P and $V(G) - P$ are global secure sets of G . We show, that for any $v \in P$, the set $S = (V(G) - P) \cup \{v\}$ is a global strong secure set of G of cardinality $k + 1$. Let w be a neighbour of v , and X be a subset of S . If $v \in X$ or $w \in X$ then, $|N[X] \cap S| \geq |X|$ and $|N[X] - S| < |X|$ since neither v nor w has neighbours in $V - S$; otherwise since $\langle S \rangle$ is connected and $v, w \notin X$, $|N[X] \cap S| \geq |X| + 1$ and $|N[X] - S| = |X|$. Hence, S is a global strong secure set of G . \square

The next lemma shows that any vertex of a global secure set can be replaced with an universal vertex.

Lemma 3.2. *Let G be a graph with a global secure set SD and v be a vertex that belongs to SD . If there exists a vertex v' such that $v' \notin SD$ and $v' \in \mathcal{U}(G)$, then $SD' = (SD - \{v\}) \cup \{v'\}$ is a global secure set of G .*

Proof. Since $v' \in SD'$, SD' is a dominating set, so it is sufficient to show that it is also a secure set. Once again we will use Theorem 2.2. Let $X \subseteq SD'$. If $v' \in X$, then obviously $|N[X] \cap SD'| \geq |N[X] - SD'|$. So let us suppose that $v' \notin X$. If $v \in N[X]$, then $|N[X] \cap SD'| = |N[X] \cap SD| + |\{v'\}| - |\{v\}|$ and $|N[X] - SD'| = |N[X] - SD| - |\{v'\}| + |\{v\}|$, so $|N[X] \cap SD'| \geq |N[X] - SD'|$. We have to consider the last case when $v \notin N[X]$. Then $|N[X] \cap SD'| = |N[X] \cap SD| + |\{v'\}|$ and $|N[X] - SD'| = |N[X] - SD| - |\{v'\}|$. Thus $|N[X] \cap SD'| \geq |N[X] - SD'|$, and this observation completes the proof. \square

Now we show that if a graph G is a join of graphs G_1 and G_2 , then we can obtain a global (strong) secure set of G as a union of some (strong) secure sets of G_1 and G_2 .

Lemma 3.3. *Let k, l be positive integers and let G_1 and G_2 be disjoint graphs of order $n_1 \geq 0$ and $n_2 \geq 0$, respectively. If $k \geq \frac{n_1}{2}$ and $l \geq \frac{n_2}{2}$, then $S(G_1)_k \cup S(G_2)_l$ is a global secure set of cardinality $k + l$ in the graph $G = G_1 + G_2$.*

Proof. Let S_1 and S_2 denote the sets $S(G_1)_k$ and $S(G_2)_l$, respectively. From the definition of the join of graphs and the fact that $k, l \geq 1$, it follows that the set $S' = S_1 \cup S_2$ is a dominating set of the graph $G = G_1 + G_2$. We will show that S' is secure. From Theorem 2.2, it follows that it is sufficient to show that $|N[X] \cap S'| \geq |N[X] - S'|$ for every set $X \subseteq S'$. If $X \subseteq S_1$, then $|N[X] \cap S'| = |N[X] \cap S_1| + l$ and $|N[X] - S'| = |N[X] - S_1| + n_2 - l$. Since $|N[X] \cap S_1| \geq |N[X] - S_1|$ and $l \geq \frac{n_2}{2}$, we have a desired inequality. Similarly, we can prove that $|N[X] \cap S'| \geq |N[X] - S'|$, for every $X \subseteq S_2$. Finally, we have to consider the case when $X \cap S_1 \neq \emptyset$ and $X \cap S_2 \neq \emptyset$. Now $|N[X] \cap S'| = k + l$ and $|N[X] - S'| = n_1 + n_2 - k - l$, so clearly $|N[X] \cap S'| \geq |N[X] - S'|$. Thus the set S' is a global secure set in G . \square

Lemmas 3.2 and 3.3 are included in [30]. Similarly as Lemma 3.3, we can prove the next result.

Lemma 3.4. *Let k, l be positive integers and let G_1 and G_2 be disjoint graphs of order $n_1 \geq 0$ and $n_2 \geq 0$, respectively. If $k > \frac{n_1}{2}$ and $l > \frac{n_2}{2}$, then $S(G_1)_k \cup S(G_2)_l$ is a global strong secure set of cardinality $k + l$ in the graph $G = G_1 + G_2$.*

The above lemmas lead to the following corollary.

Corollary 3.5. *Let G_1 and G_2 be disjoint graphs of even order n_1 and n_2 , respectively. Then, $\gamma_s(G_1 + G_2) \leq \gamma_s(G_1) + \gamma_s(G_2)$ and $\gamma_s(G_1 + G_2) \leq \gamma_s(G_1) + \gamma_s(G_2)$.*

The knowledge of the global security number of two graphs, allows us to give an upper bound on the global security number of their Cartesian product. To simplify the proofs, a global secure set of a graph G of cardinality $\gamma_s(G)$, we call a $\gamma_s(G)$ -set.

Theorem 3.6. *Let G_1 and G_2 be connected graphs of order n_1 and n_2 , respectively. Then, $\gamma_s(G_1 \times G_2) \leq \min\{\gamma_s(G_1)n_2, \gamma_s(G_2)n_1\}$.*

Proof. Since $G_1 \times G_2 = G_2 \times G_1$, without loss of generality, we can assume that $\gamma_s(G_1)n_2 \leq \gamma_s(G_2)n_1$. Let S be a $\gamma_s(G_1)$ -set. We claim that the set $W = \{(v, u) : v \in S \text{ and } u \in V(G_2)\} \subseteq V(G)$ is a global secure set of G of cardinality $\gamma_s(G_1)n_2$. We know that the graph $G_1 \times G_2$ can be obtained by a simple procedure. Step 1: we make n_2 disjoint copies of G_1 , where each copy corresponds to a vertex of G_2 (we obtain the graph n_2G_1). Step 2: if two vertices of G_2 are adjacent, then we add edges between the vertices of the corresponding copies of G_1 . Namely, we join each vertex of one copy with its counterpart in the other copy of G_1 . Let S_i denote the set S in the i -th copy of G_1 . Then, $W = S_i \cup \dots \cup S_{n_2}$. Clearly, before we make Step 2 of the procedure, W is a global secure set of n_2G_1 . However, observe that in Step 2 we add only edges whose both end-vertices are either in or outside W , hence W is a global secure set of G . \square

The above proposition can be used, e.g., to determine the global security number of an important class of graphs called *hypercubes*. A hypercube R_1 is the graph K_2 , and for $k \geq 2$, $R_k = R_{k-1} \times K_2$.

Corollary 3.7. *Let G be a hypercube of order n . Then $\gamma_s(G) = \frac{n}{2}$.*

3.2 Global secure sets in trees

First part of this section is devoted to trees, i.e., graphs that have no cycles. We give an upper bound on their global security number. Before that, let us recall some results concerning global defensive alliances. In [23] the authors gave an upper and lower bound on global alliance number of a tree. The upper bound is attained by the class of trees denoted by the authors by \mathcal{T}_1 . A tree $T \in \mathcal{T}_1$ is obtained from the disjoint union of t ($t \geq 1$) copies of $K_{1,4}$ by adding $t - 1$ edges between leaves of the copies in such a way that the center of each $K_{1,4}$ is adjacent to exactly three leaves in T . Additionally, a path on five vertices (P_5) also belongs to \mathcal{T}_1 .

Theorem 3.8. [23] *If T is a tree of order $n \geq 4$, then $\gamma_a(T) \leq \frac{3}{5}n$, with equality if and only if $T \in \mathcal{T}_1$.*

Theorem 3.9. [23] *If T is a tree of order n , then $\gamma_a(T) \geq \frac{n+2}{4}$, and this bound is sharp.*

The results in this section were published in [28]. We start with two exact values of $\gamma_s(T)$. A *star* S_k is a graph $K_{1,k}$, where $k \geq 1$, and a *center* of a star is a vertex of degree k , whereas a *double star* $S_{r,k}$ is a graph which we construct by joining with an edge the centers of a star S_r and S_k .

Lemma 3.10.

1. For any star S_k , $k \geq 1$, $\gamma_s(S_k) = 1 + \lfloor \frac{k}{2} \rfloor$.
2. For any double star $S_{r,k}$, $r, k \geq 1$, $\gamma_s(S_{r,k}) = \lceil \frac{k+r+2}{2} \rceil$.

For any vertex v of a graph G , denote the set of leaves (vertices of degree 1) adjacent to v by $L(v)$, and let $l(v) = |L(v)|$. Furthermore, we say that v is a *terminal vertex* if $l(v) \geq \deg v - 1$.

Theorem 3.11. *If T is a tree of order $n \geq 2$, then $\gamma_s(T) \leq \frac{2}{3}n$, with equality if and only if $T = P_3$.*

Proof. By Lemma 3.10 the theorem holds for stars and double stars, hence it holds for all trees of order at most four or diameter at most three. Assume the theorem is true for all trees of order at most k , where $k \geq 4$, and let T be a rooted tree with $\text{diam } T \geq 4$ and order $n \geq 5 > k$. Let v be a terminal vertex of T and let u be the unique neighbour of T that is not a leaf. Furthermore, let $T' = T - v - L(v)$ and denote the order of T' by n' . Then $\text{diam } T' \geq 2$ and thus $n' \geq 3$. If there exists a $\gamma_s(T')$ -set that contains u , let S' be such a set; otherwise let S' be any $\gamma_s(T')$ -set.

If $n' = 3$, then $T' = P_3$, $u \in S'$ by the choice of S' , and $|S'| = 2 = \frac{2}{3}n'$. Form S by adding v and $\lfloor \frac{l(v)-1}{2} \rfloor$ of its leaf neighbours to S' . Then S is a global secure set of T and $|S| < \frac{2}{3}n$. Hence we may assume that $n' \geq 4$. By the induction hypothesis, $\gamma_s(T') < \frac{2}{3}n'$.

If $u \in S'$, form S by adding v and $\lfloor \frac{l(v)}{2} \rfloor$ of its leaf neighbours to S' . Let $t = |S| - |S'|$. In all cases $|\{v\} \cup L(v)|/2 \leq t \leq 2|\{v\} \cup L(v)|/3$, hence S is a global secure set of T and $|S| < \frac{2}{3}n$.

Now assume $u \notin S'$. Form S by adding v and $\lfloor \frac{l(v)}{2} \rfloor$ of its leaf neighbours to S' . As above S is a global secure set of T . Moreover, if $l(v) \neq 3$, then $|S| < \frac{2}{3}n$. If $l(v) = 3$, let $T'' = T - L(v)$ and let $|V(T'')| = n''$. Then $n'' > n' \geq 4$ and by the induction hypothesis, $\gamma_s(T'') < \frac{2}{3}n''$. Let S'' be any $\gamma_s(T'')$ -set. If $v \in S''$, form S by adding two vertices in $L(v)$ to S'' . If $v \notin S''$, then $u \in S''$ and as above we form S by adding v and one of its leaf neighbours to S'' . In each case S is a global secure set of T and $|S| < \frac{2}{3}n$. \square

We proved that there exists only one graph for which the upper bound is reached. However, there exists a family of trees \mathcal{T} such that for any positive real number r , there is a tree $T \in \mathcal{T}$ such that $\frac{2}{3}n - r < \gamma_s(T) < \frac{2}{3}n$, where n is the order of T .

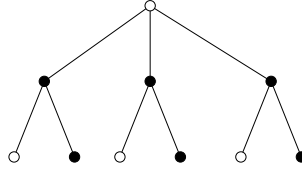


FIGURE 3.1: Example of a tree $T \in \mathcal{T}$. The black vertices form the $\gamma_s(T)$ -set.

We construct the trees which belong to \mathcal{T} by making k copies of P_3 . Next we add one vertex which we connect with the center of each copy of P_3 . The more copies of P_3 we make, the closer to the upper bound $\gamma_s(T)$ is. Therefore, if we exclude P_3 from the family of trees, we still cannot decrease the upper bound on $\gamma_s(T)$. However, there is a nontrivial class of trees for which the improvement can be made. It is a simple matter to see the correctness of the following remark.

Remark 3.12. For every tree T there exists a $\gamma_s(T)$ -set S that contains all terminal vertices of degree greater than or equal to 4. Furthermore, for every such terminal vertex we can choose one leaf neighbour that also belongs to S .

A *broom* is a tree which we obtain by connecting a center of a star with a vertex of degree 1 of a path.

Lemma 3.13. *If T is a broom or a path of order $n \geq 4$, then $\gamma_s(T) \leq \frac{3}{5}n$.*

Let \mathcal{T}_2 denote the class of trees such that $T \in \mathcal{T}_2$ if and only if $l(v) \neq 2$ for every terminal vertex v and $T \neq K_1$.

Theorem 3.14. *If $T \in \mathcal{T}_2$, then $\gamma_s(T) \leq \frac{3}{5}n$.*

Proof. From Lemma 3.10 follows that the theorem is true for all trees of diameter equal to 2 or 3 ($T \neq P_3$). So we have to consider only trees of diameter greater than or equal to 4. Suppose that our theorem is true for all trees from \mathcal{T}_2 of order n'' , where $2 \leq n'' < n$. Let T be a tree of order n and $\text{diam } T \geq 4$. If T is a broom or a path, it follows from Lemma 3.13 that the theorem is true, so we can assume that T is neither a broom nor a path. Let r be a root of T , and let v be a

terminal vertex such that $d(r, v) = \text{diam } T - 1$. Furthermore, let u be a parent of v , and w be the parent of u . Throughout this proof we use the following drawing convention. Namely, the vertices lying outside the dashed boundary are presented with their complete neighbourhood. Inside the boundary is a part of a tree which is not in the figure. The rest of the proof is organized as follows. We start with two easy cases ,i.e., when we can choose r and v in such a way that $\deg_T v \geq 6$ or $\deg_T v = 5$. Then we show that if we can choose r and v in such a way that $\deg_T v = 2$, then the theorem holds. Since $T \in \mathcal{T}_2$, T has no terminal vertices of degree 3. Thus, in the last step of the proof, we assume that for any choice of r and v , v is of degree 4.

Suppose there exist r and v such that $\deg_T v \geq 6$. Let $T' = T - \{x, y\}$, where $\{x, y\} \subset L(v)$. By Remark 3.12 there exists a $\gamma_s(T')$ -set S' that contains v . By the inductive hypothesis $|S'| \leq \frac{3}{5}n'$, where n' is the order of T' . Hence $S = S' \cup \{x\}$ is a global secure set of T and $|S| < \frac{3}{5}n$.

Now assume there exist r and v such that $\deg_T v = 5$. If $T'' \in \mathcal{T}_2$, where $T'' = T - \{v\} - L(v)$, then let S' be a $\gamma_s(T'')$ -set of T'' . Form S by adding v and two of its leaf neighbours to S' . Then S is a global secure set of T and $|S| \leq \frac{3}{5}n$. If $T'' \notin \mathcal{T}_2$, then in T we have a structure B_1 or B_2 (see Fig. 3.2). Let $T' \in \mathcal{T}_2$ be a tree obtained by removing the leaf neighbours of v , and let S' be any $\gamma_s(T')$ -set. Suppose we have the structure B_1 . If $v \in S'$ then form S by adding two of the removed leaves to S' . If $v \notin S'$ then u and w must be in it. What is more w has only one neighbour that is not in S' . Form S by adding v and one of the removed leaves to S' . In both cases S is a global secure set of T and $|S| \leq \frac{3}{5}n$. Hence we may assume that we have the structure B_2 . Then by Remark 3.12 there exists a $\gamma_s(T')$ -set S' such that v belongs to it. Form S by adding two leaf neighbours of v . Then S is a global secure set of T and $|S| < \frac{3}{5}n$.

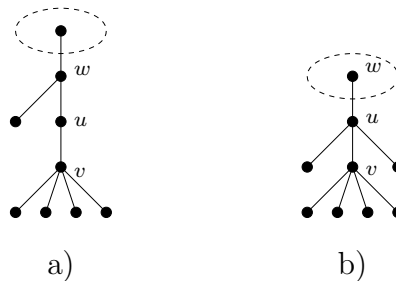


FIGURE 3.2: a) structure B_1 b) structure B_2 .

Suppose that there exist r and v such that $\deg_T v = 2$. Let z denote a leaf neighbour of v . First assume that a tree $T' = T - \{v, z\}$ belongs to \mathcal{T}_2 , and let S' be a $\gamma_s(T')$ -set. If $u \in S'$, then form S by adding v to S' ; otherwise add z to S' . The resulting set S is a global secure set of T , and $|S| \leq \frac{3}{5}n$. Hence assume that $T' \notin \mathcal{T}_2$. In this case we consider two structures: C and C' (see Fig. 3.3). Suppose

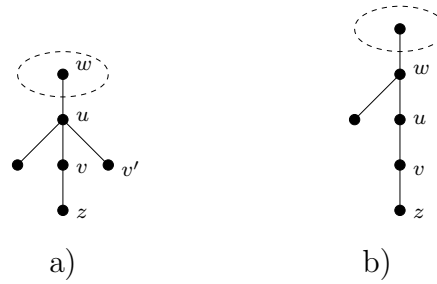


FIGURE 3.3: a) structure C b) structure C' .

that T has a structure C . Let T' be a tree obtained by removing u and all its descendants from T , and let S' be any $\gamma_s(T')$ -set. If $T' \in \mathcal{T}_2$, then clearly the set $S = S' \cup \{u, v, v'\}$ is a global secure set of T such that $|S| \leq \frac{3}{5}n$, where v' is a leaf neighbour of u . Hence let us assume that $T' \notin \mathcal{T}_2$. If T is isomorphic to a graph C_1 presented in Figure 3.4, then the theorem holds, so assume that $T \neq C_1$. Let T' be a tree obtained by removing all the descendants of u (we do not remove u). Clearly, $T' \in \mathcal{T}_2$. First suppose that T has a structure C_2 . By Remark 3.12 there exists a $\gamma_s(T')$ -set S' such that u belongs to it. Form S by adding vertices v and v' to S' , where $v' \in L(u)$. Then S is a global secure set of T and $|S| < \frac{3}{5}n$. The

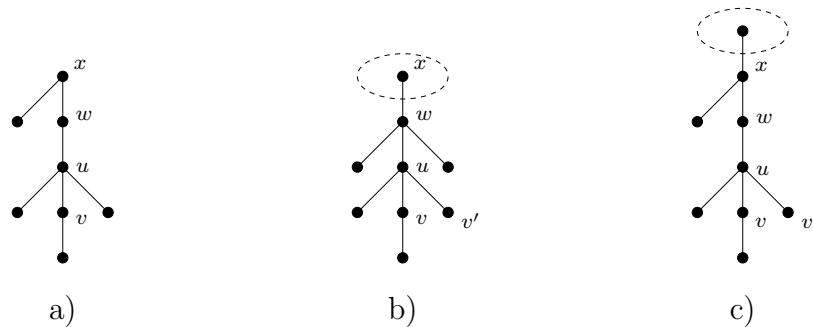


FIGURE 3.4: a) $T = C_1$ $\gamma_s(C_1)$ -set = $\{x, w, u, v\}$ b) structure C_2 c) structure C_3 .

same way as above we obtain S if we deal with the structure C_3 and u belongs to some $\gamma_s(T')$ -set. Hence let us assume that there does not exist a $\gamma_s(T')$ -set that

contains u . Let S' be any $\gamma_s(T')$ -set and x from now on denote the parent of w . Since $u \notin S'$, w , x and one of the other neighbours of x belong to S' . Form S by adding u and v to S' . One can observe that w can defend u in any attack on S . Thus S is a global secure set of T and $|S| < \frac{3}{5}n$.

Suppose now that T has a structure C' . If we can remove w and all its descendants from T and the resulting tree T' belongs to \mathcal{T}_2 , then the union of any $\gamma_s(T')$ -set and $\{u, v, w\}$ gives us a global secure set of T of the required cardinality. If T is isomorphic to C'_1 , then the theorem holds (see Fig. 3.5). Hence we can assume that $T \neq C'_1$. Now we consider the cases when T has structure C'_2 or C'_3 . Let T' be a tree obtained by removing all the descendants of w (we do not remove w). Furthermore, if there exists a $\gamma_s(T')$ -set that contains w , let S' be such a set; otherwise let S' be any $\gamma_s(T')$ -set. If in T there is a structure C'_2 , then by Remark 3.12, $w \in S'$. Form S by adding u and v to S' . If we deal with the structure C'_3 , then if $w \in S'$, let $S = S' \cup \{u, v\}$; otherwise let $S = S' \cup \{w, z\}$. In all above cases the resulting set S is a global secure set of T and $|S| < \frac{3}{5}n$.

