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Linear Combinations of Volatility Forecasts for the WIG20 and Polish Exchange Rates

Abstrakt. As is known forecast combinations may be better forecasts then forecasts obtained with single models. The purpose of the research is to check if linear combination of forecasts from models for of the WIG20 Index and different currency exchange rates is a good solution when searching for the best forecasts. We check if the forecasting models are highly correlated with response variable and poorly correlated with each other so if they fulfill the Hellwig assumptions.

Key words: volatility, forecasts , linear regression, MCS

JEL classification: C52, C53

1. INTRODUCTION

According to Stock and Watson (2004) the combination of the models generates better forecast then the single model. A combination of forecasts is a good choice when it is not possible to distinguish one dominant model (Timmermann 2006). Another argument for a combination is that the combinations of forecasts are more stable then individual forecasts (Stock, Watson 2004)

The aim is to verify if linear combination of forecasts of volatility for WIG20 nad different exchange rates are a good solution when searching for best volatility forecasting models. We check if the forecasts are highly correlated with response variable and poorly correlated with each other so if they fulfill the

“Hellwig’s assumptions”. We compare the volatility forecasts with daily realized volatility. We investigate the results for different measures of realized volatility and different best forecasting models for different functions of error.

2. FORECASTS COMBINATIONS

The simplest combination is linear with the identical coefficients and the sum of the weights equels one.

$$g\left(\hat{y}_{t+h};\omega_{t+h,t}\right)=\frac{1}{N}\sum_{j=1}^{N}\hat{y}_{t+h,t,j} \tag{1}$$

where $\hat{y}_{t+h,t}$ is the forecast, and $\omega_{t+h,t}$ is the weight

The forecast error is defined by:

$$e^{c}_{t+h,t}=y_{t+h}-g\left(\hat{y}_{t+h,t};\omega_{t+h,t}\right) \tag{2}$$

The parametrs of the optimal combinations of the forecasts in this case are the solution of the following problem

$$\omega^{*}=\underset{\omega\ \in W_t}{\arg\min}\, E\left[L\left(e^{c}(\omega)\right)\right] \tag{3}$$

where L denotes mean squared error (MSE) loss.

Under MSE the combination weights only depend on the first two moments of the joint distribution of y_{t+h} and $\hat{y}_{t+h,t}$

$$\begin{pmatrix} y_{t+h} \\ \hat{y}_{t+h} \end{pmatrix} \sim \begin{pmatrix} \mu_{y\,t+h,t} \\ \mu_{\hat{y}\,t+h,t} \end{pmatrix} \begin{pmatrix} \sigma^{2}_{y\,t+h,t} & \sigma'_{y\hat{y}\,t+h,t} \\ \sigma_{y\hat{y}\,t+h,t} & \Sigma_{\hat{y}\hat{y}\,t+h,t} \end{pmatrix} \tag{4}$$

For MSE Timmermann (2006) obtained the following optimal weights:

$$\omega_0^* = \mu_{y_{t+h,t}} - \omega^* \mu_{\hat{y}_{t+h,t}}, \omega^* = \Sigma^{-1}_{\hat{y}\hat{y}_{t+h}} \sigma_{y\hat{y}_{t+h,t}} \quad (5)$$

Consider the combination of two forecasts $\hat{y}_1, \hat{y}_2$. Let e_1 i e_2 denote the forecast errors. Assume $e_1 \sim (0, \sigma_1^2)$, $e_2 \sim (0, \sigma_2^2)$, where $\sigma_1^2 = Var(e_1)$, $\sigma_2^2 = Var(e_2)$. and $\sigma_{12} = \rho_{12}\sigma_1\sigma_2$ is the covariance between e_1 and e_2 and ρ_{12} is their correlation

The optimal weights for this combination by Timmermann (2005) have the form

$$\omega^* = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}, \ 1 - \omega^* = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}^2}. \quad (6)$$

