AN INTRODUCTION TO NON-ARCHIMEDEAN FUNCTIONAL ANALYSIS

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VALUED FIELDS

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Remark. We denote by 0 the additive identity element of K and the real number zero. Similarly, we will denote by 1 the multiplicative identity element of K and the real number one.

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is a metric on K, that induces a topology on K for which K is a topological field.

We say that K is *complete*, if it is complete with respect to this metric.

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There are two possibilities:

(1) 1 is not an accumulation point of $|K^\times|$. Then $|K^\times| = \{d^k : k \in \mathbb{Z}\}$ for some $d \geq 1$, so $|K^\times|$ is a discrete subset of $(0, \infty)$.

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We say that K is *non-archimedean* if the set $\{n \cdot 1 : n \in \mathbb{N}\} = \{1, 1+1, 1+1+1, \cdots\}$ is bounded in K (i.e. $\sup_{n\in\mathbb{N}} |n \cdot 1| < \infty$);

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otherwise we say that K is *Archimedean*.

We have the following ([6], Theorem 1.1).

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Theorem 1

For any valued field $K = (K, |\cdot|)$ the following conditions are equivalent:
1) K is non-archimedean;
2) $|a+b| \leq \max\{|a|, |b|\}$ for all $a, b \in K$ (the strong triangle inequality);
3) $|a+b| = \max\{|a|, |b|\}$ for all $a, b \in K$ with $|a| \neq |b|$;
4) $|n \cdot 1| \leq 1$ for every $n \in \mathbb{N}$.

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for all $x, y, z \in K$.

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1) Every point of a ball B in X is a center of B
(i.e. for every $b \in B[a, r] = \{x \in X : d(x, a) \leq r\}$ we have $B[a, r] = B[b, r]$ and for every $b \in B(a, r) = \{x \in X : d(x, a) < r\}$ we have $B(a, r) = B(b, r)$.

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4) The topology of X is zero-dimensional (i.e. it has a base consisting of clopen sets).

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We have the following ([6], Lemma 2.3).

Theorem 2

An ultrametric space X is spherically complete if and only if any nonempty family of pairwise non-disjoint balls $(B_i)_{i \in I}$ in X has nonempty intersection.

Examples of complete non-archimedean valued fields

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For any prime number p the completion $\mathbb{Q}_p$ of the field $\mathbb{Q}$ of rational numbers with respect to the p-adic valuation $|\cdot|_p$ given by $|a|_p = p^{-r}$ if $a = p^r m/n$ and the integers m, n are not divided by p, is a valued field.

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The valued field $\mathbb{Q}_p$ of p-adic numbers is non-archimedean and spherically complete.

It is locally compact and not algebraically closed. Its valuation is discrete.

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The valued field $\mathbb{C}_p$ of p-adic complex numbers is non-archimedean and complete but not spherically complete.
It is algebraically closed and not locally compact. Its valuation is dense.

Let $(K, |\cdot|)$ be a non-archimedean valued field. Then $B[0,1]$ is a subring of K and $B(0,1)$ is a maximal ideal in $B[0,1]$, so the quotient ring $k = B[0,1]/B(0,1)$ is a field.

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2) If the valuation of $\mathbb{K}$ is discrete, then $\mathbb{K}$ is spherically complete.

BANACH SPACES

Seminorms

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A seminorm $\|\cdot\|$ on E is *strongly solid* if $\|E\| \subset |\mathbb{K}|$.

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Principle of van Rooij. Let $\|\cdot\|$ be a seminorm on E and let $t \in (0,1]$. If $\|x+y\| \geq t\|x\|$, then $\|x+y\| \geq t\|y\|$ (so $\|x+y\| \geq t\max\{\|x\|, \|y\|\}$).

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Proof. We have $t\|y\| = t\|-x+(x+y)\| \leq t\max\{\|x\|, \|x+y\|\} \leq \|x+y\|$.
□

Normed spaces

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A normed space $(E, \|\cdot\|)$ is an ultrametric space with the metric induced by the norm:

$$d : E \times E \to [0, \infty), d(x, y) = \|x - y\|.$$

This metric induces a topology on E for which E is a topological vector space.

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If $\mathbb{K}$ is non-separable, then any non-zero normed space is non-separable.

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Proposition 3

A series $\sum_{n=1}^{\infty} x_n$ in a Banach space E is convergent if and only if the sequence (x_n) is convergent to 0 in E.

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The vector space $c_0(X)$ of all functions $f : X \to \mathbb{K}$ such that for every $c > 0$ the set $\{x \in X : |f(x)| > c\}$ is finite, is a closed vector subspace of $l_\infty(X)$; so $c_0(X)$ with the sup-norm is a Banach space.

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We put $c_0 = c_0(\mathbb{N}), l_\infty = l_\infty(\mathbb{N})$ and $\mathbb{K}^n = l_\infty(\{1, \dots, n\})$ for any $n \in \mathbb{N}$.

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THE STRUCTURE OF BANACH SPACES

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For any t-orthogonal basis X of E and every function $f : X \to \mathbb{K}^{\times}$ the set $Y = \{f(x)x : x \in X\}$ is a t-orthogonal basis of E;

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clearly, the sets X and Y have the same cardinal number.

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Theorem 4

Let $t \in (0,1]$ and let E be a Banach space with a t-orthogonal basis X. Then E is isomorphic to $c_0(X)$. If $t = 1$ and the norm of E is strongly solid, then E is isometrically isomorphic to $c_0(X)$.

Proof of Theorem 4

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Let $\mathcal{A}$ denote the family of all finite and non-empty subsets of X directed by the inclusion relation.
Let $f \in c_0(X)$. Put $f_A = \sum_{x \in A} f(x)x$ for any $A \in \mathcal{A}$. It is easy to check that $(f_A)_{A \in \mathcal{A}}$ is a Cauchy net in E; so it is convergent to some T_f in E.

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Clearly the map

$$T : c_0(X) \to E, f \to T_f,$$

is linear.

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$t|b|\max_{x\in A}|f(x)| = t|b|\|f\|_\infty$, for $f \in c_{00}(X)$.

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It follows that $c_0(X)$ and the range $R(T)$ of T are isomorphic, thus $R(T)$ is complete, so it is a closed subspace of E.

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On the other hand, $R(T)$ is dense in E, since $R(T) \supset \{T(\delta_x) : x \in X\} = X$. Thus $R(T) = E$.

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If $t = 1$ and $\|E\| \subset |\mathbb{K}|$, then $t|b| = 1$, so T is an isometry. $\square$

Now we shall prove the following ([3], Theorem 2.3.7)

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Theorem 5

Let $E = (E, \|\cdot\|)$ be a normed space of countable type. Let $t \in (0,1)$. Then E has a countable t-orthogonal basis X.

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For any $n > 1$ we can choose a $z_n \in E_n$ with $t_n\|z_n - y_n\| \leq \text{dist}(E_n, y_n)$.

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Let $x_1 = y_1$ and $x_n = z_n - y_n$ for any $n > 1$.
The sequence (x_n) is linearly independent and linearly dense in E, since $\text{lin}\{x_k : 1 \leq k < n\} = E_n$ for $n > 1$.

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It means that the set $X = \{x_n : n \in \mathbb{N}\}$ is a t-orthogonal basis in E. $\square$

If the scalar field is spherically complete we get a stronger result ([3], Theorem 2.3.25).

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Theorem 6

Assume that the scalar field $\mathbb{K}$ is spherically complete. Then every normed space E of countable type has a countable 1-orthogonal basis X.

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Theorem 6

Assume that the scalar field $\mathbb{K}$ is spherically complete. Then every normed space E of countable type has a countable 1-orthogonal basis X.

It follows by the proof of the previous theorem and by the following lemma.

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It follows by the proof of the previous theorem and by the following lemma.

Lemma 7

Assume that the scalar field $\mathbb{K}$ is spherically complete. Let E be a finite-dimensional normed space. Then E is spherically complete and for every proper vector subspace F of E there is a non-zero element $y \in E$ which is 1-orthogonal to F i.e. $\|ay + z\| = \max\{\|ay\|, \|z\|\}$ for all $a \in \mathbb{K}, z \in F$.

Proof of Lemma 7

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The sequence of closed balls $(B_m) = (B[z_m, r_m])$ in F is decreasing.

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Assume that for some $n \in \mathbb{N}$ the conclusion is true if $k = n$.
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Assume that $\dim E = n + 1$. Without loss of generality we can assume that $\dim F = n$. By the inductive assumption, F is spherically complete.
Let $s \in (E \setminus F)$. Then there exists a sequence $(z_m) \subset F$ such that the sequence $(r_m) = (\|s - z_m\|)$ is decreasing and convergent to $\operatorname{dist}(s, F) = \inf\{\|s - z\| : z \in F\}$.
The sequence of closed balls $(B_m) = (B[z_m, r_m])$ in F is decreasing. Indeed, if $z \in B[z_{m+1}, r_{m+1}]$, then

$$\|z - z_m\| \leq \max\{\|z - z_{m+1}\|, \|z_{m+1} - s\|, \|s - z_m\|\} \leq r_m,$$

so $z \in B[z_m, r_m]$.

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Thus, E is isometrically isomorphic to the product $\mathbb{K}y \times F$ (with respect to the max-norm $\|(x, z)\| = \max\{\|x\|, \|z\|\}$).

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It is easy to check that the product of two spherically complete normed spaces is spherically complete (with respect to the max-norm). Thus E is spherically complete. $\square$

Remark 8

If the scalar field $\mathbb{K}$ is not spherically complete then there is a two-dimensional normed space E (even with a strongly solid norm) which has no 1-orthogonal basis.

By Theorems 4,5 and 6 we get the following ([6], Th. 3.16).

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Corollary 9

Every infinite-dimensional Banach space E of countable type is isomorphic to c_0; if the scalar field $\mathbb{K}$ is spherically complete and the norm of E is strongly solid then E is isometrically isomorphic to c_0.

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Corollary 9

Every infinite-dimensional Banach space E of countable type is isomorphic to c_0; if the scalar field $\mathbb{K}$ is spherically complete and the norm of E is strongly solid then E is isometrically isomorphic to c_0.

Corollary 10

Every finite-dimensional normed space E is isomorphic to $\mathbb{K}^n$, where $n = \dim E$; if the scalar field $\mathbb{K}$ is spherically complete and the norm of E is strongly solid, then E is isometrically isomorphic to $\mathbb{K}^n$.

