MAREK KALUBA

CONSTRUCTIONS OF SMOOTH EXOTIC ACTIONS ON HOMOTOPY COMPLEX PROJECTIVE SPACES AND PRODUCTS OF MANIFOLDS

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March 2014

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KONSTRUKCJE EGZOTYCZNYCH DZIAŁAŃ GŁADKICH NA HOMOTOPIJNYCH ZESPOLONYCH PRZESTRZENIACH RZUTOWYCH ORAZ PRODUKTACH ROZMAITOŚCI

Rozprawa Doktorska z Matematyki
przygotowana pod opieką
Prof. dra hab. Krzysztofa Pawałowskiego
na Wydziale Matematyki i Informatyki
Uniwersytetu im. Adama Mickiewicza w Poznaniu.

Marzec 2014

ABSTRACT

The first part of this thesis presents a construction of smooth actions with exotic fixed point set of finite perfect groups on manifolds homotopy equivalent to complex projective space. The construction is rooted in the equivariant surgery theory and uses wide range of tools, including the Absorption Technique, Burnside rings, Equivariant Transversality, Surgery Obstruction L-groups, and many more. For A_5, the alternating group on five symbols, we show that the fixed point set of a smooth action on a sphere can can also appear as the fixed point set of a smooth action on $h\mathbb{C}P^n$. A similar result is obtained for many other perfect groups, although methods of the proof do not apply to A_5.

In the second part, using the same Absorption Technique, we construct exotic actions of cyclic groups on products of manifolds. In particular actions on $M \times S^2$ are studied, where M is an asymmetric manifold. The problem of recognition of a diagonal action is addressed later, and by means of Leray-Serre spectral sequence solved for free circle actions on $M \times S^1$.

STRESZCZENIE

Pierwsza część rozprawy doktorskiej przedstawia konstrukcję egzotycznych działań gładkich, doskonałych grup skończonych na rozmaitościach gładkich $h\mathbb{C}P^n$ o typie homotopii zespolonej przestrzeni rzutowej. Konstrukcja ta jest oparta na ekwiwariantnej teorii chirurgii, oraz używa szerokiej gamy narzędzi takich jak: techniki wchłaniania, pierścieni Burnside'a, ekwiwariantnej konstrukcji transwersalnej, L-grup przeszkód Walla oraz wiele innych. Dla grupy alternującej A_5 pokazujemy, że zbiór punktów stałych działania gładkiego na sferze S^{2n} może także pojawić się jako zbiór punktów stałych działania gładkiego na $h\mathbb{C}P^n$. Podobny wynik jest również dowiedziony dla wielu innych skończonych grup doskonałych, lecz metody tego dowodu nie mogą zostać zastosowane do przypadku grupy A_5.

W drugiej części, używając tej samej techniki wchłaniania, konstruujemy egzotyczne działania grup cyklicznych na kartezjańskich produktach rozmaitości. Szczególny nacisk jest położony na rozmaitości postaci $M \times S^2$, gdzie M jest rozmaitością asymetryczną. Ponadto korzystając z ciągu spektralnego Leray-Serre'a pokazujemy, że wszystkie działania wolne okręgu S^1 na $M \times S^1$ są równoważne z diagonalnym.

Hofstadter's Law:

It always takes longer than you expect,
even when you take Hofstadter's Law into account.

DOUGLAS HOFSTADTER
«Gödel, Escher, Bach: An Eternal Golden Braid»

ACKNOWLEDGMENTS

I am greatly indebted to many people that turned up during my studies in the right time to make this thesis possible.

In chronological order, I am grateful to my advisor, Krzysztof Pawałowski, for the continuous stream of new and great ideas he poured over my little head every month. Some of them overwhelmed me, a few made me feel dizzy, but others turned out to be just the right thing I needed.

I would especially express my gratitude to Matthias Kreck for his kind invitation to HIM in Bonn in 2011, where the caring eye upon me also had Diarmuid Crowley. Under their patient guidance and constant encouragement I struggled through very basics of classical surgery theory. I doubt it would be so smooth without their help. My special thanks should also include the musical activity in Bonn provided by Matthias Kreck. I hope I used it well.

My special thanks also go to Zbigniew Błaszczyk for many discussions during preparation of the second part of the thesis.

Last but not least, I would like to express my sincere appreciation to Masaharu Morimoto who kindly provided me with advice on equivariant surgery for perfect groups. Without his help this thesis would have had a completely different shape. I am particularly indebted to him.

CONTENTS

1

INTRODUCTION

Let G be a finite group. A great deal of research in the subject of transformation groups is dedicated to realizing an invariant $I(X)$ where X varies within a category of smooth G-manifolds all homotopy equivalent to a manifold Y, and all having some additional properties of the manifold Y. For example, if Y is a sphere, a disc, or Euclidean space, then the manifolds X are supposed to be the homotopy spheres, discs, or Euclidean spaces, respectively. In these cases, numerous results describe the possible values of the invariant $I(X) = X^G$, the fixed point set of the G-action on X. We shall study the rich variety of this invariant in the case Y is a complex projective space and the manifolds X are the homotopy complex projective spaces.

RIGIDITY In the subject of transformation groups, one may consider actions of uncomplicated groups G (e.g. $G = \mathbb{Z}/_p$) on spaces X of the homotopy type of acyclic spaces, spheres, projective spaces, etc. The situation when for every action the set X^G inherits certain (co)homological properties from X may be regarded as a phenomena of rigidity.

Mathematicians' interest in the topic sprung after the initial works of P.A. Smith [55, 56] on p-groups acting on homology discs and spheres where it became apparent that the homotopy (homology) type of the space X plays the fundamental role. In [55] P.A. Smith obtained a necessary condition for F to occur as the $\mathbb{Z}/_p$-fixed point set of a cellular action on a $\mathbb{Z}/_p$-acyclic complex. In 1971 L. Jones showed [23] that the necessary condition is also sufficient for every cellular action on a finite contractible G-CW-complex. Finally in 1975 S. Willson [60] provided a converse for P.A. Smith's Theorem for homology spheres. Around the same time G.E. Bredon [8] conducted research on $\mathbb{Z}/_p$-cohomology complex projective spaces much in the spirit of P.A. Smith. In 1990 W. Browder [9] obtained a partial answer to the converse of G.E. Bredon's results, although the general situation seems much more complicated than in the case of actions on acyclic complexes or homology spheres.

EXOTICNESS On the other end of the scale is research devoted to exotic actions (i.e. actions being far from resembling the linear ones). Examples of such exoticness include actions on discs without fixed points, and on spheres with exactly one fixed point. With this aim in mind, we are forced to consider more complicated groups, but (preferably) stay in a relatively simple homotopy type of the space acted upon.

In 1975, for any group G not of prime power order, R. Oliver [44] found a construction of finite contractible G-CW-complexes with the fixed point set in the given homeomorphism type. Using the Embedding Theorem of Palais-Mostow, he obtained also smooth G-actions on discs with prescribed fixed point sets (up to the homotopy type). These results were later augmented by K. Pawałowski in [46], where a method of thickening of G-CW-complexes is presented aiding the transition from the G-CW-world to the world of smooth G-manifolds. The problem of constructing smooth finite groups actions on discs with prescribed fixed point sets has been answered completely by B. Oliver in 1996 [43], see Theorem 2.1.

Around the same time much effort have been devoted to understand smooth one fixed point actions of finite groups on spheres. In particular in [27] E. Laitinen and M. Morimoto found a complete characterisation of finite groups acting smoothly with precisely one fixed point on a sphere. A series of articles of M. Morimoto and K. Pawałowski [39, 40, 41, 38] has led to a complete answer to the question which manifolds appear as the fixed point sets of smooth actions of finite *perfect groups* on spheres, see Theorem 2.2.

This thesis divides naturally into two parts. The first one is devoted to smooth actions of finite groups on manifolds homotopy equivalent to complex projective spaces. We assume reader's familiarity with the classical surgery theory and from the beginning we develop necessary tools for the equivariant surgery. The second part deals with S^1-actions or cyclic group actions on products of manifolds. This part is mostly self-contained and requires only the standard algebraic topologist's toolbox.

PART ONE Finite group actions on $h\mathbb{C}P^n$'s, homotopy complex projective spaces, has been studied previously (we refer the reader to the survey by K.H. Dovermann *et al.* [15] or a different one by Y. Suh [57]). However, most of the research was focused on realising so called *defects*, or proving *algebraic rigidity* in the context of Petrie's Conjecture [48, 49]. To my knowledge all previously constructed actions on $h\mathbb{C}P^n$'s are either G-homotopy equivalent to linear ones, or have the G-poset structure of a linear action, e.g. the fixed point set is a disjoint union of spaces homotopy equivalent to complex projective spaces (of possibly different dimensions). In this thesis we are mainly concerned with the diversity of the fixed point sets possible.

We apply the following Absorbing Technique. Let a finite group G act smoothly on a sphere S^{2n} with the fixed point set $(S^{2n})^G = F \sqcup F_0$. Consider the tangential G-representation module $T_x S^{2n}$ at $x \in F_0$. If the module is unitary, then the complex projectivisation $P_{\mathbb{C}}\left(T_x S^{2n} \oplus \mathbf{1}_G\right)$ provides a G-space diffeomorphic to $\mathbb{C}P^n$, the complex projective space. Moreover, there exists a fixed point $y \in \mathbb{C}P^n$ such that the tangential G-module at y is isomorphic to $T_x S^{2n}$ as a G-module. Therefore we may identify local neighbourhoods of the corresponding points in the S^{2n} and in the $\mathbb{C}P^n$ and form the equivariant connected sum. The new action of G on $\mathbb{C}P^n \# S^{2n} \cong \mathbb{C}P^n$ contains the manifold F as a connected component of the fixed point set. This transfers the 'exotic' part of the fixed point set from spheres to complex projective spaces.

The procedure has a side effect: although the fixed point set $(\mathbb{C}P^n)^G$ contains F, there may be several other components: the 'exceptional' component $F_0 \# \mathbb{C}P^k$, and some other of the form $\mathbb{C}P^k$ coming from the linear action. Therefore we are definitely very far from realising the invariant F. One idea to remedy the situation is to use the equivariant surgery as developed by T. Petrie, K.H. Dovermann and M. Rothenberg [16, 17], and later refined by W. Lück and I. Madsen [29, 30], and M. Morimoto [34, 36]. The summary of the theory forms the body of Chapter 3.

In the equivariant surgery we start with a G-map $f\colon X \to Y$ form a G-manifold such that the fixed point set $X^G = F$ is the desired one. The map $f\colon X \to Y$ 'tangentially' approximates a homotopy equivalence. Then, exercising the control over X provided by f we perform a sequence of modifications (called surgeries) on $f\colon X \to Y$, producing manifolds $X^{(i)}$

$$\begin{tikzcd}
X \arrow[r, rightsquigarrow] \arrow[drr, "f"'] & X^{(1)} \arrow[r, rightsquigarrow] \arrow[dr, "f^{(1)}"] & \cdots \arrow[r, rightsquigarrow] \arrow[d, phantom, "\cdots"] & X^{(m-1)} \arrow[r, rightsquigarrow] \arrow[dl, "f^{(m-1)}"'] & X^{(m)} \arrow[dll, "f^{(m)}"] \\
 & & Y & &
\end{tikzcd}$$

such that the fixed point set of the G-action on $X^{(i)}$ is preserved, i.e.

$$\left(X^{(i)}\right)^G = X^G = F.$$

These surgeries allows us to modify the 'approximating' function $f^{(i)}\colon X^{(i)} \to Y$ along the sequence, with each subsequent step being closer to a homotopy equivalence. The process ends (after a finite number of steps) when

$$f^{(m)}\colon X^{(m)} \to Y$$

is a homotopy equivalence.

Note that although in the process we preserve the desired fixed point set, we also loose the diffeomorphism type of Y. In our case the resulting manifold $X^{(n)}$ is *homotopy equivalent* to the complex projective space $\mathbb{C}P^n$. In order to see the variety of possible manifolds upon which our actions are constructed, following Madsen and Milgram [31] we discuss (in Appendix A) classification of homotopy complex projective spaces. The result was a significant milestone of high-dimensional geometric topology. To facilitate understanding of the material we provide a complete proof of the classification theorem.

In Chapter 2 we recall the highlights of the theory of smooth finite transformation groups on discs and spheres. Classes $\mathcal{A} - \mathcal{F}$ of finite groups not of prime power order are introduced there. We also describe linear actions on complex projective spaces which form the base of our constructions. The Absorption Technique is introduced in Section 2.3, which allows us to draw two corollaries (2.4 and 2.5) concerning the transfer of the fixed point set of a smooth action from the sphere S^{2n} to the complex projective space $\mathbb{C}P^n$.

Then we summarise the Surgery Programme in Section 2.3.1. The easier part of the results, for *finite perfect groups* in classes $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$, is stated therein as Theorem 2.6, although the proof is delayed until Section 4.1. The theorem states that the sufficient conditions for F to occur as the fixed point set of a smooth action on the true $\mathbb{C}P^n$'s are merely the same conditions as are required in the case of smooth actions on spheres. The proof relies solely on the results concerning actions on discs and spheres. This, however, does not apply to perfect groups in class $\mathcal{E}$. In Section 4.2 the theory of equivariant surgery described in Chapter 3 is applied to prove an analogous theorem (4.11) for A_5, the alternating group on five symbols, which is the smallest perfect group in class $\mathcal{E}$.

Throughout the first part, unless explicitly stated otherwise, we share the following conventions.

- G is a finite group;
- All manifolds are closed and smooth;
- All G-actions are smooth.

PART TWO We consider actions on products of manifolds. A supposedly easy task to decide whether an action on a product $M \times N$ of manifolds decomposes as the product of actions on each of the factors turns out to be a daunting task.

We show that although it is easy to construct actions with exotic fixed point set on a product of manifolds of high degree of symmetry (like spheres, or projective spaces), the situation changes dramatically as one of the factors becomes less symmetric. We prove that in the extreme case (i.e. M is an asymmetric manifold), even if the second factor is symmetric, there are some serious restrictions on effective actions on the product.

We have two genera of results in this part:

- We construct a (smooth) G-action on $M \times S^n$ which is not equivalent to the diagonal one.
- We prove that if n is small enough, every (smooth or even tame topological) action of a group G is equivalent to a diagonal action.

In particular, on $M \times S^2$, there exist exotic (i.e. not equivalent to the diagonal) actions of $G = S^1$, or $G = \mathbb{Z}/m$. However every free S^1-action on $M \times S^1$ is equivalent to the diagonal action. The results are presented in Theorems 5.12, 5.14 and 5.16.

Throughout the second part, unless explicitly stated otherwise, we share the following conventions.

- G is a finite cyclic group,
- all manifolds mentioned are closed (smooth or topological),
- G-actions are either smooth or topological.

For standard notions and theorems of the theory of transformation groups we refer the reader to the classic textbooks: by G.E. Bredon [8] and T. tom Dieck [14]. In the reading of the historical sections of the second part the book by C. Allday and V. Puppe [1] might be of some help.

Part I

ACTIONS ON HOMOTOPY COMPLEX PROJECTIVE SPACES

After reading which William of Ockham would be
convulsively clasping his razor, as the whole power and
might of the Equivariant Surgery Theory
is called to arms, only to identify actions
of the smallest perfect group.

2

ACTIONS ON SPHERES AND PROJECTIVE SPACES

2.1 ACTIONS ON DISCS AND SPHERES

The following paragraphs summarise the most complete (as of today) results concerning realisation of manifolds as fixed point sets of finite group actions on spheres.

ACTIONS ON DISCS Following B. Oliver [43], consider (the reduced) K-theory rings: $\widetilde{KO}(F)$, $\widetilde{KU}(F)$, $\widetilde{KSp}(F)$ (for real, complex and quaternionic K-theories) and the following diagram of realification, complexification and quaternionisation maps.

$$\begin{tikzcd}
\widetilde{KO}(F) \arrow[r, "c_{\mathbb{R}}"] & \widetilde{KU}(F) \arrow[r, "q_{\mathbb{C}}"] & \widetilde{KSp}(F), \\
\widetilde{KSp}(F) \arrow[r, "c_{\mathbb{H}}"] & \widetilde{KU}(F) \arrow[r, "r_{\mathbb{C}}"] & \widetilde{KO}(F).
\end{tikzcd}$$

Let p, q denote two different prime numbers. Now divide all groups into six disjoint classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ defined as follows.

$\mathcal{A}$: G has a pq-dihedral subquotient[1].

$\mathcal{B}$: G has **no** pq-dihedral subquotient, but G has an element of order pq conjugate to its inverse.

$\mathcal{C}$: G has elements of order pq, but **none** of them is conjugate to its inverse, and G_2, the 2-Sylow subgroup is **not** normal in G.

$\mathcal{D}$: G has elements of order pq, but **none** of them is conjugate to its inverse, moreover $G_2 \lhd G$.

$\mathcal{E}$: G has **no** element of order pq, $G_2 \ntriangleleft G$, and G is not of prime power order.

$\mathcal{F}$: G has **no** element of order pq, $G_2 \lhd G$, and G is not of prime power order.

Theorem 2.1 (B. Oliver, [43]). *Let G be a finite group not of prime power order. Then a compact manifold F is diffeomorphic to the fixed point set of a smooth action of G on a disc if and only if*

$$\chi(F) \equiv 1 \ (mod\ n_G),$$

---

[1] That is: there exist subgroups $H, K \leqslant G$ such that $K \lhd H$ and $H/_K \cong D_{2pq}$.

(*where* $n_G \geqslant 0$ *is an integer depending only on intrinsic properties of* G) *and the tangent bundle* $\tau(F)$ *satisfies one of the following conditions.*

$G \in \mathcal{A}$: *no further conditions.*

$G \in \mathcal{B}$: $c_{\mathbb{R}}([\tau(F)]) \in c_{\mathbb{H}}\left(\widetilde{KSp}(F)\right) + \operatorname{Tor}\left(\widetilde{KU}(F)\right)$.

$G \in \mathcal{C}$: $[\tau(F)] \in r_{\mathbb{C}}\left(\widetilde{KU}(F)\right) + \operatorname{Tor}\left(\widetilde{KO}(F)\right)$.

$G \in \mathcal{D}$: $[\tau(F)] \in r_{\mathbb{C}}\left(\widetilde{KU}(F)\right)$ (*i.e.* F *is stably complex*).

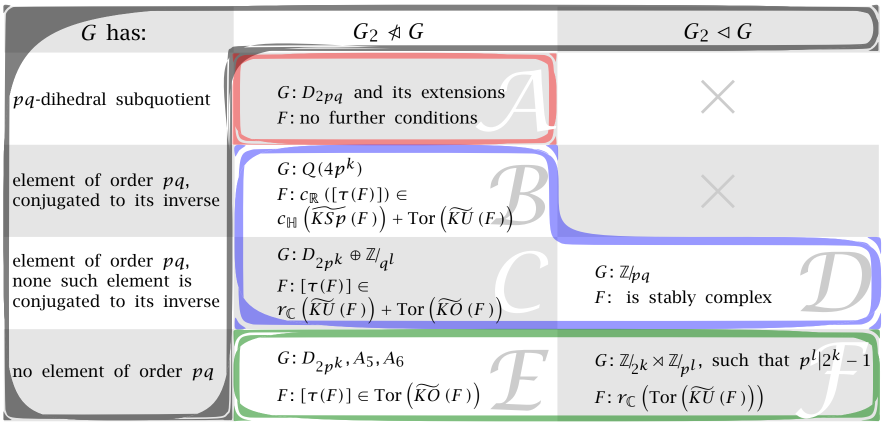
$G \in \mathcal{E}$: $[\tau(F)] \in \operatorname{Tor}\left(\widetilde{KO}(F)\right)$.

$G \in \mathcal{F}$: $[\tau(F)] \in r_{\mathbb{C}}\left(\operatorname{Tor}\left(\widetilde{KU}(F)\right)\right)$.

For groups we use the following notation:

- D_{2n} – the dihedral group of symmetries of regular n-polygon,
- $Q(n)$ – the generalised quaternionic group of order n,
- $\mathbb{Z}/_n$ – the cyclic group of order n,
- A_n – the alternating group on 5 symbols,
- $H \rtimes G$ – the semi-direct product where G acts on H.

The following table summarises groups and properties of the fixed point set. We fix notation: p, q are two distinct primes, and k, l are natural number greater than 0.



G has:	$G_2 \ntriangleleft G$	$G_2 \triangleleft G$
pq-dihedral subquotient	G: D_{2pq} and its extensions F: no further conditions $\mathcal{A}$	×
element of order pq, conjugated to its inverse	G: $Q(4p^k)$ F: $c_{\mathbb{R}}([\tau(F)]) \in c_{\mathbb{H}}\left(\widetilde{KSp}(F)\right) + \operatorname{Tor}\left(\widetilde{KU}(F)\right)$ $\mathcal{B}$	×
element of order pq, none such element is conjugated to its inverse	G: $D_{2p^k} \oplus \mathbb{Z}/_{q^l}$ F: $[\tau(F)] \in r_{\mathbb{C}}\left(\widetilde{KU}(F)\right) + \operatorname{Tor}\left(\widetilde{KO}(F)\right)$ $\mathcal{C}$	G: $\mathbb{Z}/_{pq}$ F: is stably complex $\mathcal{D}$
no element of order pq	G: D_{2p^k}, A_5, A_6 F: $[\tau(F)] \in \operatorname{Tor}\left(\widetilde{KO}(F)\right)$ $\mathcal{E}$	G: $\mathbb{Z}/_{2^k} \rtimes \mathbb{Z}/_{p^l}$, such that $p^l \mid 2^k - 1$ F: $r_{\mathbb{C}}\left(\operatorname{Tor}\left(\widetilde{KU}(F)\right)\right)$ $\mathcal{F}$

Figure 1

From the properties of $\tau(F)$ one can infer more information about the dimensions of the connected components F_α of F. Colour of the outline in the table above corresponds to the following conditions.

RED The dimensions of the F_α's are arbitrary.

BLUE The dimensions of the F_α's are of the same parity.

GREEN The dimensions of the F_α's are the same.

Example. Suppose that a group G belongs to class $\mathcal{A}$. Then the Euler characteristic is the only obstruction for a compact manifold F to occur as the fixed point set of a G-action on a disc D. Hence if there is a G-action on a disc D_1 such that $D_1^G = F$, then there also exists a G-action on some (not necessarily of the same dimension) disc D_2 such that

$$D_2^G = F \sqcup \coprod_{n_G} x_i.$$

Example. Let G belong to any of the above classes and suppose that F is the fixed point set of a G-acion on a disc. We can modify the action and insert to F (as a connected component) any number of odd dimensional spheres, or an even dimensional sphere together with $(n_G - 2)$-fold disjoint union of points.

ACTIONS ON SPHERES We will always tacitly assume that G-actions mentioned do not "reduce" to any of p-groups of G, i.e. that

$$\boxed{\begin{array}{c} S^P \neq S^G \text{ for all Sylow} \\ \text{subgroups } P \text{ of } G. \end{array}}$$

Observe that for any finite perfect group G, the 2-Sylow subgroup G_2 is not normal. Indeed, would $G_2 \lhd G$ then G/G_2 is of odd order, hence is solvable. Moreover p-groups are solvable, therefore G itself would be solvable thus not perfect. As we are going to discuss smooth actions of finite perfect groups, we may exclude classes $\mathcal{D}$ and $\mathcal{F}$ from our considerations.

The joint work of Laitinen, Morimoto, and Pawałowski [27, 28, 39, 40, 41] led to answering the question of which manifolds F occur as the fixed point sets of smooth G-actions on spheres for finite perfect groups G. Below, we quote the result just in the case G is perfect, in the form presented by Morimoto [38, Theorem 1.1], without any unnecessary restrictions on the connected components of F imposed in [41]. We should note that $n_G = 1$ for a finite perfect group G, and therefore there is no restriction on the Euler characteristic of F occurring in Theorem 2.1.

Theorem 2.2. *Let G be a non-trivial finite* ***perfect group****. Then a compact manifold F is diffeomorphic to the fixed point set of a smooth action of G on a sphere if and only if the tangent bundle $\tau(F)$ satisfies one of the following conditions.*

$G \in \mathcal{A}$: *no restrictions on F.*

$G \in \mathcal{B}$: $c_{\mathbb{R}}([\tau(F)]) \in c_{\mathbb{H}}\left(\widetilde{KSp}(F)\right) + \operatorname{Tor}\left(\widetilde{KU}(F)\right).$

$G \in \mathcal{C}$: $[\tau(F)] \in r_{\mathbb{C}}\left(\widetilde{KU}(F)\right) + \mathrm{Tor}\left(\widetilde{KO}(F)\right)$.

$G \in \mathcal{E}$: $[\tau(F)] \in \mathrm{Tor}\left(\widetilde{KO}(F)\right)$.

Remark 2.3. A similar theorem for groups from class $\mathcal{D}$ with a (pqr)-cyclic quotient is also true. We refer the reader to [38, Theorem 1.2].

2.2 LINEAR ACTIONS ON PROJECTIVE SPACES

Let G be a finite group and let V be a complex representation space of G (a complex G-module) of (complex) dimension n and denote by $\mathbf{1}_G$ the complex irreducible trivial representation. By $\mathbb{C}VP^n$ we denote the complex projectivisation of $V \oplus \mathbf{1}_G$, i.e.

$$\mathbb{C}VP^n \overset{\text{def.}}{=} (V \oplus \mathbf{1}_G \setminus \{0\})/\!\sim ,$$

where $z_1 \sim z_2$ if and only if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda z_1 = z_2$. As a topological space $\mathbb{C}VP^n$ is obviously diffeomorphic to $\mathbb{C}P^n$; the G-action on $\mathbb{C}VP^n$ induced from V will be referred to as **linear**. Note that in the definition above we could have taken a unitary representation V and define

$$\mathbb{C}VP^n \overset{\text{def.}}{=} P_{\mathbb{C}}(V \oplus \mathbb{C}) = S(V \oplus \mathbb{C})/S^1 .$$

These two definitions are equivalent.

2.2.1 *The fixed point data*

THE FIXED POINT SET Let $\widehat{G}$ denote the group of irreducible, complex, 1-dimensional representations of G (under tensor product) and express (uniquely) a complex G-module V as the direct sum of irreducible (complex) representations:

$$V = \bigoplus_{\psi \in \widehat{G}} n_\psi \psi \;\oplus\; \bigoplus_{\varphi \notin \widehat{G}} n_\varphi \varphi,$$

where $n_\chi = n_\chi^V \in \mathbb{N} \cup \{0\}$ denote the multiplicity of χ in the unique decomposition of V. We will supress V in our notation of the multiplicity if it does not give rise to a confusion. The fixed point set can be written as

$$\begin{aligned}(\mathbb{C}VP^n)^G &\cong P_{\mathbb{C}}\left((n_{\mathbf{1}_G} + 1)\mathbf{1}_G\right) \sqcup \coprod_{\substack{\psi \in \widehat{G} \\ \psi \neq \mathbf{1}_G}} P_{\mathbb{C}}\left(n_\psi \psi\right) \\ &\cong \mathbb{C}P^{n_{\mathbf{1}_G}} \sqcup \coprod_{\substack{\psi \in \widehat{G} \\ \psi \neq \mathbf{1}_G}} \mathbb{C}P^{n_\psi - 1},\end{aligned}$$

where we treat $P_{\mathbb{C}}(0\varphi) = \mathbb{C}P^{-1}$ as an empty set.

Let $x \in P_{\mathbb{C}}\left((n_{\mathbf{1}_G}+1)\mathbf{1}_G\right) \subset \mathbb{C}VP^n$ denote a point in the connected component coming from the trivial subrepresentation of V, e.g. set

$$x = [1:0:\ldots:0:0]$$

in the projective coordinates. Then the tangential G-module at x is (as the G-module) isomorphic to

$$T_x\mathbb{C}VP^n \cong V.$$

NORMAL BUNDLES Observe that

$$\nu\left(\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}\right) \cong \gamma_n,$$

where γ_n denotes the tautological bundle over $\mathbb{C}P^n$. Analogously, via pullbacks one may establish

$$\nu\left(\mathbb{C}P^k \hookrightarrow \mathbb{C}P^n\right) \cong (n-k)\gamma_k.$$

Let $(\mathbb{C}VP^n)^G_\chi = P_{\mathbb{C}}(n_\chi\chi)$ denote a connected component of the fixed point set corresponding to the 1-dimensional non-trivial G-representation χ. Then the induced G-action on the tangent space at any point $p \in (\mathbb{C}VP^n)^G_\chi$ is given by the following isomorphism.

$$\begin{aligned} T_p\mathbb{C}VP^n &\cong T_x\mathbb{C}VP^n\chi^{-1} \\ &\cong (n_\chi-1)\mathbf{1}_G \oplus (n_{\mathbf{1}_G}+1)\chi^{-1} \\ &\quad \oplus \bigoplus_{\substack{\psi\in\widehat{G} \\ \psi\neq\chi^{-1},\,\psi\neq\mathbf{1}_G}} n_\psi\left(\psi\otimes\chi^{-1}\right) \oplus \bigoplus_{\varphi\notin\widehat{G}} n_\varphi\left(\varphi\otimes\chi^{-1}\right). \end{aligned}$$

Therefore, the equivariant normal bundle of $(\mathbb{C}VP^n)^G_\chi$ decomposes into G-bundles

$$\begin{aligned} \nu\left((\mathbb{C}VP^n)^G_\chi \hookrightarrow \mathbb{C}VP^n\right) &\cong (n_{\mathbf{1}_G}+1)\gamma(\chi^{-1}) \\ &\quad \oplus \bigoplus_{\substack{\psi\in\widehat{G} \\ \psi\neq\chi,\psi\neq\mathbf{1}_G}} n_\psi\gamma\left(\psi\otimes\chi^{-1}\right) \quad \oplus \bigoplus_{\varphi\notin\widehat{G}} n_\varphi\gamma\left(\varphi\otimes\chi^{-1}\right) \end{aligned}$$

where $\gamma(\psi)$ over $(\mathbb{C}VP^n)^G_\chi$ denotes an S^1-twisted product bundle

$$\gamma(\psi) \overset{\text{def.}}{=} S(\mathbb{C}^{n_\chi}) \times_{S^1} \psi$$

for any complex G-module ψ.

The connected components $(\mathbb{C}VP^n)^G_\chi$ of the fixed point set with such decomposition of the normal bundle will be later referred to as the “linear components”.

2.3 ABSORBING ACTIONS

Assume that G acts smoothly on a sphere S^{2n} and the fixed point set can be described as $(S^{2n})^G = F_0 \sqcup F_1$ for some non-empty manifolds F_0 and F_1. Let $p \in F_0$. Suppose that $V = T_p S^{2n}$ is a complex G-module and follow the construction of the linear G-action on the complex projectivisation of V to obtain $\mathbb{C}VP^n$. Observe that

$$T_{[1,\dots,0,0]}\mathbb{C}VP^n \cong V = T_p S^{2n},$$

hence we can form an equivariant connected sum

$$\mathbf{C}P^n \overset{\text{def.}}{=} \mathbb{C}VP^m \# S^{2n}.$$

For the G-action induced on $\mathbf{C}P^n$ we have the following description of the fixed point set components and theirs G-normal bundles.