The identical weights are optimal if the forecast variances are the same independently of the correlation between forecasts on condition that the forecasts are unbiased (Timmermann 2006). The natural example is the following scheme of two forecasts:

$$(1/2) \cdot (\hat{y}_1 + \hat{y}_2). \quad (7)$$

When the forecast are unbiased Timmermann (2006) propose the combination that gives the inverse weights to the forecasts with the assumption that the correlation is zero:

$$\omega_{inv} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, 1 - \omega_{\text{inv}} = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}. \quad (8)$$

For N forecasts one can assume $0 \le \omega_{ni} \le 1,\ i = 1,\ldots,N$ to make the values of the combination forecasts be in the interval of values of the individual forecasts.

Let $\hat{y}_c = \omega\hat{y}_1 + (1-\omega)\hat{y}_2$, $y - \hat{y}_1 = e_1 \sim (0, \sigma^2)$, $y - \hat{y}_2 = e_2 \sim (\mu_2, \sigma^2)$, so $\hat{y}_2$ is the biased forecast and assume $\text{cov}(e_1, e_2) = \sigma_{12} = \rho_{12}\sigma^2$.

Using the formulas

$$e_c = \omega e_1 + (1-\omega)e_2$$

$$\sigma_c^2(\omega) = \omega^2\sigma^2 + (1-\omega)^2(\sigma^2+\mu^2) + 2\omega(1-\omega)\sigma_{12}.$$

Timmermann obtained

$$MSE(\hat{y}_c) - MSE(\hat{y}_1) = (1-\omega)\sigma^2\left((1-\omega)\left(\frac{\mu_2}{\sigma}\right)^2 - 2\omega(1-\rho_{12})\right) \tag{9}$$

So if $\left(\frac{\mu_2}{\sigma}\right)^2 > \frac{2\omega(1-\rho_{12})}{1-\omega^2}$, then $MSE(\hat{y}_c) > MSE(\hat{y}_1)$.

The condition allways holds for $\rho_{12} = 1$. In this case the forecast of the combination of models doesn't outperform the unbiased forcest of the simple model. What is more the bigger is the bias of the forecast the smaller is the advantage of the combination. If the forecasts are biased then identical weights are optimal when the forecast errors have the same variance and identical correlation between forecasts (Timmermann 2006) .

The optimal weights problem may be formulated as the optimalization task of minimalization of expected fotrecast error variance

$\Sigma_e = E[ee']$, where $e = y - l\hat{y}$ with the condition that the sum of weights is one and the individual forecasts are unbiased:

$$\min \omega' \Sigma_e \omega. \tag{10}$$

$$\omega' l = 1. \tag{11}$$

where l is the vector of ones.

For the invertible covariance matrix Σ_e Timmerman, (2005) obtains the following optimal weights:

$$\omega^* = (l' \Sigma_e^{-1} l)^{-1} \Sigma_e^{-1} l. \tag{12}$$

The problem of the optimal combination can be solved as the following test

$$H_0 : E[L(\hat{\sigma}_t^2, h_t^A)] = E[L(\hat{\sigma}_t^2, f(h_t^A, h_t^B, \theta))] \quad (13)$$

$$H^A : E[L(\hat{\sigma}_t^2, h_t^A)] > E[L(\hat{\sigma}_t^2, f(h_t^A, h_t^B, \theta))].$$

The test statistic of Diebold-Mariano and West (DMW) can be used in the test. Let define the difference

$$d_t = L\left(\hat{\sigma}_t^2, h_t^A\right) - L\left(\hat{\sigma}_t^2, f\left(h_t^A, h_t^B, \theta\right)\right) \quad (14)$$

Then the DMW test statistic is the following:

$$DMW_T = \frac{\sqrt{T}\, d_T}{\sqrt{a\hat{V}ar\left[T d_t\right]}}, \quad (15)$$

where

$$\bar{d}_T \equiv \frac{1}{T}\sum_{t=1}^{T} d_t . \quad (16)$$

Under the null hypothesis the test statistic has normal distribution.