If the scalar field has a discrete valuation we have the following ([6]).

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Theorem 11

Assume that the scalar field $\mathbb{K}$ has a discrete valuation. Let E be a normed space. Then E has a t-orthogonal basis for some $t \in (0, 1]$. If the norm of E is strongly solid, then E has a 1-orthogonal basis.

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Thus $F = E$ and $X = Z \setminus \{0\}$ is a 1-orthogonal basis of E.

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It is easy to see that $|||\cdot|||$ is a strongly solid norm on E and $\|x\| \leq |||x||| \leq d\|x\|$ for all $x \in E$.

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By the first part of this proof, $(E, |||\cdot|||)$ has a 1-orthogonal basis X.
Then X is a t-orthogonal basis in $(E, \|\cdot\|)$ for $t = 1/d$. □

Let F be a closed subspace of a Banach space E. A closed subspace G of E is called a *complement* of F in E if $G + F = E$ and $G \cap F = \{0\}$.

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We say that F is complemented in E if it has a complement in E (or equivalently, if there is a continuous linear projection from E onto F).

We have the following ([8], Propositions 10.1, 10.5).

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Theorem 12

Assume that the scalar field $\mathbb{K}$ has a discrete valuation. Let E be a Banach space. Then E is isomorphic to the Banach space $c_0(X)$ for some set X and every closed subspace of E is complemented in E. If the norm of E is strongly solid, then E is isometrically isomorphic to $c_0(X)$ for some X and every closed subspace F of E is 1-orthocomplemented in E i.e. F has a complement G such that $\|x+y\| = \max\{\|x\|, \|y\|\}$ for all $x \in F, y \in G$.

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Moreover, by Theorem 11 and its proof, F has a 1-orthogonal basis Y and E has a 1-orthogonal basis X that contain Y.

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Using Theorem 4 we infer that E is isometrically isomorphic to $c_0(X)$.

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Moreover, by Theorem 11 and its proof, F has a 1-orthogonal basis Y and E has a 1-orthogonal basis X that contain Y.
Using Theorem 4 we infer that E is isometrically isomorphic to $c_0(X)$.
It is easy to see that the closure G of the linear span of $(X \setminus Y)$ is a 1-orthogonal complement of F in E. $\square$

A sequence (x_n) in a normed space E is a *Schauder basis* in E if for each element $x \in E$ there exists exactly one sequence $(a_n) \subset \mathbb{K}$ such that $x = \sum_{n=1}^{\infty} a_n x_n$ and the coefficient functionals $x_n^* : E \to \mathbb{K}, x \to a_n (n \in \mathbb{N})$ are continuous.

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The standard sequence $(e_n) \subset c_0$ (i.e. $e_1 = (1, 0, 0, \ldots), e_2 = (0, 1, 0, \ldots), \cdots$) is a Schauder basis in the space c_0. By the proof of Theorem 4 we get the following.

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Proposition 13

If $X = \{x_n : n \in \mathbb{N}\}$ is a t-orthogonal basis in a normed space E for some $t \in (0, 1]$, then the sequence (x_n) is a Schauder basis in E.

Now we shall prove the following very useful result ([3], Theorem 2.3.13).

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Theorem 14

Let E be a Banach space of countable type. Then each closed subspace D of E is complemented in E. In fact, for every $s > 1$ there exists a continuous linear projection P from E onto D with $\|P\| \leq s$.

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Let $t \in (0, 1)$. F with the quotient norm $|||z||| = \inf\{\|x\| : x \in z\}, z \in F$, is an infinite-dimensional Banach space of countable type;

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Let $t \in (0, 1)$. F with the quotient norm $|||z||| = \inf\{\|x\| : x \in z\}, z \in F$, is an infinite-dimensional Banach space of countable type; so it has a t-orthogonal basis $Z = \{z_n : n \in \mathbb{N}\}$ and (z_n) is a Schauder basis in F.

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$$T : F \to E, \sum_{n=1}^{\infty} a_n z_n \to \sum_{n=1}^{\infty} a_n x_n,$$

is well defined; clearly T is linear and $Tz \in z$ for $z \in F$.

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$$|||\sum_{n=1}^{\infty} a_n z_n|||/t^2 = |||z|||/t^2.$$

Thus T is continuous and $\|T\| \leq 1/t^2$.
Let $Q : E \to F$ be the quotient map. It is easy to see that the map

$$P : E \to E, x \to x - TQx,$$

is a continuous linear projection from E onto D with $\|P\| \leq \|T\|$;

Proof of Theorem 14

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is a continuous linear projection from E onto D with $\|P\| \leq \|T\|$; taking $t = 1/\sqrt{s}$ we get $\|P\| \leq s$. □

THE HAHN-BANACH THEOREMS AND DUALITY

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$\|f\| = \sup\{|f(x)|/\|x\| : x \in E, x \neq 0\}, f \in E'$.
E' is a Banach space. The dual E'' of E' is called the bidual of E.
A normed space E is called *reflexive* if the canonical map $J_E : E \longrightarrow E'', x \longrightarrow J_E x$, where $J_E x : E' \longrightarrow \mathbb{K}, f \longrightarrow f(x)$, is an isomorphism.

We start with the Hahn-Banach theorem for seminormed spaces ([3], Theorem 4.1.1).

Theorem 15

Assume that the scalar field $\mathbb{K}$ is spherically complete. Let p be a seminorm on a vector space E. Then every linear functional f on a subspace D of E such that $|f| \leq p|D$ can be extended to a linear functional g on E with $|g| \leq p$.

Proof of the Theorem 15

By a standard application of Zorn's Lemma we may assume that the quotient space E/D is one-dimensional.

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For any $x \in D$ denote by B_x the closed ball $B[f(x), p(x - z)]$ in $\mathbb{K}$. For any $x, y \in D$ we have $B_x \cap B_y \neq \emptyset$, since

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For any $x \in D$ denote by B_x the closed ball $B[f(x), p(x-z)]$ in $\mathbb{K}$. For any $x, y \in D$ we have $B_x \cap B_y \neq \emptyset$, since
$|f(x) - f(y)| = |f(x-y)| \leq$
$p(x-y) = p((x-z) - (y-z)) \leq \max\{p(x-z), p(y-z)\}$.

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$|f(x) - f(y)| = |f(x-y)| \leq$
$p(x-y) = p((x-z)-(y-z)) \leq \max\{p(x-z), p(y-z)\}$.
The field $\mathbb{K}$ is spherically complete, so the intersection B of the family $\{B_x : x \in D\}$ is non-empty.

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$|f(-x/a) - b| \leq p((-x/a) - z)$ for all $x \in D, a \in \mathbb{K}^{\times}$.

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Thus $(*)$ $\quad |f(x) + ab| \leq p(x + az)$ for all $x \in D, a \in \mathbb{K}$.

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Thus $(*)$ $\quad |f(x) + ab| \leq p(x + az)$ for all $x \in D, a \in \mathbb{K}$.

Clearly, the functional
$g : E = D + \mathbb{K}z \to \mathbb{K}, x + az \to f(x) + ab$ is linear and
$g|D = f$. By $(*)$ we get $|g| \leq p$. $\square$

Hence we get the Hahn-Banach theorem for normed spaces ([3], Corollary 4.1.2).

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Theorem 16

Assume that the scalar field $\mathbb{K}$ is spherically complete. Let D be a subspace of a normed space E. Then every continuous linear functional on D can be extended to a continuous linear functional g on E with the same norm i.e. $\|g\| = \|f\|$.

If the scalar field $\mathbb{K}$ is not spherically complete then it is known that for every infinite-dimensional normed space E there exists a continuous linear functional f on some subspace D of E such that f can not be extended to a continuous linear functional g on E with $\|g\| = \|f\|$. ([4])

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In fact we will prove that in this case there exist Banach spaces (of not countable type) having trivial dual ([3], Theorem 4.1.12).

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In fact we will prove that in this case there exist Banach spaces (of not countable type) having trivial dual ([3], Theorem 4.1.12).

Theorem 17

Assume that the scalar field $\mathbb{K}$ is not spherically complete. Then the quotient Banach space l_∞ / c_0 has a trivial dual.

Proof of Theorem 17

Let $E = l_\infty / c_0$ and let $Q : l_\infty \to E$ be the quotient map.

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Proof of Theorem 17

Let $E = l_\infty / c_0$ and let $Q : l_\infty \to E$ be the quotient map. Assume that there is a non-zero continuous linear functional f on E. Clearly, $M = \ker f \circ Q$ is a closed subspace of l_∞ and $M \supset c_0$. We shall prove that the quotient space $F = l_\infty / M$ is spherically complete.

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$\|Pa\| = \inf_{b \in M} \|a - b\|_\infty \leq \inf_{b \in c_0} \|a - b\|_\infty = \limsup_n |a_n|.$

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Thus $Px \in B[y_n, r_{n-1}] = B[y_{n-1}, r_{n-1}]$ for $n > 1$,

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Thus $Px \in B[y_n, r_{n-1}] = B[y_{n-1}, r_{n-1}]$ for $n > 1$, so $\bigcap_{n=1}^{\infty} B[y_n, r_n] \neq \emptyset$.

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On the other hand, F is (isometrically) isomorphic to the quotient space $E/\ker f$, so it is one-dimensional.

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It is easy to see that any one-dimensional normed space over $\mathbb{K}$ is not spherically complete.

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Thus F is not spherically complete, a contradiction.

Proof of Theorem 17

$$\|Px - y_n\| = \|P(x - x_n)\| \leq \limsup_k |x_{k,k} - x_{n,k}| \leq$$

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Thus $Px \in B[y_n, r_{n-1}] = B[y_{n-1}, r_{n-1}]$ for $n > 1$, so $\bigcap_{n=1}^\infty B[y_n, r_n] \neq \emptyset$. It follows that the space F is spherically complete.

On the other hand, F is (isometrically) isomorphic to the quotient space $E/\ker f$, so it is one-dimensional.

It is easy to see that any one-dimensional normed space over $\mathbb{K}$ is not spherically complete.

Thus F is not spherically complete, a contradiction.

It follows that E has a trivial dual. □

Using the last theorem we get the following characterization of spherical completeness of the scalar field $\mathbb{K}$ ([3], Corollary 4.1.13).

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Theorem 18

The following properties of the scalar field $\mathbb{K}$ are equivalent.

(a) $\mathbb{K}$ *is spherically complete.*

(b) *The functional* $f : c_0 \to \mathbb{K}, f(x) = \sum_{n=1}^{\infty} x(n)$, *can be extended to a continuous linear functional* g *on* l_∞.