- The fixed point set is diffeomorphic to

$$(\mathbf{C}P^n)^G \cong F_1 \sqcup F_0 \# (\mathbb{C}VP^n)^G_{\mathbf{1}_G} \sqcup \coprod_{\substack{\chi \in \widehat{G} \\ \chi \neq \mathbf{1}_G}} (\mathbb{C}VP^n)^G_\chi$$

- The equivariant normal bundles are isomorphic to
  - $\nu(F_1 \hookrightarrow \mathbf{C}P^n) \cong \nu(F_1 \hookrightarrow S^{2n})$
  - For all χ

$$\nu\left((\mathbb{C}VP^n)^G_\chi \hookrightarrow \mathbf{C}P^n\right) = \nu(P_{\mathbb{C}}(n_\chi \chi) \hookrightarrow \mathbb{C}VP^n).$$

Corollary 2.4. *Suppose that G acts smoothly on an even-dimensional sphere with the fixed point set containing (as a connected component) an isolated point or an even-dimensional sphere*

$$\left(S^{2m}\right)^G = \begin{cases} \{pt\} \sqcup F_1 \text{ or} \\ S^{2k} \sqcup F_1. \end{cases}$$

Then there exists a smooth G-action on a complex projective space such that the fixed point set is diffeomorphic to the disjoint union of F and a number of complex projective spaces.

$$\left(\mathbb{C}P^m\right)^G = F_1 \sqcup \coprod \{\text{number of complex projective spaces}\}.$$

Corollary 2.5. *Assume that a group G acts smoothly on a sphere S^{2n} with the fixed point set diffeomorphic to $F_1 \sqcup \{pt\}$. If the tangential G-representation $T_x S^{2n}$ does not contain (as a direct summand) any complex, 1-dimensional G-representation (which holds when G is a perfect group) then there exists a smooth G-action on $\mathbb{C}P^n$ such that*

$$\left(\mathbb{C}P^n\right)^G = F_1.$$

2.3.1 *Surgery Programme*

We briefly sketch our Programme for construction of exotic actions of perfect groups on complex projective spaces.

1. We start with a smooth action of a perfect group G on a sphere with the given fixed point set F.
2. Using theorems of this chapter we perform necessary modifications of the action to claim that
   - the action occurs on an even-dimensional sphere S^{2n};
   - the fixed point set $(S^{2n})^G$ contains additional connected component: a point pt if possible, if not, an even-dimensional sphere S^{2k};
   - the tangential G-module V at the isolated point, or at any point in the added S^{2k} carries a complex structure.
3. We create the complex projectivisation of V and form equivariant connected sum $\mathbb{C}VP^n \# S^{2n}$.
4. Finally we hope to perform equivariant surgery on $\mathbb{C}VP^n \# S^{2n}$ to delete the superfluous connected components and obtain a smooth G-action a complex projective space.

Theorem 2.6. *Let G be a finite perfect group in class $\mathcal{A}$, $\mathcal{B}$, or C.*

- *If G belongs to class $\mathcal{A}$, there are no further assumptions.*
- *If G belongs to class $\mathcal{B}$ or C, we suppose that the dimension of a connected component of F is even. Moreover we assume that F satisfies the appropriate bundle condition (see Theorem 2.2).*

Then F occurs as the fixed point set of a smooth G-action on a complex projective space.

Proof. In the case of a perfect group $G \in \mathcal{A} \cup \mathcal{B} \cup C$ it is possible to add just a single point to the fixed point set of a smooth G-action on a sphere (provided that the dimension of F is even in case $G \in \mathcal{B} \cup C$). Then it is sufficient to perform only the first three steps of the Surgery Programme, by Corollary 2.5. These steps are described in detail in Section 4.1. □

Note that this proof does not apply to perfect groups in class $\mathcal{E}$, as we can not add an isolated point to the fixed point set of a smooth action on S^{2n}, see Figure 1 and the following discussion.

In Section 4.2 we continue the Programme. We perform the necessary surgeries and finally prove that the analogous theorem holds also for for $G = A_5$, the alternating group on 5 symbols which belongs to class $\mathcal{E}$. We also indicate why we limit the scope of the Programme to perfect groups, and that it is quite implausible that it may work for other groups.

It seems very likely that the theorem holds for all perfect groups but so far we were unable to prove it.

3

EQUIVARIANT SURGERY

In this section we present a very brief introduction to basic concepts of surgery theory with an outlook on equivariant surgery. We will not rigorously develop the theory, but just give an outline of the process and methods involved, as well as stress the difference between non-equivariant surgery and the equivariant one. The interested reader is encouraged to read the book D. Crowley, W. Lück, and T. Macko [13] for extremely nice exposition of non-equivariant surgery theory. On the other hand the book of T. Petrie and J. Randall [51] shades some light on early developments of equivariant surgery[1].

Surgery modifies a map with some additional information (on the structure of the normal bundle)

$$(f\colon X \to Y, b\colon T(X) \to f^*\eta)$$

to a new map

$$(f'\colon X' \to Y, b'\colon T(X') \to f'^*\eta)\,,$$

which (in some terms), is closer to a homotopy equivalence. We will be performing a geometric operation on X and f but we want to be able to control the algebraic properties of the resulting space X' and map f'. The main concepts of surgery theory are bordisms on the **geometric** side and obstructions on the ALGEBRAIC. The so called fundamental theorem of surgery may be templated as follows.

Theorem (Meta-theorem)**.** *Suppose that we are given a* ***well behaved map*** *$f\colon X \to Y$. Then there exist an* OBSTRUCTION *(an element)*

$$\sigma(f) \text{ in SURGERY OBSTRUCTION GROUP}$$

such that f can be modified by the process of surgery to a homotopy equivalence if and only if $\sigma(f) = 0$.

Moreover the OBSTRUCTION *depends only on the* ***geometric bordism class*** *of f.*

---

[1] Unfortunately there are no introductory books on the matter and the mentioned book is the most “to the groud” exposition (although not entirely error-free, the reader should be warned!). It requires a general knowledge of surgery theory and a great deal of patience to follow somehow non-continuous exposition of the material. Other books include[16, 17], but these are focused on surgery to G-homotopy equivalence and are overly general for applications here.

3.1 MOTIVATION AND BASIC DEFINITIONS

We present material in this section to motivate the normal information carried along the geometry step of surgery. Although the algebraic step has the same importance (and, especially in equivariant setting, presents more difficulties than the geometric step) we generally resolve to known theorems when tackling it. At the same time the normal structures consume a considerable effort in our constructions, hence the decision to include the motivation for it.

3.1.1 *Non-Equivariant Handle Addition*

Non-equivariant handle addition is a manifold analogue of cell attachment we know from the first course of algebraic topology. Let $\overline{\mu}\colon S^k \times D^{n-k} \to M$ be an embedding[2]. Suppose that $\overline{\mu}$ represents a non-zero class $\mu \in \pi_k(M)$. Consider

$$X' \overset{\text{def.}}{=} X \setminus \operatorname{int}\left(\mu\left(S^k \times D^{n-k}\right)\right) \cup_{\mu|_{S^k \times S^{n-k-1}}} \left(D^{k+1} \times S^{n-k-1}\right).$$

The result on the homotopy groups of the operation is

$$\pi_i(X') = \begin{cases} \pi_j(X) & \text{if } j < \min(k, n-k) \\ \pi_k(X)/\langle\mu\rangle & \text{if } i = k \text{ and } k < n-k, \end{cases}$$

where $\langle\mu\rangle$ denotes a normal subgroup generated by μ.

This can be easily seen as follows. Consider

$$W = X \times I \cup_{\mu(S^k \times D^{n-k}) \times \{1\}} \left(D^{k+1} \times D^{n-k}\right).$$

As a manifold W can be regarded as (a deformation retract of) $X \times I \cup_{\mu|_{S^k \times \{0\}}} D^{k+1}$, or as $X' \times I \cup_{\mu'} D^{n-k}$. Thus $X' \hookrightarrow W$ induces isomorphism on homotopy groups up to $n-k-1$ and $X \hookrightarrow W$ up to k (by reducing to cell attachment). Usually W is referred to as **the trace of surgery on $\overline{\mu}$**. Note that $\partial W = X \sqcup X \cup_\mu \left(D^{k+1} \times S^{n-k-1}\right) = X'$.

Suppose that we are given a map $f\colon X \to Y$ which we want to improve (modifying the base space X) to a homotopy equivalence. Assume that f induces isomorphisms on homotopy groups for $i < k$. If $f_*\colon \pi_k(X) \to \pi_k(Y)$ is not an epimorphism, we can modify X by forming $X' = X \#_l S^k \times S^{n-k}$, the appropriate connected sum. Now, the induced homomorphism may have a non-trivial kernel, so we need to kill non-zero elements in

$$\pi_k(f) \overset{\text{def.}}{=} \ker\left(f_*\colon \pi_k(X) \to \pi_k(Y)\right).$$

---

[2] We will also sometimes say that $i\colon S^k \to M$ is a framed embedding.

NORMAL STRUCTURES To be sure that the complete process is possible and that it leads to killing the entire kernel $\pi_k(f)$ we need two further assumptions.

1. Homotopy classes $[\mu] \in \pi_k(f)$ can be represented by framed embeddings.
2. We can repeat the process on X' to kill other non-trivial elements in the homotopy group, i.e. we need to be able to define $f' \colon X' \to Y$ with smaller kernel, namely $\pi_k(f') = \pi_k(f)/\langle G\mu \rangle$.

We may reduce the first assumption to the following theorem.

Theorem (Hirsch). *Let X be a manifold of dimension n, and let $k \leqslant n-2$. Let $i \colon S^k \times D^{n-k}$ be a map. Then every stable vector bundle isomorphism*[3]

$$s \colon \left(T(S^k \times D^{n-k})\right) \to i^*(TX)$$

defines an immersion $\overline{\mu_i} \colon S^k \times D^{n-k} \to X$ which is homotopic to i.

The immersion is unique up to normal homotopy (i.e. homotopy through framed embeddings). Moreover the differential of $\overline{\mu_i}$ is (stably) normally homotopic to s.

Remark. If $2k < n$ then (by a general position argument) the immersion in the theorem can be replaced by an embedding.

This can be subsequently reduced to the fundamental geometric lemma of surgery, which we will not prove here. The interested reader may look for the proof in any book on surgery theory.

Lemma (Geometric Lemma of Surgery). *Suppose that $f \colon X \to Y$ is a map and choose a non-zero element μ in $\pi_k(f)$, and that $2k < \dim X = \dim Y$. Let ξ be a vector bundle over Y, and $\beta \colon T(X) \to f^*\xi$ a stable vector bundle isomorphism. Then there exist*

- *an embedding $\overline{\mu} \colon S^k \times D^{n-k} \to X$ such that $\overline{\mu}$ represents μ,*
- *an extension $f' \colon X' = X \cup \operatorname{ind}_H^G \left(S^k \times D^{n-k} \times D(V)\right) \to Y$, and*
- *a stable vector bundle isomorphism $b \colon T(X') \to f'^*\xi$.*

Now we would like to use the relative Hurewicz homomorphism and Poincaré Duality and proceed with k as far as $n/2$ to arrive at a homotopy equivalence. It turns out that to obtain the algebraic part we need to start with a degree one normal map. In particular if we assume that f, we began with, has degree 1, then $\deg f' = 1$ as well. This will supply us with the Poincaré Duality isomorphisms on *homology kernels*. We will not proceed to explain this here (again, the interested reader may look up the topic in any book on surgery theory).

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[3] for definition see Section 3.3.1

We want to point out that even though in this thesis we will use surgery on degree one maps, sometimes this is not a necessary condition. For example when Y is a sphere, the assumption on degree is not necessary, since surgery kernels (as beeing the whole (co)homology groups in the middle dimensions) inherit the standard Poincaré Duality from Y.

3.2 THE LANGUAGE OF EQUIVARIANT SURGERY

Let G be a finite group acting smoothly on a compact manifold Y. Since the set $S(G)$ of all subgroups of G has partial order induced by inclusion, we have also a stratification of Y into H-fixed point sets Y^H for any subgroup $H \in S(G)$. We will use notation Y^H_α to indicate a connected component of Y^H.

3.2.1 *Elementary Step*

The elementary step is to kill an element in

$$\pi_{k+1}\left(f^H\right) \overset{\text{def.}}{=} \ker\left(f^H \colon \pi_k(X^H) \to \pi_k(Y^H)\right).$$

It has two steps: geometric one and algebraic one. The geometric step produces a G-normal bordism, whereas the algebraic step computes the effect of the geometric step on homology (and by the relative Hurewicz isomorphism on homotopy). We remark that the algebraic step here is significantly more technical than the algebraic step in ordinary surgery, as even in the simply connected case surgery kernels are $\mathbb{Z}[G/H]$-modules (coming from the group action), and in the non-simply connected case we have to deal with intertwined actions on the kernels of both $\pi_1(M)$ and G.

The geometric step is the equivariant handle addition: a modification of the equivariant cell attachment from the G-CW-world to live in the manifold-world. Here we provide an (extremely) brief description of both steps.

DEGREE ONE G-NORMAL MAPS In the classical surgery the process begins with a degree one normal map. Here we need to start with a G-manifold X with the desired fixed point set F and an equvariant degree one G-normal map

$$(f,b) \overset{\text{def.}}{=} (f\colon X \to Y, b\colon TX \oplus \eta \to f^*\xi \oplus \eta),$$

where f is a G-map, b is a G-bundle isomorphism for some G-vector bundle η over X. We remark that the definition of an equivariant normal map is much more technical, e.g. involving G-normal conditions. Then we may perform surgery on X away from the fixed point set trying to make (f, b) G-normally bordant to a homotopy equivalence.

In the equivariant case we need the equivariant degree. The equivariant degree of a G-map is an element of

$$\operatorname{Hom}(\Pi(Y) \to \mathbb{Z})$$

gathering all information about degrees of maps $f^H \colon X^H \to Y^H$. Here

$$\Pi(Y) = \{\pi_0(Y^H)\}_{H \in S(G)}$$

denotes G-poset given by the fundamental grupoids of the H-isotropy submanifolds in X, ordered by inclusion.

In the more “to the ground” approach, if $f \colon X \to Y$ is a G-map, we consider maps $f^H_\alpha \overset{\text{def.}}{=} f|_{X^H_\alpha} \colon X^H_\alpha \to Y^H$ and gather information about degrees of these maps. Hence the equivariant degree is a collection of weights (ordinary degrees) attached to vertices of the poset $\Pi(X)$.

Let $\alpha \in \Pi(X)$ be a connected component of an H-fixed point set. Then $f(X^H_\alpha)$ belongs to a unique connected component Y^H_β. Mapping $\alpha \mapsto \beta$ is a natural way to define an induced map $\Pi(X) \to \Pi(Y)$. Set

$$f^{-1}\left(Y^H_\beta\right) = X^H_{\alpha_1} \sqcup \ldots \sqcup X^H_{\alpha_k}.$$

We finally may define the equivariant degree map as

$$\deg f \colon \Pi(Y) \to \mathbb{Z}$$
$$\deg f(\beta) = \sum_{i=1}^{k} \deg f^H_{\alpha_i}$$

Definition 3.1. We say that a G-map $f \colon X \to Y$ is of **equivariant degree one** if for every $H \in \operatorname{Iso}(X)$

1. every connected component of X^H is oriented so that f^H_α preserves orientation, and

2. for all $\beta \in \Pi(Y)$ we have $\deg f(\beta) = 1$.

Remark. Equivariant degree is constant on conjugacy classes of (the natural) G-action on $\Pi(Y)$, hence we may sometimes refer to degree as a map

$$\deg f \colon \operatorname{Conj}(\Pi(Y)) \to \mathbb{Z}.$$

THE EQUIVARIANT CELL ATTACHMENT First let us start with equivariant cell attachment. Let $H < G$ be a subgroup and consider $X^H \subset X$. Given an element $[\mu] \in \pi_{k+1}(f^H)$, we may represent it as a map $S^k \xrightarrow{\mu} X^H$. View S^k and D^{k+1} as trivial H-spaces and define

$$O(\mu) \overset{\text{def.}}{=} X \cup_{G\mu} \operatorname{ind}_H^G D^{k+1}.$$

Ignoring G-action we may say that

$$O(\mu) = X \cup \underbrace{D^{k+1} \cup \cdots \cup D^{k+1}}_{|G/H|\text{-times}},$$

where the attaching maps can be described as $g\mu\colon S^k \to X^H$. The effect of attachment of G/K-cell on homotopy groups can be read directly:

$$\pi_j(O(\mu)) = \begin{cases} \pi_i(X), & \text{for } i < \min(k, n-k). \\ \pi_k(X)/\langle G\mu \rangle & \text{for } i = k, \end{cases}$$

where $\langle G\mu \rangle$ is the normal group in $\pi_k(X)$ generated by classes $[g\mu]$ for all $g \in G$ (effectively there are just $|G/H|$ of them).

EQUIVARIANT HANDLE ADDITION The manifold analogue of equivariant cell attachment is a little bit more involved. Suppose that $\dim X^H = n$ and consider $S^k \times D^{n-k}$ as a trivial H-space. Suppose that $[\mu] \in \pi_{k+1}(f^H)$ has a representative μ which is an embedding and let V be an H-representation (we may think of it as the normal slice representation to $\mu(S^k \times D^{n-k}) \subset X^k$). Suppose that

- $\overline{\mu}\colon S^k \times D^{n-k} \times D(V) \to X$ is an H-embedding,
- the induced G-map

  $$\operatorname{ind}_H^G \overline{\mu}\colon \operatorname{ind}_H^G \left(S^k \times D^{n-k} \times D(V)\right) \to X$$

  is also an embedding[4].

We define

$$X' \overset{\text{def.}}{=} X \cup_{\operatorname{ind}_H^G \overline{\mu}} \left(S^k \times D^{n-k} \times D(V)\right).$$

This (non-equivariantly) can be viewed as the effect of simultaneously adding $|G/H|$-copies of a k-handle.

Observation. Since we embed $S^k \times D^{n-k}$ treated as a *trivial* H-space there is no need for an “equivariant Hirsch theorem” to represent elements of the homotopy kernel by equivariant framed embeddings.

---

[4] This in particular implies $\dim V^H = 0$.

Definition. The process of modifying X by adding a G/H-handle we call **G-surgery on X of type** (H).

Lemma. *Let X' be a manifold obtained from X by G-surgery of type (H) on an embedding $\overline{\mu}\colon S^k \times D^{n-k} \times D(V) \to X$, where $n = \dim X^H$. Then*

- *If L is not conjugated to a subgroup of H then $X^L = (X')^L$,*
- $$\pi_j(X'^H) = \begin{cases} \pi_j(X^H) & \text{for } j < \min(k, n-k-1) \\ \pi_k(X^H)\big/\langle G\mu\rangle & \text{if } j = k, j < n-k-1, \end{cases}$$

  where $\mu = \overline{\mu}|_{S^k}$
- *if $n = 2k+1$ then $\pi_k(X'^H)\big/\langle G\mu'\rangle = \pi_k(X^H)\big/\langle G\mu\rangle$, where*

  $$\mu'\colon S^k \to S^k \times D^{k+1} \to X'$$

  is the canonical inclusion.

The whole process of G-surgery can be interpreted as G-bordism[5] between X and X'. Set the notation for thickened spaces

$$\mathbb{S}^k = \left(S^k \times D^{n-k} \times D(V)\right) \quad \text{and} \quad \mathbb{D}^{k+1} = \left(D^{k+1} \times D^{n-k} \times D(V)\right).$$

We define

$$W \overset{\text{def.}}{=} X \times I \setminus \operatorname{int}\left(\operatorname{ind}_H^G \overline{\mu}\left(\mathbb{S}^k\right)\right) \cup_{\operatorname{ind}_H^G \overline{\mu}} \operatorname{ind}_H^G\left(\mathbb{D}^{k+1}\right),$$

where $\overline{\mu}$ targets $X \times \{1\}$ and $X \times I$ is endowed with the diagonal action (trivial on the interval I). As in the non-equivariant case W is a bordism between X and X'.

SURGERY BELOW THE MIDDLE DIMENSION Recall that f_α^H is the restriction of f to the α-connected component of the H-fixed point set.

First we make $(f_\alpha^H)_* : \pi_j\left(X_\alpha^H\right) \to \pi_j\left(Y_\alpha^H\right)$ surjective by performing G-surgery of type (K) on framed *homotopically trivial* spheres embedded in X_α^H. This results in the equivariant connected sum of X and $\operatorname{ind}_H^G\left(S^j \times S^{2n-j} \times D(V)\right)$. If $j < n$ we may perform G-surgery of type (H) on each generator of $\pi_{j+1}\left(f_\alpha^H\right)$ to kill the kernel. This procedure is called “surgery below the middle dimension”. Note that by our assumption on the degree of the map we have Poincaré Duality on the homology kernel

$$K_{n+1}\left(f_\alpha^H\right) \overset{\text{def.}}{=} \ker\left(H_n\left(X_\alpha^H\right) \to H_n\left(Y_\alpha^H\right)\right).$$

---

[5] For definition see Section 3.3.2.

Killing the homology surgery kernel in dimension j also might create some cycles in kernels in dimension $2n-j$. However, if we are able to kill the middle dimensional kernels, then Hurewicz homomorphism and Poincaré Duality tells us that all kernels vanish.

Thus we may assume that

$$f^H_\alpha \colon X^H_\alpha \to Y^H_\alpha$$

is **connected up to the middle dimension**, i.e. the map induces isomorphisms on homotopy groups

$$\left(f^H_\alpha\right)_* \colon \pi_j\left(X^H_\alpha\right) \to \pi_j\left(Y^H_\alpha\right)$$

for $j < n$ and is an epimorphism in the middle dimension (for $j = n$). Then we are left with just one non-trivial homotopy kernel group

$$\pi_{n+1}\left(f^H_\alpha\right) = \ker\left(\pi_n\left(X^H_\alpha\right) \to \pi_n\left(Y^H_\alpha\right)\right).$$

The homology kernel

$$K_{n+1}\left(f^H_\alpha\right) \overset{\text{def.}}{=} \ker\left(H_n\left(X^H_\alpha\right) \to H_n\left(Y^H_\alpha\right)\right)$$

is self-dual. The important point here is that an attempt of surgery on an element in $\pi_{n+1}\left(f^H_\alpha\right) = K_{n+1}\left(X^H_\alpha\right)$ creates the dual element in the same kernel, so we cannot conclude that performing surgery kills every generator (as some new may be created at the same time). The surgery step in the middle dimension is a subtle play of geometric and algebraic data determined by self-intersections.

3.2.2 *Two Flavours of Equviariant Surgery*

The equivariant degree assumption and GAP-conditions allows us to define appropriate G-surgery obstructions associated with a degree one G-normal map (f, b). The surgery obstruction groups arise when trying to modify f by surgery to a homotopy equivalence.

G-SURGERY TO HOMOTOPY EQUIVALENCE Since our aim is to maintain the desired fixed point set on X, while rebuilding other isotropy structures, we want to make $f \colon X \to Y$ a G-map which is a homotopy equivalence and is *not* a G-homotopy equivalence.

Suppose we have a G-action on Y with $F \sqcup F'$ as the fixed point set and a normal G-map $f \colon X \to Y$ such that $X^G = F$. For a large family of subgroups (to be described later), The Equivariant Transversality Construction supplies a H-normal bordisms W_H from the map $\operatorname{res}^G_H f$ to the identity normal map $\operatorname{res}^G_H \mathrm{id}_Y$. Then we may use the Reflection Method (as described in Section 3.5.4) to modify f by G-surgery of type (K), relative to set of points with larger isotropy group (e.g. without changing X^G), to a G-map $f'\colon X' \to Y$ which, when restricted to the K-fixed point sets

$$(f')^K : (X')^K \to Y^K,$$

is a homotopy equivalence. This can be done for all subgroups $K < G$ which are sub-normal in H.

Suppose now that Y is simply connected, and that X', Y, W_H and $Y \times I$ all satisfy the *strong* gap condition. Then we hope to prove that one can perform G-surgery on f' so that the Euler characteristic of X'^H coincides with that of Y^H for all non-trivial cyclic subgroups H. Then the final G-surgery obstruction $\sigma(f') = \sigma(f)$ is defined in the classical Wall L-group $L^h_\bullet(\mathbb{Z}[G], w)$. This is the obstruction to a homotopy equivalence, hence we have

$$\sigma(f') = 0 \quad \Longleftrightarrow \quad \begin{array}{l} f' \text{ can be modified by free } G\text{-surgery to} \\ \text{a homotopy equivalence } f''\colon X'' \to Y. \end{array}$$

We note that the Dress Induction theorem (see Section 3.5.3) asserts that the natural restriction map

$$L^h_\bullet(\mathbb{Z}[G], w) \to \prod_{H \in \mathcal{H}(G)} L^h_\bullet(\mathbb{Z}[H], w)$$

is an injection, where product runs over $\mathcal{H}(G)$, the family of all hyperelementary[6] subgroups of G. Therefore it suffices to prove that

$$\operatorname{res}^G_H\big(\sigma(f)\big) = \sigma\left(\operatorname{res}^G_H f\right) = 0$$

for all $H \in \mathcal{H}(G)$.

G-SURGERY TO G-HOMOTOPY EQUIVALENCE For historical reasons we also include a summary of work on smooth actions of finite groups on homotopy complex projective spaces by M. Hughes [21, 22]. Under more restrictive conditions[7], he considers linear actions of odd order groups (and later, dihedral groups) on $\mathbb{C}P^n$. He obtains a G-homotopy equivalence $f\colon X_k \to CVP^n$, for many different X_k's, manifolds homotopy equivalent to $\mathbb{C}P^n$ which differ by their Pontrjagin classes[8]. We remark that since the diversity of the fixed point set was not his aim, all of actions constructed by him have a discrete set as the fixed point set. Here we summarise the proof.

---

[6] A subgroup H in G is **hyperelementary** if there exist a short exact sequence

$$0 \to C \to H \to P \to 0,$$

where C is a cyclic group and P is a group of prime power order.

[7] M. Hughes works with G-stable manifolds and G-stable bundles which imply the strong GAP-condition. We will not discus G-stable manifolds here.

[8] A surgery classification of these spaces is contained in Appendix A.

A theorem of M. Matumoto [33, Theorem 5.3] asserts that a G-map $f\colon X \to Y$ is a G-homotopy equivalence if and only if $f^H\colon X^H \to Y^H$ is an ordinary homotopy equivalence for all subgroups $H \leqslant G$. Based on this result, the equivariant surgery is an inductive process.

For simplicity assume that all connected components Y^H_α are simply connected. Knowing that $f^K\colon X^K \to Y^K$ is a homotopy equivalence for all subgroups K such that $H < K \leqslant G$ we may define H-surgery obstructions

$$\sigma_H(f^H_\alpha) \in L^h_\bullet(\mathbb{Z}[W(\alpha)], w_\alpha)$$

for every every connected component X^H_α of the H-fixed set. The group $W(\alpha) \leqslant N_G(H)$ is defined as the stabiliser of the component X^H_α in the $N_G(H)$-action on $\pi_0(X^H)$. Note that X^H_α is a $W(\alpha)$-manifold hence the inductive surgery step will consist of adding (free) $W(\alpha)$-handles into the free part of X^H_α.

The component-wise surgery obstructions can be combined into

$$\sigma_H(f) = \coprod \sigma_H(f^H_\alpha),$$

where the product runs over orbits of the $W(\alpha)$-acion on $\pi_0(X^H)$ (if we can modify $f^H_\alpha\colon X^H_\alpha \to Y^H_\alpha$, then using the same surgeries we can modify $f^H_{g\alpha}$). The main theorem of the equivariant surgery theory states that G-surgery on f to a homotopy equivalence is possible if and only if $\sigma_H(f) = 0$ for all $H \leqslant G$.

The Rothenberg exact sequence connecting L^s and L^h for simple and ordinary homotopy obstruction groups in the case of orientation preserving actions[9] is of the following form.

$$\begin{aligned} \cdots \to H^{n+1}\Big(\mathbb{Z}/2, \mathrm{Wh}'(W(\alpha))\Big) &\longrightarrow \\ \longrightarrow L^s_n(\mathbb{Z}[W(\alpha)], 1) \to L^h_n(\mathbb{Z}[W(\alpha)], 1) &\xrightarrow{p} H^n\Big(\mathbb{Z}/2, \mathrm{Wh}'(W(\alpha))\Big), \end{aligned}$$

where $\mathrm{Wh}'(W(\alpha))$ is the free part of the Whitehead group. By exactness, if $p(\sigma_H) = 0$ then σ_H must come from ${}^s\sigma_H \in L^s_n(\mathbb{Z}[W(\alpha), 1])$. We may consider now the Atiyah-Singer equivariant signature (the multisignature)

$$\mathrm{Sign}\colon L^s_n(\mathbb{Z}[W(\alpha)], 1) \to R(W(\alpha))$$

given by the formula

$$\mathrm{Sign}({}^s\sigma, H) = \mathrm{Sign}(W(\alpha), X^H_\alpha) - \mathrm{Sign}(W(\alpha), Y^H_\alpha).$$

Its kernel is detected by the Arf invariant $c\colon \mathrm{Ker}(\mathrm{Sign}) \to \mathbb{Z}/2$. Therefore the obstruction σ_H vanishes if and only if

1. $p(\sigma_H) = 0 \in H^n(\mathbb{Z}/2, \mathrm{Wh}'(W(\alpha))$,

---

[9] If G is of odd order, any action preserves orientation.

2. $\mathrm{Sign}(W(\alpha), X_\alpha^H) = \mathrm{Sign}(W(\alpha), Y_\alpha^H)$ and
3. $c({}^s\sigma_H) = c(b) = 0$.

M. Hughes obtains the vanishing of all three invariants: for 1, by choosing appropriate class of groups (e.g. of odd order), for 2, by equipping V and Y with a free involution, and by choosing an appropriate G-map $a\colon Y \times V \to Y \times V$ for 3.

3.2.3 GAP-*conditions*

The GAP-conditions impose dimension jumps for the H-fixed point strata. The reason for assuming this conditions will be explained later in the section devoted to equivariant intersection forms and surgery obstruction groups. Remark 3.21 pin-points the exact reason for (SGC) (the strong gap condition), and indicates a method to weaken it.

- Strong GAP-condition:

$$\text{(SGC)} \quad \begin{cases} \bullet \ \dim M_\alpha^H \geqslant 5 \text{ for all } \alpha \\ \bullet \ \text{if } K < H \text{ and } M_\alpha^H \subsetneq M_\beta^K \text{ then} \\ \qquad 2(\dim M_\alpha^H + 1) < \dim M_\beta^K. \end{cases}$$

- GAP-condition:

$$\text{(GC)} \quad \begin{cases} \bullet \ \dim M_\alpha^H \geqslant 5 \text{ for all } \alpha \\ \bullet \ \text{if } K < H \text{ and } M_\alpha^H \subsetneq M_\beta^K \text{ then} \\ \qquad 2\dim M_\alpha^H < \dim M_\beta^K. \end{cases}$$

- Weak GAP-condition:

$$\text{(WGC)} \quad \begin{cases} \bullet \ \dim M_\alpha^H \geqslant 5 \text{ for all } \alpha \\ \bullet \ \text{if } K < H \text{ and } M_\alpha^H \subsetneq M_\beta^K \text{ then} \\ \qquad 2\dim M_\alpha^H \leqslant \dim M_\beta^K, \\ \text{and equality happens only if } [H:K] = 2 \end{cases}$$

We say that M **satisfies** the appropriate **GAP-condition** if the condition is satisfied for all $K, H \in \mathrm{Iso}(M)$, where $\mathrm{Iso}(M)$ denotes the family of proper isotropy subgroups occurring on M.

