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On Reeb graphs and related objects

A doctoral dissertation in mathematical sciences in the area of mathematics

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O grafach Reeba i powiązanych obiektach

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Abstract

This thesis presents results concerned with Reeb graphs of smooth functions on manifolds. One of the fundamental problems in this topic, investigated also by other authors, is to distinguish and characterize graphs which could be the Reeb graphs of functions from a given class. More difficult problem is a question on the set of possible Reeb graphs of functions on a fixed manifold. In this way from a function invariant we obtain features describing a manifold itself. We use Morse theory and combinatorial methods to provide a series of realization theorems for Reeb graphs, starting from determining possible cycle ranks of Reeb graphs and ending with the description of their homeomorphism or isomorphism types.

Two kinds of objects are naturally associated with Reeb graphs: epimorphisms onto free groups, called the Reeb epimorphisms, which are induced on fundamental groups by the quotient maps from manifolds to Reeb graphs, and systems of hypersurfaces in manifolds, corresponding to edges in Reeb graphs outside spanning trees. We present number of properties of these objects and their connections with Reeb graphs. In particular, we prove that any epimorphism from the fundamental group of manifold onto free group is induced by a system of hypersurfaces non-separating the manifold. We also show the relationship between cobordism classes of systems of hypersurfaces modulo diffeomorphisms of manifold and strong equivalence classes of epimorphisms onto free groups. The full computation of these classes is made for surfaces. The obtained results allow us to extend realization theorems to characterize not only Reeb graphs but also the Reeb epimorphisms of Morse functions. In the case of surfaces we provide complete characterization of Reeb epimorphisms of simple Morse functions and we show their utility in topological conjugacy of functions.

Streszczenie

Niniejsza rozprawa przedstawia wyniki dotyczące grafów Reeba funkcji gładkich na rozmaitości. Jednym z podstawowych problemów w tej tematyce, badanym przez różnych autorów, jest wyróżnienie i scharakteryzowanie grafów, które mogą być grafami Reeba określonych klas funkcji. Znacznie trudniejszym problemem jest pytanie o zbiór dopuszczalnych grafów Reeba funkcji na ustalonej rozmaitości. W ten sposób z niezmiennika funkcji otrzymujemy właściwości charakteryzujące rozmaitość samą w sobie. Korzystając z teorii Morse'a oraz metod kombinatorycznych podajemy szereg twierdzeń realizacyjnych dla grafów Reeba, począwszy na wyznaczeniu możliwych rang cyklicznych występujących grafów Reeba, a skończywszy na opisaniu ich typów homeomorfizmu czy izomorfizmu.

Z grafem Reeba w naturalny sposób związane są dwa rodzaje obiektów: epimorfizmy na grupę wolną, zwane epimorfizmami Reeba, które są indukowane na grupach podstawowych przez odwzorowania ilorazowe z rozmaitości do grafu Reeba, oraz systemy hiperpowierzchni w rozmaitości, odpowiadające krawędziom w grafie Reeba poza drzewami rozpinającymi. Przedstawiamy szereg własności tych obiektów i ich związków z grafami Reeba. W szczególności dowodzimy, że każdy epimorfizm z grupy podstawowej rozmaitości na grupę wolną jest indukowany przez system hiperpowierzchni, który nie rozspójnia tej rozmaitości. Pokazujemy także związek klas kobordyzmu obramowanego systemów hiperpowierzchni modulo dyfeomorfizmy rozmaitości z klasami silnej równoważności epimorfizmów na grupy wolne. Dokonujemy pełnego wyliczenia tych klas dla powierzchni. Uzyskane wyniki pozwalają rozszerzyć twierdzenia realizacyjne w celu scharakteryzowania nie tylko grafów Reeba, ale także epimorfizmów Reeba funkcji Morse'a. W przypadku powierzchni podajemy pełną charakteryzację epimorfizmów Reeba prostych funkcji Morse'a oraz ich związek z topologiczną sprzężonością funkcji.

Podziękowania

Pragnę wyrazić głęboką wdzięczność wszystkim osobom, bez których powstanie tej rozprawy nie byłoby możliwe.

Przede wszystkim chciałbym serdecznie podziękować mojemu promotorowi, Profesorowi Wacławowi Marzantowiczowi, za wszelką okazaną pomoc i ogromną życzliwość przez wszystkie lata wspólnej pracy, za nieocenione wskazówki w trakcie prowadzenia badań, a także za nieskończoną cierpliwość w stosunku do mnie, szczególnie podczas opracowywania tej rozprawy.

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Contents

Introduction

Quoting from V. Arnold's paper [1]:

The topological structures of the generic smooth functions on a smooth manifold belong to the small quantity of the most fundamental objects of study both in pure and applied mathematics.

The Reeb graph $\mathcal{R}(f)$ of a smooth function $f\colon M \to \mathbb{R}$ on a closed manifold M is an invariant which encodes topological structure of function. It is obtained by contracting the connected components of level sets of f. For functions with finitely many critical points, or even with finitely many critical values, it is indeed a finite topological graph ([58]). The notion of Reeb graph was defined by G. Reeb [56] in 1946 in the context of Morse theory. A similar concept was also studied by A. Kronrod [34], thus it is sometimes called Kronrod–Reeb graph (see [32, 33, 61]). In 1991 the Reeb graph was introduced in computer graphics by Y. Shinagawa et al. [62] and since then it has been proposed to solve different problems related to shape matching, morphing and coding. It has been an object of intensive studies of applied topology for the last twenty years, also in topological data analysis (see [5, 8, 11] and references therein). Now, it plays a fundamental role in computational topology for shape analysis [5].

Reeb graphs have a lot of applications in pure mathematics which include, among others: description of surface embeddings up to isotopy [62]; classification of functions, e.g. they classify up to conjugacy simple Morse functions [36, 38, 60] and simple Morse–Bott functions on closed surfaces [40]; in singularity theory they provide a complete topological invariant of finitely determined map germs $(\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ [2, 3]; approximation of compact length spaces under the Gromov–Hausdorff distance [15, 43]; classification of Casimirs in two-dimensional hydrodynamics [24]; description of foliations defined by a Morse form [14]; they are also used in the study of second order scalar conservative differential equations [22].

The Reeb graph $\mathcal{R}(f)$ of function with finitely many critical points admits a specific orientation of edges consistent with the gradient of f, which is called a good orientation of a graph (see Definition 1.3). For every graph Γ with good orientation V. Sharko [61] constructed a surface with a function having finitely many critical points, whose Reeb graph is isomorphic to Γ. Afterwards, Y. Masumoto and O. Saeki [42] extended this construction to arbitrary finite graphs without loops, realizing them as Reeb graphs of functions on surfaces with finitely many critical values.

This leads to a natural problem:

Problem. For a given manifold M, which graph Γ can be realized as the Reeb graph of a function $f\colon M \to \mathbb{R}$ with finitely many critical points?

This problem is usually investigated without fixing a manifold, trying to characterize graphs which could be the Reeb graphs of functions from a given class (cf. [16, 18, 29, 30, 42, 61]). The question stated above is much more difficult. In this way from a function invariant we obtain features describing the manifold M itself, encoded in the set of possible Reeb graphs of functions on M. The problem is naturally divided into two parts: the case of surfaces and the case of manifolds of dimension at least 3. The former is simpler due to the classification of surfaces.

We start with giving a complete answer to this problem for surfaces (Theorem 2.20 et seq.). Let K_2 be the connected graph with two vertices and one edge. A graph $\Gamma \neq K_2$ can be realized as the Reeb graph of a function with finitely many critical points on a surface Σ if and only if it admits a good orientation and the cycle rank of Γ (i.e. its first Betti number $\beta_1(\Gamma)$) is not greater than the maximal attainable, the Reeb number $\mathcal{R}(\Sigma)$. Therefore the realizability of an oriented graph depends, with one exception, only on its homotopy type and whether its orientation is a good orientation. Next we describe graphs which can be realized by Morse functions (Theorem 2.23) and simple Morse functions (Theorem 2.24) on a given surface. It depends additionally on the number of vertices of degree 2 in a graph. These results come from the paper [45] of the author.

In the construction we need the information on the Reeb number $\mathcal{R}(M)$ of a manifold M, which is defined as the maximum cycle rank among Reeb graphs of functions with finitely many critical points on M. We show that the Reeb number is attained by simple Morse functions (Lemma 2.7). A smooth function is simple, if for every critical level it has exactly one critical point. The question about possible cycle rank of the Reeb graph was attracting an attention of other authors ([8, 14, 15, 27]). K. Cole-McLaughlin, H. Edelsbrunner et al. [8] showed a celebrated lemma describing the cycle ranks of Reeb graphs of simple Morse functions on surfaces (Lemma 2.1). As a consequence, $\mathcal{R}(\Sigma) = \lfloor \frac{k}{2} \rfloor$ for a closed surface Σ of the Euler characteristic $\chi(\Sigma) = 2 - k$, where $\lfloor x \rfloor$ is the floor of a real number x. It turns out that closed orientable surfaces has a unique property that Reeb graphs of simple Morse functions have cycle rank always equal to the genus of the surface. Higher-dimensional manifolds, as well as non-orientable surfaces, allow to admit arbitrary cycle rank of Reeb graph of simple Morse function between 0 and $\mathcal{R}(M)$ (cf. Corollary 4.19).

In general, a significant amount of work on Reeb graphs is concerned with functions on surfaces (cf. [8, 12, 27, 32, 33, 35, 40, 42, 45]). Thus we started in [46] a systematic study of Reeb graphs of higher-dimensional manifolds. The description of their cycle ranks was the first step in the settle of realization problem. We provide in Corollary 4.19 the equivalence of the following three conditions for a closed manifold M:

- the existence of epimorphism $\pi_1(M) \to F_r$ onto the free group of rank r,
- the existence of non-separating system of hypersurfaces in M of size r, i.e. r disjoint submanifolds $N_1, \dots, N_r \subset M$ of codimension one with product neighbourhoods, removal of whose does not disconnect M,
- the existence of a Morse function on M (simple, if M is not an orientable surface) whose Reeb graph has cycle rank equal to r.

This result is an extension of O. Cornea [10, Theorem 1] and W. Jaco [26, Theorem 2.1], where the first two conditions are considered (note that W. Jaco [26] works in the category of combinatorial manifolds). Independently, I. Gelbukh in [14, Theorem 13] proved the equivalence of the first and third condition by use of foliation theory, but restricting to orientable manifolds.

We conclude that the Reeb number $\mathcal{R}(M)$ is equal to the corank of fundamental group of M, defined as the maximum rank of a free group onto which there is an epimorphism from the group. It is also equal to the number $C(M)$, considered in [10], which is the maximum size of a non-separating system of hypersurfaces in M. As a consequence, any number not greater than $\mathcal{R}(M)$ occurs as the cycle rank of the Reeb graph of a Morse function, which can be simple, if M is not an orientable surface.

For higher-dimensional manifolds we provide the realization theorem (Theorem 5.2) up to homeomorphism of graphs, not the combinatorial isomorphism, so we lose information about the vertices of degree 2. It turns out that, as in the case of surfaces, a good orientation of a graph Γ and its cycle rank $\beta_1(\Gamma)$ not exceeding $\mathcal{R}(M)$ are necessary and sufficient conditions to construct a Morse function on M whose Reeb graph is orientation-preserving homeomorphic to Γ.

The first step in the proof is to reduce the problem to realizability of graphs with vertices of degrees 1 and 3 by simple Morse functions. Then the construction begins with the existence of a particular Reeb graph with a given cycle rank, called the initial graph, which is provided by the proof of Theorem 4.11. From this graph we can obtain each homeomorphism type of graphs using a finite number of combinatorial modifications of Reeb graphs realized by change of simple Morse functions. These modifications come from handle and Morse theory (see Section 4.2). We prove in Proposition 4.8 that each Reeb graph can be transformed to a canonical form using a finite number of such modifications. In the case of orientable surfaces similar operations have been used by Kudryavtseva [35, Theorem 1] and Fabio–Landi [12, Lemma 2.6] to prove an analogous reduction to a canonical form.

The above considerations can be extended to compact manifolds W with boundary and functions, which are constant on connected components of ∂W. For technical reasons we use functions on smooth triads (W, W_-, W_+), where $\partial W = W_- \sqcup W_+$. Manifolds with boundary occur when we deal with a system of hypersurfaces — the complement of their open tubular neighbourhoods is compact with boundary.

Since the Reeb graph of a function with finitely many critical points is a finite graph, its fundamental group is a free group F_r of a finite rank $r \geq 0$. M. Kaluba, W. Marzantowicz and N. Silva [27] provided the construction of an embedding $\iota\colon \Gamma(f) \to W$ such that the composition $q_f \circ \iota\colon \Gamma(f) \to \mathcal{R}(f)$ is a homotopy equivalence, where $q_f\colon W \to \mathcal{R}(f)$ is the quotient map (see Theorem 1.21). As a consequence, q_f induces on fundamental groups the so-called Reeb epimorphism $\varphi_f\colon \pi_1(W) \to \pi_1(\mathcal{R}(f)) \cong F_r$ of f, where $r = \beta_1(\mathcal{R}(f))$ is the cycle rank of $\mathcal{R}(f)$.

We consider the another natural question: is any epimorphism $\pi_1(M) \to F_r$ represented as Reeb epimorphism of a Morse function? We give the positive answer to this question in Theorem 5.6, which comes from the joint work with W. Marzantowicz [41].

