

On solutions of quadratic integral equations

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INTRODUCTION

Linear and nonlinear integral equations form an important class of problems in mathematics. There are different motivations for their study. Some equations describe mathematical models in physics, engineering or biology. There are also such equations whose interest lies in other branch of pure mathematics.

Bearing in mind both mentioned aspects we are interested on a special class of integral equations, namely on quadratic ones. In this case the unknown function is treated by some operators, then a pointwise multiplication of such operators is applied. The study of such a kind of problems was begun in early 60's due to mathematical modeling of radiative transfer (Chandrasekhar [44], Crum [49]). From the mathematical point of view they are interesting because of lack of compactness for considered operators. Thus some of the classical methods for proving existence theorems are not allowed. There is one more reason which makes this type of equations interesting. In contrast to the case of standard integral equations only continuous solutions were considered. It seems to be strange from application point of view (as will be described below) as well as it prevents a common treatment for both quadratic and non-quadratic equations.

This dissertation is devoted to study quadratic integral equations and different classes of their solutions. We concentrate on the aspect of possible discontinuity of solutions and the best possible assumptions ensuring the existence of solutions. This leads us to the Lebesgue spaces and some class of Orlicz spaces. Our approach allow, for the first time, to consider simulatneously quadratic and "classical" integral equations. We stress on strongly nonlinear problems, which leads to mentioned function spaces, but require a new method of the proof. We prove several results for such a class of equations (existence, monotonicity) on finite and infinite intervals, including functional-integral problems.

Let us begin a prototype for this theory, that is the Chandrasekhar integral equation (Chandrasekhar [44, 43])

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \psi(s)x(s) ds.$$

It describe a scattering through a homogeneous semi-infinite plane atmosphere. In particular, solutions for this equations need not to be continuous. Nevertheless, till now only such a kind of solutions was investigated. Note, that for non-homogeneous problems only approximated methods is known (Hollis and Kelley [74]). Different types of quadratic integral equations will be described later (applicable in plasma corners [40, 86], kinetic theory of gases, the theory of neutron transport, in the traffic theory or in mathematical biology). We need only to stress, that the continuity of

considered solutions does not follow from applicability of problems (on the contrary), but only from some mathematical unsolved questions (operators in functions spaces and their properties, fixed point theorems). In particular, we solve a problem from [40].

Quadratic integral equations was investigated by many authors. Initial studies by Chandrasekhar ([44] in 1947, cf. also a book [43] in 1960) form only a beginning for this theory, mainly made by astrophysicists. Then research was conducted by mathematicians. They found some interesting open questions in this theory. Let us mention some papers by Anichini, Conti [6], Cahlon, Eskin [42], Banaś, Argyros [11], Caballero, Mingarelli, Sadarangani [40, 41], Nussbaum [96], Gripenberg [72], Mullikin [94, 95], Rus [107], Shrikhandt, Joshi [113], Schillings [111] and many others.

In the first Chapter we collect all necessary definitions and theorems. We present some function spaces, linear and nonlinear operators and their properties are described. Among others, we present some new studies on a.e. monotonic functions in Lebesgue and Orlicz spaces.

Our Chapter 2 is devoted to study quadratic integral equations on unbounded intervals. We present our motivations for the study of presented equation and preceded by a historical background we investigate the quadratic integral equation on a half-line. Here we are looking for a.e. monotonic locally integrable solutions. An illustrative example completes this Chapter (and all others).

In the next Chapter 3, we study some functional integral equations of quadratic type. This aspect of the theory is not sufficiently investigated due some restrictions on functional part. Since we try to unify both quadratic and non-quadratic cases, we need to investigate functional equations. Here we study a.e. monotonic L^1 and L^p solutions for the considered problem.

In the Chapter 4 we unify our research by considering quadratic functional integral equations on a unbounded intervals. Note, that in this Chapter we do not assume, that the operators preserve monotonicity properties. A different method of the proof is then used, which allow us to locate our results among earlier ones.

The last Chapter 5 contains our main theorems and conclusions. We consider strongly nonlinear functions, which lead us to solutions in Orlicz spaces. This is well-known for classical (non-quadratic) equations, but it is completely new in the context of quadratic integral equations. We study the pointwise multiplication in Orlicz spaces and in a class of such spaces we solve considered equations. By reducing our problem to an operator equation we present an existence result for a large class of function spaces. The idea of the proof is not only to prove our theorems but also to fully cover the theory for classical equations, which was impossible in the

previous approach. We need to stress, that this allow us to prove new fixed point theorem for product of two operators.

Chapter 1

Preliminaries

1.1 Introduction

This chapter is devoted to recall some notations and known results that will be needed in the sequel. We start in section 1.2 by setting basic notations and definitions that are observed throughout the work. In section 1.3 we study some important linear and nonlinear operators of various types and the multiplication of the operators. We discuss the monotonicity of the functions in section 1.4. Section 1.5 deals with the strong and weak measure of noncompactness. We end this chapter by section 1.6 in which we introduce some important fixed point theorems.

1.2 Notation and auxiliary facts

Let \mathbb{R} be the field of real numbers, \mathbb{R}^+ be the interval $[0, \infty)$ and by $I = [a, b]$ denotes an interval subset \mathbb{R} .

Assume that $(E, \|\cdot\|)$ is an arbitrary Banach space with zero element θ . Denote by $B_r(x)$ the closed ball centered at x and with radius r . The symbol B_r stands for the ball $B(\theta, r)$. When necessary we will also indicate the space by using the notation $B_r(E)$. If X is a subset of E , then \bar{X} and $\text{conv}X$ denote the closure and convex closure of X , respectively. We denote the standard algebraic operations on sets by the symbols $k \cdot X$ and $X + Y$.

1.2.1 Lebesgue Spaces

Define $L_p = L_p(I)$, $1 \leq p < \infty$ be the space of Lebesgue integrable functions (equivalence classes of functions) on a measurable subset I of \mathbb{R} , with the norm

$$\|x\|_{L_p(I)} = \left(\int_I |x|^p dt \right)^{\frac{1}{p}}.$$

For $p = \infty$, $L_\infty(I)$ denotes the Banach space of essentially bounded functions on I with the norm

$$\|x\|_{L_\infty} = \operatorname{ess\,sup}_{t \in I} |x(t)| < \infty.$$

Recall that the essential supremum is defined as

$$\operatorname{ess\,sup}_{t \in I} |x(t)| = \inf\{a : \text{the set } \{t : |x(t)| > a\} \text{ has measure } 0\}.$$

Let $L_1(I)$ denote the space of Lebesgue integrable functions on the fixed interval $I \subset \mathbb{R}$, bounded or not.

Further, denote by $BC(\mathbb{R}^+)$ the Banach space of all real functions defined, continuous and bounded on \mathbb{R}^+ . This space is furnished with the standard norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}^+\}$.

Let us fix a nonempty and bounded subset X of $BC(\mathbb{R}^+)$ and a positive number T . For $x \in X$ and $\varepsilon \geq 0$ let us denote by $\omega^T(x, \varepsilon)$ the modulus of continuity of the function x , on the closed and bounded interval $[0, T]$ (cf. [35]) defined by

$$\omega^T(x, \varepsilon) = \sup\{|x(t_2) - x(t_1)| : t, s \in [0, T], |t_2 - t_1| \leq \varepsilon\}.$$

1.2.2 Young and N -functions

A function $M : [0, +\infty) \rightarrow [0, +\infty)$ is called a Young function if it has the form

$$M(u) = \int_0^u a(s) du \quad \text{for } u \geq 0,$$

where $a : [0, +\infty) \rightarrow [0, +\infty)$ is an increasing, left-continuous function which is neither identically zero nor identically infinite on $[0, +\infty)$. In particular, if M is finite-valued, where $\lim_{u \rightarrow 0} \frac{M(u)}{u} = 0$, $\lim_{u \rightarrow \infty} \frac{M(u)}{u} = \infty$ and $M(u) > 0$ if $x > 0$ ($M(u) = 0 \iff u = 0$), then M is called an N -function.

The functions M and N are called complementary N -functions. If

$$N(x) = \sup_{y \geq 0} (xy - M(y)).$$

Further, the N -function M satisfies the Δ_2 -condition, i.e.

$$(\Delta_2) \quad \text{there exist } \omega, t_0 \geq 0 \text{ such that for } t \geq t_0, \text{ we have } M(2t) \leq \omega M(t).$$

Let us observe, that an N -function $M(u) = \exp u^2 - 1$ satisfies this condition, while the function $M(u) = \exp |u| - |u| - 1$ does not.

An N -function M is said to satisfy Δ' -condition if there exist $K, t_0 \geq 0$ such that for $t, s \geq t_0$, we have $M(ts) \leq KM(t)M(s)$.

If the N -function M satisfies the Δ' -condition, then it also satisfies Δ_2 -condition. Typical examples: $M_1(u) = \frac{|u|^\alpha}{\alpha}$ for $\alpha > 1$, $M_2(u) = (1 + |u|) \ln(1 + |u|) - |u|$ or $M_3(u) = |u|^\alpha (|\ln |u|| + 1)$ for $\alpha > 3 + \frac{\sqrt{5}}{2}$.

The last class of N -functions, interesting for us, consists of functions which increase more rapidly than power functions.

An N -function M is said to satisfy Δ_3 -condition if there exist $K, t_0 \geq 0$ such that for $t \geq t_0$, we have $tM(t) \leq M(Kt)$.

1.2.3 Orlicz spaces

The Orlicz class, denoted by \mathcal{O}_P , consists of measurable functions $x : I \rightarrow \mathbb{R}$ for which

$$\rho(x; M) = \int_I M(x(t)) dt < \infty.$$

We shall denote by $L_M(I)$ the *Orlicz space* of all measurable functions $x : I \rightarrow \mathbb{R}$ for which

$$\|x\|_M = \inf_{\lambda > 0} \left\{ \int_I M\left(\frac{x(s)}{\lambda}\right) ds \leq 1 \right\}.$$

The N -function $M(u) = \frac{|u|^p}{p}$, $1 < p < \infty$ leads to the classical Lebesgue space $L_p(I)$ with the norm mentioned before.

Let $E_M(I)$ be the closure in $L_M(I)$ of the set of all bounded functions. Note that $E_M \subseteq L_M \subseteq \mathcal{O}_M$. The inclusion $L_M \subset L_P$ holds if, and only if, there exists positive constants u_0 and a such that $P(u) \leq aM(u)$ for $u \geq u_0$.

An important property of E_M spaces lies in the fact that this is a class of functions from L_M having absolutely continuous norms.

Moreover, we have $E_M = L_M = \mathcal{O}_M$ if M satisfies the Δ_2 -condition.

Sometimes, we will use more general concept of function spaces i.e. ideal spaces.

Definition 1.2.1. [118] A normed space $(X, \|\cdot\|)$ of (classes of) measurable functions $x : I \rightarrow U$ (U is a normed space) is called pre-ideal if for each $x \in X$ and each measurable $y : I \rightarrow U$ the relation $|y(s)| \leq |x(s)|$ (for almost all $s \in I$) implies $y \in X$ and $\|y\| \leq \|x\|$. If X is also complete, it is called an ideal space.

Ideal spaces are a very general class of normed spaces of measurable functions, which includes Lebesgue, Orlicz, Lorentz, and Marcinkiewicz spaces as well as weighted and combined forms of these spaces. Sometimes these spaces are also called Banach function spaces or (normed) Köthe spaces.

1.3 Linear and nonlinear operators.

In this section we define and discuss some properties of the nonlinear superposition operators and many integral operators that are needed throughout this dissertation such as Fredholm, Volterra and Urysohn operators in $L_p(I)$, $p \geq 1$, $L_\infty(I)$ and $L_M(I)$ spaces. We will distinguish between two different cases: when the operators take their values in Lebesgue (Orlicz) spaces $L_p(I)$ ($L_M(I)$) or in a space of essentially bounded functions $L_\infty(I)$.

1.3.1 The superposition operators.

One of the most important operator studied in nonlinear functional analysis is the so-called superposition operator [10].

Definition 1.3.1. Assume that a function $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions i.e. it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then to every function $x(t)$ being measurable on I we may assign the function

$$F_f(x)(t) = f(t, x(t)), \quad t \in I.$$

The operator F_f in such a way is called the superposition (Nemytskii) operator generated by the function f .

Furthermore, for every $f \in L_1$ and every $\phi : I \rightarrow I$ we define the superposition operator generated by the functions f and ϕ , $F_{\phi, f} : L_1(I) \rightarrow L_1(I)$ as

$$F_{\phi, f}(t) = f(t, x(\phi(t))) \quad , \quad t \in I$$

Lemma 1.3.1. ([10, Theorem 17.5]) Assume that a function $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions. Then the superposition operator F transforms measurable functions into measurable functions.

Lemma 1.3.2. ([84, Lemma 17.5] in S and [102] in L_M) *Assume that a function $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions. The superposition operator F maps a sequence of functions convergent in measure into a sequences of functions convergent in measure.*

We will be interested in the case when F acts between some Lebesgue (Orlicz) spaces.

In $L_p(I)$ we have the "automatic" continuity of the Nemytskii operator ([10, 81]):

Theorem 1.3.1. *Let f satisfies the Carathéodory conditions. The superposition operator F generated by the function f maps continuously the space $L_p(I)$ into $L_q(I)$ ($p, q \geq 1$) if and only if*

$$|f(t, x)| \leq a(t) + b \cdot |x|^{\frac{p}{q}}, \quad (1.1)$$

for all $t \in I$ and $x \in \mathbb{R}$, where $a \in L_q(I)$ and $b \geq 0$.

This theorem was proved by Krasnoselskii [81] in the case when I is a bounded interval. The generalization to the case of an unbounded interval I was given by Appell and Zabrejko [10].

Remark 1.3.1. *It should be also noted that the superposition operator F takes its values in $L_\infty(I)$ iff the generating function f is independent of x (cf. [10, Theorem 3.17]).*

Lemma 1.3.3. ([83, Theorem 17.5]) *Assume that a function $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions. Then*

$$M_2(f(s, x)) \leq a(s) + bM_1(x),$$

where $b \geq 0$ and $a \in L^1(I)$, if and only if the superposition operator F acts from $L_{M_1}(I)$ to $L_{M_2}(I)$.

In Orlicz spaces there is no automatic continuity of superposition operators like in L^p spaces, but the following lemma is useful (remember, that the Orlicz space L_M is ideal and if M satisfies Δ_2 condition it is also regular cf. [7, Theorem 1]):

Lemma 1.3.4. ([118, Theorem 5.2.1]) *Let f be a Carathéodory function, X an ideal space, and W a regular ideal space. Then the superposition operators $F : X \rightarrow W$ is continuous.*

Let us note, that in the case of functions of the form $f(t, x) = g(t)h(x)$, the superposition operator F is continuous from the space of continuous functions $C(I)$ into $L_M(I)$ even when M does not satisfies Δ_2 condition ([7]). Since $E_M(I)$ is a regular part of an Orlicz space $L_M(I)$ (cf. [119, p.72]), in the context of Orlicz spaces, we will use the following (see also Lemma 1.3.3):

Lemma 1.3.5. *Let f be a Carathéodory function. If the superposition operator F acts from $L_{M_1}(I)$ into $E_{M_2}(I)$, then it is continuous.*

The problem of boundedness of such a type of operators will be described in the proofs of our main results.

Remark 1.3.2. Let us recall, that the acting condition from Lemma 1.3.3 is not sufficient for taking $E_{M_1}(I)$ into $E_{M_2}(I)$ (cf. [10, p.95]), especially for the continuity of this operator. For the case considered when $M_1 = M_2$ we can put, for example, $f(t, x) = x$ to fulfil this requirement. But this is true also for an arbitrary Carathéodory function f when M_1 satisfies the Δ_2 -condition. For a general result of this type see [83, Th. 17.7].

Remark 1.3.3. *Let X, Y be ideal spaces. A superposition operator $F : X \rightarrow Y$ is called improving if it takes bounded subsets of X into the subsets of Y with equiabsolutely continuous norms.*

The following two theorems "Lusin and Dragoni" [56, 109], which explain the structure of measurable functions and functions satisfying Carathéodory conditions, where D^c denotes the complement of D and the symbol $meas(D)$ stands for the Lebesgue measure of the set D .

Theorem 1.3.2. *Let $m : I \rightarrow \mathbb{R}$ be a measurable function. For any $\varepsilon > 0$ there exists a closed subset D_ε of the interval I such that $meas(D_\varepsilon^c) \leq \varepsilon$ and $m|_{D_\varepsilon}$ is continuous.*

Theorem 1.3.3. *Let $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying Carathéodory conditions. Then for each $\varepsilon > 0$ there exists a closed subset D_ε of the interval I such that $meas(D_\varepsilon^c) \leq \varepsilon$ and $f|_{D_\varepsilon \times \mathbb{R}}$ is continuous.*

1.3.2 Fredholm integral operator.

Assume that $k : I \times I \rightarrow \mathbb{R}$ be measurable with respect to both variables. For an arbitrary $x \in L_p(I)$ let

$$(K_0x)(t) = \int_I k(t, s)x(s) ds, \quad t \in I. \quad (1.2)$$

This operator K is linear and is called Fredholm integral operator (cf. [84, 122]). The next theorem gives a sufficient conditions which which ensure that K maps from L_p into L_q and is continuous.

Theorem 1.3.4. [122] *Let $k : I \times I \rightarrow \mathbb{R}$ be measurable with respect to both variables. Let the linear integral operator K_0 with kernel $k(t, s)$ map L_p into L_q . Then it is continuous.*

Lemma 1.3.6. [78] Let $k : I \times I \rightarrow \mathbb{R}$ be measurable with respect to both variables. Let the linear integral operator K_0 with kernel $k(\cdot, \cdot)$ maps $L^p(I)$ into $L_\infty(I)$ i.e. either

$$\operatorname{ess\,sup}_{t \in [a, b]} \left(\int_a^b |K(t, s)|^p ds \right)^{\frac{1}{p}} < \infty$$

or

$$\left(\int_a^b (\operatorname{ess\,sup}_{s \in [a, b]} |K(t, s)|)^p dt \right)^{\frac{1}{p}} < \infty.$$

Then it is continuous.

The necessary results concerning the properties of such a kind of operators in Orlicz spaces can be found in [83], let we mention Zaanen's theorem [83] which shows that the operator (1.2) acts between Orlicz spaces.

Let, the N -functions M_1 and M_2 are the complementary functions to the N -functions N_1 and N_2 respectively.

Lemma 1.3.7. Suppose the kernel $k(x, y)$ satisfies either one of the following two conditions:

- (a) for almost all $t \in I$ the kernel $k(t, s)$, as a function of s , belongs to the space L_{N_1} , where the function $\varphi(t) = \|k(t, s)\|_{N_1}$ belongs to the space L_{M_2} ,
- (b) for almost all $s \in I$ the kernel $k(t, s)$, as a function of t , belongs to the space L_{M_2} , where the function $\Psi(s) = \|k(t, s)\|_{M_2}$ belongs to the space L_{N_1} .

Then the operator (1.2) maps L_{M_1} into L_{M_2} and is continuous.

1.3.3 Volterra integral operator.

Suppose $k : \Delta \rightarrow \mathbb{R}$ is a given function and measurable with respect to both variables where $\Delta = \{(t, s) : 0 \leq s \leq t \leq \infty\}$. For an arbitrary function $x \in L_1(\mathbb{R}^+)$ define

$$(Vx)(t) = \int_0^t k(t, s)x(s) ds, \quad t \geq 0.$$

The linear integral operator defined above is the well known linear Volterra integral operator (cf. [84, 122]).

When we consider this operator on the space $L_p([a, b])$, then it is a special case of the Fredholm operator investigated in previous section, where

$$(Vx)(t) = \int_a^b \chi_{[0, t]} k(t, s)x(s) ds, \quad t \geq 0.$$

1.3.4 Urysohn integral operator.

The most important nonlinear integral operators are the Urysohn operators [122]:

$$U(x)(t) = \int_I u(t, s, x(s)) ds. \quad (1.3)$$

Here, the kernel $u : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all $(t, s) \in I \times I$. Moreover, for arbitrary fixed $s \in I$ and $x \in \mathbb{R}$ the function $t \rightarrow u(t, s, x(s))$ is integrable.

A particular case of a Urysohn operator (1.3) is the Hammerstein integral operator $H = K_0 \circ F$:

$$H(x)(t) = \int_I k(t, s) f(s, x(s)) ds. \quad (1.4)$$

Note, that for Urysohn operators the continuity is not "automatic" as in the case of superposition operators (for Nemytskii operators see Theorem 1.3.1).

Let us recall an important sufficient condition:

Theorem 1.3.5. [84, Theorem 10.1.10] *Let $u : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all $(t, s) \in I \times I$. Assume that $U(x)(t) = \int_I u(t, s, x(s)) ds$ maps $L_p(I)$ into $L_q(I)$ ($q < \infty$) and for each $h > 0$ the function*

$$R_h(t, s) = \max_{|x| \leq h} |u(t, s, x)|$$

is integrable on s for a.e. $t \in I$. If moreover for each $h > 0$ this operator satisfies

$$\lim_{meas(D) \rightarrow 0} \sup_{|x| \leq h} \left\| \int_D u(t, s, x(s)) ds \right\|_{L_q(I)} = 0$$

and for arbitrary non-negative $z(t) \in L_p(I)$

$$\lim_{D \rightarrow 0} \sup_{|x| \leq z} \left\| \int_D u(t, s, x(s)) ds \right\|_{L_q(I)} = 0,$$

then U is a continuous operator.

The first two conditions are satisfied when $\int_I R_h(t, s) ds \in L_q(I)$, for instance.

We will use also the majorant principle for Urysohn operators (cf. [84, Theorem 10.1.11]). The following theorem which is a particular case of much more general result ([84, Theorem 10.1.16]), will be very useful in the proof of the main result for operators in $L_\infty(I)$:

Theorem 1.3.6. [84] Let $u : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t, s) . Assume that

$$|u(t, s, x)| \leq k(t, s) \cdot (a(s) + b \cdot |x|),$$

where the nonnegative function k is measurable in (t, s) , a is a positive integrable function, $b > 0$ and such that the linear integral operator with the kernel $k(t, s)$ maps $L_1(I)$ into $L_\infty(I)$. Then the operator U maps $L_1(I)$ into $L_\infty(I)$. Moreover, if for arbitrary $h > 0$

$$\lim_{\delta \rightarrow 0} \left\| \int_D \max_{|x_1| \leq h, |x_1 - x_2| \leq \delta} |u(t, s, x_1) - u(t, s, x_2)| ds \right\|_{L_\infty(I)} = 0,$$

then U is a continuous operator.

We mention also that some particular conditions guaranteeing the continuity of the operator U may be found in [116, 122].

1.3.5 The multiplication operator.

We need to describe the multiplication operator which is the key point of our work. We will denote the pointwise multiplication operator by $A(x)(t)$ of the form:

$$A(x)(t) = F(x)(t) \cdot U(x)(t),$$

where $U(x)$ is a Urysohn integral operator (1.3), in some chapters replaced by the the Hammerstein integral operator $H = K_0 F$, where K_0 is the linear integral operator and F as in definition 1.3.1.

Generally speaking, the product of two functions $x, y \in L_p(I)[L_M(I)]$ is not in $L_p(I) [L_M(I)]$. However, if x and y belongs to some particular Lebesgue (Orlicz) spaces, then the product $x \cdot y$ belong to a third Lebesgue (Orlicz) space. Let us note, that one can find two functions belonging to Lebesgue (Orlicz) spaces: $u \in L_p (L_U)$ and $v \in L_p (L_V)$ such that the product uv does not belong to any Lebesgue (Orlicz) space (this product is not integrable).

We will use the technique of factorization for some operators acting on Lebesgue (Orlicz) spaces through another Lebesgue (Orlicz) spaces. We can mention, that by using in this place different ideal spaces it is possible to obtain some extensions of our results and then we try to facilitate this approach. To stress the connection of our results with the growth condition we restrict ourselves to the case of Lebesgue (Orlicz) spaces.

Remark 1.3.4. For the so-called pre-ideal spaces (cf. [118]) if $x \in E$ and $y \in L_\infty$ implies that $xy \in E$ and $\|xy\|_E \leq \|x\|_E \|y\|_{L_\infty}$ i.e. the elements from L_∞ are pointwise multipliers for E . For more details in L_1 space see [46].

Nevertheless, we have:

Lemma 1.3.8. ([83, Lemma 13.5]), [92, Theorem 10.2] Let φ_1, φ_2 and φ are arbitrary N -functions. The following conditions are equivalent:

1. For every functions $u \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}$, $u \cdot w \in L_\varphi(I)$.
2. There exists a constant $k > 0$ such that for all measurable u, w on I we have $\|uw\|_\varphi \leq k \|u\|_{\varphi_1} \|w\|_{\varphi_2}$.
3. There exists numbers $C > 0$, $u_0 \geq 0$ such that for all $s, t \geq u_0$, $\varphi\left(\frac{st}{C}\right) \leq \varphi_1(s) + \varphi_2(t)$.
4. $\limsup_{t \rightarrow \infty} \frac{\varphi_1^{-1}(t)\varphi_2^{-1}(t)}{\varphi(t)} < \infty$.

Let us recall the following simple sufficient condition for the above statements hold true.

Lemma 1.3.9. ([83, p. 223]) If there exist complementary N -functions Q_1 and Q_2 such that the inequalities

$$Q_1(\alpha u) < \varphi^{-1}[\varphi_1(u)]$$

$$Q_2(\alpha u) < \varphi^{-1}[\varphi_2(u)]$$

hold, then for every functions $u \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}$, $u \cdot w \in L_\varphi(I)$. If moreover φ satisfies the Δ_2 -condition, then it is sufficient that the inequalities

$$Q_1(\alpha u) < \varphi_1[\varphi^{-1}(u)]$$

$$Q_2(\alpha u) < \varphi_2[\varphi^{-1}(u)]$$

hold.

An interesting discussion about necessary and sufficient conditions for product operators can be found in [83, 92].

Remark 1.3.5. An ideal space E is called regular if for every $x \in E$ we have $\lim_{m \in sD \rightarrow 0} \|x \cdot \chi_D\|_E = 0$. A set of all elements x with this property is called a regular part of E . Thus this a set of all $x \in E$ with absolutely continuous norm. A space is called perfect if the Fatou lemma holds for E .

1.4 Monotone functions.

Let $S = S(I)$ denote the set of measurable (in Lebesgue sense) functions on I and let $meas$ stand for the Lebesgue measure in \mathbb{R} . Identifying the functions equals almost everywhere the set S furnished with the metric

$$d(x, y) = \inf_{a>0} [a + meas\{s : |x(s) - y(s)| \geq a\}]$$

becomes a complete space. Moreover, the space S with the topology convergence in measure on I is a metric space, because the convergence in measure is equivalent to convergence with respect to d (cf. Proposition 2.14 in [119]).

For σ -finite subsets of \mathbb{R} we say that the sequence x_n is convergent in finite measure to x if it is convergent in measure on each set T of finite measure.

The compactness in such spaces we will call a "compactness in measure" and such sets have important properties when considered as subsets of some Orlicz spaces (ideal spaces). Let us recall, in metric spaces the set U_0 is compact if and only if each sequence from U_0 has a subsequence that converges in U_0 (i.e. sequentially compact).

In this dissertation, we need to investigate some properties of sets and operators in such a class of spaces instead of the space S . Some of them are obvious, the rest will be proved.

We are interested in finding of almost everywhere monotonic solutions for our problems. We will need to specify this notion in considered solution spaces.

Let X be a bounded subset of measurable functions. Assume that there is a family of subsets $(\Omega_c)_{0 \leq c \leq b-a}$ of the interval I such that $meas(\Omega_c) = c$ for every $c \in [0, b-a]$, and for every $x \in X$, $x(t_1) \geq x(t_2)$, $(t_1 \in \Omega_c, t_2 \notin \Omega_c)$.

It is clear, that by putting $\Omega_c = [0, c) \cup Z$ or $\Omega_c = [0, c) \setminus Z$, where Z is a set with measure zero, this family contains nonincreasing functions (possibly except for a set Z). We will call the functions from this family "a.e. nonincreasing" functions. This is the case, when we choose a measurable and nonincreasing *function* y and all functions equal a.e. to y satisfies the above condition. This means that such a notion can be also considered in the space S . Thus we can write, that *elements* from $L_1(I)$, $L_M(I)$ belong to this class of functions. Further, let Q_r stand for the subset of the ball B_r consisting of all functions which are a.e. nonincreasing on I . Functions a.e. nondecreasing are defined by similar way.

It is known, that such a family constitute a set which is compact in measure in S . We are interested, if the set is still compact in measure as a subset of subspaces of S . In general, it is not true, but for the case of Lebesgue spaces $L_1(I)$, $L_p(I)$, $p > 1$ and Orlicz spaces $L_M(I)$, we have the following:

Due to the compactness criterion in the space of measurable functions (with the topology of the convergence in measure) (see Lemma 4.1 in [18]) we have a desired theorem concerning the compactness in measure of a subset X of $L_1(I)$ (cf. Corollary 4.1 in [18] or Section III.2 in [60]).

Theorem 1.4.1. *Let X be a bounded subset of $L_1(I)$ consisting of functions which are a.e. nonincreasing (or a.e. nondecreasing) on the interval I . Then X is compact in measure in $L_1(I)$.*

In the following theorems, denote by E the spaces $L_p(I), p \geq 1$ or $L_M(I)$ (cf. [47, 48]).

We have a new characterization of compactness in measure for subspaces of S .