(c) *The dual of the quotient Banach space* l_∞ / c_0 *is non-trivial.*

Proof of the Theorem 18

(a) implies (b) by the Hahn-Banach theorem.
We shall show that (b) implies (c). The map

$$T : l_\infty \to l_\infty, (x_1, x_2, x_3, \cdots) \to (0, x_1, x_2, \cdots)$$

is linear and continuous. Thus $h = g - g \circ T$ is a continuous linear functional on l_∞. For $e = (1, 1, 1, \cdots)$ we get $h(e) = g(1, 1, 1, \cdots) - g(0, 1, 1, \cdots) = g(1, 0, 0, \cdots) = 1$.
For $x = (x_1, x_2, x_3, \cdots) \in c_0$ we have

$$h(x) = g(x_1, x_2, x_3, \cdots) - g(0, x_1, x_2, \cdots) = \sum_{n=1}^{\infty} x_n - \sum_{n=1}^{\infty} x_n = 0.$$

Proof of the Theorem 18

Thus the functional

$$G : l_\infty / c_0 \to \mathbb{K}, x + c_0 \to h(x)$$

is well defined, linear and $G(e + c_0) = h(e) = 1$, so $G \neq 0$. Let $Q : l_\infty \to l_\infty / c_0$ be the quotient map. Then $G \circ Q = h$, so $G^{-1}(U) = Q(h^{-1}(U)$ for $U \subset \mathbb{K}$. Thus G is continuous, since Q is an open map. It follows that the dual of l_∞ / c_0 is non-trivial.
(c) implies (a) by the last theorem. □

For normed spaces of countable type we have the following Hahn-Banach theorem ([3], Theorem 4.2.4, Corollary 4.2.5).

Theorem 19

Let E be a normed space of countable type. Let D be a subspace of E. Then for every $f \in D'$ and for every $c > 1$ there is an extension $g \in E'$ of f with $\|g\| \leq c\|f\|$.

Proof of the Theorem 19

Without loss of generality we may assume that the normed space E is complete and D is a closed subspace of E. Thus, by Theorem 14, there exists a linear projection Q from E onto D with $\|Q\| \leq c$. Then the linear functional $g = f \circ Q$ on E is an extension of f and $\|g\| \leq c\|f\|$. $\square$

Clearly, if the scalar field $\mathbb{K}$ is spherically complete and E is a normed space, then the conclusion of the above theorem is satisfied for any subspace D of E.

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One of the most important open problems in functional analysis is the following one ([3]).

Problem 20

Assume that the scalar field $\mathbb{K}$ is not spherically complete. Let E be a normed space. Suppose that the conclusion of the above theorem is satisfied for any subspace D of E. Does it follow that E is of countable type?

Now we determine the dual of c_0. We get the following ([3], Theorem 2.5.11).

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Theorem 21

Denote by B the map $c_0 \times l_\infty \to \mathbb{K}, (x, y) \to \sum_{n=1}^{\infty} x(n)y(n)$. The map $T : l_\infty \to c_0', T(y) = B(\cdot, y)$, is an isometrical isomorphism. Thus the dual of c_0 is isometrically isomorphic to l_∞.

Proof of Theorem 21

The map B is well defined and bilinear. Let $y \in l_\infty$. The functional $f_y : c_0 \longrightarrow \mathbb{K}, f_y(x) = B(x, y)$, is linear, continuous and $\|f_y\| \leq \|y\|_\infty$,

Proof of Theorem 21

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$$|f_y(x)| = |\sum_{n=1}^{\infty} x(n)y(n)| \leq \max_n |x(n)||y(n)| \leq \|x\|_\infty \|y\|_\infty$$

for $x \in c_0$.

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for $x \in c_0$. On the other hand, we have

$$\|f_y\| \geq \sup_n |f_y(e_n)|/\|e_n\|_\infty = \sup_n |y(n)| = \|y\|_\infty.$$

Proof of Theorem 21

The map B is well defined and bilinear. Let $y \in l_\infty$. The functional $f_y : c_0 \to \mathbb{K}, f_y(x) = B(x, y)$, is linear, continuous and $\|f_y\| \leq \|y\|_\infty$, since

$$|f_y(x)| = |\sum_{n=1}^{\infty} x(n)y(n)| \leq \max_n |x(n)||y(n)| \leq \|x\|_\infty \|y\|_\infty$$

for $x \in c_0$. On the other hand, we have

$$\|f_y\| \geq \sup_n |f_y(e_n)|/\|e_n\|_\infty = \sup_n |y(n)| = \|y\|_\infty.$$

Thus $\|Ty\| = \|f_y\| = \|y\|_\infty$. It follows that the map T is well defined, linear and isometrical.

Proof of Theorem 21

To show surjectivity of T, let $g \in c_0'$.

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To see that $g = Tz$, let $x \in c_0$.
Then $g(x) = g(\sum_{n=1}^\infty x(n)e_n) = \sum_{n=1}^\infty x(n)g(e_n) = \sum_{n=1}^\infty x(n)z(n) = B(x,z) = (Tz)(x)$.

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Thus T is an isometrical isomorphism. $\square$

Similarly one can show that for every set X the dual of $c_0(X)$ is isometrically isomorphic to $l_\infty(X)$.

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Using the Hahn-Banach theorem and the last theorem we get the following ([3]).

Proposition 22

Assume that the scalar field $\mathbb{K}$ is spherically complete. Let E be an infinite-dimensional Banach space. Then the dual of E is not of countable type.

Proof of Proposition 22

Clearly, E contains an infinite-dimensional closed subspace F of countable type. We know that F is isomorphic to c_0. Thus the dual of F is isomorphic to l_∞, so it is not of countable type. The map $T : E' \to F', g \to g|F$ is linear and continuous. By the Hahn-Banach theorem the map T is surjective. It follows that E' is not of countable type. $\square$

Corollary 23

Assume that the scalar field $\mathbb{K}$ is spherically complete. Then the dual of l_∞ is not isomorphic to c_0. Thus c_0 is not isomorphic to its bidual, so the Banach space c_0 is not reflexive.

Now we shall describe the dual of l_∞, when the scalar field $\mathbb{K}$ is not spherically complete. If $\mathbb{K}$ is spherically complete then it is not known a satisfactory description of the dual of l_∞. We have the following ([3], Theorem 5.5.5).

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Theorem 24

Assume that the scalar field $\mathbb{K}$ is not spherically complete. Denote by B the map $c_0 \times l_\infty \to \mathbb{K}, (x, y) \to \sum_{n=1}^{\infty} x(n)y(n)$. The map $S : c_0 \to l'_\infty, S(x) = B(x, \cdot)$ is an isometrical isomorphism. Thus the dual of l_∞ is isometrically isomorphic to c_0.

Proof of Theorem 24

The map B is well defined and bilinear. Let $x \in c_0$. The functional $g_x : l_\infty \to \mathbb{K}, g_x(y) = B(x, y)$, is linear, continuous and $\|g_x\| \leq \|x\|_\infty$, since

$$|g_x(y)| = |\sum_{n=1}^{\infty} x(n)y(n)| \leq \max_n |x(n)||y(n)| \leq \|x\|_\infty \|y\|_\infty$$

for $y \in l_\infty$. On the other hand we have $\|g_x\| \geq \sup_n |g_x(e_n)|/\|e_n\|_\infty = \sup_n |x(n)| = \|x\|_\infty$. Thus $\|Sx\| = \|g_x\| = \|x\|_\infty$. It follows that the map S is well defined, linear and isometrical.

To show surjectivity of S, let $f \in l'_\infty$. Since $f|c_0 \in c'_0$ and the map

$$T : l_\infty \to c'_0, T(y) = B(\cdot, y),$$

is an isometrical isomorphism,

Proof of Theorem 24

we obtain $y \in l_\infty$ such that $f(x) = B(x, y)$ for all $x \in c_0$. We prove that $y \in c_0$. Suppose that for some $c > 0$ the set $J = \{n \in \mathbb{N} : |y(n)| > c\}$ is infinite. For $z \in l_\infty(J)$ define $h_z \in l_\infty$ by

$$h_z(n) = \begin{cases} z(n)/y(n) & \text{for } \; n \in J; \\ 0 & \text{otherwise .} \end{cases}$$

The functional $h : l_\infty(J) \to \mathbb{K}, z \to f(h_z)$ is well defined, linear and continuous. If $z \in c_0(J)$ then $h_z \in c_0$ and

$$h(z) = f(h_z) = \sum_{n=1}^{\infty} h_z(n)y(n) = \sum_{n \in J} z(n).$$

Proof of Theorem 24

Thus the functional

$$g : c_0(J) \to \mathbb{K}, g(z) = \sum_{n \in J} z(n)$$

can be extended to a continuous linear functional on $l_\infty(J)$. By Theorem 18 and its proof we infer that the scalar field $\mathbb{K}$ is spherically complete, a contradiction. It follows that $y \in c_0$.

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The map $G : l_\infty \to \mathbb{K}, G(z) = f(z) - B(y, z)$ is linear, continuous and $G(x) = f(x) - B(y, x) = f(x) - B(x, y) = 0$ for $x \in c_0$.

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It follows that $G = 0$, since the dual of the quotient space l_∞ / c_0 is trivial. Thus $f(z) = B(y, z)$ for all $z \in l_\infty$, so $f = S(y)$. Thus S is surjective. □

Corollary 25

Assume that the scalar field $\mathbb{K}$ is not spherically complete. Then the Banach spaces c_0 and l_∞ are reflexive.

We say that a sequence (x_n) in a normed space E is weakly convergent to $x \in E$, if for every continuous linear functional f on E the scalar sequence $f(x_1), f(x_2), f(x_3), \cdots$ is convergent to $f(x)$.

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It is well known that any weakly convergent sequence in the Archimedean Banach space l_1 is convergent (in the norm topology).

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In our case we have a similar result for c_0 ([3], Theorem 5.5.1).

Theorem 26

Every weakly convergent sequence (x_n) in c_0 is convergent.

Proof of Theorem 26

It is enough to consider the case when (x_n) is convergent to 0.

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Without loss of generality we may assume that $\|x_n\|_\infty > 1$ for any $n \in \mathbb{N}$.

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Without loss of generality we may assume that $\|x_n\|_\infty > 1$ for any $n \in \mathbb{N}$. Then the set $J(n) = \{i \in \mathbb{N} : |x_n(i)| > 1\}$ is non-empty and finite for any $n \in \mathbb{N}$.