One of the main ingredients in the proof of this theorem is the correspondence between homomorphisms onto free groups and systems of framed hypersurfaces. First, any homomorphism $\pi_1(W) \to F_r$ is induced, using an extended Pontryagin–Thom construction, by a system of hypersurfaces $\mathcal{N} = (N_1, \ldots, N_r)$ consisting of framed and properly embedded submanifolds N_i of codimension one in a compact manifold W. A system $\mathcal{N}$ is independent if it is non-separating, i.e. its complement is connected, and it is regular if each N_i is connected. It is an easy observation that an independent system of hypersurfaces induces a surjective homomorphism. We show the converse of this fact in Theorem 3.11. It provides for any epimorphism $\varphi\colon \pi_1(W) \to F_r$ a construction of a regular and independent system of hypersurfaces which induces φ.

Having these geometric tools, we study the problem of classification of epimorphisms $G \to F_r$ up to equivalence and strong equivalence relations defined in [19–21]. Briefly, on the set of homomorphisms $\mathrm{Hom}(G, F_r)$ there are the natural actions of automorphisms groups $\mathrm{Aut}(G)$ and $\mathrm{Aut}(F_r)$ given by compositions. Two homomorphisms are strongly equivalent (resp. equivalent) if they are in the same orbit of the action of $\mathrm{Aut}(G)$ (resp. $\mathrm{Aut}(G) \times \mathrm{Aut}(F_r)$). First, note that two systems induce the same homomorphism if and only if they are framed cobordant as systems of hypersurfaces (see Definition 3.4). It leads to a correspondence between strong equivalence classes of epimorphisms $\pi_1(M) \to F_r$ and elements of $\mathcal{H}_r^{fr}(M)/_{\mathrm{Diff}_\bullet(M)}$, the set of framed cobordism classes of independent and regular systems of size r in M up to diffeomorphisms which preserve the basepoint. It is a one-to-one correspondence if the natural homomorphism $\mathrm{Diff}_\bullet(M) \to \mathrm{Aut}(\pi_1(M))$ is surjective. For example, it holds when M is a closed surface (by Dehn–Nielsen Theorem) or when M is a hyperbolic manifold of dimension at least 3 (by Mostow Rigidity Theorem). As an application of developed methods, in Theorem 3.40 we determine the elements of $\mathcal{H}_r^{fr}(\Sigma)/_{\mathrm{Diff}_\bullet(\Sigma)}$ for a closed surface Σ. It gives a classification up to strong equivalence of epimorphisms $\pi_1(\Sigma) \to F_r$, which was obtained earlier by R. Grigorchuk, P. Kurchanov and H. Zieschang [19–21] by using more algebraic, but also topological methods (see Theorem 3.18).

A transition from strong equivalence classes to equivalence classes is obtained by considering the action of $\mathrm{Aut}(F_r)$, which is generated by elementary Nielsen transformations. We define the analogous operations on $\mathcal{H}_r^{fr}(M)$ which cause the same change of an inducing epimorphism as its composition with the corresponding Nielsen transformation. These operations allow us to compute equivalence classes of epimorphisms $\pi_1(\Sigma) \to F_r$ (see Theorem 3.45) as in Grigorchuk–Kurchanov–Zieschang Theorem.

We must add that the set of homomorphisms $\mathrm{Hom}(G, F_r)$ for G finitely generated group has been intensively studied by several authors by use of the Makanin–Razborov diagrams theory (cf. [39], [55]). It led to the solution of Tarski problem of the existence of solution to a system of equation in a free finitely generated group and culminated in the Sela's works (cf. [59] and more recent articles of this author; see also [4] for nice and relatively uncomplicated introduction to this theory). The Makanin–Razborov diagram of a group G consists of its quotients and quotient maps in such a way that every homomorphism $\varphi\colon G \to F_r$ is *M–R factorized* through some branch of quotients $G \xrightarrow{q_1} L_1 \xrightarrow{q_2} \ldots \xrightarrow{q_k} L_k$, where L_k is a free group. It means that there is an element $\psi \in \mathrm{Hom}(L_k, F_r)$ and modular automorphisms $\alpha \in \mathrm{Mod}(G)$ and $\alpha_i \in \mathrm{Mod}(L_i)$ for $1 \leq i < k$ such that $\varphi = \psi \circ q_k \circ \alpha_{k-1} \circ \ldots \circ \alpha_1 \circ q_1 \circ \alpha$. It is worth to emphasize that our geometrical approach is different in nature. The equivalence and strong equivalence relations in $\mathrm{Hom}(G, F_r)$ cannot be derived from Makanin–Razborov diagrams in general.

Next, we show the relations with Reeb graph theory. In Theorem 4.11 we assign a Morse function f on W and its Reeb graph to any system of hypersurfaces without boundary in such a way that its induced homomorphism is factorized by the Reeb epimorphism of f. Moreover, if the system is independent, it gives the construction of the initial graph (see Figure 4.5) as the Reeb graph such that submanifolds from the system are components of the same level set of f. Subsequently, for a regular and independent system $\mathcal{N}$ of hypersurfaces and a graph Γ with natural necessary conditions, Theorem 5.4 provides the construction of a Morse function realizing Γ as its Reeb graph such that submanifolds from $\mathcal{N}$ correspond to prescribed edges of Γ outside a spanning tree. Prescribed component of level sets are an additional ingredient to the realization theorems for Reeb graphs. It leads to representability of any epimorphism $\varphi\colon \pi_1(M) \to \pi_1(\Gamma)$ as the Reeb epimorphism of a Morse function whose Reeb graph is homeomorphic to Γ (Corollary 5.7). An epimorphism $\pi_1(W) \to \pi_1(\Gamma)$ is represented as the Reeb epimorphism if and only if it is induced by a system of hypersurfaces without boundary (Theorems 4.18 and 5.6). Equivalently, it is factorized through $\pi_1(W)/\langle \pi_1(\partial W)\rangle^{\pi_1(W)}$, where $\langle \pi_1(\partial W)\rangle^{\pi_1(W)}$ is the smallest normal subgroup of $\pi_1(W)$ containing all classes of loops from ∂W. These results shows that

$$\mathcal{R}(W) = \mathrm{corank}\left(\pi_1(W)/\langle \pi_1(\partial W)\rangle^{\pi_1(W)}\right) .$$

Note that the problem of representability of an epimorphism as the Reeb epimorphism was independently considered by O. Saeki in [58], where for a finite graph Γ without loops and a closed manifold he constructs a smooth function with finitely many critical values such that its Reeb graph is isomorphic to Γ and under this identification its Reeb epimorphism is φ. Thus he realizes not only topological structure of Γ, but also the combinatorial one, in return losing the non-degeneracy of critical points. Note that the number of vertices of degree 2 in the Reeb graph of Morse function cannot be arbitrary (see Theorem 2.24 and Section 5.5), thus we focus on the homeomorphism type. Our results are also different in that way they deal with manifolds with boundary and allow us to control the system of hypersurfaces in connected components of level sets of the constructed function.

The next step is the classification of Reeb epimorphisms of simple Morse functions. There no restrictions other than necessary conditions on a graph Γ to represent epimorphsim $\pi_1(M) \to \pi_1(\Gamma)$ as the Reeb epimorphism of simple Morse function for manifolds M of dimension at least 3. The situation is more interesting for surfaces (see Corollary 5.11). As we know, for orientable surfaces we need to assure that the cycle rank of Γ is equal to the genus of surface. For non-orientable surfaces of odd genus there are no additional conditions. However, it turns out that simple Morse functions f on non-orientable surfaces of even genus have also some unique property, namely either cycle rank of $\mathcal{R}(f)$ is equal to the half of the genus or its Reeb epimorphism belongs to a unique strong equivalence class. This characterization can be used in the problem of conjugacy of simple Morse function (see Section 5.4).

At the end of the thesis, in Section 5.5, we point out the properties of degree 2 vertices in Reeb graphs of simple Morse functions on manifolds M of dimension at least 3. We provide lower bounds for the minimum number of them, denoted by $\Delta_2(M)$, in terms of fundamental group, homology groups, Lusternik– Schnirelmann category and Heegaard genus $g(M)$ in the case of orientable 3-manifolds. We indicate some partial results and open problems we consider for further studies. In particular, the obtained inequality

$$\operatorname{corank}(\pi_1(M)) \geq g(M) - \frac{\Delta_2(M)}{2}$$

for orientable 3-manifolds brings to mind the corank conjecture posed by T. Kerler and J. H. Przytycki who were studying connections with quantum invariants of 3-manifolds. It states that $\operatorname{corank}(\pi_1(M)) \geq \beta_1(M)/3$. However, there are counterexamples to this conjecture constructed by Ch. Leininger and A. W. Reid, A. Sikora, and S. Harvey (see [63] and references therein).

The thesis follows closely the published papers [45, 46] and preprint [41] with small generalizations, but it differs mainly in exposition in order to avoid repetitions. It is organized as follows. In Chapter 1 we introduce basic notions and properties of Reeb graphs and set up terminology. Moreover, we provide an exposition of Morse and handle theory used in the work. Chapter 2, mostly based on [45], presents realization theorems for Reeb graphs in the case of surfaces. In Chapter 3 we establish, following [41], the relation between independent systems of hypersurfaces and epimorphisms onto free groups and their properties, which are used in the next chapters. Chapter 4 is devoted to the characterizations of cycle ranks of Reeb graphs and the technique of combinatorial modifications of Reeb graphs. Finally, in Chapter 5 we settle realization problems for general manifolds and epimorphisms onto free groups. The last two chapters are a mixture of [46] and [41], with the exception that Section 5.5 is a part of forthcoming work. In addition, at the beginning of each chapter there is an exposition of its structure.

Chapter 1

Preliminaries

In this chapter we present basic definitions and properties of objects used in the work. We also set up notation and conventions. In Section 1.1 we introduce the notion of Reeb graph. One of the main tools of its study is the Morse theory which describes a function near non-degenerate critical points and provides a handle decomposition of manifold. Its brief exposition is given in Section 1.2. Then, in Section 1.3 we establish the correspondence between the index of critical point of a simple Morse function and the degree of the vertex of the Reeb graph related to the critical point. Finally, Section 1.4 introduces basic objects which are closely related to Reeb graphs and used in their studies.

1.1 Basic properties of Reeb graphs

Throughout the work by a manifold we understand a smooth manifold with boundary of finite dimension. Hereafter, M and W denote connected and compact manifolds of dimension $n \geq 2$ with fixed basepoints; in addition, M is closed. Moreover, all considered graphs are connected.

Let us start with the fundamental definition for this work.

Definition 1.1. Let X and Y be topological spaces and $f\colon X \to Y$ be a map. We say that two points $x, x' \in X$ are in **Reeb relation** $\sim_{\mathcal{R}}$ if and only if x and x' belong to the same connected component of a level set of f. The quotient space $X/_{\sim_{\mathcal{R}}}$ is denoted by $\mathcal{R}(f)$ and it is called the **Reeb space** of f.

We are mainly interested in the study of smooth manifolds and smooth functions on them. For this purpose we will deal with the following objects.

A **smooth triad** is a triple (W, W_-, W_+), where W is a compact manifold and its boundary $\partial W = W_- \sqcup W_+$ is the disjoint union of W_- and W_+ (possibly $W_\pm = \varnothing$). Thus the connected components of ∂W are divided into two parts. A function on a smooth triad (W, W_-, W_+) is a smooth function $f\colon W \to [a, b]$ such that $f^{-1}(a) = W_-$, $f^{-1}(b) = W_+$, and all critical points of f are contained in $\operatorname{Int} W$, the interior of W.

It is known that for a smooth function f with finitely many critical points on a smooth triad (W, W_-, W_+), the quotient space $\mathcal{R}(f)$ is homeomorphic to a finite graph, i.e. to a one-dimensional finite CW-complex (see [44, 58, 61]), known as the **Reeb graph** of f. The vertices of $\mathcal{R}(f)$ correspond to the critical components of the level sets of f (i.e. to the components containing critical points) and to the components of ∂W. We consider the combinatorial structure on $\mathcal{R}(f)$ induced by the homeomorphism as the standard one.

It is straightforward that the connected components of Reeb graph $\mathcal{R}(f)$ are in one-to-one correspondence with connected components of the manifold W. Thus it is reasonable to restrict our considerations to connected manifolds and connected graphs.

Remark 1.2. The notions of Reeb graphs and Reeb spaces are also studying for larger classes of spaces and maps, e.g. for Morse–Bott functions [40], continuous functions on non-compact surfaces [52, 53] or on a connected and locally path-connected topological spaces with additional assumption [15], or for circle-valued functions [2, 3]. The Reeb space is also used in the study of so-called special generic maps to euclidean spaces, where it is known as the Stein factorization (see [7, 31, 37, 54, 57]).

Every function f induces the continuous function $\overline{f}\colon \mathcal{R}(f) \to \mathbb{R}$ which satisfies $f = \overline{f} \circ q_f$, where $q_f\colon W \to W/_{\sim_\mathcal{R}} = \mathcal{R}(f)$ is the quotient map. The function $\overline{f}$ gives an orientation to the edges of $\mathcal{R}(f)$ as described by Sharko [61] and is called the good orientation of a graph following Masumoto–Saeki [42].

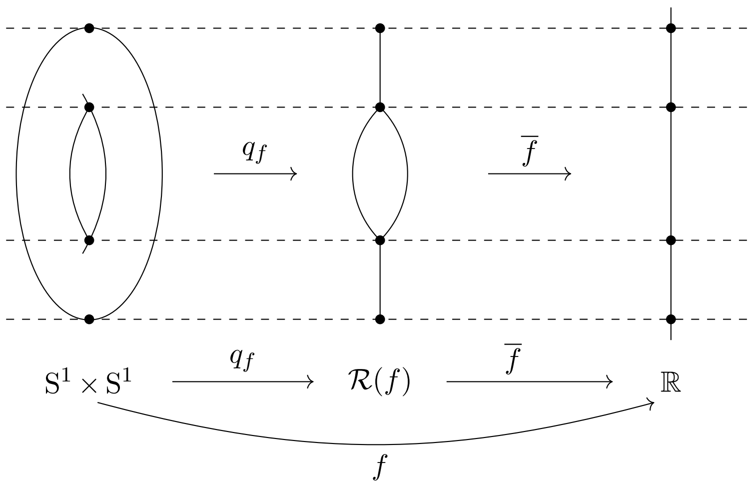


Figure 1.1: The Reeb graph of height function $f\colon S^1 \times S^1 \to \mathbb{R}$ on two-dimensional torus. Four critical points of f have different values and correspond to four vertices of $\mathcal{R}(f)$.