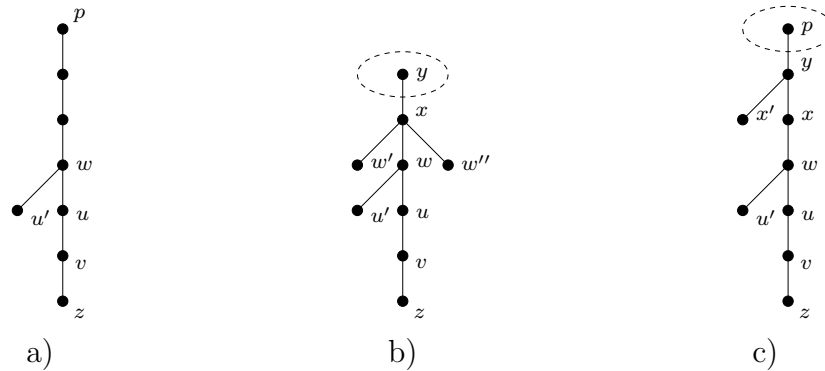


FIGURE 3.5: a) structure C'_1 $\gamma_s(C'_1)$ -set = $\{p, u', w, z\}$ b) structure C'_2 c) structure C'_3 .

There is only one more case to consider. Namely, for any choice of v and r , a vertex v is of degree 4. Suppose that u has at least one more child v' that is a terminal vertex of degree 4 or $l(u) \geq 3$. Then we remove v and all its leaf neighbours from T and one can see, that in the resulting tree T' , there exists a $\gamma_s(T')$ -set containing u . If we add to this set v and one of its leaf neighbours, then we obtain a global secure set of T of cardinality less than $\frac{3}{5}n$. Hence we may assume now that v is the only child of u that is not a leaf and $l(u) \leq 2$.

First suppose that $\deg_T u = 2$. Let T' be a tree obtained by removing u and all its descendants from T . If $T' \in \mathcal{T}_2$, then let S' be any $\gamma_s(T')$ -set. Form S by adding u , v and one of the removed leaves to S' . Clearly, S is a global secure set of

T and $|S| \leq \frac{3}{5}n$. Hence we may assume that $T' \notin \mathcal{T}_2$. Then in T there is a structure D_1 or D_2 (see Fig. 3.6). If we have a structure D_1 , then let $T' = T - \{v\} - L(v)$. By Remark 3.12 T' has a $\gamma_s(T')$ -set S' that contains u . Thus the set S , obtained by adding v and one of its leaf neighbours to S' , is a global secure set of T and $|S| < \frac{3}{5}n$. If we have a structure D_2 , then let T' be a tree of order $n' = n - 5$ which we obtain as follows. Let w' be a leaf neighbour of x (parent of w) and z be a leaf neighbour of v (see Fig. 3.6). We remove w and all of its descendants and we add a vertex w'' that is a leaf neighbour of w' in T' . The resulting tree T' belongs to \mathcal{T}_2 . Let S' be a $\gamma_s(T')$ -set. If $w' \in S'$ then form S as union of S' and $\{u, v, z\}$. If $w'' \in S'$, then remove w'' from S . Clearly S is a dominating set of T and $|S| \leq \frac{3}{5}n$. The security of S follows from the fact that when $x \in S$, w' can always defend x against w . Hence we may assume that $w' \notin S'$. Then $w'' \in S'$. If $x \in S'$ then let $S = S' \cup \{u, v, z, w\} - \{w''\}$; otherwise let $S = S' \cup \{u, v, z, w'\} - \{w''\}$. In each case S is a global secure set of T and $|S| \leq \frac{3}{5}n$. Now assume u is adjacent to

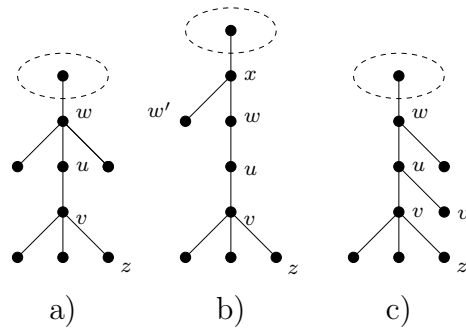


FIGURE 3.6: a) structure D_1 b) structure D_2 c) structure D_3 .

either one or two leaves v', v'' . Let T' be a tree obtained by removing v', v and $L(v)$, and let S' be any $\gamma_s(T')$ -set. If $T' \in \mathcal{T}_2$, then form S by adding $v, z \in L(v)$ and v' (if $u \in S'$) or u (if $u \notin S'$) to S' . The set S is a global secure set of T and $|S| \leq \frac{3}{5}n$. Hence suppose that $T' \notin \mathcal{T}_2$. It follows that in T there is a structure D_3 , see Fig. 3.6. From previous considerations we know that $\deg_T w = 3$ and w is adjacent to one leaf. Let T' be a tree obtained by removing v', v and $L(v)$ from T . Furthermore, we add one leaf u' adjacent to w to T' . Thus $\deg_{T'} w = 4$. By Remark 3.12 there exists a $\gamma_s(T')$ -set S' which contains u and w . If $u' \in S'$, then let $S = S' - \{u'\} \cup \{v', v, z\}$; otherwise let $S' \cup \{v, z\}$, where $z \in L(v)$. In each case S is a global secure set of T and $|S| \leq \frac{3}{5}n$. \square

Since $\gamma_a(G) \leq \gamma_s(G)$ and $\mathcal{T}_1 \subset \mathcal{T}_2$, for any $T \in \mathcal{T}_1$, $\gamma_a(T) = \gamma_s(T) = \frac{3}{5}n$.

Let G be a graph and A be a subset of $V(G)$. By $\gamma_{s_A}(G)$ we denote the minimum cardinality of a global secure set of G that contains A . We call a minimum global secure set that contains A a $\gamma_{s_A}(G)$ -set. Now, we consider global secure sets that contain a certain subset of vertices of a given tree.

Theorem 3.15. *If T is a tree of order $n \geq 2$, then for every nonempty set $A \subseteq V(T)$,*

$$\gamma_{s_A}(T) \leq \begin{cases} \frac{2}{3}n + |A| - 2 & \text{if there exist vertices } u, v \in A \text{ such that} \\ & uv \in E(T) \text{ and } \deg u \geq 2, \deg v \geq 2, \\ \frac{2}{3}n + |A| - 1 & \text{otherwise.} \end{cases}$$

Proof. If $|A| = 1$, then we can prove this theorem similarly to Theorem 3.11. Thus we may assume that $|A| \geq 2$. We use induction on the order of T . It is obvious that if $n = 2$, then the theorem is true. Hence assume the theorem is true for all trees of order at most n' , where $n' \geq 2$. Furthermore, let T be a tree of order $n \geq 3 > n'$.

First suppose that there exist vertices $u, v \in A$ such that $uv \in E(T)$ and $\deg u \geq 2, \deg v \geq 2$. Let T_u (T_v , respectively) denote the component of $T - uv$ that contains u (v , respectively). Furthermore, let $A_u = A \cap V(T_u)$ and $A_v = A \cap V(T_v)$. By the inductive hypothesis $\gamma_{s_{A_v}}(T_v) \leq \frac{2|V(T_v)|}{3} + |A_v| - 1$ and $\gamma_{s_{A_u}}(T_u) \leq \frac{2|V(T_u)|}{3} + |A_u| - 1$. One can see that the union of any $\gamma_{s_{A_v}}(T_v)$ -set and any $\gamma_{s_{A_u}}(T_u)$ -set is a global secure set of T of cardinality less or equal to $\frac{2}{3}n + |A| - 2$ and it contains all vertices of A .

Thus suppose that there are no vertices $u, v \in A$ such that $uv \in E(T)$ and $\deg u \geq 2, \deg v \geq 2$. First assume that there exists a vertex $v \in A$ such that v is a leaf in T . Let T' be a tree obtained by removing v from T . By the inductive hypothesis $\gamma_{s_{A-v}}(T') \leq \frac{2}{3}(n-1) + |A - \{v\}| - 1$. Let S' be any $\gamma_{s_{A-v}}(T')$ -set. Form S by adding v to S' . Clearly, S is a global secure set of T that contains A , and $|S| < \frac{2}{3}n + |A| - 1$. Hence suppose that there are no leaves in A . Let $P = p_1, \dots, p_m$, where $m \geq 1$, be a path between $u \in A$ and $v \in A$, and $up_1, vp_m \in E(T)$. Let us remove edges up_1, vp_m from T , and let T_v, T_u, T_P denote the resulting trees that contain u, v and P , respectively. Furthermore, let A_u, A_v and A_P denote the sets which contain the vertices that belong to A and to T_u, T_v, T_P , respectively. By the inductive hypothesis $\gamma_{s_{A_v}}(T_v) \leq \frac{2|V(T_v)|}{3} + |A_v| - 1$ and $\gamma_{s_{A_u}}(T_u) \leq \frac{2|V(T_u)|}{3} + |A_u| - 1$. If $|V(T_P)| = 1$ then form S as union of any $\gamma_{s_{A_v}}(T_v)$ -set, any $\gamma_{s_{A_u}}(T_u)$ -set and $\{p_1\}$.

Then S is a global secure set of T , $A \subseteq S$ and $|S| \leq \frac{2}{3}n + |A| - 1$. If $|V(T_P)| \geq 2$ then by the inductive hypothesis $\gamma_{s_{A_P \cup \{p_1, p_m\}}}(T_P) \leq \frac{2}{3}|T_P| + |A_P \cup \{p_1, p_m\}| - 1$. Form S as union of a $\gamma_{s_{A_v}}(T_v)$ -set, a $\gamma_{s_{A_u}}(T_u)$ -set and $\gamma_{s_{A_P \cup \{p_1, p_m\}}}(T_P)$. Then S is a global secure set of T , $A \subseteq S$ and $|S| \leq \frac{2}{3}n + |A| - 1$. \square

3.3 Global secure sets in cactus trees

A cactus tree is a graph in which every edge belongs to at most one cycle. We consider rooted cactus trees. A *connector vertex* of a cycle is a vertex which lies on this cycle and also belongs to an edge or a cycle that is closer to the root of the cactus tree (see Fig. 3.7). A *leaf cycle* of a cactus tree is a cycle such that all its vertices, except the connector vertex, are of degree two. A cactus tree may not have a leaf cycle. A *tree branch* of a cactus tree G is a maximal induced subgraph T that is a tree and has exactly one vertex that belongs to a cycle in G and $G - T$ is connected. The length of a tree branch is the maximum length of a path from the leaf to the vertex of the tree branch that is closest to the root.

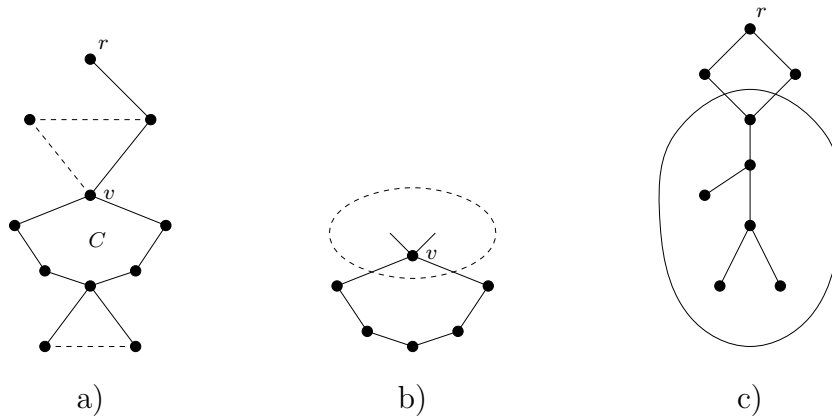


FIGURE 3.7: a) v is a connector vertex of C b) a leaf cycle (deg v equals 3 or 4) c) tree branch of length 3.

The results in this section were published in [28]. We start with a simple proposition.

Proposition 3.16. *Let x and y be end-vertices of a path P_n of order $n \geq 4$. There exists a global secure set $S_1 \subset V(P_n)$ such that $x, y \in S_1$, and there exists a global secure set S_2 such that $x, y \notin S_2$, and for $i \in \{1, 2\}$, $|S_i| \leq \frac{2}{3}n$ with equality if and only if $n = 6$.*

Now we present the theorem that gives an upper bound on the global security number of cactus trees.

Theorem 3.17. *If G is a cactus tree of order $n \geq 2$, then $\gamma_s(G) \leq \frac{2}{3}n$, with equality if and only if $G \in \{C_3, C_6, P_3\}$.*

Proof. The proof is inductive and its most important steps are the following. First we eliminate from further considerations the cases when T is a tree, a cycle or has a tree branch of length greater or equal to two. Then we show that the theorem holds if a cactus tree has a leaf cycle. In the next step of the proof we assume that G has no leaf cycles and every leaf in G is connected to a cycle. We choose a cycle such that all its descendants are leaves and we analyze case by case the structures that might appear.

It is obvious that the theorem holds for cactus trees of order two and three, hence assume it is also true for all cactus trees of order at most k , where $k \geq 3$. Let G be a rooted cactus tree of order $n \geq 4 > k$. If G is a tree, then, by Theorem 3.11, the theorem holds. Clearly, it is also true if G is a cycle. Hence we may assume that G is neither a tree nor a cycle. One can see that if we add an edge between two leaves adjacent to the same vertex, then it does not affect $\gamma_s(G)$. Consequently we can remove all edges between vertices of degree two in leaf cycles of length three. If there exists a tree branch of length greater or equal to two, then, as in the proof of Theorem 3.11, we can show that the theorem is true. Hence suppose that such a branch does not exist.

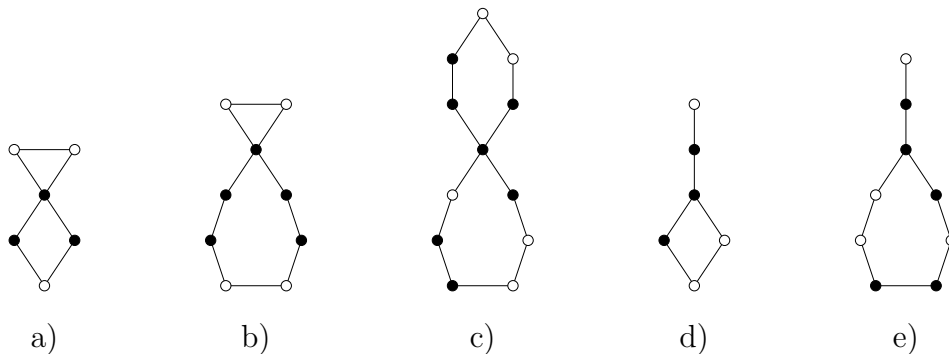


FIGURE 3.8: The black vertices belong to the global secure set of presented cactus trees.

Let us assume that there exists a leaf cycle C_k^* such that $k \geq 4$. Let G' be a cactus tree obtained by removing vertices that belong to $V(C_k^*) - \{v\}$, where v is a connector vertex. The removed vertices induce a path and by Proposition 3.16 if $k \in \{5, 6\}$ or $k \geq 8$, then we can correctly extend the $\gamma_s(G')$ -set to obtain a global secure set of G of cardinality less than $\frac{2}{3}n$. It is easy to see that if $G' \notin \{C_3, C_6, P_3\}$ and $k \in \{4, 7\}$, then the theorem also holds. The remaining special cases are presented in Figure 3.8.

Now we may assume that G has no leaf cycles and every leaf in G is connected to a cycle. Let us choose a cycle C such that all its descendants are leaves. If $G \neq C$ then let v_c denote a connector vertex that connects C with its parent; otherwise let v_c be a vertex of maximum degree. Since the number of cases we have to consider is large, some of the cases are presented only in figures. In every figure, below the solid lines lie the vertices that we remove from G to obtain a cactus tree G' of order n' , where $2 \leq n' < n$. The vertex v_c from now on always belongs to G' . The vertices that are not surrounded with the dashed line are presented with their complete neighbourhood. The boundary symbolizes the rest of the graph that is not presented. In all the figures we colour in black the vertices that belong or must be added to $\gamma_s(G')$ -set to obtain a global secure set of G of required cardinality. Furthermore, if the vertex is coloured in white then it means that it belongs neither to the $\gamma_s(G')$ -set nor to a global secure set of G . A vertex is coloured grey if its belonging to the $\gamma_s(G')$ -set is indifferent. It follows from the choice of v_c that $n' \geq 2$. In every case when $G' \in \{C_3, P_3, C_6\}$ it is easy to see that we can find a global secure set of G of cardinality less than $\frac{2}{3}n$, so we leave some cases for the reader. The simplicity of these cases follows from the next remark.

Remark 3.18. In every case when $G' \in \{C_3, P_3, C_6\}$ we can always choose a minimum global secure set of G' that contains a vertex v_c , and v_c does not have to defend any vertex (including itself) in any of the attacks.

First let us suppose that there exists a vertex $v \in V(C) - \{v_c\}$ such that $l(v) = 6$ or $l(v) \geq 8$. Let G' be a cactus tree obtained by removing $L(v)$ from G , and let S' be any $\gamma_s(G')$ -set. Notice that by the choice of v_c , $G' \notin \{C_3, C_6, P_3\}$. Let X be a subset of $\left\lfloor \frac{l(v)+1}{2} \right\rfloor$ vertices of $L(v)$. If $v \notin S'$ then let $S = S' \cup X \cup \{v\}$; otherwise let $S = S' \cup X$. Since $\deg_{G'} v = 2$, if $v \notin S'$ then at least one of its neighbours must be in S' . Hence S is a global secure set of G and $|S| < \frac{2}{3}n$. Now assume that there exists a vertex $v \in V(C) - \{v_c\}$ such that $l(v) \in \{2, 5, 7\}$. Let G' be a graph obtained by removing v and $L(v)$ from G . Then G' is a connected cactus

tree. In Figure 3.9 we can see the cut and the cases for $l(v) = 2$. If $l = 5$ or 7 we must add to $\gamma_s(G')$ -set at most 4 (from 6 removed) and 5 (from 8 removed) vertices, respectively, to get a global secure set of G of cardinality less than $\frac{2}{3}n$. Consequently, we may assume that every vertex $v \in V(C) - \{v_c\}$ has none or 1, 3 or 4 leaf neighbours.

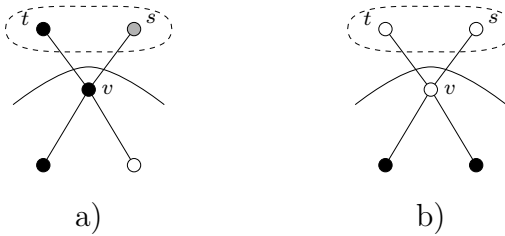


FIGURE 3.9: $v \in (V(C) - \{v_c\})$, $l(v) = 2$, $t, s \in C$ and $t \neq s$.

We can also prove that if we have a structure H_1 then the theorem holds.

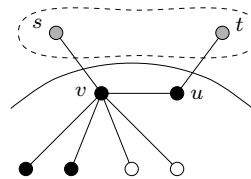


FIGURE 3.10: Structure H_1 .

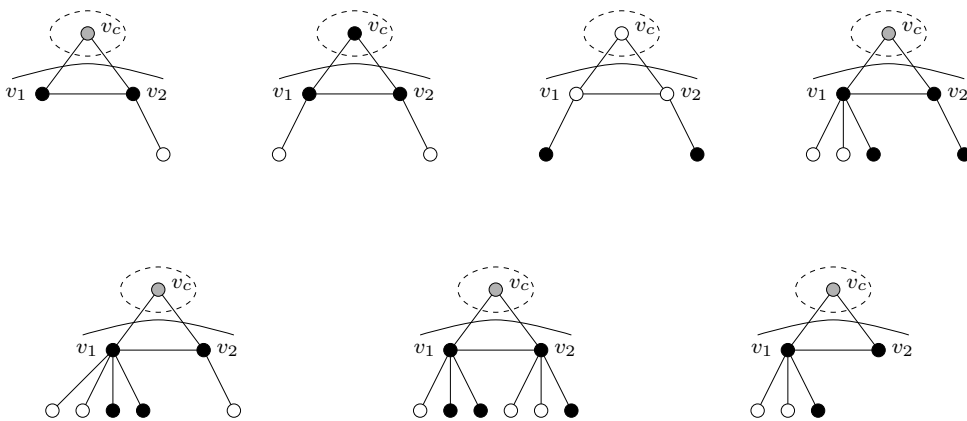


FIGURE 3.11: $|C| = 3$ and $l(v_1) + l(v_2) \leq 6$.