Lemma 1.4.1. *Let X be a bounded subset of E consisting of functions which are a.e. nondecreasing (or a.e. nonincreasing) on the interval I . Then X is compact in measure in E .*

Proof. Let $r > 0$ be such that $X \subset B_r \subset E$. It is known (cf. [84, 18]), that X is compact in measure as a subset of S . By taking an arbitrary sequence (x_n) in X we obtain that there exists a subsequence (x_{n_k}) convergent in measure to some $x \in S$. Since Orlicz spaces are perfect (cf. [118]), the balls in E are closed in the topology of convergence in measure. Thus $x \in B_r \subset E$ and then $x \in X$. \square

Remark 1.4.1. The above lemma remains true for subsets of arbitrary perfect ideal spaces ([118]).

If we consider the set of indices $c \geq 0$ in the definition of the family of a.e. nonincreasing functions, we are able to extend this result for the space $L_1(\mathbb{R}^+)$. For simplicity, we will denote such a space by L_1 . Due to some results of V ath we are able to extend the desired result from the interval $I = [a, b]$ into the σ -finite subsets of \mathbb{R} and the topology of the convergence in finite measure.

Theorem 1.4.2. *Let X be a bounded subset of $L_1(\mathbb{R}^+)$ consisting of functions which are a.e. nonincreasing (or a.e. nondecreasing) on the half-line \mathbb{R}^+ . Then X is compact in finite measure in $L_1(\mathbb{R}^+)$.*

Proof. If we consider the space $L_1(T)$ for σ -finite measure space T , then there is some equivalent finite measure ν ($\nu(\mathbb{R}^+) < \infty$) (Proposition 2.1. in [119] or Corollary 2.20 in [119]). Then the convergence of sequences in S are the same for the metric d and for

$$d_\nu(x, y) = \inf_{a>0} [a + \nu\{s : |x(s) - y(s)| \geq a\}]$$

(Proposition 2.2 in [117]). Take an arbitrary bounded sequence $(x_n) \subset X$. As a subset of a metric space $X = (L_1(\mathbb{R}^+), d_\nu)$ the sequence is compact in this metric

space (Theorem 1.4.1). Then there exists a subsequence (x_{n_k}) of (x_n) which is convergent in the space X to some x i.e.

$$d_\nu(x_{n_k}, x) \xrightarrow{k \rightarrow \infty} 0.$$

As claimed above the two metrics have the same convergent sequences, then

$$d(x_{n_k}, x) \xrightarrow{k \rightarrow \infty} 0.$$

This means that X is compact in $L_1(\mathbb{R}^+)$. □

We have also an important

Lemma 1.4.2. (Lemma 4.2 in [15]) *Suppose the function $t \rightarrow f(t, x)$ is a.e. non-decreasing on a finite interval I for each $x \in \mathbb{R}$ and the function $x \rightarrow f(t, x)$ is a.e. nondecreasing on \mathbb{R} for any $t \in I$. Then the superposition operator F generated by f transforms functions being a.e. nondecreasing on I into functions having the same property.*

We will use the fact, that the superposition operator takes the bounded sets compact in measure into the sets with the same property.

Thus we can prove the following (cf. [46, Proposition 4.1]):

Proposition 1.4.1. *Assume that a function $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions and the function $t \rightarrow f(t, x)$ is a.e. nondecreasing on a finite interval I for each $x \in \mathbb{R}$ and the function $x \rightarrow f(t, x)$ is a.e. nondecreasing on \mathbb{R} for any $t \in I$. Assume, that $F : L_M(I) \rightarrow E_M(I)$. Then $F(V)$ is compact in measure for arbitrary bounded and compact in measure subset V of $L_M(I)$.*

Proof. Let V be a bounded and compact in measure subset of $L_M(I)$. By our assumption $F(V) \subset E_M(I)$. As a subset of S the set $F(V)$ is compact in measure (cf. [18]). Since the topology of convergence in measure is metrizable, the compactness of the set is equivalent with the sequential compactness. By taking an arbitrary sequence $(y_n) \subset F(V)$ we get a sequence (x_n) in V such that $y_n = F(x_n)$. Since $(x_n) \subset V$, as follows from Lemma 1.3.2 F transforms this sequence into the sequence convergent in measure. Thus (y_n) is compact in measure, so is $F(V)$. □

For the integral operator (1.2), we have the following theorem due to Krzyż ([85, Theorem 6.2]):

Theorem 1.4.3. *The operator K_0 preserve the monotonicity of functions iff*

$$\int_0^b k(t_1, s) ds \geq \int_0^b k(t_2, s) ds$$

for $t_1 < t_2$, $t_1, t_2 \in I$ and for any $b \in I$.

1.5 Measures of noncompactness.

Now we present the concept of a regular measure of noncompactness (or of weak noncompactness), we denote by \mathcal{M}_E the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E, \mathcal{N}_E^W$ its subfamily consisting of all relatively compact and weakly relatively compact sets, respectively. The symbol \bar{X}^W stands for the weak closure of a set X while \bar{X} denotes its closure.

Definition 1.5.1. [23] *A mapping $\mu : \mathcal{M}_E \rightarrow [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:*

- (1) *the family $\ker\mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathcal{N}_E$, where $\ker\mu$ is called the kernel of the measure μ .*
- (2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
- (3) $\mu(\text{conv}X) = \mu(X)$
- (4) $\mu[\lambda X + (1 - \lambda)Y] \leq \lambda \mu(X) + (1 - \lambda) \mu(Y), \lambda \in [0, 1]$.
- (5) *If $X_n \in \mathcal{M}_E, X_n = \bar{X}_n$ and $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and if*

$$\lim_{n \rightarrow \infty} \mu(X_n) = 0, \text{ then } X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \phi.$$

Definition 1.5.2. [29] *A mapping $\gamma : \mathcal{M}_E \rightarrow [0, \infty)$ is said to be a measure of weak noncompactness in E if it satisfies conditions (2)-(4) of definition 1.5.1 and the following two conditions (being counterparts of (1) and (5)) hold:*

- (1') *the family $\ker\gamma = \{X \in \mathcal{M}_E : \gamma(X) = 0\}$ is nonempty and $\ker\gamma \subset \mathcal{N}_E^W$, where $\ker\gamma$ is called the kernel of the measure γ .*
- (5') *If $X_n \in \mathcal{M}_E, X_n = \bar{X}_n^W$ and $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and if*

$$\lim_{n \rightarrow \infty} \gamma(X_n) = 0, \text{ then } X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \phi.$$

In addition the measure of noncompactness μ (or of weak noncompactness γ) is called

- *Measure with maximum property if $\mu(X \cup Y) = \max[\mu(X), \mu(Y)]$.*
- *Homogeneous measure if $\mu(\lambda X) = |\lambda| \mu(X), \lambda \in R$.*
- *Subadditive measure if $\mu(X + Y) \leq \mu(X) + \mu(Y)$.*
- *Sublinear measure if it is homogeneous and subadditive.*
- *Complete (or full) if $\ker \mu = \mathcal{N}_E$ ($\ker \gamma = \mathcal{N}_E^W$).*
- *Regular measure if it is full, sublinear and has a maximum property.*

An classical example of measure of noncompactness is the following:

Definition 1.5.3. [23] *Let X be a nonempty and bounded subset of E . The Hausdorff measure of noncompactness $\beta_H(X)$ is defined as*

$$\beta_H(X) = \inf\{\epsilon > 0 : X \text{ can be covered with a finite number of balls of a radii less than } \epsilon\}$$

It is worthwhile to mention that the first important example of measure of weak noncompactness has been defined by De Blasi [52] by:

$$\beta(X) = \inf\{r > 0 : \text{there exists a weakly compact subset } W \text{ of } E \text{ such that } x \subset W + B_r\}.$$

Both the Hausdorff measure β_H and the De Blasi measure β are regular in the sense of the above definitions.

Another regular measure of noncompactness was defined in the space $L_1(I)$ (cf. [28]). For any $\epsilon > 0$, let c be a measure of equiintegrability of the set X (the so-called Sadvskii functional [10, p. 39]) i.e.

$$c(X) = \limsup_{\epsilon \rightarrow 0} \left\{ \sup_{x \in X} \left[\sup_D \int_D |x(t)| dt, D \subset I, meas(D) \leq \epsilon \right] \right\}.$$

Restricted to the family compact in measure subsets of this space it forms a regular measure of noncompactness (cf. [66]).

However, by considering this measure of noncompactness instead of usually considered ones based on Kolomogorov or Riesz criteria of compactness (cf. [23]) we are able to examine by the same manner the case of $L_p(I)$ spaces, where χ_D denotes the characteristic function of D .

Let us also denote by c a measure of equiintegrability of the set X in an Orlicz space $L_M(I)$ (cf. Definition 3.9 in [119] or [67, 66]):

$$c(X) = \limsup_{\epsilon \rightarrow 0} \sup_{meas D \leq \epsilon} \sup_{x \in X} \|x \cdot \chi_D\|_{L_M(I)},$$

where χ_D denotes the characteristic function of D .

Then we have the following theorem, which clarify the connections between different coefficients in Orlicz spaces. Since Orlicz spaces $L_M(I)$ are regular, when M satisfies Δ_2 condition, then Theorem 1 in [66] read as follows:

Proposition 1.5.1. *Let X be a nonempty, bounded and compact in measure subset of an ideal regular space Y . Then*

$$\beta_H(X) = c(X).$$

As a consequence, we obtain that bounded sets which are additionally compact in measure are compact in $L_M(I)$ iff they are equiintegrable in this space (i.e. have equiabsolutely continuous norms cf. [5]).

The contraction of the measure of weak non compactness is a bit more complicated when I is an unbounded interval. Let us, we recall the following criterion for weak noncompactness due to Dieudonné [55, 60], which is of fundamental importance in our subsequent analysis.

Theorem 1.5.1. *A bounded set X is relatively weakly compact in $L_1(\mathbb{R}^+)$ if and only if the following two conditions are satisfied:*

- (a) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\text{meas}(D) < \delta$ then $\int_D |x(t)| dt \leq \varepsilon$ for all $x \in X$,*
- (b) *for any $\varepsilon > 0$ there is $T > 0$ such that $\int_T^\infty |x(t)| dt \leq \varepsilon$ for any $x \in X$.*

Now, for a nonempty and bounded subset X of the space $L_1(\mathbb{R}^+)$ let us define:

$$c(X) = \lim_{\varepsilon \rightarrow 0} \{ \sup_{x \in X} \{ \sup [\int_D |x(t)| dt, D \subset \mathbb{R}^+, \text{meas}(D) \leq \varepsilon] \} \}, \quad (1.5)$$

and

$$d(X) = \lim_{T \rightarrow \infty} \{ \sup [\int_T^\infty |x(t)| dt : x \in X] \}. \quad (1.6)$$

Put

$$\gamma(X) = c(X) + d(X). \quad (1.7)$$

Then we have the following theorem, which clarify the connections between these two measures $\beta_H(x)$ and $\gamma(x)$ ([22]).

Theorem 1.5.2. *Let X be a nonempty, bounded and compact in measure subset of $L^1(\mathbb{R}^+)$. Then*

$$\beta_H(x) \leq \gamma(x) \leq 2\beta_H(x).$$

1.6 Fixed point theorems.

Fixed point theorems have always a major role in various fields, specially, in fields of differential, integral and functional equations. Fixed point theorems constitute a topological tool for the qualitative investigations of solution of linear and nonlinear equations. The theory of fixed points is concerned with the conditions which guarantee that a map $T : X \rightarrow X$ of a topological space X into it self admits one or more fixed points, that is, points x of X for which $x = Tx$.

Here we give a brief history of fixed point theorems.

The following definition states some types of mapping in a metric space (X, ρ) [71].

Definition 1.6.1. Let (X, ρ) be a metric space. The mapping $T : X \rightarrow X$ is called Lipschitz map, if there exist a number $\gamma \geq 0$, such that

$$\rho(Tx, Ty) \leq \gamma \rho(x, y), \forall x, y \in X.$$

The mapping T is called contraction if $\gamma < 1$, and is called non expensive, if $\gamma \leq 1$. Furthermore, T is contractive if

$$\rho(Tx, Ty) < \gamma \rho(x, y), \forall x \neq y.$$

Problems concerning the existence of fixed point for Lipschitz map have been of considerable interest in non linear operator theory. In 1922, the so-called Banach contraction mapping principle was given to obtain solutions for several problems.

Theorem 1.6.1. (Banach contraction mapping principle, [71])

Let X be a complete metric space and let $T : X \rightarrow X$ be a contraction map. Then T has a unique fixed point in X . Moreover, for any $x_0 \in X$, the sequence $\{T^n(x_0)\}_{n=0}^{\infty}$ converges to the fixed point.

This theorem is perhaps the most useful fixed point theorem, which is involved in many of the existence and uniqueness proofs in ordinary differential equations. The mapping T is the Banach contraction mapping principle still has a unique fixed point in any closed subset M of X . There are some conditions for a continuous mapping T in X , that guarantee the existence of a unique fixed point, such as the contraction of T^n or if there exist a function $\phi : X \rightarrow R^+$, such that for all $x \in X$, $\rho(Tx, Ty) = \phi(x) \phi(Tx)$.

In a normed space, the next fixed point theorem, is concerned with continuous mapping and has an advantage over Banach Contraction Mapping Principle in that is applied to a large class of functions.

Theorem 1.6.2. (Brouwer [71])

Let Q be a nonempty, convex, closed and bounded subset of a finite dimensional Banach space E_n and let $T : Q \rightarrow Q$ be continuous. Then T has at least one fixed point in the set Q .

A generalization of Brouwer's result to any Banach space was due to Schauder.

Theorem 1.6.3. (Schauder, [71])

Let Q be a convex subset of a Banach space X , and $T : Q \rightarrow Q$ is compact, continuous map. Then T has at least one fixed point in Q .

Next, we need the following definition.

Definition 1.6.2. [71] A mapping $H : E \rightarrow E$ is called completely continuous if H is continuous and $H(Y)$ is relatively compact for every bounded subset of Y .

Theorem 1.6.4. (Schauder-Tychonoff, [71])

Let C be a nonempty, convex, closed and bounded subset of a Banach space E . Let $H : C \rightarrow C$ be a completely continuous mapping. Then H has at least one fixed point in C .

When the concept of measure of noncompactness appeared, some fixed point theorems based on such measure were given. Among these is the Darbo fixed point theorem. Such theorem is used for a contraction mapping with respect to the Hausdorff measure of non compactness, that is, there exist a constant $\alpha \in (0, 1)$, such that $\chi(HX) \leq \alpha \chi(X)$, for any nonempty bounded subset X of G .

An importance of such a kind of functions can be clarified by using the contraction property with respect to this measure instead of compactness in the Schauder fixed point theorem. Namely, we have a theorem ([23]).

Theorem 1.6.5. (Darbo, [50])

Let Q be a nonempty, bounded, closed and convex subset of E and let $V : Q \rightarrow Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ , i.e. there exists $k \in [0, 1)$ such that

$$\mu(V(X)) \leq k\mu(X),$$

for any nonempty subset X of E . Then V has at least one fixed point in the set Q and the set $FixV$ of all fixed points of V satisfies $\mu(FixV) = 0$.

Emmanuele gives the corresponding version of Darbo fixed point theorem in the weak sense.

Theorem 1.6.6. (Emmanuele, [65])

Let Q be a nonempty, closed, convex and bounded subset of a Banach space E , Assume that $F : Q \rightarrow Q$ be a weakly continuous operator having the property that, there is a constant $\alpha \in (0, 1)$, such that $\beta(F(X)) \leq \alpha \beta(X)$, for any nonempty subset X of Q , where $\beta(X)$ is the measure of noncompactness. Then F has at least one fixed point in the set Q .

Theorem 1.6.7. [82]

Let M be a nonempty, closed, and convex subset of E . Suppose, that A, B be two operators such that

i) $A(M) + B(M) \subseteq M,$

ii) A is a contraction mapping,

iii) $B(M)$ is relatively compact and B is continuous. Then there exists a $y \in M$ with $Ay + By = y$.

Next we state a nonlinear alternative of Leray-Schauder type fixed point theorem (cf. [51]).

Theorem 1.6.8. (the Leray-Schauder alternative)

Let C be an open subset of a convex set Q in a Banach space E . Assume $0 \in C$ and the map $T : \bar{C} \rightarrow Q$ is continuous and compact. Then either

(i) T has a fixed point in \bar{C} , or

(ii) there exist $\lambda \in (0, 1)$ and $u \in \partial C$ such that $u = \lambda Tu$, where ∂C is a boundary of U .

The relative compactness for a subset in $L_p(0, 1)$ can be proved by a several methods, among these, Kolmogorov compactness criterion stated in the following theorem [59].

Theorem 1.6.9. (the Kolmogorov compactness criterion)

Let $\Omega \subseteq L^p(0, 1)$, $1 \leq p < \infty$. If

(i) Ω is bounded in $L^p(0, 1)$,

(ii) $x_h \rightarrow x$ as $h \rightarrow 0$ uniformly with respect to $x \in \Omega$, then Ω is relatively compact in $L^p(0, 1)$, where

$$x_h(t) = \frac{1}{h} \int_t^{t+h} x(s) ds.$$

Theorem 1.6.10. (the Arzela-Ascoli theorem , [80])

Let E be a compact metric space and $C(E)$ be the Banach space of real or complex valued continuous functions normed by

$$\| f \| = \sup_{t \in E} | f(t) |.$$

If $A = \{f_n\}$ is a sequence in $C(E)$ such that f_n is uniformly bounded and equicontinuous, then \bar{A} is compact.

Theorem 1.6.11. (the Lebesgue dominated convergence theorem, [80])

Let $\{f_n\}$ be a sequence of functions converging to a limit f on A , and suppose that

$$| f_n(t) | \leq \phi(t), \quad t \in A, \quad n = 1, 2, \dots,$$

where ϕ is integrable on A . Then f is integrable on A and

$$\lim_{n \rightarrow \infty} \int_A f_n(t) d\mu = \int_A f(t) d\mu.$$

Chapter 2

Monotonic integrable solutions for quadratic integral equations on a half line.

2.1 Motivations and historical background.

This Thesis is devoted to study so-called quadratic integral equations. This is a kind of problems of the form

$$x(t) = g(t) + F(x)(t) \cdot \int_{\alpha}^{\beta} u(t, s, x(s)) ds,$$

where $t \in I \subset \mathbb{R}_+$ and F is an operator. Some generalizations for the presented equations are also considered. Such a kind of problems is of mathematical and practical interests and has a long history. A classical theory of Urysohn integral equations does not include the above problem. Since for equations of this type an approach via the Schauder fixed point theorem is not useful and the Banach contraction principle is too restrictive in many applications, we need to investigate such equations very carefully.

The first considered equation of this type is the Chandrasekhar equation

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \varphi(s)x(s) ds.$$

It is an important example, because it show some of our motivations. This equation describe a radiative transfer through a homogeneous stellar atmosphere. It was investigated by many authors. The solutions was considered only in the space $C(I)$ or in Banach algebras (cf. [34]). However, such a class of solutions seems to be inadequate for integral problems and leads to several restrictions on functions. In order to apply earlier results we have to impose an additional condition that the so-called

"characteristic" function ψ is a polynomial (as in the book of Chandrasekhar [43, Chapter 5]) or at least continuous (cf. [40, Theorem 3.2]). This function is immediately related to the angular pattern for single scattering and then our results allow to consider some peculiar states of the atmosphere. In astrophysical applications of the Chandrasekhar equation the only restriction, that $\int_0^1 \psi(s) ds \leq 1/2$ is treated as necessary (cf. [40, Chapter VIII; Corollary 2 p. 187] or [69]). The continuity assumption for ψ implies the continuity of solutions for the considered equation (cf. [40]) and then seems to be too restrictive even from the theoretical point of view. About nonhomogeneous (discontinuous "characteristic" ψ in the Chandrasekhar equation) stellar atmosphere: it is only a discretization for the equation (Hollis and Kelley 1986 [74]) - till now there is no analytical methods (unless our results). An interesting discussion about the continuity of solutions for the Chandrasekhar equation and the relation between the kernel of an integral operator can be found in [114, Proposition 4.1, Theorem 4.3] - cf. also [69].

More general problem (motivated by some practical interests in plasma physics (cf. Stuart [114]) was investigated in [86]

$$x^2(t) = t^2 - \frac{J}{4\pi} \int_0^1 K(t, s, x(t), x(s)) ds.$$

Let us list some of considered previously particular cases of quadratic integral equations with their applications:

a) biology: model of spread of a disease (epidemic model) (Gripenberg [72])

$$x(t) = k \left(P - \int_{-\infty}^t A(t-s)x(s)ds \right) \cdot \int_{-\infty}^t a(t-s)s(s)ds,$$

b) physics: kinetic theory of gases (Hu, Khavanin and Zhuang [75])

$$x(t) = a(t) + \left[f(t, x(t)) + \int_0^\infty g(t, s)x(s)ds \right] \cdot \int_0^\infty h(t, s)K(s, x(s))ds,$$

c) physics: statistical mechanics, the Percus-Yevick equation (Nussbaum [96], Wertheim [121], Pimbley [101], Ramalho [104], Rus [107])

$$x(t) = 1 + \lambda \int_t^1 x(s) \cdot x(s-t)ds,$$

c) the Chandrasekhar equation: in astrophysics (Chandrasekhar, Fox, Argyros, Crum, Cahlon, Rus, Shrikhandt, Joshi, Schillings, Leonard and Mullikin [94, 95, 88], Stuart [114] and many others):

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \psi(s)x(s) ds.$$

It is worthwhile to mention, that our equation cover as special cases among others the following ones:

1. $f_1(t, x) = g(t)$, $f_2(t, x) = \lambda$ the functional Urysohn integral equation ([14, 15, 24]),
2. $f_1(t, x) = g(t)$, $f_2(t, x) = x$, $\phi_2(t) = t$ the functional-integral equation ([91]),
3. $f_2(t, x) = 0$ the abstract functional equation ([15], for instance),
4. for continuous solutions with $\phi_1(t) = \phi_2(t) = t$ and $u(t, s, x) = \frac{u_1(t, s, x)}{\Gamma(\alpha) \cdot (t-s)^{1-\alpha}}$ see [35, 61],
5. $f_2(t, x) = \lambda$ the functional integral equation (for continuous solutions see [3, 21, 54]),
6. $f_2(t, x) = x$ the quadratic (functional) Urysohn integral equation ([27, 26], for instance).

Note, that the choice of spaces allow us to consider less restrictive growth conditions, which will be clarified in next chapters.

2.2 Introduction.

In this chapter we study the following functional integral equation

$$x(t) = g(t) + f\left(t, x(t) \cdot \int_{\alpha}^{\beta} u(t, s, x(s)) ds\right). \quad (2.1)$$

The particular cases of our equation, were investigated for existence for both continuous (cf. [6, 32, 35, 62] and integrable solutions ([20, 30, 31]). The existence of different subclasses of solutions were proved (nonnegative functions, monotone, having limit at infinity etc.).

Let us note, that the problem is investigated for finite or infinite intervals. We extend the existing results dealing the monotonicity problem in a half-line for the most complicated problem of the Urysohn operators. For continuous solutions such a property was recently investigated in [62], for instance.

By applying Darbo fixed point theorem associated with the measure of noncompactness, we obtain the sufficient conditions for the existence of monotonic solutions of equation (2.1), which are integrable. The results presented in this chapter are motivated by the recent works of Banaś and Chlebowicz [20], Banaś and Rzepka [32, 33] and extend these papers in many ways.

2.3 Main result

Denote by L^1 for $L^1(\mathbb{R}^+)$ and H the operator associated with the right hand side of equation (2.1) which takes the form

$$x = Hx,$$

where

$$(Hx)(t) = g(t) + f\left(t, x(t) \cdot \int_{\alpha}^{\beta} u(t, s, x(s)) ds\right), \quad t \geq 0.$$

The operator H will be written as the product $Hx(t) = g(t) + FKx(t)$ where

$$F(x)(t) = f(t, x(t)), \quad Kx(t) = x(t) \cdot U(x)(t)$$

and $U(x)$ is the Urysohn integral operator of the form

$$(Ux)(t) = \int_{\alpha}^{\beta} u(t, s, x(s)) ds.$$

Thus equation (2.1) becomes

$$x(t) = g(t) + FKx(t). \quad (2.2)$$

We shall treat the equation (2.1) under the following assumptions which are listed below.

- (i) $g \in L^1(\mathbb{R}^+)$ and is a.e. nonincreasing on \mathbb{R}^+ .
- (ii) $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions and there are a positive function $a \in L^1$ and a constant $b \geq 0$ such that

$$|f(t, x)| \leq a(t) + b|x|,$$

for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Moreover, $f(t, x) \geq 0$ for $x \geq 0$ and f is assumed to be nonincreasing with respect to both variable t and x separately.

- (iii) $u : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t, s) . The function u is nonincreasing with respect to each variable, separately. Moreover, for arbitrary fixed $s \in \mathbb{R}^+$ and $x \in \mathbb{R}$ the function $t \rightarrow u(t, s, x(s))$ is integrable.

- (iv) There exists a measurable function k such that:

$$|u(t, s, x)| \leq k(t, s)$$

for all $t, s \geq 0$ and $x \in \mathbb{R}$. A measurable nonnegative function $k : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is supposed to be nonincreasing with respect to each variable separately and such that the linear integral operator K_0 with kernel $k(t, s)$ maps L^1 into L^∞ . Moreover, for each non-negative $z \in L^1$ let

$$\lim_{meas(D) \rightarrow 0} \sup_{|x| \leq z} \left\| \int_D u(t, s, x(s)) ds \right\|_{L^\infty} = 0$$

and assume that for arbitrary $h > 0$ ($i = 1, 2$)

$$\lim_{\delta \rightarrow 0} \left\| \int_D \max_{|x_1| \leq h, |x_1 - x_2| \leq \delta} |u(t, s, x_1) - u(t, s, x_2)| ds \right\|_{L^\infty} = 0.$$

$$(v) \quad b \cdot \|K_0\|_\infty < 1.$$

Then we can prove the following theorem.

Theorem 2.3.1. *Let the assumptions (i) - (v) be satisfied. Then the equation (2.1) has at least one solution a.e. nonincreasing on \mathbb{R}^+ which is locally integrable.*

Proof. First of all observe that by Assumption (ii) and Theorem 1.3.1 F is a continuous operator from L^1 into itself. Moreover, by (iv) U is a continuous operator from L^1 into L^∞ (see Theorem 1.3.6) and then by Hölder inequality the operator K maps L^1 into itself. Finally, for a given $x \in L^1$ the function Hx belongs to L^1 and is continuous.

Using (2.2) together with assumptions (iii) and (iv), we get

$$\begin{aligned} \|Hx\| &\leq \|g\| + \|FKx(t)\| \\ &\leq \|g\| + \int_0^\infty [a(t) + b|x(t)| \int_\alpha^\beta |u(t, s, x(s))| ds] dt \\ &\leq \|g\| + \|a\| + b \int_0^\infty |x(t)| \left[\int_\alpha^\beta k(t, s) ds \right] dt \\ &\leq \|g\| + \|a\| + b \int_0^\infty [|x(t)| \cdot \|K_0(t)\|_\infty] dt \\ &= \|g\| + \|a\| + b \cdot \|K_0\|_\infty \cdot \|x\|. \end{aligned}$$

From the above estimate it follows, that there is a constant $r > 0$ such that H maps the ball B_r into itself. Indeed, by (v) we get

$$\begin{aligned} \|Hx\| &\leq \|g\| + \|a\| + b \cdot \|K_0\|_\infty \cdot \|x\| \\ &\leq \|g\| + \|a\| + b \cdot \|K_0\|_\infty \cdot r \end{aligned}$$

and then we obtain that $H(B_r) \subset B_r$, where

$$r = \frac{\|g\| + \|a\|}{1 - b\|K_0\|_\infty}.$$

Further, let Q_r stand for the subset of B_r consisting of all functions which are a.e. nonincreasing on \mathbb{R}^+ . This set is nonempty ($x(t) = e^{-\frac{t}{r}} \in B_r \cap Q_r$, for instance), bounded (by r), convex (direct calculation from the definition) and closed in $L^1(\mathbb{R}^+)$ similarly as claimed in [17]. To prove the last property, let (y_n) be a sequence of elements in Q_r convergent in L^1 to y . Then the sequence is convergent in finite measure and as a consequence of the Vitali convergence theorem and of the characterization of convergence in measure (the Riesz theorem) we obtain the existence of a subsequence (y_{n_k}) of (y_n) which converges to y almost uniformly on \mathbb{R}^+ . Moreover, y is still nonincreasing a.e. on \mathbb{R}^+ which means that $y \in Q_r$ and so the set Q_r is closed. Now, in view of Theorem 1.4.1 the set Q_r is compact in measure. To see this it suffices to put $\Omega_c = [0, c] \setminus P$ for any $c \geq 0$, where P denotes a suitable set of with $meas(P) = 0$.

Now, we will show, that H preserve the monotonicity of functions. Take $x \in Q_r$, then $x(t)$ is a.e. nonincreasing on \mathbb{R}^+ and consequently $Kx(t)$ is also of the same type in virtue of the assumption (iii) and Theorem 1.4.2. Further, $FKx(t)$ is a.e. nonincreasing on \mathbb{R}^+ thanks for assumption (ii). Moreover, assumption (i) permits us to deduce that $Hx = g(t) + FKx(t)$ is also a.e. nonincreasing on \mathbb{R}^+ . This fact, together with the assertion $H : B_r \rightarrow B_r$ gives that H is also a self-mapping of the set Q_r . From the above considerations it follows that H maps continuously Q_r into Q_r .