The **degree** $\deg(v)$ of a vertex v in a graph Γ is the number of edges incident to v. If Γ is an oriented graph (i.e. directed — each edge has a chosen direction), then the **indegree** (**outdegree**) of v is the number of edges incoming (outgoing) to v and it is denoted by $\deg_{in}(v)$ ($\deg_{out}(v)$). Thus $\deg(v) = \deg_{in}(v) + \deg_{out}(v)$.

Definition 1.3. A **good orientation** of a graph Γ is the orientation induced by a continuous function $\Gamma \to \mathbb{R}$ which has extrema only in the vertices of degree 1 and which is strictly monotonic on the edges.

A basic observation is that a graph with good orientation does not have **loops**, i.e. edges that connect a vertex to itself. Moreover, the edges join vertices from different levels. There are no horizontal edges.

It is easy to see that the function $\overline{f}$ on $\mathcal{R}(f)$ induces indeed a good orientation. This important concept was studied comprehensively by Sharko [61] which used an equivalent definition: an oriented graph has a good orientation if and only if the following conditions are satisfied:

- it has at least two vertices of degree 1, one with incoming and one with outgoing edge,
- any vertex of degree at least 2 has both incoming and outgoing edge,
- it does not have oriented cycles.

Sharko provided an example of a graph which cannot admit a good orientation (see Figure 1.2 (a)). Any attempt to orient this graph causes failure of one of the above conditions. However, the graph presented in Figure 1.2 (b) can be oriented in a good way.

Figure 1.2: (a) Graph which does not admit a good orientation [61, Figure 1] and (b) graph which can admit it.

Remark 1.4. Recently I. Gelbukh [17] generalized the notion of good orientation to the so-called S-good orientation for any set S of non-negative integers. A directed graph which is acyclic (it has no directed cycles) has an **S-good orientation** if all its sinks (vertices with outdegree 0) and sources (vertices with indegree 0) have degrees in S. Thus a good orientation of a graph is a $\{1\}$-good orientation. I. Gelbukh provided a criterion for a graph to admit an S-good orientation in term of leaf blocks, i.e. biconnected components which contain at most one cut-vertex of the graph. In particular, a graph admits a good orientation if and only if it has no loops and all its leaf blocks are K_2, the complete graph on 2 vertices.

Since we consider graphs with orientations induced by functions, we will use the following terminology. A vertex which is a source (a sink), i.e. it has no incoming edges (outgoing edges) is called a **minimum** (**maximum**). For graphs with good orientations such vertices are of degree 1.

Let Γ be a graph with good orientation induced by $g\colon \Gamma \to \mathbb{R}$. A path $\tau\colon [0,1] \to \Gamma$ is **increasing**, if $g(\tau(t)) < g(\tau(t'))$ for $0 \leq t < t' \leq 1$. Similarly we define a **decreasing** path. For two vertices v and w in Γ we say that v is **below** w and w is **above** v if there is an increasing path from v to w. It is clear that these definitions do not depend on the choice of the function g. According to this notation, orientations of graphs presented in figures in this paper are from the bottom to the top, as we see in Figure 1.1, without indicating arrows.

By $\Delta(\Gamma)$ we denote the maximum degree of a vertex in a graph Γ, and by $\Delta_k(\Gamma)$ the number of its vertices of degree k.

Recall that a **tree** T is a connected graph without cycles. Thus as a topological space it is contractible. If V and E are sets of its vertices and edges, respectively, then the equality $|E| = |V| - 1$ holds. A **spanning tree** of graph Γ is a tree T which is a subgraph of Γ containing all its vertices.

We assume that isomorphism of oriented graphs keeps not only combinatorial structures (sends adjacent vertices to adjacent vertices), but also preserves orientations (directions) of edges. Any isomorphism of oriented graphs can be considered as an orientation-preserving homeomorphism. However, a homeomorphism of graphs does not preserves vertices of degree 2 in general.

1.2 Morse and handle theory

In this section we recall basic definitions and facts about Morse functions and handle decomposition of a manifold used in the work. For more details and proofs we refer to [48, 50, 51], but the needed results can be also found in many books on differential topology. We will not introduce the concept of gradient-like vector fields deliberately, as it is not needed for our purposes.

Let W be a compact manifold of dimension $\dim W = n$. A critical point p of a function $f\colon W \to \mathbb{R}$ on a smooth triad (W, W_-, W_+) is **non-degenerate** if the matrix of second partial derivatives at p, its Hessian matrix, is non-singular, so for any local coordinates $(x_1, \ldots, x_n)$ around p

$$\det \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)_{i,j} \neq 0.$$

The **index** of a non-degenerate critical point p, denoted by $\operatorname{ind}(p)$, is the number of negative eigenvalues of its Hessian matrix. Thus it is an integer between 0 and n.

The following fundamental result shows the behaviour of the function near a non-degenerate critical point (for a proof see e.g. [50, Lemma 2.2]).

Lemma 1.5 (Morse Lemma)**.** *Let p be a non-degenerate critical point of a smooth function $f\colon W \to \mathbb{R}$. Then there is a chart $\varphi\colon U \to \mathbb{R}^n$ around p such that $\varphi(p) = 0$ and*

$$f \circ \varphi^{-1}(x_1, \ldots, x_n) = f(p) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2,$$

where k is the index of f at p. □

Definition 1.6. A smooth function $f\colon W \to \mathbb{R}$ is called a **Morse function** if all its critical points are non-degenerate.

As a conclusion from the Morse Lemma, non-degenerate critical points are isolated and, in consequence, a Morse function on a compact manifold has finitely many critical points.

For a manifold W, a function $f\colon W \to \mathbb{R}$, $c \in \mathbb{R}$ and any interval $I \subset \mathbb{R}$ we use the following notation:

$$W_c := f^{-1}(c), \qquad W^c := f^{-1}((-\infty, c]), \qquad W^I := f^{-1}(I).$$

Theorem 1.7 ([48, Theorems 3.4 and 3.13] or [51, Theorems 9.2.10 and 9.2.13, Proposition 10.1.7])**.** *Let $f\colon W \to \mathbb{R}$ be a Morse function on a smooth triad.*

(1) If $W^{[a,b]}$ does not contain any critical point, then there is a diffeomorphism

$$W^{[a,b]} \cong W_a \times [a, b]$$

and, in consequence, $W^a \cong W^b$.

(2) If $W^{[a,b]}$ contains exactly one critical point p of index k such that $a < f(p) < b$, then

$$W^b \cong W^a \cup_\varphi h^k,$$

i.e. W^b is diffeomorphic to the manifold obtained by attaching an n-dimensional k-handle to W^a. □

Let D^n denote an n-dimensional disc with the boundary $\partial\,\mathrm{D}^n = \mathrm{S}^{n-1}$, an $(n-1)$-dimensional sphere. Note that $S^0 = \{-1, 1\}$ has two points, and by the convention $\mathrm{S}^{-1} = \varnothing$. An n-dimensional k-handle $h^k = \mathrm{D}^k \times \mathrm{D}^{n-k}$ is attached to W^a along an embedding

$$\varphi\colon\ \mathrm{S}^{k-1} \times \mathrm{D}^{n-k} \to W_a \subset \partial W^a.$$

Then $W^a \cup_\varphi h^k$ is obtained from $W^a \sqcup h^k$ by identifying any point $x \in \mathrm{S}^{k-1} \times \mathrm{D}^{n-k}$ with $\varphi(x)$. In fact, the resulting space $W^a \cup_\varphi h^k$ is a *manifold with corners*. However, it can be *straightening*, producing a smooth manifold W' and a homeomorphism $W^a \cup_\varphi h^k \to W'$ which is a diffeomorphism outside the corners. This smooth manifold is unique up to diffeomorphism (see [66, Proposition 2.6.2]).

Note that in the case (2) of Theorem 1.7 there is a diffeomorphism

$$W^{[a,b]} \cong (W_a \times [a, a+\varepsilon]) \cup_\varphi h^k$$

for some sufficiently small $\varepsilon > 0$. It turns out that the converse of this situation also holds.

Theorem 1.8 ([48, Theorem 3.12])**.** *If $W \cong (W_- \times [0,1]) \cup_\varphi h^k$ is obtained from $W_- \times [0,1]$ by attaching a k-handle, then there is a Morse function $f\colon W \to \mathbb{R}$ on (W, W_-, W_+) with exactly one critical point of index k.*

For a Morse function $f\colon W \to \mathbb{R}$ we say that it is

- **simple** if on every critical level there is exactly one critical point,
- **$\mathcal{R}$-simple** if each vertex of $\mathcal{R}(f)$ corresponds exactly to a one critical point of f,
- **self-indexing** if there exist values $c_0 < c_1 < \ldots < c_n$ (where $n = \dim W$) such that for any critical point p if $\mathrm{ind}(p) = i$, then $f(p) = c_i$,
- **ordered** if for any two critical points p and p' if $\mathrm{ind}(p) < \mathrm{ind}(p')$, then $f(p) < f(p')$.

Roughly speaking, ordered Morse functions have the property that critical points of a smaller index are below the critical points of a larger index. Self-indexing Morse functions are obviously ordered. In fact, they can be defined as ordered Morse functions for which all critical points of the same index have the same value.

Remark 1.9. In some cases it is convenient to use $\mathcal{R}$-simple Morse functions instead of usual simple Morse functions. Equivalently, a Morse function is $\mathcal{R}$-simple if any connected component of its level set contains at most one critical point. Obviously, a simple Morse function is $\mathcal{R}$-simple. By the following theorem, having an $\mathcal{R}$-simple Morse function we can easily make its small perturbation to get a simple Morse function without changing isomorphism type of its Reeb graph.

Theorem 1.10 ([48, Lemma 2.8 and Theorem 4.8], cf. [51, Corollary 9.1.5 and Lemma 10.1.5])**.** *For any compact manifold W with boundary $\partial W = W_- \sqcup W_+$ there exists a Morse function $f\colon W \to [a,b]$ on the triad (W, W_-, W_+). Moreover, if f is given, then there exist a self-indexing Morse function and a simple Morse function on (W, W_-, W_+), both with the same critical points each with the same index as f. Such a simple Morse function can be obtained by an arbitrary small perturbation of f.* □

A **handle decomposition** of a smooth triad (W, W_-, W_+) is a sequence of manifolds

$$W_0 \subset W_1 \subset \ldots \subset W_m = W,$$

where $W_0 = W_- \times [0,1]$ and for each $i = 1, \ldots, m$ the manifold W_i is obtained from W_{i-1} by attaching a handle h^{k_i} along an embedding disjoint from $W_- \times \{0\} \subset \partial W_i$.

We may write

$$W \cong (W_- \times [0,1]) \cup_{\varphi_1} h^{k_1} \cup \ldots \cup_{\varphi_m} h^{k_m}.$$

keeping in mind that each h^{k_i} is attached to W_{i-1} in general, not necessary to $W_0 \times [0,1]$.

Theorems 1.7 and 1.10 imply the following corollary.

Corollary 1.11. *Any smooth triad admits a handle decomposition.* □

It can be shown that a manifold with a k-handle attached deformation retracts onto the same manifold with a k-dimensional cell attached (see [50, Theorem 3.2] or [51, Corollary 9.3.5]). It is induced by the retraction of a handle $h^k = \mathrm{D}^k \times \mathrm{D}^{n-k}$ onto its core $\mathrm{D}^k \times \{0\}$.

Theorem 1.12. *Let M be a closed manifold and $f\colon M \to \mathbb{R}$ be a Morse function. Then M is homotopy equivalent to a finite CW-complex whose k-dimensional cells correspond bijectively to critical points of index k of f.* □

Thus Morse functions can be used to study homological properties of manifolds. In particular, we will use the following fact describing the Euler characteristic $\chi(M)$ of M, which is immediate from the above theorem.

Corollary 1.13. *If $f\colon M \to \mathbb{R}$ is a Morse function on a closed manifold M and k_i is the number of its critical points of index i, then*

$$\chi(M) = \sum_i (-1)^i k_i.$$

□

We will need the techniques of manipulating the critical points of Morse function and the handles in a handle decomposition of manifold. One of them is Rearrangement Theorem, which allows us to change the order of handles.

Theorem 1.14 ([48, Theorem 4.1 and 4.4], cf. [51, Proposition 10.4.1])**.** *If (W, W_-, W_+) is a smooth triad with handle decomposition*

$$W \cong (W_- \times [0,1]) \cup_{\varphi_m} h^m \cup_{\varphi_k} h^k,$$

where $k \leq m$, then by isotopies of φ_m and φ_k the handles may be attached simultaneously or in the reverse order.

In terms of Morse functions, if $f\colon W \to [a,b]$ is a Morse function on (W, W_-, W_+) with two critical points p and p' such that $f(p) > f(p')$ and $\mathrm{ind}(p) \leq \mathrm{ind}(p')$, then for any $c, c' \in (a,b)$ there is a Morse function $g\colon W \to [a,b]$ on (W, W_-, W_+) with the same critical points and of the same index as f such that $g(p) = c$, $g(p') = c'$ and g is equal to f in the neighbourhood of ∂W. □

The above theorem can be applied to the case of several handles/critical points of index k lying (in the sense of order) above handles/critical points of index m, if $k \leq m$.