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On the other hand, we have $\lim_n x_n(i) = \lim_n e_i^*(x_n) = 0$ for any $i \in \mathbb{N}$, since (x_n) is weakly convergent to 0. Thus for any $i \in \mathbb{N}$ the set $\{n \in \mathbb{N} : i \in J(n)\}$ is finite.

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On the other hand, we have $\lim_n x_n(i) = \lim_n e_i^*(x_n) = 0$ for any $i \in \mathbb{N}$, since (x_n) is weakly convergent to 0. Thus for any $i \in \mathbb{N}$ the set $\{n \in \mathbb{N} : i \in J(n)\}$ is finite.
Therefore there is a strictly increasing sequence $(m_k) \subset \mathbb{N}$ such the sets $J(m_k)$, $k \in \mathbb{N}$, are pairwise disjoint. Let $i_k \in J(m_k)$ for $k \in \mathbb{N}$.

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The functional

$$f : c_0 \longrightarrow \mathbb{K}, x \longrightarrow \sum_{k=1}^{\infty} x(i_k)$$

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is well defined, linear and continuous.
Let $l \in \mathbb{N}$. Then $|x_{m_l}(i_l)| > 1$ and $|x_{m_l}(i_k)| \leq 1$ for any $k \neq l$, so $|f(x_{m_l})| = |\sum_{k=1}^{\infty} x_{m_l}(i_k)| > 1$.

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is well defined, linear and continuous.
Let $l \in \mathbb{N}$. Then $|x_{m_l}(i_l)| > 1$ and $|x_{m_l}(i_k)| \leq 1$ for any $k \neq l$, so $|f(x_{m_l})| = |\sum_{k=1}^{\infty} x_{m_l}(i_k)| > 1$.
Thus the sequence $f(x_1), f(x_2), \ldots$ does not converge to 0, so the sequence (x_n) is not weakly convergent to 0; a contradiction. $\square$

Since the completion of any infinite-dimensional normed space of countable type is isomorphic to c_0, we get the following.

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Let E be a normed space of countable type. Then every weakly convergent sequence (x_n) in E is convergent.

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Hence, using the Hahn-Banach theorem we get the following result that deviates much from the Archimedean case ([3]).

Corollary 28

Assume that the scalar field $\mathbb{K}$ is spherically complete. Let E be a normed space. Then every weakly convergent sequence (x_n) in E is convergent.

Proof of Corollary 28

Let (x_n) be a sequence in E that is weakly convergent to 0.

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By the Hahn-Banach theorem, (x_n) is weakly convergent to 0 in D.
Thus, by the previous corollary, (x_n) is convergent to 0 in D, so in E. $\square$

FRÉCHET SPACES

We recall that $\mathbb{K}$ is a complete non-archimedean non-trivially valued field and all vector spaces in this mini-course are over the scalar field $\mathbb{K}$.

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Denote by $\mathbb{B}$ the unit closed ball $B[0, 1]$ of the scalar field $\mathbb{K}$. Clearly $\mathbb{B}$ is a subring of $\mathbb{K}$.

Absolutely convex sets

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The set X is *absolutely convex* if $ax + by \in X$ for all $a, b \in \mathbb{B}, x, y \in X$ (or equivalently, if X is a $\mathbb{B}$-module).

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The set
$\operatorname{aco} X = \{\sum_{i=1}^{n} a_i x_i : n \in \mathbb{N}, a_1, \ldots, a_n \in \mathbb{B}, x_1, \ldots, x_n \in X\}$
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Clearly, X is absolutely convex if and only if $\operatorname{aco} X = X$.
The set X is called *convex* if for some $y \in E$ the set $X + y$ is absolutely convex.
For any seminorm p on E and any $c > 0$ the sets $\{x \in E : p(x) < c\}$ and $\{x \in E : p(x) \leq c\}$ are absolutely convex.

Locally convex space

A Hausdorff topology τ on a vector space E is called a *locally convex topology* if there exists a collection $\mathcal{P}$ of seminorms on E that induces τ i.e. such that a subset U of E is τ-open if and only if for each $x \in U$ there exist $n \in \mathbb{N}, c > 0$ and $p_1, \ldots, p_n \in \mathcal{P}$ such that

$$\{y \in E : \max_{1 \leq i \leq n} p_i(y - x) < c\} \subset U.$$

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$$\{y \in E : \max_{1 \leq i \leq n} p_i(y - x) < c\} \subset U.$$

The pair (E, τ) is called a *locally convex space*. Frequently we will write E instead of (E, τ).
Clearly, E has a base of zero neighbourhoods consisting of absolutely convex sets.

Let $E = (E, \tau)$ be a locally convex space.

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The family of all continuous seminorms on E is denoted by $\mathcal{P}(E)$. A family $\mathcal{B} \subset \mathcal{P}(E)$ is called a *base* of $\mathcal{P}(E)$ if any $q \in \mathcal{P}(E)$ is dominated by some $p \in \mathcal{B}$ (i.e. $q \leq p$). Clearly, any base of $\mathcal{P}(E)$ induces on E the topology τ.

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By a quotient of E we mean any quotient space E/D, where D is a closed subspace of E, with the quotient topology.

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By a subspace of E we understand any vector subspace of E with the topology induced from E.

By a quotient of E we mean any quotient space E/D, where D is a closed subspace of E, with the quotient topology.

Clearly all subspaces and quotients of E are locally convex spaces.

Let $E = (E, \tau)$ be a locally convex space.
E is called *normable* if its topology τ is generated by one norm.
E is called *metrizable* if there is a metric on E inducing the topology τ.

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E is called *normable* if its topology τ is generated by one norm.
E is called *metrizable* if there is a metric on E inducing the topology τ.
As in the Archimedean case we have the following.

Theorem 29

A locally convex space E is metrizable if and only if there exists a non-decreasing sequence $(p_k) \subset \mathcal{P}(E)$ which forms a base of $\mathcal{P}(E)$.

A complete metrizable locally convex space is called a *Fréchet space.*

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Using the strong triangle inequality we get the following.

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Proposition 30

A series $\sum_{n=1}^{\infty} x_n$ in a Fréchet space E is convergent if and only if the sequence (x_n) is convergent to zero in E.

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Proposition 30

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As in the Archimedean case we have the following three fundamental results ([3]).

The Uniform Boundedness Theorem

Every pointwise bounded family of continuous linear maps from a Fréchet space E to a locally convex space F is equicontinuous.

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A linear map T from a Fréchet space E to a Fréchet space F is continuous if and only if the graph of T is closed.

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The Closed Graph Theorem

A linear map T from a Fréchet space E to a Fréchet space F is continuous if and only if the graph of T is closed.

The Open Map Theorem

Every continuous linear map T from a Fréchet space E onto a Fréchet space F is open.

THE HAHN-BANACH THEOREMS AND THE WEAK TOPOLOGY

If the scalar field $\mathbb{K}$ is spherically complete, then using the Hahn-Banach theorem for seminormed spaces, we get the following Hahn-Banach theorem for locally convex spaces ([3]).

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Theorem 31

Assume that the scalar field $\mathbb{K}$ is spherically complete. Let D be a subspace of a locally convex space E. Then every continuous linear functional f on D can be extended to a continuous linear functional g on E; if $|f| \leq p|D$ for some continuous seminorm p on E, then we can additionally claim that $|g| \leq p$.

For closed absolutely convex subsets we have the following Hahn-Banach theorem.

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Theorem 32

Assume that the scalar field $\mathbb{K}$ is spherically complete. Let A be a closed absolutely convex subset of a locally convex space E and let $x \in (E \setminus A)$. Then there is a continuous linear functional f on E with $f(x) = 1$ such that $|f|A| < 1$ i.e. $|f(y)| < 1$ for all $y \in A$.

Locally convex spaces of countable type

A locally convex space E is said to be *of countable type* if for every continuous seminorm p on E the normed space $E_p = (E/\ker p, \overline{p})$, where

$$\overline{p} : (E/\ker p) \to [0, \infty), x + \ker p \to p(x)$$

is of countable type (i.e. E_p contains a linearly dense countable subset).

Locally convex spaces of countable type

A locally convex space E is said to be *of countable type* if for every continuous seminorm p on E the normed space $E_p = (E/\ker p, \overline{p})$, where

$$\overline{p} : (E/\ker p) \to [0, \infty), x + \ker p \to p(x)$$

is of countable type (i.e. E_p contains a linearly dense countable subset).
It is easy to check the following.

Proposition 33

Let E be a metrizable locally convex space. Then the following conditions are equivalent.
(1) E is of countable type;
(2) E contains a countable subset whose linear span is dense in E;
(3) E is isomorphic to a subspace of the Fréchet space $c_0^{\mathbb{N}}$ (with the product topology).

Proposition 33

Let E be a metrizable locally convex space. Then the following conditions are equivalent.
(1) E is of countable type;
(2) E contains a countable subset whose linear span is dense in E;
(3) E is isomorphic to a subspace of the Fréchet space $c_0^{\mathbb{N}}$ (with the product topology).

Corollary 34

A locally convex space E is a Fréchet space of countable type if and only if E is isomorphic to a closed subspace of the Fréchet space $c_0^{\mathbb{N}}$.

For locally convex spaces of countable type we have the following Hahn-Banach theorem ([3]).

Theorem 35

Let E be a locally convex space of countable type. Then every continuous linear functional f on a subspace D of E can be extended to a continuous linear functional g on E; if $c > 1$ and $|f| \leq p|D$ for some continuous seminorm p on E, then we can additionally claim that $|g| \leq cp$.

Clearly, if the scalar field $\mathbb{K}$ is spherically complete, then for every locally convex space E the conclusion of the above theorem is satisfied.

One of the most important open problems in functional analysis is the following one ([3]).

Open problem

Assume that the scalar field $\mathbb{K}$ is not spherically complete. Suppose that the conclusion of the previous theorem is satisfied for some locally convex space E. Does it follow that E is of countable type?

Polar spaces

A seminorm p on a vector space E is called *polar* if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$, where E^* denotes the vector space of all linear functionals on E. A locally convex space E is *polar* (or a *polar space*) if the family of all polar continuous seminorms on E is a base of $\mathcal{P}(E)$.

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By the Hahn-Banach theorems we get the following ([3]).

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By the Hahn-Banach theorems we get the following ([3]).