From now we will assume that X is a nonempty subset of Q_r and the constant $\epsilon > 0$ is arbitrary, but fixed. Then for an arbitrary $x \in X$ and for a set $D \subset \mathbb{R}^+$, $meas(D) \leq \epsilon$ we obtain

$$\begin{aligned} \int_D |(Hx)(t)| dt &\leq \int_D [|g(t)| + a(t) + b \cdot |x(t)| \cdot \int_\alpha^\beta |u(t, s, x(s))| ds] dt \\ &= \|g\|_{L^1(D)} + \|a\|_{L^1(D)} + b \cdot \|x\|_{L^1(D)} \cdot \left\| \int_\alpha^\beta k(t, s) ds \right\|_{L^\infty} \\ &\leq \|g\|_{L^1(D)} + \|a\|_{L^1(D)} + b \cdot \|K_0\|_\infty \cdot \|x\|_{L^1(D)}. \end{aligned}$$

Hence, taking into account the obvious equality

$$\lim_{\epsilon \rightarrow 0} \left\{ \sup \left[\int_D |g(t)| dt + \int_D a(t) dt : D \subset \mathbb{R}^+, meas(D) \leq \epsilon \right] \right\} = 0$$

and by the definition of $c(X)$ (cf. Section 1.5) we get

$$c(HX) \leq b \cdot \|K_0\|_\infty \cdot c(X). \quad (2.3)$$

Furthermore, fixing $T > 0$ we arrive at the following estimate

$$\begin{aligned}
\int_T^\infty |(Hx)(t)| dt &\leq \int_T^\infty [|g(t)| + a(t) + b|x(t)| \int_\alpha^\beta |u(t, s, x(s))| ds] dt \\
&\leq \int_T^\infty [|g(t)| + a(t) + b|x(t)| \int_\alpha^\beta k(t, s) ds] dt \\
&\leq \int_T^\infty |g(t)| dt + \int_T^\infty a(t) dt + b\|K_0\|_\infty \int_T^\infty |x(t)| dt.
\end{aligned}$$

As $T \rightarrow \infty$, the above inequality yields

$$d(HX) \leq b \cdot \|K_0\|_\infty \cdot d(X), \quad (2.4)$$

where $d(X)$ has been also defined in Section 1.5.

Hence, combining (2.3) and (2.4) we get

$$\gamma(HX) \leq b \cdot \|K_0\|_\infty \cdot \gamma(X),$$

where γ denotes our measure of noncompactness defined in Section 1.5.

The inequality obtained above together with the properties of the operator H and the set Q_r established before allow us to use Theorem 1.5.2 and as a consequence, apply Theorem 1.6.5. This completes the proof. \square

Remark 2.3.1. *If we assume that the functions g and $t \rightarrow u(t, s, x)$ are a.e. nondecreasing and negative then applying the same argumentation, we can show that there exists a solution of our equation being a.e. negative and nondecreasing. Moreover, let us remark, that the monotonicity conditions in the main theorem (and examples given below) seems to be restrictive, but they are necessary as claimed in ([32] Example 2).*

2.4 Examples

We need to show two examples of problems for which our main result is useful and allow to extend the existing theorems. Let us recall, that we are looking for monotonic solutions for the considered problems in a half-line.

Let us start with a classical Chandrasekhar integral equation.

Example 2.4.1. In the case $g(t) = 1$ and $f(t, x) = x(t) \int_0^1 \frac{t}{t+s} \phi(s)x(s) ds$, equation (2.1) takes the form

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \phi(s)x(s) ds. \quad (2.5)$$

Equation (2.5) is the famous quadratic integral equation of Chandrasekhar which is considered in many papers and monographs (cf. [11, 26, 43, 75] for instance).

In this case we have $k(t, s) = \frac{t}{t+s}\phi(s)$ and as $k(\cdot, s)$ is increasing, we can put $m(s) = \phi(s)$ and then for some sufficiently good functions ϕ our result applies ($\phi(s) = e^{-s}$, for instance).

In order to illustrate the results proved in Theorem 2.3.1, let us consider the following examples

Example 2.4.2. Let us consider the following equation

$$x(t) = e^{-t} + x(t) \int_{\alpha}^{\beta} \frac{t}{t^2 + s^2 + (x(s))^2} ds. \quad (2.6)$$

By putting $g(t) = e^{-t}$, $f(t, x) = x$ and $u(t, s, x) = \frac{t}{t^2+s^2+x^2}$ it is easy to see, that u is nonincreasing with respect to each variable separately and the integrability condition is also satisfied (Assumptions (i),(ii) and (iii) are satisfied).

We have the following functions: $k(t, s) = \frac{1}{t^2+s^2}$ and since

$$\int_{\alpha}^{\beta} k(t, s) ds = \arctan \frac{\beta}{t} - \arctan \frac{\alpha}{t}, \quad \Rightarrow \left| \int_{\alpha}^{\beta} k(t, s) ds \right| \leq |\beta - \alpha|.$$

Thus the expected property (Assumption (v)) for K_0 holds (for sufficiently small parameter b dependent on α and β).

Moreover, given arbitrary $h > 0$ and $|x_2 - x_1| \leq \delta$ we have

$$|u(t, s, x_1) - u(t, s, x_2)| \leq \left| \frac{t(x_2^2 - x_1^2)}{(t^2 + s^2 + x_1^2)(t^2 + s^2 + x_2^2)} \right| \leq \frac{2ht\delta}{(t^2 + s^2 + x_1^2)(t^2 + s^2 + x_2^2)}$$

and the Assumption (iv) is satisfied.

Taking into account all the above observations and Theorem 2.3.1 we conclude that the equation (2.6) has at least one solution $x = x(t)$ defined, integrable and a.e. nonincreasing on \mathbb{R}^+ .

Chapter 3

On some integrable solutions for quadratic functional integral equations

3.1 Introduction

The object of this chapter is to study the solvability of a nonlinear Urysohn functional integral equation

$$x(t) = f_1(t, x(\phi_1(t))) + f_2(t, x(t)) \cdot \int_0^1 u(t, s, x(\phi_2(t))) ds, \quad t \in I. \quad (3.1)$$

Special cases for considered equation (quadratic integral equations) were investigated in connection with some applications of such a kind of problems in the theories of radiative transfer, neutron transport and in the kinetic theory of gases (cf. [12, 26, 40, 43]). More general problem (motivated by some practical interests in plasma physics) was investigated in [86]. The existence of continuous solutions for particular cases of the considered problem was investigated since many years (see [33, 79] or a very recent paper [4]). On the other hand, different kind of integral equations (including quadratic integral equations) should be investigated in different function spaces. This was remarked, for instance, in [86, Theorem 3.14] for the case of $L_p(I)$ -solutions, for the Hammerstein integral equation see also [79, 93] for L_p -solutions or [13, 64, 108] for integrable solutions. A very interesting survey about different classes of solutions (not only in $C(I)$ or $L_p(I)$, but also in Orlicz spaces $L_\varphi(I)$ or even in ideal spaces) for a class of integral equations related to our equation can be found in [8].

Next, let us recall that the equations involving the functional dependence have still growing number of applications (cf. [73]). We try to cover the results of this type. Let us mention, for example, the results from [14, 24].

We are interested in monotonic solutions of the above problem. The considered problem can cover, for instance, as particular cases:

1. $f_1(t, x) = g(t)$, $f_2(t, x) = \lambda$ the functional Urysohn integral equation ([14, 15, 24]),
2. $f_1(t, x) = g(t)$, $f_2(t, x) = x$, $\phi_2(t) = t$ the functional-integral equation ([91]),
3. $f_2(t, x) = 0$ the abstract functional equation ([15], for instance),
4. for continuous solutions with $\phi_1(t) = \phi_2(t) = t$ and $u(t, s, x) = \frac{u_1(t, s, x)}{\Gamma(\alpha) \cdot (t - \tau)^{1 - \alpha}}$ see [35, 61],
5. $f_2(t, x) = \lambda$ the functional integral equation (for continuous solutions see [3, 21, 54]),
6. $f_2(t, x) = x$ the quadratic (functional) Urysohn integral equation ([27, 26], for instance).

Our problem, as well as, the particular cases was investigated mainly in cases when the solutions are elements of the space of continuous functions. Thus the proofs are based on very special properties of this space (the compactness criterion, in particular), cf. [35, 89].

On the other hand, by the practical interest it is worthwhile to consider discontinuous solutions. Here we are looking for integrable solutions. Thus the operators F_1 , F_2 and U should take their values in the space $L_1(I)$. Let us recall that we are interested in finding monotonic solutions (a.e. monotonic in the case of integrable solutions). In such a case discontinuous solutions are expected even in a simplest case i.e. when

$$f_1(t, x) = h(t) = \begin{cases} 0 & t \text{ is rational,} \\ t & t \text{ is irrational} \end{cases}$$

An interesting example of discontinuous solutions for integral equations is taken from [86, Example 3.5]:

$$\chi_{[1/2, 1]}(t) \cdot (2t - 1) \cdot x(t) + \chi_{[0, 1/2]}(t) \cdot (1 - 2t) \cdot (x(t) - 1) \int_0^1 (1 - x(s)) ds = 0.$$

In contrast to the previous chapter, we extend the earlier result by considering functional integral equation in a more general form. Moreover, we prove the existence of solutions in some subspaces of $L_1(0, 1)$.

Let us add a few comments about functional dependence, i.e. functions ψ_1 and ψ_2 . Our set of assumptions is based on the paper [24]. Functions of the form $\psi_i(t) = t^\alpha$ ($\alpha > 0$) or $\psi_i(t) = t - \tau(t)$ with some set of assumptions for τ are most

important cases covered in our chapter. Let us note that functional equations with state dependent delay are very useful in many mathematical models including the population dynamics, the position control or the cell biology. A very interesting survey about such a theory and their applications can be found in [73].

The last aspect of our results is to investigate the monotonicity property of solutions. This is important property and there are many papers devoted to its study. Let us note some recent ones [27, 28, 46, 61], for instance.

The results obtained in the current chapter create some extensions for several known ones i.e. in addition to those mentioned previously also for the results from earlier papers or books ([10, 15, 38, 51, 76, 97, 99, 122], for example).

3.2 Main result

Denote by H the operator associated with the right hand side of equation (3.1) which takes the form

$$x = H(x),$$

where

$$H(x)(t) = f_1(t, x(\phi_1(t))) + f_2(t, x(t)) \cdot \int_0^1 u(t, s, x(\phi_2(s))) ds. \quad (3.2)$$

This operator will be written as $H(x) = F_{\phi_1, f_1}(x) + A(x)$,

$$A(x)(t) = F_{f_2}(x)(t) \cdot U(x)(t) = F_{f_2}(x)(t) \cdot \int_0^1 u(t, s, x(\phi_2(s))),$$

and the superposition operator F as in Definition 1.3.1. Thus equation (3.1) becomes

$$x(t) = F_{\phi_1, f_1}(x)(t) + A(x)(t).$$

3.2.1 The existence of L_1 -solution

We shall treat the equation (3.1) under the following assumptions listed below

- (i) $f_i : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions and there are a positive integrable on I functions a_i and constants $b_i \geq 0$ such that

$$|f_i(t, x)| \leq a_i(t) + b_i|x| \quad , \quad i = 1, 2,$$

for all $t \in [0, 1]$ and $x \in \mathbb{R}$. Moreover, $f_i(t, x) \geq 0$ for $x \geq 0$ and f_i is assumed to be nonincreasing with respect to both variable t and x separately for $i = 1, 2$.

(ii) $u : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t, s) . The function u is nonincreasing with respect to each variable, separately. Moreover, for arbitrary fixed $s \in I$ and $x \in \mathbb{R}$ the function $t \rightarrow u(t, s, x(s))$ is integrable.

(iii) Assume that

$$|u(t, s, x)| \leq k(t, s)(a_3(s) + b_3|x|), \quad \text{for all } t, s \geq 0 \quad \text{and} \quad x \in \mathbb{R},$$

where the function k is measurable in (t, s) , $a_3 \in L_1(I)$ and a constant $b_3 > 0$. Assume that the linear integral operator K_0 with the kernel $k(t, s)$ maps $L_1(I)$ into $L_\infty(I)$. Moreover, assume that for arbitrary $h > 0$ ($i = 1, 2$)

$$\lim_{\delta \rightarrow 0} \left\| \int_D \max_{|x_i| \leq h, |x_1 - x_2| \leq \delta} |u(t, s, x_1) - u(t, s, x_2)| ds \right\|_{L_\infty(I)} = 0.$$

(iv) $\phi_i : I \rightarrow I$ are increasing, absolutely continuous functions (for $i = 1, 2$). Moreover, there are constants $M_i > 0$ such that $\phi'_i \geq M_i$ a.e on $(0, 1)$ (for $i = 1, 2$).

(v) $\int_0^b k(t_1, s) ds \geq \int_0^b k(t_2, s) ds$ for $t_1, t_2 \in I$ with $t_1 < t_2$ and for any $b \in [0, 1]$.

(vi) let $W > \sqrt{\frac{4b_2b_3\|K_0\|_{L_\infty(I)}}{M_2} (\|a_1\|_1 + \|K_0\|_{L_\infty(I)}\|a_2\|_1\|a_3\|_1)}$, where

$$W = \left(\frac{b_1}{M_1} + \frac{b_3}{M_2} \|K_0\|_{L_\infty(I)} \|a_2\|_1 + b_2 \|K_0\|_{L_\infty(I)} \|a_3\|_1 \right) - 1$$

and let R denotes a positive solution of the quadratic equation

$$\begin{aligned} & \frac{b_2b_3\|K_0\|_{L_\infty(I)}}{M_2} \cdot t^2 \\ & - \left[1 - \left(\frac{b_1}{M_1} + \frac{b_3}{M_2} \|K_0\|_{L_\infty(I)} \|a_2\|_1 + b_2 \|K_0\|_{L_\infty(I)} \|a_3\|_1 \right) \right] \cdot t \\ & + (\|a_1\|_1 + \|K_0\|_{L_\infty(I)} \|a_2\|_1 \|a_3\|_1) = 0. \end{aligned}$$

Then we can prove the following theorem.

Theorem 3.2.1. *Let the assumptions (i) - (vi) be satisfied. Put*

$$L = \left[\frac{b_1}{M_1} + b_2 \|K_0\|_{L_\infty(I)} \left[\|a_3\|_1 + \frac{b_3}{M_2} R \right] \right].$$

If $L < 1$, then the equation (3.1) has at least one integrable solution a.e. nonincreasing on I .

Proof. First of all observe that by the assumption (i) and Theorem (1.3.1) implies that F_{ϕ_1, f_1} and F_{f_2} are continuous mappings from $L_1(I)$ into itself. By assumption (iii) and Theorem 1.3.6 we can deduce that U maps $L_1(I)$ into $L_\infty(I)$. From the Hölder inequality the operator A maps $L_1(I)$ into itself continuously. Finally, for a given $x \in L_1(I)$ the function $H(x)$ belongs to $L_1(I)$ and is continuous. Thus

$$\begin{aligned}
\|H(x)\|_1 &\leq \|F_{\phi_1, f_1}x\|_1 + \|Ax\|_1 \\
&\leq \int_0^1 [a_1(t) + b_1|x(\phi_1(t))|]dt \\
&\quad + \int_0^1 [a_2(t) + b_2|x(t)|] \int_0^1 |u(t, s, x(\phi_2(s)))| ds dt \\
&\leq \int_0^1 [a_1(t) + b_1|x(\phi_1(t))|]dt \\
&\quad + \int_0^1 [a_2(t) + b_2|x(t)|] \int_0^1 k(t, s)[a_3(s) + b_3|x(\phi_2(s))|] ds dt \\
&\leq \|a_1\|_1 + \frac{b_1}{M_1} \int_0^1 |x(\phi_1(t))|\phi_1'(t)dt \\
&\quad + \int_0^1 \int_0^1 k(t, s)a_2(t)[a_3(s) + b_3|x(\phi_2(s))|]dsdt \\
&\quad + b_2 \int_0^1 \int_0^1 k(t, s)|x(t)|[a_3(s) + b_3|x(\phi_2(s))|]dsdt \\
&\leq \|a_1\|_1 + \frac{b_1}{M_1} \int_{\phi_1(0)}^{\phi_1(1)} |x(u)|du \\
&\quad + \int_0^1 [a_3(s) + b_3|x(\phi_2(s))|] \int_0^1 k(t, s)a_2(t)dt ds \\
&\quad + b_2 \int_0^1 [a_3(s) + b_3|x(\phi_2(s))|] \int_0^1 k(t, s)|x(t)| dt ds \\
&\leq \|a_1\|_1 + \frac{b_1}{M_1} \int_0^1 |x(t)|dt \\
&\quad + \|K_0\|_{L_\infty(I)} \|a_2\|_1 \int_0^1 [a_3(s) + b_3|x(\phi_2(s))|] ds \\
&\quad + b_2 \|K_0\|_{L_\infty(I)} \|x\|_1 \int_0^1 [a_3(s) + b_3|x(\phi_2(s))|] ds \\
&\leq \|a_1\|_1 + \frac{b_1}{M_1} \|x\|_1 \\
&\quad + \|K_0\|_{L_\infty(I)} \|a_2\|_1 \int_0^1 [a_3(s) + \frac{b_3}{M_2}|x(\phi_2(s))|\phi_2'(s)] ds \\
&\quad + b_2 \|K_0\|_{L_\infty(I)} \|x\|_1 \int_0^1 [a_3(s) + \frac{b_3}{M_2}|x(\phi_2(s))|\phi_2'(s)] ds
\end{aligned}$$

$$\begin{aligned}
&\leq \|a_1\|_1 + \frac{b_1}{M_1}\|x\|_1 + \|K_0\|_{L_\infty(I)}\|a_2\|_1[\|a_3\|_1 + \frac{b_3}{M_2}\|x\|_1] \\
&\quad + b_2\|K_0\|_{L_\infty(I)}\|x\|_1[\|a_3\|_1 + \frac{b_3}{M_2}\|x\|_1] \\
&= \|a_1\|_1 + \|K_0\|_{L_\infty(I)}\|a_2\|_1\|a_3\|_1 + \left[\frac{b_1}{M_1} + \frac{b_3}{M_2}\|K_0\|_{L_\infty(I)}\|a_2\|_1\right. \\
&\quad \left.+ b_2\|K_0\|_{L_\infty(I)}\|a_3\|_1\right] \cdot \|x\|_1 + \left[\frac{b_2b_3\|K_0\|_{L_\infty(I)}}{M_2}\right] \cdot (\|x\|_1)^2.
\end{aligned}$$

By our assumption (vi) , it follows that there exists a positive constant R being the positive solution of the equation from the assumption (vi) and such that H maps the ball B_R into itself.

Further, let Q_R stand for the subset of B_R consisting of all functions which are a.e. nonincreasing on I . Similarly as claimed in [17] we are able to show that this set is nonempty, bounded (by R), convex and closed in $L_1(I)$. Only the last property needs some comments. Let (y_n) be a sequence of elements in Q_R convergent in $L_1(I)$ to y . Then the sequence is convergent in measure and as a consequence of the Vitali convergence theorem and of the characterization of convergence in measure (the Riesz theorem) we obtain the existence of a subsequence (y_{n_k}) of (y_n) which converges to y almost uniformly on I . Moreover, y is nonincreasing a.e. on I which means that $y \in Q_R$ and so the set Q_R is closed. Now, in view of Theorem 1.4.1 the set Q_R is compact in measure. To see this it suffices to put $\Omega_c = [0, c] \setminus P$ for any $c \geq 0$, where P denotes a suitable set with $meas(P) = 0$.

Now, we will show that H preserve the monotonicity of functions. Take $x \in Q_R$, then $x(t)$ and $x(\phi_i(t))$ are a.e. nonincreasing on I and consequently each f_i is also of the same type by virtue of the assumption (i) and Theorem 1.4.2. Further, $Ux(t)$ is a.e. nonincreasing on I due to assumption (ii). Moreover, F_{ϕ_1, f_1} , $A(x)(t)$ are also of the same type. Thus we can deduce that $H(x) = F_{\phi_1, f_1} + A(x)$ is also a.e. nonincreasing on I . This fact, together with the assertion $H : B_R \rightarrow B_R$ gives that H is also a self-mapping of the set Q_R . From the above considerations it follows that H maps continuously Q_R into Q_R .

From now we will assume that X is a nonempty subset of Q_R and the constant $\varepsilon > 0$ is arbitrary, but fixed. Then for an arbitrary $x \in X$ and for a set $D \subset I$, $meas(D) \leq \varepsilon$ we obtain

$$\begin{aligned}
\int_D |(H(x))(t)| dt &\leq \int_D [a_1(t) + b_1|x(\phi_1(t))|] dt \\
&\quad + \int_D [a_2(t) + b_2|x(t)|] \int_0^1 |u(t, s, x(\phi_2(s)))| ds dt \\
&\leq \|a_1\chi_D\|_1 + \frac{b_1}{M_1} \int_D |x(\phi_1(t))|\phi_1'(t) dt \\
&\quad + \int_D \int_0^1 k(t, s) a_2(t) [a_3(s) + b_3|x(\phi_2(s))|] ds dt \\
&\quad + b_2 \int_D \int_0^1 k(t, s) |x(t)| [a_3(s) + b_3|x(\phi_2(s))|] ds dt \\
&\leq \|a_1\chi_D\|_1 + \frac{b_1}{M_1} \|x\chi_D\|_1 \\
&\quad + \|K_0\|_{L_\infty(I)} \|a_2\chi_D\|_1 [\|a_3\|_1 + \frac{b_3}{M_2} R] \\
&\quad + b_2 \|K_0\|_{L_\infty(I)} \|x\chi_D\|_1 [\|a_3\|_1 + \frac{b_3}{M_2} R].
\end{aligned}$$

Hence, taking into account the equalities

$$\lim_{\varepsilon \rightarrow 0} \left\{ \sup \left[\int_D a_i(t) dt : D \subset I, \text{meas}(D) \leq \varepsilon \right] \right\} = 0, \quad i = 1, 2,$$

and by the definition of $c(X)$ (cf. Section 1.2) we get

$$c(H(X)) \leq \left[\frac{b_1}{M_1} + b_2 \|K_0\|_{L_\infty(I)} [\|a_3\|_1 + \frac{b_3}{M_2} R] \right] \cdot c(X). \quad (3.3)$$

Recall that $L = \frac{b_1}{M_1} + b_2 \|K_0\|_{L_\infty(I)} (\|a_3\|_1 + \frac{b_3}{M_2} R) < 1$ and then the inequality obtained above together with the properties of the operator H and since the set Q_R is compact in measure we are able to apply Theorem 1.6.5 which completes the proof. \square

Remark. Let us recall that in the proof we utilize the following fact: U maps $L_1(I)$ into $L_\infty(I)$ and F_2 maps $L_1(I)$ into itself. This allows us to use the Hölder inequality. In this situation, we prove the existence of a.e. monotonic solutions which are integrable. Sometimes we need more information about the solution, namely if a solution is in some subspace of $L_1(I)$ (the space L_p , for instance). In such a case we are able to use also the same type of inequality. Namely we need only to modify the growth conditions and consequently the spaces in which our operators act. As claimed in the introductory part of our chapter we can repeat our proof with appropriate changes of domains for considered operators: F_2 maps $L_p(I)$ into $L_q(I)$ and U maps $L_p(I)$ into $L_r(I)$, where $\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$. Whence we obtain an existence result for L_p -solutions.

3.2.2 The existence of L_p -solution $p > 1$

It should be noted that in some papers, their authors consider the existence of solutions in L_p spaces simultaneously for $p \geq 1$. As claimed above it cannot be done for quadratic equations. Here is a version for $p > 1$. An interesting (and motivating) remark about the solutions in L_p spaces for integral equations (by using similar method of the proof) can be found in [64, page 93]. However, by considering the measure of noncompactness $c(X) = \limsup_{meas(D) \rightarrow 0} \{\sup_{x \in X} \|x\chi_D\|_{L_p(I)}\}$ introduced by Appl and De Pascale [9] (cf. also Erzakova [66]) (restricted to the family of sets compact in measure) instead of usually considered ones based on Kolmogorov or Riesz criteria of compactness (cf. [23]) we are able to examine by the same manner the case of $L_p(I)$ spaces.

Assume that $p > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Denote by q the value $\min(p_1, p_2)$ and by r the value $\max(p_1, p_2)$. This implies, in particular, that $q \leq 2p$. We shall treat the equation (3.1) under the following set of assumptions presented below.

- (i)' Assume that functions $f_i : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy Carathéodory conditions and there are positive constants b_i ($i = 1, 2$) and positive functions $a_1 \in L_p(I)$, $a_2 \in L_q(I)$ such that

$$\begin{aligned} |f_1(t, x)| &\leq a_1(t) + b_1|x|, \\ |f_2(t, x)| &\leq a_2(t) + b_2|x|^{\frac{p}{q}}, \end{aligned}$$

for all $t \in I$ and $x \in \mathbb{R}$. Moreover, f_i ($i = 1, 2$) are assumed to be nonincreasing with respect to both variable t and x separately.

- (ii)' $u : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions. The function u is nonincreasing with respect to each variable, separately. Suppose that for arbitrary non-negative $z(t) \in L_q(I)$

$$\lim_{D \rightarrow 0} \sup_{|x| \leq z} \left\| \int_D u(t, s, x(s)) ds \right\|_{L_r(I)} = 0$$

and that

$$|u(t, s, x)| \leq k(t, s)(a_3(s) + b_3|x|^{\frac{p}{q}}), \quad \text{for all } t, s \geq 0 \quad \text{and} \quad x \in \mathbb{R},$$

where the function k is measurable in (t, s) , $a_3 \in L_q(I)$ and a constant $b_3 > 0$. Assume that the linear integral operator K_0 with the kernel $k(t, s)$ maps $L_q(I)$ into $L_r(I)$.

- (iii)' $\phi_i : I \rightarrow I$ are increasing, absolutely continuous functions (for $i = 1, 2$). Moreover, there are constants $M_i > 0$ such that $\phi_i' \geq M_i$ a.e on $(0, 1)$ (for $i = 1, 2$).

(iv)' $\int_0^b k(t_1, s) ds \geq \int_0^b k(t_2, s) ds$ for $t_1, t_2 \in I$ with $t_1 < t_2$ and for any $b \in [0, 1]$.

(v)' Assume, that the following equation

$$\begin{aligned} \frac{b_2 b_3 \|K_0\|}{M_2^{1/q}} \cdot t^{\frac{2p}{q}} + [b_2 \|a_3\|_{L_q(I)} + \frac{b_3 \|a_2\|_{L_q(I)}}{M_2^{1/q}}] \|K_0\| t^{\frac{p}{q}} + \left(\frac{b_1}{M_1^{1/p}} - 1\right) \cdot t \\ + (\|a_1\|_{L_p(I)} + \|K_0\| \|a_2\|_{L_q(I)} \|a_3\|_{L_q(I)}) = 0 \end{aligned}$$

has a solution in $(0, 1]$ and denote by s its positive solution.