We will also use the following special casae of Cancellation of handles (see [48, Theorem 5.4] or [51, Theorem 10.4.4]).

Proposition 1.15. *If (W, W_-, W_+) is a smooth triad with simple Morse function with exactly two critical points of index* 0 *and* 1 *(n and $n-1$), then (W, W_-, W_+) is a product triad, i.e. $W \cong W_- \times [0,1]$. Thus it admits a Morse function without critical points.* □

1.3 Index and degree correspondence

The correspondence between the index of critical point of a simple Morse function and the degree of the corresponding vertex in the Reeb graph was stated in the original paper of Reeb [56, Théorème 3]. Here, we present a refinement version of it. This is one of the basic ingredients needed in the realization theorems and for introducing combinatorial modifications of Reeb graphs. It is reasonable to consider separately the case of surfaces and manifolds of higher dimension.

Let us make a small preparation. Let $f\colon W \to \mathbb{R}$ be a function with finitely many critical points, $\overline{f}\colon \mathcal{R}(f) \to \mathbb{R}$ the induced function and let v be a vertex of $\mathcal{R}(f)$ with value $c = \overline{f}(v)$. Take sufficiently small $\varepsilon > 0$ such that the connected component K of $\overline{f}^{-1}([c-\varepsilon, c+\varepsilon])$ containing v has no other vertices of $\mathcal{R}(f)$. Define the submanifold $Q^v := q_f^{-1}(K)$ with boundary $\partial Q^v = Q^v_- \sqcup Q^v_+$, where $Q^v_\pm := Q^v \cap W_{c\pm\varepsilon}$.

Definition 1.16. We call the smooth triad (Q^v, Q^v_-, Q^v_+) **associated with the vertex** v.

Recall that $\pi_0(X)$ is the set of path components of a space X and their number is denoted by $|\pi_0(X)|$. Since all discussed by us spaces are locally path connected, the set $\pi_0(X)$ of path components of a space X is equal to the set of connected components of X.

It is evident that if (Q^v, Q^v_-, Q^v_+) is associated with v, then $\deg_{in}(v) = |\pi_0(Q^v_-)|$ and $\deg_{out}(v) = |\pi_0(Q^v_+)|$. Moreover, Q^v is connected.

Now, suppose that Q^v contains only one critical point p. If p is non-degenerate of index k, then by Theorem 1.7 the handle decomposition of Q^v consists of only one handle h^k, so there is an embedding $\varphi\colon \mathrm{S}^{k-1} \times \mathrm{D}^{n-k} \to Q^v_- \times \{c - \frac{\varepsilon}{2}\}$ such that

$$Q^v \cong \left(Q^v_- \times \left[c-\varepsilon, c-\frac{\varepsilon}{2}\right]\right) \cup_\varphi h^k.$$

Note that $\operatorname{Im}\varphi$ intersects each component of $Q^v_- \times \{c - \frac{\varepsilon}{2}\}$ since Q^v is connected. Otherwise, if V is a connected component of Q^v_- omitting $\operatorname{Im}\varphi$ under the canonical diffeomorphism $Q^v_- \cong Q^v_- \times \{c - \frac{\varepsilon}{2}\}$, then Q^v is the disjoint union of $V \times [c-\varepsilon, c+\varepsilon]$ and $\big((Q^v_- \setminus V) \times \big[c-\varepsilon, c-\frac{\varepsilon}{2}\big]\big) \cup_\varphi h^k$, a contradiction.

Proposition 1.17 (cf. [45, Proposition 3.3])**.** *Let $f\colon \Sigma \to \mathbb{R}$ be an $\mathcal{R}$-simple Morse function on a surface Σ. Let p be a critical point of f and $v := q_f(p)$ be the vertex of $\mathcal{R}(f)$ corresponding to p. Then* $\operatorname{ind}(p) = 0$ *or* 2 *if and only if* $\deg(v) = 1$*;* $\operatorname{ind}(p) = 1$ *if and only if*

$$\deg(v) = \begin{cases} 3 & \text{if } \Sigma \text{ is orientable,} \\ 2 \text{ or } 3 & \text{if } \Sigma \text{ is non-orientable.} \end{cases}$$

Proof. Critical points of indices 0 or 2 are extrema of the function. It is straightforward from the construction of Reeb graph that they correspond to vertices of degree 1 in $\mathcal{R}(f)$.

For $\operatorname{ind}(p) = 1$ take a triad (Q^v, Q^v_-, Q^v_+) associated with v. Then p is the only one critical point of $f|_{Q^v}$ and as above there is an embedding $\varphi\colon \mathrm{S}^0 \times \mathrm{D}^1 \to Q^v_- \times \{c - \frac{\varepsilon}{2}\}$ such that $Q^v \cong \big(Q^v_- \times \big[c-\varepsilon, c-\frac{\varepsilon}{2}\big]\big) \cup_\varphi h^1$, where $c = f(p)$. Since the domain of φ has two connected components, consider the following cases. If $\operatorname{Im}\varphi$ intersects Q^v_- in two different components, then $\deg_{in}(v) = |\pi_0(Q^v_-)| = 2$ and in the result of attaching the handle these components merge, so $\deg_{out}(v) = |\pi_0(Q^v_+)| = 1$. Otherwise, $Q^v_- \cong \mathrm{S}^1$ is connected and $\deg_{in}(v) = 1$. In this case, if φ preserves the orientations, Q^v_- splits into two components and $\deg_{out}(v) = |\pi_0(Q^v_+)| = 2$. In both the cases $\deg(v) = 3$. However, if φ is not orientation-preserving, then attaching the handle we obtain one connected component, so $\deg_{out}(v) = 1$ and $\deg(v) = 2$. The last situation can occur only for non-orientable surfaces. □

Remark 1.18. Note that following the original paper of Reeb [56, Théorème 3], some authors (e.g. Biasotti et al. in [1]) overestimate the bound on the degree of a vertex in the Reeb graph of a simple Morse function on a non-orientable surface. The straightforward conclusion from the Morse Lemma bounds the degree by 4, but the estimate given by Proposition 1.17 is sharp.

Proposition 1.19. *Let $f\colon W \to \mathbb{R}$ be an $\mathcal{R}$-simple Morse function on an n-dimensional smooth triad (W, W_-, W_+), $n \geq 3$. Let p be a critical point of f and $v := q_f(p)$ be the vertex in $\mathcal{R}(f)$ which corresponds to p. Then*

$$\deg(v) = \begin{cases} 1 & \textit{if } \operatorname{ind}(p) = 0 \textit{ or } n, \\ 2 \textit{ or } 3 & \textit{if } \operatorname{ind}(p) = 1 \textit{ or } n-1, \\ 2 & \textit{in other cases.} \end{cases}$$

$$\operatorname{ind}(p) = \begin{cases} 0 \textit{ or } n & \textit{if } \deg(v) = 1, \\ 1 & \textit{if } \deg(v) = 3 \textit{ and } \deg_{in}(v) = 2, \\ n-1 & \textit{if } \deg(v) = 3 \textit{ and } \deg_{out}(v) = 2, \\ 1 \textit{ or } \dots \textit{ or } n-1 & \textit{if } \deg(v) = 2. \end{cases}$$

Proof. The proof follows in the same way as before. Note that the domain of $\varphi\colon \mathrm{S}^{k-1} \times \mathrm{D}^{n-k} \to Q^v_- \times \{c - \frac{\varepsilon}{2}\}$ is connected for $k = \operatorname{ind}(p) \in \{2, \dots, n-2\}$, so then $\deg_{in}(v) = \deg_{out}(v) = 1$ and $\deg(v) = 2$. For $k = 1$ if Q^v_- has two components, then $\deg_{in}(v) = 2$ and $\deg_{out}(v) = 1$. However, if Q^v_- is connected, look at the function $-f|_{Q^v}$. It has one critical point p of index $n-1$. Thus we obtain a handle decomposition of (Q^v, Q^v_+, Q^v_-) consists of one handle of index $n-1$ attached along an embedding of $\mathrm{S}^{n-2} \times \mathrm{D}^1$ into Q^v_+. We conclude similarly that the embedding intersects all components of Q^v_+, so Q^v_+ is connected since $n \geq 3$. Thus $\deg_{in}(v) = \deg_{out}(v) = 1$ and $\deg(v) = 2$. The case for $k = n-1$ is analogous. □

1.4 Objects related to Reeb graphs

In this section we present two kinds of objects which arise from Reeb graphs: epimorphisms onto free groups and systems of hypersurfaces, whose systematic study in Chapter 3 can be used in the realization problem. These objects have the one common quantity, which for Reeb graph is its first Betti number. This number is one of the main features characterizing finite and connected graphs from topological point of view and it describes them up to homotopy

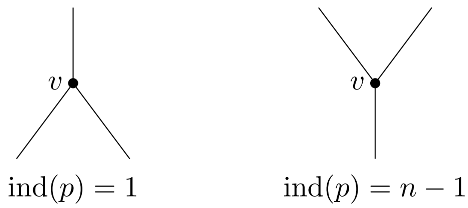


Figure 1.3: Possible neighbourhoods of a vertex of degree 3 in the Reeb graph of a simple Morse function on a manifold of dimension at least 3. We use our convention that the orientation is from the bottom to the top.

equivalence. Considering a Reeb graph as a sub-complex of a manifold will be useful in obtaining the connection between these objects.

Recall that for a finite CW-complex X the first Betti number $\beta_1(X)$ is the rank over $\mathbb{Z}$ of the first homology group, i.e. $\beta_1(X) := \operatorname{rank}_{\mathbb{Z}} \mathrm{H}_1(X;\mathbb{Z})$. For a finite and connected graph Γ its first Betti number $\beta_1(\Gamma)$ is called the **cycle rank** of Γ.

We write $\Gamma = \Gamma(V, E)$ when V is the set of vertices and E is the set of edges of Γ. Then the cycle rank of Γ is equal to $\beta_1(\Gamma) = |E| - |V| + 1$ and its fundamental group is $\pi_1(\Gamma) \cong F_{\beta_1(\Gamma)}$, where F_r is the free group of rank $r \geq 0$ (F_0 is the trivial group).

Remark 1.20. The cycle rank of a graph is also called the *number of loops* [8, 14] or the *number of cycles* [45].

There is a natural question if the Reeb graph can be embedded in a manifold as a sub-complex in such a way that the quotient map to the Reeb graph restricted to this sub-complex is an isomorphism. It is easy to seen that the answer is positive for simple Morse functions. However, in general setting of functions with finitely many critical points it is necessarily to consider trees on connected components of level sets containing critical points. The paper [27] of M. Kaluba, W. Marzantowicz and N. Silva shows that it suffices to obtain the desired sub-complex.

Theorem 1.21 ([27]). *If $f\colon W \to \mathbb{R}$ is a smooth function with isolated critical points on (W, W_-, W_+), then there exist a graph $\Gamma(f)$ and an embedding $\iota\colon \Gamma(f) \to W$ such that the composition $q_f \circ \iota\colon \Gamma(f) \to \mathcal{R}(f)$ is a homotopy equivalence which is bijective on edges joining different levels and which contracts trees on critical levels to the points. Moreover, the vertices of $\Gamma(f)$ correspond bijectivelly to critical points of f and connected components of ∂W.* □

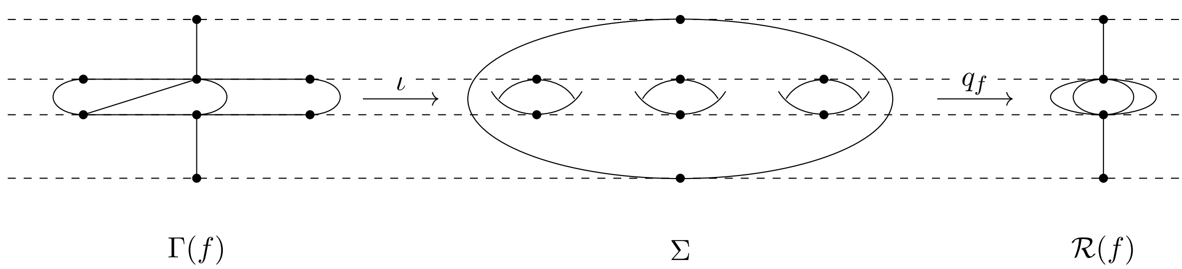


Figure 1.4: Graph $\Gamma(f)$ as a sub-complex of a surface Σ for height function f with eight critical points.

As a consequence, the quotient map $q_f\colon W \to \mathcal{R}(f)$ is surjective morphism for any homotopy functor. Therefore

$$\beta_1(\Gamma) \leq \beta_1(W)$$

from first homology groups. Moreover, on fundamental groups it gives an epimorphism

$$\varphi_f = (q_f)_\# \colon \pi_1(W) \to \pi_1(\mathcal{R}(f)) \cong F_{\beta_1(\mathcal{R}(f))}$$

onto a free group, which is called the **Reeb epimorphism** of f.

Now, let $r = \beta_1(\mathcal{R}(f))$ be the cycle rank of the Reeb graph f. Thus for a spanning tree $T \subset \mathcal{R}(f)$ the quotient $\mathcal{R}(f)/T$ is the wedge product of r circles, which correspond to r edges $e_1, \ldots, e_r$ of $\mathcal{R}(f)$ outside T. For each i take one point $a_i \in e_i$ in the interior of the edge. By the definition of Reeb relation $N_i := q_f^{-1}(a_i)$ is a connected component of regular level of f, so N_i is a submanifold of codimension 1 without boundary. Moreover, if we take a small neighbourhood $[a_i - \varepsilon, a_i + \varepsilon] \subset e_i$, then $P(N_i) = q_f^{-1}([a_i - \varepsilon, a_i + \varepsilon])$ is a tubular neighbourhood of N_i and $P(N_i) \cong N_i \times [a_i - \varepsilon, a_i + \varepsilon]$. Note that the sum of submanifolds N_i does not separate W, i.e. the complement of their sum is connected. Thus they form the so-called **regular and independent system of hypersurfaces** of size r (see Definition 3.1).