3.3 EQUIVARIANT NORMAL DATA

We begin this section with general remarks on G-vector bundles. As for almost every notion in surgery theory there are definitions of normal maps, normal cobordisms, and degree of a map for relative objects

$$(f, \partial f)\colon (X, \partial X) \to (Y, \partial Y).$$

However, we will not use it, hence we omit all the unnecessary details to save distractions for the reader. The interested reader may check books on G-surgery theory ([51, 16]), or for more modern treatment articles by W. Lück and I. Madsen [29, 30] and articles of M. Morimoto [37, 38].

Recall that a G-vector bundle $\xi\colon E \to B$ is a vector bundle with a G-action, where every element $g \in G$ acts on ξ as a vector bundle isomorphism. In the following we will always assume that a G-vector bundle is equipped with a G-invariant Riemannian metric. This assumption allows us to consider G-orthogonal complements of subbundles.

By ξ^H we denote the set of vectors in ξ which are fixed by the action of elements of a subgroup $H < G$. In particular this means that the base space of ξ^H is B^H. Note that ξ^H is, in a natural way, a $N_G(H)$-vector bundle, where $N_G(H)$ denotes the normalisator of H in G, i.e. the largest subgroup in G in which H is normal.

We may use the above assumption and decompose bundle ξ over B^H as

$$\xi|_{B^H} = \xi^H \oplus \xi_H.$$

There are two special cases of this decomposition.

Let us denote the product vector bundle $\xi = B \times V$ by $\varepsilon_B(V)$ or simply $\underline{V}$ if B is clear from the context. The above decomposition becomes

$$\xi|_{B^H} = \varepsilon_{B^H}(V^H) \oplus \varepsilon_{B^H}(V_H), \quad \text{or simply} \quad \underline{V^H} \oplus \underline{V_H}.$$

Suppose that B is a smooth G-manifold and $\xi = T(B)$ is its tangent bundle. If $M \subset B$ is a submanifold fixed by H, then we have a decomposition

$$\xi|_M = T(M) \oplus \nu(M \hookrightarrow B),$$

where $T(M)$ and $\nu(M \hookrightarrow B)$ are $N_G(H)$-bundles.

3.3.1 *Equivariant Normal Maps*

Let τ, ξ and η be G-vector bundles over B. If

$$b\colon \tau \oplus \eta \to \xi \oplus \eta$$

is a G-vector bundle map (i.e. a vector bundle map which is also a G-map) we may describe b as given by a matrix

$$\begin{pmatrix} b_{\tau,\xi} & b_{\tau,\eta} \\ b_{\eta,\xi} & b_{\eta,\eta} \end{pmatrix}, \quad \text{where } b_{\alpha,\beta} = b|_{\alpha}\colon \alpha \to \beta.$$

For our considerations, there is a very important decomposition of the form above. In view of the $N_G(H)$-decomposition we may say that $\tau|_{B^H} = \tau^H \oplus \tau_H$ and $\xi|_{B^H} = \xi^H \oplus \xi_H$. Then the G-vector bundle map b decomposes over B^H

$$b|_{B^H} = b^H \oplus b_H,$$

where $b^H\colon \tau^H \to \xi^H$ and $b_H\colon \tau_H \to \xi_H$ are $N_G(H)$-vector bundle maps.

Definition 3.2.

- A G-vector bundle map $b\colon \tau \to \xi$ is called a G-**vector bundle isomorphism** if b is a G-vector bundle map which is (non-equivariantly) a vector bundle isomorphism and covers id_B, the identity map on B.
- If
  $$b\colon \tau \oplus \eta \to \xi \oplus \eta$$
  is a G-vector bundle isomorphism we say that
  $$b_s\colon \tau \to \xi$$
  is a **stable** G**-vector bundle isomorphism from** τ **to** ξ. However, we will often abuse the notation slightly and blend the distinction writing b instead of b_s if the context is clear.

Definition 3.3. We say that b satisfies the η-**normal condition** if for every $H < G$ the map on the orthogonal complement $b_H\colon \tau_H \oplus \eta_H \to \xi_H \oplus \eta_H$ decomposes as

$$\begin{pmatrix} c_H & 0 \\ 0 & \mathrm{id}_{\eta_H} \end{pmatrix},$$

where $c_H\colon \tau_H \to \xi_H$.

Definition 3.4. A G-**normal map** is a pair (f, b) consisting of

- a G-map
  $$f\colon X \to Y$$
  between closed, oriented, smooth manifolds X and Y, and
- a stable G-vector bundle isomorphism
  $$b\colon T(X) \to f^*(\xi),$$
  satisfying the η-normal condition for some G-vector bundle η stabilising b.

There is a more general notion, where we require that b satisfies only the η-**quasinormal condition**, i.e. there exists a decomposition

$$\begin{pmatrix} c_H & d_H \\ 0 & \mathrm{id}_{\eta_H} \end{pmatrix},$$

for some G-vector bundle map $d_H \colon \tau_H \to \eta_H$. However, in our applications these definitions are equivalent (see [37, Theorem 3.5]).

The ultimate characterisation of a G-normal map is provided by W. Lück and I. Madsen in the following theorem (c.f. [29, Proposition A2]). Although this description is not needed in our applications, it may ease understanding of the analogy between normal maps and G-normal maps.

Proposition 3.5. *Let $b \colon \tau \oplus \eta \to \xi \oplus \eta$ be a G-vector bundle map satisfying the η-quasinormal condition. Then there exists a G-vector bundle isomorphism $\widehat{b} \colon \tau \oplus \underline{\mathbb{R}^m} \to \xi \oplus \underline{\mathbb{R}^m}$ for some m. Here $\mathbb{R}^m$ is understood as the m-fold direct sum of the trivial G-representation.*

3.3.2 *Equivariant Normal Cobordisms*

Suppose that a smooth G-manifold W satisfies

- $\partial W = -X \sqcup X'$,
- there exist a G-diffeomorphism between a G-collar neighbourhood $N(X)$ of X in W and the product $X \times [0, 1)$,
- the same condition holds for X replaced by X'.

Then W is called a G-**cobordism** between X and Y.

Let us translate the collar neighbourhood condition to the language of the previous section.

- there exists a G-vector bundle isomorphism

  $$T(X) \oplus \underline{\mathbb{R}} \to i^* T(W),$$

  where $i \colon X \to W$ is the inclusion.

In the process of surgery, not only do we create cobordisms between maps, but actually, by tracking the normal structures along the process, we create normal cobordisms. As cobordisms connect manifolds, normal cobordisms connect normal maps.

Let $f \colon X \to Y$ and $f' \colon X' \to Y$ be two “normal structures”, i.e. maps of manifolds which carry some additional information (like stable isomorphism class of the tangent bundle of the source). We may intuitively think about a normal cobordism W between f and f' as a cobordism (between manifolds X and X') together with a map $W \to Y \times I$ which extends the “normal structures” given on its ends.

The notion of G-normal cobordism plays the same (fundamental) role in G-surgery as normal cobordisms play in ordinary surgery. Although the definition seems much longer and complicated, yet it virtually follows the same idea. The increase in length can be attributed to the fact that more information needs to be carried.

Definition 3.6. Suppose that

$$\begin{aligned}&\big(f\colon X \to Y\ ,\ b\colon T(X) \oplus \eta \to f^*\xi \oplus \eta\big), \text{ and}\\ &\big(f'\colon X' \to Y\ ,\ b'\colon T(X') \oplus \eta' \to (f')^*\xi \oplus \eta'\big)\end{aligned}$$

are two G-normal maps from X and X' with the same target bundle ξ over Y. A G-normal cobordism between (f,b) and (f',b') is a pair

$$\big(F\colon W \to Y \times I\ ,\ B\colon T(W) \oplus H \to (\pi_Y \circ F)^*\xi \oplus \underline{\mathbb{R}} \oplus H\big),$$

where

- W is a compact, oriented smooth G-manifold such that
  $$\partial W = -X \sqcup X'$$
  ($-X$ denotes the reversed orientation),
- $F\colon (W,-X,X') \to \big(Y \times I, Y \times \{0\}, Y \times \{1\}\big)$ is a G-map such that
  $$F|_X = f', \quad \text{and} \quad F|_{X'} = f',$$
  (F restricts to f and f' on the components of the boundary)
- Ξ is a G-vector bundle over W such that
  $$\Xi|_X = \eta \quad \text{and} \quad \Xi|_{X'} = \eta'$$
  (the stabilising bundle Ξ restricts to stabilising bundles η and η' over the components of the boundary), and finally
- B is a G-vector bundle isomorphism satisfying the H-normal condition and restricting
  $$\begin{aligned}B|_{-X} &= \mathrm{id}_{\mathbb{R}} \oplus b\colon \underline{\mathbb{R}} \oplus T(-X) \oplus \eta \to \underline{\mathbb{R}} \oplus f^*(\xi) \oplus \eta\\ B|_{X'} &= \mathrm{id}_{\mathbb{R}} \oplus b'\colon \underline{\mathbb{R}} \oplus T(X') \oplus \eta' \to \underline{\mathbb{R}} \oplus f^*(\xi) \oplus \eta'\end{aligned}$$
  (the vector bundle isomorphism restricts (stably) to b and b').

3.4 LARGE SUBGROUPS AND TRANSVERSALITY

The existence of equivariant degree one G-normal maps has been proved in general setting by W. Lück and I. Madsen in [29]. A method called the Equivariant Transversality Construction to obtain such map was presented much earlier by T. Petrie in [50]. A very coarse description follows.

EQUIVARIANT TRANSVERSALITY Begin with the product G-vector bundle $Y \times V$ for an appropriate complex G-module V(e.g. $V = \mathbb{C}[G]^m$ the m-fold sum of the complex regular representation of G, or $V = U(G)$, a module specially tailored by E. Laitinen for surgery, see Section 3.4.1). Under appropriate assumptions, if

$$a\colon Y \times V \to Y \times V$$

is a proper G-map then it can be deformed by a proper G-homotopy to a map b which is transverse to $Y \times \{0\}$ in the target. We obtain a degree one G-normal map (f, b) by setting

$$X \overset{\text{def.}}{=} b^{-1}(Y \times \{0\}) \text{ and } f \overset{\text{def.}}{=} b|_X\colon X \to Y.$$

It is unfortunate that the potential properties of X and f are occluded by the non constructive “deform a to a transverse map b” phrase. Hence instead of trying to fix f we will work hard to make b as good as possible before the transversality step. The freedom of choice of the initial G-module V and the map a gives us this possibility to find an appropriate map b, which provides the desired fixed point set in X and is suitable for killing surgery obstructions at the same time.

THE BURNSIDE RING T. Petrie’s equivariant transversality construction has been refined by E. Laitinen, K. Pawałowski and M. Morimoto in [28] with final touches by M. Morimoto in [37], where idempotents in the Burnside Ring of G are used to construct G-normal maps between G-modules and hence shift the preimage of the zero section in $Y \times V$ away from the unwanted isotropy types. The method uses the Equivariant Segal Conjecture and the Localisation Theorem for equivariant cohomotopy.

The simplified method described in Section 3.4.3 allows us to control the number of intersections of the preimage $f^{-1}(Y^G)$ and $Y^G \times \mathbb{C}$, understood as the trivial submodule, and hence the *number of copies of the fixed point set components*. In particular, we may choose to delete those unwanted connected components coming from the linear action on $\mathbb{C}P^n$ or from an action on the sphere S^{2n}.

3.4.1 *Representation $U(G)$*

In the process of G-surgery on the absorbed action a special place takes the following representation especially tailored for equivariant surgery by E. Leitinen and M. Morimoto (see [27]).

For a finite group G and a prime p denote by $\mathcal{O}^p(G)$ the ‘co-Sylow’ subgroup, i.e. the smallest normal subgroup in G such that $G/\mathcal{O}^p(G)$ is a p-group.

Let $U(G)$ be the following G-module.

$$U(G) \overset{\text{def.}}{=} (r(G) - \mathbf{1}_G) - \bigoplus_{p \mid |G|} \left(r(G/\mathcal{O}^p(G)) - \mathbf{1}_{G/\mathcal{O}^p(G)}\right).$$

The minus sign denotes the orthogonal complement (with respect to a G-invariant metric). There is the canonical monomorphism of G-modules

$$r(G/\mathcal{O}^p(G)) \hookrightarrow r(G)$$

with the natural G-action given by the map

$$(g', g\mathcal{O}^p(G)) \mapsto g'g \sum_{g_i \in \mathcal{O}^p(G)} g_i,$$

so the formula on $U(G)$ makes sense. We remark that the embedding is the same as taking

$$\operatorname{ind}_{\mathcal{O}^p(G)}^G \mathbf{1}_{\mathcal{O}^p(G)}.$$

We will be using this embedding and sometimes regard $r(G/\mathcal{O}^p(G))$ as G or even as H-representation ($H < G$) without writing all the necessary symbols.

Observation. For every subgroup $H < G$ we have

$$n_{\mathbf{1}_H}^{\operatorname{res}_H^G U(G)} = \dim U(G)^H = [G \colon H] - 1 - \sum_{p \mid |G|} \left([G : H\mathcal{O}^p(G)] - 1\right),$$

where $H\mathcal{O}^p(G)$ is subgroup generated by elements in H or in $\mathcal{O}^p(G)$.

To consider the GAP-condition on $U(G)$ first we have to identify family of isotropy subgroups in $U(G)$.

Definition 3.7. A subgroup $H < G$ is called **large** if $\mathcal{O}^p(G) \subseteq H$ for some p dividing order of G. Define also the family

$$\mathcal{L}(G) \overset{\text{def.}}{=} \{H \in \mathcal{S}(G) \colon H \text{ is large}\}.$$

We end this section with two statements relating $\mathcal{S}(G)$ and $\mathcal{L}(G)$.

Proposition 3.8 ([27]). *For any (finite) group G we have*

$$\mathcal{M}(G) \overset{\textit{def.}}{=} \operatorname{Iso}(U(G)) = \mathcal{S}(G) \setminus \mathcal{L}(G).$$

Remark 3.9. If G is a finite perfect group then $\mathcal{L}(G) = \{G\}$, or equivalently $\mathcal{M}(G) = \mathcal{S}(G) \setminus \{G\}$.

3.4.2 *The Burnside Ring*

In this section we will recall a definition of the Burnside ring $\Omega(G)$. Moreover we provide theorems relating family $\mathcal{M}(G)$ and idempotents in $\Omega(G)$. It will only become apparent in the Equivariant Transversality Constructions in Section 3.4.3 how these two ingredients are intertwined.

The Burnside ring for a finite group G has two[10] equivalent definitions, as the Grothendieck group of finite G-sets and as a set of equivalence classes of all finite G-CW-complexes.

Definition 3.10. Let X and Y be finite G-CW-complexes. We say that X **is equivalent to** Y when the Euler characteristics of X^H and Y^H agree for all subgroups $H < G$.

The set of equivalence classes with addition and multiplication defined as the disjoint union and Cartesian product, respectively, will be denoted by $\Omega(G)$ and will be called the **Burnside ring of** G.

We have a few trivial examples of elements in $\Omega(G)$. The inverse of $[X]$ is the class $[-X] = [X \times X']$, where X' is a G-CW-complex with trivial G-action satisfying $\chi(F) = -1$. Clearly $[X] = 0 \in \Omega(G)$ if $\chi(X^H) = 0$ for all $H < G$. Finally $1 \in \Omega(G)$ can be represented by a G-CW-complex X such that $\chi(X^H) = 1$.

The last two examples are obvious idempotent elements in $\Omega(G)$, however if G is complicated enough, these are surely not the only ones.

Proposition (T. Dieck, [14, Chapter IV, Proposition 7.7])**.** *A finite group G is solvable if and only if 0 and 1 are the only idempotent elements in $\Omega(G)$.*

From [28, Proposition 2.4] we can obtain the following proposition.

Proposition 3.11. *Let G be a finite, non-solvable group. There exists an idempotent element $\alpha \in \Omega(G)$ such that*

$$\chi\left(\alpha^H\right) = \begin{cases} 0 & \text{for } H \in \mathcal{M}(G) \\ 1 & \text{for } H = G. \end{cases}$$

For other subgroups we do not specify the value.

Corollary 3.12. *Let G be a non-trivial, perfect group. Then there exists an idempotent element $[\alpha] \in \Omega(G)$ such that*

$$\chi\left(\alpha^H\right) = \begin{cases} 0 & \text{for } H \neq G \\ 1 & \text{for } H = G. \end{cases}$$

---

[10] Actually many more if we use the Equivariant Segal Conjecture, or spaces of locally constant maps on conjugacy classes of G.

3.4.3 *Equivariant Transversality Construction*

In this section we finally bind all the notions and theorems defined so far. We first give an insight into the properties of transverse preimage, together with the techniques used. Then we state a simplification of the Equivariant Transversality Construction (the most general version is given in [37, Theorem 4.4]) tailored for our purposes.

LOCALISATION Given an element $\alpha \in \Omega(G)$ of the Burnside ring, we use the Equivariant Segal Conjecture (which has been proved to be a theorem) to identify α with a stable equivariant homotopy class (this is supposed to represent the stable part of the normal map)

$$\alpha \in \omega_G^0(pt) \overset{\text{def.}}{=} \varinjlim \, [(mV)^\bullet \to (mV)^\bullet]_G$$

for $mV = \bigoplus_m \mathbb{C}[G]$, the m-fold direct sum of $\mathbb{C}[G]$, the complex regular representation of G, and $V^\bullet$ denoting the one-point compactification of V. We define equivariant cohomotopy group as

$$\omega_G^k(Y) \overset{\text{def.}}{=} \varinjlim \left[(Y \sqcup \mathrm{pt}) \wedge (mV)^\bullet, S^k \wedge (mV)^\bullet\right],$$

where the sphere S^k is understood with trivial action.

Observation. Note that $\omega_G^0(Y^G) = \bigoplus_c \omega_G^0\left(Y_c^G\right)$, where sum runs over the connected components of Y^G.

Let A denote a multiplicatively closed set $\{1, \alpha, \alpha^2, \dots\}$, and by the localisation at α, denoted $A^{-1}R$, we mean a ring where we invert every power of α.

Theorem 3.13 (Localisation Theorem)**.** *Let $j\colon Y^G \hookrightarrow Y$ be the inclusion map and let $A = \{\alpha^m, m \geqslant 0\}$ be the multiplicatively closed subset of $\Omega(G)$, where α is as in Proposition 3.11. Then the localised restriction homomorphism*

$$A^{-1}j^*\colon A^{-1}\omega_G^0(Y) \to A^{-1}\omega_G^0(Y^G)$$

is an isomorphism.

FROM ω_G^0 TO A NORMAL MAP We will construct an identity-covering map

$$Y \times mV \to Y \times mV$$

out of a chosen element $\beta \in \omega_G^0(Y^G)$. By the Localisation Theorem, there exist an element $\omega \in A^{-1}\omega_G^0(Y)$ such that $A^{-1}j^*(\omega) = \beta$. By the very definition of localisation, equivalence class of ω contains a representative ϱ such that $j^*(\varrho)$ and β differ by a factor α^n for some n. Represent ϱ by a G-map $(Y \sqcup \mathrm{pt}) \wedge (mV)^\bullet \to (mV)^\bullet$. The map extends to a G-map

$$\varphi\colon Y \times (mV)^\bullet \to (mV)^\bullet$$

such that $\varphi(y,\infty)=\infty$ for all $y\in Y$. Finally we obtain an identity-covering map

$$\psi\colon Y\times(mV)^{\bullet}\to Y\times(mV)^{\bullet}$$

by setting

$$(y,v)\mapsto(y,\varphi(y,v))\,.$$

Deform ψ by an ε-close G-homotopy relative to $Y^G\times mV$ to a map (denoted by b) which is transverse to $Y\subset Y\times mV$ understood as the 0-section. We set

$$X\overset{\text{def.}}{=}b^{-1}(Y)\quad\text{and}\quad f\overset{\text{def.}}{=}b|_X\colon X\to Y.$$

EXPLICIT EXAMPLE Defining our map we will be closely following the construction above, however, to achieve final properties of b and X, we will have to add a few modifications as we proceed.

We start with the element

$$\beta\overset{\text{def.}}{=}\prod_c\left((1-n_c)\mathbf{1}_{Y_c^G}\right)\in\bigoplus_c\omega_G^0\left(Y_c^G\right),$$

where $\mathbf{1}_{Y_c^G}\in\omega_G^0\left(Y_c^G\right)$ is the map $(Y\sqcup\mathrm{pt})\wedge(mV)^{\bullet}\to(mV)^{\bullet}$ given by

$$[y,v]\mapsto v\quad\text{and}\quad[\mathrm{pt},v]\mapsto\infty.$$

By the Localisation Isomorphism, there exists a unique element $\omega\in A^{-1}\omega_G^0(Y)$, such that

$$A^{-1}j^*(\omega)=\alpha\beta\left(=\alpha\cdot\prod_c\left((1-n_c)\,\mathbf{1}_{Y_c^G}\right)\right)\in A^{-1}\left(\bigoplus_c\omega_G^0\left(Y_c^G\right)\right).$$

By the very definition of localisation, there exists an element $\varrho\in\omega_G^0(Y)$ such that[11]

$$j^*(\alpha\varrho)=\alpha^k\cdot\prod_c\left((1-n_c)\mathbf{1}_{Y_c^G}\right).$$

Choose a representative of class $\mathbf{1}_Y-\alpha\varrho$, and extend it to a G-map

$$\varphi\colon Y\times(mV)^{\bullet}\to(mV)^{\bullet}\,.$$

The map can be further extended (by identity) to

$$\psi\colon Y\times(mV)^{\bullet}\to Y\times(mV)^{\bullet}\,.$$

Before we proceed we need the following lemma.

---

[11] $\alpha\varrho$ in the argument is not a misprint!

Lemma 3.14 ([37, Lemma 4.6])**.** *Via isomorphism $\Omega(G) \cong \omega_G^0$ every element*

$$\sum_H z_H[G/H] = \sum_H \left(\varphi_H^+ - \varphi_H^-\right)[G/H] \in \Omega(G)$$

($\varphi_H^\pm$ are non-negative integers) can be represented by a base-point preserving G-map $h\colon (mV)^\bullet \to (mV)^\bullet$ such that

- *h is transverse to $\{0\} \in mV$,*
- *the number of points fixed by H in $h^{-1}(0)$ is equal to z_H, moreover we have presentation*

$$h^{-1}(0) = \coprod_H \left(\coprod^{\varphi_H^+}[G/H] \sqcup \coprod^{\varphi_H^-} -[G/H] \right)$$

- *for every point in $h^{-1}(0)$ fixed by H the map h is orientation preserving or orientation reversing, depending on the sign of z_H,*
- *the H-normal derivatives of h (maps on the H-normal slices) at every point in $h^{-1}(0)$,*

$$(mV)_H = \left(T_x\,(mV)^\bullet\right)_H \to \left(T_0\,(mV)^\bullet\right)_H = (mV)_H$$

are the identity maps.

Using this lemma we can obtain a few properties of ψ. Most importantly, the restriction of ψ to $Y_c^G \times (mV)^\bullet$ is equal to

$$\mathrm{id} \times j^*\left(\mathbf{1}_Y - \alpha\varrho\right) = \mathrm{id} \times \left(\mathbf{1}_{Y_c^G} - \alpha^k\,(1-n_c)\,\mathbf{1}_{Y_c^G}\right) \in \omega_G^0\left(Y_c^G\right),$$

hence it does not depend on $y \in Y_c^G$. Therefore

$$\left(\psi|_{Y_c^G}\right)^{-1}\left(Y_c^G\right) = Y_c^G \times \varphi^{-1}(0).$$

By the lemma above, in the class of φ we can find a map (also denoted by φ) such that $\varphi^{-1}(0)$ is the disjoint union of

$$\chi\left(\left(\left(1 - \alpha^k\,(1-n_c)\right)\mathbf{1}_{Y_c^G}\right)^G\right) = n_c$$

points. Thus we may assume that $\psi^{-1}\left(Y_c^G\right) = \coprod_{n_c} Y_c^G$.

We may now choose an ε-G-homotopy of ψ to a map

$$b\colon Y \times (mV)^\bullet \to Y \times (mV)^\bullet$$

transverse regular to Y. The homotopy may be chosen relative $Y^G \times (mV)^\bullet$, since ψ is already in general position on this set, by the last statement of the above lemma.

Finally we set as it was advertised earlier:

$$X = b^{-1}(Y) \text{ and } f = b|_X\colon X \to Y,$$

which gives the commuting diagram below.

$$\begin{tikzcd}
Y \times (mV)^{\bullet} \arrow[r, "b"] \arrow[d] & Y \times (mV)^{\bullet} \arrow[d] \\
X = b^{-1}Y \times \{0\} \arrow[r, "f"] & Y
\end{tikzcd}$$

Observation. Observe that X^G (by the second property in the lemma above) satisfies

$$f^{-1}\left(Y_c^G\right) = \coprod_{n_c} \{\mathrm{pt}\} \times Y_c^G = \coprod_{n_c} Y_c^G .$$

We are now ready to state the full Equivariant Transversality Theorem ([37, Theorem 4.4]).

Theorem (Equivariant Transversality Construction)**.** *Let G be a finite Oliver group*[12] *and let Y be a compact, connected, oriented, smooth G-manifold. Suppose that $Y^G = \coprod_c Y_c^G$ is the decomposition of Y^G into connected components and for every c choose a non-negative integer $n_c \geqslant 0$. Then there exists a G-normal map*

$$(f,b) = \left(f\colon X \to Y, b\colon T(X) \oplus m\underline{\mathbb{C}[G]} \to f^*T(Y) \oplus m\underline{\mathbb{C}[G]}\right)$$

satisfying the following properties.

1. *Locally, around each component Y_c^G f is a smooth n_c-fold covering, i.e. there exists U_c, a regular G-neighbourhood of Y_c^G such that*
$$f^{-1}(U_c) = U_{c,1} \sqcup \ldots \sqcup U_{c,n_c}$$
(n_c-fold disjoint union), and $f|_{U_{c,j}}$ is a G-diffeomorphism.

2. $\deg\left(f^H\colon X^H \to Y^H\right) = 1$ *for all $H \in \mathcal{M}(G)$,*

3. *For each $H \in \mathcal{M}(G)$ there exists an H-normal cobordism*
$$(F_H, B_H) : \left(\mathrm{res}_H^G f, \mathrm{res}_H^G b\right) \sim \left(\mathrm{id}_{\mathrm{res}_H^G Y}, \mathrm{id}_{\mathrm{res}_H^G (T(Y) \oplus m\underline{\mathbb{C}[G]})}\right)$$
(this is just a normal bordism between (f,b), and the identity normal map
$$\left(\mathrm{id}_Y, \mathrm{id}_{T(Y)\oplus m\underline{\mathbb{C}[G]}}\right) : Y \times m\mathbb{C}[G] \to Y \times m\mathbb{C}[G]$$
when both maps are understood as H-maps).

---

[12] A finite group G is said to be an Oliver group if G does not contain a sequence of subgroups $P \trianglelefteq H \trianglelefteq G$ such that P is of prime power order, H/p is a cyclic group, and G/H is of prime power order. For our purposes it's enough to conclude that every perfect group is an Oliver group.

3.5 OBSTRUCTION GROUPS FOR EQUIVARIANT SURGERY

In the following we will focus on the case $\dim X^H_\alpha = 2n$. A similar theory was also developed for the odd-dimensional case[13]. We mentioned before that the obstruction groups are groups of quadratic forms modulo some relations arising when considering self-intersections of framed immersions of the middle dimensional spheres.

In this section we will provide geometric motivation, and precise definition of these groups.

3.5.1 *Equivariant Intersection Form*

We will follow Wall's definition of intersection form and self-intersection number on even dimensional manifolds, see [59, p. 44]. Most of the section shares the following assumption.

Group G acts freely on a simply connected,
$2n$-dimensional manifold X.

Suppose that we have a G-normal map (f, b) which is connected up to the middle dimension. Let α, β be two framed immersions $S^n \looparrowright X$ representing elements in the homology surgery kernel

$$\pi_{k+1}(f) \cong K_n(f) = \ker\left(f_* \colon H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})\right).$$

By the general position arguments we may assume that

- they intersect transversally,
- their intersection $\mathcal{B} = \alpha(S^n) \cap \beta(S^n)$ consists of a finite number of points b_j,
- preimages $\alpha^{-1}(b_j) = \{x_i\}$ and $\beta^{-1}(b_j) = \{y_j\}$ both consist of a single point, for all j.

Let $\kappa \colon K_n(f) \times K_n(f) \to \mathbb{Z}$ denote the standard (non-equivariant) intersection form, namely $\kappa(\alpha, \beta) = \sum_{\mathcal{B}} \pm 1$, where the sign depends on the orientation of tangent spaces at b_j in the following way.

Connect the base point x_0 of X with the initial point $a = \alpha(x)$ in the image of immersed sphere $\alpha(S^n)$. We call this path u_α. Let $z \in \mathcal{B}$, the intersection of images of both immersions α, β. For each such point we can find the unique point $y_\alpha \in S^n$ such that $\alpha(y_\alpha) = z$. Choose any path S^n connecting x_α to y_α, and call it v_α. The same can be done for $\beta(S^n)$ and y_β, resulting in u_β and v_β.

Parallel transport of $T_{x_0}X$, the tangent space to X at point x_0, along paths $\alpha(v_\alpha) * u_\alpha$ (or $\beta(v_\beta) * u_\beta$) gives us an orientation of the tangent space $T_z X$. On the other hand, parallel transport of $T_{x_\alpha}S^n$ ($T_{x_\beta}S^n$, respectively) along v_α (v_β, respectively) and mapping via α

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[13] Although the definition of the obstruction groups is completely different.

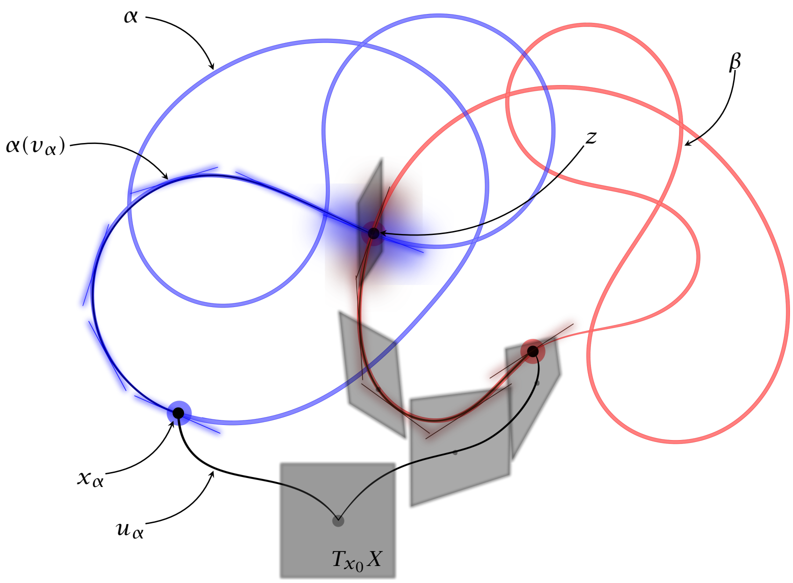

Figure 2: Intersection Form

(β, respectively) gives us orientation of two complementary (since images of α and β were transverse) subspaces of dimension n in T_zX. Hence we have an automorphism

$$T_xS^n \oplus T_yS^n \longrightarrow T_zX.$$

Set the sign to $+1$ if this automorphism is orientation preserving or -1 otherwise.