Proposition 36

Every locally convex space of countable type is polar. If the scalar field $\mathbb{K}$ is spherically complete, then every locally convex space is polar.

For polar spaces we have the following Hahn-Banach theorem ([3]).

Theorem 37

Let E be a polar space. Then every linear functional f on a finite-dimensional subspace D of E can be extended to a continuous linear functional g on E; if $c > 1$ and $|f| \leq p|D$ for some continuous polar seminorm p on E, then we can additionally claim that $|g| \leq cp$.

Corollary 38

Every polar space E is dual-separating i.e. for every non-zero element x of E there is a continuous linear functional f on E with $f(x) \neq 0$.

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Corollary 39

If the scalar field $\mathbb{K}$ is not spherically complete then the Banach space l_∞ / c_0 is not polar.

The weak topology

Let E be a dual-separating locally convex space. The weak topology of E is the locally convex topology on E induced by the family $\{|f| : f \in E'\}$ of seminorms. It is denoted by $\sigma(E, E')$ and it is the smallest topology on E for which all $f \in E'$ are continuous. Clearly, the locally convex space $(E, \sigma(E, E'))$ is of countable type.

The weak topology

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As in the Archimedean case one can show the following.

Theorem 40

Assume that the scalar field $\mathbb{K}$ is spherically complete. Then every closed convex subset B of a locally convex space E is weakly closed (i.e. closed in the weak topology of E). In particular, every closed subspace of a locally convex space is weakly closed.

If the scalar field $\mathbb{K}$ is not spherically complete, then the subspace c_0 of l_∞ is closed but not weakly closed and c_0 contains an absolutely convex subset which is closed but not weakly closed. Nevertheless we have the following.

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Theorem 41

Let E be a locally convex space of countable type. Then every closed subspace of E is weakly closed.

Bounded sets

A subset B of a locally convex space E is called *bounded* if any continuous seminorm on E is bounded on B.

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For bounded sets we have the following.

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For bounded sets we have the following.

Theorem 42

A subset B of a polar space E is bounded if and only if B is weakly bounded (i.e. any continuous linear functional on E is bounded on B).

NUCLEARITY AND REFLEXIVITY

Let E and F be locally convex spaces. The vector space of all continuous linear maps from E to F is denoted by $L(E, F)$. We will write $L(E)$ instead of $L(E, E)$.

A map $T \in L(E, F)$ is an *isomorphism* if T is injective, surjective and the inverse map T^{-1} is continuous.

E is isomorphic to F if there exists an isomorphism $T : E \rightarrow F$.

Compactoid sets

Let E be a locally convex space.

Compactoid sets

Let E be a locally convex space.
Recall that a subset X of E is *precompact* if for every zero neighbourhood U in E there is a finite subset Y of E such that $X \subset U + Y$. It is easy to see that any non-empty precompact convex subset of E has only one element, if the scalar field $\mathbb{K}$ is not locally compact. To overcome this difficulty we "convexify" the notion of precompactness:

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A subset B of E is called *compactoid* (or a *compactoid*) if for every zero neighbourhood U in E there is a finite subset Y of E such that $X \subset U + \operatorname{aco} Y$.

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A subset B of E is called *compactoid* (or a *compactoid*) if for every zero neighbourhood U in E there is a finite subset Y of E such that $X \subset U + \operatorname{aco} Y$.
If the scalar field $\mathbb{K}$ is locally compact then a subset B of E is compactoid if and only if B is precompact.

Fréchet-Montel spaces

Clearly, any compactoid set in a locally convex space is bounded. A Fréchet space is called a *Fréchet-Montel space* if every bounded subset of it is compactoid.

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Theorem 43

Let E be a Fréchet space. E is a Fréchet-Montel space if and only if E has no subspace isomorphic to c_0.

Fréchet-Montel spaces

Clearly, any compactoid set in a locally convex space is bounded. A Fréchet space is called a *Fréchet-Montel space* if every bounded subset of it is compactoid.

Theorem 43

Let E be a Fréchet space. E is a Fréchet-Montel space if and only if E has no subspace isomorphic to c_0.

Theorem 44

Any Fréchet-Montel space is of countable type.

For compactoid sets we have also the following ([3])

Theorem 45

Let E be a Fréchet space. The initial topology of E and any weaker locally convex topology on E coincide on compactoid subsets of E.

For compactoid sets we have also the following ([3])

Theorem 45

Let E be a Fréchet space. The initial topology of E and any weaker locally convex topology on E coincide on compactoid subsets of E.

Theorem 46

Let E be a polar space. The initial topology of E and the weak topology of E coincide on compactoid subsets of E. Any closed compactoid subset of E is weakly closed.

Compactoid maps

Let E and F be locally convex spaces.

A linear map $T : E \longrightarrow F$ is called *compactoid* if the range $T(U)$ of some zero neighbourhood U in E is compactoid in F.

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A linear map $T : E \to F$ is called *compactoid* if the range $T(U)$ of some zero neighbourhood U in E is compactoid in F.

We denote by $C(E, F)$ the space of all compactoid linear maps from E to F.

Compactoid maps

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A linear map $T : E \to F$ is called *compactoid* if the range $T(U)$ of some zero neighbourhood U in E is compactoid in F.

We denote by $C(E, F)$ the space of all compactoid linear maps from E to F.

Clearly $C(E, F) \subset L(E, F)$.

Nuclear spaces

A locally convex space E is *nuclear* if every continuous linear map from E to any normed space F is compactoid.

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We have the following ([3], Theorem 8.5.1).

Theorem 47

Let E be a locally convex space. Then the following conditions are equivalent.
(1) E is nuclear;
(2) For every continuous seminorm p on E there is a continuous seminorm q on E such that the canonical map $\varphi_{p,q} : E_q \longrightarrow E_p$ is compactoid;
(3) E is of countable type and every continuous linear map from E to c_0 is compactoid.

Nuclear spaces have the following properties ([3], Theorem 8.5.7).

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Theorem 48

(1) Subspaces and quotients of nuclear spaces are nuclear;
(2) The completion of a nuclear space is nuclear;
(3) Products, projective limits and inductive limits of nuclear spaces are nuclear.

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Theorem 48

(1) Subspaces and quotients of nuclear spaces are nuclear;
(2) The completion of a nuclear space is nuclear;
(3) Products, projective limits and inductive limits of nuclear spaces are nuclear.

Theorem 49

In a nuclear space any bounded subset is compactoid.

Nuclear Fréchet spaces

We have the following characterization of nuclear Fréchet spaces ([16], Theorem 2).

Nuclear Fréchet spaces

We have the following characterization of nuclear Fréchet spaces ([16], Theorem 2).

Theorem 50

A Fréchet space E is nuclear if and only if E is of countable type and has no quotient isomorphic to c_0.

The strong dual

Let E be a locally convex space. The *strong topology* on the dual E' of E is the locally convex topology on E' induced by the seminorms

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$$p_B : E' \to [0, \infty), p_B(f) = \sup_{x \in B} |f(x)|,$$

where B runs through the family of all bounded subsets of E.

The strong dual

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where B runs through the family of all bounded subsets of E.

This topology is denoted by $b(E, E')$. The vector space E' with the strong topology $b(E, E')$ is called the *strong dual* of E.

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where B runs through the family of all bounded subsets of E.

This topology is denoted by $b(E, E')$. The vector space E' with the strong topology $b(E, E')$ is called the *strong dual* of E. It is a polar locally convex space and it is denoted by E'_b.

Reflexive spaces

We define the natural map

$$j_E : E \longrightarrow (E'_b)'_b, x \longrightarrow j_E(x),$$

where

$$j_E(x) : E'_b \longrightarrow \mathbb{K}, f \longrightarrow f(x).$$

Reflexive spaces

We define the natural map

$$j_E : E \longrightarrow (E'_b)'_b, x \longrightarrow j_E(x),$$

where

$$j_E(x) : E'_b \longrightarrow \mathbb{K}, f \longrightarrow f(x).$$

A locally convex space E is called *reflexive* if the linear map j_E is an isomorphism.

We have the following characterization of reflexive Fréchet spaces ([3], Theorem 7.4.19, Corollaries 7.4.20, 7.4.30, [8], Proposition 20.7).

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Theorem 51

If the scalar field $\mathbb{K}$ is spherically complete, then the following conditions are equivalent for any Fréchet space E.
(1) E is reflexive;
(2) E is a Fréchet-Montel space
(i.e. every bounded subset of E is compactoid);
(3) No subspace of E is isomorphic to c_0;
(4) The strong dual E'_b of E is nuclear.

Theorem 52

If the scalar field $\mathbb{K}$ is spherically complete, then every reflexive Fréchet space is of countable type and every reflexive Banach space is finite-dimensional.

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Theorem 53

If the scalar field $\mathbb{K}$ is not spherically complete, then every Fréchet space of countable type is reflexive.

Theorem 52

If the scalar field $\mathbb{K}$ is spherically complete, then every reflexive Fréchet space is of countable type and every reflexive Banach space is finite-dimensional.

Theorem 53

If the scalar field $\mathbb{K}$ is not spherically complete, then every Fréchet space of countable type is reflexive.

Corollary 54

Every Fréchet-Montel space is reflexive.
In particular, every nuclear Fréchet space is reflexive.

THE STRUCTURE OF FRÉCHET SPACES OF COUNTABLE TYPE

Let E be a Fréchet space.

Schauder basis

A sequence (x_n) in E is a *Schauder basis* of E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} a_n x_n$ with $(a_n) \subset \mathbb{K}$ and the coefficient functionals $x_n^* : E \to \mathbb{K}, x \to a_n,\ n \in \mathbb{N}$, are continuous.

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Schauder basic sequence

A sequence (x_n) in E is a *Schauder basic sequence* in E if it is a Schauder basis of its closed linear span in E.

Schauder decomposition

A sequence (P_n) of continuous linear non-zero projections on E is a *Schauder decomposition* of E if $x = \sum_{n=1}^{\infty} P_n x$ for all $x \in E$ and $P_n \circ P_m = 0$ for all $n, m \in \mathbb{N}, n \neq m$.

A Schauder decomposition (P_n) of E is *finite-dimensional* if $\dim P_n(E) < \infty$ for all $n \in \mathbb{N}$, and *strong finite-dimensional* if $\sup_n \dim P_n(E) < \infty$.

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A Schauder decomposition (P_n) of E is *finite-dimensional* if $\dim P_n(E) < \infty$ for all $n \in \mathbb{N}$, and *strong finite-dimensional* if $\sup_n \dim P_n(E) < \infty$.