By L' we will denote a number

$$\frac{b_1}{M_1^{1/p}} + b_2 s^{\frac{2p}{q}-1} \|K_0\| \left(\|a_3\|_{L_q(I)} + \frac{b_3}{M_2^{1/p}} s^{\frac{p}{q}} \right).$$

Theorem 3.2.2. *Let the assumptions (i)' - (v)' be satisfied. If $L' < 1$, then the equation (3.1) has at least one $L_p(I)$ -solution a.e. nonincreasing on I .*

Proof. By assumption (i) and Theorem 1.3.1 implies that F_{ϕ_1, f_1} maps $L_p(I)$ into itself continuously and F_{f_2} maps $L_p(I)$ into $L_q(I)$ and is continuous. By assumption (ii) and Theorem 1.3.5 we can deduce that U is a continuous map from $L_p(I)$ into $L_r(I)$. From the Hölder inequality the operator A maps $L_p(I)$ into itself continuously. Finally, for a given $x \in L_p(I)$ the function $H(x)$ belongs to $L_p(I)$ and is continuous. Thus for $\frac{1}{q} + \frac{1}{q'} = 1$

$$\begin{aligned} \|H(x)\|_p &\leq \|F_{\phi_1, f_1} x(t)\|_p + \|Ax(t)\|_p \\ &\leq \|a_1 + b_1 |x(\phi_1)|\|_p + \|F_{f_2} x\|_q \|Ux\|_r \\ &\leq \|a_1\|_p + b_1 \left(\int_0^1 |x(\phi_1(t))|^p dt \right)^{\frac{1}{p}} \\ &\quad + \|a_2 + b_2 |x|^{\frac{p}{q}}\|_q \cdot \left\| \int_0^1 k(t, s) [a_3(s) + b_3 |x(\phi_2(s))|^{\frac{p}{q}}] ds \right\|_r \\ &\leq \|a_1\|_p + \frac{b_1}{M_1^{\frac{1}{p}}} \left(\int_0^1 |x(\phi_1(t))|^p \phi_1'(t) dt \right)^{\frac{1}{p}} \\ &\quad + [\|a_2\|_q + b_2 \|x\|_p^{\frac{p}{q}}] \cdot \| \|k(t, \cdot)\|_{q'} [\|a_3\|_q + b_3 \| |x(\phi_2)|^{\frac{p}{q}} \|_q] \|_r, \end{aligned}$$

where

$$\|x^{\frac{p}{q}}\|_q = \left(\int_0^1 \left(|x(s)|^{\frac{p}{q}} \right)^q ds \right)^{\frac{1}{q}} = \|x\|_p^{\frac{p}{q}} \quad \text{and} \quad \|K_0\|_{r, q'} \equiv \|t \rightarrow \|k(t, \cdot)\|_{q'}\|_r.$$

$$\begin{aligned}
\|H(x)\|_p &\leq \|a_1\|_p + \frac{b_1}{M_1^{\frac{1}{p}}} \left(\int_{\phi_1(0)}^{\phi_1(1)} |x(v)|^p dv \right)^{\frac{1}{p}} \\
&+ [\|a_2\|_q + b_2 \|x\|_p^{\frac{p}{q}}] \cdot \|K_0\|_{r,q'} [\|a_3\|_q + b_3 \left(\int_0^1 (|x(\phi_2(s))|^{\frac{p}{q}})^q ds \right)^{\frac{1}{q}}] \\
&\leq \|a_1\|_p + \frac{b_1}{M_1^{\frac{1}{p}}} \left(\int_0^1 |x(v)|^p dv \right)^{\frac{1}{p}} \\
&+ [\|a_2\|_q + b_2 \|x\|_p^{\frac{p}{q}}] \cdot \|K_0\|_{r,q'} [\|a_3\|_q + \frac{b_3}{M_2^{\frac{1}{q}}} \left(\int_0^1 (|x(\phi_2(s))|^{\frac{p}{q}})^q \phi_2'(s) ds \right)^{\frac{1}{q}}] \\
&\leq \|a_1\|_p + \frac{b_1}{M_1^{\frac{1}{p}}} \|x\|_p \\
&+ [\|a_2\|_q + b_2 \|x\|_p^{\frac{p}{q}}] \cdot \|K_0\|_{r,q'} [\|a_3\|_q + \frac{b_3}{M_2^{\frac{1}{q}}} \left(\int_{\phi_2(0)}^{\phi_2(1)} (|x(v)|^{\frac{p}{q}})^q dv \right)^{\frac{1}{q}}] \\
&\leq \|a_1\|_p + \frac{b_1}{M_1^{\frac{1}{p}}} \|x\|_p \\
&+ [\|a_2\|_q + b_2 \|x\|_p^{\frac{p}{q}}] \cdot \|K_0\|_{r,q'} [\|a_3\|_q + \frac{b_3}{M_2^{\frac{1}{q}}} \left(\int_0^1 (|x(v)|^{\frac{p}{q}})^q dv \right)^{\frac{1}{q}}] \\
&\leq \|a_1\|_p + \frac{b_1}{M_1^{\frac{1}{p}}} \|x\|_p + [\|a_2\|_q + b_2 \|x\|_p^{\frac{p}{q}}] \cdot \|K_0\|_{r,q'} [\|a_3\|_q + \frac{b_3}{M_2^{\frac{1}{q}}} \|x\|_p^{\frac{p}{q}}] \\
&\leq \|a_1\|_p + \|K_0\|_{r,q'} \|a_2\|_q \|a_3\|_q + \frac{b_1}{M_1^{\frac{1}{p}}} \|x\|_p \\
&+ \|K_0\|_{r,q'} [b_2 \|a_3\|_q + \frac{b_3 \|a_2\|_q}{M_2^{\frac{1}{q}}}] \cdot \|x\|_p^{\frac{p}{q}} + \frac{b_2 b_3 \|K_0\|_{r,q'}}{M_2^{\frac{1}{q}}} \|x\|_p^{\frac{2p}{q}} \\
&\leq \|a_1\|_p + \|K_0\|_{r,q'} \|a_2\|_q \|a_3\|_q + \frac{b_1}{M_1^{\frac{1}{p}}} R' \\
&+ \|K_0\|_{r,q'} [b_2 \|a_3\|_q + \frac{b_3 \|a_2\|_q}{M_2^{\frac{1}{q}}}] R'^{\frac{p}{q}} + \frac{b_2 b_3 \|K_0\|_{r,q'}}{M_2^{\frac{1}{q}}} R'^{\frac{2p}{q}} \leq R'.
\end{aligned}$$

From assumption (v)' it follows there exists a positive solution R' of the equation, which implies that H maps the ball $B_{R'}$ into it self.

As before in Theorem 3.2.1, we can construct a subset $Q_{R'}$ of $B_{R'}$ is nonempty, bounded, convex and closed in $L_p(I)$ consisting of all functions which are a.e. non-increasing on I . Such that H maps $Q_{R'}$ into it self.

From now we will assume that X is a nonempty subset of $Q_{R'}$ and the constant $\varepsilon > 0$ is arbitrary, but fixed. Then for an arbitrary $x \in X$ and for a set $D \subset I$,

$meas(D) \leq \varepsilon$ we obtain

$$\begin{aligned}
\|H(x)\chi_D\|_p &\leq \| [a_1 + b_1|x(\phi_1)]\chi_D \|_p + \|F_{f_2}x\chi_D\|_q \|Ux\chi_D\|_r \\
&\leq \|a_1\chi_D\|_p + \frac{b_1}{M_1^{\frac{1}{p}}}\|x\chi_D\|_p \\
&\quad + [\|a_2\chi_D\|_q + b_2\|x\chi_D\|_q^{\frac{p}{q}}] \cdot \left\| \int_0^1 k(t,s)[a_3(s) + b_3|x(\phi_2(s))|^{\frac{p}{q}}] \right\|_r \\
&\leq \|a_1\chi_D\|_p + \frac{b_1}{M_1^{\frac{1}{p}}}\|x\chi_D\|_p \\
&\quad + [\|a_2\chi_D\|_q + b_2\|x\chi_D\|_q^{\frac{p}{q}}] \cdot \|k(t,\cdot)\|_{q'} [\|a_3\|_q + b_3\|x(\phi_2)\|_q^{\frac{p}{q}}] \\
&\leq \|a_1\chi_D\|_p + \frac{b_1}{M_1^{\frac{1}{p}}}\|x\chi_D\|_p \\
&\quad + [\|a_2\chi_D\|_q + b_2\|x\chi_D\|_q^{\frac{p}{q}}] \cdot \|K_0\|_{r,q'} [\|a_3\|_q + \frac{b_3}{M_2^{\frac{q}{p}}}\|x\|_p^{\frac{p}{q}}] \\
&\leq \|a_1\chi_D\|_p + \frac{b_1}{M_1^{\frac{1}{p}}}\|x\chi_D\|_p \\
&\quad + [\|a_2\chi_D\|_q + b_2R^{\frac{p}{q}-1}\|x\chi_D\|_p] \cdot \|K_0\|_{r,q'} [\|a_3\|_q + \frac{b_3}{M_2^{\frac{q}{p}}}R^{\frac{p}{q}}].
\end{aligned}$$

Since $a_1 \in L_p$ and $a_2 \in L_q$,

$$\lim_{\varepsilon \rightarrow 0} \{ \sup [\|a_1\chi_D\|_p : D \subset I, meas(D) \leq \varepsilon] \} = 0$$

and

$$\lim_{\varepsilon \rightarrow 0} \{ \sup [\|a_2\chi_D\|_q : D \subset I, meas(D) \leq \varepsilon] \} = 0.$$

Then by the definition of $c(x)$ in L_p space, we have

$$c(H(X)) = \left[\frac{b_1}{M_1^{\frac{1}{p}}} + b_2s^{\frac{p}{q}-1} \|K_0\|_{r,q'} (\|a_3\|_q + \frac{b_3}{M_2^{\frac{q}{p}}}s^{\frac{p}{q}}) \right] \cdot c(X).$$

Recall that $L' = [\frac{b_1}{M_1^{\frac{1}{p}}} + b_2s^{\frac{p}{q}-1} \|K_0\|_{r,q'} (\|a_3\|_q + \frac{b_3}{M_2^{\frac{q}{p}}}s^{\frac{p}{q}})] < 1$ and then the inequality obtained above together with the properties of the operator H and since the set Q_R is compact in measure we are able to apply Theorem 1.6.5 which completes the proof. □

Let us note, that in the assumption (v)' we consider the equation of the type $A + Bt + Ct^{\frac{p}{q}} + Dt^{\frac{2p}{q}} = t$. The case $p = q$ leads to the quadratic equation (considered in our first theorem). Although the case $p < q$ seems to be more complicated, it

should be noted that since $\frac{p}{q} < 1$ and $\frac{2p}{q} < 2$ this equation has a solution in $(0, 1]$. In some papers the assumption of this type is described by using auxiliary functions. In such a formulation the problem of existence of functions is unclear. Let us note, that for arbitrary pair of spaces $L_p(I)$ and $L_q(I)$ we are able to solve our problem.

Indeed, if $\frac{2p}{q} \geq 1$, then for $t \in I$ we have $A + Bt + Ct^{\frac{p}{q}} + Dt^{\frac{2p}{q}} \leq A + Bt + C + Dt$ and our inequality has a solution in $(0, 1]$ whenever $\frac{A+C}{1-B-D} < 1$. In the case $\frac{2p}{q} < 1$, we have the following estimation: $A + Bt + Ct^{\frac{p}{q}} + Dt^{\frac{2p}{q}} \leq A + Bt + C + D$ and then $\frac{A+C+D}{1-B} < 1$ form a sufficient condition for the existence of solutions of our inequality in $(0, 1]$. Thus the set of functions satisfying our assumptions is nonempty (cf. also some interesting Examples in [16]). Let us remind that the first case is considered in the chapter.

We would like to pay attention, that the condition (ii)' implies that the kernels $k(t, s)$ are of Hille-Tamarkin classes i.e. $\| \|k(t, \cdot)\|_q \|_r$ and $\| \|k(\cdot, s)\|_q \|_{r'}$ are finite being at the same time the upper bounds for $\|K_0\|$, where q' and r' are conjugated with q and r , respectively.

Moreover, it is worthwhile to note that by the same manner we can extend our main result for other subspaces of $L_1(I)$ for which we are able to check the required properties of considered operators (some Orlicz spaces, for instance) cf. [47].

Remark 3.2.1. Till now, we are interested in finding monotonic solutions of our problem. Assume that we have the decomposition of the interval I into the disjoint subsets T_1 and T_2 with $T_1 \cup T_2 = I$, such that $f_i(\cdot, x)$ are a.e. nondecreasing on T_1 and a.e. nonincreasing on T_2 . By an appropriate change of the monotonicity assumptions we are able to prove the existence of solutions belonging to the class of functions described above (similarly like in [15]). In such a case we need to consider the operators preserving this property, too.

3.3 Examples

We need to show an example for which our main result is useful and allow to extend the existing theorems. Let us recall that we are looking for monotonic solutions for the considered problems in the interval I .

But first, let us recall that the quadratic equations have numerous applications in the theories of radiative transfer, neutron transport and in the kinetic theory of gases [12, 26, 40, 43]. In order to apply earlier results of the considered type, we have to impose an additional condition that the so-called "characteristic" function ψ is continuous (cf. [40, Theorem 3.2]) or even Hölder continuous ([12]). In the theory of radiative transfer this function is immediately related to the angular pattern

for single scattering and then our results allow to consider some peculiar states of the atmosphere. In astrophysical applications of the Chandrasekhar equation $x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \psi(s) x(s) ds$ the only restriction that $\int_0^1 \psi(s) ds \leq 1/2$ is treated as necessary (cf. [40, Chapter VIII; Corollary 2 p. 187]). An interesting discussion about this condition and the applicability of such equations can be found in [40]. Recall that to ensure the existence of solutions normally one assumes that $\psi(t)$ is an even polynomial (as in the book of Chandrasekhar [43, Chapter 5]) or continuous ([40]). The using of different solution spaces in the current chapter allow us to remove this restriction and then we give a partial answer to the problem from [40]. The continuity assumption for ψ implies the continuity of solutions for the considered equation (cf. [40]) and then seems to be too restrictive even from the theoretical point of view.

Let us consider now the following integral equation

$$x(t) = a(t) + \frac{-\ln(1 + x^2(\frac{t}{3} + \frac{t^2}{2}))}{3+t} + \arctan\left(\frac{1+h(x)}{\sqrt{t+2}}\right) \int_0^1 \frac{\lambda}{t^2+s^2} \left[\frac{1}{\sqrt{s+1}} + \frac{x(s)}{1+x^2(s)} \right] ds, \quad (3.4)$$

where

$$a(t) = \begin{cases} 0 & t \text{ is rational,} \\ 1-t & t \text{ is irrational} \end{cases}, \quad h(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{\sin x}{1+e^x} & \text{for } x > 0. \end{cases}$$

It can be easily seen that equation (3.4) is a particular case of the equation (3.1), where

$$f_1(t, x) = a(t) + \frac{-\ln(1 + x^2(\frac{t}{3} + \frac{t^2}{2}))}{3+t}, \quad f_2(t, x) = \arctan\left(\frac{1+h(x)}{\sqrt{t+2}}\right)$$

and

$$u(t, s, x) = \frac{\lambda}{t^2+s^2} \left[\frac{1}{\sqrt{s+1}} + \frac{x(s)}{1+x^2(s)} \right].$$

In view of the inequalities $\ln(1+x^2) \leq x$ ($x > 0$) and $\arctan\left(\frac{1+h(x)}{\sqrt{t+2}}\right) < \frac{1+h(x)}{\sqrt{t+2}}$, the functions f_1 , f_2 and u are nonincreasing in each variable separately. Moreover, $|f_1(t, x)| \leq a(t) + \frac{1}{4}|x|$, $|f_2(t, x)| \leq \frac{1}{\sqrt{t+2}} + \frac{1}{3}h(x)$ and

$$|u(t, s, x)| \leq \frac{\lambda}{t^2+s^2} \left[\frac{1}{\sqrt{s+1}} + \frac{1}{2}|x| \right],$$

with $a_1(t) = a(t)$, $a_2(t) = \frac{1}{\sqrt{t+2}}$, $a_3(s) = \frac{1}{\sqrt{s+1}}$ and $k(t, s) = \frac{\lambda}{t^2+s^2}$. Here we have the constants $b_1 = \frac{1}{4}$, $b_2 = \frac{1}{3}$ and $b_3 = \frac{1}{2}$.

Since $\int_0^1 \frac{\lambda}{t^2+s^2} ds = \lambda \arctan \frac{1}{t}$, $|k(t, s)| \leq \lambda$. Thus the expected property for the operator K_0 holds true. Moreover, given arbitrary $h > 0$ and $|x_2 - x_1| \leq \delta$ we have

$$\begin{aligned} |u(t, s, x_1) - u(t, s, x_2)| &= \frac{1}{t^2 + s^2} \left| \frac{x_1(1 + x_2^2) - x_2(1 + x_1^2)}{(1 + x_1^2)(1 + x_2^2)} \right| \\ &= \frac{1}{t^2 + s^2} \left| \frac{(x_1 - x_2) + x_1x_2(x_2 - x_1)}{(1 + x_1^2)(1 + x_2^2)} \right| \\ &\leq \frac{1}{t^2 + s^2} \frac{\delta(1 + h^2)}{(1 + x_1^2)(1 + x_2^2)}. \end{aligned}$$

Put $\phi_1(t) = \frac{t}{3} + \frac{t^2}{2}$ and $\phi_2(t) = t$, then

$$\phi_1'(t) = \frac{1}{3} + t > \frac{1}{3} = M_1 \quad \text{and} \quad \phi_2'(t) = 1 > \frac{1}{2} = M_2$$

.

Thus our assumptions (i)-(iv) are satisfied. Since $\frac{3}{4} + \frac{1}{3}\lambda\frac{\pi}{2}(1 + R) < 1$ for small $\lambda > 0$, assumption (v) holds true for sufficiently small λ .

Taking into account all the above observations we are able to deduce from Theorem 3.2.1 that for sufficiently small λ the equation (3.4) has at least one integrable solution x which is a.e. nonincreasing on I .

Chapter 4

Functional quadratic integral equations with perturbations on a half line

4.1 Introduction

We study the solvability of the following functional integral equation. Let $t \in \mathbb{R}^+$

$$x(t) = g(t, x(\varphi_3(t))) + f_1 \left(t, f_2(t, x(\varphi_2(t))) \cdot \int_0^t u(t, s, x(\varphi_1(s))) ds \right). \quad (4.1)$$

This equation has been studied for non quadratic integral equation in [19] with $g = 0, f_2 = 1$ using Schauder fixed point theorem and in [115] with a perturbation term. In [24], it was checked the existence of monotonic solutions, where $g(t, x(t)) = h(t), f_2 = 1$, improved also by Emmanuele (cf. [63]). The authors used the general Krasnoselskii fixed point theorem to obtain the existence result (cf. [57, 87, 115]). In [124] the author studied a special case of our equation in a general Banach space X by using the classical Krasnoselskii fixed point theorem.

In the case of the sum of two sufficiently regular operators the contraction condition is easily verified. Nevertheless, a construction for the set M make the above theorem more restrictive. The presence of the perturbation term $g(t, x(t))$ in the integral equation make the Schauder fixed point theorem unavailable. Given operators A and B , it may be possible to find the bounded domains M_A and M_B in such a way that $A : M_A \rightarrow M_A$ and $B : M_B \rightarrow M_B$, but it is often impossible to arrange matters so that $M_A = M_B = M$ and $Ax + By \in M_A$ for $x, y \in M$. Actually, the Krasnoselskii fixed point theorem allow us to avoid these problems for obtaining the result of the solution.

The results presented in this chapter are motivated by extending the recent results to the functional quadratic integral equation with a perturbation term by using

classical Krasnoselskii fixed point theorem and the measure of weak noncompactness.

Let us stress, that we will dispense the monotonicity assumptions that mentioned in the previous chapters, so we need to use different method of the proof.

4.2 Main result

Equation (4.1) takes the following form

$$x = Ax + Bx,$$

where

$$Ax(t) = F_{\varphi_3, g}x(t) - g(t, 0),$$

$$\begin{aligned} (Bx)(t) &= f_1 \left(t, f_2(t, x(\varphi_2(t))) \cdot \int_0^t u(t, s, x(\varphi_1(s))) ds \right) + g(t, 0) \\ &= F_{f_1}(Kx)(t) + g(t, 0), \end{aligned}$$

$$Kx(t) = F_{\varphi_2, f_2}x \cdot Ux(\varphi_1)(t) \quad \text{and} \quad Ux(t) = \int_0^t u(t, s, x(s)) ds.$$

We shall treat the equation (4.1) under the following assumptions listed below, where $L_1 = L_1(\mathbb{R}^+)$.

- (i) $g, f_i : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions and there are positive functions $a_i \in L_1$ and constants $b_i \geq 0$ for $i = 1, 2, 3$. such that

$$|f_i(t, x)| \leq a_i(t) + b_i |x|, \quad i = 1, 2, \quad \text{and} \quad |g(t, 0)| \leq a_3(t),$$

for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Moreover, the function g is assumed to satisfy the Lipschitz condition with constant b_3 for almost all t :

$$|g(t, x) - g(t, y)| \leq b_3|x - y|.$$

- (ii) $u : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t, s) . Moreover, for arbitrary fixed $s \in \mathbb{R}^+$ and $x \in \mathbb{R}$ the function $t \rightarrow u(t, s, x(s))$ is integrable.
- (iii) There exists a function $k(t, s) = k : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ which satisfies Carathéodory conditions such that:

$$|u(t, s, x)| \leq k(t, s)$$

for all $t, s \geq 0$ and $x \in \mathbb{R}$, such that the linear integral operator K_0 with kernel $k(t, s)$ maps L_1 into L_∞ . Moreover, assume that for arbitrary $h > 0$ ($i = 1, 2$)

$$\lim_{\delta \rightarrow 0} \left\| \int_D \max_{|x_i| \leq h, |x_1 - x_2| \leq \delta} |u(t, s, x_1) - u(t, s, x_2)| ds \right\|_{L^\infty} = 0.$$

(iv) $\varphi_i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is increasing absolutely continuous function and there are positive constants M_i such that $\varphi'_i \geq M_i$ a.e. on \mathbb{R}^+ for $i = 1, 2, 3$.

$$(v) \quad q = \left(\frac{b_3}{M_3} + \frac{b_2 b_1}{M_2} \|K_0\|_\infty \right) < 1,$$

$$(vi) \quad p = \frac{b_3}{M_3} < 1.$$

Then we can prove the following theorem.

Theorem 4.2.1. *Let the assumptions (i) - (vi) are satisfied, then equation (4.1) has at least one integrable solution on \mathbb{R}^+ .*

Proof. The proof will be given in six steps.

- **Step 1.** The operator $A : L_1 \rightarrow L_1$ is a contraction mapping.
- **Step 2.** We will construct the ball B_r such that $A(B_r) + B(B_r) \subseteq B_r$, where r will be determined later.
- **Step 3.** We will proof that $\mu(A(Q) + B(Q)) \leq q\mu(Q)$ for all bounded subset Q of B_r , where μ as in Definition 1.5.2.
- **Step 4.** We will construct a nonempty closed convex weakly compact set M in on which we will apply fixed point theorem to prove the existence of solutions.
- **Step 5.** $B(M)$ is relatively strongly compact in L_1 .
- **Step 6.** We will check out the conditions needed in Krasonskii's fixed point theorem are fulfilled.

Step 1. From assumption (i), we have

$$\begin{aligned} ||g(t, x)| - |g(t, 0)|| &\leq |g(t, x) - g(t, 0)| \leq b_3|x| \\ |g(t, x)| - a_3(t) &\leq b_3|x| \Rightarrow \\ |g(t, x)| &\leq a_3(t) + b_3|x|. \end{aligned}$$

The inequality obtained above with Theorem 1.3.1 permits us to deduce that the operator A maps L_1 into itself.

Now,

$$\begin{aligned}
\int_0^\infty |(Ax)(t) - (Ay)(t)|dt &= \int_0^\infty |g(t, x(\varphi_3(t))) - g(t, y(\varphi_3(t)))|dt \\
&\leq b_3 \int_0^\infty |x(\varphi_3(t)) - y(\varphi_3(t))|dt \\
&\leq \frac{b_3}{M_3} \int_0^\infty |x(\varphi_3(t)) - y(\varphi_3(t))|\varphi_3'(t)dt \\
&= \frac{b_3}{M_3} \int_{\varphi_3(0)}^{\varphi_3(\infty)} |x(v) - y(v)|dv \\
&\leq \frac{b_3}{M_3} \int_0^\infty |x(v) - y(v)|dv,
\end{aligned}$$

which implies that

$$\|Ax - Ay\|_{L_1} \leq \frac{b_3}{M_3} \|x - y\|_{L_1}. \quad (4.2)$$

Assumption (vi) permits us to deduce that the operator A is a contraction mapping.

Step 2. Let x and y be arbitrary functions in $B_r \subset L_1(\mathbb{R}^+)$. In view of our assumptions we get a priori estimation

$$\begin{aligned}
\|Ax + By\|_{L_1} &\leq \int_0^\infty |g(t, x(\varphi_3(t)))|dt \\
&\quad + \int_0^\infty |f_1 \left(t, f_2(t, y(\varphi_2(t))) \cdot \int_0^t u(t, s, y(\varphi_1(s))) ds \right)|dt \\
&\leq \int_0^\infty [a_3(t) + b_3|x(\varphi_3(t))|]dt \\
&\quad + \int_0^\infty [a_1(t) + b_1 \cdot |f_2(t, y(\varphi_2(t)))| \cdot \int_0^t |u(t, s, y(\varphi_1(s)))| ds]dt \\
&\leq \int_0^\infty [a_3(t) + b_3|x(\varphi_3(t))|]dt \\
&\quad + \int_0^\infty [a_1(t) + b_1 \cdot (a_2(t) + b_2|y(\varphi_2(t))|) \cdot \int_0^t k(t, s) ds]dt \\
&\leq \|a_1\|_{L_1} + \|a_3\|_{L_1} + \frac{b_3}{M_3} \int_0^\infty |x(\varphi_3(t))|\varphi_3'(t)dt \\
&\quad + b_1 \cdot \|K_0\|_{L_\infty} \int_0^\infty (a_2(t) + b_2|y(\varphi_2(t))|)dt \\
&\leq \|a_1\|_{L_1} + \|a_3\|_{L_1} + \frac{b_3}{M_3} \int_{\varphi_3(0)}^{\varphi_3(\infty)} |x(v)|dv \\
&\quad + b_1 \cdot \|K_0\|_{L_\infty} [\|a_2\|_{L_1} + \frac{b_2}{M_2} \int_0^\infty |y(\varphi_2(t))|\varphi_2'(t)dt]
\end{aligned}$$

$$\begin{aligned}
&\leq \|a_1\|_{L_1} + \|a_3\|_{L_1} + \frac{b_3}{M_3} \int_0^\infty |x(t)| dt \\
&\quad + b_1 \cdot \|K_0\|_{L_\infty} [\|a_2\|_{L_1} + \frac{b_2}{M_2} \int_0^\infty |y(\varphi_2(t))| \varphi_2'(t) dt] \\
&\leq \|a_1\|_{L_1} + \|a_3\|_{L_1} + \frac{b_3}{M_3} \|x\|_{L_1} \\
&\quad + b_1 \|a_2\|_{L_1} \cdot \|K_0\|_{L_\infty} + \frac{b_2}{M_2} \|K_0\|_{L_\infty} \cdot \|y\|_{L_1} \\
&\leq \|a_1\|_{L_1} + \|a_3\|_{L_1} + \frac{b_3}{M_3} \cdot r \\
&\quad + b_1 \|a_2\|_{L_1} \cdot \|K_0\|_{L_\infty} + \frac{b_1 b_2}{M_2} \|K_0\|_{L_\infty} \cdot r \leq r.
\end{aligned}$$

From the above estimate, we have that $A(B_r) + B(B_r) \subseteq B_r$ provided

$$r = \frac{\|a_1\|_{L_1} + \|a_3\|_{L_1} + b_1 \|a_2\|_{L_1} \cdot \|K_0\|_{L_\infty}}{1 - (\frac{b_3}{M_3} + \frac{b_1 b_2}{M_2} \|K_0\|_{L_\infty})} > 0.$$

Step 3. Take an arbitrary number $\varepsilon > 0$ and a set $D \subset \mathbb{R}^+$ such that $meas(D) \leq \varepsilon$. For any $x, y \in Q$, we have

$$\begin{aligned}
\int_D |Ax(t) + By(t)| dt &\leq \int_D |Ax(t)| dt + \int_D |By(t)| dt \\
&= \int_D |F_{g, \varphi_3} x(t)| dt + \int_D |F_{f_1} K y(t)| dt \\
&\leq \int_D [a_3(t) + b_3 |x(\varphi_3(t))|] dt \\
&\quad + \int_D [a_1(t) + b_1 |f_2(t, y(\varphi_2(t)))| \int_0^t |u(t, s, y(\varphi_1(s)))| ds] dt \\
&\leq \int_D [a_3(t) + b_3 |x(\varphi_3(t))|] dt \\
&\quad + \int_D [a_1(t) + b_1 \cdot (a_2(t) + b_2 |y(\varphi_2(t))|)] \cdot \int_0^t k(t, s) ds dt \\
&\leq \int_D a_1(t) dt + \int_D a_3(t) dt + \frac{b_3}{M_3} \int_D |x(\varphi_3(t))| \varphi_3'(t) dt \\
&\quad + b_1 \|K_0\|_{L_\infty} \int_D [a_2(t) + b_2 |y(\varphi_2(t))|] dt \\
&\leq \int_D a_1(t) dt + \int_D a_3(t) dt + \frac{b_3}{M_3} \int_{\varphi_3(D)} |x(v)| dv \\
&\quad + b_1 \|K_0\|_{L_\infty} \int_D [a_2(t) + \frac{b_2}{M_2} |y(\varphi_2(t))| \varphi_2'(t)] dt
\end{aligned}$$

$$\begin{aligned}
&\leq \int_D a_1(t)dt + \int_D a_3(t)dt + \frac{b_3}{M_3} \int_{\varphi_3(D)} |x(v)|dv \\
&\quad + b_1 \|K_0\|_{L_\infty} \left[\int_D a_2(t)dt + \frac{b_2}{M_2} \int_{\varphi_2(D)} |y(v)|dv \right] \\
&\leq \int_D a_1(t)dt + \int_D a_3(t)dt + \frac{b_3}{M_3} \int_{\varphi_3(D)} |x(v)|dv \\
&\quad + b_1 \|K_0\|_{L_\infty} \left[\int_D a_2(t)dt + \frac{b_2}{M_2} \int_{\varphi_2(D)} |y(v)|dv \right],
\end{aligned}$$

where the symbol $\|K_0\|_{L_\infty(D)}$ denotes the norm of the operator K_0 acting from the space $L_1(D)$ into $L_\infty(D)$.

Now, using the fact that

$$\lim_{\varepsilon \rightarrow \infty} \sup \left[\int_D a_i(t)dt : D \subset \mathbb{R}^+, m(D) \leq \varepsilon \right] = 0, \quad \text{for } i = 1, 2, 3.$$

From Definition 1.5 it follows that

$$c(A(Q) + B(Q)) \leq [q = \left(\frac{b_3}{M_3} + \frac{b_1 b_2}{M_2} \|K_0\|_{L_\infty} \right)] c(Q). \quad (4.3)$$

For $T > 0$ and any $x, y \in Q$, we have

$$\begin{aligned}
\int_T^\infty |Ax(t) + By(t)|dt &\leq \int_T^\infty a_1(t)dt + \int_T^\infty a_3(t)dt + \frac{b_3}{M_3} \int_{\varphi_3(T)}^\infty |x(v)|dv \\
&\quad + b_1 \|K_0\|_{L_\infty} \left[\int_T^\infty a_2(t)dt + \frac{b_2}{M_2} \int_{\varphi_2(T)}^\infty |y(v)|dv \right],
\end{aligned}$$

where $\varphi_i(T) \rightarrow \infty$ as $T \rightarrow \infty$ for $i = 1, 2$. Then as $T \rightarrow \infty$ and by the Definition 1.6 we get

$$d(A(Q) + B(Q)) \leq [q = \left(\frac{b_3}{M_3} + \frac{b_1 b_2}{M_2} \|K_0\|_{L_\infty} \right)] d(Q). \quad (4.4)$$

By combining equation 4.3 and 4.4 and Definition 1.7, we have

$$\mu(A(Q) + B(Q)) \leq [q = \left(\frac{b_3}{M_3} + \frac{b_1 b_2}{M_2} \|K_0\|_{L_\infty} \right)] \mu(Q).$$

Step 4. Let $B_r^1 = \text{Conv}(A(B_r) + B(B_r))$, where B_r is defined in step 1, $B_r^2 = \text{Conv}(A(B_r^1) + B(B_r^1))$ and so on. We then get a decreasing sequence $\{B_r^n\}$, that is $B_r^{n+1} \subset B_r^n$ for $n = 1, 2, \dots$. Obviously all sets belonging to this sequence are closed and convex, so weakly closed. By the fact proved in step 2.