Definition 3.15. We define the **equivariant intersection form**

$$\lambda\colon K_n(f) \times K_n(f) \to \mathbb{Z}[G]$$

by the formula

$$(\alpha, \beta) \mapsto \sum_{g \in G} \kappa\left(\alpha, g^{-1}\beta\right) g.$$

Remark. The reader should be warned that the proof that λ is well defined is actually quite hard. The classical proof requires passing through groups of immersions $(S^n \looparrowright X)$ modulo regular (normal) homotopy[14]. The definition is substantially more complicated if we drop assumption that X is simply connected.

To understand properties of the self intersection form $\mu(\alpha) = \lambda(\alpha, \alpha)$ we have to ponder for a moment its geometric interpretation. One might try do define the *self-intersection* simply by computing $\lambda(\alpha, \alpha)$. However, in this approach the definition depends not only

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[14] We will not prove this here as it would require another paragraph. For more detail we refer the interested reader to [13, Chapter 4.2]

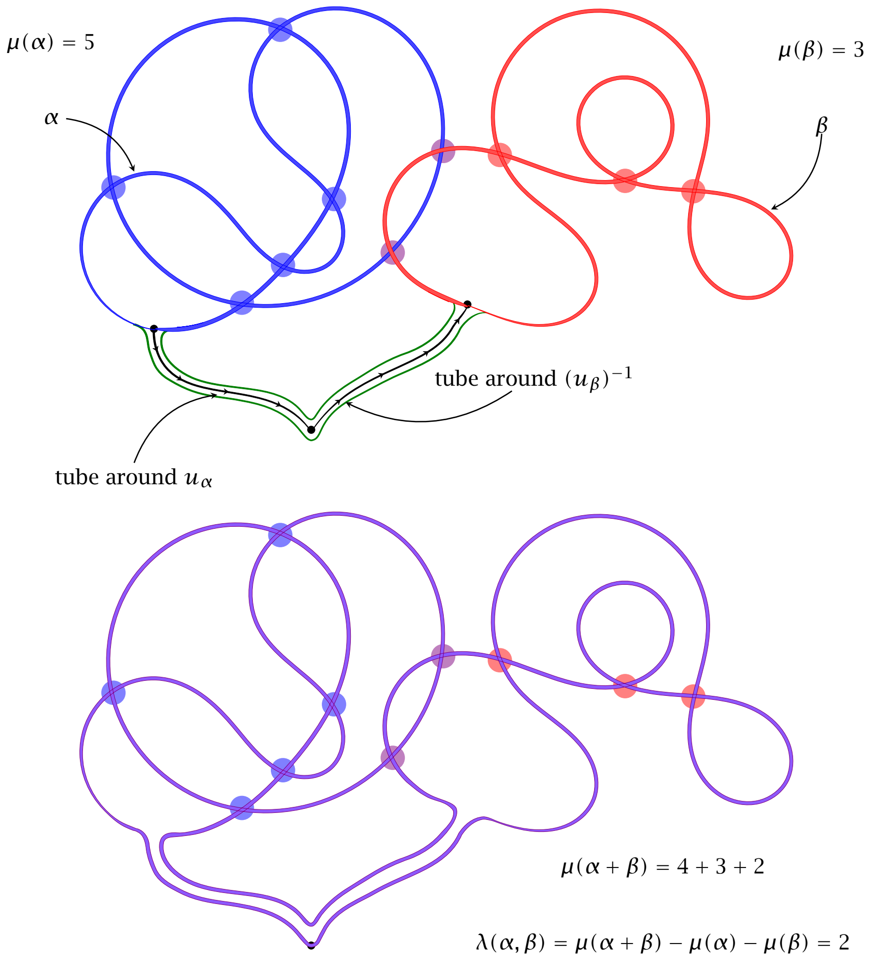

Figure 3: Self-intersection form

on α, but also on an order of the points in $\alpha^{-1}(b)$. Observe that altering the order of these points not only should send g to g^{-1}, but the sign (depending on orientation) might get reversed. Intuitively, changing the order of immersions should be an anti-involution of some kind.

Let us define

$$\Gamma \overset{\text{def.}}{=} \{x - (-1)^n \overline{x} \colon x \in \mathbb{Z}[G]\},$$

where $\overline{\cdot}$ is the natural anti-involution on $\mathbb{Z}[G]$ given by the formula

$$\overline{\sum_{g \in G} n_g g} \overset{\text{def.}}{=} \sum_{g \in G} \pm n_g g^{-1},$$

and the sign depends on whether g (understood as a diffeomorphism $X \to X$) preserves or reverses the orientation. Thus the equivariant self-intersection should not be able to distinguish different elements in Γ, so we anticipate $\mu(\alpha) \in \mathbb{Z}[G]/\Gamma$.

Definition 3.16. The **self-intersection** $\mu(\alpha)$ is defined as the composition of $\lambda(\alpha, \alpha)$ and the projection $\pi \colon \mathbb{Z}[G] \to \mathbb{Z}[G]/\Gamma$.

To understand μ algebraically decompose G as $\{1\} \sqcup G(2) \sqcup G(>2)$, where $G(2)$ ($G(>2)$ respectively) is the set of elements in G of order 2 (of order > 2, respectively). Then we have

$$\mathbb{Z}[G]/\Gamma = \mathbb{Z}/\varepsilon\langle 1\rangle \oplus \bigoplus_{a\in A} \mathbb{Z}/2\langle a\rangle \oplus \bigoplus_{b\in B} \mathbb{Z}\langle b\rangle \oplus \bigoplus_{c\in C} \mathbb{Z}\langle c\rangle,$$

where ε is 0 or 2 (depending on parity of n), every $a \in A \subset G(2)$ reverses the orientation, $B = G(2) \setminus A$ (i.e. $b \in G(2)$ preserves the orientation), and $C \subset G(>2)$ is the set of representatives of pairs $\{g, g^{-1} : g \in G(>2)\}$.

Remark. We may define addition on the set of regular homotopy classes of immersions $S^n \looparrowright X$, by the connected sum along the boundary of the tubular neighbourhood of the path $u_\alpha^{-1} * u_\beta$ connecting the base points of the spheres. Observe, that each intersection of α and β becomes a self-intersection of $\alpha + \beta$. Additionally self-intersections of α or β persist the connected sum construction, therefore we should have the following equality

$$\lambda(x, y) = \mu(x + y) - \mu(x) - \mu(y),$$

as can be observed in the figure above.

3.5.2 *Wall Classical L-groups*

Let us start with some notions coming from algebra. In the definition below n, m are elements of $\mathbb{Z}[G]$.

Definition 3.17. A $(-1)^n$**-quadratic form** is the triple (K, λ, μ) where K is a finitely generated, (stably) free $\mathbb{Z}[G]$-module,

$$\lambda : K \times K \to \mathbb{Z}[G]$$

is a $\overline{\cdot}$-twisted symmetric bilinear form on K, i.e.

$$\begin{aligned}\lambda(\alpha, n\beta_1 + m\beta_2) &= n\lambda(\alpha, \beta_1) + m\lambda(\alpha, \beta_2)\\ \lambda(n\alpha_1 + m\alpha_2, \beta) &= \lambda(\alpha_1, \beta)\overline{n} + \lambda(\alpha_2, \beta)\overline{m}\\ \lambda(\alpha, \beta) &= (-1)^n\overline{\lambda(\beta, \alpha)}\end{aligned}$$

and a map

$$\mu : K \to \mathbb{Z}[G]/\Gamma$$

satisfying

$$\begin{aligned}\mu(n\alpha) &= n\mu(\alpha)\overline{n},\\ \lambda(\alpha, \alpha) &= \mu(\alpha) + (-1)^n\overline{\mu(\alpha)}\\ \mu(\alpha + \beta) - \mu(\alpha) - \mu(\beta) &= \lambda(\alpha, \beta) \pmod{\Gamma}\end{aligned}$$

We used a similar notation in the previous section to suggest that the geometric data given by the homology surgery kernel, the equivariant intersection and the self-intersection forms fit precisely into this algebraic description.

Definition 3.18. The $(-1)^n$-hyperbolic form H is defined as

$$H \overset{\text{def.}}{=} \left(\mathbb{Z}[G] \oplus \mathbb{Z}[G], \lambda = \begin{bmatrix} 0 & 1 \\ (-1)^n & 0 \end{bmatrix}, \mu \equiv 0 \right).$$

By H^r we denote the orthogonal sum of r copies of H.

Definition 3.19. We define the (Wall's) **Surgery Obstruction L-group** in dimension $2n$, denoted by $L_{2n}(\mathbb{Z}[G], w)$, as the abelian group of the equivalence classes of non-degenerate $(-1)^n$-quadratic forms. The homomorphism $w\colon G \to \mathbb{Z}/2$ is the orientation character we used to set the appropriate signs while defining λ and μ.

Two such forms (K, λ, μ), (K', λ', μ') are called equivalent if they are stably isomorphic by a $\mathbb{Z}[G]$-isomorphism preserving λ's and μ's. In precise symbols this means that there exist natural numbers r, r' and a $\mathbb{Z}[G]$-module isomorphism

$$\Phi\colon K \oplus H^r \to K' \oplus H^{r'}$$

such that

$$\begin{aligned} \Phi(\mu(\cdot)) &= \mu'(\Phi(\cdot)) \text{ and} \\ \Phi(\lambda(\cdot,\cdot)) &= \lambda'(\Phi(\cdot), \Phi(\cdot)). \end{aligned}$$

Stabilisation in the definition is justified by the following lemma.

Lemma. *Let $(f, b)\colon X \to Y$ be a normal map of degree one, which is connected up to n, the middle dimension, where* $\dim X = \dim Y = 2n$. *Suppose that the non-degenerate $(-1)^n$-quadratic form $(K_n(f), \lambda, \mu)$ is stably isomorphic to H^r, a direct sum of the hyperbolic form H. Then we can perform a finite number of surgery steps on X to obtain a normal map*

$$(f', b')\colon X' \to Y$$

such that $f'\colon X' \to Y$ is a homotopy equivalence.

By the virtue of the lemma above, the zero element in the L-group is represented by class of the hyperbolic form H, whereas the inverse of (K, λ, μ) equals $(K, -\lambda, -\mu)$.

Definition 3.20. Given a degree 1-normal map (f, b) we define its **surgery obstruction** $\sigma(f, b)$ as follows.

By a finite number of surgery steps we may say that (f, b) is normally cobordant to a map (f', b') connected up to the middle dimension. In the non-equivariant case we may assume that homology surgery kernel $K_n(f')$ of (f', b') is stably free as a $\mathbb{Z}[\pi]$-module.

We perform the appropriate surgery on the trivial spheres to make $K_n(f')$ actually free. Then $(K_n(f'), \lambda, \mu)$ carries the structure of a non-degenerate $(-1)^n$-quadratic form. Finally we **define**

$$\sigma(f, b) \overset{\text{def.}}{=} [K_n(f'), \lambda, \mu] \quad \in L_{2n}(\mathbb{Z}[G], w).$$

It turns out that the class of $(K_n(f'), \lambda, \mu)$ is a normal bordism invariant, hence the definition actually makes sense.

3.5.3 *Obstruction Groups for Equivariant Surgery*

So far we assumed that G acts freely on X. We will lift this assumption now. First let us introduce some notations. Set

$$\begin{aligned} X^{>H} &\overset{\text{def.}}{=} \left\{x \in X^H \colon G_x > H, G_x \neq H\right\}, \\ X^{=H} &\overset{\text{def.}}{=} \left\{x \in X^H \colon G_x = H\right\}, \text{ and} \\ X^{=H} &\overset{\text{def.}}{=} X^H \setminus X^{>H}. \end{aligned}$$

Suppose that $f^L\colon X^L \to Y^L$ is already a homotopy equivalence for all $L > H$. Recall that X satisfies the strong GAP-condition (see Section 3.2.3) if

$$2\left(\dim X^L_\alpha + 1\right) < \dim X^H_\alpha$$

for $H < L$ and if $X^L_\alpha \subsetneq X^H_\alpha$.

In particular this means that we can use G-surgery of type (H) below the middle dimension on

$$X^{=H} = X^H \setminus X^{>H}$$

without altering the L-fixed point set for all overgroups L of H. We should pay special attention here to make all necessary homotopies stay in $X^{=H}$.

Suppose for a moment, that X^H is connected. Then the Weyl group $W(H) = N_G(H)/H$ acts freely on $X^{=H}$, hence we have a similar situation in the middle dimension as in the free case. The homology kernel then is $\mathbb{Z}[W(H)]$-module and we perform $N_G(H)$-surgeries of type (H) on $X^{=H}$. These surgeries can be later on induced (i.e. taking ind_H^G instead of $\operatorname{ind}_H^{N_G(H)}$ and thickening the normal disc bundle $D(V)$) to G-surgeries that keep $X^{>H}$ intact.

If X^H is not connected, then for every component α of Y^H_α we have to consider a separate "surgery problem". Since f is of degree one we may assume that $X^H_\alpha = f^{-1}(Y^H_\alpha)$ is non-empty. Let $N(\alpha) < N_G(H)$ denote the subgroup of G which preserves component X^H_α as a whole. Similarly $W(\alpha) = N(\alpha)/H$ acts freely on H^H_α. From a different point of view, $W(\alpha)$ is the stabiliser of X^H_α in the G-action on $\Pi(Y)$. Self intersections of the immersed spheres representing generators of $K^n(f^H_\alpha)$ determine whether the G-surgery of type (H) to a homotopy equivalence on f^H_α is possible.

Remark 3.21. Note that while trying to perform surgery in the middle dimension on $X^{=H} = X^H \setminus X^{>H}$ we have actually two problems to overcome. Firstly, the spheres of $K_n(f_\alpha^H)$ may link in a non trivial way with each other, however this is supposed to be dealt with by the surgery obstruction. Secondly, if the difference

$$2n(= \dim X^H) - \dim X^{>H}$$

is less than or equal to $n+1$ then we might have a situation where $S^n \looparrowright X^{=H}$ links non-trivially with $X^{>H}$ (as there is not enough space to perform all necessary regular homotopies). To avoid the second problem we may want to assume that

$$n + 1 < 2n - \dim X^{>H}$$

or equivalently

$$\dim X^{>H} + 1 < n.$$

This is precisely the strong GAP-condition.

In particular, (see [30, Proposition 2.10]) if G is of odd order or Y satisfies the strong GAP-condition, and Y_α^H is simply connected, then the obstruction

$$\sigma_\alpha^H(f) \overset{\text{def.}}{=} \sigma(f_\alpha^H)$$

for converting $f_\alpha^H\colon X_\alpha^H \to Y_\alpha^H$ to a homotopy equivalence belongs to the classical Wall L-group

$$L_{2n}^h(\mathbb{Z}[W(\alpha)], \text{triv.}),$$

If these obstructions vanish consequently for all subgroups $H < G$, then one may proceed with surgery on the poset $\Pi(X)$ (on each H-isotropy set separately and away from $X^{>H}$) to modify f to a homotopy equivalence (for an alternative description using the splitting of obstruction groups, see [30, Proposition 2.10]).

BAK GROUPS However, in our applications G is a perfect group hence of even order. Moreover, Y does not satisfy (SGC), but as can be seen in Section 4.1.3, a condition slightly stronger than the GAP-condition, but still weaker than the strong GAP-condition. Thus we can not hope[15] for the obstructions to belong to the Wall L-groups. The proper obstruction groups for this situation were defined by A. Bak in [2] and recognized by M. Morimoto in [34, 36]. Further refinements of these groups were obtained by the authors in the joint papers [3, 4, 5]. The Bak groups are traditionally denoted by

$$W_{2n}(\mathbb{Z}[G], \Gamma G(Y); w).$$

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[15] This was actually a common belief in the eighties, corrected only around 1990 in papers of M. Morimoto [34, 36], W. Lück and I. Madsen in [29, 30].

Here we want to indicate that despite more complicated definition, these groups have quite similar properties to Wall's groups and serve exactly the same purpose, but have broader applications. For example the manifolds we consider need only to satisfy the (ordinary) GAP-condition (and in some cases only weak GAP-condition) to define the surgery obstruction to the equivariant surgery leading to a homotopy equivalence. The following theorem summarises the role of surgery obstructions in the equivariant surgery.

Definition 3.22. Let $G(Y) \subset G$ denote the set of 2-torsion elements in G such that

$$2(\dim Y^g + 1) = \dim Y.$$

- We define

  $$\Gamma G(Y) \overset{\text{def.}}{=} \langle x - (-1)^n \overline{x}, h \colon x \in \mathbb{Z}[G],\ h \in G(Y) \rangle_{\mathbb{Z}}$$

  to be the additive subgroup of $\mathbb{Z}[G]$ generated by elements in Γ and $G(Y)$.

- A $\Gamma G(Y)$-quadratic module is a triple (K, λ, μ) such that
  - K is a stably free $\mathbb{Z}[G]$-module,
  - $\lambda \colon K \times K \to \mathbb{Z}[G]$ is a $(-1)^n$-quadratic form, and
  - $\mu \colon K \to \mathbb{Z}[G]/\Gamma G(Y)$ is a homomorphism

  satisfying the familiar properties, as in definition 3.17.

Definition 3.23. We define the **Bak Surgery Obstruction group** in dimension $2n$, denoted by $W_{2n}(\mathbb{Z}[G], \Gamma G(Y), w)$, as the abelian group of stable isomorphism classes of non-degenerate $\Gamma G(Y)$-quadratic modules.

The stabilisation is defined again by the direct sum with the hyperbolic form.

The Bak groups can be seen as a generalisation of L-groups. For example in the case of odd order groups we have $\Gamma G(Y) = \Gamma \varnothing = \Gamma$ and $w =$ triv., hence the Bak group reduces to the ordinary L-group. The following theorem summarises the role of the Bak groups in the equivariant surgery.

Theorem 3.24 (A. Bak, M. Morimoro, [5, Theorem 1.1])**.** *Let Y be a smooth simply-connected G-manifold of dimension $2n \geq 6$. Let*

$$(f, b) \colon (X, TX) \to (Y, f^*\xi)$$

be a degree one G-normal map. Suppose that

- *Y satisfies* (GC)*,*

- *X satisfies* (SGC) *for every subgroup $L < G$ containing $\langle 1, g\rangle$ for some $g \in G(Y)$ as a proper subgroup, i.e.*
$$2(\dim X^L + 1) < \dim X = 2n,$$
- *and that $K_n(f) = \ker(f_*\colon H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z}))$ is stably free as a $\mathbb{Z}[G]$-module.*

Then the obstruction $\sigma(f,b)$ is defined in $W_{2n}(\mathbb{Z}[G], \Gamma G(Y); w)$ such that if $\sigma(f,b) = 0$, then f can be converted by (free) G-surgery to a degree one G-normal map (f', b') such that f' is a homotopy equivalence.

Moreover if $(f', b')\colon (X', TX') \to (Y, f^*\xi)$ is a different degree one G-normal map, such that (f', b') is G-normally bordant to (f,b), then
$$\sigma(f', b') = \sigma(f, b) \quad \text{in } W_{2n}(\mathbb{Z}[G], \Gamma G(Y); w).$$

We will refer to this property as the ***bordism invariance of obstructions.***

It seems that using this theorem we could not bother about intermediate surgery obstructions and jump directly into free surgery. However guaranteeing the third assumption requires knowledge of the fixed point sets X^P for p-groups, as well as the Euler characteristic for the fixed point sets of cyclic groups. Moreover killing the final obstruction by Dress' Induction (see the next section) requires some knowledge of obstructions for different subgroups of G. Hence the significant part of the proof will be devoted to H-surgery for subgroups $H < G$.

Remark 3.25. It is also worth noting that larger dimensional gaps between H-isotropy strata does not automatically prove that surgery is easier. For example suppose that $G = \mathbb{Z}/2 = \langle g\rangle$ acts preserving orientation on $4n + 2$ dimensional manifold X with the fixed point set $X^G = X^{\{g\}}$ of dimension $2n$. Then there are no obstructions for G-surgery, as $W_{4n+2}(\mathbb{Z}[\mathbb{Z}/2], \Gamma\{g\}, \text{triv.}) = 0$. On the other hand, if X^G is of dimension strictly less than $2n$, then the obstruction belongs to the Wall group $L_{4n+2}(\mathbb{Z}[\mathbb{Z}/2], \text{triv.}) = \mathbb{Z}/2$, so to perform surgery we need some more arguments (e.g. based on geometric properties of X), to claim that the obstruction vanishes.

INDUCTION THEOREMS In this paragraph we quote induction theorems for the Surgery Obstruction Groups. These theorems allow us to compute surgery obstruction $\sigma(f,b)$ in terms of
$$\operatorname{res}^G_H \sigma(f,b) = \sigma\left(\operatorname{res}^G_H (f,b)\right).$$

As could be anticipated, our aim is to prove that certain obstruction vanish, hence it would be useful to find an injective homomorphism form the obstruction group $L_{2n}(\mathbb{Z}[G], w)$ to some groups which are easier to compute. Such monomorphism is provided by Dress' Induction Theorem below.

Recall that a subgroup H of G is called 2-**hyperelementary** if it is an extension of odd order cyclic group by a 2-group, i.e. there exist a short exact sequence

$$0 \to C \to H \to P \to 0,$$

where C is a cyclic group of odd order and P is a 2-group.

Theorem (A. Dress [18, Theorems 1 and 3], A. Bak [2, Theorems 12.9, 12.10], M. Morimoto [34, Proposition 7.3]). *Let $\mathcal{OG}(G)$ denote either Wall's L-group $L_{2n}(\mathbb{Z}[G], w)$ or Bak group $W_{2n}(\mathbb{Z}[G], \Gamma G(Y), w)$. The restriction homomorphism*

$$\mathcal{OG}(G) \to \prod_{H \in \mathcal{H}} \mathcal{OG}(H)$$

is injective, where product runs over $\mathcal{H}$ - the family of all 2-hyperelementary subgroups of G.

By the theorem, to prove that the surgery obstruction $\sigma(f, b)$ vanishes, it suffices to prove that in the same setting, when we forget G-actions and treat the underlying geometrical objects (manifolds, cobordisms, normal maps, etc) as objects with H-actions, the corresponding obstruction vanishes.

3.5.4 *Reflection Method*

The Reflection Method is a method to improve a G-normal map and an H-normal cobordism to a ones more suitable for surgery. As this is the crucial step in our construction we will expand this section to explain it well.

We start with a non-equivariant example. Set $I = [0, 1]$. Let

$$(F, B)\colon (W, \partial W) \to (Y \times I, Y \sqcup Y)$$

be a normal bordism between

$$(f, b)\colon X \to Y \times \{0\} \quad \text{and} \quad (\mathrm{id}_Y, \mathrm{id}_{\tau(Y)})\colon Y \to Y \times \{1\}.$$

and $(F, B)|_{\partial W} = (f, b) \sqcup (\mathrm{id}_Y, \mathrm{id}_{\tau(Y)})$. Denote by

$$F_-\colon (W_-, \partial(W_-)) \to (Y \times [-1, 0], Y \sqcup Y)$$

the reversed normal cobordism, i.e. $\partial(W_-) = Y \sqcup X$,

$$F_-|_Y(Y) = \mathrm{id}_Y(Y) = Y \times \{-1\}, \quad \text{and} \quad F_-|_X(X) = f(X) = Y \times \{0\}.$$

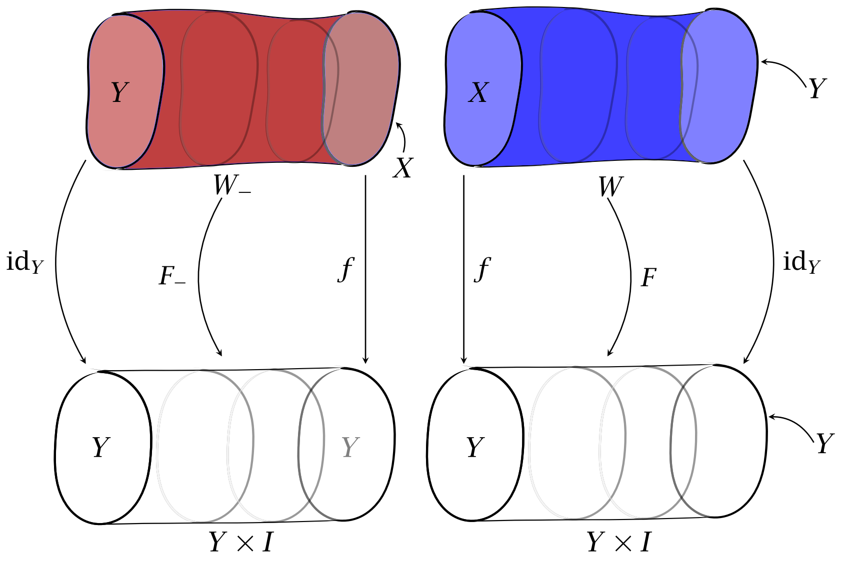


Figure 4: Reflection Method

We may glue these cobordisms along $f\colon X \to Y \times \{0\}$ to obtain a new cobordism $F_\cup\colon W_\cup \to Y \times [-1,1]$ between id_Y and id_Y. The main point of the construction is that the Cartesian product $W \times [-1,1]$ is a normal cobordism (relative to the boundary)

$$\text{from} \quad F_\cup = F_- \cup \mathrm{id}_{X\times[-1,1]} \cup F \quad \text{to} \quad \mathrm{id}_Y \times [-1,1].$$

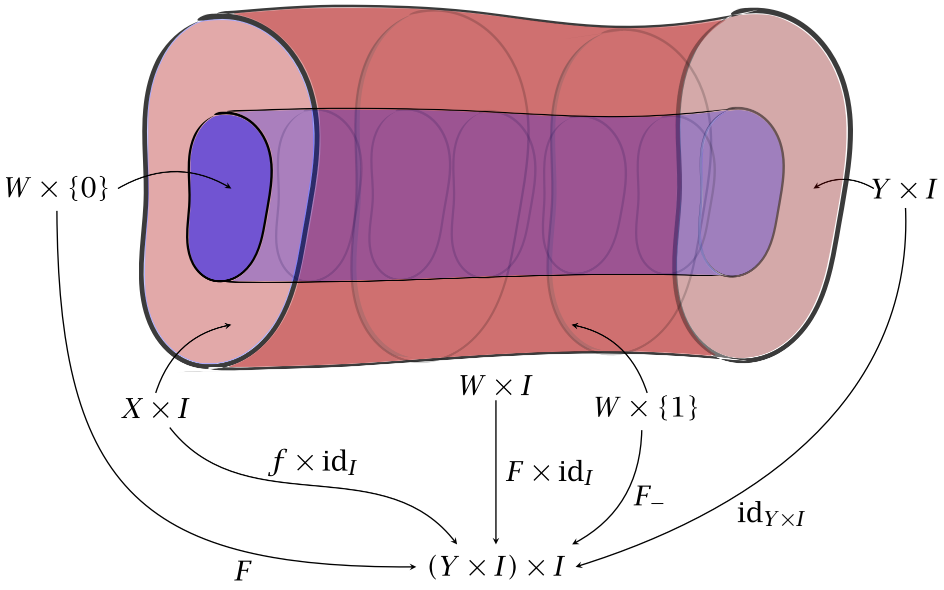


Figure 5

The Reflection Method in G-surgery is a play between G-surgery, G-normal bordisms, and H-normal bordisms (relative boundary which have G-bordisms as their boundaries. It is quite easy to get confused, so as a general rule one may remember that in the following G-normal bordisms are always traces of G-surgery on X or Y, whereas H-normal bordisms are either given (e.g. by the Equivariant Transversality Construction) or are restrictions of G-bordisms.

Let $f\colon X \to Y$ be a G-normal map, $H < G$ and

$$F_H\colon W_H \to \operatorname{res}^G_H(Y \times I)$$

an H-normal cobordism between $\operatorname{res}^G_H f$ and $\operatorname{res}^G_H \mathrm{id}_Y$.

Theorem 3.26 (Reflection Method, [35, Theorems 4.2, 4.7]). *Consider the following four assumptions.*

(A1) *K is a subgroup of H such that $N_G(K) \leqslant H$;*

(A2) *There exist a closed H-regular neighbourhood U of H-orbit of $W_H^{>K} \subset W_H$ which looks like a product, i.e. there exists an H-embedding*

$$\psi\colon (X \cap U) \times I \hookrightarrow W_H,$$

which is the canonical inclusion when restricted to $(X \cap U) \times \{0\}$;

(A3) $2(\dim X_\alpha^{>K} + 1) \leqslant \dim X_\alpha^K$*;*

(A4) $\pi_j\left((W_H)_\beta^K, Y_\alpha^K\right) = 0$ *for all* $j \leqslant \dim X_\alpha^{>K}$.

As usual the α-subscript denotes the α-th connected component of a space. If all spaces under consideration are connected we may drop subscripts uniformly. If they are disconnected then we require that these assumptions are satisfied for all α, β such that $X_\alpha^{>K} \subset (W_H)_\beta^K$.

Then the following three conclusion hold.

(C1) *One can perform G-surgery of type (K) on $f\colon X \to Y$ to obtain a G-normal cobordism from $(f\colon X \to Y)$ to $(f'\colon X' \to Y)$ such that*

$$(f')^K : X^K \to Y^K$$

is a homotopy equivalence;

(C2) *One can perform H-surgery of type (K) on $F_H\colon W_H \to Y \times I$ to obtain an H-normal cobordism from $(F_H\colon W_H \to Y \times I)$ to $(F'_H\colon W'_H \to Y \times I)$ such that*

$$(F'_H)^K : (W'_H)^K \to Y^K \times I$$

is a homotopy equivalence.

Moreover, these surgeries are performed on $X^{=H}$ ($W_H^{=H}$, respectively) hence they do not alter X^L (W_H^L, respectively) for all subgroups L of G not conjugated to a subgroup of K.

Finally,

(C3) *there exist a closed H-regular neighbourhood U' of*

$$H\,(W'_H)^K \subset W'_H$$

containing U which looks like a product, i.e. there exists an H-embedding

$$\psi'\colon (X' \cap U') \times I \hookrightarrow W'_H,$$

extending ψ, which is the canonical inclusion when restricted to $(X' \cap U') \times \{0\}$.

Proof. We will only sketch the proof.

First, using Assumptions (A3) and (A4) we will extend the H-diffeomorphism ψ of (A2) to an H-diffeomorphism

$$\Psi\colon (N \cap X) \times I \to N,$$

where N is a product H-neighbourhood of $HX^{>K} \cup HW_H^{>K} \subset W_H$. Note that if $L < G$, is a subgroup properly containing H, such that $L \cap H = K$, then $X^L \subset X^{>}K$, but $X^L \subset W_H^K \setminus W_H^{>K}$.

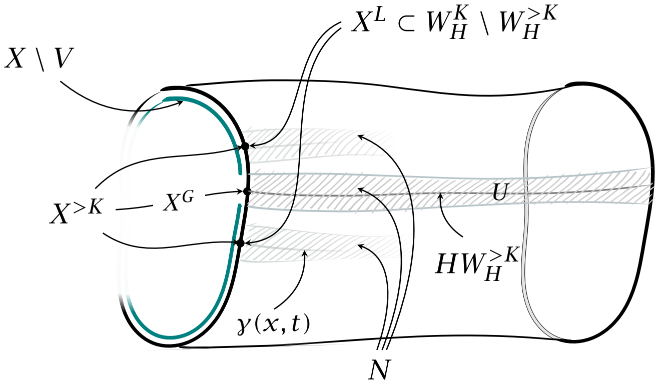


Figure 6

Then we will use the prescribed above geometric procedure to modify f and F_H according to Conclusions (C1) and (C2).