Clearly, any Fréchet space with a Schauder basis has a strong finite-dimensional Schauder decomposition.

Bounded approximation property

E has the *bounded approximation property* if there exists a sequence $(P_n) \subset L(E)$ with $\dim P_n(E) < \infty, n \in \mathbb{N}$ such that $\lim_n P_n x = x$ for all $x \in E$.

Bounded approximation property

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Of course any Fréchet space with a finite-dimensional Schauder decomposition has the bounded approximation property.

We know that any infinite-dimensional Banach space of countable type has a Schauder basis.

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For Fréchet spaces it is not true ([11], Theorem 3).

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Theorem 55

There exist infinite-dimensional Fréchet spaces of countable type without a Schauder basis.

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For Fréchet spaces it is not true ([11], Theorem 3).

Theorem 55

There exist infinite-dimensional Fréchet spaces of countable type without a Schauder basis.

In fact we have the following ([11], Theorems 3, 7 and 11).

Theorem 56

There exist nuclear Fréchet spaces with a strong finite-dimensional Schauder decomposition but without a Schauder basis.

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Theorem 57

There exist nuclear Fréchet spaces with a finite-dimensional Schauder decomposition but without a strong finite-dimensional Schauder decomposition.

Theorem 58

There exist nuclear Fréchet spaces with a Schauder decomposition but without a finite-dimensional Schauder decomposition (even without the bounded approximation property).

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There exist nuclear Fréchet spaces with a Schauder decomposition but without a finite-dimensional Schauder decomposition (even without the bounded approximation property).

Open problems

- Does every Fréchet space of countable type have a Schauder decomposition?
- Does every Fréchet space with the bounded approximation property have a finite-dimensional Schauder decomposition?

We know that any infinite-dimensional Banach space has a Schauder basic sequence.

We know that any infinite-dimensional Banach space has a Schauder basic sequence.

We shall prove a similar result for Fréchet spaces.

Orthogonal sequence

A sequence (x_n) in a vector space E is called *orthogonal with respect to a seminorm p on E* if

$$p\Big(\sum_{i=1}^{n} a_i x_i\Big) = \max_{1\le i\le n} p(a_i x_i)$$

for all $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{K}$.

A sequence (x_n) in a locally convex space E is called *orthogonal* if the family of all $p \in \mathcal{P}(E)$ such that (x_n) is orthogonal with respect to p is a base of $\mathcal{P}(E)$.

Orthogonal basic sequence

Let E be a locally convex space.

An orthogonal sequence (x_n) in E is called *orthogonal basic sequence* if $x_n \neq 0, n \in \mathbb{N}$.

An orthogonal basic sequence (x_n) in E is called *orthogonal basis* if it is linearly dense (i.e. the linear span of the set $\{x_n : n \in \mathbb{N}\}$ is dense in E).

It is known the following ([2], Propositions 1.4 and 1.7).

Theorem 59

Every orthogonal basic sequence in a locally convex space is a Schauder basic sequence and every Schauder basic sequence in a Fréchet space is an orthogonal basic sequence.

Let $t \in (0, 1]$ and let p be a seminorm on a vector space E.

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An element $x \in E$ is *t-orthogonal to a subspace M of E with respect to p* if

$$p(ax + y) \geq t \max\{p(ax), p(y)\}$$

for all $a \in \mathbb{K}, y \in M$.

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A sequence $(x_n) \subset E$ is *t-orthogonal with respect to p* if

$$p(\sum_{i=1}^{n} a_i x_i) \geq t \max_{1 \leq i \leq n} p(a_i x_i)$$

for all $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{K}$.

It is known the following ([2], Proposition 2.6).

Theorem 60

A sequence (x_n) in a locally convex space E is orthogonal if and only if the family of all $p \in \mathcal{P}(E)$ such that (x_n) is t_p-orthogonal with respect to p for some $t_p \in (0, 1]$ is a base of $\mathcal{P}(E)$.

We know that for every closed subspace of a Banach space E of countable type and for every $t > 1$ there is a continuous linear projection P from E onto D with $\|P\| < t$ (Theorem 14). Using this result one can show the following lemma ([9], Lemma 1).

We know that for every closed subspace of a Banach space E of countable type and for every $t > 1$ there is a continuous linear projection P from E onto D with $\|P\| < t$ (Theorem 14). Using this result one can show the following lemma ([9], Lemma 1).

Lemma 61

Let M be a finite-dimensional subspace of vector space F with $\dim F = \aleph_0$ and let $q_1, \ldots, q_n$ be norms on F. Then for every $t \in (0, 1)$ there exists a non-zero element x of F which is t-orthogonal to M with respect to q_i for $1 \leq i \leq n$.

Applying the last lemma we can prove the existence of orthogonal basic sequences in metrizable locally convex spaces.

Theorem 62

Any infinite-dimensional metrizable locally convex space E has an orthogonal basic sequence.

Sketch of proof of Theorem 62

By metrizability of E, there exists a non-decreasing sequence $(p_k) \subset \mathcal{P}(E)$ which forms a base of $\mathcal{P}(E)$.

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By metrizability of E, there exists a non-decreasing sequence $(p_k) \subset \mathcal{P}(E)$ which forms a base of $\mathcal{P}(E)$.

If $\dim(E/\ker p_n) < \infty$ for all $n \in \mathbb{N}$, then E is isomorphic to a dense subspace of the Fréchet space $\mathbb{K}^{\mathbb{N}}$, so has an orthogonal basis ([2], Theorem 3.5).

Sketch of proof of Theorem 62

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Suppose that there exists $k \in \mathbb{N}$ with $\dim(E/\ker p_k) = \infty$.

Sketch of proof of Theorem 62

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If $\dim(E/\ker p_n) < \infty$ for all $n \in \mathbb{N}$, then E is isomorphic to a dense subspace of the Fréchet space $\mathbb{K}^{\mathbb{N}}$, so has an orthogonal basis ([2], Theorem 3.5).

Suppose that there exists $k \in \mathbb{N}$ with $\dim(E/\ker p_k) = \infty$. We can assume that $k = 1$. Let $\{x_n + \ker p_1 : n \in \mathbb{N}\}$ be a linearly independent set in $E/\ker p_1$ and put $F = \mathrm{lin}\{x_n : n \in \mathbb{N}\}$. Clearly $\dim F = \aleph_0$ and $q_n = p_n|F$ is a norm on F for each $n \in \mathbb{N}$.

Sketch of proof of Theorem 62

Let $(s_n) \subset (0,1)$ be a sequence with $s := \prod_{n=1}^{\infty} s_n > 0$.

Sketch of proof of Theorem 62

Let $(s_n) \subset (0,1)$ be a sequence with $s := \prod_{n=1}^{\infty} s_n > 0$.
By the previous lemma we can construct inductively a sequence (y_n) of non-zero elements of F such that for every $n \in \mathbb{N}$ y_{n+1} is s_{n+1}-orthogonal to $\mathrm{lin}\{y_1, \ldots, y_n\}$ with respect to q_i for $1 \leq i \leq n$.

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One can prove that for every $m \in \mathbb{N}$ the sequence (y_n) is t_m-orthogonal with respect to q_m for some $t_m \in (0,1]$.
Then (y_n) is t_m-orthogonal with respect to p_m for any $m \in \mathbb{N}$.

Sketch of proof of Theorem 62

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One can prove that for every $m \in \mathbb{N}$ the sequence (y_n) is t_m-orthogonal with respect to q_m for some $t_m \in (0,1]$.
Then (y_n) is t_m-orthogonal with respect to p_m for any $m \in \mathbb{N}$.

Thus (y_n) is an orthogonal basic sequence in E. $\square$

Corollary 63

Any infinite-dimensional Fréchet space has a Schauder basic sequence.

In fact one can show a stronger result ([12], Theorem 2.7).

Theorem 64

Any non-normable Fréchet space F has an infinite-dimensional nuclear closed subspace G with a Schauder basis.
If F is not isomorphic to the product of a Banach space X and the Fréchet space $\mathbb{K}^{\mathbb{N}}$ we can additionally claim that G has a continuous norm.

We know that every closed subspace of an infinite-dimensional Banach space of countable type is complemented.

We know that every closed subspace of an infinite-dimensional Banach space of countable type is complemented.

For Fréchet spaces it is not true. We have the following ([11], Theorem 7 and Proposition 9, [13], Proposition 3 and Theorem 10).

Theorem 65

The following conditions are equivalent for any infinite-dimensional Fréchet space of countable type.
(a) *Any closed subspace of F is complemented.*
(b) *Any closed subspace of F with a continuous norm is normable.*
(c) *Any infinite-dimensional closed subspace of F has a Schauder basis.*
(d) *Any infinite-dimensional closed subspace of F has a strong finite-dimensional Schauder decomposition.*
(e) *F is isomorphic to one of the following spaces:*
$c_0, \mathbb{K}^{\mathbb{N}}, c_0 \times \mathbb{K}^{\mathbb{N}}$.

Open problems

- Does every infinite-dimensional Fréchet space of countable type have a complemented subspace with a Schauder basis?
- Does every infinite-dimensional Fréchet space E of countable type have a complemented subspace D such that D and E/D are infinite-dimensional?
- Does every infinite-dimensional complemented subspace of a Fréchet space with a Schauder basis have a Schauder basis?

Moreover we have the following result ([12], Theorems 1.5 and 2.2, Corollaries 1.11 and 1.12).

Theorem 66

A Fréchet space F of countable type is nuclear (respectively: reflexive, normable, a Fréchet-Montel space) if and only if each of its closed subspaces with a Schauder basis is nuclear (respectively: reflexive, normable, a Fréchet-Montel space).

Köthe spaces

A Fréchet space with a Schauder basis and with a continuous norm is called a *Köthe space.*

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One can show the following ([19], Proposition 1; [3], Theorem 9.3.7).

Theorem 67

Any Fréchet space E with a Schauder basis is isomorphic to either $\mathbb{K}^{\mathbb{N}}, \mathbb{K}^{\mathbb{N}} \times F$ or F, where F is the product of a countable family of Köthe spaces.

By a *Köthe matrix* we mean an infinite matrix $B = (b_{k,n})$ of positive real numbers such that $\forall k, n \in \mathbb{N} : b_{k,n} \leq b_{k+1,n}$.