If

$$\sigma\left(y - \hat{y}_1\right) > \sigma\left(y - \hat{y}_2\right) \quad (17)$$

$$\mathrm{cov}\left(y - \hat{y}_1, y - \hat{y}_2\right) \neq \sigma\left(y - \hat{y}_2\right)\sigma\left(y - \hat{y}_1\right), \quad (18)$$

the optima model is the combination of forecasts, Timmermann (2006).

Another scheme can be created on the base of the ranking of models by Aiolfi and Timmermann (2006). Let R_i be the position of the *i*-model in ranking. The weighs od the combination are the following:

$$\hat{\omega} = R_i^{-1} / \left(\sum_{i=1}^{N} R_i^{-1}\right). \quad (19)$$

3. HELLWIG'S IDEA

In goood linear regression model:

1. explanatory variables are highly correlated with response variable.
2. explanatory variables are poorly correlated with each other.

What is more big correlations between exsplanatory variables cause big parametrs average errors.

4. DATA

In the empirical investigation we used daily observations of the WIG20 Index, from May 8, 2001 till May 8, 2009 for model estimation. On the next 256 data from 29 April 2008 till 8 May, 2009 we calculated 1 day volatility forecasts. To evoluate the quality of our forecasts we compared them with daily realized volatility calculated for 5, 10 and 30 minute intraday returns.

We considered the following types of GARCH (1, 1) with different distributions of error: RiskMetrics, GARCH, EGARCH, GJR, APARCH, IGARCH, FIGARCH-BBM, FIGARCH-CHUNG, FIEGARCH, FIAPARCH-BBM, FIAPARCH-CHUNG, HYGARCH. The models estimated with different distributions of error: GAUSSIAN, STUDENT-t, and GED, SKEWED – STUDENT

5. THE TEALIZED VOLATILITY

The realized volatility can be calculated by summing the squares of intraday returns. With the use of the equation which allow for the night return it is defined as follow:

$$\sigma_{2,t}^{2} = \sum_{i=0}^{N} r_{t,i}^{2}, \tag{20}$$

where the intraday return in the day n and in the moment d is :

$$r_{n,d} = 100\left(\ln P_{n,d} - \ln P_{n,d-1}\right), \quad r_{n,0} = 100\left(\ln P_{n,1} - \ln P_{n-1,N}\right), \tag{21}$$

N is the numer of periods in a day.

The alternative approach was proposed by Andersen and Bollerslev in 1997. They suggested reprezenting the daily volatility as the sum of intraday returns

$$\sigma_{1,t}^{2} = \sum_{i=1}^{N} r_{t,i}^{2}. \tag{22}$$

They suggest multiplying $\sigma_{1,t}^2$ by $(1+c)$, where c is the positive constant (Martens 2002). They choose $\left(\sigma_{\mathrm{co}}^2+\sigma_{\mathrm{oc}}^2\right)/\sigma_{\mathrm{oc}}^2$ as the constant c, where $\sigma_{\mathrm{co}}^2 = Var(r_{t,0})$ and $\sigma_{\mathrm{oc}}^2 = Var(\sum_{t=1}^{N} r_{t,n})$, Koopman i et al, (2005). Then the realized volatility can be expressed:

$$\sigma_{3,t}^2 = \frac{\sigma_{\mathrm{oc}}^2+\sigma_{\mathrm{co}}^2}{\sigma_{\mathrm{oc}}^2}\sum_{i=1}^{N} r_{t,i}^2 \tag{23}$$

In the article MSE means the mean squared error and MAD means mean absolute deviation, where *N* is the number of forecasts.