Denote by α the composite

$$\alpha \overset{\text{def.}}{=} \psi \circ i\colon \left(HX^{>K} \cap U\right) \times I \hookrightarrow (X \cap U) \times I \to U \subset W_H,$$

where the first map is the inclusion. We will show that α extends to an H-embedding $\beta\colon HX^{>H} \times I \hookrightarrow W_H$. Then the tubular neighbourhood of the embedding will form N (and canonically Ψ), as stated above.

Let $V = \operatorname{Int}(X \cap U) \times I$. To show the existence of the extension β it is enough to show that we can extend the $N_H(K)$-embedding $\alpha^K|_\partial$ from

$$\left((X \setminus V)^{>K} \cap (U \setminus V)^K\right) \times I \to (U \setminus V)^K$$

(which is the boundary of $(X^{>K} \cap U) \times I$), to

$$(X \setminus V)^{>K} \times I \to (W_H \setminus V)^K,$$

the remaining part of $X^{>K} \times I$.

Since $\dim X^K = \dim Y^K$ for all $K < G$ (by the assumption on the degree of map f), from assumption (A3) we obtain

$$\begin{aligned} j + 1 + \dim W_H^{>K} &< \dim W_H^K \quad \text{and} \\ j + 1 + \dim Y^{>K} &< \dim Y^K + 1 \end{aligned}$$

for all $j \leqslant \dim X^{>K}$. Therefore the same equalities are valid if we subtract V from $W_H^{>K}$, W_H^K, $Y^{>K}$, and Y^K.

Moreover, by the homotopy excision property and Assumption (A4), we have

$$\pi_j\left(W_H^K \setminus V, Y^K \setminus V\right) \cong \pi_j\left(W_H^K, Y^K\right) = 0$$

for $j \leqslant \dim X^{>K}$. Thus we can extend $\alpha^K|_\partial$ to an $N_H(K)$-equivariant map

$$\gamma' \colon (X \setminus V)^{>K} \to (W_H \setminus V)^K.$$

Again using (A3) we can take an $N_H(K)$-embedding $\gamma \colon (X \setminus V)^K \times I \to (W_H \setminus V)^K$ approximating γ'. We can finally define

$$\beta \overset{\text{def.}}{=} \alpha \cup H\gamma \colon \left(HX^{>K} \cap V\right) \times I \cup (X \setminus V)^{>K} \times I \to W_H^K.$$

Set N to be an H-tubular neighbourhood of the image of β and denote $\Psi \colon (X \cap N) \times I \to W_H^K$ as the embedding of the neighbourhood.

Using diffeomorphism Ψ we may write

$$W_H^K = X \times I \cup \bigcup \operatorname{ind}_K^{N_H(K)} D^{k_i} \times D^{2n-k_i+1} \times D(V_i),$$

as the trace of some $N_H(K)$-surgeries of type (K) on X performed away from N. This means that Y^K is the result of $N_H(K)$-surgeries on X.

We may regard $F_H \colon W_H \to Y \times I$ as the result of glueing H-normal cobordism $F_H' \colon W_H' \to Y \times I$ from $\operatorname{res}_H^G f$ to $\operatorname{res}_H^G \mathrm{id}_Y'$ and an H-normal homotopy equivalence $\mathrm{Id}_H \colon Y \times I \to Y \times I$, such that $\mathrm{Id}_H|_{Y \times \{0\}} = \mathrm{id}_Y'$ and $\mathrm{Id}_H|_{Y \times \{1\}} = \operatorname{res}_H^G \mathrm{id}_Y$.

Perform the surgeries leading from $\operatorname{res}_H^G f^K$ to id_K', thickened to G-surgeries of type (K) starting from X (recall that these surgeries may be performed away from N). This leads to a new G-normal bordism

$$F_+ \colon W_+ \to Y \times I, \quad \text{from} \quad (f \colon X \to Y) \quad \text{to} \quad (f' \colon X' \to Y).$$

Observe that, by the choice of surgeries,

$$f'^K = {\mathrm{id}'_Y}^K \simeq \left(\mathrm{res}^G_H \mathrm{id}_Y\right)^K$$

which proves Conclusion (C1).

To prove Conclusion (C2) denote by $F_- : W_- \to Y \times I$ the reversed cobordism F_+, i.e. the G-normal bordism W_- between f' and f such that $F_-(X') = Y \times \{-1\}$ and $F_-(X) = Y \times \{0\}$. Glue together $\mathrm{res}^G_H F_-$, and F_H along $\mathrm{res}^G_H f$. The resulting H-normal bordism

$$F_\cup : W_\cup \overset{\text{def.}}{=} W_- \cup_X W'_H \to Y \times [-1, 1]$$

is a bordism between

$$\mathrm{res}^G_H f' : \mathrm{res}^G_H X' \to Y \quad \text{and} \quad \mathrm{id}'_Y : \mathrm{res}^G_H Y \to \mathrm{res}^G_H Y.$$

By the same observation as in the non-equivariant case

$$F^K_\cup : W^K_\cup \to Y^K \times [-1, 1]$$

is $N_G(K)$-normally cobordant to the product normal map ${\mathrm{id}'_Y}^K \times I$ (this is cobordism of cobordisms relative boundary). The cobordism connecting $F^K_\cup$ and $\mathrm{id}^K_{Y \times I}$ is precisely $(F_H)^K \times I$ and can be seen as H-normal cobordism, as in Figure 5.

Now we to apply the same trick as before: starting from $F_\cup$ we perform surgeries (on the interior of $W_\cup$) leading to Γ, the cobordism above, but in the reversed order. Then we glue the result along $F_\cup$ to obtain a new cobordism Γ between $\Gamma_- \cup_{W_\cup} \Gamma$. The interested reader may look up the details in [35, Proof of Theorem 4.2].

□

Conclusion (C3) allows us to proceed with an inductive argument. Here we briefly sketch the initial step. In this case $H = G$.

- We start with a maximal (proper) subgroup K which is self normalising (this is (A1)).
- Assume that that the fixed point set X^G together with a tubular neighbourhood U is already G-diffeomorphic to a part of the fixed point set Y^G. This will be obvious from the Equivariant Transversality Construction. Then we can embed H-equivariantly $U \times I$ into W_H which guarantees (A2).
- We use the GAP-conditions and surgery below the middle dimension to obtain (A3) and (A4).
- Now we use the Reflection Method and modify X to X' such that $X'^K \to Y^K$ is a homotopy equivalence (this is Conclusion (C1))

- since (by (C2)) we can modify the normal H-bordism between $(f'\colon X' \to Y)$ and $(\mathrm{id}_Y\colon Y \to Y)$ such that the H-normal bordism connecting X'^H and Y^H is H-diffeomorphic to $X \times I$ (possibly with concatenation of a homotopy equivalence at the $X \times \{1\}$), Assumption (A2) is fulfilled for $H = K$.
- Finally Conclusion (C3) allows us to take a subgroup K' of K such that $N_G(K') \leqslant K$ and carry on the inductive step once again.

4

RESULTS

Recall the main theme of our Programme: absorb the fixed point set of a group action on a sphere S^{2n} and then use surgery to delete the superfluous connected components created by the projectivisation of a unitary representation.

4.1 PREPARING THE SETTING

Our plan for this section is as follows. Given a smooth action of a group G on the sphere S^j with $F \cong (S^j)^G$ we modify the action of G on S^j in the following fashion.

1. We introduce a new connected component F_0 to the fixed point set;
2. Since we want to create a complex projectivisation of $T_pS^M = V$ for $p \in F_0$, therefore we require that
   - M is be even, say $M = 2n$, and
   - $\dim F_0$ is even;
3. We create the equivariant connected sum
$$S^{2n} \# \mathbb{C}VP^n.$$
4. Assume that $\mathrm{Iso}(S^{2n})$ is *suitable* for equivariant surgery (as S^{2n} was created via equivariant surgery).
5. If we want to use equivariant surgery then
   - we require also $\mathrm{Iso}(\mathbb{C}VP^n)$ to be *suitable*, and
   - we check that $S^{2n} \# \mathbb{C}VP^n$ satisfies the GAP-conditions.

After completing these steps we are ready to start the process of equivariant surgery. This is done in the next section.

If we are done with the surgery step, we have found a smooth manifold X homotopy equivalent to $\mathbb{C}P^n$ equipped with a smooth G-action such that $X^G = F$.

We begin with identifying the normal bundle of the fixed point set obtained from the construction of M. Morimoto, see Theorem 2.2.

Consider a smooth action of a finite group G on a sphere. Suppose that the fixed point set is diffeomorphic to a (not necessarily connected) manifold F. Using Theorem 2.2 we may modify the action to add an even-dimensional sphere S^{2k} to the fixed point set. Since we want to create the complex projectivisation of the tangent space T_pS^N for $p \in S^{2k}$ we are interested whether it can be endowed with a complex structure.

4.1.1 *Complex structure on T_pS^n*

Suppose that a closed, smooth manifold

$$F = F_0 \sqcup F_1$$

can be realized as the fixed point set of a smooth G-action on a sphere S^j. Choose $p \in F_0 \subset S^j$ and consider the tangential G-module T_pS^j.

The tangential G-module decomposes as

$$T_pS^j = \tau(F_0)|_p \oplus \nu(F_0)|_p \cong \mathbb{R}^{\dim F_0} \oplus \nu(F_0)|_p$$

where the G-action on $\nu(F_0)|_p$, the fibre over p of the normal bundle of F_0, is without fixed points (except the origin). Since $\tau(F_0)|_p$ has the trivial action, then (as a necessary condition to the existence of a complex strucure) we need $\dim F_0 = 2k$. Assume so.

The process of obtaining actions on spheres with a given fixed point set follows through results of B. Oliver ([43], actions on discs) and then taking the double of the disc. By the work of B. Oliver, the only[1] obstructions are the K_G-theoretical structure of the fixed point set F. Namely, the class $[\tau(F)]$ must belong to the appropriate part of the reduced K-theory ring (see Theorem 2.1).

Recall that for a G-module W we denote by $\underline{W}$ the product G-vector bundle

$$X \times W \to X.$$

Consider the G-bundle over F_0

$$\tau(F_0) \oplus \nu(F_0) \oplus \underline{\nu(F_0)|_p} \longrightarrow F_0.$$

This bundle has an obvious extension to a bundle over the whole F, namely

$$\eta \overset{\text{def.}}{=} \left(\tau(F) \oplus \nu(F) \oplus \underline{\nu(F_0)|_p} \longrightarrow F\right).$$

We claim that $\tau(F) \oplus \nu(F) \oplus \underline{\nu(F_0)|_p}$ is a good candidate for the normal bundle of the G-fixed point set of a smooth G-action on the disc D^{2n}. To prove the claim we need to introduce another piece of theory.

---

[1] In the case of perfect groups.

Let $\operatorname{res}^G_H \xi$ denote the element of the reduced H-equivariant KO-theory of F determined by the vector bundle $\operatorname{res}^G_H \xi$ obtained from ξ by restricting the action to a subgroup H of G. Let p be a prime dividing the order of G and P a p-subgroup of G. Consider the subgroup $p\text{-}div < \widetilde{KO}_P(F)_{(p)}$, of all infinitely p-divisible elements in the (localised at p) K-thory ring. B. Oliver in [43] has defined the obstruction

$$\mathcal{O}\ell(\xi) \overset{\text{def.}}{=} \operatorname{res}^G_{\{e\}}(\xi) + \sum_{P \neq \{e\}} \left[\operatorname{res}^G_P(\xi)\right] \in \widetilde{KO}(F) \oplus \widetilde{KO}_P(F)_{(p)}/p\text{-}div\ ,$$

for extending the G-vector bundle ξ over F to a G-vector bundle Ξ over a contractible G-CW-complex. The following theorem is a consequence of Theorem 2.1 of B. Oliver and the Equivariant Thickening Theorem due to K. Pawałowski [46].

Theorem 4.1 (see [47, Theorem 8.2])**.** *Let G be a finite perfect group. Let F be a smooth compact manifold. Let ν be a real G-vector bundle over F such that $\dim \nu^G = 0$. Then the following two statements are equivalent.*

- *There exists a smooth action of G on some disc D such that $D^G \cong F$ and as G-vector bundles $\nu(F \hookrightarrow D) \cong \nu \oplus l\,U(G)$.*
- *$\mathcal{O}\ell(\tau(F) \oplus \nu) = 0$.*

If $\tau(F) \oplus \nu(F)$ satisfies the second condition of the theorem, then $\eta = \tau(F) \oplus \nu(F) \oplus \nu(F_0)|_p$ satisfies the condition as well. Indeed, η and $\tau(F) \oplus \nu(F)$ differ only by a direct summand which is a product bundle, hence K-theory classes $[\tau(F) \oplus \nu(F)]$ and $[\eta]$ are equal. Therefore we can obtain a G-action on a disc D^M, and then (by forming its double) on the sphere S^M which realises $F \sqcup F$ as the fixed point set. Note that, by construction, $M = 2k + 2\dim \nu(F) + 2l \dim_{\mathbb{R}} (r_{\mathbb{C}}(U(G)) = 2n$, where $r_{\mathbb{C}}$ denotes the realification of a complex G-vector bundle.

From now on we assume that
G is a finite perfect group.

We remark, that the assumption is justified later (or rather forced upon us) by Proposition 4.4.

By Theorem 2.2, we may assume that the sphere S^{2n} contains precisely F as the fixed point set. Over a single point $x \in F \subset S^{2n}$ the tangent fibre $\tau\,(S^{2n})|_x$ decomposes (as G-module)

$$\tau(F)|_x \oplus \nu(F)|_x \oplus \nu(F_0)|_p \oplus l\,U(G)$$

hence over $p \in F_0$ we have the following isomorphisms

$$\begin{aligned}\tau\left(S^{2n}\right)\Big|_p &= \tau(F_0)|_p \oplus \nu(F_0)|_p \oplus \nu(F_0)|_p \oplus lU(G)\\ &\cong \mathbb{R}^{2k} \oplus \left\langle \nu(F_0)|_p\right\rangle \oplus i\left\langle \nu(F_0)|_p\right\rangle \oplus l\,U(G)\\ &\cong \mathbb{C}^k \oplus \left(\nu(F_0)|_p \otimes \mathbb{C}\right) \oplus l\,U(G).\end{aligned}$$

This proves the following lemma.

Lemma 4.2. *Let G be a finite perfect group and suppose that G acts smoothly on a disc with the fixed point set F such that $\partial F = \varnothing$. Suppose moreover that $\dim F_0 = 2k$ for a connected component $F_0 \subset F$. Then G acts on an even-dimensional sphere S^{2n} with the fixed point set F and for every $x \in F_0$, the tangent space $T_x S^{2n}$ can be endowed with a complex structure.*

Knowing that the connected component F_0 has a complex structure on a fibre of the normal bundle we want to describe explicitly the tangential representation at some $p \in F_0 \subset S^{2n}$.

Lemma 4.3. *In the setting as above the G-action on S^{2n} can be chosen in such a way that, for a point $p \in F_0$, we have*

$$T_p S^{2n} \cong k\mathbf{1}_G \oplus mU(G)$$

for a positive integer r.

Proof. Since $T_p S^{2n}$ has a complex structure, by the previous lemma, we may decompose it as the direct sum of complex irreducible representations

$$T_p S^{2n} \cong k\mathbf{1}_G \oplus \bigoplus_{\chi \neq \mathbf{1}_G} n_\chi \chi.$$

Recall that for a perfect group G, we have $U(G) \cong r(G) - \mathbf{1}_G$ thus, for a sufficiently large s (i.e. $s \geqslant \max_\chi\{n_\chi\}$), we may treat $T_p S^{2n}$ as a direct summand of the G-module $sU(G) \oplus k\mathbf{1}_G$. Consider the representation

$$W \overset{\text{def.}}{=} (sU(G) \oplus k\mathbf{1}_G) - T_p S^{2n},$$

where the minus sign denotes taking the orthogonal complement in some G-invariant metric.

By the same argument as above we have

$$\mathcal{O}\ell(\tau(F) \oplus \nu(F) \oplus \underline{W}) = \mathcal{O}\ell(\tau(F) \oplus \nu(F)) = 0.$$

By Theorem 4.1 we may realise F as the fixed point set of a G-action on an even-dimensional disc with the normal bundle isomorphic to $\tau(F) \oplus \nu(F) \oplus \underline{W} \oplus \underline{lU(G)}$, hence on the sphere $S^{2k+2s+2l}$ of the same dimension. By the construction, the G-representation on the tangent space at $p \in F_0$ is isomorphic to

$$T_p S^{2k+2s+2l} \cong k\mathbf{1}_G \oplus \nu(F_0)|_p \oplus W \oplus lU(G) \cong k\mathbf{1}_G \oplus (s+l)U(G).$$

□

We refer the interested reader to [38] for the details construction of the action on a sphere which we used above. In Section 6 therein, one can find the precise construction and a different argument for the normal bundle of the fixed point set.

4.1.2 $\mathcal{L}(G)$-free linear actions on $\mathbb{C}VP^n$

Next we identify the family of (proper) isotropy subgroups of $\mathbb{C}VP^n$. Recall that for the equivariant surgery the suitable family is

$$\mathcal{M}(G) = \mathcal{S}(G) \setminus \mathcal{L}(G).$$

In this section we describe groups G for which $P_{\mathbb{C}}(U(G) \oplus \mathbf{1}_G)$ contains no large subgroup as a proper isotropy subgroup (apart from G itself). First we start with general remarks on the isotropy subgroups of linear actions on $\mathbb{C}P^n$'s. Then in a series of lemmas we prove

Proposition 4.4. *The projectivisation $P_{\mathbb{C}}(U(G) \oplus \mathbf{1}_G)$ has no proper large subgroup as an isotropy subgroup if and only if G is a perfect group.*

The proposition indicates that the assumption in Section 4.1.1 is not limiting, as the surgery Programme can be applied only to perfect groups.

GENERAL REMARKS The connected components of the H-fixed point set of the linear G-action on $\mathbb{C}VP^n$ correspond bijectively to the complex 1-dimensional H-submodules of $\mathrm{res}^G_H V$. Suppose that $K < H \leqslant G$ and that a 1-dimensional, complex K-representation ψ descends from an H-represenation χ, i.e.

$$\mathrm{res}^H_K \chi = \psi.$$

Then we have the canonical inclusion

$$(P_{\mathbb{C}}(V))^H_\chi \subseteq (P_{\mathbb{C}}(V))^K_\psi.$$

If $n_\chi(\mathrm{res}^G_H V)$ (the multiplicity of χ-summand in $\mathrm{res}^G_H V$) is strictly smaller than $n_\psi(\mathrm{res}^G_K V)$ then the inclusion is proper. Note that we always have the inequality $n_\chi(\mathrm{res}^G_H V) \leqslant n_\psi(\mathrm{res}^G_K V)$.

In the following we will silently apply $\mathrm{res}^G_* \langle \cdot \rangle$ when neccessary (and clear from the context). Let $\widehat{H}$ denote the group of linear, complex characters of H (i.e. the group of complex, 1-dimensional representations of H). and let $\chi \in \widehat{H}$. To simplify the notation denote the tensor product of the H-representations $\mathrm{res}^G_H V$ and χ by $V\chi \overset{\text{def.}}{=} \mathrm{res}^G_H V \otimes \chi$.

We are especially interested in n^V_χ, since

$$n_\chi(V) = n_{\mathbf{1}_H}\left(V\chi^{-1}\right) = \dim_{\mathbb{C}}\left(V\chi^{-1}\right)^H = \dim_{\mathbb{C}}\left(P_{\mathbb{C}}(V)\right)^H_\chi + 1,$$

the dimension we want to estimate.

Set $V = U(G) \oplus \mathbf{1}_G$. To analyse the isotropy structure of $P_{\mathbb{C}}(U(G) \oplus \mathbf{1}_G)$ we have to analyse the action of $\widehat{H}$ on representations of H (by the tensor product), and in particular the orbit of $\operatorname{res}^G_H(U(G) \oplus \mathbf{1}_G)$. In our discussion we will always assume that $\chi \neq \mathbf{1}_H$, as the dimension of the fixed point set component corresponding to the trivial representation is covered above.

Observe that

$$\begin{aligned}&\operatorname{res}^G_H(U(G) \oplus \mathbf{1}_G)\chi^{-1}\\ &\qquad = \operatorname{res}^G_H r(G)\chi^{-1} - \bigoplus_{p \,\big|\, |G|} \operatorname{res}^G_H\Big(r\left(G/\mathcal{O}^p(G)\right) - \mathbf{1}_{G/\mathcal{O}^p(G)}\Big)\chi^{-1}.\end{aligned}$$

The multiplicity of χ in the first summand is equal to

$$n_\chi(\operatorname{res}^G_H r(G)) = n_\chi([G\colon H]r(H)) = [G\colon H]n_\chi(r(H)) = [G\colon H].$$

Moreover, χ^{-1} leaves $r(H)$ invariant, thus the first term contributes exactly $[G\colon H]$ to $n_\chi(U(G) \oplus \mathbf{1}_G)$.

We will analyse the H-fixed point set of the latter term for a single prime p. We have

$$\operatorname{res}^G_H\Big(r\left(G/\mathcal{O}^p(G)\right) - \mathbf{1}_{G/\mathcal{O}^p(G)}\Big) = \operatorname{res}^G_H\left(\operatorname{ind}^G_{\mathcal{O}^p(G)}\mathbf{1}_{\mathcal{O}^p(G)}\right) - \mathbf{1}_H,$$

and we may forget about the "$-\mathbf{1}_H$" part, since it does not contribute to the χ-summand at the end. Consider $g \ker\chi \lhd H$, the generator of the cyclic group $H/\ker\chi$. We may choose a lift of the generator to G such that $g\ker\chi \in H/\ker\chi$ corresponds to $g \in G$.

If $(\mathcal{O}^p(G) \cap H) \nleq \ker\chi$, (or equivalently $\langle g\rangle \cap (H \cap \mathcal{O}^p(G)) \neq \{e\}$), then χ does not occur in the decomposition of $\operatorname{res}^G_H\left(r\left(G/\mathcal{O}^p(G)\right)\right)$. Indeed

$$\begin{aligned}\operatorname{res}^G_H\left(\operatorname{ind}^G_{\mathcal{O}^p(G)}\mathbf{1}_{G/\mathcal{O}^p(G)}\right) &= \operatorname{res}^G_H\Big(r\left(G/\mathcal{O}^p(G)\right)\Big)\\ &= [G\colon H\mathcal{O}^p(G)]r\Big(H/(H \cap \mathcal{O}^p(G))\Big),\end{aligned}$$

and since a non-zero power of g belongs to $(H \cap \mathcal{O}^p(G))$, the representation χ is not a summand of $r\left(H/(H \cap \mathcal{O}^p(G))\right)$ by an argument on the order. Therefore we obtain

$$\dim\left[\left(\operatorname{res}^G_H\left(r\left(G/\mathcal{O}^p(G)\right)\right) - \mathbf{1}_H\right)\chi^{-1}\right] = 0.$$

If $(\mathcal{O}^p(G) \cap H) \leqslant \ker\chi$ (or equivalently $\langle g\rangle \cap (H \cap \mathcal{O}^p(G)) = \{e\}$), then the characteristic subgroup of χ survives untouched to $H/(H \cap \mathcal{O}^p(G))$ and therefore to

$$\operatorname{res}^G_H\left(r\left(G/\mathcal{O}^p(G)\right)\right) = [G\colon H\mathcal{O}^p(G)]r\left(H/(H \cap \mathcal{O}^p(G))\right).$$

We finally obtain

$$\dim\left[\left(\operatorname{res}^G_H\left(r\left(G/\mathcal{O}^p(G)\right)\right) - \mathbf{1}_H\right)\chi^{-1}\right] = [G\colon H\mathcal{O}^p(G)].$$

Remark 4.5. For any non-trivial $\chi \in \widehat{H}$, $\ker \chi$ contains $(\mathcal{O}^p(G) \cap H)$ for at most one prime p.

Indeed, observe that $\mathcal{O}^p(H) \leqslant (\mathcal{O}^p(G) \cap H)$. Assume the contrary, that $\ker \chi$ contains $\mathcal{O}^p(G) \cap H$ and $\mathcal{O}^q(G) \cap H$ for two distinct primes p, q. Then $\ker \chi$ contains both $\mathcal{O}^p(H)$ and $\mathcal{O}^q(H)$, hence it also contains the subgroup generated by both of them. Thus we have

$$H = \mathcal{O}^p(H)\mathcal{O}^q(H) \leqslant \ker \chi,$$

a contradiction.

Using all of these results we obtain the following characterisation.

Lemma 4.6. *Suppose that $\chi \in \widehat{H}$ is a non-trivial 1-dimensional complex representation.*

- *If $\ker \chi$ does not contain a subgroup of the form $(\mathcal{O}^p(G) \cap H)$ for any prime p, then*

$$\begin{aligned} &n_\chi\Big(\mathrm{res}^G_H(U(G) \oplus \mathbf{1}_G)\Big) \\ &\quad= [G\colon H] - \sum_{p \mid |G|} \underbrace{\dim\Big[\Big(\mathrm{res}^G_H\Big(r\Big(G/\mathcal{O}^p(G)\Big)\Big) - \mathbf{1}_H\Big)\chi^{-1}\Big]}_{=0 \text{ for all } p} \\ &\quad= [G\colon H]. \end{aligned}$$

- *If $\ker \chi$ contains a (unique, by the remark above) subgroup $(\mathcal{O}^q(G) \cap H)$ for a prime q, then*

$$\begin{aligned} &n_\chi\Big(\mathrm{res}^G_H(U(G) \oplus \mathbf{1}_G)\Big) \\ &\quad= [G\colon H] - \sum_{p \mid |G|} \underbrace{\dim\Big[\Big(\mathrm{res}^G_H\Big(r\Big(G/\mathcal{O}^p(G)\Big)\Big) - \mathbf{1}_H\Big)\chi^{-1}\Big]}_{=0 \text{ for } p \neq q, \text{ and } =[G:H\mathcal{O}^q(G)] \text{ for } p = q} \\ &\quad= [G\colon H] - [G\colon H\mathcal{O}^q(G)]. \end{aligned}$$

As an example we may look at a subgroup

$$H \leqslant \bigcap_{p \mid |G|} \mathcal{O}^p(G).$$

The restriction is trivial for all p, hence for all $\chi \in \widehat{H}$ we have

$$n_\chi\Big(\mathrm{res}^G_H(U(G) \oplus \mathbf{1}_G)\Big) = [G\colon H].$$

Lemma 4.7. *Let G be a finite group, and let H be a large subgroup. Fix p, a prime such that $\mathcal{O}^p(G) \leqslant H$. The following conditions are equivalent.*

1. *$\mathcal{O}^p(G) \leqslant [H, H]$ (the commutator subgroup)*
2. *$H \notin \mathrm{Iso}(P_{\mathbb{C}}(U(G) \oplus \mathbf{1}_G))$.*

Proof. As usual we may exclude the case $\chi = \mathbf{1}_H$, since

$$\dim_{\mathbb{C}} \left(P_{\mathbb{C}}(U(G) \oplus \mathbf{1}_G)\right)^H_{\mathbf{1}_H} = n_{\mathbf{1}_G}(U(G) \oplus \mathbf{1}_G) = 1$$

for any large subgroup H of G. To prove that the first condition implies the second, observe that

$$\mathcal{O}^p(G) \leqslant [H,H] = \bigcap_{\chi \in \widehat{H}} \ker \chi.$$

Hence for any $\chi \in \widehat{H}$ we have $\mathcal{O}^p(G) \leqslant \ker \chi$. By the second case in the lemma above we have

$$\begin{aligned} n_\chi \overset{\text{def.}}{=} n_\chi\left(\operatorname{res}^G_H(U(G) \oplus \mathbf{1}_G)\right) &= [G\colon H] - \sum_{\substack{q \,\mid\, |G| \\ \mathcal{O}^q(G) \cap H \leqslant \ker \chi}} [G\colon H\mathcal{O}^q(G)] \\ &= [G\colon H] - [G\colon H\mathcal{O}^p(G)] = 0. \end{aligned}$$

Recall that in the projectivisation of $U(G) \oplus \mathbf{1}_G$ the H-fixed point set component $(P_{\mathbb{C}}(U(G) \oplus \mathbf{1}_G))^H_\chi$ corresponding to the χ-summand is diffeomorphic to $\mathbb{C}P^{n_\chi - 1} = \mathbb{C}P^{-1} = \varnothing$ (by convention). Since χ was chosen arbitrarily, the second condition follows.

For the opposite implication assume that $H \notin \operatorname{Iso}\left(P_{\mathbb{C}}(U(G) \oplus \mathbf{1}_G)\right)$. Fix a non-trivial $\chi \in \widehat{H}$. The assumption implies that n_χ vanishes. Since

$$n_\chi = [G\colon H] - \sum_{\substack{q \,\mid\, |G| \\ \mathcal{O}^q(G) \cap H \leqslant \ker \chi}} [G\colon H\mathcal{O}^q(G)]$$

and the sum on the right has at most one non-vanishing summand we have $\mathcal{O}^p(G) \cap H \leqslant \ker \chi$ and $[G\colon H\mathcal{O}^p(G)] = [G\colon H]$ for some prime p. This further implies that $\mathcal{O}^p(G) \leqslant H$. Recall that for $H \neq G$ there is at most one prime p such that $\mathcal{O}^p(G) \leqslant H$. Hence the same conclusion holds for all χ with *the same* p. Therefore for all $\chi \in \widehat{H}$ we have $O^p(G) \cap H = \mathcal{O}^p(G) \leqslant \ker \chi$. This finally can be rephrased as

$$\mathcal{O}^p(G) \leqslant \bigcap_{\chi \in \widehat{H}} \ker \chi = [H,H].$$

□

Finally the following lemma fully characterises groups for which the complex projectivisation of $U(G)$ is $\mathcal{L}(G)$-free.

Proposition. *The following conditions are equivalent.*

- $\operatorname{Iso}\left(P_{\mathbb{C}}(U(G) \oplus \mathbf{1}_G)\right) = \mathcal{M}(G) \cup \{G\}$
- *G is a perfect group.*

Proof. It suffices to check that $H \notin \mathrm{Iso}(P_{\mathbb{C}}(U(G) \oplus \mathbf{1}_G))$ for any large **proper** subgroup.

By the lemma above this is equivalent to $\mathcal{O}^p(G) \leqslant [H, H]$ for all H. In particular for $H = \mathcal{O}^p(G)$ we have $\mathcal{O}^p(G) \leqslant [\mathcal{O}^p(G), \mathcal{O}^p(G)]$ i.e. $\mathcal{O}^p(G)$ is a perfect group, for every prime p.

Suppose that $\mathcal{O}^p(G)$ and $\mathcal{O}^q(G)$ are perfect groups for distinct primes p and q. Let $i = p, q$ and observe that $\mathcal{O}^i(G) \cap [G, G]$ is a normal subgroup in $\mathcal{O}^i(G)$ which moreover contains $[\mathcal{O}^i(G), \mathcal{O}^i(G)] = \mathcal{O}^i(G)$. Thus $\mathcal{O}^i(G) \leqslant [G, G]$, and therefore

$$G = \mathcal{O}^p(G)\mathcal{O}^q(G) \leqslant [G, G],$$

i.e. G is perfect.