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The *Köthe space associated with the Köthe matrix* B is the Köthe space

$$K(B) = \{(x_n) \in \mathbb{K}^{\mathbb{N}} : b_{k,n}|x_n| \to_n 0 \text{ for any } k \in \mathbb{N}\}$$

with the base (p_k) of continuous norms:
$p_k((x_n)) = k \max_n b_{k,n}|x_n|, k \in \mathbb{N}$.

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with the base (p_k) of continuous norms:
$p_k((x_n)) = k \max_n b_{k,n}|x_n|, k \in \mathbb{N}$.
The sequence (e_n) of coordinate vectors is a Schauder basis of $K(B)$. Thus for any $x = (x_n) \in K(B)$ we have $x = \sum_{n=1}^{\infty} x_n e_n$.

It is known the following ([1], Propositions 2.4 and 3.5).

Theorem 68

Any Köthe space is isomorphic to a space $K(B)$ for some Köthe matrix B.

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Theorem 68

Any Köthe space is isomorphic to a space $K(B)$ for some Köthe matrix B.

Theorem 69

The Köthe space $K(B)$ is nuclear if and only if

$$\forall k \in \mathbb{N} \, \exists m \in \mathbb{N} \ : \lim_n (b_{k,n}/b_{m,n}) = 0.$$

.

Let $a = (a_n)$ be a non-decreasing unbounded sequence of positive real numbers.

Then the following Köthe spaces are nuclear ([1], Corollary 3.6):

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(1) $A_1(a) = K(B)$ with $B = (b_{k,n})$, $b_{k,n} = (\frac{k}{k+1})^{a_n}$;

Let $a = (a_n)$ be a non-decreasing unbounded sequence of positive real numbers.

Then the following Köthe spaces are nuclear ([1], Corollary 3.6):

(1) $A_1(a) = K(B)$ with $B = (b_{k,n}), b_{k,n} = (\frac{k}{k+1})^{a_n}$;
(2) $A_\infty(a) = K(B)$ with $B = (b_{k,n}), b_{k,n} = k^{a_n}$.

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Then the following Köthe spaces are nuclear ([1], Corollary 3.6):

(1) $A_1(a) = K(B)$ with $B = (b_{k,n}), b_{k,n} = (\frac{k}{k+1})^{a_n}$;
(2) $A_\infty(a) = K(B)$ with $B = (b_{k,n}), b_{k,n} = k^{a_n}$.

$A_1(a)$ and $A_\infty(a)$ are called the *power series spaces* (*of finite* and *infinite type*, respectively).

A Köthe matrix $B = (b_{k,n})$ is called *stable* if

$$\forall k \in \mathbb{N}\ \exists l \in \mathbb{N} : \sup_n \left(\frac{b_{k,n+1}}{b_{l,n}} + \frac{b_{k,n}}{b_{l,n+1}} \right) < \infty.$$

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For example, for an increasing unbounded sequence $(a_n) \subset (0, +\infty)$ the Köthe matrixes $A = (a_{k,n}) = (k^{a_n})$ and $C = (c_{k,n}) = ((\frac{k}{k+1})^{a_n})$ are stable, provided $\sup_n(a_{n+1}/a_n) < \infty$.

A Fréchet space is called *twisted* if it is not isomorphic to the product of a countable family of Fréchet spaces with continuous norm.

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By Theorem 67 no Fréchet space with a Schauder basis is twisted.

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By Theorem 67 no Fréchet space with a Schauder basis is twisted.

We have the following ([19], Theorem 7 and Proposition 8).

Theorem 70

There exist twisted nuclear Fréchet spaces.

Sketch of proof of Theorem 70

Let X and Y be nuclear Fréchet spaces with continuous norms such that there exists a continuous map Q from X onto Y which kernel $\ker Q$ is not complemented in X.

Sketch of proof of Theorem 70

Let X and Y be nuclear Fréchet spaces with continuous norms such that there exists a continuous map Q from X onto Y which kernel $\ker Q$ is not complemented in X.
For example, we can take $X = A_\infty(a)$ and $Y = A_1(b)$ provided $a = (a_n)$ and $b = (b_n)$ are increasing unbounded sequences of positive numbers with $\lim_n(a_n/b_n) = 0$ and $\sup_n(a_{2n}/a_n) < \infty$.

Sketch of proof of Theorem 70

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For example, we can take $X = A_\infty(a)$ and $Y = A_1(b)$ provided $a = (a_n)$ and $b = (b_n)$ are increasing unbounded sequences of positive numbers with $\lim_n (a_n/b_n) = 0$ and $\sup_n (a_{2n}/a_n) < \infty$.
Let (r_k) and (p_k) be non-decreasing bases of continuous norms on X and Y, respectively.

Sketch of proof of Theorem 70

Let $B = (b_{k,n})$ be a stable Köthe matrix such that the Köthe space $K(B)$ is nuclear.

Sketch of proof of Theorem 70

Let $B = (b_{k,n})$ be a stable Köthe matrix such that the Köthe space $K(B)$ is nuclear.
The space

$$K(B, Y) = \{(y_n) \subset Y : \lim_n b_{k,n} p_k(y_n) = 0 \text{ for every } k \in \mathbb{N}\}$$

with the base of norms $q_k((y_n)) = \max_n b_{k,n} p_k(y_n), k \in \mathbb{N}$, is a nuclear Fréchet space.

Sketch of proof of Theorem 70

The linear space

$$Z \doteq \{(x_n) \in X^{\mathbb{N}} : (Qx_n) \in K(B, Y)\}$$

with the locally convex topology induced by the seminorms

$$s_k : Z \to [0, \infty), s_k((x_n)) = \max\{\max_{1 \le n \le k} r_k(x_n), q_k((Qx_n))\},$$

$k \in \mathbb{N}$ is a Fréchet space.

Sketch of proof of Theorem 70

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$k \in \mathbb{N}$ is a Fréchet space.

Using the theory of projective and inductive limits one can show that Z is a twisted nuclear Fréchet space. □

We know that a Fréchet space E has a subspace isomorphic (1) to $\mathbb{K}^{\mathbb{N}}$ if an only if E has no continuous norm and (2) to c_0 if an only if E is not a Fréchet-Montel space ([3]).

We know that a Fréchet space E has a subspace isomorphic (1) to $\mathbb{K}^{\mathbb{N}}$ if an only if E has no continuous norm and (2) to c_0 if an only if E is not a Fréchet-Montel space ([3]).
For quotients we have the following ([16], Theorem 2, [20], Theorems 3.1 and 3.4, Corollary 3.6).

Theorem 71

An infinite-dimensional Fréchet space E of countable type has a quotient isomorphic
(1) to $\mathbb{K}^{\mathbb{N}}$ if and only if E is not normable.
(2) to c_0 if and only if E is not nuclear.
(3) to $c_0 \times \mathbb{K}^{\mathbb{N}}$ if and only if E is non-normable and non-nuclear.
(4) to $c_0^{\mathbb{N}}$ if and only if E is non-normable, non-nuclear and non-isomorphic to $c_0 \times \mathbb{K}^{\mathbb{N}}$.

For quotients we have also the following ([20], Theorem 3.7).

Theorem 72

For any infinite-dimensional Fréchet space E of countable type the following conditions are equivalent.
(1) Any quotient of E with a continuous norm is normable.
(2) For any continuous seminorm p on E the quotient space $E/\ker p$ (with the quotient topology) is normable.
(3) E is isomorphic to the product of a countable family of Banach spaces i.e. to one of the following spaces
$c_0, c_0 \times \mathbb{K}^{\mathbb{N}}, \mathbb{K}^{\mathbb{N}}, c_0^{\mathbb{N}}$.

Corollary 73

Any quotient of $c_0^{\mathbb{N}}$ is isomorphic to the product of a countable family of Banach spaces. In particular, any complemented closed subspace of $c_0^{\mathbb{N}}$ is isomorphic to the product of a countable family of Banach spaces.

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Any quotient of $c_0^{\mathbb{N}}$ is isomorphic to the product of a countable family of Banach spaces. In particular, any complemented closed subspace of $c_0^{\mathbb{N}}$ is isomorphic to the product of a countable family of Banach spaces.

The last corollary is surprising since any Fréchet space of countable type is isomorphic to a closed subspace of $c_0^{\mathbb{N}}$.

Nuclear Köthe spaces are very important Fréchet spaces.

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We know that a non-normable Fréchet space E contains a nuclear Köthe subspace if and only if E is not isomorphic to the product of a Banach space X and the Fréchet space $\mathbb{K}^{\mathbb{N}}$.

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We know that a non-normable Fréchet space E contains a nuclear Köthe subspace if and only if E is not isomorphic to the product of a Banach space X and the Fréchet space $\mathbb{K}^{\mathbb{N}}$.

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We know that a non-normable Fréchet space E contains a nuclear Köthe subspace if and only if E is not isomorphic to the product of a Banach space X and the Fréchet space $\mathbb{K}^{\mathbb{N}}$.

In particular, every non-normable Köthe space has a nuclear Köthe subspace.
For quotients we have the following ([16], [23], [21]).

Theorem 74

Any non-normable Köthe space has a nuclear Köthe quotient.

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Any non-normable Köthe space has a nuclear Köthe quotient.

Theorem 75

An infinite-dimensional Fréchet-Montel space E has a nuclear Köthe quotient if and only if it is not isomorphic to $\mathbb{K}^{\mathbb{N}}$.

Theorem 76

Assume that the scalar field $\mathbb{K}$ is spherically complete. Then there exist non-normable Fréchet spaces of countable type with a continuous norm that have no nuclear Köthe quotient.

Sketch of proof of Theorem 76

Let X be an infinite-dimensional Banach space and let W be a subspace of the dual space X' of X.
We say that W is total if it is dense in $(X', \sigma(X', X))$.

Sketch of proof of Theorem 76

Let X be an infinite-dimensional Banach space and let W be a subspace of the dual space X' of X.
We say that W is total if it is dense in $(X', \sigma(X', X))$.
By W^1 we denote the set of all elements $x' \in X'$ such that there exists a bounded net (x'_α) in W which converges to x' in $(X', \sigma(X', X))$. Clearly W^1 is a subspace in X'.

Sketch of proof of Theorem 76

Let X be an infinite-dimensional Banach space and let W be a subspace of the dual space X' of X.
We say that W is total if it is dense in $(X', \sigma(X', X))$.
By W^1 we denote the set of all elements $x' \in X'$ such that there exists a bounded net (x'_α) in W which converges to x' in $(X', \sigma(X', X))$. Clearly W^1 is a subspace in X'.
We put $W^0 = W$ and $W^n = (W^{n-1})^1$ for $n \in \mathbb{N}$. We say that W is strongly non-norming if $W^n \subsetneq X'$ for all $n \in \mathbb{N}$.