We see that the cycle rank of a Reeb graph, rank of free group in a Reeb epimorphism and the size of associated regular and independent system of hypersurfaces are equal. In the next chapters we will study the relations between these three objects.

Chapter 2

Reeb graphs of functions on surfaces

In this chapter we start to investigate the problem of realization of a graph as the Reeb graph, mainly in the case of functions on surfaces. However, we obtain some basic results for manifolds of arbitrary dimension. In Section 2.1 we determine some formulas on cycle ranks of Reeb graphs of simple Morse functions. Then, in Section 2.2 we show that the Reeb number of a manifold, the maximum cycle rank among Reeb graphs of its functions, is attained by simple Morse functions. Next, in Section 2.3 we use the procedure of attaching the handles to construct a manifold of a given dimension and a Morse function whose Reeb graph is isomorphic to a desired graph. We use this construction in Section 2.4 to give a complete characterization of graphs which can arise as Reeb graphs of functions on surfaces with finitely many critical points..

2.1 Simple Morse functions and cycle ranks of their Reeb graphs

Let us begin with investigation of cycle ranks of Reeb graphs of simple Morse functions on surfaces. Cole-McLaughlin, Edelsbrunner et al. [8] established the following celebrated lemma. Their results are also true for $\mathcal{R}$-simple Morse functions.

By $\Sigma_g := \#_{i=1}^{g} T^2$ we denote the closed orientable surface of genus g which is the connected sum of g copies of the two-dimensional torus $T^2 = \mathrm{S}^1 \times \mathrm{S}^1$ and by $S_g := \#_{i=1}^{g} \mathbb{R}\mathrm{P}^2$ the closed non-orientable surface of genus g, the connected sum of g copies of the real projective plane $\mathbb{R}\mathrm{P}^2$.

Lemma 2.1 ([8, Lemma A and C]). *Let $f \colon \Sigma \to \mathbb{R}$ be an $\mathcal{R}$-simple Morse function on a closed surface Σ.*

(1) If $\Sigma = \Sigma_g$, then $\beta_1(\mathcal{R}(f)) = g$.

(2) If $\Sigma = S_g$, then $\beta_1(\mathcal{R}(f)) \leq \lfloor \frac{g}{2} \rfloor$, where $\lfloor x \rfloor$ is the floor of x.

Their proof in the orientable case follows by collapsing vertices of degree 1 in $\mathcal{R}(f)$, merging arcs across vertices of degree 2 and a simple calculation. In the non-orientable case the composition of a particular orientable 2-sheeted covering map and f is perturbed to obtain a simple Morse function. They state that it does not alter the Reeb graph. While the argument may be correct in their situation, it is worth pointing out that it is not the case for perturbations of arbitrary Morse functions.

Example 2.2. Consider a self-indexing Morse function $f \colon \Sigma_g \to \mathbb{R}$ for $g \geq 1$. The function has only three critical levels 0, 1 and 2 and all the vertices of $\mathcal{R}(f)$ are on these levels. Since the vertices on levels 0 and 2 have degree 1 and the edges join vertices from different levels, the Reeb graph $\mathcal{R}(f)$ is a tree. However, by Lemma 2.1, any perturbation of f which is a simple Morse function has in its Reeb graph exactly $g \geq 1$ cycles.

We give another elementary proof of Lemma 2.1 (cf. [45, Lemma 3.2]). Let V and E be the sets of vertices and edges of $\mathcal{R}(f)$, respectively, and k_i be the number of critical points of f of index i.

Proof of Lemma 2.1. Since f is an $\mathcal{R}$-simple Morse function, $|V| = k_0 + k_1 + k_2$ and by Corollary 1.13 we have $\chi(\Sigma) = k_0 - k_1 + k_2$. By Euler's handshaking lemma $2|E| = \sum_{v \in V} \deg(v)$.

The index-degree correspondence (Proposition 1.17) yields $2|E| = k_0 + 3k_1 + k_2$ if $\Sigma = \Sigma_g$ is orientable. Then

$$\begin{aligned} 2\beta_1(\mathcal{R}(f)) &= 2\left(|E| - |V| + 1\right) \\ &= k_0 + k_2 + 3k_1 - 2\left(k_0 + k_1 + k_2\right) + 2 \\ &= 2 - \left(k_0 - k_1 + k_2\right) = 2 - \chi(\Sigma_g) = 2g. \end{aligned}$$

However, if $\Sigma = S_g$ is a non-orientable surface, Proposition 1.17 gives only $2|E| \leq k_0 + 3k_1 + k_2$ and so $2\beta_1(\mathcal{R}(f)) \leq 2 - \chi(S_g) = g$. □

The above simple calculations can be improved to obtain some form of the converse of Lemma 2.1, i.e. we would like to know under which conditions on $\mathcal{R}(f)$ a Morse function f is simple or $\mathcal{R}$-simple. By Proposition 1.17 the necessary condition is that the maximum degree of vertex in $\mathcal{R}(f)$ is not greater than 3. The following theorem gives a complete answer to this question. It states that for Morse functions on surfaces the property that critical points belong to different connected components of level sets is determined by combinatorial and homotopical features of their Reeb graphs. The result partially comes from [41, Lemma 4.12]. It will be useful later in the characterization of Reeb graphs and Reeb epimorphisms of simple Morse functions.

Theorem 2.3. *Let $f \colon \Sigma \to \mathbb{R}$ be a Morse function on a closed surface Σ such that $\Delta(\mathcal{R}(f)) \leq 3$.*

(1) If $\Sigma = \Sigma_g$, then f is $\mathcal{R}$-simple if and only if $\beta_1(\mathcal{R}(f)) = g$.

(2) If $\Sigma = S_g$, then f is $\mathcal{R}$-simple if and only if $g = 2\beta_1(\mathcal{R}(f)) + \Delta_2(\mathcal{R}(f))$.

Moreover, if $\Sigma = S_{2g}$ is non-orientable of even genus $2g$, then $\beta_1(\mathcal{R}(f)) = g$ if and only if f is $\mathcal{R}$-simple and $\Delta_2(\mathcal{R}(f)) = 0$.

Proof. As before, $\beta_1(\mathcal{R}(f)) = |E| - |V| + 1, \chi(\Sigma) = k_0 - k_1 + k_2$ and the number of vertices of degree 1 in $\mathcal{R}(f)$ is equal to $k_0 + k_2$. Let $\Delta_k := \Delta_k(\mathcal{R}(f))$. Note that $2|E| = \sum_{v \in V} \deg(v) = k_0 + k_2 + 2\Delta_2 + 3\Delta_3$ by Euler's handshaking lemma. Combining these equalities we obtain

$$2|E| - 2|V| + \chi(\Sigma) = \Delta_3 - k_1.$$

Assume that $\Sigma = \Sigma_g$ or $\Sigma = S_{2g}$. Then $\chi(\Sigma) = 2 - 2g$ and

$$2\left(\beta_1(\mathcal{R}(f)) - g\right) = 2|E| - 2|V| + \chi(\Sigma) = \Delta_3 - k_1.$$

Thus $\beta_1(\mathcal{R}(f) = g$ if and only if $k_1 = \Delta_3$. However, $k_1 \geq \Delta_2 + \Delta_3$, so $k_1 = \Delta_3$ if and only if $\Delta_2 = 0$ and each vertex of $\mathcal{R}(f)$ corresponds to a single critical point of f, so f is $\mathcal{R}$-simple. In the orientable case $\mathcal{R}$-simplicity alone implies $\beta_1(\mathcal{R}(f)) = g$ since $\Delta_2 = 0$ by by Proposition 1.17.

Now, let $\Sigma = S_g$. Then

$$2\beta_1(\mathcal{R}(f)) - g + \Delta_2 = 2|E| - 2|V| + \chi(\Sigma) + \Delta_2 = \Delta_3 + \Delta_2 - k_1$$

Therefore $g = 2\beta_1(\Gamma) + \Delta_2$ if and only if $k_1 = \Delta_2 + \Delta_3$, what means that f is $\mathcal{R}$-simple. □

Remark 2.4. Note that an $\mathcal{R}$-simple Morse function on a non-orientable surface of odd genus has always a vertex of degree 2 in its Reeb graph.

A similar calculations can be done for higher-dimensional manifold, but they do not provide such interesting results.

Lemma 2.5. *Let $f\colon M \to \mathbb{R}$ be an $\mathcal{R}$-simple Morse function on an n-dimensional manifold M, $n \geq 2$, and k_i be the number of critical points of index i. Then*

$$\beta_1(\mathcal{R}(f)) = -\frac{k_0 + k_n}{2} + \frac{\Delta_3(\mathcal{R}(f))}{2} + 1.$$

Furthermore, if we denote by Δ_3^{in} (by Δ_3^{out}) the number of vertices of $\mathcal{R}(f)$ with indegree 2 (outdegree 2), then

$$\Delta_3^{\text{in}} - k_0 + 1 = \beta_1(\mathcal{R}(f)) = \Delta_3^{\text{out}} - k_n + 1$$

Proof. The desired formula is derived from equalities $|V| = \sum_i k_i$, $2|E| = \sum_{v \in V} \deg(v)$ and $\beta_1(\mathcal{R}(f)) = |E| - |V| + 1$ as before. The second part is just a careful investigation of graphs with vertices of degrees 1, 2 and 3. □

2.2 Maximizing the cycle rank – the Reeb number

In this section we indicate the problem of maximizing the cycle rank in Reeb graphs. We show that the maximum, called the Reeb number of a manifold, is attained by simple Morse functions.

Definition 2.6. The **Reeb number** $\mathcal{R}(W, W_-, W_+)$ of a smooth triad (W, W_-, W_+) is the maximum cycle rank among all Reeb graphs of smooth functions on (W, W_-, W_+) with finitely many critical points.

For a closed manifold M we simply write $\mathcal{R}(M)$ for its Reeb number.

As we mentioned in Section 1.4, Theorem 1.21 implies that $\beta_1(\mathcal{R}(f)) \leq \beta_1(W)$ for every smooth function $f\colon W \to \mathbb{R}$ with finitely many critical points, hence $\mathcal{R}(W, W_-, W_+)$ is well-defined and it is bounded from above by $\beta_1(W)$.

We will show that Reeb graphs of simple Morse functions maximize the cycle rank.

Lemma 2.7. *Let $f\colon W \to \mathbb{R}$ be a function with finitely many critical points on a triad (W, W_-, W_+). Then there exists a simple Morse function $g\colon W \to \mathbb{R}$ on (W, W_-, W_+) such that $\beta_1(\mathcal{R}(g)) \geq \beta_1(\mathcal{R}(f))$.*

Proof. Let v be a vertex of $\mathcal{R}(f)$, $c := \overline{f}(v)$ and take a triad (Q^v, Q^v_-, Q^v_+) associated with v.

By Theorem 1.10 there exists a simple Morse function $h\colon Q^v \to [c - \varepsilon, c + \varepsilon]$ on the smooth triad (Q^v, Q^v_-, Q^v_+) which has critical values different from critical values of $f|_{W \setminus Q^v}$.

Let us define the function $g\colon W \to \mathbb{R}$ by

$$g(x) = \begin{cases} h(x) & \text{if } x \in Q^v, \\ f(x) & \text{if } x \in W \setminus \operatorname{Int} Q^v. \end{cases}$$

Clearly, g is smooth, with the same critical points as f and h. Since $V := q_f(Q^v)$ is contractible as a small neighbourhood of a vertex, we have homotopy equivalence $\mathcal{R}(f) \simeq \mathcal{R}(f)/V$, which is in turn equivalent to $\mathcal{R}(g)/\mathcal{R}(h)$. Let T be a spanning tree of the graph $\mathcal{R}(h)$. It is easily seen that $\mathcal{R}(g)/T$ is homeomorphic to $\mathcal{R}(g)/\mathcal{R}(h)$ after adding the remaining edges from $\mathcal{R}(h)/T$. Hence

$$\beta_1(\mathcal{R}(g)) = \beta_1(\mathcal{R}(g)/T) \geq \beta_1(\mathcal{R}(g)/\mathcal{R}(h)) = \beta_1(\mathcal{R}(f)).$$

After performing the procedure as described above for each vertex v of $\mathcal{R}(f)$ we obtain the desired simple Morse function. □

Remark 2.8. For a Morse function f, the simple Morse function g in the above lemma can be obtained without changing the critical points of f and their indices. Moreover, g may differ from f only in small neighbourhoods of the critical points as in Theorem 1.10 (cf. [48, Lemma 2.8.]).

Corollary 2.9. *The Reeb number of a triad is attained by simple Morse functions, i.e.*

$$\mathcal{R}(W, W_-, W_+) = \max \{ \beta_1(\mathcal{R}(f)) \mid f\colon W \to \mathbb{R} \textit{ is a simple Morse function on } (W, W_-, W_+) \}.$$

By the above corollary and Lemma 2.1 we conclude that $\mathcal{R}(\Sigma_g) = g$ and $\mathcal{R}(S_g) \leq \lfloor \frac{g}{2} \rfloor$. To obtain equality in the case of non-orientable surfaces it suffices to provide an explicit example of a function which realizes maximum cycle rank. It is easy to construct a simple Morse function on a Klein bottle $K = S_2$ whose Reeb graph has cycle rank equal to 1. It may come from the handle decomposition of K with four handles. Take k copies of Klein bottle and $g - 2k$ projective planes with any simple Morse function and perform the connected sum operation near extrema of functions, obtaining S_g. The function which is the piecewise extension of the functions has Reeb graph with cycle rank equal to k (cf. also [8] and Theorem 2.20). Taking $k = \lfloor \frac{g}{2} \rfloor$ we get the maximum cycle rank.