$$\mathrm{MSE} = N^{-1}\sum_{t=1}^{N}\left(\sigma_{l,t}^2-\hat{\sigma}_{k,t}^2\right)^2, \tag{24}$$

$$\mathrm{MAD} = N^{-1}\sum_{t=1}^{N}\left|\sigma_{l,t}^2-\hat{\sigma}_{k,t}^2\right|, \tag{25}$$

where $l\in\{1,2,3\}$, $k\in\{1,\ldots,m\}$ is the numer of models from the considered set. In the following formula $\hat{\sigma}_{k,t}^2$ is the forecast of volatility from the model k on the moment *t*, $\sigma_{l,t}^2$ is the value of the realized volatility of the type *l* in the moment *t.*

6. EMPIRICAL RESULTS

The best models obtained with Model Confidence Set method (MCS) for MAD loss function , realized volatility $\sigma_{1,t}^2$, $\sigma_{2,t}^2$, $\sigma_{3,t}^2$ and 5 minute frequency of returns are:

1 GARCH (1,1) with Gaussian distribution of error

2 AR(1)-GARCH with Gaussian distribution of error

3 MA(1)-GARCH with Gaussian distribution of error

4 HYGARCH with Gaussian distribution of error

5 AR(1)-HYGARCH with Gaussian distribution of error

6 MA(1)-HYGARCH with Gaussian distribution of error

The matrics of correlations:

Table 1. The values of correlations between forecasts

	1	2	3	4	5	6
1	1	0.999678	0.999691	0.985923	0.98323	0.983293
2		1	0.999999	0.984996	0.982814	0.982866
3			1	0.984982	0.982782	0.982836
4				1	0.999253	0.999267
5					1	0.999999
6						1

The bests model obtained with MCS method for MSE loss function, realized volatility $\sigma^2_{1,t}$, $\sigma^2_{2,t}$, $\sigma^2_{3,t}$ and 5 minute frequency of returns is: *RiskMetrics with skewed Student t distribution of error.*

The MCS for MAD, realized volatility $\sigma^2_{1,t}$, $\sigma^2_{3,t}$ and 10 minute frequency of returns is:

1 GARCH (1,1) with Gaussian distribution of error

2 AR(1)-GARCH with Gaussian distribution of error

3 MA(1)-GARCH with Gaussian distribution of error

4 HYGARCH with Gaussian distribution of error

The matrics of correlations

Table 2. The values of correlations between forecasts

	1	2	3	4
1	1	0.999678	0.999691	0.985923
2		1	0.999999	0.984996
3			1	0.984982
4				1

The MCS for MSE , realized volatility $\sigma^2_{1,t}$, $\sigma^2_{2,t}$, $\sigma^2_{3,t}$ and 10 minute frequency of returns is:

1. FIGARCH with GED

2. AR(1)-RiskMetrics with Gaussian distribution of error

3. RiskMetrics with skewed Student distribution of error

4. GARCH with skewed Student – t distribution of error

The matrics of correlations :

Table 3. The values of correlations between forecasts

	1	2	3	4
1	1	0.985936	0.987563	0.987633
2		1	0.99972	0.995763
3			1	0.996994
4				1

The best models obtained with MCS method for MAD loss function , realized volatility $\sigma^2_{1,t}$, $\sigma^2_{3,t}$ and 30 minute frequency of returns are:

1 GARCH (1,1) with Gaussian distribution of error

2 AR(1)-GARCH with Gaussian distribution of error

3 MA(1)-GARCH with Gaussian distribution of error

4 HYGARCH with Gaussian distribution of error

5 AR(1)-HYGARCH with Gaussian distribution of error

6 MA(1)-HYGARCH with Gaussian distribution of error

The matrics of correlations :

Table 4. The values of correlations between forecasts

	1	2	3	4	5	6
1	1	0.999678	0.999691	0.985923	0.98323	0.983293
2		1	0.999999	0.984996	0.982814	0.982866
3			1	0.984982	0.982782	0.982836
4				1	0.999253	0.999267
5					1	0.999999
6						1

The best models obtained with MCS method for MAD loss function, realized volatility $\sigma^2_{2,t}$ and 30 minute frequency of returns are:

1. GARCH with GED

2. FIGARCH with GED

3. ARNA(1,1) – GARCH with GED

4. GARCH with skewed Student t

The matrics of correlations:

Table 5. The values of correlations between forecasts

	1	2	3	4
1	1	0.98712	0.999998	0.999795
2		1	0.987216	0.987633
3			1	0.999815
4				1

The MCS for MSE , realized volatility $\sigma_{1,t}^2$, $\sigma_{2,t}^2$, $\sigma_{3,t}^2$ and 30 minute frequency of returns is:

1. AR(1) – RiskMetrics with Gaussian distribution of error

2. RiskMetrics with skewed Student t distribution of error

The matrics of correlations:

Table 6. The values of correlations between forecasts

	1	2
1	1	0.99972
2		1

7. THE ESTIMATES OF THE PARAMETERS OF THE BEST MODELS

Table 7. The estimates of the parameters of the best models

Model	GARCH	AR(1)-GARCH	MA(1)-GARCH	HYGARCH	AR(1)-HYGARCH	MA(1)-HYGARCH
Distribution	Gauss	Gauss	Gauss	Gauss	Gauss	Gauss
Parameters						
μ				0.07171 (0.03309)	0.07254 (0.03499)	0.07262 (0.03486)
a_1		0.05284 (0.02422)			0.05588 (0.02349)	
b_1			-0.05239 (0.02398)			-0.05554 (0.02325)
ω	0.07077 (0.0562)	0.06925 (0.0542)	0.06933 (0.05242)	0.2277 (0.1327)	0.23065 (0.1292)	0.23096 (0.1294)
α_1	0.06009 (0.0132)	0.06007 (0.01281)	0.06006 (0.01281)	-0.050178 (0.09521)	-0.50707 (0.08936)	-0.50696 (0.08956)
β_1	0.90831 (0.03599)	0.90895 (0.03475)	0.90892 (0.03477)	0.68262 (0.10113)	0.6759 (0.09717)	0.67568 (0.09744
				0.86847	0.87852	0.87831

k				(0,0712)	(0,0741)	(0,0713)
d				0.59709 (0.0658)	0.5981 (0.0697)	0.59698 (0.066)

Table 8. The estimates of the parameters of the best models

Model	GARCH(1,1)	FIGARCH(1,d,1)	ARMA(1,1) - GARCH(1,1)	GARCH(1,1)
Distribution	GED	GED	GED	skewed –Student - *t*
Parameters				
μ				0.06983 (0.03343)
a_1			0.78602 (0.12383)	
b_1			0.78602 (0.13456)	
ω	0.01502 (0.0048)	0.08545 (0.095)	0.04616 (0.0259)	0.03765 (0.019)
α_1		-0.47563 (0.23607)	0.05527 (0.00976)	0.05744 (0.00992)
β_1		0.72345 (0.17126)	0.9252 (0.01786)	0.92862 (0.01394)
d		0.50567 (0.2454)		
ν	1.35756 (0.0838)	1.38342 (0.0721)	1.40532 (0.0783)	7.51165 (1.402)
ξ				1.04176 (0.0305)

Table 9. The estimates of the parameters of the best models

Model	RiskMetrics	AR(1)- RiskMetrics	GARCH(1,1)
Distribution	skewed - Student *t*	Gauss	skewed - Student *t*

Parameters			
μ	0.06906 (0.03287)		0.06983 (0.003343)
a_1		0.05468 (0.02302)	
ω	0.016 (0.005)		0.03765 (0.019)
α_1			0.05744 (0.00992)
β_1 or λ		0.94	0.92862 (0.01394)
ν	6.7712 (1.2878)		7.51165 (1.4009)
ξ	1.04346 (0.0305)		1.04176 (0.0305)

8. CONCLUSION

W conclude that linear combination of volatility forecasts doesn't outperform the forecast from the single model, because of the big correlations between forecasts for WIG20 Index. The deduction is the same for main Polish exchange rates volatility forecasts, not prezented in the article.

9. THE REFERENCES

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