That $\mu(A(Q) + B(Q)) \leq q\mu(Q)$ for all bounded subset Q of B_r , we have

$$\mu(B_r^n) \leq q^n \mu(B_r),$$

which yields that $\lim_{n \rightarrow \infty} \mu(B_r^n) = 0$.

Denote $M = \bigcap_{n=1}^\infty B_r^n$, and then $\mu(M) = 0$. By the definition of the measure of weak

noncompactness we know that M is nonempty. From the definition of the operator A , we can deduce that $B(M) \subset M$.

M is just nonempty closed convex weakly compact set which we need in the following steps.

Step 5. Let $\{x_n\} \subset M$ be arbitrary sequence. Since $\mu(M) = 0$, $\exists T, \forall n$, the following inequality is satisfied:

$$\int_T^\infty |x_n(t)| dt \leq \frac{\varepsilon}{4}. \quad (4.5)$$

Considering the function $f_i(t, x)$ on $[0, T]$, ($i = 1, 2$), $u(t, s, x)$ on $[0, T] \times [0, T] \times \mathbb{R}$, and $k(t, s)$ on $[0, T] \times [0, T]$, in view of Theorem 1.3.3 we can find a closed subset D_ε of the interval $[0, T]$, such that $\text{meas}(D_\varepsilon^c) \leq \varepsilon$, and such that $f_i|_{D_\varepsilon \times \mathbb{R}}$ ($i = 1, 2$), $u|_{D_\varepsilon \times D_\varepsilon \times \mathbb{R}}$, and $k|_{D_\varepsilon \times [0, T]}$ are continuous. Especially $k|_{D_\varepsilon \times [0, T]}$ is uniformly continuous.

Let us take arbitrary $t_1, t_2 \in D_\varepsilon$ and assume $t_1 < t_2$ without loss of generality. For an arbitrary fixed $n \in \mathbb{N}$ and denoting $H_n(t) = (F_{\varphi_2, f_2} x_n) \cdot (U x_n)(t)$ we obtain:

$$\begin{aligned} |H_n(t_2) - H_n(t_1)| &= |f_2(t_2, x_n(\varphi_2(t_2))) \int_0^{t_2} u(t_2, s, x_n(\varphi_1(s))) ds \\ &\quad - f_2(t_1, x_n(\varphi_2(t_1))) \int_0^{t_1} u(t_1, s, x_n(\varphi_1(s))) ds| \\ &\leq |f_2(t_2, x_n(\varphi_2(t_2))) - f_2(t_1, x_n(\varphi_2(t_1)))| \int_0^{t_2} |u(t_2, s, x_n(\varphi_1(s)))| ds \\ &\quad + |f_2(t_1, x_n(\varphi_2(t_1))) \int_0^{t_2} u(t_2, s, x_n(\varphi_1(s))) ds \\ &\quad - f_2(t_1, x_n(\varphi_2(t_1))) \cdot \int_0^{t_1} u(t_1, s, x_n(\varphi_1(s))) ds| \\ &\leq |f_2(t_2, x_n(\varphi_2(t_2))) - f_2(t_1, x_n(\varphi_2(t_1)))| \int_0^{t_2} |u(t_2, s, x_n(\varphi_1(s)))| ds \\ &\quad + |f_2(t_1, x_n(\varphi_2(t_1))) \int_0^{t_2} u(t_2, s, x_n(\varphi_1(s))) ds \\ &\quad - f_2(t_1, x_n(\varphi_2(t_1))) \int_0^{t_1} u(t_2, s, x_n(\varphi_1(s))) ds| \\ &\quad + |f_2(t_1, x_n(\varphi_2(t_1))) \int_0^{t_1} u(t_2, s, x_n(\varphi_1(s))) ds \\ &\quad - f_2(t_1, x_n(\varphi_2(t_1))) \cdot \int_0^{t_1} u(t_1, s, x_n(\varphi_1(s))) ds| \end{aligned}$$

$$\begin{aligned}
&\leq |f_2(t_2, x_n(\varphi_2(t_2))) - f_2(t_1, x_n(\varphi_2(t_1)))| \int_0^{t_2} k(t_2, s) ds \\
&+ |f_2(t_1, x_n(\varphi_2(t_1)))| \int_{t_1}^{t_2} |u(t_2, s, x_n(\varphi_1(s)))| ds \\
&+ |f_2(t_1, x_n(\varphi_2(t_1)))| \int_0^{t_1} |u(t_2, s, x_n(\varphi_1(s))) - u(t_1, s, x_n(\varphi_1(s)))| ds \\
&\leq |f_2(t_2, x_n(\varphi_2(t_2))) - f_2(t_1, x_n(\varphi_2(t_1)))| \int_0^{t_2} k(t_2, s) ds \\
&+ [a_2(t_1) + b_2|x_n(\varphi_2(t_1))|] \int_{t_1}^{t_2} k(t_2, s) ds + [a_2(t_1) + b_2|x_n(\varphi_2(t_1))|] \\
&\times \int_0^{t_1} |u(t_2, s, x_n(\varphi_1(s))) - u(t_1, s, x_n(\varphi_1(s)))| ds.
\end{aligned}$$

Then we have

$$\begin{aligned}
|H_n(t_2) - H_n(t_1)| &\leq \omega^T(f_2, |t_2 - t_1|)T\tilde{k} + [a_2(t_1) + b_2|x_n(\varphi_2(t_1))|](t_2 - t_1)\tilde{k} \\
&+ [a_2(t_1) + b_2|x_n(\varphi_2(t_1))|]T\omega^T(u, |t_2 - t_1|), \tag{4.6}
\end{aligned}$$

where $\omega^T(f_2, \cdot)$ and $\omega^T(u, \cdot)$ denotes the modulus continuity of the functions f_2 and u on the sets $D_\varepsilon \times \mathbb{R}$ and $D_\varepsilon \times D_\varepsilon \times \mathbb{R}$ respectively and $\tilde{k} = \max\{|k(t, s)| : (t, s) \in D_\varepsilon \times [0, T]\}$. The last inequality (4.6) is obtained since $M \subset B_r$.

Taking into account the fact that $\mu(\{x_n\}) \leq \mu(M) = 0$, we infer that the number $t_2 - t_1$ is small enough, then the right hand side of (4.6) tends to zero independently of x_n as $t_2 - t_1$ tends to zero. We have $\{H_n\}$ is equicontinuous in the space $C(D_\varepsilon)$.

Moreover,

$$\begin{aligned}
|H_n(t)| &\leq |f_2(t, x_n(\varphi_2(t)))| \cdot \int_0^t |u(t, s, x_n(\varphi_1(s)))| ds \\
&\leq [|a_1(t)| + b_2|x_n(\varphi_2(t))|] \cdot \int_0^t k(t, s) ds \\
&\leq \tilde{k}T[d_1 + b_2d_2],
\end{aligned}$$

where $|a_1(t)| \leq d_1$, $|x_n(\varphi_2(t))| \leq d_2$ for $t \in D_\varepsilon$. From the above, we have that $\{H_n\}$ is equibounded in the space $C(D_\varepsilon)$.

Next, let us put

$$Y = \sup\{|H_n(t)| : t \in D_\varepsilon, n \in \mathbb{N}\}.$$

Obviously Y is finite in view of the choice of D_ε . Assumption (i) conclude that the function $f_1|_{D_\varepsilon \times [-Y, Y]}$ is uniformly continuous. So $\{B(x_n)\} = \{F_{f_1}H_n + g(t, 0)\}$

is equibounded and equicontinuous in the space $C(D_\varepsilon)$. Hence, by Ascoli-Arzelà theorem [80], we obtain that the sequence $\{B(x_n)\}$ forms a relatively compact set in the space $C(D_\varepsilon)$.

Further observe that the above reasoning does not depend on the choice of ε . Thus we can construct a sequence D_l of closed subsets of the interval $[0, T]$ such that $meas(D_l^c) \rightarrow 0$ as $l \rightarrow \infty$ and such that the sequence $\{B(x_n)\}$ is relatively compact in every space $C(D_l)$. Passing to subsequences if necessary we can assume that $\{B(x_n)\}$ is a Cauchy sequence in each space $C(D_l)$, for $l = 1, 2, \dots$.

In what follows, utilizing the fact that the set $B(M)$ is weakly compact, let us choose a number $\delta > 0$ such that for each closed subset D_δ of the interval $[0, T]$ such that $meas(D_\delta^c) \leq \delta$, we have

$$\int_{D_\delta^c} |Bx(t)| dt \leq \frac{\varepsilon}{4}, \quad (4.7)$$

for any $x \in M$.

Keeping in mind the fact that the sequence $\{Bx_n\}$ is a Cauchy sequence in each space $C(D_l)$ we can choose a natural number l_0 such that $meas(D_{l_0}^c) \leq \delta$ and $meas(D_{l_0}) > 0$, and for arbitrary natural numbers $n, m \geq l_0$ the following inequality holds

$$|(B(x_n))(t) - (B(x_m))(t)| \leq \frac{\varepsilon}{4meas(D_{l_0})} \quad (4.8)$$

for any $t \in D_{l_0}$.

Now use the above facts together with (4.5), (4.7), (4.8) we obtain

$$\begin{aligned} \int_0^\infty |(Bx_n)(t) - (Bx_m)(t)| dt &= \int_T^\infty |(Bx_n)(t) - (Bx_m)(t)| dt + \int_{D_{l_0}} |(Bx_n)(t) \\ &\quad - (Bx_m)(t)| dt + \int_{D_{l_0}^c} |(Bx_n)(t) - (Bx_m)(t)| dt \leq \varepsilon, \end{aligned}$$

which means that $\{B(x_n)\}$ is a Cauchy sequence in the space $L_1(\mathbb{R}^+)$. Hence we conclude that the set $B(M)$ is relatively strongly compact in the space $L_1(\mathbb{R}^+)$.

Step 6. We now can show that:

- (1) From step 4, we obtain that $A(M) + B(M) \subseteq M$, where M is the set constructed in step 3.
- (2) Step 1 allow us to know that A is a contraction mapping.
- (3) By step 5, $B(M)$ is relatively compact and by assumptions (i), (iii) B is continuous.

We can apply Theorem 1.6.7, and have that equation (4.1) has at least one integrable solution in \mathbb{R}^+ . □

4.3 Examples

We need to show an example for which our main result is useful and allow to extend the existing theorems.

Example 4.3.1. Consider the following quadratic integral equation. Let $t \in \mathbb{R}^+$

$$x(t) = \frac{1}{1+t^2} + \frac{1}{4}x\left(\frac{t}{3} + \frac{t^2}{2}\right) + \arctan \left(\left[e^{-t} + \frac{1}{6}x\left(\frac{t}{3} + \frac{t^2}{2}\right) \right] \cdot \int_0^t \frac{t \cos(ts)}{1+(x(s))^2} ds \right)^2. \quad (4.9)$$

It can be seen that equation (4.9) is a particular case of equation (4.1), where

$$g(t, x) = \frac{1}{1+t^2} + \frac{1}{4}x, \quad f_2(t, x) = e^{-t} + \frac{1}{6}x, \quad f_1(t, x) = \arctan x^2 \leq 2x$$

and

$$u(t, s, x) = \frac{t \cos(ts)}{1+(x(s))^2}.$$

Let us note that

$$|u(t, s, x)| \leq t \cos(ts),$$

since $\int_0^t t \cos(ts) ds = \sin t^2$, then $|\int_0^t k(t, s) ds| \leq 1$, which implies that $\|K_0\|_{L^\infty} \leq 1$.

Moreover, given arbitrary $h > 0$ and $|x_2 - x_1| \leq \delta$ we have

$$\begin{aligned} |u(t, s, x_1) - u(t, s, x_2)| &\leq |t \cos(ts)| \left| \frac{x_2^2 - x_1^2}{(1+x_1^2)(1+x_2^2)} \right| \\ &\leq \left| \frac{2th\delta}{(1+x_1^2)(1+x_2^2)} \right|. \end{aligned}$$

In view of Theorem 4.2.1, we can deduce

- g, f_1, f_2 satisfy assumption (i) with $a_1(t) = 0$, $a_2(t) = e^{-t}$, $a_3(t) = \frac{1}{1+t^2}$ and with constants $b_1 = 2$, $b_2 = \frac{1}{6}$ and $b_3 = \frac{1}{4}$.
- Assumptions (ii), (iii) are satisfied,
- $\varphi_1 = t$, $\varphi_2 = (t + \frac{t^2}{2})$, $\varphi_3 = (\frac{t}{2} + \frac{t^2}{2})$ satisfied assumption (iv) with $M_1 = M_2 = 1$, $M_3 = \frac{1}{2}$,
- $p = \frac{1}{2} < 1$, $q = (\frac{1}{2} + \frac{1*2}{6}) = \frac{5}{6} < 1$.

Thus all the assumptions of Theorem 4.2.1 are satisfied so the quadratic functional integral (4.9) has at least one integrable solution in \mathbb{R}^+ .

Chapter 5

On quadratic integral equations in Orlicz spaces

5.1 Introduction

The chapter is devoted to study the following quadratic integral equation

$$x(t) = g(t) + G(x)(t) \cdot \lambda \int_a^b K(t, s) f(s, x(s)) ds. \quad (5.1)$$

We will deal with problems in which either the growth of the function f or the kernel K is not polynomial. This is motivated, for instance, by some mathematical models in physics. An interesting discussion about such a kind of problems can be found in [45] or [120]. The considered thermodynamical problem lead to the integral equation $x(t) + \int_I k(t, s) \exp x(s) ds = 0$ and thus the integral equations with exponential nonlinearities turn out important from an application point of view. Let us also note, that such a kind of problems can be applied for integral equations associated (making use of the Green kernel) for the operator $-\Delta u + \exp u$ on a bounded regular subsets of \mathbb{R}^2 (see [37]) and that the solutions in Orlicz spaces are also sometimes studied in partial differential equations ([36], for instance).

Our theorems allow to consider the cases of integral equations when the kernel function K is more singular than in previously considered cases. Moreover, we are able to consider strongly nonlinear functions f . Both extensions seems to be important from the applications point of view (cf. [36, 37, 45, 120], for instance). Let us note, that our results are motivated by the paper of Cheng and Kozak [45]. We generalize some of their assumptions and we consider more complicated quadratic integral equations.

An operator G is supposed to be continuous on a required space of solutions. The problem is modeled on some quadratic integral equations (G is usually identity operator or the Nemytskii superposition operator), but our approach allows to include

also standard integral equations. For standard integral equations different classes of solutions are considered and this is mainly dependent on growth restrictions for f and K . Similar investigations for quadratic integral equations relate mainly to continuous solutions. The key point is to ensure, that an operator of pointwise multiplication is well-defined and has some compactness properties. We propose an approach to this problem allowing us to consider a wide class of integral equations with solutions in some spaces of discontinuous functions.

We are interested in a different class of solutions than the earlier papers. Such a kind of integral equations was investigated in spaces of continuous or integrable functions. Similar problems are also important when L^p -solutions are checked. The currently considered case is less restrictive and include large class of real problems. Whenever one has to deal with some problems involving strong nonlinearities (of exponential growth, for instance), it is a useful device to look for solutions not in Lebesgue spaces, but in Orlicz spaces.

In the literature, mostly solutions of integral equations are sought in $C[0, 1]$ and $L^p[0, 1]$ with $p > 1$. The results obtained for $L^p[0, 1]$ invariably assume a polynomial growth (in x) on the nonlinearity $f(t, x)$. On the other hand, seeking solutions in other Orlicz spaces will lead to restrictions that are not of polynomial type, and hence will allow us to consider new classes of equations. All very basic types of integral or differential equations were satisfactorily examined (cf. [83, 98, 100, 103, 112]). Some additional properties of solutions (in a simplest case of the Δ_2 -condition) are also investigated (constant-sign solutions, for instance [1, 2]). An interesting discussion about advantages of integral equations in Orlicz spaces can be found in [84, Section 40] (see also [7]).

Nevertheless, for the quadratic integral equations the operators generated by the right-hand side of the equation are more complicated and was not investigated in this case. Let us note, that in this case the methods based on properties of some Banach algebras are usually applied (cf. [12, 34, 89, 90]). This approach seems to be strictly related with continuous or Hölder continuous solutions (the product is an inner operation in the Banach algebra of continuous functions and at the same time is an operator used in the integral equation). Let us note, that this method is dependent on some properties of $C(0, 1)$ and cannot be easily applied to different classes of spaces.

This suggest an operator oriented approach. In a class of Orlicz spaces we consider spaces associated with growth conditions for G and f . For a moment denote by X an Orlicz space of solutions for our problem and by F the Nemytskii superposition operator generated by f . Thus we have $G : X \rightarrow W_1$, $F : X \rightarrow U$ and finally the linear integral operator H with a kernel K is acting from U into

W_2 . The space U is depending on some growth assumptions of f - not necessarily of polynomial type. In a typical case of quadratic problems the spaces W_1 and W_2 are supposed both to be the space of continuous functions and then some properties of this Banach algebra allow to solve the problem. Unfortunately, this is really restrictive assumption. We started to replace this assumption by considering $X = L^1(I)$ and $W_2 = L^\infty(I)$ ([46]).

Here we present a complete theory for such problems. In general, allowing U be an Orlicz space depending on f we consider the triples of Orlicz spaces (not: Banach algebras) for which the pointwise multiplication take a pair of functions from W_1 and W_2 into X .

For a case of L^p -solutions we propose to use a factorization and we will assume the Hammerstein integral operator which is multiplied by the function $G(x)$ has values in the space conjugated to the space of solutions. Such an idea was used by Brézis and Browder for Hammerstein integral equations ([39]) by considering conjugated Lebesgue L^p spaces. We extend this procedure for Orlicz (or: ideal) spaces and for a triple of spaces (two for Hammerstein operator and one more for a multiplication operator). This allows us to prove the existence theorems under much more general conditions than previously considered ones. Our method leads to extensions for both types of results (quadratic and non-quadratic ones).

We concentrate on the property of monotonicity of solutions for the equation (5.9). This notion is broaden to some function spaces and the basic properties of families of such functions are investigated.

Especially, the quadratic integral equation of Chandrasekhar type

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \varphi(s)x(s) ds$$

can be very often encountered in many applications (cf. [11, 12, 31, 26, 40, 43]).

The results of this chapter are divided into a few parts. This is because the proofs are depending on the choice of considered spaces. We stress on the "size" of solution spaces and we try to relate the growth assumptions of functions and the expected space of solutions.

Since for equations of this type an approach via the Schauder fixed point theorem is not useful and the Banach contraction principle is too restrictive in many applications, we prefer to investigate the properties of operators with respect to the topology of convergence in measure. We check this topology on considered Orlicz spaces and then we use the Darbo fixed point theorem for proving main results. To show a detailed theory we need to consider a few different cases (for different classes of Orlicz spaces). The theorems proved by us extend, in particular, that presented in [6, 11, 27, 33] considered in the space $C(I)$ or in Banach algebras (cf. [34]). In the

context of non-quadratic integral equations in Orlicz spaces, which are also covered by our theorems, let us mention the papers [1, 2, 98, 100, 103, 112].

5.2 The case of operators with values in $L^\infty(I)$.

This part will be devoted to present some new results, which are related to our theorems from previous chapters. Nevertheless, we are looking for solutions in some Orlicz spaces instead of Lebesgue ones. Functional dependence is not considered here, but our current approach is presented in such a way to cover non-quadratic integral equations as particular cases. Earlier results for quadratic integral equations cannot be compared with standard ones.

Denote B the operator associated with the right-hand side of the equation (5.2)

$$x(t) = g(t) + \lambda f_1(t, x(t)) \int_a^b K(t, s) f_2(s, x(s)) ds. \quad (5.2)$$

i.e. $B(x) = g + U(x)$, where $U(x)(t) = F_1(x)(t) \cdot A(x)(t)$ such that

$$F_2(x)(t) = f_2(t, x(t)), \quad A = H \circ F_2, \quad H(x)(t) = \lambda \int_I K(t, s) x(s) ds, \quad F_1 = f_1(s, x(s)).$$

By considering different spaces of solutions (i.e. different growth conditions) we need to investigate pair of spaces constituting the domain and the range for the operator U . To facilitate the reading of (technical) assumptions we will consider different pairs of spaces, including most typical ones. We stress on applicability of our results. The general case will be presented in an abstract form in the next Section of this chapter.

Let $F_1 : V_1 \rightarrow W_1$ and $A : V_1 \rightarrow W_2$, where V_1, W_1 and W_2 are some function spaces. Since U stands for the pointwise multiplication operator and the range of U should be in a space of solutions W , we need to investigate pair of spaces (W_1, W_2) such that $x(t) \cdot y(t) \in W$ for all $t \in [a, b]$ and $x \in W_1, y \in W_2$.

We need to consider the case when either W_1 or W_2 is a space of all essentially bounded functions $L_\infty[a, b]$.

5.2.1 The case of $W_1 = L_\infty(I)$.

The case of $W_1 = L_\infty(I)$ is trivial. The unique situation in which the Nemytskii operator takes values in this space is when f_1 is autonomous. Namely, we have:

Lemma 5.2.1. [10, Theorem 3.17] *Let $p \geq 1$. The superposition operator F_1 generated by f_1 maps $L^p(I)$ into $L_\infty(I)$ if and only if*

$$|f(s, u)| \leq a(s) \quad \text{for } u \in \mathbb{R}$$

for some $a \in L_\infty(I)$, in this case, F_1 is always bounded; F_1 is continuous if and only if F_1 is constant (i.e. f_1 does not depend on u).

Let us briefly discuss the case of Orlicz spaces. We have the following lemma

Lemma 5.2.2. *Let $L_M(I)$ be an Orlicz space. Then F_1 generated by a Carathéodory function f_1 is continuous from $L_M(I)$ into $L_\infty(I)$ if and only if F_1 is constant (i.e. f_1 does not depend on u).*

Proof. Since we consider a finite measure, an arbitrary function x from L_M can be obtained as a pointwise limit of a sequence of simple functions, a fortiori this sequence is convergent in measure. Simple functions belong to E_M and then L_M is a quasi-regular ideal space (cf. [10, Theorem 2.8, Theorem 3.17]).

On the other hand the space $L_\infty(I)$ is completely irregular i.e. the regular part of the space consists only $\{\theta\}$. Since both spaces are ideal, by [10, Theorem 2.8] F_1 is continuous if and only if F_1 is constant. \square

Note that a domain for F_1 cannot be arbitrarily small. Some assumptions on the domain of superposition operators are always expected (the identity on $L_\infty(I)$ is continuous, for instance).

Thus, this is the standard non-quadratic case and can be easily reduced to known theorems (cf. [100, 103, 112] for most interesting results).

5.2.2 The case of $W_2 = L_\infty(I)$.

We will consider now the new case when $W_2 = L_\infty(I)$. It is more interesting and we stress on growth conditions for both f_1 and f_2 allowing to have nonlinear growth. This is closely related to the choice of an intermediate space for A i.e. the domain of H . It can be made because of the assumptions on f_2 or the kernel K depending on the practical problem described by our equation. Thus the choice of this space allows to consider more general growth conditions for f_2 or K . Theorem given below is formulated for arbitrary Lebesgue space $L^p(I)$ ($p \geq 1$) taken as an intermediate space.

Theorem 5.2.1. *Let $\frac{1}{p} + \frac{1}{q} = 1$ and assume, that φ is an N -function and that:*

(C1) $g \in E_\varphi(I)$ is nondecreasing a.e. on I ,

(C2) $f_i : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions and $f_i(t, x)$ is assumed to be nondecreasing with respect to both variable t and x separately, for $i=1, 2$.

(C3) $\|F_1(x)\|_\varphi \leq G_1(\|x\|_\varphi)$, and $\|F_2(x)\|_p \leq G_2(\|x\|_\varphi)$, for all $x \in E_\varphi(I)$, where G_i are positive, continuous and nondecreasing for $i = 1, 2$. Moreover there exist $\gamma > 0$ such that $G_1(u) \leq \gamma|u|$ for $|u| \leq 1$. Assume, that the superposition operator F_1 generated by f_1 is acting from $E_\varphi(I)$ into itself.

(C4) Assume that the function K is measurable in (t, s) and assume that the linear integral operator H with kernel $K(t, s)$ maps $L^p(I)$ into $L_\infty(I)$, $s \mapsto |K(\cdot, s)| \in L^q(I)$, $k(t) = |K(t, \cdot)| \in L_\infty(I)$ and H is continuous with a norm

$$\|K_0\|_{L_\infty} := \operatorname{esssup}_{t \in I} \left(\int_I |K(t, s)|^q ds \right)^{\frac{1}{q}}.$$

(K1) $\int_I K(t_1, s) ds \geq \int_I K(t_2, s) ds$ for $t_1, t_2 \in I$ with $t_1 < t_2$.

Assume, that there exists a positive number $r \leq 1$ such that

$$\|g\|_\varphi + \lambda \|K_0\|_{L_\infty} G_1(r) \cdot G_2(r) \leq r. \quad (5.3)$$

Then there exists a number $\rho > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$ and for every $g \in E_\varphi$, there exists a solution $x \in E_\varphi(I)$ of (5.2) which is a.e. nondecreasing on I .

Proof. We need to divide the proof into a few steps.

- I. The operator B is well-defined from $L_\varphi(I)$ into itself. In particular, for operator H we need to check the properties of this operator.
- II. We will construct an invariant ball B_r for B in $L_\varphi(I)$.
- III. We construct a subset Q_r of this ball which contains a.e. nondecreasing functions and we investigate the properties Q_r .
- IV. We check the continuity and monotonicity properties of B in Q_r , so

$$B : Q_r \rightarrow Q_r.$$

- V. We prove that B is a contraction with respect to a measure of noncompactness.
- VI. We use the Darbo fixed point theorem to find a solution in Q_r .

I. First of all observe that by the assumptions (C2) and (C3) and Lemma 1.3.5 implies that F_2 is continuous mappings from $L_\varphi(I)$ into L^p . Note, that L^p can be treated as an Orlicz space L_{M_p} for $M_p(x) = \frac{|x|^p}{p}$. It is clear, that this space satisfies the Δ_2 -condition and therefore it is a regular space. Recall, that by Lemma 1.3.2 F_2

is sequentially continuous with respect to convergence in measure and continuous by Lemma 1.3.5. For the operator F_1 the situation is described in Remark 1.3.2.

By assumption (C4) the operator A maps $E_\varphi(I)$ into $L_\infty(I)$ and is continuous. F_1 maps $E_\varphi(I)$ into itself continuously (thanks to Assumption (C3) and Lemma 1.3.5). Thus U is a continuous mapping from $E_\varphi(I)$ into itself. Finally, Assumption (C1) permits us to deduce that the operator B maps $E_\varphi(I)$ into itself and is continuous.

II. If φ satisfies the Δ_2 -condition then F_1 is bounded in $L_\varphi(I)$. In a general case we need restrict this operator to the (regular) space $E_\varphi(I)$. We will prove the boundedness of the operator B on this space, namely we will construct the invariant set for this operator.

Let Γ be a set of all positive numbers λ such that

$$\|g\|_\varphi + \lambda \|K_0\|_{L_\infty} G_1(r) \cdot G_2(r) \leq r.$$

Put $\rho = \min\left(\sup \Gamma, \frac{1}{\gamma \|K_0\|_{L_\infty} G_2(1)}\right)$ and fix λ with $|\lambda| < \rho$.

Let r be a positive number $r \leq 1$ such that

$$\|g\|_\varphi + |\lambda| \|K_0\|_{L_\infty} \cdot G_1(r) \cdot G_2(r) \leq r. \quad (5.4)$$

Since we provided a continuity of operators F_1 and B only in $E_\varphi(I)$, as a domain for the operator B we will consider the ball $B_r(E_\varphi(I))$ in this space.

Recall, that $L_\varphi(I)$ is an ideal space, $F_1(x) \in L_\varphi(I)$ and $A(x) \in L_\infty(I)$, so $U(x) \in L_\varphi(I)$ and $\|U(x)\|_\varphi \leq \|F_1(x)\|_\varphi \|A(x)\|_\infty$.

We have

$$\begin{aligned} \|B(x)\|_\varphi &\leq \|g\|_\varphi + \|U(x)\|_\varphi \\ &= \|g\|_\varphi + \|F_1(x) \cdot A(x)\|_\varphi \\ &\leq \|g\|_\varphi + |\lambda| \cdot \|F_1(x)\|_\varphi \cdot \left\| \int_I K(t, s) |f_2(s, x(s))| ds \right\|_\infty. \end{aligned}$$

But by (C4) H is continuous and then the norm of $H(x)$ is estimated by (cf. [78, Theorem XI.1.6])

$$\|H(x)\|_\infty \leq |\lambda| \|K_0\|_{L_\infty} \|x\|_p.$$

Thus for $x \in E_\varphi(I)$ with $\|x\|_\varphi \leq r$

$$\begin{aligned} \|B(x)\|_\varphi &\leq \|g\|_\varphi + \|F_1(x)\|_\varphi \cdot |\lambda| \|K_0\|_{L_\infty} \cdot \|F_2(x)\|_p \\ &\leq \|g\|_\varphi + |\lambda| \|K_0\|_{L_\infty} G_1(\|x\|_\varphi) \cdot G_2(\|x\|_\varphi) \\ &\leq \|g\|_\varphi + |\lambda| \|K_0\|_{L_\infty} G_1(r) \cdot G_2(r) \leq r. \end{aligned}$$

Then we have $B : B_r(E_\varphi(I)) \rightarrow B_r(E_\varphi(I))$. Moreover, B is continuous on $B_r(E_\varphi(I))$ (see the part I of the proof).