Corollary 2.10. $\mathcal{R}(\Sigma_g) = g$ *and* $\mathcal{R}(S_g) = \lfloor \frac{g}{2} \rfloor$. *In other words, if a closed surface* Σ *has the Euler characteristic* $\chi(\Sigma) = 2 - k$, *then* $\mathcal{R}(\Sigma) = \lfloor \frac{k}{2} \rfloor$. □

The upper bound $\mathcal{R}(\Sigma_g) \leq g$ can also be found in [27, Theorem 5.6.], where the proof relies on the maximal number of non-separating circle embeddings in an orientable surface.

2.3 Construction of a manifold with function

Sharko [61, Theorem 2.1.] proved that for a graph Γ with good orientation there exist a surface Σ and a function $f\colon \Sigma \to \mathbb{R}$ with finitely many critical points such that $\mathcal{R}(f)$ is isomorphic to Γ. In general, f may have degenerate critical points, so it is not a Morse function. Sharko stated (without a proof) that methods of Takens in [65] lead to a version of this theorem for n-dimensional manifolds. We will give an explicit construction, in terms of handle decomposition, of such a manifold M of an arbitrary given dimension $n \geq 2$ and a Morse function on M such that its Reeb graph is isomorphic to Γ.

Lemma 2.11. *Let $n \geq 2$, $k_-, k_+ \geq 1$ and $t \geq 0$ (k_- or $k_+ \geq 2$ for $t = 0$) be integers and let $c \in (a, b)$. Then there exist a connected and compact n-manifold W with boundary $\partial W = W_- \sqcup W_+$, where $W_\pm := \bigsqcup_{i=1}^{k_\pm} \mathrm{S}^{n-1}$ is the disjoint union of $(n-1)$-dimensional spheres with the standard smooth structure, and a Morse function $f\colon W \to [a, b]$ on the smooth triad (W, W_-, W_+) with only one critical value c and $(k_+ + t - 1)$ critical points of index $(n-1)$ and $(k_- + t - 1)$ critical points of index 1.*

Recall that for an n-manifold W an embedding $\varphi\colon\ \mathrm{S}^{k-1} \times \mathrm{D}^{n-k} \to \partial W$ is **non-trivial**, if $[\varphi|_{\mathrm{S}^{k-1} \times \{0\}}] \in \pi_{k-1}(\partial W)$ is a non-trivial element of the $(k-1)$th homotopy group of ∂W (taking basepoints as appropriate). In the case $k = 1$ the non-triviality means that the image of φ belongs to separate connected components of ∂W.

We will use the correspondence between attaching k-handles and critical points of index k (Theorems 1.7 and 1.8). From [28, Lemma 2.1.] we know that the sphere S^{n-1} with the standard smooth structure serves as identity element of the connected sum operation. As the result of attaching a 1-handle along a non-trivial embedding $\mathrm{S}^0 \times \mathrm{D}^{n-1} \to \partial W \cong \mathrm{S}^{n-1} \sqcup \mathrm{S}^{n-1} \sqcup P$ with the image contained in $\mathrm{S}^{n-1} \sqcup \mathrm{S}^{n-1}$ we obtain a new manifold W' with $\partial W' \cong (\mathrm{S}^{n-1} \# \mathrm{S}^{n-1}) \sqcup P \cong \mathrm{S}^{n-1} \sqcup P$. The operation thus reduces the number of spheres in the boundary of W. The dual operation to the above (attachment of $(n-1)$-handle via trivial embedding) increases the number of connected components — one copy of S^{n-1} splits into two copies of S^{n-1}.

Proof. Let $W_\pm$ be as in the statement. We start with $W_- \times [a, a+\varepsilon]$ and attach $(k_+ + t - 1)$ $(n-1)$-handles along trivial embeddings with the correct orientation to $W_- \times \{1\}$, as described above. This produces a manifold W' with $\partial W' = W_- \sqcup P$, where P is the disjoint union of $(k_- + k_+ + t - 1)$ copies of S^{n-1}. On $W_- \times [a, a+\varepsilon]$ the projection on the second factor is a Morse function without critical points, so attaching the handles corresponds (by Theorem 1.8) to a Morse function $h\colon W' \to [a, a+2\varepsilon]$ with $(k_+ + t - 1)$ critical points all of index $(n-1)$.

Next we attach $(k_- + t - 1)$ 1-handles to W' along non-trivial embeddings to P, linking different components of W' (the manifold W' has k_- connected components), obtaining a connected manifold W with boundary $\partial W = W_- \sqcup W_+$. Hence there exists a Morse function $h'\colon W \to [a, b]$ with $(k_+ + t - 1)$ critical points of index $(n-1)$ and $(k_- + t - 1)$ critical points of index 1.

Since we attached $(n-1)$-handles before 1-handles, the critical points of the function h' of index $(n-1)$ are below the critical points of index 1. Therefore we can apply the method of Theorem 1.14 to obtain a new Morse function $f\colon W \to [a, b]$ on the smooth triad (W, W_-, W_+) with all critical points moved to one critical level c. □

Remark 2.12. For $n = 2$ the produced surface W in the above lemma is clearly orientable. In the same way we can obtain a non-orientable surface W (for $n = 2$ and $t = 0$) with Morse function with unique critical value and $k_- + k_+ + r - 2$ critical points of index 1, by attaching r additional 1-handles along trivial embeddings with non-compatible orientations.

Theorem 2.13. *Let Γ be a finite graph with good orientation and $n \geq 2$. Then there exist a smooth closed n-manifold M and a Morse function $f\colon M \to \mathbb{R}$ such that the Reeb graph $\mathcal{R}(f)$ is isomorphic to Γ.*

Proof. Let $g\colon \Gamma \to \mathbb{R}$ be a function inducing the good orientation of Γ. Since Γ is finite we can define the positive real number

$$\varepsilon := \frac{1}{3} \min \{|g(v) - g(w)| : v \text{ and } w \text{ are ends of an edge of } \Gamma\}.$$

Let v be a vertex of Γ of degree one. Set $Q^v := \mathrm{D}^n$ and define $f_v\colon Q^v \to [g(v)-\varepsilon, g(v)+\varepsilon]$ by $f_v(x_1,\ldots,x_n) = g(v) \pm \varepsilon(x_1^2 + \ldots + x_n^2)$, where sign $\pm$ depends whether v is minimum or maximum of g. Obviously, f_v has one critical point in the centre of the disc.

The function g has extrema only in vertices of degree 1, hence for each vertex v of degree at least two we have inequalities $k_- := \deg_{in}(v) \geq 1$ and $k_+ := \deg_{out}(v) \geq 1$. By Lemma 2.11 (for $t = 1$) we obtain an n-manifold Q^v with boundary $\partial Q^v = Q^v_- \sqcup Q^v_+$, where $Q^v_\pm := \bigsqcup_{i=1}^{k_\pm} \mathrm{S}^{n-1}$ and a Morse function $f_v\colon Q^v \to [g(v)-\varepsilon, g(v)+\varepsilon]$ such that $f_v^{-1}(g(v)\pm\varepsilon) = Q^v_\pm$ and the only critical value of f_v is $g(v)$ (it has $k_- + k_+ + 2t - 2 \geq 2$ critical points).

In both the cases, since f_v has one critical level and $\mathcal{R}(f_v)$ is connected (as the image of the connected manifold Q^v), it has one internal vertex corresponding to one critical value of f_v and has $k_\pm$ ends of edges on the level $g(v) \pm \varepsilon$. Therefore $\mathcal{R}(f_v)$ is homeomorphic to a small neighbourhood of v in Γ.

For each oriented edge e in Γ from a vertex v to w we form the manifold $Q^e = \mathrm{S}^{n-1} \times [g(v)+\varepsilon, g(w)-\varepsilon]$ and the Morse function $f_e\colon Q_e \to [g(v)+\varepsilon, g(w)-\varepsilon]$, by the projection on the second factor.

We glue smoothly the manifolds Q^v and Q^e using a diffeomorphism between (the corresponding to e) component S^{n-1} of Q^v_+ and $\mathrm{S}^{n-1} \times \{g(v)+\varepsilon\}$. Similarly, we glue Q^w and Q^e. On the resulting smooth manifold we define a function which is the piecewise extension of f_v, f_e and f_w and has the same critical points.

Performing the operation as described above for each edge of Γ we obtain a smooth closed n-manifold M and a Morse function $f\colon M \to \mathbb{R}$. It is clear from the construction that the Reeb graph $\mathcal{R}(f)$ is isomorphic to Γ. ☐

2.4 Reeb graphs of functions on surfaces

In this section we characterize Reeb graphs of various classes of functions on surfaces.

Lemma 2.14. *For $n = 2$ the surface W obtained in Lemma 2.11 has the Euler characteristic $\chi(W) = 2 - (k_- + k_+ + 2t)$.*

Proof. Recall that W is formed from $W_- \times [0,1] = \bigsqcup_{i=1}^{k_-} \mathrm{S}^1 \times [0,1]$, $(k_+ + t - 1)$ handles of index $n-1 = 1$ and $(k_- + t - 1)$ handles of index 1, i.e. $r := k_- + k_+ + 2t - 2$ handles of index 1 in total. We have $\chi(W_- \times [0,1]) = 0$, $\chi(\mathrm{D}^1 \times \mathrm{D}^1) = 1$ and $\chi(\mathrm{S}^0 \times \mathrm{D}^1) = 2$. Since the 1-handles $\mathrm{D}^1 \times \mathrm{D}^1$ are attached along $\mathrm{S}^0 \times \mathrm{D}^1$ we obtain

$$\chi(W) = \chi(W_- \times [0,1]) + r \cdot \chi(\mathrm{D}^1 \times \mathrm{D}^1) - r \cdot \chi(\mathrm{S}^0 \times \mathrm{D}^1) = -r = 2 - (k_- + k_+ + 2t).$$

☐

We need to modify the construction of the surface in the proof of Theorem 2.13 (for $n = 2$) to obtain desirable properties. The following result can also be found in [42, 61], but the constructions are different in parts.

Corollary 2.15. *Let $\Gamma = \Gamma(V,E)$ be a graph with good orientation and let $g := \beta_1(\Gamma)$. Then there exist a closed surface Σ and a function $f\colon \Sigma \to \mathbb{R}$ with finitely many critical points such that $\mathcal{R}(f)$ is isomorphic to Γ and Σ can be taken to be either orientable of genus g or non-orientable of genus $2g$ (for $g \geq 1$). If Γ is a tree, then Σ is diffeomorphic to S^2.*

Proof. First, in the construction in the proof of Theorem 2.13 (for $n = 2$) we change the surfaces Q^v and functions f_v by using $t = 0$ in Lemma 2.11. Moreover, for each vertex w of Γ such that $\deg_{in}(w) = \deg_{out}(w) = 1$ let Q^w be $S^1 \times [-\varepsilon, \varepsilon]$. We take a function $f_w\colon Q^w \to [g(w) - \varepsilon, g(w) + \varepsilon]$ on the triad $(Q^w, S^1 \times \{-\varepsilon\}, S^1 \times \{\varepsilon\})$ with one (degenerate) critical point, which in local coordinates can be written as $f_w(x, y) = -x^2 + y^3 + g(w)$. Its Reeb graph is homeomorphic to a small neighbourhood of w in Γ. The detailed construction of f_w can be found in [42, Theorem 2.1. Case (b)]. Let Σ be the resulting closed surface.

By the previous lemma each surface Q^v has the Euler characteristic $\chi(Q^v) = 2-(\deg_{in}(v)+\deg_{out}(v)) = 2 - \deg(v)$ and each Q^e has $\chi(N_e) = 0$, where $v \in V$ and $e \in E$. To obtain Σ we attach Q^e along copies of S^1, hence

$$\begin{aligned}\chi(\Sigma) &= \sum_{v\in V} \chi(Q^v) = \sum_{v\in V} (2 - \deg(v)) = 2|V| - \sum_{v\in V} \deg(v) \\ &= 2|V| - 2|E| = 2 - 2\,(|E| - |V| + 1) = 2 - 2g,\end{aligned}$$

where $g = \beta_1(\Gamma)$.

If Γ is a tree, then $g = 0$, so $\chi(\Sigma) = 2$ and, in consequence, $\Sigma \cong S^2$. For $g \geq 1$ it is clear that the constructed surface in Lemma 2.11 (for $n = 2$) is orientable, so Σ can be orientable or not, what depends on the way of attaching Q^e for edges e outside a spanning tree of Γ. Therefore the corollary follows from the classification of closed surfaces. □

Before we present realization theorems, let us consider the case of compact surfaces with boundary. It is convenient to distinguish the following natural relation between graphs and smooth triads.

Definition 2.16. A finite oriented graph Γ is **admissible** for a smooth triad (W, W_-, W_+) if among its vertices there are at least $|\pi_0(W_-)|$ minima, and at least $|\pi_0(W_+)|$ maxima.

Note that by the definition if $f\colon W \to \mathbb{R}$ is a smooth function with isolated critical points on (W, W_-, W_+), then $\mathcal{R}(f)$ is admissible for (W, W_-, W_+). Clearly, it is a necessary condition for a graph to be realized as the Reeb graph of function on (W, W_-, W_+). For a closed manifold M any Γ is admissible. It turns out that in the reasonable class of smooth function, realization theorems for closed surfaces are also true for compact surfaces with boundary if we add the assumption on admissibility of a graph for a given smooth triad. This fact can be presented in the form of the following proposition.