The other direction is easy. Suppose that G is perfect. Then for all p, $G/_{\mathcal{O}^p(G)}$ is perfect. On the other hand $G/_{\mathcal{O}^p(G)}$ is a p-group, hence $G/_{\mathcal{O}^p(G)} = \{e\}$.

□

4.1.3 GAP-*conditions for perfect groups*

Let us remark that perfect groups are quite complicated in the sense that any quotient of a perfect group is still perfect. Thus perfect groups have no large subgroups beside G itself. Thus if G is a perfect group, then

$$U(G) = r(G) - \mathbf{1}_G.$$

We have used the fact *implicite* in the proof of Proposition 4.4. This means that the tangential G-module at $p \in S^{2k} \subset S^{2n}$ is isomorphic to

$$k\mathbf{1}_G \oplus mU(G).$$

Proposition 4.8.

$$P_{\mathbb{C}}(m(r(G) - \mathbf{1}_G) \oplus k\mathbf{1}_G \oplus \mathbf{1}_G) = P_{\mathbb{C}}(mU(G) \oplus k\mathbf{1}_G \oplus \mathbf{1}_G)$$

satisfies (GC) (*the* GAP-*condition*) *for all sufficiently large integers m, and for all finite groups G.*

Proof. Since (by definition) $\mathrm{Iso}(r(G)) = S(G)$, we have to prove the appropriate inequalities for all subgroups H of G. Recall that the G-action on $r(G)$ is given by the permutation representation (permutation of a basis consisting of elements $g \in G$), hence a complex, 1-dimensional representation of H occurs in $\mathrm{res}^G_H r(G)$ exactly $[G\colon H]$-times. As it is shown in Section 2.2, these representations correspond bijectively to the connected components of $P_{\mathbb{C}}(r(G))^H$, therefore the H-fixed point set connected components are all of the same dimension.

In the case of $m(r(G)-\mathbf{1}_G)\oplus(k+1)\mathbf{1}_G$, for $\chi \neq \mathbf{1}_H$ we have

$$\begin{aligned}\dim_{\mathbb{R}} P_{\mathbb{C}}(m(r(G)-\mathbf{1}_G)\oplus k\mathbf{1}_G\oplus\mathbf{1}_G)^H_\chi \\ = 2(m[G\colon H]-1) = 2m[G\colon H]-2,\end{aligned}$$

whereas for $\chi = \mathbf{1}_H$,

$$\dim_{\mathbb{R}} P_{\mathbb{C}}(m(r(G)-\mathbf{1}_G)\oplus k\mathbf{1}_G\oplus\mathbf{1}_G)^H_{\mathbf{1}_H} = 2(m([G\colon H]-1)+k)$$

Let $K<H$, $X^H \subsetneq X^K$ and assume that $\chi\in\hat{H}$. We have to consider the following three cases.

- If $\chi=\mathbf{1}_H$ then obviously $\operatorname{res}^H_K\chi=\mathbf{1}_K$ and

$$\begin{aligned}2\dim_{\mathbb{R}} P_{\mathbb{C}}(m(r(G)-\mathbf{1}_G)\oplus k\mathbf{1}_G\oplus\mathbf{1}_G)^H_{\mathbf{1}_H} \\ &= 4(m([G\colon H]-1)+k) \\ &= 2m([G\colon H]\cdot 2-2)+2k \\ &\leqslant 2m([G\colon H][H:K]-2)+2k \\ &= 2(m([G\colon K]-1)+k)-2m \\ &< \dim_{\mathbb{R}} P_{\mathbb{C}}\Big(m(r(G)-\mathbf{1}_G)\oplus k\mathbf{1}_G\oplus\mathbf{1}_G\Big)^K_{1_K}.\end{aligned}$$

- If $\chi\neq\mathbf{1}_H$ and $\operatorname{res}^H_K\chi\neq\mathbf{1}_K$ then

$$\begin{aligned}2\dim_{\mathbb{R}} P_{\mathbb{C}}(n(r(G)-\mathbf{1}_G)\oplus k\mathbf{1}_G\oplus\mathbf{1}_G)^H_\chi \\ &= 4m[G\colon H]-4 \\ &= 2m[G\colon H]\cdot 2-4 \\ &\leqslant (2m[G\colon H][H:K]-2)-2 \\ &= 2(m[G\colon K]-1)-2 \\ &= \dim_{\mathbb{R}} P_{\mathbb{C}}(m(r(G)-\mathbf{1}_G)\oplus k\mathbf{1}_G\oplus\mathbf{1}_G)^K_{\operatorname{res}^H_K\chi}-2.\end{aligned}$$

- If $\chi\neq\mathbf{1}_H$ and $\operatorname{res}^H_K\chi=\mathbf{1}_K$ then

$$\begin{aligned}2\dim_{\mathbb{R}} P_{\mathbb{C}}(n(r(G)-\mathbf{1}_G)\oplus k\mathbf{1}_G\oplus\mathbf{1}_G)^H_\chi \\ &= 4(m[G\colon H]-1) \\ &= 2m[G\colon H]\cdot 2-4 \\ &\leqslant 2m[G\colon H][H:K]-4 \\ &\leqslant 2m[G\colon K]-(2n-2k),\ \text{(as long as } m-k\geqslant 2) \\ &= \dim_{\mathbb{R}} P_{\mathbb{C}}(mr(G))^K_{\mathbf{1}_K}.\end{aligned}$$

□

Corollary 4.9. *If G is a group of odd order then $P_{\mathbb{C}}(m(r(G)-\mathbf{1}_G)\oplus k\mathbf{1}_G\oplus\mathbf{1}_G)$ satisfies* (SGC) *for all sufficiently large integers m.*

Proof. By choosing m large enough (SGC) holds for the first and the third case. The estimate in the second case does not depend on m, but the inequality may be rephrased as

$$2(\dim P_{\mathbb{C}}(mr(G))^H + 1) \leqslant \dim P_{\mathbb{C}}(mr(G))^K$$

and the equality holds if and only if $[H\colon K] = 2$. □

GAP-CONDITION FOR PROJECTIVE SPACES As shown in Section 2.2, establishing the GAP-conditions for $\mathbb{C}VP^n$ requires analysis of not only dimensions of the fixed point sets V^H, but rather dimensions of the summands created by 1-dimensional H-representations in $\operatorname{res}^G_H V$. Here we obtain a characterisation of complex G-modules V which satisfy (GC) after projectivisation. The impatient reader may skip this lemma, as it is included just for general knowledge and not used in the following part.

Lemma 4.10. *Let V be a complex, $2n$-dimensional G-representation with the following decomposition into irreducibles:*

$$V = \bigoplus_{\chi \in \hat{G}} n^V_\chi \chi \quad \oplus \bigoplus_{\varphi \in irr_{\mathbb{C}}(G) - \hat{G}} n^V_\varphi \varphi.$$

Set $W \overset{def.}{=} \operatorname{res}^G_H V$ and suppose that the following conditions hold.

- *$n_\chi = 0$ or $n_\chi \geqslant 4$ for all $\chi \in \hat{G} - \{\mathbf{1}_G\}$;*
- *$n_{\mathbf{1}_G} = 0$ or $n_{\mathbf{1}_G} \geqslant 3$;*
- *For all $K < H \in S(G)$ and for all $\chi \in \hat{H}$: if $n^W_\chi \neq n^{\operatorname{res}^H_K W}_{\operatorname{res}^H_K \chi}$ then*

$$2n^W_\chi \leqslant n^{\operatorname{res}^H_K W}_{\operatorname{res}^H_K \chi}.$$

Then $\mathbb{C}VP^n$ satisfies (GC).

Proof. The first condition says that a component

$$(\mathbb{C}VP^n)^G_\chi \subset (\mathbb{C}VP^n)^G$$

has the dimension greater than or equal 5. Similarly the second implies that $(\mathbb{C}VP^n)^G_{\mathbf{1}_G}$ satisfies the same dimension bound.

Indeed consider a submodule $n_\chi \chi \subset V$ ($n_{\mathbf{1}_G}\mathbf{1}_G \subset V$, respectively). The subspace gives rise to a connected component of the fixed point set

$$(\mathbb{C}VP^n)^G_\chi \cong \mathbb{C}P^{n_\chi - 1} \quad \left(\text{or} \quad (\mathbb{C}VP^n)^G_{\mathbf{1}_G} \cong \mathbb{C}P^{n_{\mathbf{1}_G}+1}\right)$$

(recall that we have enlarged V by adding a trivial summand in the process). By the first (the second, respectively) assumption, dimensions of these spaces are greater than or equal to 6.

The third condition can be read as follows. The dimension of the χ-summand in W and the $\operatorname{res}^H_K\chi$-summand in res^H_K is either the same, or has to increase at least twice. The latter case happens when different H-representations become isomorphic to $\operatorname{res}^H_K\chi$ after restriction. This may mean that either two different 1-dimensional H-representations become isomorphic when treated as K-representations, or that a higher-dimensional irreducible H-representation decomposes into linear summands.

Let $(\mathbb{C}WP^n)^H_\chi$ be the connected component of H-fixed point set. If multiplicity of χ increase after applying res^H_K then we have a proper inclusion

$$(\mathbb{C}WP^n)^H_\chi \subset (\mathbb{C}WP^n)^K_{\operatorname{res}^H_K\chi}.$$

The real dimension of the former space is $2n^W_\chi - 2$, whereas of the latter is $2n^{\operatorname{res}^H_K W}_{\operatorname{res}^H_K\chi} - 2$. The GAP-condition now follows from the inequality assumed:

$$\begin{aligned}2\left(\dim_{\mathbb{R}}(\mathbb{C}WP^n)^H_\chi\right) = 2\left(2n^W_\chi - 2\right) < 2\left(2n^W_\chi\right) - 2 \leqslant \\ \leqslant 2n^{\operatorname{res}^H_K W}_{\operatorname{res}^H_K\chi} - 2 = \dim_{\mathbb{R}}(\mathbb{C}WP^n)^K_{\operatorname{res}^H_K\chi}.\end{aligned}$$

□

4.2 SURGERY STEP

Fix the following notation and assumptions.

- $G = A_5$, the alternating group on 5 symbols;
- we are given a smooth G-action on S^{2n} such that

$$\left(S^{2n}\right)^G = F \sqcup S^{2k};$$

- if $p \in S^{2k} \subset S^{2n}$ then $T_pS^{2n} = k\mathbf{1}_G \oplus (n-k)U(G)$. We denote the tangent space by V.

Then we can construct a smooth G-action on $Y \overset{\text{def.}}{=} \mathbb{C}VP^n \# S^{2n}$ such that the fixed point set is diffeomorphic to

$$Y^G = \left(\mathbb{C}VP^n \# S^{2n}\right)^G = F \sqcup \mathbb{C}P^k.$$

In this section

1. We construct a degree one G-normal map

$$(f,b)\colon (X, Y \times mU(G)) \to (Y, Y \times m(U(G))$$

via the Equivariant Transversality Construction;

2. We modify (f, b) by G-surgery to (f', b') such that
    - the map
    $$f'^P : X'^P \to Y^P$$
    is a homotopy equivalence for all prime power order subgroups P of G.
    - $\chi(X'^C) = \chi(Y^C)$ for all cyclic subgroups of G.

    We will actually show more, that for all non-trivial, proper subgroups H, f^H can be modified to a homotopy equivalence.

By the work of T. Petrie and B. Oliver [45, Lemma 2.1 and 2.4] these conditions imply that the surgery kernel $K_n(f)$ is stably free as a G-module. Thus by Theorem 3.24 we have a properly defined surgery obstruction $\sigma(f, b)$ to (free) G-surgery to a homotopy equivalence.

3. Finally Dress' Induction Theorem will provide vanishing of the final obstruction.

This is a sketch of a proof of the following theorem.

Theorem 4.11. *Let $G = A_5$ be the alternating group on 5 symbols. Let F be a closed even-dimensional manifold such that every connected component is of the same dimension. Moreover suppose that the tangent bundle*
$$[\tau_F] \in \operatorname{Tor}\left(\widetilde{KO}(F)\right)$$
belongs to the torsion part of the reduced real K-theory group. Then F occurs as the fixed point set of a smooth action of G on a manifold homotopy equivalent to a complex projective space.

Remark 4.12. The condition $[\tau_F] \in \operatorname{Tor}\left(\widetilde{KO}(F)\right)$ in the theorem is necessary and sufficient for F to occur as the fixed point set of a smooth action of G on a sphere, respectively, a disc. The constructed homotopy complex projective space is of the same (real) dimension as the sphere and the disc.

We remark that all assumptions as well as the construction of the normal map, hold true for an arbitrary perfect group. Based on the work presented above we strongly believe that the following conjecture is also true.

Recall that a closed smooth manifold F occurs as the fixed point set of a smooth action of G on a sphere, respectively, a disc, if and only if the bundle condition that $\tau(F)$ belongs to the appropriate part of the reduced real K-theory is satisfied (see Theorem 2.1 or 2.2 for the bundle conditions).

Conjecture. *Let G be a perfect group and let F be a closed, even-dimensional smooth manifold. If F satisfies the appropriate bundle condition, then F occurs as the fixed point set of a smooth action of G on a manifold homotopy equivalent to a complex projective space.*

The next two sections are devoted to proving Theorem 4.11.

4.2.1 *The degree one G-normal map*

We will use the material from Section 3.4. Since G is perfect, by Proposition 3.11 we obtain non-trivial idempotent α in the Burnside ring $\Omega(G)$ such that

$$\chi(\alpha^H) = \begin{cases} 0, & \text{when } H \neq G \\ 1, & \text{when } H = G. \end{cases}$$

We use the Equivariant Segal Conjecture to identify α with a stable equivariant homotopy class (also denoted by α) in $\omega_G^0(\mathrm{pt})$. Now we use the Localisation Theorem (3.13) and identify

$$A^{-1}\omega_G^0(Y) \cong A^{-1}\omega_G^0\left(Y^G\right) = A^{-1}\omega_G^0(F) \oplus A^{-1}\omega_G^0\left(\mathbb{C}P^k\right),$$

where the localising set $A = \{1, \alpha\}$. As we choose $n_F = 1$ and $n_{\mathbb{C}P^k} = 0$, the element β defined in Section 3.4.3 becomes

$$\beta = (0, \mathbf{1}_{\mathbb{C}P^k}) \in \omega_G^0(F) \oplus \omega_G^0\left(\mathbb{C}P^k\right).$$

By the Localisation Theorem there exists an element $\omega \in A^{-1}\omega_G^0(Y)$ such that

$$A^{-1}j^*(\omega) = \alpha\left(0, \mathbf{1}_{\mathbb{C}P^k}\right),$$

thus by the very definition of a localised ring $\alpha^i j^*(\varrho) = \alpha^j \alpha\beta$ for some $\varrho \in [\omega]$ (an equivalence class in $A^{-1}\omega_G^0(Y)$). Since α is an idempotent we may write $j^*(\varrho) = \alpha^{m+1}\beta$ where $m = 0$ or 1. Choose k such that $j^*(\alpha^k\varrho) = \alpha\beta$.

Regard $(\mathbf{1}_{\mathbf{Y}} - \alpha^k\varrho)$ as a G-map

$$\varphi\colon Y \times (mr(G))^\bullet \to (mr(G))^\bullet.$$

This map can be lifted to (the identity covering) map

$$\psi\colon Y \times (mr(G))^\bullet \times Y \times (mr(G))^\bullet$$

by setting $\psi(y, v) = (y, \varphi(y, v))$. Note that when restricted to $Y^G \times (mr(G))^\bullet$ the map no longer depends on the first coordinate and becomes

$$j^*\left(\mathrm{id}_Y, \mathbf{1}_Y - \alpha^k\varrho\right) = (\mathrm{id}_{Y^G}, \mathbf{1}_{Y^G} - \alpha\beta) = (\mathrm{id}_Y, (\mathbf{1}_F, \mathbf{1}_{\mathbb{C}P^k} - \alpha\mathbf{1}_{\mathbb{C}P^k})).$$

Now we want to use Lemma 3.14 to choose the appropriate representative of ψ.

Observe ψ (or φ) can be prescribed by the initial element from the Burnside ring $\mathbf{1}_{Y^G} - \alpha\beta = (\mathbf{1}_F, \mathbf{1}_{\mathbb{C}P^k} - \alpha\mathbf{1}_{\mathbb{C}P^k})$. Recall that the map

$$\mathbf{1}_Y\colon Y \times mr(G) \to mr(G)$$

is given by

$$(y, v) \mapsto v.$$

Therefore the map $\mathbf{1}_{Y^G} - \alpha\beta$ is represented by

$$\begin{cases} (1-0)\cdot[G/G] + \sum_{(H)\neq(G)}(0-0)[G/H]), \\ 1[G/G] - \chi(\alpha^G)[G/G] + \sum_{(H)\neq(G)}(0-\chi(\alpha)^H)[G/H], \end{cases}$$

on the first and the second coordinate, respectively. By to Lemma 3.14 we obtain $\psi^{-1}(Y^G) = F$.

Take an ε-G-homotopy (relative to $\psi^{-1}(Y^G)$) from ψ to a map

$$b\colon Y \times (mr(G))^\bullet \to Y \times (mr(G))^\bullet$$

which is transverse to the 0-section $Y \times \{0\}$ and define (as has already been advertised before)

$$X \overset{\text{def.}}{=} b^{-1}(Y \times \{0\}), \quad \text{and} \quad f \overset{\text{def.}}{=} b|_X\colon X \to Y.$$

It is clear that $X^G = F$. Taking an advantage of the Equivariant Transversality Construction to the full extent we also may claim that

- the normal bundles $\nu(X^G \hookrightarrow X)$ and $\nu(F^G \hookrightarrow Y)$ are isomorphic as G-vector bundles (this is equivalent to the existence of a G-diffeomorphism of neighbourhoods of $F \subset X$ and $F \subset Y$);
- $\deg(f^H\colon X^H \to Y^H) = 1$ for all subgroups of G (here we use perfectness of G);
- for each proper subgroup $H < G$ there exists an H-normal cobordism

$$(F_H, B_H) : \left(\operatorname{res}^G_H f, \operatorname{res}^G_H b\right) \sim \left(\operatorname{id}_{\operatorname{res}^G_H Y}, \operatorname{id}_{\operatorname{res}^G_H (T(Y)\oplus m\underline{\mathbb{C}[G]})}\right).$$

4.2.2 *Surgery Performed*

We start with the following proposition.

Proposition 4.13. *The degree one normal G-map*

$$(f,b)\colon (X, T(X) \oplus m\underline{\mathbb{C}[G]}) \to (Y, f^*T(Y) \oplus m\underline{\mathbb{C}[G]})$$

obtained in the previous section can be modified by G-surgery of type H (where H is a proper and non-trivial subgroup of G) to a map $(f', b')\colon X' \to Y$ such that

(1) $X'^G \cong F$

(2) $f'^H\colon X'^H \to Y^H$ *is a homotopy equivalence for all proper and non-trivial subgroups* $1 \lneq H \lneq G$*;*

Furthermore, the H-normal cobordisms

$$(F_H, B_H) : \left(W_H, T(W_H) \oplus m\underline{\mathbb{C}[G]}\right) \to \\ \to \left(\operatorname{res}^G_H (Y \times I), \operatorname{res}^G_H \left(T(Y) \oplus m\underline{\mathbb{C}[G]}\right)\right) \\ \textit{from} \quad \left(\operatorname{res}^G_H f', \operatorname{res}^G_H b'\right) \textit{ to } \left(\operatorname{id}_{\operatorname{res}^G_H Y}, \operatorname{id}_{\operatorname{res}^G_H (T(Y)\oplus m\underline{\mathbb{C}[G]})}\right)$$

for all 2-hyperelementary subgroups H of G, can be modified by H-surgery of type $L \geq \{e\}$ to (F'_H, B'_H) such that

(3) *for all primes which divide the order of H and for all non-trivial p-groups P in H*

$$F'^P_H : W'^P_H \to Y^P \times I$$

are homotopy equivalences.

Postponing the proof of the proposition till the next section we will prove Theorem 4.11 assuming it.

Proof of Theorem 4.11. Observe that (f', b') satisfies assumptions of Theorem 3.24, hence the surgery obstruction is well defined. Note that $\dim X_{sing} < 3 \dim X$, where X_{sing} denote set of points in X with non-trivial isotropy subgroup, thus we may claim that the final obstruction $\sigma(f', b')$ belongs to the Wall L-group $L_{2n}(\mathbb{Z}[G]; w)$. Note that $\sigma(f, b) = \sigma(f', b')$ by the normal bordism invariance of surgery obstructions.

Application of the normal bordism invariance to bordisms (F'_H, B'_H) proves that $\operatorname{res}^G_H \sigma(f', b') = 0$. Since

$$\sigma\left(\operatorname{res}^G_H (f', b')\right) = \operatorname{res}^G_H \left(\sigma(f', b')\right)$$

by Dress' Induction Theorem we have $\sigma(f, b) = 0$. Thus we may perform the (free) G-surgery on $f' : X' \to Y$ such that the resulting map $f'' : X'' \to Y$ is a homotopy equivalence. □

4.2.3 *Proof of Proposition 4.13*

In the proof of Proposition 4.13 we will use the Reflection Method (Section 3.5.4). A typical application of the technique has been presented therein. Here we supply more details for $G = A_5$. The proof follows by induction (starting form G) over the descending poset of conjugacy classes of subgroups of G. The lattice of conjugacy classes of subgroups in A_5 is presented in the Figure 7.

To start the induction we need the following lemma.

Lemma 4.14. *Let G be a perfect group. Then $N_G(H) = H$ for all maximal proper subgroups.*

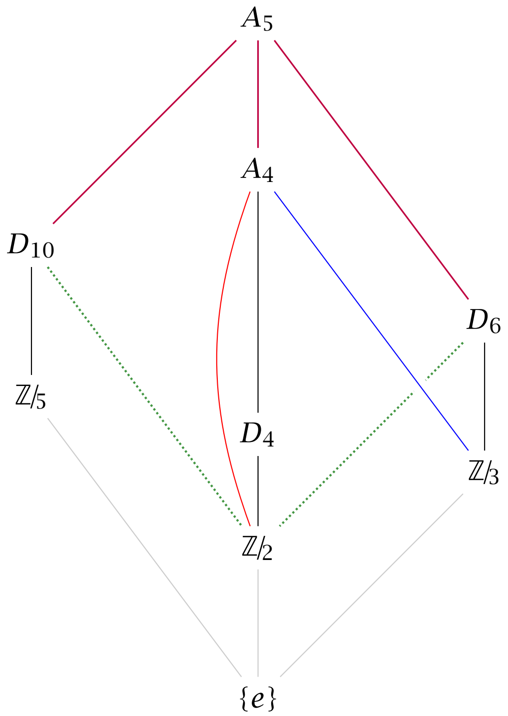


Figure 7: Conjugacy Classes Lattice of A_5

Proof. Tentatively suppose that $N_G(H) = G$, i.e. H is a normal subgroup. Choose a cyclic subgroup $\langle gH\rangle < G/H$. Then $K = \bigcup_i g^i H$ is an intermediate subgroup $H \leqslant K \leqslant G$. If $K \neq G$ then K is proper what contradicts maximality of H. If $K = G$ then G/H is cyclic hence G is not perfect, contrary to the assumption. Thus $N_G(H) = H$.

Recall that for $\mathbb{C}VP^n$ we have $[G\colon H]\dim(\mathbb{C}VP^n)^H \leqslant 2n$. Thus this proves also that $3\dim X_{sing} + 2 < \dim X$, the claim we used in the proof of Theorem 4.11. □

LET $H = K = A_4, D_{10}$, OR D_6. Then H is a maximal proper subgroup in G, thus by the lemma above Assumption (A1) of the Reflection Method is satisfied. Note that $X^{>K} = X^{A_5}$ and $W_H^{>K} = W_H^{>H} = \varnothing$. Recall that the transversal deformation we used in the Equivariant Transversality Construction was taken relative X^G. Thus we may regard X^{A_5} together with its G-invariant tubular neighbourhood U as a subspace of Y. Furthermore the H-bordism W_H granted by the theorem contains $\mathrm{res}^G_H(N \times I)$ treated as H-neighbourhood and $F_H|_N = \mathrm{id}_N$ provides the H-diffeomorphism required by Assumption (A2) of the Reflection Method. Recall that $\mathbb{C}VP^n$ satisfies (SGC) condition for pairs (G, H) (note that $[G\colon H] > 2$ and apply the corollary that follows Proposition 4.8), whereas the sphere which we glued to $\mathbb{C}VP^n$ satisfies (SGC) globally. Thus Assumptions (A3) and (A4) of the Reflection Method are satisfied and we can modify (f,b) by G-surgery of type (K), and (F_H, B_H) by (H)-surgery of type (K) to maps which are homotopy equivalences when restricted to the K-fixed point set.

Moreover, by Conclusion (C3) of the Reflection Method there exists a product H-neighbourhood U'_H of W_H^H in W_H, namely there exists an H-diffeomorphism

$$\psi'_H \colon (U'_H \cap X) \times I \to U'_H \subset W_H$$

which is the canonical inclusion on $(U' \cap X) \times \{0\}$. This finishes the initial step. Note that these product neighbourhoods and bordisms are different for every group H. The main point here is that although bordisms are different, we may perform all of G-surgeries to the same G-normal map f.

SET $H = A_4$ AND $K = D_4 (\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2)$ Note that $N_{A_5}(D_4) = A_4$ hence Assumption (A1) is satisfied. Observe that since the only group in A_4 containing D_4 is A_4 itself, we have $W_{A_4}^{>D_4} = W_{A_4}^{A_4}$. Setting $\psi = \psi'_{A_4}$ and $U = U'_{A_4}$ which were obtained in the first case satisfies Condition (A2).

By a similar argument as above Assumptions (A3) and (A4) are satisfied too. The Reflection Method allows us to modify (f,b) and (F_{A_4}, B_{A_4}) by G-surgery of type (D_4) and A_4-surgery of type (D_4) to maps which are homotopy equivalences when restricted to the D_4-fixed point set.

CONSIDER $H = D_{10}$ AND $K = \mathbb{Z}/5$ As before we have $N_{A_5}(\mathbb{Z}/5) = D_{10}$, $X^{>\mathbb{Z}/5} = X^{D_{10}}$ and $W_{D_{10}}^{>\mathbb{Z}/5} = W_{D_{10}}^{D_{10}}$. Regard U and ψ as D_{10}-neighbourhood $U'_{D_{10}}$ and D_{10}-diffeomorphism $\psi'_{D_{10}}$ obtained in the first case. By Proposition 4.8 X satisfies Assumption (A3).

To prove that (A4) is satisfied note that both

$$f^{\mathbb{Z}/5} \colon X^{\mathbb{Z}/5} \to Y^{\mathbb{Z}/5} \quad \text{and} \quad F_{D_{10}} \colon W_{D_{10}}^{\mathbb{Z}/5} \to Y^{\mathbb{Z}/5} \times I$$

can be made (by a G-, or a D_{10}-surgery of type $(\mathbb{Z}/5)$) connected up to the middle dimension, which is precisely $\dim X^{>\mathbb{Z}/5}$. Comparing the long sequences of homotopy groups of pairs

$$\left(W_{D_{10}}^{\mathbb{Z}/5}, Y^{\mathbb{Z}/5}\right) \quad \text{and} \quad \left(Y^{\mathbb{Z}/5} \times I, Y^{\mathbb{Z}/5}\right)$$

via the Five Lemma, yields

$$\pi_j\left(W_{D_{10}}^{\mathbb{Z}/5}, Y^{\mathbb{Z}/5}\right) \cong \pi_j\left(Y^{\mathbb{Z}/5} \times I, Y^{\mathbb{Z}/5}\right) = 0,$$

as long as $\pi_j\left(W_{D_{10}}^{\mathbb{Z}/5}\right) \cong \pi_j\left(Y^{\mathbb{Z}/5} \times I\right)$. Thus Assumption (A4) is satisfied for $j \leqslant \dim X^{>\mathbb{Z}/5}$.

LET $H = D_6$ AND $K = \mathbb{Z}/3$ Since $N_{A_5}(\mathbb{Z}/3) = D_6$, Assumption (A1) is satisfied. We have the D_6-product bordism $\psi'_{D_6}\colon (U'_{D_6} \cap X) \times I \to U'_{D_6} \subset W_{D_6}$ obtained in the first case hence (A2) is satisfied. As before we may use the GAP-conditions and argue for (A3). However Assumption (A4) requires a little bit more care.

Indeed $\mathbb{Z}/3$ is a proper subgroup of three different proper subgroups of A_5: D_6, $A(1)$ and $A(2)$, where $A(i)$ is isomorphic to A_4. Thus

$$X^{>\mathbb{Z}/3} = X^{D_6} \cup X^{A(1)} \cup X^{A(2)}.$$

In each of this cases the dimension is less than or equal to $\dim X^{D_6}$. Note that $A(1) \cap D_6 = A(2) \cap D_6 = \mathbb{Z}/3$, thus we may regard $Y^{A(i)}$ as embedded in $W^{\mathbb{Z}/3}_{D_{10}}$ (for dimensional reasons), and

$$\pi_j\left(W^{\mathbb{Z}/3}_{D_{10}}, Y^{A(i)}\right) = 0$$

by an argument as above.

CONSIDER $H = A_4$ AND $K = \mathbb{Z}/2$ Since $N_{A_5}(\mathbb{Z}/2) = A_4$, (A1) is satisfied. We have $W^{>\mathbb{Z}/2}_{A_4} = W^{D_4}_{A_4}$ and to check (A2) we may set $\psi = \psi'_{D_4}$ and $U = U'_{D_4}$ as obtained in the second case. Assumption (A3) is satisfied by the same argument as above. Assumption (A4) requires a similar argument as in the previous case. There are six proper subgroups of A_5 which properly include $\mathbb{Z}/2$ as subgroups, namely

$$A_4,\ D_4,\ D_6^a,\ D_6^b,\ D_{10}^a,\ D_{10}^b,$$

where D_6^i and D_{10}^j are isomorphic to D_6 and D_{10}, respectively. Therefore we have

$$X^{>\mathbb{Z}/2} = X^{D_4} \cup X^{D_6^a} \cup X^{D_6^b} \cup X^{D_{10}^a} \cup X^{D_{10}^b}.$$

As above we may regard $Y^{D_k^i}$ as embedded into $Y^{\mathbb{Z}/2}$ and the dimension gap is large enough to claim that $\pi_j\left(W^{\mathbb{Z}/2}_{A_4}, Y^{>\mathbb{Z}/2}\right) = 0$ for $j \leqslant \dim X^{>\mathbb{Z}/2}$. Thus we may perform G-surgeries of type $(\mathbb{Z}/2)$ on X and A_4-surgeries on the interior of W_{A_4} without interfering with $W^{>\mathbb{Z}/2}_{A_4}$ to obtain new normal maps (f, b) and (F_H, B_H) which are homotopy equivalences when restricted to $\mathbb{Z}/2$-fixed point sets.