III. Let Q_r stand for the subset of $B_r(E_\varphi(I))$ consisting of all functions which are a.e. nondecreasing on I . Similarly as claimed in [17] this set is nonempty, bounded (by r), convex (direct calculation from the definition) and closed in $L_\varphi(I)$.

To prove the last property, let (y_n) be a sequence of elements in Q_r convergent in $L_\varphi(I)$ to y . Then the sequence is convergent in measure and as a consequence of the Vitali convergence theorem for Orlicz spaces and of the characterization of convergence in measure (the Riesz theorem) we obtain the existence of a subsequence (y_{n_k}) of (y_n) which converges to y almost uniformly on I (cf. [100]). Moreover, y is still nondecreasing a.e. on I which means that $y \in Q_r$ and so the set Q_r is closed. Now, in view of Lemma 1.4.1 the set Q_r is compact in measure.

IV. Now, we will show, that B preserve the monotonicity of functions. Take $x \in Q_r$, then x is a.e. nondecreasing on I and consequently F_1 and F_2 are also of the same type in virtue of the assumption (C2) and Theorem 1.4.2. Further, $A(x) = H \circ F_2(x)$ is a.e. nondecreasing on I (thanks for the assumption (K1)). Since the pointwise product of a.e. monotone functions is still of the same type, the operator U is a.e. nondecreasing on I . Moreover, the assumption (C1) permits us to deduce that $B(x)(t) = g(t) + U(x)(t)$ is also a.e. nondecreasing on I . This fact, together with the assertion that $B : B_r(E_\varphi(I)) \rightarrow B_r(E_\varphi(I))$ gives us that B is also a self-mapping of the set Q_r .

V. We will prove that B is a contraction with respect to a measure of strong noncompactness. Since

$$\begin{aligned} |A(x)(t)| &= \left| \lambda \int_I K(t, s) f_2(s, x(s)) ds \right| \leq |\lambda| \int_I |K(t, s)| |f_2(s, x(s))| ds \\ &\leq |\lambda| \cdot \int_I |K(t, s)| |F_2(x)(s)| ds \\ &\leq |\lambda| \cdot \|K(t, \cdot)\|_q \cdot |F_2(x)|_p \\ &\leq |\lambda| \cdot \|K(t, \cdot)\|_q \cdot G_2(\|x\|_\varphi) \end{aligned}$$

for a.e. $t \in I$, whence $\|A(x)\chi_D\|_{L^\infty} \leq |\lambda| \|K_0\|_{L^\infty} \cdot G_2(\|x\|_\varphi)$.

Assume that X is a nonempty subset of Q_r and let the fixed constant $\varepsilon > 0$ be arbitrary. Then for an arbitrary $x \in X$ and for a set $D \subset I$, $\text{meas}(D) \leq \varepsilon$ and $t \in D$ we have $F_1(x)(t) = F_1(x\chi_D)(t)$ and for $t \notin D$ $F_1(x)(t) \cdot \chi_D(t) = 0$ and $F_1(x\chi_D)(t) = f_1(t, 0)$. Whence

$$\begin{aligned} |F_1(x)(t)\chi_D(t)| &= |f_1(t, x(t))| = |f_1(t, x(t)) - f_1(t, 0) + f_1(t, 0)| \\ &\leq |f_1(t, x(t)) - f_1(t, 0)| + |f_1(t, 0)| \\ &\leq |F_1(x\chi_D)(t) - F_1(0)(t)\chi_D(t)| + |F_1(0)(t)\chi_D(t)|. \end{aligned}$$

Thus for all $t \in I$ we have

$$|F_1(x)(t)\chi_D(t)| \leq |F_1(x\chi_D)(t) - F_1(0)(t)\chi_D(t)| + |F_1(0)(t)\chi_D(t)|.$$

Finally

$$\|F_1(x) \cdot \chi_D\|_\varphi \leq \|F_1(x\chi_D) - F_1(0)\|_\varphi + \|F_1(0) \cdot \chi_D\|_\varphi \leq \|F_1(x\chi_D)\|_\varphi + 2\|F_1(0) \cdot \chi_D\|_\varphi.$$

For the operator B we get the following estimation

$$\begin{aligned} \|B(x) \cdot \chi_D\|_\varphi &\leq \|g \cdot \chi_D\|_\varphi + \|U(x) \cdot \chi_D\|_\varphi \\ &= \|g \cdot \chi_D\|_\varphi + \|F_1(x) \cdot A(x) \cdot \chi_D\|_\varphi \\ &= \|g \cdot \chi_D\|_\varphi + \|(F_1(x) \cdot \chi_D) \cdot A(x)\|_\varphi \\ &\leq \|g \cdot \chi_D\|_\varphi + \|F_1(x) \cdot \chi_D\|_\varphi \cdot \|A(x)\|_{L_\infty} \\ &\leq \|g \cdot \chi_D\|_\varphi + (\|F_1(x\chi_D)\|_\varphi + 2\|F_1(0) \cdot \chi_D\|_\varphi) |\lambda| \|K_0\|_{L_\infty} G_2(\|x\|_\varphi) \\ &\leq \|g \cdot \chi_D\|_\varphi + (G_1(\|x\chi_D\|_\varphi) + 2\|F_1(0) \cdot \chi_D\|_\varphi) |\lambda| \|K_0\|_{L_\infty} G_2(\|x\|_\varphi) \\ &\leq \|g \cdot \chi_D\|_\varphi + (\gamma \cdot \|x\chi_D\|_\varphi + 2\|F_1(0) \cdot \chi_D\|_\varphi) |\lambda| \|K_0\|_{L_\infty} G_2(\|x\|_\varphi). \end{aligned}$$

Since $g \in E_\varphi$ and $F_1(0) \in E_\varphi$,

$$\lim_{\varepsilon \rightarrow 0} \sup_{meas(D) \leq \varepsilon} \|g \cdot \chi_D\|_\varphi = 0,$$

$$\lim_{\varepsilon \rightarrow 0} \sup_{meas(D) \leq \varepsilon} \|F_1(0) \cdot \chi_D\|_\varphi = 0.$$

Thus by definition of $c(x)$ and by taking the supremum over all $x \in X$ and all measurable subsets D with $meas(D) \leq \varepsilon$ we get

$$c(B(X)) \leq \gamma |\lambda| \|K_0\|_{L_\infty} \cdot G_2(r) \cdot c(X).$$

Since $X \subset Q_r$ is a nonempty, bounded and compact in measure subset of an ideal regular space $E_\varphi(I)$, we can use Proposition 5.5.1 and get

$$\beta_H(B(X)) \leq \gamma \cdot |\lambda| \cdot \|K_0\|_{L_\infty} \cdot G_2(1) \cdot \beta_H(X).$$

The constant in the above inequality is smaller than 1, so the properties of the operator B and assumption (C4) allow us to apply the Darbo Fixed Point Theorem 1.6.5, which completes the proof. \square

Remark 5.2.1. In this chapter we consider continuous linear operators of the form H acting on $L^p(I)$ with values in the space $L_\infty(I)$. It is known, that the continuity

property is depending on the kernel K . In a particular case of Riemann-Liouville fractional integral operators i.e. for $K(t, s) = \frac{1}{\Gamma(\alpha)}(t - s)^{\alpha-1}\chi_{[a,t]}(s)$

$$J_\alpha x(s) = \frac{1}{\Gamma(\alpha)} \int_a^s (s - t)^{\alpha-1} x(t) dt \quad , \quad s \in [a, b]$$

is not continuous from $L^p(I)$ into $L_\infty(I)$ when $p = \frac{1}{\alpha}$ ([70, Remark 4.1.2]), but continuous for $p < \frac{1}{\alpha}$ (cf. also [7]).

Remark 5.2.2. Let us add some comments about assumption (C3). Our acting condition from Lemma 1.3.3 has the form $|f_2(t, x)|^p \leq a_2(t) + b_2\varphi(x)$. This pointwise estimation implies our assumption. Indeed, this implies that for $x \in L_\varphi(I)$ with $\|x\|_\varphi \leq 1$

$$\begin{aligned} \int_I |f_2(t, x(t))|^p dt &\leq \|a_2\|_1 + b_2 \int_I \varphi(x(t)) dt \\ \|F_2(x)\|_p^p &\leq \|a_2\|_1 + b_2 \|x\|_\varphi \end{aligned}$$

and then $\|F_2(x)\|_p \leq (\|a_2\|_1 + b_2 \|x\|_\varphi)^{\frac{1}{p}}$, so our assumption holds true for a special case of $G_2(t) = (\|a_2\|_1 + b_2 \cdot t)^{\frac{1}{p}}$ which is used in [1], for instance. The growth restrictions for G_1 result from necessary and sufficient conditions for continuity of F_1 (see [10]) and since $L_\infty(I)$ is completely irregular i.e. θ is the only singleton with absolutely continuous norm.

The boundedness of the Nemytskii operators on "small" balls was firstly proved by Shrigin in [110] and then used to investigate the Hammerstein integral equations in Orlicz spaces by Vainberg and Shrigin.

5.3 The existence of L_p -solution.

Let us present a special case of L^p -solutions. This will be still more general result than the earlier ones.

Theorem 5.3.1. *Assume, that $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

(C1') $g \in L^p(I)$ is nondecreasing a.e. on I ,

(C2') $f_i : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions and $f_i(t, x)$ is assumed to be nondecreasing with respect to both variable t and x separately, for $i=1, 2$.

(C3') $|f_1(t, x)| \leq a_1(t) + b_1|x|$, and $|f_2(t, x)| \leq a_2(t) + b_2|x|^{\frac{p}{q}}$, for all $t \in I$ and $x \in \mathbb{R}$, where $a_1 \in L^p(I)$, $a_2 \in L^q(I)$ and some constants $b_i \geq 0$ for $i = 1, 2$.

(C4') Assume that the function K is measurable in (t, s) and that the linear integral operator K_0 with kernel $K(\cdot, \cdot)$ maps $L^q(I)$ into $L_\infty(I)$,

$$\operatorname{ess\,sup}_{t \in [a, b]} \left(\int_a^b |K(t, s)|^q ds \right)^{\frac{1}{q}} < \infty$$

and is continuous.

(K1) $\int_a^b K(t_1, s) ds \geq \int_a^b K(t_2, s) ds$ for $t_1, t_2 \in [a, b]$ with $t_1 < t_2$.

Assume, that there exists a positive number $r < 1$ such that

$$\|g\|_p + |\lambda| \|K_0\|_{L_\infty} \cdot [\|a_1\|_{L^1} + b_1 \cdot r] [\|a_2\|_{L^1} + b_2 \cdot r^{p/q}] \leq r \quad (5.5)$$

and choose λ in such a way to get $\|K_0\|_{L_\infty} < \frac{1}{|\lambda| \cdot b_1 b_2 \cdot r^{\frac{p}{q}}}$.

Then there exists a number $\rho > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$ and for every $g \in L^p(I)$, there exists a solution $x \in L^p(I)$ of (5.2) which is a.e. nondecreasing on I .

Proof. I. First of all observe that by assumptions (C2'), (C3') and Theorem 1.3.1 we have that F_1 maps $L_p(I)$ into itself and F_2 maps $L_p(I)$ into $L_q(I)$ continuously. By assumption (C4') we can deduce that A maps $L_p(I)$ into $L_\infty(I)$ and is continuous. From the Hölder inequality the operator U maps $L_p(I)$ into itself continuously. Finally, assumption (C1') permits us to deduce that B maps $L_p(I)$ into itself and is continuous.

To prove step **II**. Put

$$\rho = \frac{1}{b_1 b_2 \|K'_0\|_{L_\infty} \|a_1\|_{L^p} \cdot \|a_2\|_{L^q}}.$$

Fix λ with $|\lambda| < \rho$.

Choose a positive number R in such a way that

$$\|g\|_{L_p} + |\lambda| \cdot [\|a_1\|_{L_p} + b_1 \cdot R] \cdot \|K'_0\|_{L_\infty} [\|a_2\|_{L_q} + b_2 \cdot R^{\frac{p}{q}}] \leq R. \quad (5.6)$$

As a domain for the operator B we will consider the ball $B_R(L_p(I))$.

$$\begin{aligned} \|B(x)\|_{L_p} &\leq \|g\|_{L_p} + \|Ux\|_{L_p} \\ &= \|g\|_{L_p} + \|F_1(x) \cdot A(x)\|_{L_p} \\ &\leq \|g\|_{L_p} + \|F_1(x)\|_{L_p} \cdot \|A(x)\|_{L_\infty} \\ &\leq \|g\|_{L_p} + |\lambda| \|f_1(t, x)\|_{L_p} \left\| \int_I K(t, s) |f_2(s, x(s))| ds \right\|_{L_\infty} \end{aligned}$$

$$\begin{aligned}
&\leq \|g\|_{L_p} + |\lambda| \cdot \|a_1 + b_1|x(s)|\|_{L_p} \\
&\times \left\| \int_I K(t, s) [a_2(s) + b_2|x(s)|^{\frac{p}{q}}] ds \right\|_{L_\infty} \\
&\leq \|g\|_{L_p} + |\lambda| [\|a_1\|_{L_p} + b_1\|x\|_{L_p}] \|K'_0\|_{L_\infty} [\|a_2\|_{L_q} + b_2\|x\|_{L_p}^{\frac{p}{q}}],
\end{aligned}$$

where

$$\|x^{\frac{p}{q}}\|_{L_q} = \|x\|_{L_p}^{\frac{p}{q}}.$$

$$\|B(x)\|_{L_p} \leq \|g\|_{L_p} + |\lambda| \cdot [\|a_1\|_{L_p} + b_1 \cdot R] \cdot \|K'_0\|_{L_\infty} [\|a_2\|_{L_q} + b_2 \cdot R^{\frac{p}{q}}] \leq R.$$

Then we have $B : B_R(L_p(I)) \rightarrow B_R(L_p(I))$. Moreover, B is continuous on $B_R(L_p(I))$ (see the part I of the proof).

Step **III** and Step **IV** are similar as in Theorem 5.2.1.

V. We will prove that B is a contraction with respect to a measure of strong noncompactness. Assume that X is a nonempty subset of Q_R and let the fixed constant $\epsilon > 0$ be arbitrary. Then for an arbitrary $x \in X$ and for a set $D \subset I$, $\text{meas}(D) \leq \epsilon$ we obtain

$$\begin{aligned}
\|B(x) \cdot \chi_D\|_{L_p} &\leq \|g \cdot \chi_D\|_{L_p} + \|Ux \cdot \chi_D\|_{L_p} \\
&\leq \|g \cdot \chi_D\|_{L_p} + \|F_1(x) \cdot \chi_D\|_{L_p} \cdot \|A(x) \cdot \chi_D\|_{L_\infty} \\
&= \|g \cdot \chi_D\|_{L_p} + |\lambda| \cdot \|f_1(t, x) \cdot \chi_D\|_{L_p} \left\| \int_D K(t, s) f_2(s, x(s)) ds \right\|_{L_\infty} \\
&\leq \|g \cdot \chi_D\|_{L_p} + |\lambda| \cdot \| [a_1(s) + b_1|x(s)|] \cdot \chi_D \|_{L_p} \\
&\times \left\| \int_D K(t, s) [a_2(s) + b_2|x(s)|^{\frac{p}{q}}] ds \right\|_{L_\infty} \\
&\leq \|g \cdot \chi_D\|_{L_p} + |\lambda| \cdot \| [\|a_1 \cdot \chi_D\|_{L_p} + b_1\|x \cdot \chi_D\|_{L_p}] \\
&\times \left\| \int_D K(t, s) a_2(s) ds + b_2 \int_I K(t, s) |x(s)|^{\frac{p}{q}} ds \right\|_{L_\infty} \\
&\leq \|g \cdot \chi_D\|_{L_p} + |\lambda| \cdot \| [\|a_1 \cdot \chi_D\|_{L_p} + b_1\|x \cdot \chi_D\|_{L_p}] \\
&\times [\|K'_0\|_{L_\infty} \|a_2 \cdot \chi_D\|_{L_p} + b_2 \|K'_0\|_{L_\infty} \|x(s)\|_{L_p}^{\frac{p}{q}}] \\
&\leq \|g \cdot \chi_D\|_{L_p} + |\lambda| \cdot \| [\|a_1 \cdot \chi_D\|_{L_p} + b_1\|x \cdot \chi_D\|_{L_p}] \\
&\times \|K'_0\|_{L_\infty} [\|a_2 \cdot \chi_D\|_{L_p} + b_2 R^{\frac{p}{q}}].
\end{aligned}$$

Since $g \in L_p$, $a_1 \in L_p$ and $a_2 \in L^q$, then we have

$$\lim_{\epsilon \rightarrow 0} \left\{ \sup_{\text{mes } D \leq \epsilon} [\sup_{x \in X} \{ \|g \cdot \chi_D\|_{L_p} \}] \right\} = 0, \quad \lim_{\epsilon \rightarrow 0} \left\{ \sup_{\text{mes } D \leq \epsilon} [\sup_{x \in X} \{ \|a_1 \cdot \chi_D\|_{L_p} \}] \right\} = 0$$

and

$$\lim_{\varepsilon \rightarrow 0} \left\{ \sup_{mes D \leq \varepsilon} [\sup_{x \in X} \{ \|a_2 \cdot \chi_D\|_{L_q} \}] \right\} = 0.$$

Thus by definition of $c(x)$,

$$c(B(X)) \leq b_1 b_2 |\lambda| \cdot \|K'_0\|_{L_\infty} R^{\frac{p}{q}} \cdot c(X).$$

Since $X \subset Q_r$ is a nonempty, bounded and compact in measure subset of an ideal regular space L_p , we can use Proposition 5.5.1 and get

$$\beta_H(B(X)) \leq b_1 b_2 |\lambda| \cdot \|K'_0\|_{L_\infty} R^{\frac{p}{q}} \cdot \beta_H(X).$$

The inequality obtained above together with the properties of the operator B and the set Q_R established before and assumption (C'4) allow us to apply the Darbo fixed point theorem (see [23]), which completes the proof. \square

5.3.1 Remarks and examples.

We need to stress on some aspects of our results. First of all we can observe, that our solutions are not necessarily continuous as in almost all previously investigated cases. In particular, we need not to assume, that the Hammerstein operator transforms the space $C(I)$ into itself. Our treatment allows to solve problems when function f_1 doesn't have sublinear growth. In this case it is sufficient to take a little bit "nicer" kernel K , but this is still weaker assumption than in the case of continuous solutions.

We need to stress, that quadratic equations are also strictly related to problems of the type

$$\left(\frac{x(t) - g(t)}{f_1(t, x)} \right)' = f_2(t, x(t)) \quad x(0) = 0,$$

where $f_1 : I \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$.

It is well-known that under typical assumptions this problem is equivalent to the integral equation

$$x(t) = g(t) + \left(f_1(t, x(t)) \cdot \left(\int_a^b \chi_{[0,t]}(s) f_2(s, x(s)) ds - \frac{g(0)}{f_1(0, 0)} \right) \right).$$

Nevertheless, when we are looking for continuous solutions for integral equation, we obtain classical solutions for differential one i.e. x is continuously differentiable. This seems to be too restrictive in many applications. In our case we investigate Caratéodory solutions for the Cauchy problem.

Another typical example is a boundary value problem ($f_1 : I \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$, f_2 and g satisfy some regularity conditions)

$$\left(\frac{x(t) - g(t)}{f_1(t, x)} \right)'' = f_2(t, x(t)) \quad x(0) = 0 \quad , \quad x'(0) = 0. \quad (5.7)$$

In this case we can consider an equivalent integral problem with a kernel G (an appropriate Green function)

$$x(t) = g(t) + \left(f_1(t, x(t)) \cdot \left(\int_a^b G(t, s) f_2(s, x(s)) ds - \frac{g(0)}{f_1(0, 0)} \right) + t \cdot \frac{g'(0)}{f_1(0, 0)} - t \cdot \frac{g(0) \cdot \frac{\partial f_1}{\partial t}(0, 0)}{(f_1(0, 0))^2} \right), \quad (5.8)$$

where $f_1 : I \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$.

In this case when we are looking for continuous solutions for the quadratic integral equations, the solutions for the above differential problems are classical. Our approach allows to investigate weaker types of solutions (in Orlicz-Sobolev spaces).

5.4 A general case of Orlicz spaces.

Here we will present a detailed theory of quadratic integral equations of the form:

$$x(t) = g(t) + G(x)(t) \cdot \lambda \int_a^b K(t, s) f(s, x(s)) ds \quad (5.9)$$

in Orlicz spaces.

Denote by B the operator associated with the right-hand side of the equation (5.9) i.e. $B(x) = g + U(x)$, where

$$U(x)(t) = G(x)(t) \cdot \lambda \int_a^b K(t, s) f(s, x(s)) ds.$$

Thus $B = g + G \cdot A = g + G \cdot H \circ F$, where $H(x)(t) = \lambda \int_a^b K(t, s) x(s) ds$ and $F = f(t, x(t))$.

We will try to choose the domains of operators defined above in such a way to obtain the existence of solutions in a desired Orlicz space $L_\varphi(I)$. We stress on conditions allowing us to consider strongly nonlinear operators.

Let us note, that our assumptions on G are referred to the case of quadratic integral equations (i.e. $G(x)(t) = q(t) \cdot x(t)$).

We need to distinguish two different cases. This allow us to obtain more general growth conditions on f (cf. [100, 103, 112, 98] for non-quadratic equations). In every case we need to describe some assumptions on "intermediate" spaces being the images of $L_\varphi(I)$ for G and F ($L_{\varphi_1}(I)$ and $L_M(I)$, respectively) and the range for H (i.e. $L_{\varphi_2}(I)$). This approach is based on a classical (non-quadratic) case as in [100, 103, 112]) and seems to be important in view of optimality of assumptions for every considered case.

5.4.1 The case of N satisfying the Δ' -condition.

Theorem 5.4.1. *Assume, that $\varphi, \varphi_1, \varphi_2$ are N -functions and that M and N are complementary N -functions. Moreover, put the following set of assumptions:*

(N1) *there exists a constant $k_1 > 0$ such that for every $u \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}(I)$ we have $\|uw\|_{\varphi} \leq k_1 \|u\|_{\varphi_1} \|w\|_{\varphi_2}$,*

(C1) *$g \in E_{\varphi}(I)$ is nondecreasing a.e. on I ,*

(C2) *$f : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions and $f(t, x)$ is assumed to be nondecreasing with respect to both variable t and x separately,*

(C3) *$|f(t, x)| \leq b(t) + R(|x|)$ for $t \in I$ and $x \in \mathbb{R}$, where $b \in E_N(I)$ and R is nonnegative, nondecreasing, continuous function defined on \mathbb{R}^+ ,*

(C4) *Let N satisfies the Δ' -condition and suppose that there exist $\omega, \gamma, u_0 \geq 0$ for which*

$$N(\omega(R(u))) \leq \gamma \varphi_2(u) \leq \gamma M(u) \quad \text{for } u \geq u_0,$$

(G1) *$G : L_{\varphi}(I) \rightarrow L_{\varphi_1}(I)$, takes continuously $E_{\varphi}(I)$ into $E_{\varphi_1}(I)$ and there exists a constant $G_0 > 0$ such that $\|G(x)\|_{\varphi_1} \leq G_0 \|x\|_{\varphi}$ and that G takes the set of all a.e. nondecreasing functions into itself,*

(K1) *$s \rightarrow K(t, s) \in L_M(I)$ for a.e. $t \in I$,*

(K2) *$K \in E_M(I^2)$ and $t \rightarrow K(t, s) \in E_{\varphi_2}(I)$ for a.e. $s \in I$ with*

$$\|K\|_M < \frac{1}{2k_1 \cdot |\lambda| \cdot G_0 \cdot R(1)}$$

,

(K3) *$\int_a^b K(t_1, s) ds \geq \int_a^b K(t_2, s) ds$ for $t_1, t_2 \in [a, b]$ with $t_1 < t_2$.*

Then there exists a number $\rho > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$ and for all g with $\|g\|_{\varphi} < 1$ there exists a solution $x \in E_{\varphi}(I)$ of (5.9) which is a.e. nondecreasing on I .

Proof. We need to divide the proof into a few steps.

- I. The operator B is well-defined from $L_{\varphi}(I)$ into itself and continuous on a domain depending on the considered case.
- II. We will construct an invariant ball B_r for B in $L_{\varphi}(I)$.
- III. We construct a subset Q_r of this ball which contains a.e. nondecreasing functions and we investigate the properties Q_r .

- IV. We check the continuity and monotonicity properties of B in Q_r , so $U : Q_r \rightarrow Q_r$.
- V. We prove that B is a contraction with respect to a measure of noncompactness.
- VI. We use the Darbo fixed point theorem to find a solution in Q_r .

I. First of all observe that under the assumptions (C2) and (C3) by Lemma 1.3.3 the superposition operator F acts from $L_\varphi(I)$ to $L_N(I)$.

In this case we will prove, that U is a continuous mapping from the unit ball in $E_\varphi(I)$ into the space $E_\varphi(I)$.

Let us recall, that $x \in E_\varphi(I)$ iff for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x\chi_T\|_\varphi < \varepsilon$ for every measurable subset T of I with the Lebesgue measure smaller than δ (i.e. x has absolutely continuous norm). First, let us observe that in view of Lemma 1.3.8, it is sufficient to check this property for the operator $A = H \circ F$.

Since N is an N -function satisfying Δ' -condition and by (C3), we are able to use [83, Lemma 19.1]. From this there exists a constant C (not depending on the kernel) such that for any measurable subset T of I and $x \in L_\varphi(I)$, $\|x\|_\varphi \leq 1$ we have

$$\|A(x)\chi_T\|_{\varphi_2} \leq C\|K\chi_{T \times I}\|_M. \quad (5.10)$$

Now, by the Hölder inequality and the assumption (C2) we get

$$|K(t, s)f(s, x(s))| \leq \|K(t, s)\| \cdot |f(s, x(s))| \leq \|K(t, s)\| \cdot (b(s) + R(|x(s)|))$$

for $t, s \in I$. Put $k(t) = 2\|K(t, \cdot)\|_M$ for $t \in I$. As $K \in E_M(I^2)$ this function is integrable on I . By the assumptions (K1) and (K2) about the kernel K of the operator H (cf. [112]) we obtain that

$$\|A(x)(t)\| \leq k(t) \cdot (\|b\|_N + \|R(|x(\cdot)|)\|_N) \text{ for a.e. } t \in I.$$

Whence for arbitrary measurable subset T of I and $x \in E_\varphi(I)$

$$\|A(x)\chi_T\|_{\varphi_2} \leq \|k\chi_T\|_{\varphi_2} \cdot (\|b\|_N + \|R(|x(\cdot)|)\|_N).$$

Finally if t is such that $K(t, \cdot) \in E_M(I)$ and $x \in E_\varphi(I)$ we have

$$\int_T \|K(t, s)f(s, x(s))\| ds \leq 2\|K(t, \cdot)\chi_T\|_M \cdot (\|b\|_N + \|R(|x(\cdot)|)\|_N) \text{ for a.e. } t \in I.$$

From this it follows that A maps $B_1(E_\varphi(I))$ into $E_{\varphi_2}(I)$.

We are in a position to prove the continuity of A as a mapping from the unit ball $B_1(E_\varphi(I))$ into the space $E_{\varphi_2}(I)$. Let $x_n, x_0 \in B_1(E_\varphi(I))$ be such that $\|x_n - x_0\|_\varphi \rightarrow 0$ as n tends to ∞ . Suppose, contrary to our claim, that A is not continuous and

the $\|A(x_n) - A(x_0)\|_{\varphi_2}$ does not converge to zero. Then there exists $\varepsilon > 0$ and a subsequence (x_{n_k}) such that

$$\|A(x_{n_k}) - A(x_0)\|_{\varphi_2} > \varepsilon \quad \text{for } k = 1, 2, \dots \quad (5.11)$$

and the subsequence is a.e. convergent to x_0 . Since (x_n) is a subset of the ball the sequence $(\int_a^b \varphi(|x_n(t)|)dt)$ is bounded. As the space $E_{\varphi_2}(I)$ is regular the balls are norm-closed in $L_1(I)$ so the sequence $(\int_a^b |x_n(t)|dt)$ is also bounded.

Moreover, by (C3) and (C4) there exist $\omega, \gamma, u_0 > 0$, s.t. (cf. [83, p. 196])

$$\begin{aligned} \|R(|x(\cdot)|)\|_N &= \frac{1}{\omega} \|\omega R(|x(\cdot)|)\|_N \\ &\leq \frac{1}{\omega} \inf_{r>0} \left\{ \int N(\omega R(|x(t)|)/r) dt \leq 1 \right\} \\ &\leq \frac{1}{\omega} \left(1 + \int_a^b N(\omega R(|x(t)|)) dt \right) \\ &\leq \frac{1}{\omega} \left(1 + N(\omega R(u_0)) \cdot (b-a) + \gamma \int_a^b \varphi_2(|x(t)|) dt \right), \end{aligned}$$

whenever $x \in L_\varphi(I)$ with $\|x\|_\varphi \leq 1$.