Sketch of proof of Theorem 76

We prove that the dual l_∞ of c_0 has a total strongly non-norming subspace M.

Using this space M we construct a non-normable Fréchet space F of countable type with a continuous norm, such that the strong dual F'_b of F is a strict LB-space.

Next we show that F has no infinite-dimensional Fréchet-Montel quotient space with a continuous norm. In particular, F has no nuclear Köthe quotient. □

The following problem is still open.

Open problem

Assume that the scalar field $\mathbb{K}$ is not spherically complete. Does every non-normable Fréchet spaces of countable type which is not isomorphic to $\mathbb{K}^{\mathbb{N}}$ have a nuclear Köthe quotient?

The following problem is still open.

Open problem

Assume that the scalar field $\mathbb{K}$ is not spherically complete. Does every non-normable Fréchet spaces of countable type which is not isomorphic to $\mathbb{K}^{\mathbb{N}}$ have a nuclear Köthe quotient?

We have the following

Theorem 77

Assume that the scalar field $\mathbb{K}$ is not spherically complete. Let E be a non-normable Fréchet space of countable type. If E has no nuclear Köthe quotient then E is isomorphic to the strong dual of a strict LB-space.

We know that any non-normable Köthe space has an infinite-dimensional closed subspace without a strong finite-dimensional Schauder decomposition (in particular, without a Schauder basis).

For quotients we have a stronger result ([15], Theorem 3).

Theorem 78

Any non-normable Köthe space has a quotient without the bounded approximation property.

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Hence we get the following ([15], Corollary 10).

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Hence we get the following ([15], Corollary 10).

Corollary 79

A Fréchet space of countable type E has an infinite-dimensional quotient without a Schauder basis if and only if E is not isomorphic to the product of a countable family of Banach spaces.

We know that every Fréchet space of countable type is isomorphic to a closed subspace of the Fréchet space $c_0^{\mathbb{N}}$ with a Schauder basis.

We know that every Fréchet space of countable type is isomorphic to a closed subspace of the Fréchet space $c_0^{\mathbb{N}}$ with a Schauder basis.
We can ask:
Does there exist a Fréchet space X with a Schauder basis such that every Fréchet space of countable type is isomorphic to a quotient of X?

We know that every Fréchet space of countable type is isomorphic to a closed subspace of the Fréchet space $c_0^{\mathbb{N}}$ with a Schauder basis.

We can ask:

Does there exist a Fréchet space X with a Schauder basis such that every Fréchet space of countable type is isomorphic to a quotient of X?

In other words:

Does there exist a Fréchet space X with a Schauder basis such that every Fréchet space of countable type is a continuous linear range of X?

The answer is positive. We have the following ([18], Theorem 5 and Corollary 6).

Theorem 80

There exists a Köthe-Montel space W such that any Fréchet space of countable type is a continuous linear range of W (i.e. it is isomorphic to a quotient of W).

For nuclear Fréchet spaces we have the following ([18], Theorem 7 and Corollary 12).

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Theorem 81

Any nuclear Fréchet space is a continuous linear range of a nuclear Köthe space.

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Theorem 81

Any nuclear Fréchet space is a continuous linear range of a nuclear Köthe space.

Theorem 82

For any nuclear Köthe space X there is a nuclear Fréchet space which is not a continuous linear range of X.

FREDHOLM OPERATORS BETWEEN FRÉCHET SPACES

We shall prove that the index of a Fredholm operator between Fréchet spaces is preserved under compactoid perturbations.

Let X and Y be vector spaces. We denote by $\mathcal{L}(X, Y)$ the vector space of all linear operators from X to Y.

We say that $T \in \mathcal{L}(X, Y)$ has an index if the spaces $\ker T$ and $Y/T(X)$ are finite-dimensional.

In this case the index of T is defined as

$$\chi(T) = \dim \ker T - \dim(Y/T(X)).$$

If $T \in \mathcal{L}(X, Y)$ has an index and $F \in \mathcal{L}(X, Y)$ is finite-dimensional (i.e. it has a finite-dimensional range), then $T + F$ has an index and $\chi(T + F) = \chi(T)$.

Let X, Y and Z be vector spaces. If two of the three operators $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{L}(Y, Z)$ and $ST \in \mathcal{L}(X, Z)$ have indexes, then the third one also has an index and
$\chi(ST) = \chi(T) + \chi(S)$.

Fredholm operators

Let X and Y be locally convex spaces. We denote by $F(X, Y)$ the space of all $T \in L(X, Y)$ with $\dim T(X) < \infty$.

A map $T \in L(X, Y)$ is a *Fredholm operator* if it has an index and $T(X)$ is a closed subspace of Y. The family of all Fredholm operators from X to Y is denoted by $\Phi(X, Y)$.

Using the open mapping theorem we get the following result: *Let X and Y be Fréchet spaces. If a continuous linear map from X to Y has an index then it is a Fredholm operator.*

We need the following two lemmas ([14], Lemmas 1 and 2).

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Lemma 83

Let X and Y be locally convex spaces. Let K be a compactoid linear map from X to Y and let D be a finite dimensional subspace of X. Then there exists an $F \in F(X, Y)$ such that $F(x) = K(x)$ for any $x \in D$.

We need the following two lemmas ([14], Lemmas 1 and 2).

Lemma 83

Let X and Y be locally convex spaces. Let K be a compactoid linear map from X to Y and let D be a finite dimensional subspace of X. Then there exists an $F \in F(X, Y)$ such that $F(x) = K(x)$ for any $x \in D$.

Lemma 84

Let X and Y be Fréchet spaces. Assume that $K \in C(X, Y)$ and $S \in L(Y, X)$. Then there exists a an $F \in F(X, Y)$ such that the linear map $(I_X + S(K - F)) : X \to X$ is an isomorphism.

Now, we shall prove our main result ([14], Theorem 4.

Theorem 85

Let X and Y be Fréchet spaces. If $T \in \Phi(X, Y)$ and $K \in C(X, Y)$, then $T + K \in \Phi(X, Y)$ and $\chi(T + K) = \chi(T)$.

Proof of Theorem 85

Denote by $\hat{X}$ the quotient space $X/\ker T$ and by Q the quotient map from X onto $\hat{X}$. Let $\hat{T} : \hat{X} \to Y$ with $\hat{T}(Qx) = Tx, x \in X$.

Proof of Theorem 85

Denote by $\hat{X}$ the quotient space $X/\ker T$ and by Q the quotient map from X onto $\hat{X}$. Let $\hat{T} : \hat{X} \to Y$ with $\hat{T}(Qx) = Tx, x \in X$.

Clearly, $Q \in \Phi(X, \hat{X})$ and $\hat{T} \in \Phi(\hat{X}, Y)$.

Proof of Theorem 85

Denote by $\hat{X}$ the quotient space $X/\ker T$ and by Q the quotient map from X onto $\hat{X}$. Let $\hat{T} : \hat{X} \to Y$ with $\hat{T}(Qx) = Tx, x \in X$.

Clearly, $Q \in \Phi(X, \hat{X})$ and $\hat{T} \in \Phi(\hat{X}, Y)$.

It is easy to see that there exists $S \in L(Y, \hat{X})$ with $S\hat{T} = I_{\hat{X}}$. Of course $K(X)$ is of countable type.

Proof of Theorem 85

By the first lemma there is an $F \in F(X, Y)$ such that $\ker T \subset \ker(K - F)$. Let $G : \hat{X} \to Y$ with $G(Qx) = (K - F)(x), x \in X$; clearly, $G \in C(\hat{X}, Y)$.

Proof of Theorem 85

By the first lemma there is an $F \in F(X, Y)$ such that $\ker T \subset \ker(K - F)$. Let $G : \hat{X} \to Y$ with $G(Qx) = (K - F)(x), x \in X$; clearly, $G \in C(\hat{X}, Y)$.

By the second lemma there exists an $H \in F(\hat{X}, Y)$ such that the operator $(I_{\hat{X}} + S(G - H)) : \hat{X} \to \hat{X}$ is an isomorphism.

Proof of Theorem 85

Since $S\hat{T} = I_{\hat{X}}, I_{\hat{X}} + S(G - H) = S(\hat{T} + G - H)$ and $\hat{T} \in \Phi(\hat{X}, Y)$, then $S \in \Phi(Y, \hat{X})$, $(\hat{T} + G - H) \in \Phi(\hat{X}, Y)$, $\chi(S) = -\chi(\hat{T})$ and $\chi(\hat{T} + G - H) = \chi(\hat{T})$.

Proof of Theorem 85

Since $S\hat{T} = I_{\hat{X}}$, $I_{\hat{X}} + S(G - H) = S(\hat{T} + G - H)$ and $\hat{T} \in \Phi(\hat{X}, Y)$, then $S \in \Phi(Y, \hat{X})$, $(\hat{T} + G - H) \in \Phi(\hat{X}, Y)$, $\chi(S) = -\chi(\hat{T})$ and $\chi(\hat{T} + G - H) = \chi(\hat{T})$.

Hence $(\hat{T} + G) \in \Phi(\hat{X}, Y)$ and $\chi(\hat{T} + G) = \chi(\hat{T} + G - H) = \chi(\hat{T})$.

Proof of Theorem 85

Since $S\hat{T} = I_{\hat{X}}$, $I_{\hat{X}} + S(G - H) = S(\hat{T} + G - H)$ and $\hat{T} \in \Phi(\hat{X}, Y)$, then $S \in \Phi(Y, \hat{X})$, $(\hat{T} + G - H) \in \Phi(\hat{X}, Y)$, $\chi(S) = -\chi(\hat{T})$ and $\chi(\hat{T} + G - H) = \chi(\hat{T})$.

Hence $(\hat{T} + G) \in \Phi(\hat{X}, Y)$ and $\chi(\hat{T} + G) = \chi(\hat{T} + G - H) = \chi(\hat{T})$.

It follows that $(T + K) - F = (\hat{T} + G)Q \in \Phi(X, Y)$ and $\chi(T + K - F) = \chi(\hat{T} + G) + \chi(Q) = \chi(\hat{T}) + \chi(Q) = \chi(\hat{T}Q) = \chi(T)$. Thus $T + K \in \Phi(X, Y)$ and $\chi(T + K) = \chi(T + K - F) = \chi(T)$. $\square$

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