SET $H = A_4$ AND $K = \mathbb{Z}/3$ By surgery performed in the case ($H = D_6$ and $K = \mathbb{Z}/3$) we know that $f^{\mathbb{Z}/3}$ is already a homotopy equivalence. Thus we will modify W_{A_4} by A_4-surgery of type $(\mathbb{Z}/3)$ to a map which is a homotopy equivalence when restricted to $W^{\mathbb{Z}/3}_{A_4}$.

Recall that the $\mathbb{Z}/3$-fixed point on the sphere is connected (by Smith theory), whereas any fixed point set component of the linear action on $\mathbb{C}VP^n$ has even dimension. Thus we may assume that $\dim X^{\mathbb{Z}/3}$ is even. Following the discussion in Section 3.5.3 the surgery obstruction for this setting belongs to $L_{odd}\left(\mathbb{Z}\left[N_{A_4}(\mathbb{Z}/3)/\mathbb{Z}/3\right], w\right)$. One can check that $N_{A_4}(\mathbb{Z}/3) = \mathbb{Z}/3$ and it is well known that $L_{odd}(\mathbb{Z}[e]; w) = 0$. Therefore there are no obstructions to completing the surgery.

LET $H = D_6$ OR $H = D_{10}$, AND $K = \mathbb{Z}/2$ By surgery performed in the fifth case we may assume that $f^{\mathbb{Z}/2}$ is already a homotopy equivalence. Thus it suffices to perform H-surgery on W_H of type $\mathbb{Z}/2$. Again the normaliser $N_H(\mathbb{Z}/2) = \mathbb{Z}/2$ and the corresponding surgery obstruction group vanishes.

Part II

ACTIONS ON PRODUCTS OF MANIFOLDS

Episode 12b:

How to recognise different types of actions from quite a long way away?

number 1: The Exotic One.

(With thanks to Zbigniew Błaszczyk)

5

ACTIONS ON PRODUCTS

5.1 PRODUCT ACTIONS

Let G be a finite group acting on manifolds M and N.

Definition 5.1. We say that an action

$$\phi\colon G \times (M \times N) \to M \times N$$

is a **product action** if it is equivalent to a **diagonal action** i.e. there exists a G-equivariant CAT-isomorphism (where CAT = *Diff* or *Top*), $f\colon M \times N \to M \times N$ such that the following diagram commutes.

$$\begin{tikzcd}
G \times (M \times N) \arrow[r, "\phi"] \arrow[d, "\mathrm{id} \times f"'] & M \times N \arrow[d, "f"] \\
G \times (M \times N) \arrow[r, "\psi = (\psi_M, \psi_N)"] & M \times N
\end{tikzcd}$$

Here $\psi_M\colon G \times M \to M$ and $\psi_N\colon G \times N \to N$ are two CAT-actions.

Recognizing whether a given G-action on a product of manifolds is a product one and describing the action on individual factors seems to be a daunting task.

There is a natural and obvious characterisation of a product action.

Observation. A G-action on $M \times N$ is a product action if and only if both projections $\pi_M\colon M \times N \to M$ and $\pi_N\colon M \times N \to N$ are G-equivariant maps.

However, verifying if the condition holds seems to be as hard as providing the G-map f itself. We will analyse the H-isotropy sets of a G-action instead.

5.2 ASYMMETRIC MANIFOLDS

Intuition tells us that products of manifolds of high degree of symmetry admits plenty of exotic (i.e. not diagonal) actions. As the example of free actions on products of spheres shows, actions of finite groups on products of manifolds can be very complicated (and the spheres are among the most symmetric manifolds). However, as the degree of symmetry of one of the manifolds in the product decreases, there might be a ‘threshold dimension’ of the second factor, below which all actions become equivalent to a product one. The following seems to be a very natural testing example.

> Let M be an asymmetric manifold. What are the possible (effective) actions of G on $M \times S^n$, depending on the group G and the natural number n?

This paper is concerned with the cases $n = 2$ and $G \leqslant S^1$ (that is, either $G = S^1$ or G is a finite cyclic group), as well as $n = 1$ and $G = S^1$.

In this thesis we will consider a class of 'almost' asymmetric manifolds prescribed by V. Puppe and M. Kreck. We give examples of exotic actions on $M \times S^2$. On the other hand for 'uncomplicated' groups we prove that actions on $M \times S^1$ are equivalent to product actions. Our methods allow to prove existence of exotic actions of $\mathbb{Z}/_p$ for all $n \geqslant 2$. Moreover if n is odd the same proof as in the case $n = 1$ will show that all free actions on $M \times S^n$ are equivalent to product actions.

In the following all manifolds are assumed to be compact.

5.2.1 *History*

In 1976 W. Browder and W-C. Hsiang [10] proposed a list of open problems in the homotopy theory, manifolds and transformation groups. The seventh problem in the section *Compact Transformation Groups* was posted by F. Raymond and R. Schultz.

> It is generally felt that a manifold 'chosen at random' will have very little symmetry. Can this intuitive notion be made more precise? In connection with this intuitive feeling, we have the following specific question.
>
>> *Question: Does there exist a closed simply connected manifold on which no finite group acts effectively? (A weaker question, no involution?)*

A manifold is said to be 'asymmetric' if it does not admit any non-trivial action of a finite group. Examples of such manifolds were known before: e.g. in dimension 3 by the work of F. Raymond and J.L. Tollefson [54], in dimension 4 by the work of E. Bloomberg [6], in dimension 7 and some others by the works of P.E. Conner and F. Raymond [11, 12]. However, all of these manifolds have non-trivial fundamental groups. In fact, except in dimension 4, all of these manifolds are $K(\pi, 1)$'s). V. Puppe in his article [52] remarks later that asymmetry is *"forced upon the manifolds by the 'complexity' of theirs fundamental groups".*

As a more general rule A. Borel proved [7] that if a group π has trivial center and the group of outer automorphisms of π is torsion free, then $K(\pi, 1)$ is asymmetric. For numerous examples and a smooth converse to the theorem of Borel, see an article of W. Malfait [32].

5.2.2 *Detection of asymmetry*

V. Puppe observed [52] that even a non-trivial, but cohomologically trivial action of a finite group $G = \mathbb{Z}/p$ on a manifold M can be detected using cohomological methods. The method is a subtle play on algebraic structures in cohomology of M. Let us sketch the proof.

Our aim is to show that $H^*(M) \cong H^*(M^G)$ as graded algebras, which in turn will imply $M^G \cong M$. However, we will start with the equivariant cohomology $H^*_G(M)$.

Consider the Borel fibration

$$M \to M \times_G EG \xrightarrow{p} BG.$$

If G acts trivially on the cohomology ring of M then the second page of the Serre Spectral Sequence is isomorphic to $H^*(BG) \otimes H^*(M)$. Observe that a non-trivial G-action on M gives rise to a non-trivial differential

$$\partial \colon H^*(M;k) \to H^*(M;k),$$

where $H^*(M;k)$ is treated as a $H^*(BG;k)$-module (coefficients in the field $k = \mathbb{F}_p$).

Since $H^*(BG;k) = k[s] \otimes k[t]$, where $t = \beta(s)$ is the image of the Bockstein homomorphism and $s^2 = 0$ we may view the second differential

$$d_2^{p,q} \colon H^p(BG;k) \otimes H^q(M;k) \to H^{p-1}(BG;k) \otimes H^{q+2}(M;k)$$

as a derivation of $H^*(M;k)[t]$. If $H^*(M;k)$ admits no such derivations, all differentials in the sequence must vanish, and hence the equivariant cohomology algebra $H^*_G(M;k)$ is isomorphic to

$$H^*(M;k) \otimes H^*(BG;k)$$

as an $H^*(BG;k)$-module (by the Leray-Hirsch theorem).

The Localisation Theorem for equivariant cohomology provides a (localised) isomorphism (S is a multiplicative subset of $H^*(BG;k)$)

$$S^{-1}H^*_G(M;k) \cong S^{-1}H^*_G(M^S;k),$$

where

$$M^S \overset{\text{def.}}{=} \{x \in X \colon j_x^* p^*(f) \neq 0 \text{ for all } f \in S\},$$

$j_x^* \colon H^*_G(X;k) \to H^*(BG_x;k)$ is a homomorphism induced by the inclusion $G(x) \hookrightarrow X$ and composed with the isomorphisms

$$H^*_G(G(x);k) \cong H^*_G(G/G_x;k) \cong H^*(BG_x;k).$$

In our case this reduces to just two cases: $G_x = \{e\}$ or $G_x = G$, and thus if we invert $t \in H^*(BG;k)$, then the isomorphism becomes

$$H^*(M;k) \cong H^*(M^G;k),$$

since $j_x^* p^*$ becomes the identity if $x \in M^G$ and 0 otherwise.

The restriction to the fibre $H_G^*(M;k) \to H^*(M;k)$ given by $t \mapsto 1$ is multiplicative, hence the cup-product on $H_G^*(M;k)$ is equal to the component-wise cup-product on $M^*(M;k) \otimes H^*(BG;k)$ modulo elements of lower gradation. This also implies that there is a filtration $\mathcal{F} = \left\{\mathcal{F}_j\right\}_{j\in\mathbb{Z}}$ on the cohomology algebra of the fixed point set

$$H^*\left(M^G;k\right) = \bigoplus_j \mathcal{F}_j\left[H^*\left(M^G;k\right)\right]$$

such that the isomorphism above becomes a filtered algebra isomorphism. Therefore there exists a negative weight deformation

$$d_j\colon H^j(M;k) \to \mathcal{F}_j\left[H^*\left(M^G;k\right)\right].$$

If the deformation is trivial, then $H^*(M;k)$ and $H^*(M^G;k)$ are isomorphic as filtered algebras. This, however, in general is not an isomorphism of graded algebras (the isomorphism may change gradation of an element). There are many algebras (realised by cohomology algebra of a manifold) isomorphic to a given one, but if $H^*(M;k)$ has the lowest possible 'formal dimension' among all of these, then the inclusion $M^G \hookrightarrow M$ induces isomorphism of graded algebras.

The above is a very sketchy proof to the following theorem:

Theorem 5.2 (V. Puppe, [53]). *Let M be a compact manifold such that:*

- *$H^*(M;k)$ has no automorphism of order p (in the case $p = 2$: which acts as $-\mathrm{id}$ on the top cohomology, i.e. no orientation reversing automorphism),*
- *$H^*(M;k)$ has no non-trivial derivation of negative degree,*
- *$H^*(M;k)$ has no non-trivial deformation of negative weight,*
- *$H^*(M;k)$ has minimal formal dimension.*

Then M does not admit any non-trivial (in the case $G = \mathbb{Z}/2$, any non-trivial orientation preserving) action of any finite group G.

Remark. Observe that the theorem above does **not** exclude existence of an *orientation reversing* involution on M.

As shown by A. Edmonds [19], every simply connected 4-manifold admits a non-trivial $\mathbb{Z}/p$-action. Also simply connected 5-manifolds have a very simple structure of the cohomology ring, hence the least dimension to which the method can be applied is 6.

5.2.3 *Most of* 6*-manifolds are asymmetric*

Let $\mathcal{M}$ denote the class of simply connected, oriented, spin 6-manifolds with $H^3(M) = 0$. These assumptions imply that the cohomology ring of M is torsion free and concentrated in even degrees. Recall the classification obtained by C.T.C Wall.

Theorem 5.3 (C.T.C. Wall, [58])**.** *The diffeomorphism classes of elements of $\mathcal{M}$ correspond bijectively to isomorphism classes of the following invariants.*

1. *a free $\mathbb{Z}$-module H of finite rank, corresponding to $H^2(M;\mathbb{Z})$,*
2. *a trilinear, symmetric form $\mu\colon H \times H \times H \to \mathbb{Z}$, corresponding to the cup product in $H^*(M;\mathbb{Z})$,*
3. *a linear map $p_1 \in \hom(H,\mathbb{Z})$, corresponding to the dual of the first Pontrjagin class,*

subject to the following conditions:

(A) $\mu(x,x,y) \equiv \mu(x,y,y) \pmod 2$ *for* $x, y \in H$,

(B) $p_1(x) \equiv 4\mu(x,x,x) \pmod{24}$ *for* $x \in H$.

Starting from this result V. Puppe in [52] tried to formalize the intuitive notion of a 'manifold chosen at random'. He considers a density measure on finite subsets of the set of all trilinear symmetric forms of dimension $m = \operatorname{rank}_{\mathbb{Z}} H^2(M,\mathbb{Z})$. Observe that the set $\mathfrak{S}^3(\mathbb{Z}^m)$ of all trilinear forms on $\mathbb{Z}^m$ is in bijection with

$$\mathfrak{S}^3(\mathbb{Z}^m) \cong \mathbb{Z}^{\binom{m+2}{3}}.$$

Let $\mathfrak{R}(m) \subset \mathfrak{S}^3(\mathbb{Z}^m)$ denote a subset realised by the cohomology rings of 6-manifolds in $\mathcal{M}$.

Definition 5.4. The **density** of $A \subset \mathfrak{R}(m)$ is defined as

$$d_m(A) \overset{\text{def.}}{=} \limsup_{N\to\infty} \frac{\#(A \cap [-N,N])}{\#(\mathfrak{R}(m) \cap [-N,N])},$$

where $[-N,N] \subset \mathfrak{R}(m)$ denotes the subset of forms with coefficients not exceeding $\pm N$.

We say that a set $A \subset B$ is dense in B if

$$\lim_{m\to\infty} \frac{d_m(A)}{d_m(B)} = 1.$$

Theorem 5.5 (V. Puppe)**.**

- *For a given prime $p \neq 2$, let $C_p(m) \subset \mathfrak{R}(m)$ denote the subset corrsponding to those manifolds, which admit a non-trivial orientation preserving $\mathbb{Z}/_p$-action. Then*

$$\lim_{m\to\infty} d_m(C_p(m)) = 0.$$

- *For* $m \geqslant 6$ *the subset of* $\mathfrak{R}(m)$ *corresponding to those manifolds which admit non-trivial* $\mathbb{Z}/_p$*-actions for infinitely many primes* p *has density* 0.

Once we know that 6-manifolds admitting symmetries are 'rare' among all 6-manifolds, we would like to see at least an example of an asymmetric one.

5.2.4 *Examples*

Example (T. Iarrobino)**.** Let

$$\begin{aligned} f(x_1, ..., x_6) = 6\Big(& x_1x_4^2 - x_1^2x_4 + x_2x_4^2 + x_2x_4^2 - x_2^2x_5 + x_2x_5^2 + \\ &+ x_3^2x_4 - x_3x_4^2 + x_3^2x_6 + x_3x_6^2 + x_5^2x_6 + x_5x_6^2 + \\ &+ x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_6 + x_2x_4x_6 + x_3x_5x_6 + \\ &+ x_4x_5x_6 + x_4x_5x_6 + x_4^3 + x_6^3\Big). \end{aligned}$$

The map f determines a trilinear symmetric form, hence also a cup-product-like structure

$$\mu\colon \mathbb{Z}^6 \oplus \mathbb{Z}^6 \oplus \mathbb{Z}^6 \to \mathbb{Z}$$

given by the formula

$$\begin{aligned} \mu(x, y, z) \overset{\text{def.}}{=} \frac{1}{6}\Big[& f(x + y + z) + (f(x) + f(y) + f(z)) - \\ & - (f(x + y) + f(y + z) + f(x + z))\Big]. \end{aligned}$$

By Wall's classification (Theorem 5.3), there exists an infinite family of smooth simply connected spin 6-manifolds $\mathcal{M}_\mu$ such that the cup-product structure on $H^*(M)$ is prescribed by the form above. All such M's are distinguished by $p_1(M) \in H^4(M)$, their first Pontryagin class. As we have the following isomorphisms,

$$H^4(M) \cong \hom(H_4(M), \mathbb{Z}) \cong \hom(H^2(M), \mathbb{Z}),$$

$p_1(M)$ can be treated as a $\mathbb{Z}$-linear map on $H^2(M)$.

Suppose that $g\colon M \to M$ is a homeomorphism of finite period which acts as the identity on the cohomology of M. Then g^* has to preserve not only μ, but also the first Pontryagin class

$$g^*p_1 = p_1\colon H^2(M) \to \mathbb{Z}.$$

By the results of Wall, all possible, and indeed realisable, choices of the Pontryagin class of M have to satisfy the congruence

$$p_1(x) \equiv 4\mu(x, x, x) \pmod{24}.$$

Let $\mathcal{P}\colon H^2(M) \to 24\mathbb{Z}$ be any homomorphism. The $\mathbb{Z}$-module of all admissible p_1's is generated by $p_1^i\colon H^2(M) \to \mathbb{Z}$ $(i = 1, \ldots, 6)$, where

$$p_1^i(x_j) \overset{\text{def.}}{=} \begin{cases} 4\mu(x_j, x_j, x_j) + \mathcal{P}(x_j) & \text{for } j = i, \\ \mathcal{P}(x_j) & \text{otherwise.} \end{cases}$$

Thus it is possible to choose p_1 so that only the identity element of $\mathrm{Aut}_{\mathbb{Z}}(H^2(M)) = GL(6, \mathbb{Z})$ keeps both μ and p_1 invariant. In fact again we have infinitely many of such choices.

Theorem 5.6 (V. Puppe, [52]). *The family $\mathcal{M}_{As} \subset \mathcal{M}_\mu$ of manifolds which admit no effective and orientation preserving action of a finite group is dense in $\mathcal{M}_\mu$ (in the sense of Definition 5.4).*

5.2.5 *Involutions on 6-manifolds*

Given the example above one is inclined to wonder about *orientation reversing* involutions. M. Olbermann showed recently that on these very manifolds there exist many smooth involutions with 3-dimensional fixed points sets.

Theorem 5.7 (M. Olbermann, [42]). *All manifolds in $\mathcal{M}_{As}$ are conjugation spaces (e.g. they admit smooth involutions with 3-dimensional fixed point set).*

To avoid this, M. Kreck stepped outside of the smooth category and proved the existence of topological, non-smoothable manifolds (with cohomology ring structure given by μ) by making use of the Kirby-Siebenmann smoothing obstruction.

Theorem 5.8 (M. Kreck, [25, 26]). *There is an infinite family of closed, asymmetric, simply connected non-smoothable 6-manifolds.*

The existence of *smooth*, simply connected asymmetric manifolds is still an open problem, hence will call manifolds which do not admit orientation preserving action of any finite group **almost** asymmetric.

5.3 ACTIONS ON $M \times S^2$

Throughout the rest of the section, G is isomorphic to either $\mathbb{Z}/p$ for an odd prime p or S^1, unless otherwise stated.

5.3.1 *Detection of exotic actions*

Proposition 5.9. *Let M be a manifold such that the fixed point set M^G is connected for every locally linear G-action on M. If G acts locally linearly on $M \times S^n$ with the fixed point set (the H-fixed point set, for some $H \leqslant G$, respectively)*

$$\left(M \times S^n\right)^G \cong X \sqcup Y$$

($(M \times S^n)^H \cong X \sqcup Y$, respectively) for X not homeomorphic to Y, then the G-action on $M \times S^n$ is exotic.

Proof. Since all subgroups $H \leqslant S^1$ are normal, we may restrict the argument for S^1-actions only to actions of $S^1/H \cong S^1$ on $M \times S^n$, which clearly have non-empty fixed point sets.

By Smith Theory, the fixed point set of any G-action on S^n has to be a $\mathbb{Z}/p$-homology (in the case $G = \mathbb{Z}/p$) or $\mathbb{Z}$-homology (in the case $G = S^1$) sphere Σ. Therefore, every fixed point set of a product action is homotopy equivalent to $M^G \times \Sigma$. Since the action is locally linear, Σ is either a connected manifold, or just two isolated points. Therefore $M^G \times \Sigma$ is either connected or (in the case $\Sigma = \Sigma^0$) homeomorphic to $M^G \sqcup M^G$. The conclusion follows now immediately. □

Proposition 5.9 will allow us to construct exotic actions on manifolds of the form $M \times S^2$.

5.3.2 *Absorbing actions*

Assume that G acts locally linearly on M with non-empty fixed point set F and that at some point $x \in F$ the tangential G-module $T_x M$ is isomorphic to V for some G-module V.

Choose a non-trivial, irreducible, unitary G-representation χ and form the Cartesian product

$$M \times S(\chi \oplus \mathbb{R}) \cong M \times S^2.$$

The fixed point set of the action on the product consists of two copies of F with G-actions on the tangent spaces $T_{(x,z)}(M \times S^2)$ given by $V \oplus \chi$ (here $z \in S^2$ is the north or the south pole, which are fixed by G).

Now suppose that we have a manifold Σ, such that $\dim \Sigma = \dim F$ and that Σ can be realized as the fixed point set of a locally linear G-action on the sphere $S^{\dim M+2}$. By Smith Theory, Σ is a $\mathbb{Z}/p$-homology (or a $\mathbb{Z}$-homology) sphere. Suppose moreover that $y \in \Sigma \subset S^{\dim M+2}$ and that the action on the tangent space at y is given by $V \oplus \chi$. Then we can form the equivariant connected sum

$$\left(M \times S^2\right) \# S^{\dim M+2}$$

along neighbourhoods of points (x, z) and y. The fixed point set of the induced (absorbed) G-action on $M \times S^2$ is homeomorphic to,

$$(M \times S^2)^G \cong F \sqcup F \# \Sigma.$$

Thus by appropriate choices of Σ we may create exotic smooth G-actions on $M \times S^2$.

5.3.3 *Realisation of fixed point sets*

Suppose that M is an (almost) asymmetric n-manifold (for $n = 6$, e.g. M belongs to the class $\mathcal{M}_{As}$). To show that there exists an exotic G-action on $M \times S^2$ we need an appropriate homology n-sphere Σ^n and a smooth G-action on S^{n+2} realising Σ as the fixed point set.

CODIMENSION 2 FIXED POINT SETS In addition to conditions above, we would like to have some control over the tangent representation. Consider the following construction.

Suppose that X^{n+1}, for $n \geqslant 3$, is a contractible manifold with boundary F^n. We may endow $X \times D(\chi)$ with the diagonal action by setting

$$g \cdot (x, y) \mapsto (x, gy),$$

where χ is a non-trivial, irreducible, unitary G-representation. Since X is contractible and $n + 3 \geqslant 6$ the h-cobordism theorem gives $X^{n+1} \times D(\chi) \cong D^{n+3}$. The action restricted to the boundary sphere is the desired one and we have proved the following theorem.

Proposition 5.10 (c.f. W-Y. Hsiang, [20]). *Let X be a contractible manifold of dimension $(n+1)$, $n \geq 3$, with smooth boundary $\partial X = F$. Then there exists an effective smooth G-action on the sphere S^{n+2} with the fixed-point set diffeomorphic to F.*

Remark (W-Y. Hsiang). **Every** codimension 2 fixed point set S^1-action on a sphere comes from this construction.

It turns out that all homology spheres bound contractible manifolds.

Theorem 5.11 (M.A. Kervaire, [24]). *Every 4-dimensional homology sphere bounds a contractible manifold. If N^n is a smooth homology sphere with $n \geqslant 5$, then there exists a unique smooth homotopy n-sphere Σ_N such that $N \# \Sigma_N$ bounds a contractible smooth manifold.*

This leads us to the proof of the next theorem.

5.3.4 *Exotic actions*

Theorem 5.12. *Let M be an n-dimensional asymmetric manifold for $n \geqslant 4$. There exists an effective, exotic topological G-action on $M \times S^2$ for $G = \mathbb{Z}/p$ (p an odd prime) and for $G = S^1$.*

If M is smooth and $n \geqslant 5$ then the action can be chosen to be smooth.

Proof. Choose a connected n-dimensional manifold F with an infinite fundamental group bounding a contractible manifold (e.g. $F = N \# \Sigma_N$ may be a smooth homology sphere by Theorem 5.11). Let χ ($\mathbf{1}_G$, respectively) be any irreducible 2-dimensional orthogonal G-representation (the 1-dimensional trivial representation, respectively). By the construction above there exists a smooth action of G on S^{n+2} with the fixed point set diffeomorphic to F and the tangential G-module at $y \in F$ isomorphic to $\chi \oplus n\mathbf{1}_G$. We can now form the equivariant connected sum

$$\big(M \times S(\chi \oplus \mathbf{1}_G)\big) \# S^{n+2} \cong M \times S^2.$$

Note that even if M is not smoothable we can create the equivariant connected sum without locally linear neighbourhood of a point in $M \times S\,(\chi \oplus \mathbf{1}_G)$, since action on M is trivial. Since all G-actions on S^2 are linear any (non-trivial) product action on $M \times S^2$ has the fixed point set homeomorphic to $M \sqcup M$. However, the fixed point set of the action constructed consists of two components

$$M \sqcup (M \# F),$$

which have non-isomorphic fundamental groups (e.g. by Grushko-Neumann Theorem). The conclusion of the theorem follows now from Proposition 5.9. □

Remark 5.13. The construction above produces non-equivalent actions for different $\mathbb{Z}$-homology spheres, but it is worthwhile to point out that there are many more possibilities for a choice of the fixed point set (including e.g. some $\mathbb{Z}/p$-homology spheres).

5.4 ACTIONS ON $M \times S^1$

Theorem 5.14. *Let $M \in \mathcal{M}_{As}$ (M be the asymmetric, non-smoothable counterpart of a manifold in $\mathcal{M}_{As}$, respectively). Every orientation preserving free (tame, respectively) S^1-action on $M \times S^1$ is a product action.*

Observe that a product action on $M \times S^1$ in the theorem is an action equivalent (conjugate) to the action given by the formula

$$(g,(x,z)) \xrightarrow{\mathrm{id} \times \cdot} (x, g \cdot z),$$

where · means the standard complex multiplication.

Note that X, M, and the G-actions have to be either all smooth or all non-smoothable, therefore proofs in both categories are virtually parallel.

Proof. We use the fact that a free S^1-action on $M \times S^1$ yields the following fibre bundle over the orbit space $X \overset{\text{def.}}{=} M \times_{S^1} S^1$:

$$\xi \overset{\text{def.}}{=} \left(S^1 \to M \times S^1 \to X \right).$$

The bundle has a classifying map

$$X \xrightarrow{c(\xi)} BS^1$$

which can be used to compare different fibre bundles. An S^1-bundle is determined by its first Chern class

$$c_1(\xi) = c(\xi)^*(x),$$

where x is the generator of $H^2(BS^1, \mathbb{Z}) \cong H^2(\mathbb{C}P^\infty, \mathbb{Z})$. The first part of the proof is to prove vanishing of $c_1(\xi)$ so that we have the trivial bundle

$$\left(S^1 \to X \times S^1 \to X \right).$$

Let us postpone the proof of this fact to the next section and just draw the conclusions.

The first one is that there is the following commutative (up to homotopy) diagram.

$$\begin{tikzcd}
 & M \times S^1 \arrow[r, "\pi_G"] \arrow[dd, "\parallel"] & M \arrow[dd, "\wr"] \arrow[dr, "const."] & \\
S^1 \arrow[ur] \arrow[dr] & & {} \arrow[r, phantom, "\overset{h}{\circlearrowright}"] & BS^1 \\
 & X \times S^1 \arrow[r, "\pi_G"] & X \arrow[ur, "c(\xi)"'] &
\end{tikzcd} \quad (1)$$

We have a few remarks to the situation.

Note that in the lower row we may identify the total space of the trivial S^1-bundle over X and $M \times S^1$. As will be proved later, X is simply connected and hence the diagram above gives us a homotopy equivalence $M \to X$. Our aim is to improve it to a CAT-isomorphism. Finally, since the action is a product one and M is asymmetric (hence every S^1-action on M is trivial), any CAT-isomorphism $M \to X$ would commute with π_G.

This is enough to claim that the initial action on $M \times S^1$ is the product action.

□

These remarks reduce the proof of Theorem 5.14 to the following two facts.

FACT 1. *The first Chern class of the bundle ξ is trivial.*

FACT 2. *A homotopy equivalence $M \simeq X$ can be improved to a* CAT-*isomorphism.*

We will prove the first fact via identification of multiplication by $c_1(\xi)$ and a differential on the second page of the Leray-Serre spectral sequence. To obtain the second we will relay on Wall's classification of simply connected 6-manifolds (see [58]).

5.4.1 *Triviality of the Chern class*

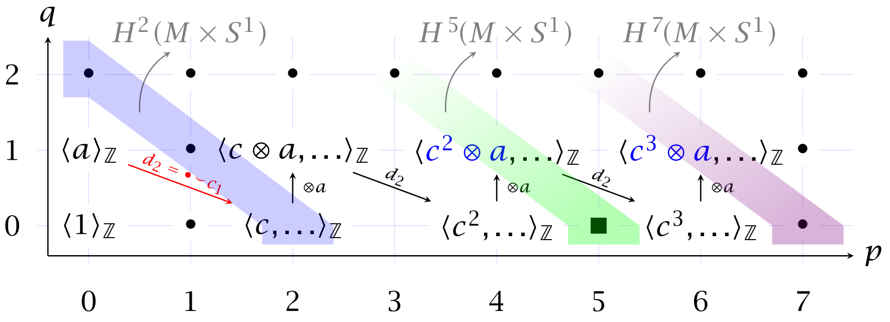
Proof of FACT 1. Recall that M is a 6-dimensional, simply connected manifold with free cohomology generated in the second gradation. By inspection of the long exact sequence of the fibration ξ

$$\cdots \to \pi_1(S^1) \to \pi_1(M \times S^1) \to \pi_1(X) \to \pi_0(S^1) \to \cdots,$$

we obtain that $\pi_1(X)$ is either trivial or finite cyclic. Since G acts preserving orientation, we can consider the Leray-Serre spectral sequence

$$E_2^{p,q} = H^p(X, H^q(S^1;\mathbb{Z})) \Longrightarrow H^{p+q}(M \times S^1;\mathbb{Z})$$

with untwisted coefficients. The following diagram shows the nontrivial part of the second page (the trivial group is denoted by •).



The differential $d_2\colon E_2^{0,1} \to E_2^{2,0}$ is the multiplication by $c_1(\xi)$. Assume that $c_1 \neq 0$ and set $d_2(a) = c \neq 0$. We claim that c is a generator of the torsion part of the second cohomology of X,

$$\operatorname{Tor}(H^2(X)) = H_1(X) = \mathbb{Z}/_k.$$

Indeed, if c were not a generator, then the generator would survive to E_3 (as being in the kernel of the differential going down and not in the image of d_2) and after passing to E_∞, $H^2(M \times S^1)$ would contain a torsion element.

By the multiplicative properties of the spectral sequence we have

$$d_2(c \otimes a) = d(c) \smile a + c \smile d(a) = c^2,$$

so we may push c up to $c^3 \otimes a \in E_2^{6,1}$. Now observe that $c^3 \otimes a$ survives to E_∞ and hence to $H^7(M \times S^1)$, as there is no differential to kill it. But $H^7(M \times S^1)$ is torsion free and thus

$$d_2(c^2 \otimes a) = c^3 = 0.$$

Now suppose that $c^2 \otimes a$ survives to E_∞. Passing from E_∞ to $H^5(M \times S^1)$ requires solving an extension problem

$$0 \to \underbrace{E_2^{4,1}}_{\ni c^2 \otimes a} \hookrightarrow H^5(M \times S^1) \to \blacksquare \to 0.$$

Again $H^5(M \times S^1)$ contains no torsion subgroup, hence already $c^2 \otimes a$ had had to vanish.

We may play the same game once again and finally arrive at the conclusion that $a \smile c = 0$ simultaneously proving that X is simply connected. □

The proof suggests that the fact is more general and it holds for all manifolds with torsion-free cohomology concentrated in even degrees.