Thus

$$\begin{aligned} \int_T \|K(t, s)f(s, x_n(s))\| ds &\leq 2\|K(t, \cdot)\chi_T\|_M \cdot (\|b\|_N + \|R(|x_n(\cdot)|)\|_N) \\ &\leq 2\|K(t, \cdot)\chi_T\|_M \cdot (\|b\|_N \\ &\quad + \frac{1}{\omega} \left[1 + N(\omega R(u_0)) \cdot (b-a) + \gamma \int_a^b \varphi_2(|x_n(t)|) dt \right]) \end{aligned}$$

and then the sequence $(\|K(t, s)f(s, x_n(s))\|)$ is equiintegrable on I for a.e. $t \in I$. By the continuity of $f(t, \cdot)$ we get $\lim_{k \rightarrow \infty} K(t, s)f(s, x_{n_k}(s)) = K(t, s)f(s, x_0(s))$ for a.e. $s \in I$. Now, applying the Vitali convergence theorem we obtain that

$$\lim_{k \rightarrow \infty} A(x_{n_k})(t) = A(x_0)(t) \quad \text{for a.e. } t \in I.$$

But the equation (5.10) implies that $A(x_{n_k})$ is a subset of $E_{\varphi_2}(I)$ and then

$$\lim_{k \rightarrow \infty} A(x_{n_k})(t) = A(x_0)(t)$$

which contradicts the inequality (5.11). Since A is continuous between indicated spaces, By our assumption (G1) the operator G is continuous from $B_1(E_\varphi(I))$ into $E_{\varphi_1}(I)$ and then by (N1) the operator U has the same property and then U is a continuous mapping from $B_1(E_\varphi(I))$ into the space $E_\varphi(I)$. Finally, by the assumption (C1) B maps $B_1(E_\varphi(I))$ into $E_\varphi(I)$ continuously.

II. We will prove the boundedness of the operator U , namely we will construct the invariant ball for this operator. By B we will denote the right-hand side of our integral equation i.e. $B = g + U$.

Set $r \leq 1$ and let

$$\rho = \frac{1 - \|g\|_\varphi}{2k_1 \cdot C \cdot G_0 \cdot \|K\|_M}.$$

Let x be an arbitrary element from $B_1(E_\varphi(I))$. Then by using the above consideration, the assumption (C3), the formula (5.10) and Proposition 1.3.8 for sufficiently small λ (i.e. $|\lambda| < \rho$) we obtain

$$\begin{aligned} \|B(x)\|_\varphi &\leq \|g\|_\varphi + \|Ux\|_\varphi \\ &= \|g\|_\varphi + \|G(x) \cdot A(x)\|_\varphi \\ &\leq \|g\|_\varphi + k_1 \|G(x)\|_{\varphi_1} \cdot \|A(x)\|_{\varphi_2} \\ &= \|g\|_\varphi + k_1 |\lambda| \cdot G_0 \cdot \|x\|_\varphi \cdot \left\| \int_a^b K(\cdot, s) f(s, x(s)) ds \right\|_{\varphi_2} \\ &\leq \|g\|_\varphi + 2k_1 \cdot |\lambda| \cdot C \cdot G_0 \cdot \|x\|_\varphi \cdot \|K\|_M \\ &\leq \|g\|_\varphi + 2k_1 \cdot |\lambda| \cdot C \cdot G_0 \cdot r \cdot \|K\|_M \\ &\leq \|g\|_\varphi + 2k_1 \cdot \rho \cdot C \cdot G_0 \cdot \|K\|_M \leq r \end{aligned}$$

whenever $\|x\|_\varphi \leq r$.

Then we have $B : B_r(E_\varphi(I)) \rightarrow B_r(E_\varphi(I))$. Moreover, B is continuous on $B_r(E_\varphi(I))$ (see the part I of the proof).

III. Let Q_r stand for the subset of $B_r(E_\varphi(I))$ consisting of all functions which are a.e. nondecreasing on I . Similarly as claimed in [17] this set is nonempty, bounded (by r) and convex (direct calculation from the definition). It is also a closed set in $L_\varphi(I)$.

Indeed, let (y_n) be a sequence of elements in Q_r convergent in $L_\varphi(I)$ to y . Then the sequence is convergent in measure and as a consequence of the Vitali convergence theorem for Orlicz spaces and of the characterization of convergence in measure (the Riesz theorem) we obtain the existence of a subsequence (y_{n_k}) of (y_n) which converges to y almost uniformly on I (cf. [100]). Moreover, y is still nondecreasing a.e. on I which means that $y \in Q_r$ and so the set Q_r is closed. Now, in view of Lemma 1.4.1 the set Q_r is compact in measure.

IV. Now, we will show, that B preserve the monotonicity of functions. Take $x \in Q_r$, then x is a.e. nondecreasing on I and consequently $F(x)$ is also of the same type in virtue of the assumption (C2) and Lemma 1.4.2. Further, $A(x) = H \circ F(x)$ is a.e. nondecreasing on I thanks for the assumption (K3). Since the pointwise product of a.e. monotone functions is still of the same type and by (G1), the operator U is a.e. nondecreasing on I .

Moreover, the assumption (C1) permits us to deduce that $Bx(t) = g(t) + U(x)(t)$ is also a.e. nondecreasing on I . This fact, together with the assertion that $B : B_r(E_\varphi(I)) \rightarrow B_r(E_\varphi(I))$ gives us that B is also a self-mapping of the set Q_r . From the above considerations it follows that B maps continuously Q_r into Q_r .

V. We will prove that B is a contraction with respect to the measure of noncompactness μ . Assume that X is a nonempty subset of Q_r and let the fixed constant > 0 be arbitrary. Then for an arbitrary $x \in X$ and for a set $D \subset I$, $\text{meas}(D) \leq \varepsilon$ we obtain

$$\begin{aligned}
\|B(x) \cdot \chi_D\|_\varphi &\leq \|g\chi_D\|_\varphi + \|U(x) \cdot \chi_D\|_\varphi \\
&= \|g\chi_D\|_\varphi + \|G(x) \cdot A(x)\chi_D\|_\varphi \\
&\leq \|g\chi_D\|_\varphi + k_1 \|G(x)\chi_D\|_{\varphi_1} \cdot \|A(x) \cdot \chi_D\|_{\varphi_2} \\
&= \|g\chi_D\|_\varphi + k_1 \cdot |\lambda| \cdot \|G(x)\chi_D\|_{\varphi_1} \cdot \left\| \int_D K(\cdot, s) f(s, x(s)) ds \right\|_{\varphi_2} \\
&\leq \|g\chi_D\|_\varphi + k_1 |\lambda| G_0 \|x\chi_D\|_\varphi \int_D |K(\cdot, s)| (b(s) + R(|x(s)|)) ds \|_{\varphi_2} \\
&\leq \|g\chi_D\|_\varphi + k_1 \cdot |\lambda| \cdot G_0 \cdot \|x\chi_D\|_\varphi \cdot 2 \|K\|_M \| [b\chi_D + R(r)] \|_N \\
&\leq \|g\chi_D\|_\varphi + 2k_1 \cdot |\lambda| \cdot G_0 \cdot \|x\chi_D\|_\varphi \cdot \|K\|_M [\|b\chi_D\|_N + R(1)].
\end{aligned}$$

Hence, taking into account that $g \in E_\varphi$, $b \in E_N$

$$\lim_{\varepsilon \rightarrow 0} \left\{ \sup_{\text{meas } D \leq \varepsilon} [\sup_{x \in X} \{ \|g\chi_D\|_\varphi \}] \right\} = 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \left\{ \sup_{\text{meas } D \leq \varepsilon} [\sup_{x \in X} \{ \|b\chi_D\|_N \}] \right\} = 0.$$

Thus by definition of $c(x)$ and by taking the supremum over all $x \in X$ and all measurable subsets D with $\text{meas}(D) \leq \varepsilon$ we get

$$c(B(X)) \leq 2k_1 \cdot |\lambda| \cdot G_0 \cdot \|K\|_M \cdot R(1) \cdot c(X).$$

Since $X \subset Q_r$ is a nonempty, bounded and compact in measure subset of an ideal regular space E_φ , we can use Proposition 5.5.1 and get

$$\beta_H(B(X)) \leq 2k_1 \cdot |\lambda| \cdot G_0 \cdot \|K\|_M \cdot R(1) \cdot \beta_H(X).$$

The inequality obtained above together with the properties of the operator B and the set Q_r established before and the inequality from the Assumption (K2) allow us to apply the Darbo Fixed Point Theorem 1.6.5, which completes the proof. \square

5.4.2 The case of N satisfying the Δ_3 -condition.

Let us consider the case of N -functions with the growth essentially more rapid than a polynomial. In fact, we will consider N -functions satisfying Δ_3 -condition. This is very large and important class, especially from an application point of view (cf. [36, 37, 106, 120]). An extensive description of this class can be found in [106, Section 2.5]. Recall, that an N -function M determines the properties of the Orlicz space $L_M(I)$ and then the less restrictive rate of the growth of this function implies the "worse" properties of the space. By ϑ we will denote the norm of the identity operator from $L_\varphi(I)$ into $L^1(I)$ i.e. $\sup\{\|x\|_1 : x \in B_1(L_\varphi(I))\}$. For the discussion about the existence of φ which satisfies our conditions see [84, p. 61].

Theorem 5.4.2. *Assume, that $\varphi, \varphi_1, \varphi_2$ are N -functions and that M and N are complementary N -functions and that (N1), (C1), (C2), (C3), (G1), (K1) and (K3) hold true. Moreover, put the following assumptions:*

- (C5) 1. N satisfies the Δ_3 -condition,
 2. $K \in E_M(I^2)$ and $t \rightarrow K(t, s) \in E_{\varphi_2}(I)$ for a.e. $s \in I$,
 3. There exist $\beta, u_0 > 0$ such that

$$R(u) \leq \beta \frac{M(u)}{u}, \quad \text{for } u \geq u_0,$$

(K4) φ_2 is an N -function satisfying

$$\iint_{I^2} \varphi_2(M(|K(t, s)|)) dt ds < \infty$$

and

$$2k_1 \cdot (2 + (b - a)(1 + \varphi_2(1))) \cdot |\lambda| \cdot G_0 \cdot \|K\|_{\varphi_2 \circ M} \cdot R(r_0) < 1,$$

where

$$r_0 = \frac{1}{\vartheta} \left[\frac{\omega}{2|\lambda| \cdot k_1 \cdot G_0 \cdot (2 + (b - a)(1 + \varphi_2(1))) \cdot \|K\|_{\varphi_2 \circ M}} - \|b\|_N \right].$$

Then there exist a number $\rho > 0$ and a number $\varpi > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$ and for all $g \in E_\varphi(I)$ with $\|g\|_\varphi < \varpi$ there exists a solution $x \in E_\varphi(I)$ of (5.9) which is a.e. nondecreasing on I .

Proof. We will indicate only the points of the proof if they differ from the previous case.

I. In this case the operator B can be considered as continuous when acting on the whole $E_\varphi(I)$.

By [84, Lemma 15.1 and Theorem 19.2] and the assumption (K4):

$$\|A(x)\chi_T\|_{\varphi_2} \leq 2 \cdot (2 + (b-a)(1 + \varphi_2(1))) \cdot \|K \cdot \chi_{T \times I}\|_{\varphi_2 \circ M} (\|b\|_N + \|R(|x(\cdot)|)\|_N) \quad (5.12)$$

for arbitrary $x \in L_\varphi(I)$ and arbitrary measurable subset T of I .

Let us note, that the assumption (C5) 3. implies that there exist constants $\omega, u_0 > 0$ and $\eta > 1$ such that $N(\omega R(u)) \leq \eta u$ for $u \geq u_0$.

Thus for $x \in L_\varphi(I)$

$$\begin{aligned} \|R(|x(\cdot)|)\|_N &\leq \frac{1}{\omega} \left(1 + \int_I N(\omega R(|x(s)|)) ds \right) \\ &\leq \frac{1}{\omega} \left(1 + \eta u_0 (b-a) + \eta \int_I |x(s)| ds \right). \end{aligned}$$

The remaining estimations can be derived as in the first main theorem and then we obtain, that $A : E_\varphi(I) \rightarrow E_{\varphi_2}(I)$, so by the properties of G we get $B : E_\varphi(I) \rightarrow E_\varphi(I)$.

II. Put

$$\rho = \frac{1}{2 \cdot k_1 \cdot G_0 \cdot (2 + (b-a)(1 + \varphi_2(1))) \cdot \|K\|_{\varphi_2 \circ M} \cdot [\|b\|_N + \frac{1}{\omega} (1 + \eta u_0 (b-a))]}.$$

Fix λ with $|\lambda| < \rho$.

Choose a positive number r in such a way that

$$\begin{aligned} \|g\|_\varphi + 2rk_1 \cdot (2 + (b-a)(1 + \varphi_2(1))) \cdot G_0 \cdot |\lambda| \cdot \|K\|_{\varphi_2 \circ M} (\|b\|_N \\ + \frac{1}{\omega} (1 + \eta u_0 (b-a) + \eta \vartheta r)) \leq r. \end{aligned} \quad (5.13)$$

As a domain for the operator B we will consider the ball $B_r(E_\varphi(I))$.

Let us remark, that the above inequality is of the form $a + (b + vr)cr \leq r$ with $a, b, c, v > 0$. Then $vc > 0$ and if we assume that $bc - 1 < 0$ and that the discriminant is positive, then Viète's formulas imply that the quadratic equation has two positive solutions $r_1 < r_2$ for sufficiently small λ . By the definition of ρ it is clear, that our assumptions guarantee the above requirements, so there exists a positive number r satisfying this inequality.

Put $C = (2 + (b-a)(1 + \varphi(1)))$. Let us note, in view of the above considerations, that the assumption about the discriminant which implies the existence of solutions for the above problem is of the form:

$$\begin{aligned} [\|b\|_N + \frac{1}{\omega} (1 + \eta u_0 (b-a)) - \frac{1}{2|\lambda| \cdot k_1 \cdot G_0 \cdot C \cdot \|K\|_{\varphi_2 \circ M}}]^2 \\ \times |\lambda| \cdot k_1 \cdot G_0 \cdot C \cdot \|K\|_{\varphi_2 \circ M} > \frac{2\|g\|_\varphi \eta \vartheta}{\omega} \end{aligned}$$

i.e.

$$\varpi = \left[\|b\|_N + \frac{1}{\omega} (1 + \eta u_0(b-a)) - \frac{1}{2|\lambda| \cdot k_1 \cdot G_0 \cdot C \cdot \|K\|_{\varphi_2 \circ M}} \right]^2 \\ \times \frac{|\lambda| \cdot k_1 \cdot G_0 \cdot C \cdot \omega \|K\|_{\varphi_2 \circ M}}{2\eta\vartheta}.$$

For $x \in B_r(E_\varphi(I))$ we have the following estimation:

$$\begin{aligned} \|B(x)\|_\varphi &\leq \|g\|_\varphi + \|Ux\|_\varphi \\ &= \|g\|_\varphi + \|G(x) \cdot A(x)\|_\varphi \\ &\leq \|g\|_\varphi + k_1 \|G(x)\|_{\varphi_1} \cdot \|A(x)\|_{\varphi_2} \\ &= \|g\|_\varphi + k_1 |\lambda| \|G(x)\|_{\varphi_1} \cdot \left\| \int_a^b K(\cdot, s) f(s, x(s)) ds \right\|_{\varphi_2} \\ &\leq \|g\|_\varphi + 2k_1 \cdot C \cdot G_0 \cdot |\lambda| \cdot \|x\|_\varphi \|K\|_{\varphi_2 \circ M} \left[\|b\|_N \right. \\ &\quad \left. + \frac{1}{\omega} (1 + N(\omega R(u_0))) \cdot (b-a) + \eta \int_I |x(s)| ds \right] \\ &\leq \|g\|_\varphi + 2k_1 \cdot C \cdot G_0 \cdot |\lambda| \cdot \|x\|_\varphi \|K\|_{\varphi_2 \circ M} \left[\|b\|_N \right. \\ &\quad \left. + \frac{1}{\omega} (1 + N(\omega R(u_0))) \cdot (b-a) + \eta \|x\|_1 \right] \\ &\leq \|g\|_\varphi + 2k_1 \cdot C \cdot G_0 \cdot |\lambda| \cdot \|x\|_\varphi \|K\|_{\varphi_2 \circ M} \left[\|b\|_N \right. \\ &\quad \left. + \frac{1}{\omega} (1 + N(\omega R(u_0))) \cdot (b-a) + \eta\vartheta \|x\|_\varphi \right] \\ &= \|g\|_\varphi + 2rk_1 \cdot C \cdot G_0 \cdot |\lambda| \cdot \|K\|_{\varphi_2 \circ M} \left(\|b\|_N \right. \\ &\quad \left. + \frac{1}{\omega} (1 + \eta u_0(b-a) + \eta\vartheta r) \right) \leq r. \end{aligned}$$

Then $B : B_r(E_\varphi(I)) \rightarrow B_r(E_\varphi(I))$.

Note, that the parts **III.** and **IV.** of the previous proof are similar to those from the first theorem, so we omit the details.

V. We will prove that B is a contraction with respect to a measure of noncompactness. Assume that X is a nonempty subset of Q_r and let the fixed constant $\varepsilon > 0$ be arbitrary. Then for an arbitrary $x \in X$ and for a set $D \subset I$, $\text{meas}(D) \leq \varepsilon$

we obtain

$$\begin{aligned}
\|B(x) \cdot \chi_D\|_\varphi &\leq \|g\chi_D\|_\varphi + \|U(x) \cdot \chi_D\|_\varphi \\
&= \|g\chi_D\|_\varphi + \|G(x) \cdot A(x)\chi_D\|_\varphi \\
&\leq \|g\chi_D\|_\varphi + k_1\|G(x)\chi_D\|_{\varphi_1} \cdot \|A(x) \cdot \chi_D\|_{\varphi_2} \\
&= \|g\chi_D\|_\varphi + k_1 \cdot |\lambda| \cdot G_0 \cdot \|x\chi_D\|_\varphi \cdot \left\| \int_D K(\cdot, s)f(s, x(s)) ds \right\|_{\varphi_2} \\
&\leq \|g\chi_D\|_\varphi + k_1|\lambda|G_0\|x\chi_D\|_\varphi \left\| \int_D |K(\cdot, s)|(b(s) + R(|x(s)|)) ds \right\|_{\varphi_2} \\
&\leq \|g\chi_D\|_\varphi + k_1 \cdot |\lambda| \cdot G_0 \cdot \|x\chi_D\|_\varphi \cdot \left(\left\| \int_D |K(\cdot, s)|b(s) ds \right\|_{\varphi_2} \right. \\
&\quad \left. + \left\| \int_D |K(\cdot, s)|R(|x(s)|) ds \right\|_{\varphi_2} \right) \\
&\leq \|g\chi_D\|_\varphi + 2 \cdot C \cdot k_1 \cdot G_0 \cdot |\lambda| \cdot \|x\chi_D\|_\varphi \cdot \vartheta \cdot \|K\|_{\varphi_2 \circ M} \|b\chi_D\|_N \\
&\quad + 2 \cdot C \cdot k_1 \cdot G_0 \cdot \|x\chi_D\|_\varphi \cdot \left\| \int_D |K(\cdot, s)|R(|x(s)|) ds \right\|_{\varphi_2} \\
&\leq \|g\chi_D\|_\varphi + 2Ck_1G_0|\lambda| \cdot \|x\chi_D\|_\varphi \|K\|_{\varphi_2 \circ M} [\|b\chi_D\|_N + R(r)] \\
&\leq \|g\chi_D\|_\varphi + 2Ck_1G_0|\lambda| \cdot \|x\chi_D\|_\varphi \|K\|_{\varphi_2 \circ M} [\|b\chi_D\|_N + R(r_0)],
\end{aligned}$$

where

$$r_0 = \frac{1}{\vartheta} \left[\frac{\omega}{2|\lambda| \cdot k_1 \cdot G_0 \cdot (2 + (b - a)(1 + \varphi_2)) \cdot \|K\|_{\varphi_2 \circ M}} - \|b\|_N \right].$$

Let us note, that r_0 is an upper bound for solutions of (5.13).

Similarly as in the previous theorem we get

$$\beta_H(B(X)) \leq 2 \cdot k_1 \cdot C \cdot G_0 \cdot |\lambda| \|K\|_{\varphi_2 \circ M} \cdot R(r_0) \cdot \beta_H(X).$$

The inequality obtained above together with the properties of the operator B and the set Q_r established before and then the assumption (K4) allow us to apply the Theorem 1.6.5, which completes the proof. \square

We need to stress on some aspects of our results. First of all we can observe, that our solutions are not necessarily continuous as in previously investigated cases. In particular, we need not to assume, that the Hammerstein operator transforms the space $C(I)$ into itself. For the examples and conditions related to Hammerstein operators in Orlicz spaces we refer the readers to [106, Chapter VI.6.1., Corollary 6 and Example 7].

We have two more information about the set of solutions: it is included in $E_\varphi(I)$ and in view of Theorem 1.6.5 it can be proved, that this set is compact as a subset of $L_\varphi(I)$.

5.4.3 Remarks on classes of solutions.

There is one more interesting question related to the case of continuous solutions. This is the question if we are able to put in our results the same Orlicz space (as in the case of $C(I)$). In other words, the case of Banach-Orlicz algebras. Note, that the presented case in this Chapter seems to be more general we need to mention, that as claimed by Kalton [77] this property is true for $\varphi(x) = x(1+x)^{-1}$ or $\varphi(x) = \log_+ x$. These spaces are not of big interest in the theory of integral equations, then let us present some remarks on operators satisfying our assumptions.

Let X, Y be ideal spaces. A superposition operator $F : X \rightarrow Y$ is called improving (cf. remark 1.3.3). The applications of such operators are based on the observation that large classes of linear integral operators

$$Hy(t) = \lambda \int_D k(t, s)y(s)ds,$$

although not being compact, map sets with equiabsolutely continuous norms into precompact sets. In contrast to the classical (non-quadratic) case, for quadratic integral equations even such a nice assumption is not sufficient for using the Schauder fixed point theorem.

Moreover, we assume in our main theorem, that G maps sets with equiabsolutely continuous norms into the same family, but we don't need to assume, that it is an improving operator. Conversely, any improving operator G can be considered in our results. Let us note, that for operators from Lebesgue spaces $L_p(I)$ into $L_r(I)$ (i.e. Orlicz spaces with $p(x) = x^p$ and $r(x) = x^r$, respectively), the characterization of improving operators is known ([123]): a superposition operator $F : L_p(I) \rightarrow L_r(I)$ is improving if and only if there exists a continuous and even function M satisfying $\lim_{u \rightarrow \infty} \frac{M(u)}{u} = \infty$ and such that $G(x)(t) = M(f(t, x(t)))$ is also an operator from $L_p(I)$ into $L_r(I)$ (for an appropriate growth condition of f see [123]).

The aspect of applicability of our results deal also with the technique of Orlicz spaces for partial differential equations, so for an appropriate class of integral equations. In this context one can consider more singular equations than in a classical case. Motivated by previously considered equations (see [36, 37, 45, 120] or [100, 112]) we extend this method to the case of quadratic integral equations.

It should be recalled that our method of the proof can be also adapted to classical equations considered in [83, 36, 45, 120]. For more information we refer the readers to the Chapter IX "Nonlinear PDEs and Orlicz spaces" in [105].

Finally, let us remark, that our results can be applied also for Lebesgue spaces $L_p(I)$ ($p \geq 1$) (cf. [89, 90]). As mentioned above this class of spaces is also included into the class of Orlicz spaces. But even in this case we allow for f or K to be strongly

nonlinear. The simplest case is that when $F : L_1(I) \rightarrow L_N(I)$, $H : L_N(I) \rightarrow L_p(I)$, $G : L_1(I) \rightarrow L_q(I)$ ($\frac{1}{p} + \frac{1}{q} = 1$). Thus we will have integrable solutions (in $L_1(I)$), but f or K can be strongly nonlinear. Of course, strong nonlinearity of one of them implies that we need to consider the weak one for another (cf. [83, Chapter IV.19]).

Let us present an example of such spaces. By N_1, N_2 we denote complementary functions for M_1, M_2 , respectively. Put $M_1(u) = \exp |u| - |u| - 1$ and $M_2(u) = \frac{u^2}{2} = N_2(u)$. Note, that M_1 satisfies the Δ_3 -condition. In this case $N_1(u) = (1 + |u|) \cdot \ln(1 + |u|) - |u|$. If we define an N -function either as $\Psi(u) = M_2[N_1(u)]$ or $\Psi(u) = N_1[M_2(u)]$, then by choosing arbitrary kernel K from the space $L_\Psi(I)$ we are able to apply [83, Theorem 15.4]. Thus $H : L_{M_1}(I) \rightarrow L_{M_2}(I)$ is continuous and we may apply our result (Theorem 5.4.2) for operators $G : L_1(I) \rightarrow L_q(I)$ and $F : L_1(I) \rightarrow L_{M_1}(I)$ (with natural growth conditions, see Lemma 1.3.3).

Let us also to pay attention to the particular case of our problem with $G(x) = a(t)x(t)$:

$$x(t) = g(t) + \lambda a(t) \cdot x(t) \int_0^1 K(t, s) f(s, x(s)) ds.$$

Since we are motivated by some study on quadratic integral equations, this is of our particular interest. Note, that a full description for acting and continuity conditions for $G(x) = a(t)x(t)$ can be found in [83, Theorem 18.2].

5.5 Conclusions and fixed point theorems.

If we are looking for the proofs of our main results, in contrast to earlier theorems we stress on some properties of spaces rather on continuity of solutions and the properties of this particular space $C(I)$. We will present a general approach for differential and integral problems by presenting a new fixed point theorem specialized to quadratic equations.

For completeness, we need to recall some necessary facts. In this section some properties of function spaces play a major role. We need to consider the triples of spaces with the following property: for a triple of spaces E, E_1, E_2 there exists a constant k such that for arbitrary $x \in E_1$ and $y \in E_2$ a product (pointwise multiplication) $x \cdot y \in E$ and

$$\|x \cdot y\|_E \leq k \cdot \|x\|_{E_1} \cdot \|y\|_{E_2}.$$

Let us recall some special cases. Most known is the case of Banach algebras i.e. the space of continuous functions. In this case $E = E_1 = E_2 = C(I, X)$ ($k = 1$). Moreover, some subalgebras of this space can be interesting. If we try to consider "bigger" spaces we need to go outside the class of Banach algebras.

For discontinuous functions, let us recall the Hölder inequality for Lebesgue spaces: $\|x \cdot y\|_{L^1} \leq \|x\|_{L^p} \cdot \|y\|_{L^q}$ whenever $\frac{1}{p} + \frac{1}{q} = 1$. Thus, a triple (L^1, L^p, L^q) is good enough.

Third important example (and most important) is for Orlicz spaces (for definitions see [83, 92], for instance). Generally speaking, the product of two functions $x, y \in L_M(I)$ is not in $L_M(I)$. However, if x and y belongs to some particular Orlicz spaces, then the product $x \cdot y$ belong to a third Orlicz space. Let us note, that one can find two functions belonging to Orlicz spaces: $u \in L_U(I)$ and $v \in L_V(I)$ such that the product uv does not belong to any Orlicz space (this product is not integrable). Nevertheless, Lemma 1.3.8 give us an interesting characterization for such a triple of spaces. An interesting discussion about necessary and sufficient conditions for product operators can be found in [83, 92]. Note, that since $L^p = L_M$ for $M(t) = \frac{t^p}{p}$ the case of Lebesgue spaces L^p is included in the mentioned Lemma.

Finally we have a special case for $E_2 = L^\infty$ and some function spaces for which ($E = E_1$) $\|x \cdot y\|_E \leq \|x\|_E \cdot \|y\|_{L^\infty}$. The class of spaces with this property is known as preideal* spaces (cf. [119, p. 66] or [118]) or Köthe function spaces. Although this case seems to be general it has one weakness from our point of view: the measure of noncompactness in L^∞ seems to be inapplicable and we do not discuss it in this Thesis.

It is possible to check this property for a given triple of spaces. An open question is if is possible to characterize all such spaces?

Recall, that for any $\varepsilon > 0$, let c be a measure of equiintegrability of the set X in a ideal space E (introduced in [9], cf. also [119, Definition 3.9], [67, 66]):

$$c(X) = \limsup_{\varepsilon \rightarrow 0} \sup_{\text{meas} D \leq \varepsilon} \sup_{x \in X} \|x \cdot \chi_D\|_E,$$

where χ_D denotes the characteristic function of D . To distinguish between measures of noncompactness μ (or c) in different spaces we will indicate an appropriate space as an index i.e. $\mu_E, c_{E_1}, \mu_{E_2}$ etc.

The following theorem clarify the connections between the two coefficients in E .

Proposition 5.5.1. ([66, Theorem 1]) *Let X be a nonempty, bounded and compact in measure subset of an ideal regular space E . Then*

$$\beta_H(X) = c(X).$$

As a consequence, we obtain that bounded sets which are additionally compact in measure are compact in E iff they are equiintegrable in this space (i.e. have equiabsolutely continuous norms, in particular when X is a subset for a regular part of E).

In contrast to the case considered in [25] we will need the following property:

Lemma 5.5.1. *Assume, that for a triple of regular ideal function spaces E, E_1, E_2 there exists a constant k such that for arbitrary $x \in E_1, y \in E_2$ and $t \in I$ a product $x \cdot y \in E$ and $\|x \cdot y\|_E \leq k \cdot \|x\|_{E_1} \cdot \|y\|_{E_2}$. Then for any set $X \subset E_1, Y \subset E_2$ we have $X \cdot Y \subset E$ and*

$$c_E(X \cdot Y) \leq k \cdot c_{E_1}(X) \cdot c_{E_2}(Y),$$

where c_V stands for a measure of equiintegrability in the space V for $V = E, E_1$ or E_2 , respectively.