Suppose that $|C| = 3$ and $C = \{v_c, v_1, v_2\}$. Let G' be a graph of order n' , obtained by removing the vertices v_1, v_2 and all their leaf neighbours from G . First we

assume that $l(v_1)+l(v_2) \geq 7$. Let D be a $\gamma_s(G')$ -set and $S = \{v_1, v_2\} \cup X_1 \cup X_2 \cup D$, where for $i \in \{1, 2\}$, $X_i \subset L(v_i)$ and $|X_i| = \lfloor \frac{l(v_i)}{2} \rfloor$. Then S is a global secure set of G and $|S| < \frac{2}{3}n$. As we can see in Figure 3.11 in all the cases when $l(v_1)+l(v_2) \leq 6$ the theorem also holds.

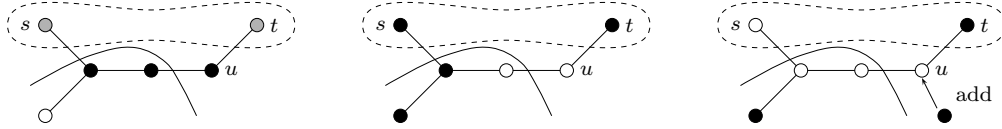


FIGURE 3.12: Structure H_3 .



FIGURE 3.13: Structure H_4 .

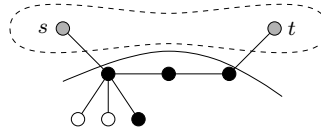


FIGURE 3.14: Structure H_5 .

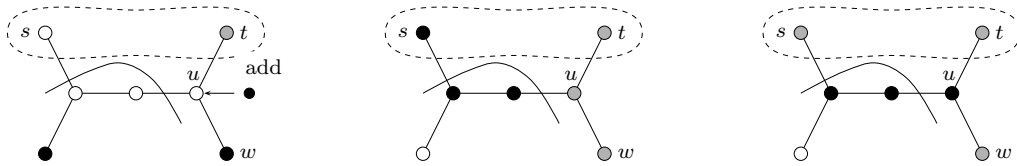


FIGURE 3.15: Structure H_6 .

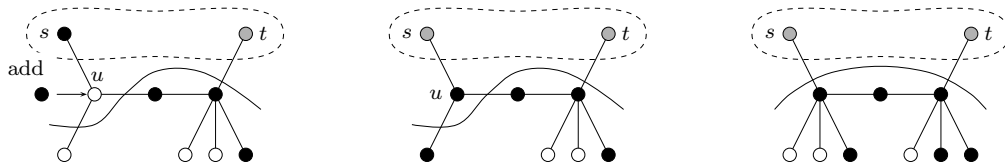


FIGURE 3.16: Structure H_7 and H_8 .

Hence we may assume that $|C| \geq 4$. Now we show that if the vertices that belong

to $C - \{v_c\}$, except the vertices $s \in C$ and $t \in C$ (the situation when $s = v_c$ or $t = v_c$ is also possible), induce one of the structures from H_3 to H_8 , then the theorem is true. In Figures 3.12, 3.15 and 3.16 we indicate which vertices of G' that do not belong to a $\gamma_s(G')$ -set must be added to obtain a global secure set of G .

From the presented cases follows that if a vertex $v \in V(C) - \{v_c\}$ is of degree 2 then it is a neighbour of v_c . The next figures present possible cases when $|C| = 4$ and two of the vertices of C are of degree 2.

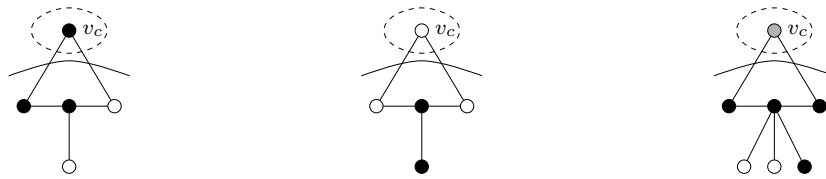
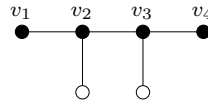


FIGURE 3.17: $|C| = 4$ and two of the vertices of C are of degree 2.

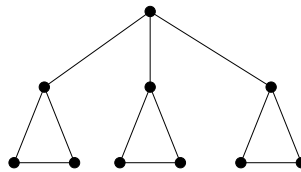
We may assume now that the above situations do not occur. Let us denote the vertices that belong to $C - \{v_c\}$ by v_1, v_2, \dots, v_p , where $p \geq 3$, $v_i v_{i+1} \in E$ ($i \in \{1, \dots, p-1\}$), and $v_c v_1, v_c v_p \in E$. Let S_P be a global secure set of the graph P induced by the vertices v_1, v_2, \dots, v_p and their descendants. Furthermore, let G' be a cactus tree of order n' that we obtain by removing the vertices that belong to P . We can divide P into p stars St_1, \dots, St_p with centers v_1, v_2, \dots, v_p , respectively. If $\deg v_1 = 2$ or $\deg v_p = 2$ then we join it with the star St_2 or St_{p-1} , respectively. For each of the stars we find a minimum global secure set that contains its center. Let S_P be the union of these sets and let S' be a $\gamma_s(G')$ -set. Furthermore, suppose that there does not exist a $\gamma_s(G')$ -set that contains v_c . Then $G' \notin \{C_3, P_3, C_6\}$ and $\gamma_s(G') < \frac{2}{3}n'$. One can see that if $\deg v_1 = 2$ or $\deg v_p = 2$ (notice that if $\deg v_1 = 2$ then $\deg v_2 \neq 6$; see structure H_1) or at least one vertex v_i has four descendants, then from previous considerations and Lemma 3.10, it follows that $S = S_P \cup S'$ is a global secure set of G and $|S| < \frac{2}{3}n$. Hence we may assume that every star St_i , $1 \leq i \leq p$, has a global secure set that contains half of its vertices and $|S_P| = \frac{1}{2}|V(P)|$. Form S by adding S_P and a vertex t to S' , where $t \in V(P) - S_P$. Then $|S| < \frac{2}{3}n' + 1 + \frac{1}{2}|V(P)|$. The previous remarks about structures that we can exclude from further consideration give us the guarantee that $|V(P)| \geq 6$, which implies that $|S| < \frac{2}{3}n$. Now assume that there exists a

$\gamma_s(G')$ -set S' containing v_c . We observe that even if $\deg v_1 = 2$ or $\deg v_p = 2$ we can always modify the set S_P in such a way that it contains v_1 and v_2 , and the cardinality of S_P is not increased. Furthermore, only if $P = H_9$ (see Figure 3.18) we have $|S_P| = \frac{2}{3}|V(P)|$; in all other cases $|S_P| < \frac{2}{3}|V(P)|$. If $G' \in \{C_3, P_3, C_6\}$

FIGURE 3.18: Graph H_9 .

and $|S_P| = \frac{2}{3}|V(P)|$ then $S = (S_P - \{v_1\}) \cup S'$ is a global secure set and $|S| < \frac{2}{3}n$. Otherwise we obtain a global secure set of G of required cardinality as the union of the sets S_P and S' . \square

Now we show that the bound presented in Theorem 3.17 cannot be improved. We present a family of cactus trees whose construction is very similar to the construction of trees that belong to the family \mathcal{T} that was defined in the previous section. Let CT be a cactus tree obtained as follows: make k copies of C_3 and add an additional vertex v ; add edges between v and one of the vertices from each copy of C_3 . Let \mathcal{CT} denote a family of all such cactus trees. One can see that for any positive real number r , there exists a cactus tree $CT \in \mathcal{CT}$ of order n such that $\frac{2}{3}n - r < \gamma_s(CT) < \frac{2}{3}n$.

FIGURE 3.19: Example of a cactus tree $CT \in \mathcal{CT}$.

Chapter 4

Possible cardinalities of alliances and secure sets in graphs

The ideas presented in this chapter were inspired by the following question. Suppose that a graph G of order n has a security number equal to $k < n$. Does this fact imply that we can find in G a secure set of any greater cardinality less than n ?

4.1 Introduction

In the previous sections we were interested in alliances and secure sets of minimum cardinality. However, we can easily imagine a situation in which we would like to know whether a coalition of particular cardinality can be built. For this purpose we define a new notion.

Definition 4.1. We say that a graph G of order n is:

1. *a-monotone*, if for any $k \in \{a(G), \dots, n\}$, G has a defensive alliance of cardinality k ;
2. *γ_a -monotone*, if for any $k \in \{\gamma_a(G), \dots, n\}$, G has a global defensive alliance of cardinality k ;
3. *s-monotone*, if for any $k \in \{s(G), \dots, n\}$, G has a secure set of cardinality k ;

4. γ_s -monotone, if for any $k \in \{\gamma_s(G), \dots, n\}$, G has a global secure set of cardinality k ;
5. \hat{s} -monotone, if for any $k \in \{\hat{s}(G), \dots, n\}$, G has a strong secure set of cardinality k ;
6. $\gamma_{\hat{s}}$ -monotone, if for any $k \in \{\gamma_{\hat{s}}(G), \dots, n\}$, G has a global strong secure set of cardinality k .

We can easily find graphs that satisfy the conditions of the above definitions.

Proposition 4.2. *Complete graphs, paths, cycles, complete bipartite graphs are a -monotone, γ_a -monotone, s -monotone, γ_s -monotone, \hat{s} -monotone and $\gamma_{\hat{s}}$ -monotone.*

On the other hand, are there graphs that are, e.g, not a -monotone? The next result gives us the answer.

Proposition 4.3. *For any positive integer $c \geq 2$,*

1. *there is a graph G such that G has no defensive alliances of cardinality $k \in \{a(G) + 1, \dots, a(G) + c\}$,*
2. *there is a graph G such that G has no secure sets of cardinality $k \in \{s(G) + 1, \dots, s(G) + c\}$,*
3. *there is a graph G such that G has no strong secure sets of cardinality $k \in \{\hat{s}(G) + 1, \dots, \hat{s}(G) + c\}$.*

Proof. Let G be a graph obtained by adding a pendant vertex v , to a clique C of order $2c+4$. Clearly, $\{v\}$ is a defensive alliance and a secure set of G . However,

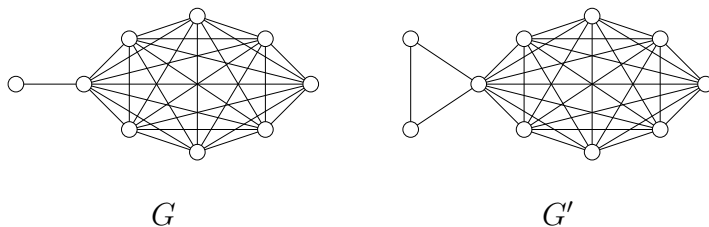


FIGURE 4.1: Graphs G and G' for $c = 2$.

any other defensive alliance and secure set must contain a vertex of C . Since C is a clique, any defensive alliance (secure set) of G that contains a vertex of C , must

contain at least $c + 2$ vertices of C . Hence, G has no defensive alliance (secure set) of cardinality $k \in \{a(G) + 1, \dots, a(G) + c\}$ ($k \in \{s(G) + 1, \dots, s(G) + c\}$).

Similarly, we can show that a graph G' , obtained from two vertex disjoint complete graphs K_3 and K_{2c+4} by identifying a vertex of K_3 with a vertex of K_{2c+4} , has no strong secure sets of cardinality $k \in \{\hat{s}(G) + 1, \dots, \hat{s}(G) + c\}$. \square

The graph constructions in the above proof are very simple. However, to show that not every graph is γ_a -monotone we have to use more complex structures. Let G_i be a complete graph with a vertex set $\{v_{i1}, \dots, v_{ik}\}$. A graph G with a vertex set $V = V(G_1) \cup \dots \cup V(G_{k+1})$, and an edge set $E = V(G_1) \cup \dots \cup V(G_{k+1}) \cup \{v_{1j}v_{ij} : j \in \{1, \dots, k\}, i \in \{2, \dots, k+1\}\}$ we call a k -flower. Moreover, we say that the set $V(G_1)$ is a *steam* of a k -flower, and the set $V(G_i)$, where $i \neq 1$, is an i -th *petal*. An example of a 3-flower is presented in Fig. 4.2.

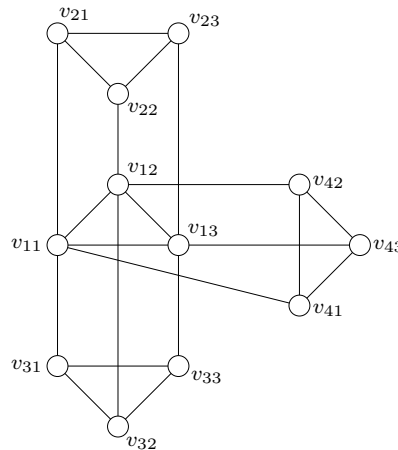


FIGURE 4.2: A 3-flower.

It is easy to see that the k -flowers have the following properties.

Remark 4.4. Let G be a k -flower with a steam T . Then,

1. any vertex of a petal has exactly one neighbour in a steam of G ,
2. for any vertex $v \in T$, in each of k petals of G , there is exactly one vertex u such that $N(u) \cap T = \{v\}$,
3. if vertices $u, v \in V(G)$ belong to the same petal, then $N(u) \cap T \neq N(v) \cap T$,
4. if two vertices $u, v \in V(G)$ belong to different petals, then $uv \notin E(G)$.

The next theorem shows, that if $k \geq 4$ then a k -flower is not $\gamma_a(G)$ -monotone.

Theorem 4.5. *If a graph G is a k -flower, where $k \geq 4$, then $\gamma(G) = \gamma_a(G) = k$, and G has no global defensive alliance of cardinality $p \in \{k + 1, \dots, k + \lfloor \frac{k}{2} \rfloor - 1\}$.*

Proof. First, we show that $\gamma(G) = \gamma_a(G) = k$. Suppose to the contrary that there is a dominating set $D \subset V(G)$ with less than k vertices. Hence, there is a petal with no vertex in D . Thus, by Remark 4.4, all k vertices of a steam must belong to D , a contradiction. Now, we can easily verify that the vertices of a steam of G form a minimum dominating set and a minimum global defensive alliance.

Now suppose that there exists a global defensive alliance B of cardinality $p \in \{k + 1, \dots, k + \lfloor \frac{k}{2} \rfloor - 1\}$. If B contains all the vertices of the steam T , then $1 \leq |B - T| \leq \lfloor \frac{k}{2} \rfloor - 1$. It follows that there is a petal with a vertex $v \in B - T$, and by Remark 4.4, $|N[v] \cap B| \leq 1 + \lfloor \frac{k}{2} \rfloor - 1 = \lfloor \frac{k}{2} \rfloor$ and $|N[v] - B| \geq \lceil \frac{k}{2} \rceil + 1$. We obtain a contradiction with the fact that B is a defensive alliance. Moreover, our considerations lead us to the following remark.

Remark 4.6. If a defensive alliance of a k -flower contain a vertex of a petal P , then it must contain at least $\lfloor \frac{|P|}{2} \rfloor$ vertices of P .

Hence let us assume that B does not contain all the vertices of T . Remark 4.6 and the fact that $k + 1 \leq |B| \leq k + \lfloor \frac{k}{2} \rfloor - 1$ implies that B contains vertices of at most 3 different petals of G . By Remark 4.4, the vertices of the remaining $k - 3$ petals, must be dominated by the vertices of the steam. However, by Remark 4.4, all vertices of the steam must belong to B , a contradiction. \square

In Section 4.3 we show that there are cographs that are not γ_s -monotone (see Theorem 4.22). However, we obtained only examples of graphs that have no global secure set of cardinality greater by one than their global security number. We also did not find any graphs that are not γ_s -monotone. We conclude this section with two important open problems.

- Is every graph γ_s -monotone?
- Does there exist a graph G such that G has no global secure sets of cardinality $t \in \{\gamma_s(G) + p, \dots, \gamma_s(G) + k\}$, where p, k are positive integers and $p < k$?

4.2 Properties of γ_s -monotone graphs

The results we present in this and the next section were published in [29] and [30]. Our first remarks are very simple but we will refer to them in many parts of the proofs presented in this chapter.

Remark 4.7. If a graph G has an order smaller or equal to 5, then it is γ_s -monotone.

Remark 4.8. Let G be a connected graph of even order n . If $\gamma_s(G) \geq \frac{n}{2} + 1$, then $n \geq 6$.

In Section 4.3 we show that not all graphs are γ_s -monotone. For this reason, we introduce two special graph classes. Namely, we say that a graph G belongs to \mathcal{MSD}_k if it is nearly γ_s -monotone, i.e., it has all the required global secure sets except the one of cardinality k , where $k > \gamma_s(G)$. Furthermore, by \mathcal{S}_m^n we denote the class of all graphs that have secure sets of cardinality k , for any $k \in \{m, m+1, \dots, n\}$. The forthcoming lemmata are technical but they will be very useful in the proofs of the theorems presented in Section 4.3.

Lemma 4.9. *Let both G_1 and G_2 be connected and γ_s -monotone graphs of order $n_1 > 0$ and $n_2 > 0$, respectively. If $\gamma_s(G_1) = \lceil \frac{n_1}{2} \rceil$ and $\gamma_s(G_2) = \lceil \frac{n_2}{2} \rceil$, then the graph $G = G_1 \cup G_2$ belongs to $\mathcal{S}_k^{n_1+n_2}$, where $k = \lceil \frac{n_1+n_2}{2} \rceil$.*

Proof. If the graph G_1 or G_2 has even order, then clearly the lemma holds. If G_1 and G_2 have both odd order, then, without loss of generality, we can assume that $n_1 \geq n_2$. We obtain the two required sets of the smallest cardinality as follows: $S_{\frac{n_1+n_2}{2}} = SD(G_1)_{\frac{n_1+n_2}{2}}$, $S_{\frac{n_1+n_2}{2}+1} = SD(G_1)_{\frac{n_1+1}{2}} \cup SD(G_2)_{\frac{n_2+1}{2}}$. Now it is easy to see how to obtain the rest of the sets. \square

Lemma 4.10. *Let both G_1 and G_2 be connected and γ_s -monotone graphs of even order $n_1 > 0$ and $n_2 > 0$, respectively. If $\gamma_s(G_1) = \frac{n_1}{2} + 1$ and $\gamma_s(G_2) = \frac{n_2}{2} + 1$, then the graph $G = G_1 \cup G_2$ belongs to $\mathcal{S}_k^{n_1+n_2}$, where $k = \frac{n_1+n_2}{2} + 2$ and there exists the set $S_{\frac{n_1+n_2}{2}}$.*

Proof. Without loss of generality we can assume that $n_1 \geq n_2$. From Remark 4.8, we know that $n_2 \geq 6$. Hence we see that as the set $S_{\frac{n_1+n_2}{2}}$ we can choose the set $SD(G_1)_{\frac{n_1+n_2}{2}}$. The remaining sets are easy to obtain, so we omit this part of the proof. \square

Lemma 4.11. *Let both G_1 and G_2 be connected and γ_s -monotone graphs of order $n_1 > 0$ and $n_2 > 0$, respectively, where n_1 is even. If $\gamma_s(G_1) = \frac{n_1}{2} + 1$ and $\gamma_s(G_2) = \lceil \frac{n_2}{2} \rceil$, then the graph $G = G_1 \cup G_2$ belongs to $\mathcal{S}_k^{n_1+n_2}$, where $k = \lceil \frac{n_1+n_2}{2} \rceil$.*

Proof. One can see that, if $n_1 \geq n_2$, then the set $SD(G_1)_{\lceil \frac{n_1+n_2}{2} \rceil}$ gives us $S_{\lceil \frac{n_1+n_2}{2} \rceil}$; otherwise, we can choose $SD(G_2)_{\lceil \frac{n_1+n_2}{2} \rceil}$ as $S_{\lceil \frac{n_1+n_2}{2} \rceil}$. Now the other required sets can be easily constructed. \square

Lemma 4.12. *Let both G_1 and G_2 be connected graphs of order $n_1 > 0$ and $n_2 > 0$, respectively, such that $G_1 \in \mathcal{MSD}_{\gamma_s(G_1)+1}$, n_1 is even and G_2 is γ_s -monotone. If $\gamma_s(G_1) = \frac{n_1}{2}$ and $\gamma_s(G_2) = \lceil \frac{n_2}{2} \rceil$, then the graph $G = G_1 \cup G_2$ belongs to $\mathcal{S}_k^{n_1+n_2}$, where $k = \lceil \frac{n_1+n_2}{2} \rceil$.*