Proposition 2.17. *Let Γ be a finite oriented graph admissible for a smooth triad (W, W_-, W_+), where W is a compact surface. Denote by Σ a closed surface obtained by attaching discs to all boundary components of W. Then there exists a function with finitely many critical points (Morse function or simple Morse function) on (W, W_-, W_+) whose Reeb graph is isomorphic to Γ if and only if such a function exists on Σ.*

Proof. It is clear that the function on (W, W_-, W_+) can be extended on Σ in such a way that vertices corresponding to ∂W will correspond to the centres of the discs attached to W, so it preserves the Reeb graph. Similarly, for $f\colon \Sigma_g \to \mathbb{R}$ consider minima $v_1, \ldots, v_{k_-}$ and maxima $w_1, \ldots, w_{k_+}$ of $\mathcal{R}(f)$, where $k_\pm = |\pi_0(W_\pm)|$. The triads Q^{v_i} and Q^{w_i} associated with these vertices are discs. It is clear that there is a diffeomorphism

$$h\colon W \to \Sigma \setminus \left(\bigcup_{i=1}^{k_-} \operatorname{Int} Q^{v_i} \cup \bigcup_{i=1}^{k_+} \operatorname{Int} Q^{w_i} \right).$$

By homogeneity we may assume that it maps components of W_- to boundaries of Q^{v_i} and components of W_+ to boundaries of Q^{w_i} Therefore $f \circ h \colon W \to \mathbb{R}$ is a desired function on (W, W_-, W_+), since h induces isomorphism of Reeb graphs of $f \circ h$ and $f|_{\operatorname{Im} h}$. □

Thus the Reeb number of a compact surface with boundary depends only on its orientability and genus, regardless of particular smooth triad. Thus we may write $\mathcal{R}(\Sigma)$ for a compact surface Σ instead of $\mathcal{R}(\Sigma, \Sigma_-, \Sigma_+)$. We will see in Chapter 4 that the Reeb number does not depend on smooth triad also for higher-dimensional manifolds.

Corollary 2.18. *Let $\Sigma_{g,h}$ and $S_{g,h}$ denote, respectively, an orientable and non-orientable surface of genus g with h open discs removed. Then $\mathcal{R}(\Sigma_{g,h}) = g$ and $\mathcal{R}(S_{g,h}) = \lfloor \frac{g}{2} \rfloor$.*

Remark 2.19. Consider the complete graph K_2 on two vertices. If $f \colon M \to \mathbb{R}$ is a function with finitely many critical points on a closed manifold M such that $\mathcal{R}(f)$ is isomorphic to K_2, then f has only two critical points. Reeb Theorem [47, Theorem 1'] asserts that M is homeomorphic to the n-dimensional sphere S^n.

It turns out that K_2 is the only graph satisfying natural necessary conditions (stated in the following theorem) which does not arise as the Reeb graph for surfaces other than the sphere.

Theorem 2.20. *Let $\Gamma \neq K_2$ be a finite oriented graph admissible for a triad $(\Sigma, \Sigma_-, \Sigma_+)$, where Σ is a compact surface. Then there exists a function $f \colon \Sigma \to \mathbb{R}$ on $(\Sigma, \Sigma_-, \Sigma_+)$ with finitely many critical points such that $\mathcal{R}(f)$ is isomorphic to Γ if and only if Γ has a good orientation and $\beta_1(\Gamma) \leq \mathcal{R}(\Sigma)$. If $\Gamma = K_2$, then it can be realized only for S^2, D^2 or $\mathrm{S}^1 \times [0,1]$.*

Proof. By Proposition 2.17 we may assume that Σ is closed. The only if part is straightforward. For the reverse implication let $k := \beta_1(\Gamma) \leq \mathcal{R}(\Sigma)$. If $\Sigma = \Sigma_g$ and $g = k$ or if $\Sigma = S_g$ and $g = 2k$, then the statement follows from Corollary 2.15.

If $\Sigma = \Sigma_g$ and $g > k$, we need to change one surface Q^v at vertex v of degree at least 2 in the proof of Corollary 2.15. Such a vertex exists, if we assume that $\Gamma \neq K_2$. In the construction of Q^v in Lemma 2.11 let us set $t = g - k \geq 1$ and let M be the resulting orientable surface with a function f with finitely many critical points such that $\mathcal{R}(f) \cong \Gamma$. Then from calculations as in Corollary 2.15 we have $\chi(M) = 2 - 2k - 2t = 2 - 2(k + t) = 2 - 2g$, so $M \cong \Sigma_g$.

For the case when $\Sigma = S_g$ is a non-orientable surface and $g > 2k$ we also need to change only one manifold Q^v at vertex v of degree at least two by setting $r = g - 2k \geq 1$ in Remark 2.12. We obtain a non-orientable surface M with a function f with finitely many critical points such that $\mathcal{R}(f) \cong \Gamma$. As in Corollaries 2.14 and 2.15 we get $\chi(M) = 2 - 2k - r = 2 - g$, so $M \cong S_g$.

If $\Gamma = K_2 = \mathcal{R}(f)$ and Σ is closed, then $\Sigma \cong \mathrm{S}^2$ by Reeb Theorem, and the Reeb graph of the height function on S^2 is K_2. Therefore if Σ has boundary, then $\mathcal{R}(f) = K_2$ implies that Σ has a one boundary component and $\chi(\Sigma) = 1$ (so Σ is a disc), or it has two boundary components and $\chi(\Sigma) = 0$ (so Σ is an annulus). □

Remark 2.21. The degenerate critical points of the function f on Σ in the above theorem come from the vertices of degree 2 in Γ.

Corollary 2.22. *Let Γ be a finite oriented graph admissible for a triad $(\Sigma, \Sigma_-, \Sigma_+)$, where Σ is a compact surface. Then there exists a Morse function $f \colon \Sigma \to \mathbb{R}$ on $(\Sigma, \Sigma_-, \Sigma_+)$ with finitely many critical points such that $\mathcal{R}(f)$ is orientation-preserving homeomorphic to Γ if and only if Γ has a good orientation and $\beta_1(\Gamma) \leq \mathcal{R}(\Sigma)$.* □

We can also describe isomorphism classes of Reeb graphs of Morse functions on surfaces.

Theorem 2.23. *Let $\Gamma \neq K_2$ be a finite oriented graph and let Σ be a compact surface of genus g (orientable or not). Assume that Γ is admissible for $(\Sigma, \Sigma_-, \Sigma_+)$. Then there exists a Morse function $f\colon \Sigma \to \mathbb{R}$ such that $\mathcal{R}(f)$ is isomorphic to Γ if and only if Γ has a good orientation and*

- $g \geq \beta_1(\Gamma) + \Delta_2(\Gamma)$, *when Σ is orientable,*
- $g \geq 2\beta_1(\Gamma) + \Delta_2(\Gamma)$, *when Σ is non-orientable.*

Proof. In the view of Proposition 2.17 there is no loss of generality in assuming that Σ is closed. Let f be as in the statement. Divide Σ into surfaces Q^v which form smooth triads associated to vertices $v \in V$ in $\Gamma = \mathcal{R}(f)$ and into surfaces $Q^e \cong \mathrm{S}^1 \times [0,1]$ corresponding to edges e in Γ. As in Corollary 2.15 we have $\chi(\Sigma) = \sum_{v \in V} \chi(Q^v)$. Obviously, $\chi(Q^v) \leq 2 - \deg(v)$, where the equality holds when Q^v is the sphere with $\deg(v)$ open discs removed. We show that it is not the case for vertices of degree 2.

Let w be a vertex of degree 2 in Γ and suppose Q^w is the sphere with two holes. The function $f_w = f|_{Q^w}$ is a Morse function with at least one critical point of index 1. This leads (after attaching two discs (0 and 2 handles) to Q^w) to a Morse function on the sphere with $k_1 \geq 1$ critical points of index 1 and exactly two extrema. Thus $2 = \chi(\mathrm{S}^2) = 2 - k_1 \leq 1$, a contradiction. Therefore $\chi(Q^w) \leq 2 - \deg(w) - 1 = -1$.

Let $\Delta_2 := \Delta_2(\Gamma)$. If Σ is non-orientable, then just as in Corollary 2.15 we have

$$2 - g = \sum_{v \in V} \chi(Q^v) \leq \sum_{v \in V} (2 - \deg(v)) - \Delta_2 = 2 - 2\beta_1(\Gamma) - \Delta_2,$$

hence $g \geq 2\beta_1(\Gamma) + \Delta_2$. If Σ is orientable, then so is Q^w and $\chi(Q^w) \leq -2$, since the Euler characteristic of a compact orientable surface with two boundary components is even. Hence $2 - 2g \leq \sum_{v \in V}(2 - \deg(v)) - 2\Delta_2 = 2 - 2\beta_1(\Gamma) - 2\Delta_2$, so $g \geq \beta_1(\Gamma) + \Delta_2$.

For the reverse implication, if $\Delta_2 = 0$, then $\mathcal{R}(\Sigma) \geq \beta_1(\Gamma)$ and the realization by a Morse function follows from Corollary 2.22. Assume that $\Delta_2 \geq 1$. We need to change in Corollary 2.15 the surfaces Q^w for which $\deg(w) = 2$.

If Σ is orientable, then we take Q^w to be from Lemma 2.11 for $n = 2$, $k_\pm = 1$ and $t = 1$ for all such vertices w except one, for which we set $t = t_0 := g - \beta_1(\Gamma) - \Delta_2 + 1 \geq 1$. Since $\chi(Q^w) = 2 - \deg(w) - 2t$ by Corollary 2.14, the resulting closed orientable surface M has $\chi(M) = 2 - 2\beta_1(\Gamma) - 2(\Delta_2 - 1) - 2t_0 = 2 - 2g$, so $M \cong \Sigma$.

Similarly, when Σ is non-orientable, we change Q^w by a surface from Remark 2.12 for $r = 1$, but for one such a vertex w we set $r = r_0 := g - 2\beta_1(\Gamma) - \Delta_2 + 1 \geq 1$. Then $\chi(Q^w) = 2 - \deg(w) - r$. In the result we obtain a closed non-orientable surface M with $\chi(M) = 2 - 2\beta_1(\Gamma) - (\Delta_2 - 1) - r_0 = 2 - g$, so $M \cong \Sigma$. □

The next step is to provide a characterization of Reeb graphs of simple Morse functions.

Theorem 2.24. *Let Γ be a finite oriented graph which is admissible for $(\Sigma, \Sigma_-, \Sigma_+)$, where Σ is a compact surface of genus g. Then there exists a simple Morse function $f\colon \Sigma \to \mathbb{R}$ such that $\mathcal{R}(f)$ is isomorphic to Γ if and only if Γ has a good orientation, $\Delta(\Gamma) \leq 3$ and*

- $g = \beta_1(\Gamma)$ *and* $\Delta_2 = 0$, *when Σ is orientable,*
- $g = 2\beta_1(\Gamma) + \Delta_2$, *when Σ is non-orientable.*

Proof. Again, we may assume that Σ is closed. Note that in both the cases: Σ is orientable, $g = \beta_1(\Gamma)$ and $\Delta_2 = 0$, or Σ is non-orientable and $g = 2\beta_1(\Gamma) + \Delta_2$, the previous theorem provides the existence of a Morse function f on Σ such that $\mathcal{R}(f) \cong \Gamma$. If $\Delta(\Gamma) \leq 3$, then Theorem 2.3 provides that f is $\mathcal{R}$-simple, and by Remark 1.9 it can be perturbed to a simple Morse function without changing isomorphism type of Reeb graph.

Conversely, if f is a simple Morse function with $\mathcal{R}(f)$ isomorphic to Γ, then Proposition 1.17 yields the bound on degree of vertices of Γ. Moreover, if Σ is orientable, then $\Delta_2 = 0$. The desired equalities follow again from Theorem 2.3. □

Remark 2.25. Martinez-Alfaro et al. [40] showed a realization theorem for simple Morse–Bott functions on orientable surfaces.

Remark 2.26. These constructions can be extended to graphs with a (not necessarily good) orientation, but without oriented cycles. Such a graph Γ may contain an extremum vertex v of degree at least two such that $\deg_{in}(v) = 0$ ($\deg_{out}(v) = 0$ respectively). For such a vertex Masumoto–Saeki in [42, Theorem 2.1. Case (c)] construct a surface Q^v and a function $f_v \colon Q^v \to \mathbb{R}$ such that

- $\chi(Q^v) = 2 - \deg(v)$,
- f_v has infinitely many critical points, but only one critical value a and $f_v^{-1}(a)$ is connected,
- the range of f_v is $[a, a+\varepsilon]$ ($[a-\varepsilon, a]$, resp.),
- $f_v^{-1}(a+\varepsilon)$ ($f_v^{-1}(a-\varepsilon)$, resp.) is the disjoint union of $\deg(v)$ copies of S^1,
- $\mathcal{R}(f_v)$ is homeomorphic to a small neighbourhood of v in Γ.

Proposition 2.27. *Let Σ be a closed surface and Γ be an oriented graph without oriented cycles such that $\beta_1(\Gamma) \leq \mathcal{R}(\Sigma)$. Then there exists $f \colon \Sigma \to \mathbb{R}$ with finitely many critical values such that $\mathcal{R}(f)$ is orientation-preserving homeomorphic to Γ. Moreover, if Γ has a vertex w such that both $\deg_{in}(w)$ and $\deg_{out}(w)$ are nonzero, then $\mathcal{R}(f)$ and Γ are in fact isomorphic.*

Proof. Adding a new vertex in the centre of an edge of Γ gives the homeomorphic graph, so we may assume that Γ contains a vertex w such that both $\deg_{in}(w)$ and $\deg_{out}(w)$ are nonzero. The orientation of Γ can be induced by a continuous function which is strictly monotonic on the edges (cf. [61, Theorem 3.1.]). Thus the construction is the same as in the proof of Theorem 2.20 with the exception of extrema vertices v of degree at least two for which we use the surface and function from Remark 2.26. □

Remark 2.28. For more results about realizing graphs as Reeb graphs of functions with finitely many critical values we refer to [16, 18, 40, 42, 58].