Proof. By the properties of spaces we obtain, that $X \cdot Y \subset E$. Take arbitrary $x \in X$ and $y \in Y$ and arbitrary measurable subset D of I . Then

$$\|(x \cdot y) \cdot \chi_D\|_E \leq k \cdot \|x \cdot \chi_D\|_{E_1} \cdot \|y \cdot \chi_D\|_{E_2}.$$

Then by the properties of supremum

$$\sup_{x \in X} \sup_{y \in Y} \|(x \cdot y) \cdot \chi_D\|_E \leq k \cdot \sup_{x \in X} \sup_{y \in Y} \|x \cdot \chi_D\|_{E_1} \cdot \|y \cdot \chi_D\|_{E_2}$$

and by the property of $\limsup_{\text{meas}(D) \rightarrow 0}$

$$\limsup_{\text{meas}(D) \rightarrow 0} \sup_{x \in X} \sup_{y \in Y} \|(x \cdot y) \cdot \chi_D\|_E \leq k \cdot \limsup_{\text{meas}(D) \rightarrow 0} \sup_{x \in X} \|x \cdot \chi_D\|_{E_1} \cdot \limsup_{\text{meas}(D) \rightarrow 0} \sup_{y \in Y} \|y \cdot \chi_D\|_{E_2},$$

which ends the proof. \square

The last property is obvious but useful:

Lemma 5.5.2. *Let E be a regular ideal space. For any bounded subset X of E we have $c(X) \leq \|X\|_E$.*

5.5.1 A fixed point theorem.

Since we are interested on a fixed points of some product operators we will assume, that our operators have values in some intermediate spaces and then the product will again turn the values into the original space.

First, let us apply our approach for the most applicable theorem of this type. Consider an arbitrary (in the sense of Definition 1.5.1) measure of noncompactness μ in $C(I, E)$. An interesting fixed point theorem in Banach algebras was proved by Banaś and Lecko (cf. [25]). Let us consider different spaces of continuous functions with a suitable choice of measures of noncompactness μ_E on $E = C(I)$, μ_{E_1} on E_1 and μ_{E_2} on E_2 . By using our approach we are able to present the following extension of the mentioned theorem:

Theorem 5.5.1. *Let E, E_1, E_2 be regular ideal function spaces. Assume that T is nonempty, bounded, closed, and convex subset of the Banach space E , and the operators $A : E \rightarrow E_1$ and $B : E \rightarrow E_2$. Moreover, assume:*

1. (A1) *A transform continuously the set T into $T_1 \subset E_1$ and $A(T)$ is bounded in E_1 ,*
2. (A2) *there exists a constant $k_1 > 0$ such that A satisfies an inequality:*

$$\mu_E(A(U)) \leq k_1 \cdot \mu_{E_1}(U)$$

for arbitrary bounded subset U of E ,

3. (B1) *B transform continuously the set T into $T_2 \subset E_2$ and $B(T)$ is bounded in E_2 ,*
4. (B2) *there exists a constant $k_2 > 0$ such that B satisfies an inequality:*

$$\mu_E(B(U)) \leq k_2 \cdot \mu_{E_2}(U)$$

for arbitrary bounded subset U of E ,

5. (E1) *for a triple of spaces E, E_1, E_2 there exists a constant k such that for arbitrary $x \in E_1, y \in E_2$ and $t \in I$ a product $x \cdot y \in E$ and $\|x \cdot y\|_E \leq k \cdot \|x\|_{E_1} \cdot \|y\|_{E_2}$,*
6. (E2) *for every $x \in T_1$ and $y \in T_2$ one has $x \cdot y \in T$,*
7. (C) $\|A(T)\|_{E_1} \cdot k_2 + \|B(T)\|_{E_2} \cdot k_1 < 1$.

Then there exists at least one fixed point for the operator $K = A \cdot B$ in the set T and that the set of all fixed points of K belongs to the kernel $\ker \mu_E$.

This theorem was proved by Banaś in a special case of Banach algebras $E = E_1 = E_2 = C(I, \mathbb{R})$ ($k = 1$) (cf. also Dhage and Kumpulainen [53], for instance).

We do not require, that the values of all operators are from the same space, but by using a property of considered spaces we are able to repeat the proof, so we omit the details. We will present a proof for a more general case. In the first theorem the Ascoli criterion of compactness in spaces of continuous functions simplify the proof, because the convergence of sequences is directly related with pointwise convergence.

Now, we will consider the case of functions spaces without such a nice property. We will consider some subspaces of a space of $L^0(I)$ of measurable functions, bigger than $C(I)$. This allows us to apply the fixed point theorem for the problems with discontinuous solutions. This proof will be based on different compactness criterion (the Dunford-Pettis theorem and the Erzakova theorem).

Theorem 5.5.2. *Assume that T is nonempty, bounded, closed, convex and compact in measure subset of a regular ideal function space E , and the operators $A : E \rightarrow E_1$ and $B : E \rightarrow E_2$. Put the following set of assumptions:*

1. (A1) *A transform continuously the set T into $T_1 \subset E_1$ and $A(T)$ is bounded in E_1 ,*
2. (A2) *there exists a constant $k_1 > 0$ such that A satisfies an inequality:*

$$c(A(U)) \leq k_1 \cdot c_{E_1}(U)$$

for arbitrary bounded subset U of E ,

3. (B1) *B transform continuously the set T into $T_2 \subset E_2$ and $B(T)$ is bounded in E_2 ,*
4. (B2) *there exists a constant $k_2 > 0$ such that B satisfies an inequality:*

$$c(B(U)) \leq k_2 \cdot c_{E_2}(U)$$

for arbitrary bounded subset U of E ,

5. (E1) *for a triple of regular ideal spaces E, E_1, E_2 there exists a constant k such that for arbitrary $x \in E_1$ and $y \in E_2$ a product $x \cdot y \in E$ and $\|x \cdot y\|_E \leq k \cdot \|x\|_{E_1} \cdot \|y\|_{E_2}$,*
6. (E2) *for every $x \in T_1$ and $y \in T_2$ one has $x \cdot y \in T$,*
7. (C1) $k \cdot k_1 \cdot \|B(T)\|_{E_2} < 1$,
8. (C2) $k \cdot \|A(T)\|_{E_1} \cdot k_2 < 1$.

Assume, that (A1), (B1), (E1), (E2) and either (A2) and (C1) or (B2) and (C2) are satisfied. Then there exists at least one fixed point for the operator $K = A \cdot B$ in the set T and the set of all fixed points $FixK$ is relatively compact in E .

Proof. Let us present the proof, when (A2) and (C1) are satisfied. The second case is similar.

It is obvious that the operator K is well-defined on T and by (E2) it acts between T into itself.

Denote $M_1 = \sup_{t \in T} \|A(t)\|_{E_1}$ and $M_2 = \sup_{t \in T} \|B(t)\|_{E_2}$. Let (x_n) be an arbitrary sequence in T tending to $x \in T$. Then

$$\begin{aligned} \|K(x_n) - K(x)\|_E &= \|A(x_n) \cdot B(x_n) - A(x) \cdot B(x)\|_E \\ &\leq \|A(x_n) \cdot B(x_n) - A(x) \cdot B(x) - A(x) \cdot B(x_n)\|_E \\ &\quad + \|A(x) \cdot B(x_n)\|_E \\ &\leq \|(A(x_n) - A(x)) \cdot B(x_n)\|_E + \|(B(x_n) - B(x)) \cdot A(x)\|_E \\ &\leq k \cdot \|A(x_n) - A(x)\|_{E_1} \cdot M_2 + k \cdot \|B(x_n) - B(x)\|_{E_2} \cdot M_1. \end{aligned}$$

From our assumptions it follows that K is continuous from T into E .

Now, we will investigate the contraction property for a measure $c(X)$.

Assume that X is a nonempty subset of T and let the fixed constant $\varepsilon > 0$ be arbitrary. Then for an arbitrary $x \in X$ and for a set $D \subset I$, $meas(D) \leq \varepsilon$ we obtain

$$\|K(x) \cdot \chi_D\|_E \leq k \|A(x) \cdot \chi_D\|_{E_1} \cdot \|B(x)\|_{E_2}.$$

Since for any non-negative real-valued functions f and g we have $\sup_I(f \cdot g) \leq \sup_I f \cdot \sup_I g$, by definition of $c(x)$ and by taking the supremum over all $x \in X$ and all measurable subsets D with $meas(D) \leq \varepsilon$ we get

$$c(K(X)) \leq k \cdot k_1 \cdot \|B(T)\|_{E_2} \cdot c(X).$$

Because $X \subset T$ is a nonempty, bounded and compact in measure subset of an ideal regular space E , we can use Proposition 5.5.1 and get

$$\beta_H(K(X)) \leq k \cdot k_1 \cdot \|B(T)\|_{E_2} \cdot \beta_H(X).$$

The inequality obtained above together with the properties of the operator K and the set T established before, allow us to apply the classical Darbo fixed point theorem for β_H . If we suppose, that $\beta_H(FixK) \neq 0$, then $K = FixK$ implies $\beta_H(FixK) = \beta_H(K) < \beta_H(K)$, a contradiction, which completes the proof. \square

Remark 5.5.1. We need to remark, that one of our assumptions can be easily relaxed. We assume, that the space is regular. Denote by ${}_0$ a regular part of E . It is

sufficient to assume, that $K : T \cap E_0 \rightarrow T \cap E_0$. This seems to be important for the case of so-called improving operators (taking bounded subsets of E into the sets with equiabsolutely continuous norms i.e. into E_0). A detailed theory of compactness in regular ideal spaces can be found in [118]. If an ideal space E has nontrivial regular part, then our result applies for any operator which is measure-compact (see [68]).

Bibliography

- [1] R.P Agarwal, D. O'Regan, P. Wong, *Constant-sign solutions of a system of Volterra integral equations in Orlicz spaces*, J. Integral Equations Appl. 20 (2008), 337–378.
- [2] R.P Agarwal, D. O'Regan, P. Wong, *Solutions of a system of integral equations in Orlicz spaces*, J. Integral Equations Appl. 21 (2009), 469–498.
- [3] A. Aghajani, Y. Jalilian, *Existence and global attractivity of solutions of a nonlinear functional integral equation*, Commun. Nonlinear Sci. Numer. Simulat. 15 (2010), 3306–3312.
- [4] A. Aghajani, J. Banaś and Y. Jalilian, *Existence of solutions for a class of nonlinear Volterra singular integral equation*, Comp. Math. Appl. 61 (2011), 1215–1227.
- [5] J. Alexopoulos, *De La Vallée Poussin's theorem and weakly compact sets in Orlicz spaces*, Quaestiones Mathematicae 17 (1994), 231–248.
- [6] G. Anichini, G. Conti, *Existence of solutions of some quadratic integral equations*, Opuscula Mathematica 28 (2008), 433–440.
- [7] J. Appell, *The importance of being Orlicz*, Banach Center Publ. 64 (2004), 21–28.
- [8] J. Appell, O.W. Diallo and P.P. Zabrejko, *On linear integro-differential equations of Barabashin type in spaces of continuous and measurable functions*, Jour. Integral Equat. Appl. 1 (1988), 227–247.
- [9] J. Appell, E. De Pascale, *Su alcuni parametri connessi con la misura di non compattezza di Hausdorff in spazi di funzioni misurabili*, Boll. Un. Mat. Ital. 6, 3-B (1984), 497-515.
- [10] J. Appell, P.P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge Tracts in Mathematics 95, Cambridge University Press, Cambridge, 1990.

- [11] I.K. Argyros, *Quadratic equations and applications to Chandrasekhar and related equations*, Bull. Austral. Math. Soc. 32 (1985), 275–292.
- [12] I.K. Argyros, *On a class of quadratic integral equations with perturbations*, Functiones et Approximatio 20 (1992), 51–63.
- [13] K. Balachandran, S. Ilamaram, *A note on integrable solutions of Hammerstein integral equations*, Proc. Indian Acad. Sci. (Math. Sci.) 105 (1995), 99–103.
- [14] J. Banaś, *An existence theorem for nonlinear Volterra integral equation with deviating argument*, Rend. Circ. Mat. Palermo 25 (1986), 82–89.
- [15] J. Banaś, *Applications of measure of weak noncompactness and some classes of operators in the theory of functional equations in the Lebesgue sense*, Nonlin. Anal. 30 (1997), 3283–3293.
- [16] J. Banaś, *Existence results for Volterra-Stieltjes quadratic integral equations on an unbounded interval*, Math. Scand. 98 (2006), 143–160.
- [17] J. Banaś, *Integrable solutions of Hammerstein and Urysohn integral equations*, J. Austral. Math. Soc. 46 (1989), 61–68.
- [18] J. Banaś, *On the superposition operator and integrable solutions of some functional equations*, Nonlinear Anal. 12 (1988), 777–784.
- [19] J. Banaś, A. Chlebowicz, *On existence of integrable solutions of a functional integral equation under Carathéodory conditions*, Nonlin. Anal. 70 (2009), 3172–3179.
- [20] J. Banaś, A. Chlebowicz *On integrable solutions of a nonlinear Volterra integral equation under Carathéodory conditions*, Bull. London Math. Soc. 41 (2009), 1073–1084.
- [21] J. Banaś, B.C. Dhage, *Global asymptotic stability of solutions of a functional integral equation*, Nonlin. Anal. 69 (2008), 949–952.
- [22] J. Banaś, W.G. El-Sayed, *Measures of noncompactness and solvability of an integral equation in the class of functions of locally bounded variation*, J. Math. Anal. Appl., 167 (1992), 133–151.
- [23] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lect. Notes in Math. 60, M. Dekker, New York - Basel, 1980.
- [24] J. Banaś, Z. Knap, *Integrable solutions of a functional-integral equation*, Rev. Mat. Univ. Compl. Madrid 2 (1989), 31–38.

- [25] J. Banaś, M. Lecko, *Fixed points of the product of operators in Banach algebra*, Panamerican Mathematical Journal 12 (2002), 101–109.
- [26] J. Banaś, M. Lecko and W.G. El-Sayed, *Existence theorems for some quadratic integral equations*, Jour. Math. Anal. Appl. 222 (1998), 276–285.
- [27] J. Banaś, A. Martinon, *Monotonic solutions of a quadratic integral equation of Volterra type*, Comp. Math. Appl. 47 (2004), 271–279 .
- [28] J. Banaś, L. Olszowy, *Measures of noncompactness related to monotonicity*, Comment. Math. Prace Matem. 41 (2001), 13–23.
- [29] J. Banaś, J. Rivero, *On measures of weak noncompactness*, Ann. Mat. Pura Appl. 151 (1988), 213–224.
- [30] J. Banaś, J. Rocha, K.B. Sadarangani *Solvability of a nonlinear integral equation of Volterra type*, Jour. Comp. Appl. Math. 157 (2003), 31–48.
- [31] J. Banaś, B. Rzepka, *On existence and asymptotic stability of a nonlinear integral equation*, J. Math. Anal. Appl. 284 (2003), 165–173.
- [32] J. Banaś, B. Rzepka, *Monotonic solutions of a quadratic integral equation of fractional order*, J. Math. Anal. Appl. 332 (2007) 1371–1379.
- [33] J. Banaś, B. Rzepka, *Nondecreasing solutions of a quadratic singular Volterra integral equation*, Math. Comp. Model. 49 (2009) 488–496.
- [34] J. Banaś, K. Sadarangani, *Solutions od some functional-integral equations in Banach algebra*, Math. Comp. Model. 38 (2003), 245–250.
- [35] J. Banaś, T. Zając, *Solvability of a functional integral equation of fractional order in the class of functions having limits at infinity*, Nonlin. Anal. 71 (2009), 5491–5500.
- [36] A. Benkirane, A. Elmahi, *An existence theorem for a strongly nonlinear elliptic problem in Orlicz spaces*, Nonlin. Anal. 36 (1999), 11–24.
- [37] J. Berger, J. Robert, *Strongly nonlinear equations of Hammerstein type*, J. London Math. Soc. 15 (1977), 277–287.
- [38] P.J. Bushell, *On the class of Volterra and Fredholm nonlinear integral equations*, Math Proc. Cambridge Philos. Soc. 79 (1976), 329–335.
- [39] H. Brézis, F. Browder, *Existence theorems for nonlinear integral equations of Hammerstein type*, Bull. Amer. Math. Soc. 81 (1975), 73–78.

- [40] J. Caballero, A.B. Mingarelli and K. Sadarangani, *Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative transfer*, Electr. Jour. Differ. Equat. 57 (2006), 1–11.
- [41] J. Caballero, D. O'Regan, K. Sadarangani, *On monotonic solutions of some integral equations*, Archivum Math (Brno) 41 (2005), 325–338.
- [42] B. Cahlon, M. Eskin, *Existence theorems for an integral equation of the Chandrasekhar H -equation with perturbation*, Jour. Math. Anal. Appl. 83 (1981), 159–171.
- [43] S. Chandrasekhar, *Radiative Transfer*, Dover Publications, New York, 1960.
- [44] S. Chandrasekhar, *The transfer of radiation in stellar atmospheres*, Bull. Amer. Math. Soc. 53 (1947), 641–711.
- [45] I.-Y. S. Cheng, J.J. Kozak, *Application of the theory of Orlicz spaces to statistical mechanics. I. Integral equations*, J. Math. Phys. 13 (1972), 51–58.
- [46] M. Cichoń, M. Metwali, *Monotonic solutions for quadratic integral equations*, Discuss. Math. Diff. Incl. 31 (2011), 157–171.
- [47] M. Cichoń, M. Metwali, *On quadratic integral equations in Orlicz spaces*, Jour. Math. Anal. Appl. 387 (2012), 419–432.
- [48] M. Cichoń, M. Metwali, *On monotonic integrable solutions for quadratic functional integral equations*, Mediterranean Jour. Math. 10 (2013), 909–926.
- [49] M.M. Crum, *On an integral equation of Chandrasekhar*, Quarterly Jour. Math. 18 (1947), 244–252.
- [50] G. Darbo, *Punti uniti in trasformazioni a codominio noncompatto*, Rend. Sem. Mat. Univ. Padova 24 (1955), 84–92.
- [51] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.
- [52] F.S. De Blasi, *On a property of the unit sphere in Banach spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie 21 (1977), 259–262.
- [53] B.C. Dhage, M. Kumpulainen, *Nonlinear functional boundary value problems involving the product of two nonlinearities*, Appl. Math. Letters 21 (2008), 537–544.
- [54] M. Diana Julie, K. Balachandran, *On solutions of a Volterra integral equation with deviating arguments*, Electr. Jour. Diff. Equat. 57 (2009), 1–8.

- [55] J. Dieudonné, *Sur les espaces de Köthe*, J. Anal. Math. (1951), 81–115.
- [56] D. Dinculeanu, *Vector Measures*, Pergamon Press, 1967.
- [57] S. Djebali, Z. Sahnoun, *Nonlinear alternatives of Schauder and Krasnoselskij types with applications to Hammerstein integral equations in L^1 spaces*, Jour. Diff. Equat. 249 (2010), 2061–2075.
- [58] H. Domke, *An equivalence theorem for Chandrasekhar's H-functions and its application for accelerating convergence*, J. Quant. Spectrosc. Radiat. Transfer 39 (1988), 283–286.
- [59] J. Dugundji, A. Granas, *Fixed Point Theory*, Monografie Matematyczne, PWN, Warszawa, 1963.
- [60] N. Dunford, J.T. Schwartz, *Linear Operators. Part I: General Theory*, Interscience Publishers, New York, 1958.
- [61] M.M. El Borai, W.G. El-Sayed and M.I. Abbas, *Monotonic solutions of a class of quadratic singular integral equations of Volterra type*, Int. J. Contemp. Math. Sci. 2 (2007), 89–102.
- [62] W.G. El-Sayed, B. Rzepka, *Nondecreasing solutions of a quadratic integral equation of Urysohn type*, Comp. Math. Appl. 51 (2006), 1065–1074.
- [63] G. Emmanuele, *About the existence of integrable solutions of functional-integral equation*, Rev. Mat. Univ. Complut.Madrid 4 (1991), 65–69.
- [64] G. Emmanuele, *Integrable solutions of a functional-integral equation*, J. Integr. Equat. Appl. 4 (1992), 89–94.
- [65] G. Emmanuele, *Measure of noncompactness and fixed point theorem*, Bull. Math. Soc. Math. R.S. Roum. 25 (1981), 253–285.
- [66] N. Erzakova, *Compactness in measure and measure of noncompactness*, Siberian Math. J. 38 (1997), 926–928.
- [67] N. Erzakova, *On measures of non-compactness in regular spaces*, Z. Anal. Anwendungen 15 (1996), 299–307.
- [68] N. Erzakova, *On measure-compact operators*, Russian Mathematics 55 (2011), 37–42.
- [69] Ch. Fox, *A solution of Chandrasekhar's integral equation*, Trans. Amer. Math. Soc. 99 (1961), 285–291.

- [70] R. Gorenflo, S. Vessela, *Abel Integral Equations*, Lect. Notes Math. 1461, Springer, Berlin-Heidelberg, 1991.
- [71] K. Goebel, W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.
- [72] G. Gripenberg, *Periodic solutions of an epidemic model*, Jour. Mathematical Biology 10 (1980), 271–280.
- [73] F. Hartung, T. Krisztin, H.-O. Walther and J. Wu, *Functional differential equations with state-dependent delays: theory and applications*, in: *Handbook of Differential Equations. Ordinary Differential Equations* vol. 3, A. Canada, P. Drábek, A Fonda Eds., Elsevier, 2006.
- [74] S.L. Hollis, C.T. Kelley, *Vector algorithms for H-equations arising in radiative transfer through inhomogeneous media*, Transport Theory and Statistical Physics 15 (1986), 33–48.
- [75] S. Hu, M. Khavani, W. Zhuang, *Integral equations arising in the kinetic theory of gases*, Appl. Analysis 34 (1989), 261–266.
- [76] G. Infante, J.R.L. Webb, *Nonzero solutions of Hammerstein integral equations with discontinuous kernels*, J. Math. Anal. Appl. 272 (2002), 30–42.
- [77] N.J. Kalton, *Subalgebras of Orlicz spaces and related algebras of analytic functions*, Arkiv för Matematik 18 (1980), 223–254.
- [78] L.V. Kantorovich, G.P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, 1982.
- [79] A. Karoui, A. Jawahdou, *Existence and approximate L_p and continuous solutions of nonlinear integral equations of the Hammerstein and Volterra types*, Appl. Math. Comput. 216 (2010), 2077–2091.
- [80] A.N. Kolmogorov, S.V. Fomin, *Introduction for Real Analysis*, Prentice-Hall Inc., 1970.
- [81] M.A. Krasnoselskii, *On the continuity of the operator $Fu(x) = f(x, u(x))$* , Dokl. Akad. Nauk 77 (1951), 185–188.
- [82] M.A. Krasnoselskii, *Two remarks on the method of successive approximations*, Uspehi. Mat. Nauk. 10 (1955), 123–127.
- [83] M.A. Krasnoselskii, Yu. Rutitskii, *Convex Functions and Orlicz Spaces*, Gröningen, 1961.

- [84] M.A. Krasnoselskii, P.P. Zabrejko, E.I. Pustyl'nik and P.E. Sobolevskii, *Integral Operators in Spaces of Summable Functions*, Nauka, Moscow, 1966 (English translation: Noordhoff, Leyden, 1976).
- [85] J. Krzyż, *On monotonicity-preserving transformations*, Ann. UMCS 6 (1952), 91–111.
- [86] M. Kunze, *On a special class of nonlinear integral equations*, Jour. Integr. Equat. Appl. 7 (1995), 329–350.
- [87] K. Latrach, M. A. Taoudi, *Existence results for a generalized nonlinear Hammerstein equation on L_1 spaces*, Nonlin. Anal. 66 (2007), 2325–2333.
- [88] A. Leonard, T.W. Mullikin, *Integral equations with difference kernels on finite intervals*, Trans. Amer. Math. Soc. 116 (1965), 465–473.
- [89] K. Maleknejad, R. Mollapourasl and K. Nouri, *Study on existence of solutions for some nonlinear functional-integral equations*, Nonlin. Anal. 69 (2008), 2582–2588.
- [90] K. Maleknejad, K. Nouri, R. Mollapourasl, *Existence of solutions for some nonlinear integral equations*, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009), 2559–2564.
- [91] K. Maleknejad, K. Nouri, R. Mollapourasl, *Investigation on the existence of solutions for some nonlinear functional-integral equations*, Nonlin. Anal. 71 (2009), 1575–1578.
- [92] L. Maligranda, *Orlicz Spaces and Interpolation*, Campinas SP Brazil: Departamento de Matemática, Universidade Estadual de Campinas, 1989.
- [93] M. Meehan, D. O'Regan, *Positive L_p solutions of Hammerstein integral equations*, Arch. Math. 76 (2001), 366–376.
- [94] T.W. Mullikin, *Radiative transfer in finite homogeneous atmospheres with anisotropic scattering. I. Linear singular equations*, The Astrophysical Journal 139 (1964), 379–395.
- [95] T.W. Mullikin, *A nonlinear integrodifferential equation in radiative transfer*, Journal of the Society for Industrial and Applied Mathematics 13 (1965), 388–410.
- [96] R. Nussbaum, *A quadratic integral equation*, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze 7 (1980), 375–480.

- [97] W. Okrański, *Nonlinear Volterra integral equations and physical applications*, Extracta Math. 4 (1989), 51–80.
- [98] D. O'Regan, *Solutions in Orlicz spaces to Urysohn integral equations*, Proceedings of the Royal Irish Academy, Section A 96 (1996), 67–78.
- [99] D. O'Regan, M. Meehan, *Existence Theory for Nonlinear Integral and Integrodifferential Equations*, Kluwer Academic Publishers, Dordrecht, 1998.
- [100] W. Orlicz, S. Szufła, *On some classes of nonlinear Volterra integral equations in Banach spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 30 (1982), 239–250.
- [101] G.H. Pimbley, *Positive solutions of a quadratic integral equation*, Archive for Rational Mechanics and Analysis 24 (1967), 107–127.
- [102] R. Płuciennik, *The superposition operator in Musielak-Orlicz spaces of vector-valued functions*, Proceedings of the 14th winter school on abstract analysis (Srń, 1986), Rend. Circ. Mat. Palermo (2) Suppl. No. 14 (1987), 411–417.
- [103] R. Płuciennik, S. Szufła, *Nonlinear Volterra integral equations in Orlicz spaces*, Demonstratio Math. 17 (1984), 515–532.
- [104] R. Ramalho, *Existence and uniqueness theorems for a nonlinear integral equation*, Mathematische Annalen 221 (1976), 35–44.
- [105] M.M. Rao, Z.D. Ren, *Applications Of Orlicz Spaces*, Marcel Dekker, New York, 2002.
- [106] M.M. Rao, Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1991.
- [107] I.A. Rus, *An abstract point of view for some integral equations from applied mathematics*, in: *Proc. od the International Conference: Analysis and Numerical Computation of Solutions of Nonlinear Systems Modelling Physical Phenomena, Especially: Nonlinear Optics, Inverse Problems, Mathematical Material Sciences and Theoretical Fluid Mechanics*, Timișoara, 1997. pp. 256-270.
- [108] N. Salhi, M.A. Taoudi, *Existence of integrable solutions of an integral equation of Hammerstein type on an unbounded interval*, Mediterr. J. Math. (2011) DOI 10.1007/s00009-011-0147-3.
- [109] G. Scorza Dragoni, *Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile*, Rend. Sem. Mat. Univ. Padova 17 (1948), 102–106.

- [110] I.V. Shragin, *The boundedness of the Nemytskii operator in Orlicz spaces*, Kišinev. Gos. Univ. Učen. Zap. 50 (1962), 119–122.
- [111] M.B. Schillings, *Existence theorems for generalized Hammerstein equations*, J. Functional Analysis 23 (1976), 177–194.
- [112] A. Sołtysiak, S. Szuffla, *Existence theorems for L_φ -solutions of the Hammerstein integral equation in Banach spaces*, Comment. Math. Prace Mat. 30 (1990), 177–190.
- [113] P.N. Srikanth, M.C. Joshi, *Existence theorems for generalized Hammerstein equations*, Proc. Amer. Math. Soc. 78 (1980), 369–374.
- [114] Ch.A. Stuart, *Existence theorems for a class of non-linear integral equations*, Mathematische Zeitschrift 137 (1974), 49–66.
- [115] M. A. Taoudi, *Integrable solutions of a nonlinear functional integral equation on an unbounded interval*, Nonlin. Anal. 71 (2009), 4131–4136.
- [116] M. Väth, *A general theorem on continuity and compactness of the Urysohn operator*, J. Integral Equations Appl. 8 (1996), 379–389.
- [117] M. Väth, *Continuity of single- and multivalued superposition operators in generalized ideal spaces of measurable functions*, Nonlin. Funct. Anal. Appl. 11 (2006), 607–646.
- [118] M. Väth, *Ideal Spaces*, Lect. Notes Math. 1664, Springer, Berlin-Heidelberg, 1997.
- [119] M. Väth, *Volterra and Integral Equations of Vector Functions*, Marcel Dekker, New York-Basel, 2000.
- [120] J.D. Weeks, S.A. Rice, J.J. Kozak, *Analytic approach to the theory of phase transitions*, J. Chem. Phys. 52 (1970), 2416–2426.
- [121] M. Wertheim, *Exact solution of the mean spherical model for fluids of hard spheres with permanent electric dipole moments*, The Journal of Chemical Physics 55 (1971), 4291–4298.
- [122] P.P. Zabrejko, A.I. Koshlev, M.A. Krasnoselskii, S.G. Mikhlin, L.S. Rakovshchik, V.J. Stecenko, *Integral Equations*, Noordhoff, Leyden, 1975.
- [123] P.P. Zabrejko, E.I. Pustyl'nik, *On continuity and complete continuity of non-linear integral operators in L_p spaces*, Uspekhi Mat. Nauk 19 (116) (1964), 204–205.

- [124] H. Zhu, *On a Nonlinear Integral Equation with Contractive Perturbation*, Advances in Difference Equations, Vol. (2011), Article ID 154742, 10 pages.