Proof. If $n_2 = 1$ or n_2 is even, then one can see that the lemma is true. So let us assume that $n_2 \geq 3$ and that n_2 is odd. From Remark 4.7, we know that $n_1 \geq 6$. We obtain the two required sets of the smallest cardinality as follows: if $n_1 \geq n_2$, then $S_{\lceil \frac{n_1+n_2}{2} \rceil} = SD(G_1)_{\lceil \frac{n_1+n_2}{2} \rceil}$. Since n_2 is odd, we have $\lceil \frac{n_1+n_2}{2} \rceil = \frac{n_1+n_2+1}{2} \geq \frac{n_1+4}{2} = \frac{n_1}{2} + 2$. So the indicated sets exist. On the other hand, if $n_1 < n_2$, then $S_{\lceil \frac{n_1+n_2}{2} \rceil} = SD(G_2)_{\lceil \frac{n_1+n_2}{2} \rceil}$. We can also see that the set $S_{\lceil \frac{n_1+n_2}{2} \rceil+1}$ can be chosen as the union of the sets $SD(G_1)_{\frac{n_1}{2}}$ and $SD(G_2)_{\lceil \frac{n_2}{2} \rceil+1}$. Now it is easy to see how to obtain the rest of the required sets. \square

Lemma 4.13. *Let both G_1 and G_2 be connected graphs of order $n_1 > 0$ and $n_2 > 0$, respectively, such that $G_1 \in \mathcal{MSD}_{\gamma_s(G_1)+1}$, G_2 is γ_s -monotone and n_2 is even. If $\gamma_s(G_1) = \lceil \frac{n_1}{2} \rceil$ and $\gamma_s(G_2) = \frac{n_2}{2} + 1$, then the graph $G = G_1 \cup G_2$ belongs to $\mathcal{S}_k^{n_1+n_2}$, where $k = \lceil \frac{n_1+n_2}{2} \rceil$.*

Proof. From Remark 4.7, we know that $n_1 \geq 6$, and from Remark 4.8, it follows that also $n_2 \geq 6$. If $n_1 \geq n_2$, then $S_{\lceil \frac{n_1+n_2}{2} \rceil} = SD(G_1)_{\lceil \frac{n_1+n_2}{2} \rceil}$; otherwise $S_{\lceil \frac{n_1+n_2}{2} \rceil} = SD(G_2)_{\lceil \frac{n_1+n_2}{2} \rceil}$. We can obtain the set $S_{\lceil \frac{n_1+n_2}{2} \rceil+1}$ as the union of the sets $SD(G_1)_{\lceil \frac{n_1}{2} \rceil}$ and $SD(G_2)_{\frac{n_2}{2}+1}$. Furthermore, the set $S_{\lceil \frac{n_1+n_2}{2} \rceil+2}$ can be chosen as the union of the sets $SD(G_1)_{\lceil \frac{n_1}{2} \rceil}$ and $SD(G_2)_{\frac{n_2}{2}+2}$. The remaining required sets are easy to obtain. \square

Lemma 4.14. *Let both G_1 and G_2 be connected graphs of even order $n_1 > 0$ and $n_2 > 0$, respectively, such that $G_1 \in \mathcal{MSD}_{\gamma_s(G_1)+1}$ and $G_2 \in \mathcal{MSD}_{\gamma_s(G_2)+1}$. If $\gamma_s(G_1) = \frac{n_1}{2}$ and $\gamma_s(G_2) = \frac{n_2}{2}$, then the graph $G = G_1 \cup G_2$ belongs to $\mathcal{S}_k^{n_1+n_2}$, where $k = \frac{n_1+n_2}{2} + 2$, and there exists a set $S(G)_{\frac{n_1+n_2}{2}}$.*

Proof. Without loss of generality, we can assume that $n_1 \geq n_2$. From Remark 4.7, it follows that $n_1, n_2 \geq 6$. Hence we have $S_{\frac{n}{2}} = SD(G_1)_{\frac{n_1+n_2}{2}}$, $S_{\frac{n}{2}+2} = SD(G_1)_{\frac{n_1}{2}+2} \cup SD(G_2)_{\frac{n_2}{2}}$, and $S_{\frac{n}{2}+3} = SD(G_1)_{\frac{n_1}{2}+3} \cup SD(G_2)_{\frac{n_2}{2}}$. It is a simple matter to find the required secure sets of greater cardinalities. \square

4.3 Cographs are not γ_s -monotone

Cographs are a well-known family of graphs with many applications. We define them as follows:

1. K_1 is a cograph,
2. the union of disjoint cographs G_1 and G_2 is a cograph,
3. the join of disjoint cographs G_1 and G_2 is a cograph.

Cographs are also graphs without induced path on four vertices [11]. Furthermore, they can be characterized by the *cotree* [12]. A cotree T is a rooted tree of the *modular decomposition* of a cograph. A *module* in a graph is a set of vertices that have the same neighbourhood outside the set. Empty set, singletons and $V(G)$ are *trivial* modules. The *modular decomposition* is a decomposition of a graph into modules. See [13], [14], [34] and [35] for more information. The cotree T can be described in the following way. The leaves of T are the vertices of G , and every interior node is either 0-node or a 1-node. A 0-node and a 1-node represent a union operation and a join operation, respectively. A cotree is uniquely determined, and no 0-node is a child of a 0-node and no 1-node is a child of a 1-node. Moreover, every interior node has at least two children.

Now we formulate the main theorem of this section, which concerns the whole class of cographs.

Theorem 4.15. *Let G be a cograph of order n .*

1. *If G is disconnected, then:*
 - (a) *if n is even, then $G \in \mathcal{S}_{\frac{n}{2}+2}^n$ and there exist a set $S_{\frac{n}{2}}$ or $S_{\frac{n}{2}+1}$,*
 - (b) *if n is odd, then $G \in \mathcal{S}_{\frac{n+1}{2}}^n$.*
2. *If G is connected, then:*

- (a) if n is even, then G is γ_s -monotone and $\gamma_s \in \{\frac{n}{2}, \frac{n}{2} + 1\}$ or $G \in \mathcal{MSD}_{\frac{n}{2}+1}$ and $\gamma_s = \frac{n}{2}$,
- (b) if n is odd, then G is γ_s -monotone and $\gamma_s = \frac{n+1}{2}$.

Proof. For the clarity of the proof, we distinguish some remarks, so that we can easily refer to them. It is easy to verify that, for all cographs of order 1,2,3 and 4, our theorem and the next remark is true.

Remark 4.16. Every disconnected cograph of order 2 or 4 has $S_{\frac{n}{2}}$ and $S_{\frac{n}{2}+1}$, and every connected cograph of order 2 or 4 has $SD_{\frac{n}{2}}$ and $SD_{\frac{n}{2}+1}$.

We use induction on the order of G . Let us suppose that the theorem is true for all cographs of order k , where $1 \leq k < n$. Let G be a cograph of order n .

Part 1. Let us suppose that G is disconnected. We must consider two cases.

a) Suppose that n is even. That implies that the number of components of odd order is even. So, if there exist components of odd order, we can join them in pairs. We do the same with the components of even order. There can be at most one component of even order without a pair. But to this component we can also apply our inductive hypothesis, and this fact together with Lemmata 4.9-4.14 and Proposition 2.4 implies that G has a secure set of cardinality $\frac{n}{2}$ or $\frac{n}{2} + 1$ and all the remaining sets $S_{\frac{n}{2}+2}, \dots, S_n$.

b) Suppose that n is odd, which implies that we have an odd number of components of odd order. As previously, we join in pairs the components of the same parity of the order. If there is one component of even order without a pair, then we join it in a pair with one remaining component of odd order, and as in previous case we can prove that the theorem is true. The next remark allows us to simplify some parts of the proof.

Remark 4.17. Every disconnected cograph G of even order n that has an even number of components has a set $S_{\frac{n}{2}}$.

Part 2. Let us suppose that G is connected. If G is a complete graph, then our theorem is true, so we can assume that it is not the case. Let $G = G_1 + G_2$. The properties of the cotree allow us to assume that G_1 or G_2 is disconnected. Suppose the following.

- a) n_1 and n_2 are odd. Then from the inductive hypothesis and Lemma 3.3 it follows that our theorem is true.
- b) n_1 is even and n_2 is odd. First, we suppose that there exists $S(G_1)_{\frac{n_1}{2}}$. If $n_2 > 1$

or there exists $S(G_1)_{\frac{n_1}{2}+1}$, then clearly our theorem is true. If $n_2 = 1$ and the set $S(G_1)_{\frac{n_1}{2}+1}$ does not exist, then by the inductive hypothesis and Lemma 3.3 we can obtain all the required sets except the set $SD_{\frac{n_1+n_2+1}{2}+1}$. If G_2 is connected, then G_1 is not, and from Remark 4.7 we know that $n_1 \geq 6$. Since $S(G_1)_{\frac{n_1}{2}+1}$ does not exist, in any partition into pairs of the components of G_1 (as was described in Part 1) there does not exist a pair that satisfies the conditions of Lemmata 4.9, 4.11, 4.12 or 4.13. Thus in G_1 are only connected components of even order and to each pair in the partition we can apply either the Lemma 4.14 or 4.10. Hence the existence of the missing set follows from following two claims, Claims 4.18 and 4.19.

Claim 4.18. *If G is a disconnected cograph of order $n \geq 12$ such that it has two connected components G_1 and G_2 of even order, and $G_1 \in \mathcal{MSD}_{\gamma_s(G_1)+1}$, $G_2 \in \mathcal{MSD}_{\gamma_s(G_2)+1}$, $\gamma_s(G_1) = \frac{n_1}{2}$, and $\gamma_s(G_2) = \frac{n_2}{2}$, then $G' = G + K_1$ of order $n' = n + 1$ is γ_s -monotone and $\gamma_s(G) = \lceil \frac{n}{2} \rceil$.*

Proof. Let $V(K_1) = \{v\}$. From the inductive hypothesis of the theorem, we know that a cograph H induced by the vertex v and $V(G_1)$ has $SD(H)_{\frac{n_1}{2}+2}$ ($n_1 \geq 6$). Moreover, by Lemma 3.2, we can choose $SD(H)_{\frac{n_1}{2}+2}$ that contains v , and so $SD_{\frac{n'+1}{2}+1} = SD(H)_{\frac{n_1}{2}+2} \cup SD(G_2)_{\frac{n_2}{2}}$. The existence of the remaining required sets follows from Lemmas 3.3 and 4.14. \square

Claim 4.19. *If G is a disconnected cograph of order $n \geq 4$ such that it has two connected components G_1 and G_2 of even order, and both components are γ_s -monotone and $\gamma_s(G_1) = \frac{n_1}{2} + 1$ and $\gamma_s(G_2) = \frac{n_2}{2} + 1$, then $G' = G + K_1$ of order $n' = n + 1$ is γ_s -monotone and $\gamma_s(G) = \lceil \frac{n}{2} \rceil$.*

Proof. Let $V(K_1) = \{v\}$. From the inductive hypothesis of the theorem, it follows that a cograph H induced by the vertex v and $V(G_1)$ has $SD(H)_{\frac{n_1+1+1}{2}}$ ($n_1 \geq 6$). Moreover, by Lemma 3.2, we can choose $SD(H)_{\frac{n_1+1+1}{2}}$ that contains v . Thus $SD_{\frac{n'+1}{2}+1} = SD(H)_{\frac{n_1+1+1}{2}} \cup SD(G_2)_{\frac{n_2}{2}+1}$. The existence of the remaining required sets follows from Lemma 4.10. \square

So let us suppose that $S(G_1)_{\frac{n_1}{2}}$ does not exist. Thus, if G_1 is disconnected, then we can find the set $S(G_1)_{\frac{n_1}{2}+1}$ according to the procedure we described in Part 1. Otherwise, from the inductive hypothesis we have $S(G_1)_{\frac{n_1}{2}+1} = SD(G_1)_{\frac{n_1}{2}+1}$. Let G_{1i} be the one component of even order which is not in a pair (if G_1 is connected then $G_{1i} = G_1$). Because $S(G_1)_{\frac{n_1}{2}}$ does not exist, then there does

not exist $S(G_{1i})_{\frac{n_{1i}}{2}}$. Let $H = G_{1i} + K_1$ and $V(K_1) = \{v\}$. From the inductive hypothesis and Lemma 3.2, we know that there exists $SD(H)_{\frac{(n_{1i}+1)+1}{2}}$ that contains v (note that, if $G_{1i} = G_1$, then G_2 is disconnected and $n_2 > 1$; thus $|V(H)| < n$).

Remark 4.20. Vertex v does not have to defend itself in any of the attacks of the vertices that belong to $V(H) - S(H)_{\frac{(n_{1i}+1)+1}{2}}$.

Of course there exists an attack in which v must defend a vertex $x \in V(G_{1i}) \subset V(H)$; otherwise, there would exist set $S(G_{1i})_{\frac{n_{1i}}{2}}$. Now we can see that, from Remark 4.17, 4.20 and Lemma 3.3, it follows that, if $W = G_1 - G_{1i}$, then $S(W)_{\frac{n_W}{2}} \cup (S(H)_{\frac{(n_{1i}+1)+1}{2}} - \{v\}) \cup S(G_2)_{\frac{n_2+1}{2}}$ gives us $SD_{\frac{n+1}{2}}$ (if $G_{1i} = G_1$, then $S(W)_{\frac{n_W}{2}} = \emptyset$). It is a simple matter to see how to obtain the rest of the required global secure sets.

c) n_1 and n_2 are even. First, let us suppose that there exists neither $S(G_1)_{\frac{n_1}{2}}$ nor $S(G_2)_{\frac{n_2}{2}}$. In this case, as in subcase b), we can prove that there exist sets $SD_{\frac{n}{2}+1}, \dots, SD_n$. Finally, if there exists $S(G_1)_{\frac{n_1}{2}}$ or $S(G_2)_{\frac{n_2}{2}}$, then we can use the inductive hypothesis and Lemma 3.3 to complete the proof. \square

Cographs of odd order always have a global secure set of cardinality equal to the lower bound on $\gamma_s(G)$. Unfortunately the situation is not so clear in the case of cographs of even order. We know that there exist cographs with $\gamma_s(G) = \frac{n}{2}$ (for example complete graphs of even order), but it does not follow from the proof of Theorem 4.15, whether there exist cographs with $\gamma_s(G) = \frac{n}{2} + 1$. To prove this fact we need to present a special class of cographs. Let Q_k denote the union of k ($k \geq 1$) disjoint graphs K_2 and let $H_k = Q_k + Q_k$. Let $\mathcal{P} = \{G \text{ is a cograph} : G = (H_k \cup K_1) + (H_k \cup K_1) \text{ and } k \text{ is even}\}$. A graph, which belongs to \mathcal{P} , is presented in Figure 4.3.

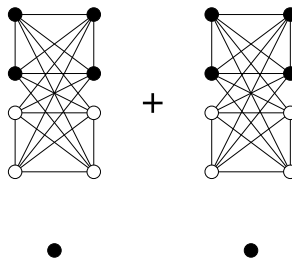


FIGURE 4.3: The graph $G = (H_2 \cup K_1) + (H_2 \cup K_1)$. The black vertices belong to $SD_{\frac{n}{2}+1}$, where n is the order of G .

Lemma 4.21. $\gamma_s(G) = \frac{n}{2} + 1$, for $G \in \mathcal{P}$.

Proof. Let $G = G_1 + G_2$, where $G_1 = (H_k \cup K_1)$, and $G_2 = (H_k \cup K_1)$, where k is even and greater than or equal to 2. Furthermore, let v denote the isolated vertex in G_1 . It is easy to see that $\gamma_s(G) \leq \frac{n}{2} + 1$ (see Fig. 4.3 for an example). Thus it is sufficient to prove that there does not exist $SD_{\frac{n}{2}}$. Suppose for the sake of contradiction that there exists a set D such that D is secure, dominating, and $|D| = \frac{n}{2}$. From the construction of G , we know that $\frac{n}{2} = 4k + 1$ is an odd number. Let $D = D_1 \cup D_2$, where $D_1 = V(G_1) \cap D$ and $D_2 = V(G_2) \cap D$. Without loss of generality, we can assume that $|D_1| > |D_2|$. Let us denote this condition by $(*)$. We must also describe the graph G_1 more precisely. Let $H_k = Q_k^1 + Q_k^2$, where the additional indices allow us to distinguish the vertices that belong to first and the second graphs which are joined in the construction of H_k . First, we assume that $v \in D_1$. Then $|D_2| = 2k$, or else v would have more attackers than defenders. It follows that $|D_1| = 2k + 1$. Let us assume that $D_1 \cap (V(G_1) - \{v\}) \subseteq Q_k^1$ ($D_1 \cap (V(G_1) - \{v\}) \subseteq Q_k^2$). It follows that there exists a vertex $x \in D_1 \cap (V(G_1) - \{v\})$ such that $|N[x] \cap D| = 2 + 2k$ and $|N[x] - D| = 2k + 2k + 1$ (all the neighbours in Q_k^2 (Q_k^1) and all the neighbours in $V(G_2) - D_2$). Thus x has more attackers than defenders, which contradicts the security of D . So let us suppose that there exist $x, y \in D_1 \cap (V(G_1) - \{v\})$ such that $x \in Q_k^1$ and $y \in Q_k^2$. Then $|N[\{x, y\}] \cap D| = 2k + 2k$ (all the vertices that belong to D_2 and $D_1 - \{v\}$) and $|N[\{x, y\}] - D| = 2k + 2k + 1$ (all the vertices that belong to $V(G_1) - \{v\} - D_1$ and to $V(G_2) - D_2$). Once again we get a contradiction to the fact that D is secure. It follows that, if there exists $SD_{\frac{n}{2}}$, then v does not belong to it. Let $A = D \cap Q_k^1$ and $B = D \cap Q_k^2$ ($A \cup B = D_1$). Without loss of generality, we can also assume that $|A| \leq |B|$. In our calculations we use the following fact (**): $|A| + |B| + |D_2| = D = 4k + 1$.

Suppose there exists a vertex $x \in B$ such that it has a neighbour $u \in Q_k^2$ and $u \notin B$. Thus $|N[x] \cap D| = |A| + |D_2| + |\{x\}|$ and $|N[x] - D| = 1 + 2k - |A| + 4k + 1 - |D_2|$ (u , the vertices in $V(Q_k^1) - A$ and $V(G_2) - D_2$). Since D is secure, $|N[x] \cap D| \geq |N[x] - D|$, and from this inequality it follows that $|B| \leq k$ (see (1)).

$$(1) |N[x] \cap D| \geq |N[x] - D| \Rightarrow |A| + |D_2| + 1 \geq 6k + 2 - |A| - |D_2| \Rightarrow 2(|A| + |D_2|) \geq 6k + 1 \Rightarrow |A| + |D_2| \geq 3k + \frac{1}{2} \stackrel{(**)}{\Rightarrow} 4k + 1 - |B| \geq 3k + \frac{1}{2} \Rightarrow |B| \leq k + \frac{1}{2} \Rightarrow |B| \leq k$$

If there does not exist a vertex x that satisfies our requirements, then surely $|B|$ is even. Let x be any vertex in B . Thus $|N[x] \cap D| = |A| + |D_2| + |\{x, u\}|$

and $|N[x] - D| = 2k - |A| + 4k + 1 - |D_2|$. From the security of D it follows that $|N[x] \cap D| \geq |N[x] - D|$, and from this inequality we get $|B| \leq k$ (see (2)).

$$(2) \quad |N[x] \cap D| \geq |N[x] - D| \Rightarrow 2 + |A| + |D_2| \geq 2k - |A| + 4k + 1 - |D_2| \Rightarrow 2(|A| + |D_2|) \geq 6k - 1 \Rightarrow |A| + |D_2| \geq 3k - \frac{1}{2} \stackrel{(**)}{\Rightarrow} 4k + 1 - |B| \geq 3k - \frac{1}{2} \Rightarrow |B| \leq k + \frac{3}{2} \stackrel{|B| \text{ is even}}{\Rightarrow} |B| \leq k$$

So, in any case, $|B| \leq k$, which means that $|A| \leq |B| \leq k$. Therefore, $|D_1| \leq 2k$, which implies that $|D_2| \geq 2k + 1$. This contradicts our earlier assumption that $|D_1| > |D_2|$. \square

Theorem 4.15 does not give the necessary and sufficient conditions for a connected cograph to be γ_s -monotone. This problem seems to be a hard task. Also finding a class of cographs which are not γ_s -monotone is not trivial. Therefore we describe here only some examples of such cographs. Let \mathcal{P}' denote a family of cographs such that for every graph $G \in \mathcal{P}'$, $G = (G' \cup G') + (G' \cup G')$, where $G' \in \mathcal{P}$.