5.4.2 *Cancellation for products*

We need to prove the following cancellation property.

Fact 2. *Let M and N be two simply connected* CAT *6-manifolds. If $M \times S^1$ is* CAT*-isomorphic to $N \times S^1$ then M is* CAT*-isomorphic to N.*

Let us remark that the fact holds in much more general setting.

Lemma 5.15. *Let* CAT *be any of Top, PL, Diff. Suppose that the Whitehead group $Wh(\pi_1(M))$ is trivial. If $M \times S^1$ is* CAT*-isomorphic to $N \times S^1$, then M is* CAT*-isomorphic to N.*

Proof. Lift the isomorphism to the $\mathbb{Z}$ cover

$$\varphi : M \times \mathbb{R} \to N \times \mathbb{R}.$$

Since $\varphi(M \times \{0\})$ is compact in $N \times \mathbb{R}$, hence the image must belong to $N \times [-a, a]$ for some real number $a > 0$. Let A denote a connected component of $N \times \mathbb{R} \setminus \varphi(M \times \{0\})$. Note that

$$W = A \cap (N \times (-\infty, a])$$

is a non-empty, connected manifold with boundary $\partial W \cong N \sqcup \varphi(M)$. Moreover the inclusions $N \hookrightarrow W$ and $\varphi(M) \hookrightarrow W$ are homotopy equivalences. Lemma follows by application of the s-cobordism theorem. □

Using the lemma above we obtain a CAT-isomorphism $M \to X$. Since any action on M is trivial (M is asymmetric) plugging an isomorphism into Diagram 1 gives us the desired equivalence of actions.

5.4.3 *Involutions on* $M \times S^1$

Recall that in Theorem 5.14 we excluded $\mathbb{Z}/p$ actions for p an odd prime. This is in contrast to the case $G = \mathbb{Z}/2$.

Theorem 5.16. *Let* M^n *be an asymmetric manifold* $(n \geqslant 4)$*. There exist exotic involutions on* $M \times S^1$*. If* M *is smooth and* $n \geqslant 5$ *then the involutions may be chosen to be smooth.*

Proof. Denote by χ the non-trivial irreducible $\mathbb{Z}/2$-representation. Assume that we have an involution on S^{n+1} with a codimension 1 fixed point set a $\mathbb{Z}/2$-homology sphere Σ. The same construction as in the proof of Theorem 5.12 realises any $\mathbb{Z}$-homology sphere as Σ. Then the connected sum

$$(M \times S(\chi \oplus \mathbf{1}_G)) \# S^{n+1}$$

provides the desired exotic involution. Indeed, the fixed point set of any product involution is equal to $M \times S(\chi \oplus \mathbf{1}_G)^G = M \sqcup M$, whereas after the connected sum we obtain

$$\left((M \times S(\chi \oplus \mathbf{1}_G)) \# S^{n+1}\right)^G = M \sqcup (M \# \Sigma).$$

The conclusion follows from Proposition 5.9. □

5.5 CONJECTURES AND FURTHER WORK

Here we share a few predictions about general theorems on actions on product manifolds.

PRODUCT ACTIONS We strongly believe that the following is true.

Conjecture. *Let* M *be an (almost) asymmetric manifold of arbitrary dimension. If* p *is an odd prime then all* $\mathbb{Z}/p$*-actions on* $M \times S^1$ *are product actions.*

This would also prove that any fixed point free S^1-action is the product action. As the proof of Theorem 5.14 suggests, the vanishing of the first Chern class should be a more general fact and applied to all asymmetric manifolds with torsion-free cohomology in even degrees. The diffeomorphism type of these manifolds should agree based on the framework of modified surgery of M. Kreck.

NON-PRODUCT ACTIONS As the method of the proof suggests, the threshold mentioned in the introduction seems to be connected with the dimension of the H-fixed point set. Observe that as soon as we grasp the H-fixed point set (with its normal bundle) of the locally linear G-action, we can modify the action to a non-product one. We believe that this relates directly to the following number. Let $n(G)$ denote the minimal dimension of a non-trivial irreducible representation of G.

Conjecture. *Let M be an asymmetric manifold and G a finite group. Suppose that $n < n(G)$. Every locally linear G-action on $M \times S^n$ is a product action.*

Part III

APPENDIX

I would like to thank Wojciech Politarczyk with whom the initial version of this part has been prepared for *Regensburg BlockSeminar – Surgery* in 2012. We had many fruitful discussions over the last three years.

Even greater thanks should go to Diarmuid Crowley for many enlightening comments and remarks preceding and during the seminar, and especially for conversations during preparation of this script.

A

CLASSIFICATION OF HOMOTOPY COMPLEX PROJECTIVE SPACES

This appendix contains the classification of homotopy complex projective spaces. It is written for readers with general surgery theory background in mind. Especially used is the surgery exact sequence and the identification of the (geometric) normal invariant set with the set of homotopy classes $[\cdot, G/PL]$. The main part is largely based on Chapter 8 in [31], and can be read as an extension of the chapter. Also, there are included (more or less trivial) remarks connecting presented material and free action on spheres.

A.1 PLUMBING ON $D(TS^n)$

At the beginning let us review the standard construction of plumbing.

To set the necessary notation let TS^n denote the tangent bundle of a sphere, and let $D(TS^n)$ be its disk bundle. Let $G = \{V, E\}$ be a connected graph and choose a vertex $v_i \in V$ in the graph take a copy of $D(TS^n)$ denoted by $D(TS^n)_i$. Then for each edge $e \in E$ beginning or ending in the vertex choose a disk $N_{i,e}$ in the 0-section sphere $S^n \subset D(TS^n)_i$ together with small closed neighbourhood

$$N_{i,e} \times D^n \cong D^n \times D^n.$$

If necessary shrink the disks chosen for all edges, to make them disjoint and perform the following identification.

If there is an edge $e \in E$ in the graph connecting vertices v_i and v_j then glue the chosen neighbourhoods of the points using the following map interchanging the coordinates.

$$\begin{aligned} N_{i,e} \times D^n &\longrightarrow N_{j,e} \times D^n \\ (x, y) &\longmapsto (y, x). \end{aligned}$$

As the result we obtain a manifold with boundary $(M, \partial M)$. Its homotopy type can be easily described as the wedge of spheres $S^n \vee \cdots \vee S^n \vee S^1 \vee \ldots \vee S^1$, one S^n for each vertex and S^1 for each loop in the graph.

Remark. We have the exact sequence of pair $(M, \partial M)$:

$$0 \to H_n(\partial M) \to H_n(M) \xrightarrow{A} H_n(M) \to H_{n-1}(M) \to 0.$$

If A is an isomorphism (i.e. $\det A = \pm 1$) then

$$H_{n-1}(\partial M) = H_n(\partial M) = 0$$

and thus ∂M is a homology $(2n-1)$-sphere. Observe that the matrix A provides the intersection form on M.

To describe the intersection form we will consider two cases.

n IS EVEN: since $2k$-dimensional sphere has double self-intersection we see that there are 2s on the diagonal of A. By construction, the i, j-element is equal ± 1 if and only if there is an edge connecting v_i and v_j, and 0s elsewhere;

n IS ODD: we have 0s on the diagonal and ± 1 or 0 as above.

Theorem A.1. *If the graph $G = \{V, E\}$ is simply connected, then ∂M is $(n-2)$-connected and the homology of ∂M is described in terms of the intersection matrix of M.*

Lemma A.2. *If the graph G is simply connected, $n \geq 3$ and $\det A = \pm 1$ then the boundary is PL-homeomorphic to the standard sphere*

$$\partial M^{2n} \cong_{PL} S^{2n-1}.$$

A.2 KERVAIRE AND MILNOR MANIFOLDS

We will make use of these two examples.

Example (Kervaire Manifold)**.** For $k = 2n + 1$ plumb together two copies of $D(TS^k)$, according to the graph

(this is the Dynkin diagram of the classical group A_2). The result is the **Kervaire manifold with boundary** $\overline{M}_A^{4n+2}$. By what we have said above, the intersection matrix is

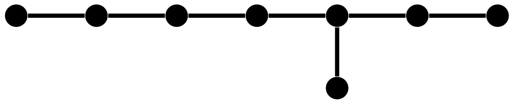
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Therefore the boundary of $\overline{M}_A^{4n+2}$ is a *PL*-sphere by the Lemma above, hence we may attach a cone on the boundary (that is glue in a *PL*-disk D^{4n+2}). The result we will call the (closed) **Kervaire manifold** and denote by

$$M_A^{4n+2} \overset{\text{def.}}{=} \overline{M}_A^{4n+2} \cup_\partial D^{4n+2}.$$

Remark. Since we attach just a simplicial disk in the construction above we are bound to leave the smooth category. However M_A^{2n} happens to be smoothable in some special cases, namely in dimensions $2, 6, 14, 30, 62$ and potentially 126, what is (of course) related to the Arf invariant 1 problem.

Example. For $k = 2n$ plumb together eight copies of $D(TS^k)$ using the graph



(this is the Dynkin diagram of the exceptional group E_8). The result is the **Milnor manifold with boundary** $\overline{M}_B^{4n}$. The intersection matrix is the famous E_8 matrix

$$B = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix},$$

which is believed to have determinant 1 (the last 2 on the diagonal corresponds to the vertex below seven others). As above we may cone of the boundary to obtain the (index 8, closed) **Milnor Manifold** which will be denoted

$$M_B^{4n} \overset{\text{def.}}{=} \overline{M}_B^{4n} \cup_\partial D^{4n}.$$

Remark. In contrast to the Kervaire manifold, M_B^{4n} is never smoothable. However, the m-fold connected sum

$$\#_m M_B^{4n} = \underbrace{M_B^{4n} \# \cdots \# M_B^{4n}}_{m \text{ times}}$$

happens to admit a smooth structure. This is related to the fact that the boundary of $\overline{M}_B^{4n}$ generates the group bP_{4k} of homotopy spheres bounding parallelizable manifolds. Smooth cases are described in the following theorem.

Theorem A.3 (Kervaire-Milnor, Quillen)**.** *The connected sum $\#_m M_B^{4n}$ is a smooth manifold if and only if m is a multiple of*

$$a_n 2^{2n-2}\left(2^{2n-1} - 1\right) \operatorname{Num}\left(\frac{B_{2n}}{4n}\right),$$

which is the order of the group bP_{4k}.

A.3 THE CONSTRUCTION

Kervaire and Milnor closed manifolds are very important in understanding *PL*-structures on spheres. In particular, the degree 1 map $\pi\colon M_B^{4n} \to S^{4n}$ is covered by a normal bundle map and we have the associated map $S^{4n} \to G/PL$ which turns out to be a generator in $\pi_{4n}(G/PL)$. Similar result holds for the Kervaire manifold in dimensions $4n+2$.

Before we begin with the construction let us recall briefly a procedure to construct $\mathbb{C}P^{n+1}$ out of $\mathbb{C}P^n$ such that the $\mathbb{C}P^n$ embeds naturally on the first $n+1$ coordinates.

Consider $\gamma_n \to \mathbb{C}P^n$ the Hopf bundle (or tautological complex line bundle) and its disk bundle $D(\gamma_n)$. The boundary of the disk bundle is actually an ordinary (smooth) sphere

$$\partial D(\gamma_n) = S(\gamma_n) = S^{2n+1},$$

hence we may glue a cone on the boundary (that is a disk D^{2n+2} along its boundary) and obtain

$$D(\gamma_n) \cup_\partial D^{2n+2} \cong \mathbb{C}P^{n+1}.$$

In the following examples we will be trying to mimic this construction.

A.3.1 *Manifolds homotopy equivalent to $\mathbb{C}P^5$*

Let us begin with a very graspable example of the connected sum $\mathbb{C}P^4 \#_m M_B^8$ (note: this is usually not a smooth manifold any more). There is a natural map $M_B^8 \to S^8$ (namely: the map collapsing $\overline{M}_B^8$ to a point) which induces a degree 1 normal map

$$f\colon \mathbb{C}P^4 \#_m M_B^8 \to \mathbb{C}P^4 \#_m S^8 \cong \mathbb{C}P^4.$$

We may now pull back the Hopf bundle γ_4 over $\mathbb{C}P^4 \#_m M_B^8$.

$$\begin{tikzcd}
D(f^*\gamma_4) \arrow[r, "f^*"] \arrow[d] & D(\gamma_4) \arrow[d, "p"] \\
\mathbb{C}P^4 \#_m M_B^8 \arrow[r, "f"] & \mathbb{C}P^4
\end{tikzcd}$$

The sphere bundle of the Hopf bundle is the true sphere, hence restriction of f to boundaries is a map

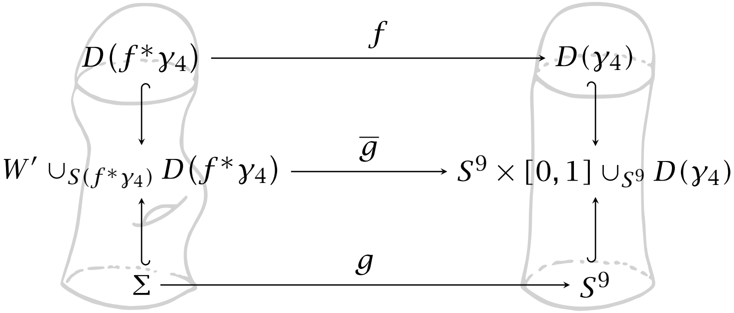
$$f|_\partial\colon S(f^*\gamma_4) \to S(\gamma_4) = S^9.$$

Lemma A.4. *The map $f|_\partial$ above is a degree 1 normal map.*

Since we are working between simply connected odd dimensional manifolds, by surgery below the middle dimension we may assume that $f|_\partial$ is normally bordant to a homotopy equivalence $g\colon \Sigma \to S^9$. Thus, by the Poincaré Conjecture $f|_\partial$ is indeed bordant to a *PL*-homeomorphism.

$$\begin{tikzcd}
S(f^*\gamma_4) \arrow[r, "f|_\partial", "\text{degree 1 normal map}"'] \arrow[d, hook] & S(\gamma_4) \arrow[d, hook'] \\
W' \arrow[r, "\overline{g}", "\text{normal bordism}"'] & S^9 \times [0,1] \\
\Sigma \arrow[u, hook'] \arrow[r, "g", "PL\text{-homeomorphism}"'] & S^9 \arrow[u, hook]
\end{tikzcd}$$

Choose $W' \to S^9 \times [0,1]$ to be such bordism and glue $D(f^*\gamma_4)$ or $D(\gamma_4)$ on top of W' or $S^9 \times [0,1]$ respectively. Now we have the following picture:



$$\begin{tikzcd}
D(f^*\gamma_4) \arrow[r, "f"] \arrow[d, hook] & D(\gamma_4) \arrow[d, hook'] \\
W' \cup_{S(f^*\gamma_4)} D(f^*\gamma_4) \arrow[r, "\overline{g}"] & S^9 \times [0,1] \cup_{S^9} D(\gamma_4) \\
\Sigma \arrow[u, hook'] \arrow[r, "g"] & S^9 \arrow[u, hook]
\end{tikzcd}$$

We slightly abuse the notation, since map $\overline{g}$ in the diagram above is extended to $D(f^*\gamma_4)$ using f.

We intend to change the normal map $\overline{g}$ on the whole bordism to a homotopy equivalence. Then since g is a *PL*-homeomorphism we may close the (lower) boundaries by a *PL*-disk.

Lemma A.5. *We can perform a surgery on the normal bordism*

$$\overline{g}\colon W' \cup_{S(f^*\gamma_4)} D\left(f^*\gamma_4\right) \to D(\gamma_4)$$

to make it a homotopy equivalence.

Proof. Proof of the lemma is just an application of Wall's realisation theorem. □

As a result we obtain a manifold $(W^{10}, \partial W)$ with boundary *PL*-homeomorphic to S^9 and a homotopy equivalence

$$h\colon W^{10} \to D(\gamma_4).$$

We may now cone off the boundary of W^{10} and call the result

$$\widetilde{\mathbb{C}P^5} \overset{\text{def.}}{=} W^{10} \cup_{S^9} D^{10}.$$

Observe that we can extend h to

$$\tilde{h}\colon \widetilde{\mathbb{C}P^5} \to \mathbb{C}P^5$$

at the same time, since the cone on the boundary is just a *PL*-disk. This is our first example of a space homotopy equivalent to the complex projective 5-space.

Lemma A.6. *$\widetilde{\mathbb{C}P^5}$ and $\mathbb{C}P^5$ are not homeomorphic.*

Proof. One possible proof is to compare the $\mathcal{L}$-polynomials of these two spaces.

The constructed homotopy equivalence, is also a degree 1 normal map and hence it corresponds to a normal invariant $\lambda \in [\mathbb{C}P^5, G/_{PL}]$. Note that (by the construction) λ is trivial if and only if $m = 0$ (recall that m is the multiplicity of the connected sum with M^8_B).

The inverse image of $\mathbb{C}P^i \subset \mathbb{C}P^5$ is equal to $\mathbb{C}P^i$ for $i = 1, 2, 3$, but for $i = 4$ we have

$$f^{-1}\left(\mathbb{C}P^4\right) = \mathbb{C}P^4 \#_m M^8_B,$$

hence the surgery obstruction of $f^{-1}(\mathbb{C}P^4)$ is equal to m. Using properties of the $\mathcal{L}$-polynomials we are able to compute it for our homotopy complex projective space

$$\mathcal{L}\left(\widetilde{\mathbb{C}P^5}\right) = \mathcal{L}\left(\mathbb{C}P^5\right)\left(1 + 8mx^4\right)$$

where x generates the second cohomology of $\widetilde{\mathbb{C}P^5}$.

Since $\mathcal{L}$-polynomials are rational polynomials in the Pontryagin classes and the rational Pontryagin classes are homeomorphism invariants, the conclusion follows. □

A.3.2 *Manifolds homotopy equivalent to $\mathbb{C}P^6$*

The construction above gives us a degree 1 normal map

$$\tilde{h}\colon \widetilde{\mathbb{C}P^5} \to \mathbb{C}P^5.$$

We may pull back the canonical line bundle $\gamma_5 \to \mathbb{C}P^5$ over $\widetilde{\mathbb{C}P^5}$.

$$\begin{tikzcd}
D(h^*\gamma_5) \arrow[r, "\tilde{h}^*"] \arrow[d] & D(\gamma_5) \arrow[d] \\
\widetilde{\mathbb{C}P^5} \arrow[r, "\tilde{h}"] & \mathbb{C}P^5
\end{tikzcd}$$

Again by the Poincaré conjecture, the sphere bundle $S\left(\tilde{h}^*(\gamma_5)\right) = \partial D\left(\tilde{h}^*\gamma_5\right)$ is *PL*-homeomorphic to the sphere S^{11}. The space

$$\widetilde{\mathbb{C}P^6} \overset{\text{def.}}{=} D\left(\tilde{h}^*\gamma_5\right) \cup_{S^{11}} D^{12}$$

is a homotopy complex projective space obtained by coning off the boundary of the disk bundle.

Similarly we may form the connected sum $\widetilde{\mathbb{C}P^5} \#_n M_A^{10}$, where M_A^{10} is the Kervaire manifold, and try to perform the same construction as in case of $\mathbb{C}P^4 \#_m M_B^8$. We take pull-back of γ_5 line bundle via the natural degree 1 normal map. Now to obtain a homotopy equivalence $\partial D\left(\tilde{h}^*\gamma_5\right) \to \partial D(\gamma_5)$ we have to use again a little bit of surgery.

Nevertheless, this all can be done (it is a nice exercise to check it!) and there exists a manifold W^{12} such that

$$\tilde{k}\colon D\left(\tilde{h}^*\gamma_5\right) \cup W^{12} \to D(\gamma_5)$$

is a homotopy equivalence of manifolds with boundary. Finally we may extend the homotopy equivalences to the cones on boundaries using the same argument. The resulting space will be denoted by

$$\widehat{\mathbb{C}P^6} \overset{\text{def.}}{=} D\left(\tilde{h}^*\gamma_5\right) \cup W^{12} \cup_{S^{11}} D^{12}$$

Remark. The same is true for $\widetilde{\mathbb{C}P^5}$ replaced by $\mathbb{C}P^5$. The result will be denoted by $\bar{h}\colon \overline{\mathbb{C}P^6} \to \mathbb{C}P^6$.

Lemma A.7. *$\mathbb{C}P^6$, $\widetilde{\mathbb{C}P^6}$, $\widehat{\mathbb{C}P^6}$ and $\overline{\mathbb{C}P^6}$ are topologically distinct homotopy projective spaces.*

The proof of the above lemma will become clear once we learn the machinery of splitting invariants which will be covered in the next section. Nevertheless we can already give sequences of the splitting invariants what will suffice as a proof after learning the classification in the next section.

These are

$\mathbb{C}P^6$	$(0,0,0,0)$
$\widetilde{\mathbb{C}P^6}$	$(0,0,m,0)$
$\overline{\mathbb{C}P^6}$	$(0,0,0,n \pmod 2)$
$\widehat{\mathbb{C}P^6}$	$(0,0,m,n \pmod 2)$

(the i-th element of the sequence is roughly speaking the surgery obstruction of the pre-image of $\mathbb{C}P^i$).

A.3.3 *Free actions on spheres*

Observe the orbit space of the standard free action of S^1 on $S^{2n+1} \subset \mathbb{C}^{2n}$ is the complex projective space. The easiest example is the original Hopf fibration

$$S^1 \hookrightarrow S^3 \to S^2 = \mathbb{C}P^1.$$

There is a theorem (see [59], Chapter 14C) stating that any free action of S^1 on the $(2n+1)$-sphere gives rise to a homotopy complex projective space $h\mathbb{C}P^n$ sitting in the obvious fibration

$$S^1 \hookrightarrow S^{2n+1} \to S^{2n+1}/S^1 = h\mathbb{C}P^n,$$

and conversely, each $h\mathbb{C}P^n$ corresponds to a unique, free S^1-action on S^{2n+1}.

We may consider the join $S^1 * S^{2n+1} \cong S^{2n+3}$. Since join is a nice quotient space of the Cartesian product $S^1 \times S^{2n+1} \times [0,1]$, from (free) actions on both spheres we have an induced (free) action on the join. In this language, the construction of $\widetilde{\mathbb{C}P^6}$ above is equivalent to the following.

Given $\widetilde{\mathbb{C}P^5}$, identify the corresponding free S^1-action on S^{11}. Take a join of the S^{11} with S^1 with the standard action. This gives us a free S^1 action on S^{13}. The quotient space is precisely $\widetilde{\mathbb{C}P^6}$ as constructed above.

A.4 THE CLASSIFICATION

Consider the surgery exact sequence for $\mathbb{C}P^n$. Since $L_{2n+1}(\mathbb{Z}[e]) = 0$ we have

$$0 \to S^{PL}(\mathbb{C}P^n) \to [\mathbb{C}P^n, G/PL] \xrightarrow{\sigma} L_{2n}(\mathbb{Z}[\searrow])$$

and this implies that the PL structure set is a subset of the set of normal invariants whose surgery obstruction is zero. In order to identify the structure set we have to compute the set of normal invariants and the surgery obstruction map. We will proceed inductively.

Consider the following cofibration sequence.

$$S^{2n-1} \to \mathbb{C}P^{n-1} \to \mathbb{C}P^n \to S^{2n}$$

It yields the following exact sequence of abelian groups

$$\pi_{2n}(G/PL) \to \left[\mathbb{C}P^n, G/PL\right] \to \left[\mathbb{C}P^{n-1}, G/PL\right] \to \pi_{2n-1}(G/PL).$$

By the generalized Poincaré conjecture, homotopy groups of G/PL are isomorphic to L-groups of $\mathbb{Z}$ in dimension 5 and greater. Thus $\pi_{2n-1}(G/PL) = 0$, and as a result, the map

$$\left[\mathbb{C}P^n, G/PL\right] \to \left[\mathbb{C}P^{n-1}, G/PL\right],$$

induced by the inclusion is surjective. It remains to identify the leftmost map of the above sequence.

For $n > 2$ the following triangle commutes.

$$\begin{tikzcd}
L_{2n}(\mathbb{Z}[e]) \cong \pi_{2n}(G/PL) \arrow[r] \arrow[dr, "\cong"'] & {[\mathbb{C}P^n, G/PL]} \arrow[d, "\sigma"] \\
 & L_{2n}(\mathbb{Z}[e])
\end{tikzcd}$$

Consequently the map $\pi_{2n}(G/PL) \to [\mathbb{C}P^n, G/PL]$ is split injective.

Although for $n = 2$ the sequence does not split, we have an explicit description of all maps, namely the vertical maps in the diagram below form an isomorphism of exact sequences.

$$\begin{tikzcd}
\pi_4(G/PL) \arrow[r] \arrow[d] & {[\mathbb{C}P^2, G/PL]} \arrow[r] \arrow[d] & {[\mathbb{C}P^1, G/PL]} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

The first and the third isomorphisms are given by the standard calculation of homotopy groups of G/PL.

The second isomorphism $[\mathbb{C}P^2, G/PL] \to \mathbb{Z}$ is given by the surgery obstruction map which maps the normal map $\mathbb{C}P^2 \# 8\overline{\mathbb{C}P}^2 \to \mathbb{C}P^2$ (overline denotes reversed orientation) to the generator of $L_4(\mathbb{Z}[e])$.

A.4.1 *Splitting invariants*

Definition A.8. Let $f\colon \mathbb{C}P^n \to G/PL$ be a map. For $1 \leq k \leq n$ we define the $2k$**-th splitting invariant** as

$$s_{2k}(f) \overset{\text{def.}}{=} \sigma\left(f|_{\mathbb{C}P^k}\right).$$

Remark. Consider a map $f\colon \mathbb{C}P^n \to G/PL$ and let $g\colon M^{2n} \to \mathbb{C}P^n$ be the corresponding normal map of degree 1. In order to compute the splitting invariants of f, deform g homotopically to a map which is transverse to $\mathbb{C}P^k \subset \mathbb{C}P^n$. This yields the following degree 1 normal map

$$g|_{g^{-1}\left(\mathbb{C}P^k\right)}\colon g^{-1}\left(\mathbb{C}P^k\right) \to \mathbb{C}P^k.$$

Surgery obstruction of this map equals $s_{2k}(f)$.

Example. Consider the degree one normal map (actually a homotopy equivalence) $\widetilde{h}\colon \widetilde{\mathbb{C}P^5} \to \mathbb{C}P^5$ constructed in one of the previous sections. From the construction it is clear that the transverse inverse image of $\mathbb{C}P^1$, $\mathbb{C}P^2$ and $\mathbb{C}P^3$ are $\mathbb{C}P^1$, $\mathbb{C}P^2$ and $\mathbb{C}P^3$ respectively. However, the transverse inverse image of $\mathbb{C}P^4$ is the connected sum $\mathbb{C}P^4 \# mM^8_B$, and the associated degree one normal map has surgery obstruction equal to m. Thus $s_2\left(\widetilde{h}\right) = s_4\left(\widetilde{h}\right) = s_6\left(\widetilde{h}\right) = 0$ but $s_8\left(\widetilde{h}\right) = m$.

Analogously, it is easy to compute splitting invariants of the degree one normal map $\overline{h}\colon \overline{\mathbb{C}P^6} \to \mathbb{C}P^6$. All splitting invariants of $\overline{h}$ vanish, with an exception of $s_{10}(\overline{h}) = n \pmod 2$. For the normal map $\widehat{\mathbb{C}P^6} \to \mathbb{C}P^6$ the splitting invariants are zero except for $s_8 = m$ and $s_{10} = n \pmod 2$.

Remark A.9. Observe that joining any free action on S^{2n+1} with the standard free action on

$$S^{2k+1} = \underbrace{S^1 * \cdots * S^1}_{k+1}$$

gives rise to a free action on $S^{2(n+k+1)+1}$. The orbit space of the action – a homotopy complex projective space $h\mathbb{C}P^{n+k+1}$ has vanishing splitting invariants s_{2l} for $l > n + 1$.

A.4.2 *Identification of the structure set*

Lemma A.10. *The map*

$$[\mathbb{C}P^n, G/PL] \longrightarrow L_4(\mathbb{Z}[e]) \oplus L_6(\mathbb{Z}[e]) \oplus \cdots \oplus L_{2n}(\mathbb{Z}[e])$$

given by the formula

$$[f] \longmapsto \big(s_4(f), s_6(f), \ldots, s_{2n}(f)\big)$$

is bijective.

Proof. The proof proceeds by induction. To establish the base case consider the commutative diagram (as above).

$$\begin{tikzcd}
\pi_4(G/PL) \arrow[r] \arrow[d, "\cong"] & {[\mathbb{C}P^2, G/PL]} \arrow[r] \arrow[d, "\cong"] & {[\mathbb{C}P^1, G/PL]} \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

Notice that the commutativity of this diagram implies that $s_4 \equiv s_2 \pmod 2$ and consequently $[\mathbb{C}P^2, G/PL]$ is determined solely by the surgery obstruction s_4. Suppose now that $n > 2$ and the conclusion is proved for $n - 1$. Since the sequence

$$0 \to L_{2n}(\mathbb{Z}[e]) \to \big[\mathbb{C}P^n, G/PL\big] \to \big[\mathbb{C}P^{n-1}, G/PL\big] \to 0$$

is split exact, it is easy to obtain the desired result. □

Using Lemma A.10 we can now identify the structure set.

Theorem A.11. *The map*

$$S^{PL}(\mathbb{C}P^n) \longrightarrow L_4(\mathbb{Z}[e]) \oplus L_6(\mathbb{Z}[e]) \oplus \cdots \oplus L_{2n-2}(\mathbb{Z}[e])$$

given by the formula

$$[f] \longmapsto \big(s_4(f), s_6(f), \ldots, s_{2n-2}(f)\big)$$

is bijective.

Corollary A.12. *All manifolds* $\mathbb{C}P^6$, $\widetilde{\mathbb{C}P^6}$, $\widehat{\mathbb{C}P^6}$, $\overline{\mathbb{C}P^6}$ *are not PL-homeomorphic to each other.*

In order to obtain the full classification of manifolds homotopy equivalent to a complex projective space, we need to quotient out the structure set by the action of $\mathcal{E}(\mathbb{C}P^n)$, the group of homotopy classes of homotopy self-equivalences of the $2n$-dimensional complex projective space.

Lemma A.13. *The group* $\mathcal{E}(\mathbb{C}P^n)$ *consists of two elements: the homotopy class of the identity and the complex conjugation.*

Proof. By the cellular approximation theorem there is a bijection

$$[\mathbb{C}P^n, \mathbb{C}P^n] = [\mathbb{C}P^n, \mathbb{C}P^\infty] = H^2(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}.$$

Homotopy classes of homotopy equivalences correspond to generators ±1. One generator is represented by the homotopy class of the identity map and the other one is represented by the homotopy class of the complex conjugation. □

We need to identify the action of $\mathcal{E}(\mathbb{C}P^n)$ on the set $[\mathbb{C}P^n, G/PL]$. Sullivan was able to compute the homotopy type of G/PL and using the results it possible to check that the complex conjugation acts trivially. Thus we obtain the following classification theorem.

Theorem A.14. *The PL-homeomorphism type of a manifold homotopy equivalent to* $\mathbb{C}P^n$*, for* $n > 2$*, is determined completely by the set of its splitting invariants* $s_4, \ldots, s_{2n-2}$.

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