Theorem 4.22. *If $G \in \mathcal{P}'$, then $\gamma_s(G) = \frac{n}{2}$ and $G \in \mathcal{MSD}_{\gamma_s(G)+1}$.*

Proof. Let $G = (G_1 \cup G_2) + (G_3 \cup G_4)$, where $G_1 = G_2 = G_3 = G_4 = G'$ and $G' \in \mathcal{P}$. Furthermore, let $G_1 = (H_k \cup K_1) + (H_k \cup K_1)$, for even $k > 0$. Clearly, $V(G_1) \cup V(G_3)$ gives us $SD(G)_{\frac{n}{2}}$. Suppose that the second part of the theorem is false. Then we can find a set D such that D is secure, dominating, and $|D| = \frac{n}{2} + 1 = 16k + 5$. Let $D_1 = V((G_1 \cup G_2)) \cap D$ and $D_2 = V((G_3 \cup G_4)) \cap D$. Because of the symmetry of the graph we can suppose, without loss of generality, that $|D_1| \geq |D_2|$ (C_1^*). Additionally, let $A = V(G_1) \cap D$ and $B = V(G_2) \cap D$. We can assume that $|A| \geq |B|$. If $|B| = 0$, then $|D_1| \leq |V(G_1)| = 8k + 2$ and $|D_2| \geq 8k + 3$, which contradicts (C_1^*). So $A \neq \emptyset$ and $B \neq \emptyset$.

Claim 4.23. *Let G be a graph that belongs to \mathcal{P} and let $G = G_1 + G_2$, where $G_1 = (H_k \cup K_1)$, $G_2 = (H_k \cup K_1)$, and k is even and greater than or equal to 2. If S is a secure set in G , then S is a dominating set that contains vertices of both G_1 and G_2 .*

Proof. Suppose that our claim is false. It follows that either $S \subseteq V(G_1)$ or $S \subseteq V(G_2)$. Without loss of generality, we can assume that $S \subseteq V(G_1)$. Let v be an

isolated vertex of G_1 . If $v \in S$, then $|N[v] \cap S| = 1$ and $|N[v] - S| = |V(G_2)| > 1$. Thus we have a contradiction to the fact that S is secure. So $v \notin S$. Let x be any vertex that belongs to S . It is easy to see that $|N[x] \cap S| \leq 2 + 2k$ and $|N[x] - S| \geq |V(G_2)| = 4k + 1$. Since $k \geq 2$, we have shown once again that S cannot be a secure set. \square

Since, by Lemma 4.21, $\gamma_s(G') = \frac{n'}{2} + 1 = 4k + 2$, and by Claim 4.23 every secure set of G' is a dominating set, it follows that if $|A| \leq 4k + 1$ or $|B| \leq 4k + 1$, then A or B is not a secure set in G_1, G_2 , respectively. Suppose, without loss of generality, that $|A| \leq 4k + 1$. Thus there exists a set $X \subseteq A$ such that $|N_{G_1}[X] \cap A| < |N_{G_1}[X] - A|$. Suppose that $|D_2| \leq 8k + 2$. Then $|N[X] \cap D| = |N_{G_1}[X] \cap A| + |D_2|$ and $|N[X] - D| = |N_{G_1}[X] - A| + |V((G_3 \cup G_4))| - |D_2|$. Now from our assumptions it follows that $|N[X] \cap D| < |N[X] - D|$. Hence $|D_2| \geq 8k + 3$, $|D_1| \leq 8k + 2$, and we get a contradiction with (C_1^*) .

Let us suppose that $|B| = z \geq 4k + 3$. Thus $|A| \geq 4k + 3$, $|D_1| \geq 8k + 6$, and $|D_2| \leq 8k - 1$. From the construction of G_2 we know that there exist vertices $x, y \in B$ such that $|N[\{x, y\}] \cap D| = z + |D_2|$ and $|N(\{x, y\}) - D| = |V(G_2)| - z + |V(G_3) \cup V(G_4)| - |D_2|$. Since D is secure, $|N[\{x, y\}] \cap D| \geq |N(\{x, y\}) - D|$.

$$(1) \quad |N[\{x, y\}] \cap D| \geq |N(\{x, y\}) - D| \Rightarrow z + |D_2| \geq 8k + 2 - z + 16k + 4 - |D_2| \\ \Rightarrow z + |D_2| \geq 24k + 6 - z - |D_2| \Rightarrow z + |D_2| \geq 12k + 3 \Rightarrow |D| - |A| \geq 12k + 3 \Rightarrow \\ 16k + 5 - |A| \geq 12k + 3 \Rightarrow |A| \leq 4k + 2$$

From this inequality, it follows that $|A| \leq 4k + 2$ (see (1)), which contradicts our previous assumptions. Thus we have to assume that $|B| = 4k + 2$. Hence $|A| \geq 4k + 2$, $|D_1| \geq 8k + 4$, and $|D_2| \leq 8k + 1$. Again, from the construction of G_2 and the cardinality of B , it follows that there must exist vertices $x, y \in V(G_2)$ such that $N_{G_2}[\{x, y\}] = V(G_2)$. Thus $|N[\{x, y\}] \cap D| = 4k + 2 + |D_2|$ and $|N[\{x, y\}] - D| = 4k + |V(G_3) \cup V(G_4)| - |D_2|$. One can see that the only possibility that D could be secure is when $|D_2| = 8k + 1$ and $|A| = |B|$. Before we make the final conclusion, we must prove one more claim.

Claim 4.24. *If $G \in \mathcal{P}$, then there does not exist $S_{\frac{n}{2}+1}$ which is also secure in $G + K_2$ or $G + \overline{K_2}$.*

Proof. Suppose that the assertion of the claim is false. Let S' be a secure set in $G + K_2$ (the proof for $G + \overline{K_2}$ stays the same) and $|S'| = \frac{n}{2} + 1$. Furthermore, let $G = G_1 + G_2$, where $G_1 = H_k \cup K_1$, $V(K_1) = \{q\}$ and $G_2 = H_k \cup K_1$, for an even

$k > 0$. Moreover, let $S' = S'_1 \cup S'_2$, where $S'_1 = V(G_1) \cap S'$ and $S'_2 = V(G_2) \cap S'$. Without loss of generality, we can assume that $|S'_1| \geq |S'_2|$ (we denote this condition by (*)). We must also describe the graph G_1 more precisely. Let $H_k = Q_k^1 + Q_k^2$, where the additional indices allow us to distinguish the vertices that belong to the first and second graphs which are joined. Let $A_1 = Q_k^1 \cap S' \subset V(G_1)$ and $A_2 = Q_k^2 \cap S' \subset V(G_1)$. Because of the symmetry of the graphs, we can assume that $|A_1| \geq |A_2|$. First, let us suppose that $q \in S'_1$. From the degree of q in $G + K_2$ and (*), it follows that $|S'_2| = |S'_1| = 2k + 1$ ($|S'| = 4k + 2$) and $|A_1| + |A_2| = 2k$. If $A_2 \neq \emptyset$, then, for any pair of vertices x, y such that $x \in A_1$ and $y \in A_2$, we have $|N[\{x, y\}] \cap S'| = |A_1| + |A_2| + |S'_2| = |S'| - 1 = 4k + 1$ and $|N(\{x, y\}) - S'| = |V(G_1) - \{q\}| - |A_1| - |A_2| + |V(G_2)| - |S'_2| + 2 = 4k - 2k + 4k + 1 - (2k + 1) + 2 = 4k + 2$ which contradicts the security of S' . If $A_2 = \emptyset$, then, for any vertex $z \in A_1$, $|N[z] \cap S'| = 2 + |S'_2| = 2 + 2k + 1$ and $|N[z] - S'| = 2 + 2k + 2k$, and we have a contradiction to the fact that S' is secure. Consequently, we can assume that $q \notin S'$. Suppose that there exists a vertex $x \in A_1$ such that it has a neighbour in Q_k^1 which does not belong to A_1 . Then $|N[x] \cap S'| = 1 + |A_2| + |S'_2|$ and $|N[x] - S'| = 1 + 2k - |A_2| + 4k + 1 - |S'_2| + 2$. We will use this fact (**): $|A_1| + |A_2| + |S'_2| = |S'| = 4k + 2$.

$$(1) \quad |N[x] \cap S'| \geq |N[x] - S'| \Rightarrow 1 + |A_2| + |S'_2| \geq 1 + 2k - |A_2| + 4k + 1 - |S'_2| + 2 \\ \Rightarrow 2(|A_2| + |S'_2|) \geq 6k + 3 \Rightarrow |A_2| + |S'_2| \geq 3k + \frac{3}{2} \stackrel{(**)}{\Rightarrow} |S'| - |A_1| \geq 3k + \frac{3}{2} \Rightarrow \\ 4k + 2 - |A_1| \geq 3k + \frac{3}{2} \Rightarrow |A_1| \leq k + \frac{1}{2} \Rightarrow |A_1| \leq k$$

Now let us suppose that such a vertex x does not exist. Thus $|A_1|$ must be even. Then there exists a vertex $r \in A_1$ such that $|N[r] \cap S'| = 2 + |A_2| + |S'_2|$ and $|N[r] - S'| = 2k - |A_2| + 4k + 1 - |S'_2| + 2$.

$$(2) \quad |N[r] \cap S'| \geq |N[r] - S'| \Rightarrow 2 + |A_2| + |S'_2| \geq 2k - |A_2| + 4k + 1 - |S'_2| + 2 \\ \Rightarrow 2(|A_2| + |S'_2|) \geq 6k + 1 \Rightarrow |A_2| + |S'_2| \geq 3k + \frac{1}{2} \stackrel{(**)}{\Rightarrow} |S'| - |A_1| \geq 3k + \frac{1}{2} \Rightarrow \\ 4k + 2 - |A_1| \geq 3k + \frac{1}{2} \Rightarrow |A_1| \leq k + \frac{3}{2} \stackrel{(|A_1| \text{ is even})}{\Rightarrow} |A_1| \leq k$$

In both cases, we have shown that $|A_2| \leq |A_1| \leq k$ (see (1) and (2)). Thus $|S'_1| \leq 2k$ and $|S'_2| \geq 2k + 2$. This gives us a contradiction to (*). \square

By Claim 4.24, we know that neither A nor B is a secure set of cardinality $4k + 2$ that is also secure in $G_2 + K_2$ or $G_2 + \overline{K_2}$. Since $|V(G_3) \cup V(G_4)| - |D_2| = 8k + 3$, we conclude that D is not secure.

The existence of the global secure sets of cardinality greater than $\gamma_s(G) + 1$ follows from Theorem 4.15. \square

Now we show that there exists nontrivial subclass of cographs such that every graph from this subclass is γ_s -monotone. Let \mathcal{C} be a class of graphs defined as follows:

1. K_1 belongs to \mathcal{C} ,
2. the union of disjoint graphs $G_1 \in \mathcal{C}$ and $G_2 \in \mathcal{C}$ belongs to \mathcal{C} ,
3. the join of disjoint graphs $G_1 \in \mathcal{C}$ and $G_2 \in \mathcal{C}$, where G_1 or G_2 is connected, belongs to \mathcal{C} .

From the definition of the cotree and the class \mathcal{C} , this remark follows.

Remark 4.25. If $G \in \mathcal{C}$, then every 1-node of a cotree is adjacent to at least one leaf.

We can make a connection with another well-known graph class, i.e., *trivially perfect graphs*. A graph is trivially perfect if and only if it contains no vertex subset of cardinality 4 that induces a path or a cycle [22]. These graphs have also a cotree characterization.

Lemma 4.26. [26] *A cograph G is a trivially perfect graph if and only if, in the cotree T of G , every 1-node has at most one child that is a 0-node.*

From the above lemma and Remark 4.25 it follows that trivially perfect graphs form a subclass of \mathcal{C} .

Theorem 4.27. *Let $G \in \mathcal{C}$ be a graph of order n .*

- (a) *If G is disconnected, then $G \in \mathcal{S}_{\lfloor \frac{n}{2} \rfloor}^n$.*
- (b) *If G is connected, then $\gamma_s(G) = \lfloor \frac{n}{2} \rfloor$, $\gamma_{\bar{s}}(G) = \lfloor \frac{n+1}{2} \rfloor$, and G is γ_s -monotone and $\gamma_{\bar{s}}$ -monotone.*

Proof. Again we use induction on the order of G . Let us suppose that the theorem is true for all graphs in \mathcal{C} of order k , where $1 \leq k < n$. Let $G \in \mathcal{C}$ be a graph of order n .

Suppose that G is disconnected. Then, as in the proof of Theorem 4.15, we

join its components in pairs with respect to their parity of orders. If we have an even number of components, then we create at most one mixed pair which contains one component of odd order and one of even order. To all pairs, even the mixed one (if it exists), we can apply Lemma 4.9. If the number of components is odd, then we have a component (at most one) without a pair. In this case, we can also use the inductive hypothesis, since every component is a graph from \mathcal{C} . Note that, if we have a component without a pair, then we do not have a mixed pair. From this considerations, it follows that the first part of our theorem is true. We prove the following property of the graph $G \in \mathcal{C}$ of odd order.

Claim 4.28. *If a disconnected graph $G \in \mathcal{C}$ has odd order, then it has a set $\hat{S}_{\lceil \frac{n}{2} \rceil}$.*

Proof. We join in pairs the components of G in accordance to the rules previously described. Let P_1, \dots, P_m denote the obtained pairs. Since G has odd order we have either one mixed-pair or a component H of odd order n_H without a pair. By the inductive hypothesis, for every pair $P_i = \{G', G''\}$ of components of orders n' and n'' , respectively, such that $n' \geq n''$, there exists a set $\hat{S}D_{\lceil \frac{n'+n''}{2} \rceil}(G')$. Let us denote this set by T_i . If there is a component H without a pair, then, by the inductive hypothesis, it has a global strong secure set $\hat{S}D(H)_{\lceil \frac{n_H}{2} \rceil}$. Clearly, if H exists, then $T_1 \cup \dots \cup T_m \cup \hat{S}D(H)_{\lceil \frac{n_H}{2} \rceil}$ is a strong secure set of G of order $\lceil \frac{n}{2} \rceil$; otherwise $T_1 \cup \dots \cup T_m$ is such a set. \square

Now we suppose that G is connected. From the definition of \mathcal{C} we know that join was the last operation in the creation of G . So let $G = G_1 + G_2$, where $G_1, G_2 \in \mathcal{C}$ and G_2 is connected. If G is a complete graph, then our theorem is true; otherwise, without loss of generality, we can assume that G_1 is disconnected. From the inductive hypothesis we know that G_2 is γ_s -monotone, $\gamma_s(G_2) = \lceil \frac{n_2}{2} \rceil$ and $G_1 \in \mathcal{S}_{\lceil \frac{n_1}{2} \rceil}^n$, where n_1, n_2 denote the order of G_1, G_2 , respectively.

Suppose that G_1 or G_2 has even order. By Lemma 3.3 and the inductive hypothesis, we can obtain the set SD_k , for $k \geq \lceil \frac{n}{2} \rceil$ as a union of $S_p(G_1)$ and $SD_t(G_2)$, where $p \geq \lceil \frac{n_1}{2} \rceil$, $t \geq \lceil \frac{n_2}{2} \rceil$, and $p + t = k$. Furthermore, if $k > \lceil \frac{n}{2} \rceil$ or n_2 is odd and $k = \lceil \frac{n}{2} \rceil$, then we formulate additional conditions, that t must be greater than $\frac{n_2}{2}$ and instead $SD_t(G_2)$ we chose $\hat{S}D_t(G_2)$. Then we can easily verify that the so-obtained set SD_k is also a global strong secure set of cardinality k . We have one more case to consider, i.e., n_1 is odd (n_2 is even) and $k = \lceil \frac{n}{2} \rceil$. By Claim 4.28, G_1 has a set $\hat{S}_{\lceil \frac{n_1}{2} \rceil}(G_1)$. Its union with $SD_{\frac{n_2}{2}}(G_2)$ gives us $\hat{S}D_k$.

So let us assume that both n_1 and n_2 are odd. If we look for the set SD_k or

$\hat{S}D_k$, where $k > \frac{n}{2}$, then we can obtain it by union of $S_p(G_1)$ and $\hat{S}D_t(G_2)$, where $p \geq \lceil \frac{n_1}{2} \rceil$, $t \geq \lceil \frac{n_2}{2} \rceil$, and $t + p = k$. To finish the proof, we need to show that there exists $SD_{\frac{n_1+n_2}{2}}$. Let G' be any component of G_1 of odd order n' . Let H be a graph that is isomorphic to K_1 and $V(H) = \{v\}$. We call v an *artificial vertex*. From the inductive hypothesis and Lemma 3.2, we know that there exists a global secure set $SD_{\frac{n'+1}{2}}(G' + H)$ that contains v . Let us denote it by W' . From the inductive hypothesis we also know that $G_1 - G'$ has a secure set $S_{\frac{n_1-n'}{2}}(G_1 - G')$ and G_2 has a set $\hat{S}D_{\lceil \frac{n_2}{2} \rceil}(G_2)$. Let us denote these sets by W_1 and W_2 , respectively. Let $B = (W' - \{v\}) \cup W_1 \cup W_2$. Clearly, B is a dominating set. Let us consider its security. Let X be a subset of B . If $X \subseteq W_1$, then $|N[X] \cap B| = |N_{G_1-G'}[X] \cap W_1| + |W_2|$ and $|N[X] - B| = |N_{G_1-G'}[X] - W_1| + |V(G_2)| - |W_2|$. Since W_1 is a secure set in $G_1 - G'$ and $|W_2| > |V(G_2)| - |W_2|$, we have that $|N[X] \cap B| \geq |N[X] - B|$. If $X \subseteq W_2$, then $|N[X] \cap B| = |N_{G_2}[X] \cap W_2| + |W_1| + |W'| - 1 = |N_{G_2}[X] \cap W_2| + \lceil \frac{n_1}{2} \rceil - 1$ and $|N[X] - B| = |N_{G_2}[X] - W_2| + |V(G_1)| - |W_2| - (|W'| - 1) = |N_{G_2}[X] - W_2| + \lceil \frac{n_1}{2} \rceil$. However, W_2 is a strong secure set, so $|N_{G_2}[X] \cap W_2| \geq |N_{G_2}[X] - W_2| + 1$, which gives us $|N[X] \cap B| \geq |N_{G_2}[X] - W_2| + 1 + \lceil \frac{n_1}{2} \rceil - 1$, and thus $|N[X] \cap B| \geq |N[X] - B|$. So let us suppose that $X \subseteq (W' - \{v\})$. We have that $|N[X] \cap B| = |N_{G'+H}[X] \cap W'| - 1 + |W_2| = |N_{G'+H}[X] \cap W'| - 1 + \lceil \frac{n_1}{2} \rceil$ and $|N[X] - B| = |N_{G'+H}[X] - W'| + |V(G_2)| - |W_2| = |N_{G'+H}[X] - W'| + \lceil \frac{n_1}{2} \rceil - 1$. Since W' is secure in $G' + H$, $|N[X] \cap B| \geq |N[X] - B|$. Let us consider the case when $X \subseteq W_1 \cup (W' - \{v\})$. Then there exist vertex disjoint sets $X_1 \subseteq W_1$ and $X_2 \subseteq W' - \{v\}$ such that $X = X_1 \cup X_2$. Thus $|N[X] \cap B| = |N_{G_1-G'}[X_1] \cap W_1| + |N_{G'+H}[X_2] \cap W'| - 1 + |W_2|$ and $|N[X] - B| = |N_{G_1-G'}[X_1] - W_1| + |N_{G'+H}[X_2] - W'| + |V(G_2)| - |W_2|$. As above, we have that $|N[X] \cap B| \geq |N[X] - B|$. To show that B is secure, we have to prove one more case when $X \cap V(G_2) \neq \emptyset$ and $X \cap V(G_1) \neq \emptyset$. It is easy to see that since there exist vertices $x \in G_1$ and $y \in G_2$ that belong to X , $|N[X] \cap B| = |W'| - 1 + |W_1| + |W_2| = \frac{n_1-1}{2} + \frac{n_2+1}{2} = \frac{n_1+n_2}{2}$ and $|N[X] - B| = \frac{n_2-1}{2} + \frac{n_1+1}{2} = \frac{n_1+n_2}{2}$. Thus $|N[X] \cap B| = |N[X] - B|$. From the above considerations it follows that B is a global secure set of cardinality $\frac{n_1+n_2}{2}$ and this completes the proof. \square

On the basis of the above proof, we can give a linear time algorithm that finds a global (strong) secure set of a graph $G \in \mathcal{C}$.

Theorem 4.29. *If $G \in \mathcal{C}$, then we can find its global secure set of cardinality k , $\gamma_s(G) \leq k \leq n$, in $\mathcal{O}(|V(G)| + |E(G)|)$ time.*

Proof. A cotree T of G can be computed in $\mathcal{O}(|V(G)| + |E(G)|)$, so we can assume that it is given. By using depth-first search algorithm we can compute for each node of T the number of leaves among its descendants; we will call it a *leaf number* of a node. If a vertex is a leaf then we set its leaf number to 1. In the next step we prepare for each node the list of leaves that are adjacent to it, and additionally for each 1-node a list of 0-nodes and for each 0-node two lists of 1-nodes, one containing only nodes with odd leaf number and the second with nodes with even leaf number. These lists can be made during one execution of breadth-first search algorithm.

T is now prepared to run an algorithm based on the proof of Theorem 4.27. In this recursive algorithm, if the root of the given tree is a 1-node, then, if all of its children are leaves then we can mark the required number of them as the vertices which belong to SD_k . If this is not the case then in $\mathcal{O}(1)$ time we can split the tree and add an artificial vertex if it is required (if there is a situation that a subgraph to which we want to add the artificial vertex is K_1 , then we do not modify T ; we can simply omit the call of the algorithm for this subtree). We run the algorithm for the obtained subtrees. If the root of the given tree is a 0-node, then after visiting every child only once (due to the previously prepared lists) we can join them in proper pairs and run the algorithm for chosen subtrees. If the root of the tree has no children (it is a leaf in T), then we simply mark it as the vertex that belongs to the global secure set that we search for. After the recursive algorithm stops, by using depth-first search algorithm we can delete all artificial vertices, and if the deleted vertex did not belong to the global secure set, then we exclude from the global secure set a leaf from the proper subtree (the root node of this subtree is the parent of the removed artificial vertex). Since a 1-node can have at most one artificial vertex, then we can store its address in the node. The above considerations show that the algorithm runs in $\mathcal{O}(|V(G)| + |E(G)|)$ time. \square

Similarly, we can prove the next theorem.

Theorem 4.30. *If $G \in \mathcal{C}$, then we can find its global strong secure set of cardinality k , $\gamma_s(G) \leq k \leq n$, in $\mathcal{O}(|V(G)| + |E(G)|)$ time.*

Now we show that there is another wide subclass of cographs that are γ_s -monotone and γ_s -monotone. *Weakly quasi-threshold graphs (wqt graphs)* were introduced in [1] by Bapat et al.. However, we present their alternative definition given in [37].

Definition 4.31. [37] The class of wqt graphs can be defined recursively as follows:

1. an edgeless graph is a wqt graph;
2. if G_1 and G_2 are wqt graphs, then $G_1 \cup G_2$ is a wqt graph;
3. if G is a wqt graph and H is an edgeless graph, then $G + H$ is a wqt graph.

We prove, that every wqt graph is γ_s -monotone and $\gamma_{\hat{s}}$ -monotone.

Theorem 4.32. *If a graph G of order n is a connected wqt graph, then G is γ_s -monotone, $\gamma_{\hat{s}}$ -monotone, $\gamma_s(G) = \lceil \frac{n}{2} \rceil$, and $\gamma_{\hat{s}}(G) = \lceil \frac{n+1}{2} \rceil$.*

Proof. Clearly, if a given wqt graph has only 1 vertex, then the theorem is true. Suppose that it is also true for every connected wqt graph of order at most b , where $b > 1$. Let G be a wqt graph of order $n > b$. Since G is connected, join was the last operation in its creation. Let $G = G_1 + H$, where H is an edgeless graph (see Definition 4.31). By n_1 and n_H we denote the order of G_1 and H , respectively.

Case 1. Assume G_1 is connected. First, let us suppose that both n_1 and n_H are odd. By the inductive hypothesis there exists $\hat{S}D_{\lceil \frac{n_1}{2} \rceil}(G_1)$, let B denote this set. Furthermore, let W be the union of B and $\lfloor \frac{n_H}{2} \rfloor$ vertices of H . Clearly, W is a dominating set and $|W| = \frac{n}{2}$. Let X be a subset of W . If $X \subseteq V(H)$ then $|N[X] \cap W| = |X| + |B| = |X| + \lceil \frac{n_1}{2} \rceil$ and $|N[X] - W| = n_1 - |B| = \lfloor \frac{n_1}{2} \rfloor$. Hence $|N[X] \cap W| > |N[X] - W|$. In the case when $X \subseteq V(G_1)$, $|N[X] \cap W| = |N[X] \cap B| + \lfloor \frac{n_H}{2} \rfloor$ and $|N[X] - W| = |N[X] \cap (V(G_1) - B)| + \lceil \frac{n_H}{2} \rceil$. B is a strong secure set. Thus, by the definition, $|N[X] \cap B| \geq |N[X] \cap (V(G_1) - B)| + 1$ which implies that $|N[X] \cap W| \geq |N[X] - W|$. If $X \cap V(G_1) \neq \emptyset$ and $X \cap V(H) \neq \emptyset$ then $|N[X] \cap W| = \frac{n}{2} = |N[X] - W|$. From the above considerations, it follows that for any X , $|N[X] \cap W| \geq |N[X] - W|$. Thus, by Theorem 2.2, W is a secure set of G of cardinality $\frac{n}{2}$, which implies that W is a minimum global secure set of G . Now we show how to obtain the global secure sets of greater cardinality. For any $m > \frac{n_1+n_H}{2}$, let W be a set of cardinality m , obtained by the union of $SD_p(G_1)$ and Y_t , where $p + t = m$, $p \geq \lceil \frac{n_1}{2} \rceil$, $t \geq \lceil \frac{n_H}{2} \rceil$, $Y_t \subseteq V(H)$ and $|Y_t| = t$. By Lemma 3.3, W is a global secure set of G . Now we show that W is also a global strong secure set of G . Let X be a nonempty subset of W . Suppose first that $X \subseteq V(G_1)$. Then $|N[X] \cap W| = |N[X] \cap SD_p(G_1)| + |Y_t|$ and $|N[X] - W| = |N[X] \cap (V(G_1) - SD_p(G_1))| + |V(H) - Y_t|$. By the security of $SD_p(G_1)$, we have that $|N[X] \cap SD_p(G_1)| \geq |N[X] \cap (V(G_1) - SD_p(G_1))|$. Since n_H

is odd and $t \geq \lceil \frac{n_H}{2} \rceil$, $|Y_t| > |V(H) - Y_t|$. Thus $|N[X] \cap W| > |N[X] - W|$. Now let us assume that $X \subseteq V(H)$. Then $|N[X] \cap W| = |X| + |SD_p(G_1)|$ and $|N[X] - W| = |V(G_1) - SD_p(G_1)|$. Since n_1 is odd and $p \geq \lceil \frac{n_1}{2} \rceil$, $|SD_p(G_1)| > |V(G_1) - SD_p(G_1)|$ and $|N[X] \cap W| > |N[X] - W|$. In the last case, assume $X \cap SD_p(G_1) \neq \emptyset$ and $X \cap Y_t \neq \emptyset$. Then $|N[X] \cap W| = |W| > \frac{n}{2}$ and $|N[X] - W| = n - |W| < \frac{n}{2}$, which implies that $|N[X] \cap W| > |N[X] - W|$. Hence, for any nonempty $X \subseteq W$, $|N[X] \cap W| > |N[X] - W|$. It follows that W is a global strong secure set of G .

Assume that n_1 or n_H is even. By Lemma 3.3, we can obtain a set $SD_m(G)$ as the union of the set $SD_p(G_1)$ and r vertices of H , where $p \geq \lceil \frac{n_1}{2} \rceil$, $r \geq \lceil \frac{n_H}{2} \rceil$ and $p + r = m \geq \lceil \frac{n}{2} \rceil$. If $p > \frac{n_1}{2}$ then instead $SD_p(G_1)$ we choose $\hat{S}D_p(G_1)$. As previously, we can prove that, for any $m > \frac{k}{2}$, the obtained set $SD_m(G)$ is also a global strong secure set of G .

Case 2. Suppose that G_1 is disconnected. Let G_{11}, \dots, G_{1q} ($q \geq 2$) be connected components of G_1 . We join in disjoint pairs the components whose orders are of the same parity. If q is odd then we leave one component without a pair. Only in case when q is even and n_1 is odd, we create one mixed pair, in which the components have different parity of orders. Observe, that we cannot have both a mixed pair and a component without a pair. Let us denote the pairs obtained in the above procedure by $P_1, \dots, P_{\lfloor \frac{q}{2} \rfloor}$.

First, we show how to obtain the minimum global secure set of G . Let G' and G'' be graphs that belong to the pair P_1 . Without loss of generality, we can assume that $n' \geq n''$, where n' and n'' denote the order of G' and G'' , respectively. By the inductive hypothesis there exists a set $\hat{S}D_{\lceil \frac{n'+n''}{2} \rceil}(G')$, let us denote it by C^1 . Analogously we obtain the set C^i for every pair P_i , where $2 \leq i \leq \lfloor \frac{q}{2} \rfloor$. If q is odd then we have one component G_{1r} ($r \in \{1, \dots, q\}$) of order n_{1r} that does not belong to any pair. If n_{1r} is odd, then there exists a set $\hat{S}D_{\lceil \frac{n_{1r}}{2} \rceil}(G_{1r})$, let C^0 be such a set; otherwise let C^0 be any $SD_{\lceil \frac{n_{1r}}{2} \rceil}(G_{1r})$. If q is even then $C^0 = \emptyset$. The most problematic situation is when n_1 is odd and $n_H = 1$. Suppose first that this is not the case. Then, we obtain the set $SD_{\lceil \frac{n_1+n_H}{2} \rceil}(G)$ as the union of the sets $C^0, \dots, C^{\lfloor \frac{q}{2} \rfloor}$ and x vertices of H , where if n_1 is odd then $x = \lfloor \frac{n_H}{2} \rfloor$; otherwise $x = \lceil \frac{n_H}{2} \rceil$. Our choice of the sets $C^0, \dots, C^{\lfloor \frac{q}{2} \rfloor}$ implies that in the case when n is odd, the obtained set is strong secure. Now let us consider the case when n_1 is odd and $V(H) = \{v\}$. We modify either C^0 or C^j , where P_j is a mixed pair. If q is odd, we replace an arbitrary vertex from C^0 with v , otherwise we do that for C^j . We leave the remaining sets C^z ($z \neq j$, $1 \leq z \leq \lfloor \frac{q}{2} \rfloor$) without a change. Now let W denote the union of the sets $C^0, \dots, C^{\lfloor \frac{q}{2} \rfloor}$. Let $X \subseteq W$. If $v \notin X$, then, by

Lemma 3.2 and the choice of $C^0, \dots, C^{\lfloor \frac{g}{2} \rfloor}$, we have $|N[X] \cap W| \geq |N[X] - W|$; otherwise, since v is a universal vertex in G , $|N[X] \cap W| = \frac{n}{2} = |N[X] - W|$. Hence W is a minimum global secure set in G .

To obtain a set SD_m , where $m > \gamma_s(G)$, we use the inductive hypothesis and Lemma 4.9 (we can apply it to the pairs of components) to find a set $S_p(G_1)$ (a secure set of cardinality p), where $n_1 \geq p \geq \lceil \frac{n_1}{2} \rceil$. By Lemma 3.3, a set $SD_m(G)$ can be obtained as the union of $S_p(G_1)$ and t vertices of H , where $p \geq \lceil \frac{n_1}{2} \rceil$ and $t \geq \lceil \frac{n_H+1}{2} \rceil$. As in the previous part of the proof, we can easily verify that the set $SD_m(G)$ is also a global strong secure set of G . \square

Again the presented proof shows a method that allows to find a global (strong) secure set of any wqt graph in linear time.

Theorem 4.33. *If G is a connected wqt graph, then we can find its global secure set of cardinality k , $\gamma_s(G) \leq k \leq |V(G)|$, in $\mathcal{O}(|V(G)| + |E(G)|)$ time.*

Theorem 4.34. *If G is a connected wqt graph, then we can find its global strong secure set of cardinality k , $\gamma_s(G) \leq k \leq |V(G)|$, in $\mathcal{O}(|V(G)| + |E(G)|)$ time.*

Chapter 5

Expansion of alliances and secure sets

In this chapter we consider the possibility of adding new members into a given alliance or a secure set.

5.1 Introduction

If we take a look at alliances in a real world, then we will see that they change, and the most common reason of this change are new members. NATO and EU are constantly in a process of the expansion. This considerations led us to a new concept in the field of the alliances in graphs. Namely, we say that an alliance (of any type) is *expandable* if there exists a vertex that can be added to it and the resulting set is still an alliance. The formal definition for a (global) defensive alliance is as follows. Let $G = (V, E)$ be a graph of order n . A (global) defensive alliance A is *expandable* if $|A| < n$, and there exists a vertex $v \in V - A$ such that the set $A \cup \{v\}$ is a (global) defensive alliance in G . Analogously, a (global) (strong) secure set S is *expandable* if $|S| < n$, and there exists a vertex $v \in V - S$ such that the set $S \cup \{v\}$ is a (global) (strong) secure set in G . Furthermore, we say that the graph G is (γ_a -*expandable*) *a-expandable* if every its (global) defensive alliance of cardinality less than n is expandable, and the graph G is (γ_s -*expandable*) ($\gamma_{\hat{s}}$ -*expandable*) (\hat{s} -*expandable*) *s-expandable* if every its (global secure set) (global strong secure set) (strong secure set) secure set of cardinality less than n is expandable. As we can see, the formulated requirements are very strong. In many

situations, we do not need such a strong property of the graph. If we consider creating a coalition within a given structure, sometimes we would only like to know whether there exists a strategy, that allows us to start with a small alliance that can be gradually expanded. This concept is described in the following definitions. We say that a graph G is:

1. *weakly a -expandable*, if there exist sets $A_{a(G)}, A_{a(G)+1}, \dots, A_n$ such that, for $a(G) < i \leq n$, $A_i = A_{i-1} \cup \{v\}$, where $v \in V - A_{i-1}$;
2. *weakly γ_a -expandable*, if there exist sets $AD_{\gamma_a(G)}, AD_{\gamma_a(G)+1}, \dots, AD_n$ such that, for $\gamma_a(G) < i \leq n$, $AD_i = AD_{i-1} \cup \{v\}$, where $v \in V - AD_{i-1}$;
3. *weakly s -expandable*, if there exist sets $S_{s(G)}, S_{s(G)+1}, \dots, S_n$ such that, for $s(G) < i \leq n$, $S_i = S_{i-1} \cup \{v\}$, where $v \in V - S_{i-1}$;
4. *weakly γ_s -expandable*, if there exist sets $SD_{\gamma_s(G)}, SD_{\gamma_s(G)+1}, \dots, SD_n$ such that, for $\gamma_s(G) < i \leq n$, $SD_i = SD_{i-1} \cup \{v\}$, where $v \in V - SD_{i-1}$.

5.2 Basic properties

In Chapter 4 we defined a problem of p -monotonicity of a graph G , where p stands for $a(G)$, $s(G)$, $\gamma_a(G)$ and $\gamma_s(G)$. Clearly, if G is not p -monotone, then it cannot be p -expandable. Moreover, by the definitions, any p -expandable graph is p -monotone. The next remark also follows directly from the definitions.

Remark 5.1. If a graph is a -expandable (s -expandable), then it is γ_a -expandable (γ_s -expandable).

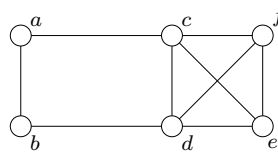


FIGURE 5.1: A graph W which is both γ_a -expandable and γ_s -expandable, but it is neither a -expandable nor s -expandable

To see that the converse implication is not true let us consider a graph W presented in Fig. 5.1. The set $B = \{a, b\} \subseteq V(W)$ is an example of a secure set (defensive

alliance) that is not expandable. However, we can easily check, that every global defensive alliance and every global secure set in W is expandable. It follows from the fact that any global defensive alliance and any global secure set must contain at least two vertices that belong to the clique induced by $\{c, d, e, f\}$. It is not hard to see, that the family of graphs that are γ_a -expandable (γ_s -expandable) but are not a -expandable (s -expandable) is infinite.

Verifying whether a given defensive alliance is expandable requires checking the condition from the following proposition.

Proposition 5.2. *Let G be a graph with a defensive alliance A . The set A is expandable if and only if there exists a vertex $v \in V - A$ such that $|N(v_i) \cap A| \geq \left\lfloor \frac{|N(v)|}{2} \right\rfloor$.*

Proof. Suppose that A is expandable and there is no vertex $v \in V - A$ such that $|N(v_i) \cap A| \geq \left\lfloor \frac{|N(v)|}{2} \right\rfloor$. Hence, for any vertex v , $A' = A \cup \{v\}$ is not a defensive alliance since $|N[v] \cap A'| < |N[v] - A'|$, a contradiction. If there exists a vertex $v \in V - A$ such that $|N(v_i) \cap A| \geq \left\lfloor \frac{|N(v)|}{2} \right\rfloor$, then we can easily verify that $A' = A \cup \{v\}$ is a defensive alliance. \square

Since every secure set is also a defensive alliance, we obtain the next corollary.

Corollary 5.3. *Let G be a graph with a secure set S . If the set S is expandable, then there exists a vertex $v \in V - S$ such that $|N(v_i) \cap S| \geq \left\lfloor \frac{|N(v)|}{2} \right\rfloor$. If there does not exist a vertex $v \in V - S$ such that $|N(v_i) \cap S| \geq \left\lfloor \frac{|N(v)|}{2} \right\rfloor$, then S is not expandable.*

Now we present several simple graph classes that are both a -expandable and s -expandable.

Proposition 5.4. *If a graph G is:*

1. *a complete graph,*
2. *a complete bipartite graph,*
3. *a path,*
4. *a cycle,*

then G is a -expandable and s -expandable.

We show that also all connected graphs with maximum degree 3 are both a -expandable and s -expandable.

Theorem 5.5. *Let G be a connected graph such that $\Delta(G) \leq 3$. Then, G is a -expandable and s -expandable.*

Proof. Observe, that if $\Delta(G) \leq 3$, then the end-vertices of any edge of G form an alliance. Hence, clearly G is a -expandable. Now, let S be any secure set of G , we show that it is expandable. If there exists a vertex $x \in N(S) - S$ such that $|N(x) \cap S| \geq \lceil \frac{\deg x}{2} \rceil$, then clearly $S \cup \{x\}$ is a secure set, and the lemma holds. So suppose that this is not the case, i.e., all the vertices that belong to $N(S) - S$ have degree 3 and exactly one neighbour in S . In this case, there is only one maximal attack. First, let us suppose that also each vertex of S has at most one neighbour outside the set. Let v be a vertex of S that has a neighbour $y \in N(S) - S$. Observe, that if we add y to S , then in any attack (not necessarily maximal) every vertex that has a neighbour in $N(S) - S$ can defend itself, and since v has now no neighbours in $N(S) - S$, it can always defend y . Hence, the set $S \cup \{y\}$ is secure. Thus, let us suppose that there is a vertex $x \in S$ with two neighbours $z, p \in N(S) - S$. Since S is secure, x has a neighbour $u \in S$, and moreover in the maximal attack both u and x defend x . We can assume that x always fights against z , and u against p . Clearly, the set $S' = S \cup \{z\}$ is secure, since now, in any attack, x can defend z , and a defence for the rest of the vertices can be organized in the same way as for the maximal attack on S . \square

The above theorem is included in [31].

5.3 Expansion of global (strong) secure sets and global defensive alliances

In this section we consider the expansion of global secure sets of k -trees and quasi-threshold graphs. The presented results (except for Theorem 5.6) were published in [29].

5.3.1 k -trees

A k -tree [38] is defined inductively as follows:

1. K_k is a k -tree;
2. let G be a k -tree on n vertices and let K be a k -clique in G ; then, the $(n + 1)$ -vertex graph G' formed from G by adding a new vertex v adjacent to all vertices of K is a k -tree.

A *perfect elimination order* is an ordering (v_1, v_2, \dots, v_n) of the vertices of the graph such that $N(v_i)$ induces a clique in a graph induced by the set of vertices $\{v_{i+1}, \dots, v_n\}$. It is clear that every k -tree has a perfect elimination order.

Theorem 5.6. *If a graph G is a k -tree, where $k \leq 2$, then G is γ_a -expandable.*

Proof. Let v_1, \dots, v_n be a perfect elimination order of G which is the reverse order of vertices added to construct G . Furthermore, let A ($|A| < n$) be a global defensive alliance in G . Let i be the smallest i such that $v_i \notin A$. Thus $i = 1$ or $v_1, \dots, v_{i-1} \in A$. Since $k \leq 2$, v_i has at most two neighbours in v_{i+1}, \dots, v_n . Moreover, A is a dominating set, which implies that v_i has at least one neighbour in A . Hence, by our choice of v_i and Proposition 5.2, $A \cup \{v_i\}$ is a global defensive alliance. \square

Theorem 5.7. *If a graph G is a k -tree where $k \leq 2$, then G is γ_s -expandable.*

Proof. We prove the above theorem for $k = 2$. Similarly, we can prove it for $k = 1$. Let v_1, \dots, v_n be a perfect elimination order of G which is the reverse order of vertices added to construct G . Furthermore, let A ($|A| < n$) be a global secure set in G . Let i be the smallest i such that $v_i \notin A$. Thus $i = 1$ or $v_1, \dots, v_{i-1} \in A$. Since A is a dominating set and G is a 2-tree, v_i has at least 1 neighbour in A and at most 2 neighbours in $\{v_{i+1}, \dots, v_n\}$ that do not belong to A . If at most 1 neighbour of v_i does not belong to A , then clearly we can add v_i to A . So suppose that $i > 1$ and v_i has 2 neighbours $v_j, v_t \notin A$, where $j, t \in \{i + 1, \dots, n\}$. Since G is a 2-tree and A is a dominating set such that $v_1, \dots, v_{i-1} \in A$, there exists a vertex v_p ($p < i$) such that $v_i v_p \in E$ and $v_j v_p \in E$ or $v_t v_p \in E$. Without loss of generality, we can assume that $v_j v_p \in E$. Since A is a dominating set, it is enough to show that $A \cup \{v_i\}$ is a secure set. From the security of A , it follows

that if in an attack the vertex v_t or v_j attacks a vertex different from v_i , then such an attack is defensible, since v_i can defend itself against one attacker. The only problematic case is when both v_t and v_j attack v_i . We show that in this situation, v_p can defend v_i . Suppose conversely that v_p cannot help v_i . It follows that v_p has a neighbour v_r , where $1 \leq r \leq p - 1$, that has at least 1 (otherwise v_r does not require any defence) and at most 2 neighbours in $V - (A \cup \{v_i\})$. Let us consider the neighbourhood of v_r . From the choice of v_i and the perfect elimination order, it follows that all the neighbours of v_r in $\{v_1, \dots, v_{r-1}\}$ belong to A . However, v_r has only two neighbours in $\{v_{r+1}, \dots, v_n\}$. One of them is v_p , let us denote the second one by y . From the construction of G , it follows that $v_p y \in E$. However, $N(v_p) \cap (V - (A \cup \{v_i\})) = \{v_j\}$. Hence, $y = v_j$ and v_j attacks v_r (otherwise v_p would not have to participate in the defence of v_r), which gives us the contradiction with our previous assumption that v_j is the attacker of v_i . This observation completes the proof. \square

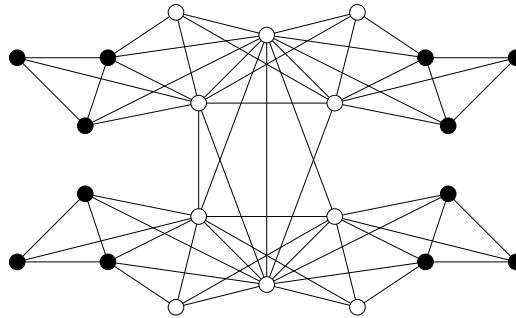


FIGURE 5.2: A 3-tree that is neither γ_s -expandable nor γ_a -expandable. The black vertices belong to a global secure set that is not expandable.

The next theorem concerns k -trees, where $k \geq 3$.

Theorem 5.8. *For every integer $k \geq 3$, there exists a k -tree that is neither γ_s -expandable nor γ_a -expandable.*

Proof. We divide the proof into two cases. First suppose that $k = 3$. Consider the graph presented in Fig. 5.2, let us denote it by G . The black vertices form a global secure set, let us denote it by W , that is not expandable. It follows from the fact that for every white vertex x , $|N[x] \cap (W \cup \{x\})| < |N[x] - (W \cup \{x\})|$. Hence, by the definition of the defensive alliances and Theorem 2.2, the set $W \cup \{x\}$ is

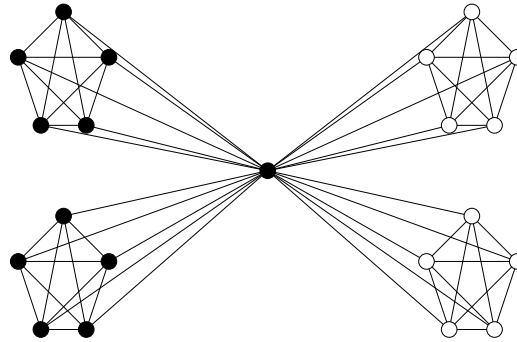
neither a defensive alliance nor a secure set.

Now assume $k \geq 4$. Let G' be a k -tree such that $V(G') = \{v_1, v_2, \dots, v_{4k}\}$. Moreover, the following sets of vertices induce cliques: $\{v_1, \dots, v_k\}$, $\{v_i, \dots, v_{i+k}\}$ for $1 \leq i \leq k$, $\{v_2, \dots, v_{k+1}\} \cup \{v_j\}$ for $2k+1 \leq j \leq 3k$, and $\{v_{k+1}, \dots, v_{2k}\} \cup \{v_p\}$ for $3k+1 \leq p \leq 4k$. Let us consider a set $W = \{v_{k+1}, \dots, v_{2k}\} \cup \{v_{3k+1}, \dots, v_{4k}\}$. From the construction of G' it follows that v_{k+1} is a universal vertex. Hence W is a dominating set. Now we show that W is secure. Let X be a subset of W . First suppose that $v_{k+1} \notin W$. If $v_t \in W$, where $t \in \{k+2, \dots, 2k\}$, then $|N[X] \cap W| = 2k$ and $|N[X] - W| \leq k$; otherwise $|N[X] \cap W| \geq |X| + k$ and $|N[X] - W| = 0$. Now suppose that $v_{k+1} \in W$. Then $|N[X] \cap W| = |N[X] - W|$. In every considered case, $|N[X] \cap W| \geq |N[X] - W|$. Thus by Theorem 2.2, the set W is secure.

In the last step of the proof we show that there does not exist $y \in (V(G') - W)$ such that $W \cup \{y\}$ is secure. Consider the vertices v_{2k+1}, \dots, v_{3k} . Each of these vertices has exactly 1 neighbour in W , and k neighbours in $V(G') - W$. Whereas each of the vertices v_1, \dots, v_k has at most k neighbours in W , and $2k-1$ neighbours in $V(G') - W$. Thus, by Theorem 2.2, none of the considered vertices can be used to expand the set W . It follows that W is a global secure set of G' that is not expandable, which implies that G' is not γ_s -expandable. Clearly, W is also an example of a global defensive alliance that is not expandable. This observation finishes the proof. \square

5.3.2 Co-quasi-threshold graphs

In Chapter 4, we investigated the γ_s -monotonicity of cographs. First, we observe that not every cograph that is γ_s -monotone is also weakly γ_s -expandable. The trivially perfect graph H presented in Fig. 5.3 is not weakly γ_s -expandable. The graph H has 1 universal vertex and 4 vertex-disjoint cliques C_1, C_2, C_3, C_4 of size 5. $\gamma_s(H) = 11$ but there does not exist a minimum global secure set that contains at least 2 vertices from each clique C_i , where $i \in \{1, \dots, 4\}$. Moreover, any set that contains only 1 vertex from a clique C_i , is not secure. Thus, in every minimum global secure set W , there is at least one clique C_k ($k \in \{1, \dots, 4\}$) such that $C_k \cap W = \emptyset$. It follows that at some point of the process of the expansion it is impossible to add a vertex from C_k . Thus H is not weakly γ_s -expandable. Basing on the cliques of odd order we can build infinitely many wqt graphs, similar to H , that are not weakly γ_s -expandable.



H

FIGURE 5.3: An example of a trivially perfect graph that is not weakly γ_s -expandable. The black vertices form a global secure set that is not expandable.

Co-quasi-threshold graphs (co-qt graphs) are graphs without induced P_4 and $2K_2$. This class of graphs is equivalent to co-trivially perfect graphs. Co-qt graphs can be recognized in linear time by an algorithm presented by Chu [9]. We prove that co-qt graphs are weakly γ_s -expandable and weakly $\gamma_{\bar{s}}$ -expandable.

Theorem 5.9. *If a graph G is a connected co-qt graph of order n , then G is weakly γ_s -expandable, weakly $\gamma_{\bar{s}}$ -expandable, $\gamma_s(G) = \lceil \frac{n}{2} \rceil$, $\gamma_{\bar{s}}(G) = \lceil \frac{n+1}{2} \rceil$, and there exist sets $SD_{\gamma_s(G)}, SD_{\gamma_s(G)+1}, \dots, SD_n$ such that for $\gamma_s(G) < i \leq n$, $SD_i = SD_{i-1} \cup \{v\}$ where $v \in V - SD_{i-1}$, and for every integer $k > \frac{n}{2}$, SD_k is also a global strong secure set.*

Proof. If G is a complete graph, then clearly the theorem holds. Let us suppose that the theorem is true for every co-qt graph of order at most n' , where $n' > 2$, and let G be a connected co-qt graph of order $n > n'$ that is not a complete graph. Since G is a connected graph, join was the last operation in its creation. Hence, let $G = G_1 + G_2$ and n_1, n_2 denote the order of G_1 and G_2 , respectively. From the properties of the cotree and the fact that G is not a complete graph it follows that, without loss of generality, we can assume that G_1 is disconnected. Since G does not have $2K_2$ as an induced subgraph, neither G_1 nor G_2 has two connected components of order greater than 1. Let $G_1 = G_{11} \cup H_1$ ($G_2 = G_{21} \cup H_2$) where H_1 (H_2) is an edgeless graph and G_{11} (G_{21}) is a connected component of G_1 (G_2) of the greatest cardinality. Furthermore, let $n_{11} > 0$ ($n_{21} > 0$) and $n_{H_1} > 0$ ($n_{H_2} \geq 0$) denote the order of G_{11} (G_{21}) and H_1 (H_2), respectively. Since being a co-qt graph is an induced hereditary property, we can apply the

inductive hypothesis to G_{11} and G_{21} . Hence, for $j \in \{1, 2\}$ there exists sets $\mathcal{A}_j = \{SD_{\gamma_s(G_{j1})}(G_{j1}), SD_{\gamma_s(G_{j1})+1}(G_{j1}), \dots, SD_{n_{j1}}(G_{j1})\}$ such that for $\gamma_s(G_{j1}) < i \leq n_{j1}$, $SD_i(G_{j1}) = SD_{i-1}(G_{j1}) \cup \{v\}$ where $v \in V(G_{j1}) - SD_{i-1}(G_{j1})$, and for every $k > \frac{n_{j1}}{2}$, $SD_k(G_{j1})$ is also a global strong secure set in G_{j1} . From now on, we choose only global secure sets of G_{11} and G_{21} that belong to \mathcal{A}_1 and \mathcal{A}_2 , respectively. First we show how to obtain the minimum global secure set of G . For $i \in \{1, 2\}$, let X_k^i denote the subset of cardinality k of the vertices of H_i . Let us assume that n_1 or n_2 is even. For $i \in \{1, 2\}$, we define the set F_i in the following way:

$$F_i = \begin{cases} X_{\lfloor \frac{n_{H_i}}{2} \rfloor}^i \cup SD_{\lceil \frac{n_{i1}}{2} \rceil}(G_{i1}) & \text{if } n_{i1} \text{ is odd,} \\ X_{\lfloor \frac{n_{H_i}}{2} \rfloor - 1}^i \cup SD_{\frac{n_{i1}}{2} + 1}(G_{i1}) & \text{if } n_{i1} \text{ is even and } n_{H_i} > 0, \\ SD_{\frac{n_{i1}}{2}}(G_{i1}) & \text{if } n_{i1} \text{ is even and } n_{H_i} = 0. \end{cases}$$

By Lemma 3.3, the set $F = F_1 \cup F_2$ is a global secure set of G of cardinality $\lfloor \frac{n}{2} \rfloor$. Furthermore, if n is odd, then it is a global strong secure set in G . Now we consider the case when both n_1 and n_2 are odd. We define the set F_2 as previously, whereas to obtain the minimum global secure set of cardinality $\frac{n}{2}$ we decrease the cardinality of F_1 . If $n_{H_1} > 1$, let $F_1 = X_{\lfloor \frac{n_{H_1}}{2} \rfloor - 1}^1 \cup SD_{\lceil \frac{n_{11}+1}{2} \rceil}(G_{11})$; otherwise n_{11} is even and $F_1 = SD_{\frac{n_{11}}{2}}(G_{11})$. Our choice of F_2 implies that even though $|F_1| < \frac{n_1}{2}$ the set $F = F_1 \cup F_2$ is a global secure set of G . To complete the proof we describe one of the possible procedures of the expansion of F .

Let $P_1 = F$.

For $i := 1$ to $n - |P_1|$ do

 If $V(G_{11}) - P_i \neq \emptyset$ then

$v := SD_{k+1}(G_{11}) - SD_k(G_{11})$, where $k = |V(G_{11}) \cap P_i|$

 else if $(V(H_1) - P_i) \neq \emptyset$ then

$v :=$ any vertex that belongs to $V(H_1) - P_i$

 else if $V(G_{21}) - P_i \neq \emptyset$ then

$v := SD_{k+1}(G_{21}) - SD_k(G_{21})$, where $k = |V(G_{21}) \cap P_i|$

 else $v :=$ any vertex that belongs to $V(H_2) - P_i$

$P_{i+1} := P_i \cup \{v\}$;

Recall that also in this procedure we use only the global (strong) secure sets that belong to \mathcal{A}_1 and \mathcal{A}_2 . The existence of the sets P_i ($1 \leq i \leq n - |P_1| + 1$) proves that

G is weakly γ_s -expandable, and the choice of the P_i guarantees that, if $|P_i| > \frac{n}{2}$, then P_i is a global strong secure set in G . Thus G is also weakly γ_s -expandable. This observation finishes the proof. \square

Clearly, the above proof gives us a linear time algorithm that can find a global (strong) secure set of any proper cardinality in a given co-qt graph.

Chapter 6

Graph partitioning and the alliances

There are many concepts in graph partitioning that are related to the alliances. Some of them were defined earlier than defensive alliances and can be seen as their precursors.

6.1 Introduction

Satisfactory partition is a bipartition of the vertex set of a graph, such that each vertex has as many neighbours in its part as in the other part. The problem of finding such bipartition was introduced in [20, 21] by Gerber and Kobler. An interesting generalization of satisfactory partition was studied in [2]. The authors of [2] developed the problem by adding a function of satisfiability f . Now each vertex is required to have at least $f(v)$ neighbours in its part of the partition. If d is a degree function and $f = \lceil d/2 \rceil$, then we get satisfactory partition, which can be also seen as a bipartition of the vertex set into two defensive alliances. Moreover, if $f = d - 1$, then the problem of finding the bipartition with such function of satisfiability is widely known as the *matching cutset problem*. In [3], Bazgan et al. introduced *balanced satisfactory partition* in which we require that sets of the bipartition have the same cardinality. Clearly, this partition is defined only for graphs of even order.

6.2 Matching cutset

6.2.1 Introduction

A set M of independent edges in a graph $G = (V, E)$ is called a *matching*. A matching M is called *perfect* if every vertex of $V(G)$ is incident to an edge of M . Moreover, a matching M is *almost perfect* if all the vertices of $V(G)$ but one are incident to an edge of M . A set S of edges (or vertices) of a graph G is called a *cutset* (in G) if $G - S$ has more components than G . Furthermore, if a cutset is a matching, then it is called a *matching cutset*. In other words, we say that a graph has a matching cutset if its vertices can be coloured in red and blue in such a way that there exists at least one vertex coloured in red and at least one vertex coloured in blue, and every vertex has at most one neighbour coloured in the opposite colour. Such colouring we call a *mc-colouring*.

The problem of recognizing graphs with a matching cutset (let MATCHING CUTSET denote this problem) is well-studied in the literature. Here are some of the most important results. Chvátal [10] proved that MATCHING CUTSET is NP-complete even for graphs with maximum degree 4. In [36], Moshi showed that MATCHING CUTSET is NP-complete, even if the input is restricted to bipartite graphs of minimum degree 2. In the same paper the author presented an $O(n^3m)$ algorithm, for graphs without induced cycles of length greater than 4, which determines whether G has a matching cutset, where $n = |V(G)|$ and $m = |E(G)|$. He also proved that if G is a line graph, then we can determine in $O(m)$ time whether G has a matching cutset.

In [4] the authors proved that if a graph $G \in \mathcal{P}(2)$ has a matching cutset M , then M is perfect or almost perfect depending on the parity of the order of G , where $\mathcal{P}(2)$ denotes the class of graphs of diameter 2 that have only trivial modules. This property implies the next result.

Proposition 6.1. *If a graph $G \in \mathcal{P}(2)$ has a matching cutset, then $\gamma_s(G) = \lceil \frac{n}{2} \rceil$.*

Borowiecki and Jesse-Józefczyk also proved that if a graph $G \in \mathcal{P}(2)$, then we can decide in polynomial time whether G has a matching cutset [4]. Now we turn our attention to graphs that possess nontrivial modules. Let G be a graph whose vertices are coloured in red and blue. Then, $R(G)$ ($B(G)$) denote the set of vertices that are coloured in red (blue), and let us denote by $E_2(G)$ the set of all bichromatic edges of G , i.e., the set of edges whose end vertices are coloured

in different colours. When no confusion is possible, we denote $R(G)$, $B(G)$ and $E_2(G)$ by R , B and E_2 , respectively. The set of the uncoloured vertices of G is denoted by $\mathcal{UC}(G)$ or briefly by \mathcal{UC} . The next result is included in [4].

Theorem 6.2. *If $\text{diam } G = 2$ and G has nontrivial modules, then we can decide in polynomial time whether G has a matching cutset.*

Proof. In our proof we use a modular decomposition of a graph G . If $\delta(G) = 1$, then clearly G has a matching cutset. Hence, let us assume that $\delta(G) \geq 2$.

Part 1. Suppose that G has a module W^* such that W^* satisfies one of the following conditions:

1. $|W^*| \geq 3$;
2. $W^* = \{x, y\}$ and $xy \in E$;
3. $W^* = \{x, y\}$, $xy \notin E$, and $|N(W^*)| \geq 3$;
4. $W^* = \{x, y\}$, $xy \notin E$, $N(W^*) = \{u, v\}$, and $uv \in E$;
5. $W^* = \{x, y\}$, $xy \notin E$, $N(W^*) = \{u, v\}$, and $(N(u) - W^*) \cap (N(v) - W^*) \neq \emptyset$;
6. $W^* = \{x, y\}$, $xy \notin E$, $N(W^*) = \{u, v\}$, and u or v is adjacent to a different module W^{**} .

It is easy to see that if W^* satisfies one of the conditions (1)–(5) then every vertex $v \in N[W^*]$ must have the same colour. Let us consider the case when W^* satisfies only the condition (6) (see Fig. 6.1). Let u be adjacent to module $W^{**} = \{w_1, w_2\}$ ($W^* \neq W^{**}$). Without loss of generality, we can suppose that we would colour x, u in red and y, v in blue. Then we have to colour w_1 and w_2 in red. Since diam

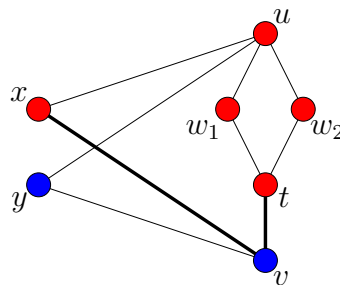


FIGURE 6.1: W^* satisfies only condition (6). Bold edges contradict the existence of the matching cutset.

$G = 2$ and W^* does not satisfy any of the conditions (1)–(5), there must exist a vertex $t \in N(v) - W^*$ that is adjacent to W^{**} (otherwise $d(w_1, v) = d(w_2, v) \geq 3$).

Since t has two neighbours w_1 and w_2 coloured in red, it must be coloured in red. Now we have the required contradiction since v has two neighbours coloured in red, i.e., x and t . Hence, without loss of generality, we can colour in red all the vertices in $N[W^*]$. According to the fact that no vertex can have two neighbours in the opposite colour, we must colour in red all the neighbours of coloured modules of order at least 2. Furthermore, we must colour in red the modules of order at least 3 that are adjacent to a coloured vertex. We also colour in red every module and vertex that has at least two coloured neighbours. We must repeat these operations as long as we can. After that, every uncoloured module or vertex has exactly one neighbour coloured in red (if this condition would not be satisfied we would get a contradiction with the fact that $\text{diam } G = 2$), and the order of these modules will be at most 2.

Claim 6.3. *Every remaining uncoloured module of order 2 must be coloured in red.*

Proof. Let $P = \{x, y\}$ be an uncoloured module of order 2 and $z = N(P) \cap R$. First, suppose that $xy \notin E$. Since $\delta(G) \geq 2$ and $|N(P) \cap R| = 1$, x, y must have an uncoloured neighbour t . Also t must have a neighbour $m \in R$, otherwise we would have a contradiction with the fact that $\text{diam } G = 2$. If $z = m$, then xz, xt, tz and zt, ty, yz induce triangles in G , so x, y, t must be coloured in red. Now suppose that $z \neq m$. If we colour x (y) in blue, then we must colour t in blue and y (x) in red. However, in this situation t would have two neighbours, m and y , coloured in red. Thus assume that $xy \in E$. Then xy, xz, yz induce a triangle in G . Hence x, y must be coloured in red. \square

It follows that we must colour every uncoloured module of order 2 and all its neighbours in red. After this operation, all the remaining uncoloured vertices do not belong to any module. It is obvious, that we can also colour in red every pair of vertices x, y such that $xy \in E$ and $N(x) \cap R = N(y) \cap R$, and every vertex v such that $|N(v) \cap R| \geq 2$. After these operations every vertex $v \in \mathcal{UC}$ has exactly one neighbour in R .

Claim 6.4. *The graph G has a matching cutset if and only if there does not exist a vertex $v \in R$ such that $|N(v) \cap \mathcal{UC}| > 1$.*

Proof. (\Rightarrow) Suppose that G has a matching cutset and there exists a vertex $v \in R$ such that $|N(v) \cap \mathcal{UC}| > 1$. Let $A = N(v) \cap \mathcal{UC}$. For every $x, y \in A$ $xy \notin E$ or

else they would be already coloured in red. Since $\delta(G) \geq 2$ and $\text{diam } G = 2$, every vertex $x \in A$ has at least one uncoloured neighbour that does not belong to A . Furthermore, if x is not adjacent to a vertex p that belongs to the set $\mathcal{U} - A$, then there exists a vertex $t \in \mathcal{U} - A$ such that $xt, tp \in E$. From these observations, it follows that, if we colour every vertex $x \in A$ in red then we have to colour in red all the other vertices. This is a clear contradiction of the fact that G has a matching cutset. Hence suppose that we would colour in blue a vertex $x \in A$. All the other vertices in A would had to be coloured in red. Let y be one of these vertices.

Suppose that x and y have a common neighbour $t \in \mathcal{UC}$ ($t \notin A$), and r be a neighbour of t that belongs to the set $(N(t) \cap R) - A$. From previous observations, we know that such vertex exists. Moreover, r is the only vertex that belongs to the set $(N(t) \cap R) - A$. Now t must be coloured in blue (otherwise x would have two neighbours coloured in red). However, then t has two neighbours coloured in red, i.e. y and r . Hence suppose that x and y do not have a common neighbour (see Fig. 6.2). Let $t \in N(x) \cap \mathcal{UC}$ ($t \notin A$). Since $\text{diam } G = 2$, there must exist a

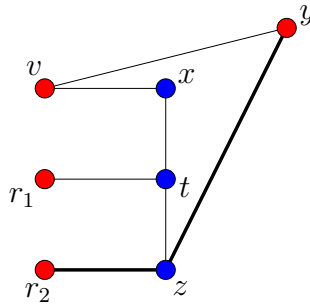


FIGURE 6.2: x and y do not have a common neighbour. Bolded edges contradict the existence of the matching cutset.

vertex $z \in N(y) \cap \mathcal{UC}$ ($z \notin A$) such that $zt \in E$. As previously, t and z must have a neighbour $r_1 \in (N(t) \cap R) - A$ and $r_2 \in (N(z) \cap R) - A$, respectively. Since x is coloured in blue, t and z must be coloured in blue. Now we have the required contradiction since z has two neighbours r_2 and y coloured in red.

(\Leftarrow) It is easy to see, that if there does not exist a vertex $v \in R$ such that $|N(v) \cap \mathcal{UC}| > 1$, then we can colour every vertex in \mathcal{UC} in blue to obtain a matching cutset. \square

Part 2. Suppose that G has no modules that satisfy any of the conditions (1)–(6) of Part 1. Then, G has exactly one module $W^* = \{x, y\}$ and $N(W^*) = \{u, v\}$. Without loss of generality, we can colour x, u in red and y, v in blue. Now

we must colour all the vertices in $N(u)$ in red and all the vertices in $N(v)$ in blue. It is obvious, that after this colouring every vertex is coloured. If E_2 is a matching cutset then we obtained the required solution. Hence, suppose that E_2 is not a matching cutset. Then we must colour every vertex in $N[W^*]$ in red. As previously, we can also colour in red every pair of vertices x, y such that $xy \in E$ and $N(x) \cap R = N(y) \cap R$, and every vertex v such that $|N(v) \cap R| \geq 2$. After that, we can use Claim 6.4 to decide whether G has a matching cutset. \square

6.2.2 Matching cutset in 5-regular graphs

We already mentioned that MATCHING CUTSET is NP-complete. Chvátal [10] proved its NP-completeness through the polynomial time reduction from a well-known problem of bicolouring of hypergraphs. We prove that the problem is also NP-complete for 5-regular graphs. In the proof we use a version of the 3SAT, which is called MONOTONE ONE-IN-THREE 3SAT (M 1-IN-3 3SAT), and is defined as follows. Let \mathcal{C} be a collection of m clauses over the set U such that each clause $C \in \mathcal{C}$ has exactly three unnegated literals. We ask whether there exists a truth assignment such that each clause has exactly one true literal. It was proven that this problem is NP-complete, see [19, 40] for details.

Theorem 6.5. *If a graph G is a 5-regular graph, then the problem of deciding whether G has a matching cutset is NP-complete.*

Proof. For an instance M 1-IN-3 3SAT where \mathcal{C} is a set of m clauses with the set of literals $X = \{x_1, x_2, \dots, x_n\}$ we make a polynomial time reduction from this problem to MATCHING CUTSET. For every literal $x_i \in X$, let C_{x_i} denote the set of indices of the clauses that contain x_i . For each clause C_j , where $j \in \{1, \dots, m\}$, we construct a graph G_{C_j} which is depicted in Fig. 6.3. In each of the resulting graphs we distinguish some vertices as it is presented in the figure. One can see the graph G_{C_j} has a matching cutset, but in any mc-colouring, the vertices y_j and z_j cannot have the same colour, and exactly two vertices from $\{x_k^j, x_l^j, x_p^j\}$ are coloured the same. Moreover, they have the same colour as z_j .

Now let us present the graph T_k which will play the role of the 'colour transmitter'. We obtain it by making k copies of the graph depicted in Fig. 6.4. By H_l we denote each copy of the graph, where l corresponds to the number of the copy.

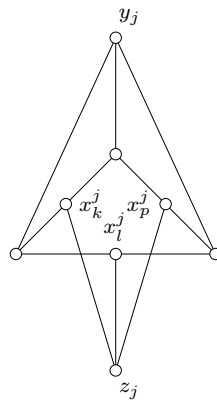


FIGURE 6.3: A graph G_{C_j} , where $C_j = \{x_k, x_l, x_p\}$.

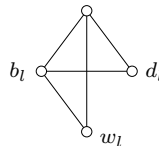


FIGURE 6.4: Graph H_l .

Then, for $l = 1, \dots, k - 1$, we identify the vertices d_l and b_{l+1} according to the indices of the vertices in Fig. 6.4. In the next step we add vertices u_1 and u_2 , and edges u_1u_2 , u_1b_1 , u_2b_1 . Finally, we add a graph K_3 and we join its vertices with every vertex from $\{u_1, u_2, d_k\}$. In Fig. 6.5 we present an example of the graph T_k , for $k = 3$. Since in any mc-colouring the vertices that belong to one triangle must have the same colour, the graph T_k does not have a matching cutset. Now we are

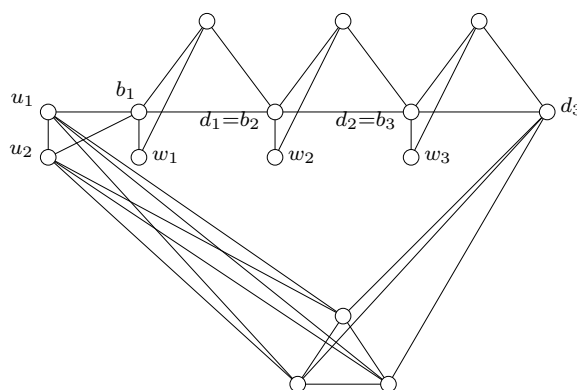


FIGURE 6.5: Graph T_3 .

ready to complete the instance for MATCHING CUTSET. Let A and B be two

vertex disjoint graphs isomorphic to T_m . For $j = 1, \dots, m$, we identify the vertex $y_j \in V(G_{C_j})$ with $w_j \in V(A)$, and the vertex $w_j \in V(B)$ with $z_j \in V(G_{C_j})$. The structure of the resulting graph guarantees that in any mc-colouring the vertices $\{y_1, \dots, y_m\}$ are in one colour, and the vertices $\{z_1, \dots, z_m\}$ are in the second colour class. Now, for every literal $x_i \in X$, we do the following. If x_i belongs to at least two clauses, then we construct a graph $T_{|C_{x_i}|}$ and we identify the vertex w_r with exactly one vertex x_i^k , where $x_i^k \in V(G_{C_k})$, $k \in C_{x_i}$ and $1 \leq r \leq |C_{x_i}|$. Furthermore, every vertex x_i^k must be identified with exactly one vertex w_r . In the graph that we obtained so far, we have only vertices of degree 3 and 5. To get rid of the vertices of degree 3, for every such vertex, we make a copy of the graph $K_6 - e$ and we connect it with two vertices of degree four in $K_6 - e$. Clearly, the graph $K_6 - e$ does not have a matching cutset, and its vertices must be coloured the same as the vertex with which it was connected. Now our graph is 5-regular. Let us denote the resulting graph by G .

Suppose that G has a matching cutset. First we make several simple observations. As we already mentioned, if three vertices induce a triangle in a graph G , then they must have the same colour. Thus, every vertex x_i^k , for $i \in \{1, \dots, n\}$ and $k \in C_{x_i}$, has the same colour. From our previous observations it also follows that, without loss of generality, we can assume that, for $j \in \{1, \dots, m\}$, the vertex y_j is coloured blue and the vertex z_j is coloured red. From the properties of the graph G_j we know that for $j \in \{1, \dots, m\}$, there is exactly one vertex $x_k^j \in V(G_{C_j})$ that is coloured blue. Hence, we conclude that if we assign a truth value to these literals x_i for which the vertices x_i^k are coloured blue for $k \in C_{x_i}$, then we have a good truth assignment for M 1-IN-3 3SAT.

Now suppose that the instance of M 1-IN-3 3SAT has a good truth assignment. In such an assignment, each clause has exactly one true literal. If the literal x_i has the truth value then let us colour all the vertices x_i^k ($k \in C_{x_i}$) in our corresponding graph in red; otherwise we colour them blue. One can see that we can easily extend this colouring in such a way that it will give us an mc-colouring. \square

6.3 Role of the graph partitioning in the expansion

In Chapter 5 we defined the concept of expansion of alliances. Now we would like to connect it with the already mentioned and some other bipartitions of the

vertices of a graph. First we show a connection of MATCHING CUTSET with determining whether the graph is a -expandable or s -expandable.

Proposition 6.6. *If a graph G with $\delta(G) \geq 4$ has a matching cutset, then it is neither a -expandable nor s -expandable.*

Proof. Let S be a set of red vertices of any mc-colouring of the graph G . Clearly, S is a defensive alliance and a secure set. Then, every vertex of $N(S) \cap (V - S)$ has exactly one neighbour in S and at least 3 neighbours in $V - S$. By Proposition 5.2 and Corollary 5.3, S is a defensive alliance and a secure set that is not expandable. \square

The existence of the matching cutset is also crucial in the expansion of strong secure sets in cubic graphs. First let us recall Moshi's result.

Theorem 6.7. [36] *Let G be a simple connected cubic graph that is different from K_4 and $K_{3,3}$. Then, G has a matching cutset.*

Sidorowicz and Jesse-Józefczyk proved in [31] the following result.

Theorem 6.8. [31] *A connected cubic graph G is \hat{s} -expandable if and only if G is one of the graphs presented in Figure 6.6.*

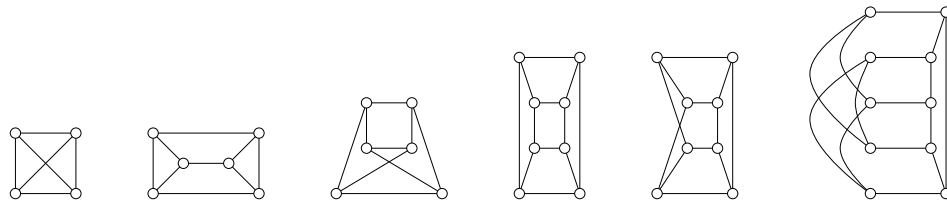


FIGURE 6.6: Cubic \hat{s} -expandable graphs.

With the expansion of alliances we can also relate another decomposition of graphs. Rusu and Spinrad [39] defined a graph bipartition called F -free decomposition. F denotes a bipartite graph with fixed bipartition of the vertices (X_F, Y_F) such that all the vertices in X_F are coloured red and all the vertices in Y_F are coloured blue. The ordered bipartition (V_1, V_2) is F -free decomposition if satisfies these four conditions:

1. $V_1 \supseteq R(G)$, $V_2 \supseteq B(G)$,

2. $E_2(G)$ is an edge cutset of G ,
3. F is not an induced subgraph of the graph $G' = (R(G) \cup B(G), E_2(G))$,
4. $|R(G)| \geq |X_F|$, $|B(G)| \geq |Y_F|$.

We can generalize the above definition as follows. Let \mathcal{F} be a set of bipartite graphs with given red-blue colouring. We say that the ordered bipartition (V_1, V_2) is \mathcal{F} -free decomposition if:

1. $V_1 \supseteq R(G)$, $V_2 \supseteq B(G)$,
2. $E_2(G)$ is an edge cutset of G ,
3. no $F \in \mathcal{F}$ is an induced subgraph of the graph $G' = (R(G) \cup B(G), E_2(G))$,
4. V_1 and V_2 are both nonempty.

We can see that $\{K_{1,2}, K_{2,1}\}$ -free decomposition is equivalent to MATCHING CUTSET, and forbidding $K_2 + K_1$ gives the notion of the module. For more examples of F -free decomposition see [39].

Theorem 6.9. *Let G be a 5-regular graph. G is a -expandable if and only if G has no $\{K_{1,4}, K_{2,1}\}$ -free decomposition.*

Proof. Let us suppose conversely that G is a -expandable, and has a $\{K_{1,4}, K_{2,1}\}$ -free decomposition with bipartition (V_1, V_2) . Since G is 5-regular and every vertex from V_1 has at most 3 neighbours in V_1 , V_1 is a defensive alliance. On the other hand, every vertex from V_2 has at most 1 neighbour in V_1 . Thus, by Proposition 5.2, V_1 is a defensive alliance that is not expandable, which gives us a contradiction. Assume, to the contrary, that G has no $\{K_{1,4}, K_{2,1}\}$ -free decomposition and is not a -expandable. Let W be a defensive alliance that is not expandable. Since W is a defensive alliance, every vertex in W has at most three neighbours in $V - W$. Furthermore, because it is not expandable, every vertex from $V - W$ has at most one neighbour in W . Thus we obtain the required contradiction, since $(W, V - W)$ is a $\{K_{1,4}, K_{2,1}\}$ -free decomposition. \square

Using similar argument we can prove the next theorem.

Theorem 6.10. *Let G be a 4-regular graph. G is a -expandable if and only if G has no $\{K_{1,3}, K_{2,1}\}$ -free decomposition.*

We can also use \mathcal{F} -free decomposition to describe γ_a -expandable graphs. We only have to formulate some additional requirements for one of the sets of the bipartition.

Theorem 6.11. *Let G be a 5-regular graph. G is γ_a -expandable if and only if G has no $\{K_{1,4}, K_{2,1}\}$ -free decomposition (V_1, V_2) such that V_2 induce a 4-regular graph.*

Proof. Suppose, to the contrary, that G is γ_a -expandable and has a $\{K_{1,4}, K_{2,1}\}$ -free decomposition (V_1, V_2) such that V_2 induce a 4-regular graph. From the fact that V_2 induce a 4-regular graph, it follows that every vertex from V_2 has exactly one neighbour in V_1 . Thus V_1 is a dominating set. Furthermore, since every vertex from V_1 has at most 3 neighbours in V_2 , V_1 is a global defensive alliance and by Proposition 5.2, V_1 is a global defensive alliance that is not expandable. Hence our first supposition is false.

Now let us assume that G has no $\{K_{1,4}, K_{2,1}\}$ -free decomposition (V_1, V_2) such that V_2 induce a 4-regular graph and is not γ_a -expandable. Let W be a global defensive alliance that is not expandable. Since W is a defensive alliance and G is 5-regular, each vertex in W has at most three neighbours in $V - W$. On the other hand, from the fact that W is a dominating set, it follows that every vertex in $V - W$ has at least one neighbour in W . If there exists a vertex $v \in (V - W)$ such that $|N(v) \cap W| \geq 2$, then $W \cup \{v\}$ is a global defensive alliance. Thus, every vertex from $V - W$ has exactly one neighbour in W , which implies that $V - W$ induce a 4-regular graph. Hence, $(W, V - W)$ is a $\{K_{1,4}, K_{2,1}\}$ -free decomposition (V_1, V_2) such that V_2 induce a 4-regular graph. This contradiction finishes the proof. \square

As previously we can formulate similar theorem for 4-regular graphs.

Theorem 6.12. *Let G be a 4-regular graph. G is γ_a -expandable if and only if G has no $\{K_{1,3}, K_{2,1}\}$ -free decomposition (V_1, V_2) such that V_2 induce a 3-regular graph.*

Many key questions remain to be investigated, in particular the following ones.

- What is the complexity of the $\{K_{1,4}, K_{2,1}\}$ -free decomposition?
- What is the complexity of the $\{K_{1,3}, K_{2,1}\}$ -free decomposition?

-
- What is the complexity of the $\{K_{1,4}, K_{2,1}\}$ -free decomposition (V_1, V_2) such that V_2 induce a 4-regular graph?
 - What is the complexity of the $\{K_{1,3}, K_{2,1}\}$ -free decomposition (V_1, V_2) such that V_2 induce a 3-regular graph?

List of Figures

2.1	A graph G with $s(G) = 1$ and $\hat{s}(G) = 4$	13
2.2	A cubic graph of order 14 with a strong security number equal to 8	19
2.3	Cubic graphs of order 8 with a strong security number equal to 6	19
3.1	Example of a tree $T \in \mathcal{T}$	25
3.2	Illustration to the proof of Theorem 3.14	26
3.3	Illustration to the proof of Theorem 3.14	27
3.4	Illustration to the proof of Theorem 3.14	27
3.5	Illustration to the proof of Theorem 3.14	28
3.6	Illustration to the proof of Theorem 3.14	29
3.7	Connector vertex, a leaf cycle and a tree branch	31
3.8	Illustration to the proof of Theorem 3.17	32
3.9	Illustration to the proof of Theorem 3.17	34
3.10	Illustration to the proof of Theorem 3.17	34
3.11	Illustration to the proof of Theorem 3.17	34
3.12	Illustration to the proof of Theorem 3.17	35
3.13	Illustration to the proof of Theorem 3.17	35
3.14	Illustration to the proof of Theorem 3.17	35
3.15	Illustration to the proof of Theorem 3.17	35
3.16	Illustration to the proof of Theorem 3.17	35
3.17	Illustration to the proof of Theorem 3.17	36
3.18	Illustration to the proof of Theorem 3.17	37
3.19	Example of a cactus tree $CT \in \mathcal{CT}$	37
4.1	Illustration to the proof of Proposition 4.3	39
4.2	A 3-flower	40
4.3	A graph $G \in \mathcal{P}$	47
5.1	A graph W which is both γ_a -expandable and γ_s -expandable, but it is neither a -expandable nor s -expandable	60
5.2	A 3-tree that is neither γ_s -expandable nor γ_a -expandable	64
5.3	An example of a trivially perfect graph that is not weakly γ_s -expandable	66
6.1	Illustration to the proof of Theorem 6.2	71
6.2	Illustration to the proof of Theorem 6.2	73
6.3	Illustration to the proof of Theorem 6.5	75

6.4	Illustration to the proof of Theorem 6.5	75
6.5	Illustration to the proof of Theorem 6.5	75
6.6	Cubic \hat{s} -expandable graphs	77

Bibliography

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