Chapter 3

Systems of hypersurfaces and epimorphisms onto free groups

In this chapter we develop the theory of systems of hypersurfaces in a manifold. By an extended Pontryagin–Thom construction any such a system induces a homomorphism from the fundamental group of the manifold to a free group. The notion of framed cobordism of systems of hypersurfaces in a closed manifold M leads to the bijection between the set $\mathrm{Hom}(\pi_1(M), F_r)$ of all homomorphisms $\pi_1(M) \to F_r$ and the set of all framed cobordism classes of systems of hypersurfaces in M of size r. We are focused on independent systems non-separating the manifold, which induces epimorphisms onto free groups. As we have seen in Section 1.4 these objects are closely related to Reeb graphs, and the results obtained in this chapter will be used later in the settle of realization problems.

In Section 3.1 we present basic definitions and facts concerned with systems of hypersurfaces. Section 3.2 establishes the crucial Theorem 3.11 that any epimorphism onto a free group can be induced by a regular and independent system. In Section 3.3 we show basic facts about corank. Next, in Section 3.4 we give an exposition of the equivalence and strong equivalence relations between homomorphisms, which provide methods of describing the structure of sets $\mathrm{Hom}(G, F_r)$ and $\mathrm{Epi}(G, F_r)$ for a group G. In the next two sections we show how to obtain the correspondence between these notions and framed cobordism of regular and independent systems of hypersurfaces. Section 3.5 provides a full calculation of these cobordism classes up to diffeomorphism in the case of surfaces. In Section 3.6 we introduce operations on systems of hypersurfaces, which are analogues of elementary Nielsen transformations for a free group. They allow us to study equivalence of epimorphisms in terms of systems of hypersurfaces.

3.1 Systems of hypersurfaces

We use the following model of F_r, the free group on r generators. Consider the circle S^1 as the quotient $[-1, 1]/\{-1, 1\}$ and take $F_r := \pi_1(\bigvee_{i=1}^{r} \mathrm{S}^1_i)$ as the fundamental group of the wedge product of r copies of circle. We use the convention that $\bigvee_{i=1}^{0} \mathrm{S}^1_i = \mathrm{pt}$, thus $F_0 = 1$ is the trivial group.

We will omit basepoints from the notation.

Let W be a compact manifold. A submanifold N of W is called **proper** if $N \cap \partial W = \partial N$. A **framing** of a submanifold N in W is a smooth function ν which assigns to each $x \in N$ a basis of the normal bundle of N at the point x. The pair (N, ν) is called a **framed submanifold**. If N is of codimension 1, then its framing is just a nonzero section of the normal bundle of N. Thus N has a closed product neighbourhood $P(N) \cong N \times [-1, 1]$ and it is called **2-sided**. We assume that $P(N)$ is compatible with the framing. Denote by $P_t(N)$ the submanifold corresponding to $N \times \{t\}$. The positive side of N containing $P_t(N)$ for $t \in (0, 1]$ agrees with the side determined by the framing.

Definition 3.1. A **system of hypersurfaces** in W is a tuple $\mathcal{N} = (N_1, \dots, N_r)$ of disjoint, proper, 2-sided submanifolds N_i together with their framings ν_i. The number r is called the **size** of the system $\mathcal{N}$.

Denote by
$$W|\mathcal{N} := W \setminus \bigcup_{i=1}^{r} \operatorname{Int} P(N_i)$$
the complement of the system $\mathcal{N}$ for sufficiently small product neighbourhoods of N_i's. It will cause no confusion if we use $\mathcal{N}$ to designate also $\bigcup_{i=1}^{r} N_i$, the sum of all submanifolds from the system. Of course, framings ν_i of submanifolds N_i form a framing ν of $\mathcal{N}$ such that $\nu|_{N_i} = \nu_i$. Unless it is necessary, we will not write a framing of a system explicitly.

A system $\mathcal{N}$ is called **independent** if $W|\mathcal{N}$ is connected, and it is called **regular** if each N_i is connected. The system $\mathcal{N}$ is **without boundary** if $\partial\mathcal{N} = \varnothing$. Note that we do not assume that submanifolds N_i are connected, unless $\mathcal{N}$ is regular.

Now we define the extended Pontryagin–Thom construction for a system of hypersurfaces.

Definition 3.2. The **homomorphism** $\varphi_{\mathcal{N}} \colon \pi_1(W) \to F_r$ **induced by a system** $\mathcal{N} = (N_1, \dots, N_r)$ omitting the basepoint is defined as follows. Fix product neighbourhoods $P(N_i) \cong N_i \times [-1, 1]$ which are disjoint. We define the map $f_{\mathcal{N}} \colon W \to \bigvee_{i=1}^{r} \mathrm{S}^1_i$ which maps $W|\mathcal{N}$ to the basepoint and each $P(N_i)$ onto i-th circle $\mathrm{S}^1_i = [-1, 1]/\{-1, 1\}$ by mapping $P_t(N_i)$ into t. It is clear that $f_{\mathcal{N}}$ is continuous, thus we put $\varphi_{\mathcal{N}} := (f_{\mathcal{N}})_\#$ to be the homomorphism induced by $f_{\mathcal{N}}$ on fundamental groups.

By the definition of a system of hypersurfaces $\varphi_{\mathcal{N}}$ is well-defined and it is clear that it does not depend on the choice of $P(N_i)$'s and a given framing, but only on the induced orientation of the normal bundle of $\mathcal{N}$.

Proposition 3.3. *Any homomorphism $\varphi \colon \pi_1(W) \to F_r$ is induced by a system of hypersurfaces. If a system $\mathcal{N}$ is independent, then $\varphi_{\mathcal{N}}$ is an epimorphism.*

Proof. Since $\bigvee_{i=1}^{r} \mathrm{S}^1_i$ is an Eilenberg–MacLane space $\mathrm{K}(F_r, 1)$, there is a map $f \colon W \to \bigvee_{i=1}^{r} \mathrm{S}^1_i$ such that $f_\# = \varphi$. Smooth it outside the inverse image of basepoint and take regular values $a_i \in \mathrm{S}^1_i$ of both f and $f|_{\partial W}$. Since W is compact, there is a neighbourhood $[a_i - \varepsilon, a_i + \varepsilon]$ consisting of regular values, and thus $N_i := f^{-1}(a_i)$ is a 2-sided, proper submanifold with product neighbourhood $f^{-1}([a_i - \varepsilon, a_i + \varepsilon]) \cong N_i \times [a_i - \varepsilon, a_i + \varepsilon]$. Take the map $h \colon \bigvee_{i=1}^{r} \mathrm{S}^1_i \to \bigvee_{i=1}^{r} \mathrm{S}^1_i$ which contracts
$$\bigvee_{i=1}^{r} \mathrm{S}^1_i \setminus \bigcup_{i=1}^{r} [a_i - \varepsilon, a_i + \varepsilon]$$
to the basepoint and maps linearly and orientation-preserving $[a_i - \varepsilon, a_i + \varepsilon]$ onto S^1_i. It is clear that $(h \circ f)_\# = \varphi$ is induced by $\mathcal{N} = (N_1, \dots, N_r)$ with framings compatible with the orientations of $[a_i - \varepsilon, a_i + \varepsilon]$.

If $\mathcal{N}$ is independent, then for any i there is a loop α_i in $(W|\mathcal{N}) \cup P(N_i)$ such that $f_{\mathcal{N}} \circ \alpha_i$ represents the generator of $\pi_1(\bigvee \mathrm{S}^1_i)$ corresponding to S^1_i. Thus $\varphi_{\mathcal{N}}$ is surjective. □

There is a quite easy characterization, using a special notion of framed cobordism, of systems in a closed manifold M which induce the same homomorphism to a free group.

Recall (cf. [49]) that submanifolds N and N' in M are **cobordant** if there exists a proper compact submanifold $W \subset M \times [0,1]$, called **cobordism** between N and N', such that $W \cap (M \times [0,\varepsilon]) = N \times [0,\varepsilon]$ and $W \cap (M \times [1-\varepsilon,1]) = N' \times [1-\varepsilon,1]$. Framed submanifolds (N,ν) and (N',ν') are **framed cobordant**, if there is a cobordism $W \subset M \times [0,1]$ between N and N' with a framing ϑ such that $\vartheta(x,t) = (\nu(x),0)$ for $(x,t) \in N \times [0,\varepsilon]$ and $\vartheta(x,t) = (\nu'(x),0)$ for $(x,t) \in N' \times [1-\varepsilon,1]$.

Definition 3.4. Let $\mathcal{N} = (N_1,\ldots,N_r)$ and $\mathcal{N}' = (N'_1,\ldots,N'_r)$ be two systems in M of the same size r. We say that $\mathcal{N}$ and $\mathcal{N}'$ are **framed cobordant** (as systems of hypersurfaces) if there are r disjoint framed cobordisms $W_i \subset M \times [0,1]$ between N_i and N'_i.

In other words, the systems $\mathcal{N}$ and $\mathcal{N}'$ are framed cobordant, if framed submanifolds $\mathcal{N}$ and $\mathcal{N}'$ are framed cobordant by the cobordism W which has a partition into r disjoint parts $W = W_1 \sqcup \ldots \sqcup W_r$ such that $\partial W_i = N_i \times \{0\} \sqcup N'_i \times \{1\}$. Clearly, it is an equivalence relation in the family of systems of hypersurfaces in M of size r. Note that the cobordisms W_i form the system $\mathcal{W} = (W_1,\ldots,W_r)$ of hypersurfaces in $M \times [0,1]$.

Note that the notion of framed cobordism between systems of hypersurfaces of size 1 is the same as an ordinary framed cobordism.

Proposition 3.5. *Systems $\mathcal{N}$ and $\mathcal{N}'$ of hypersurfaces in M are framed cobordant if and only if $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$.*

Proof. If $\mathcal{N}$ and $\mathcal{N}'$ are framed cobordant by framed cobordisms $W_1,\ldots,W_r$ which form the system $\mathcal{W}$, then as in Definition 3.2 it leads to the map $f_{\mathcal{W}}\colon M \times [0,1] \to \bigvee_{i=1}^r \mathrm{S}^1_i$ for a fixed product neighbourhood $P(\mathcal{W})$. It is clear that $f_{\mathcal{W}}|_{M\times\{0\}} = f_{\mathcal{N}}$ and $f_{\mathcal{W}}|_{M\times\{1\}} = f_{\mathcal{N}'}$ for product neighbourhoods $P(\mathcal{N}) = P(\mathcal{W}) \cap M \times \{0\}$ and $P(\mathcal{N}') = P(\mathcal{W}) \cap M \times \{1\}$, respectively. Thus $f_{\mathcal{W}}$ is a homotopy between $f_{\mathcal{N}}$ and $f_{\mathcal{N}'}$, so $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$.

Conversely, if $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$, then $f_{\mathcal{N}}$ and $f_{\mathcal{N}'}$ are homotopic by a map $f\colon M\times[0,1] \to \bigvee_{i=1}^r \mathrm{S}^1_i$ which is smooth outside the preimage of basepoint since $\bigvee_{i=1}^r \mathrm{S}^1_i$ is an Eilenberg–MacLane space $\mathrm{K}(F_r,1)$. As in the proof of Proposition 3.3 take a regular values $a_i \in \mathrm{S}^1_i$ and framed submanifolds $W_i = f^{-1}(a_i)$ which form a system of hypersurfaces in $M \times [0,1]$. They are framed cobordisms between $f_{\mathcal{N}}^{-1}(a_i) \cong N_i$ and $f_{\mathcal{N}'}^{-1}(a_i) \cong N'_i$. By the construction of $f_{\mathcal{N}}$ and $f_{\mathcal{N}'}$ it is clear that the system $(f_{\mathcal{N}}^{-1}(a_1),\ldots,f_{\mathcal{N}}^{-1}(a_r))$ is framed cobordant to $\mathcal{N}$ and $(f_{\mathcal{N}'}^{-1}(a_1),\ldots,f_{\mathcal{N}'}^{-1}(a_r))$ is framed cobordant to $\mathcal{N}'$. The statement follows by transitivity of framed cobordism. □

Remark 3.6. It is easy to check that if two systems of hypersurfaces differ only in their framings, but the determined positive sides are the same, then they are framed cobordant.

3.2 Epimorphisms and independency of inducing systems

The aim of this section is to prove that any epimorphism onto a free group is induced by an independent and regular system.

Let $\mathcal{N} = (N_1, \ldots, N_r)$ be a system of hypersurfaces in a compact and connected manifold W. Note that any class of loops in W can be represented by a loop in the interior $\operatorname{Int} W$.

Lemma 3.7. *Any class of loops $\omega \in \pi_1(W)$ either can be represented by a loop in $W|\mathcal{N}$ or there is a loop $\alpha \in \omega$ which can be written as the concatenation of paths $\alpha_1 \cdot \ldots \cdot \alpha_k$ which ends lie in $W|\mathcal{N}$ and $\alpha_i \cap P_t(\mathcal{N})$ is a single point for any $t \in [-1, 1]$. Thus putting $a_i := [\mathrm{S}^1_i]$ as the generators of $F_r = \pi_1\left(\bigvee_{i=1}^r \mathrm{S}^1_i\right)$ we have $\varphi_{\mathcal{N}}(\omega) = a_{i_1}^{\epsilon_1} \ldots a_{i_k}^{\epsilon_k}$, where $\epsilon_j \in \{-1, +1\}$ and i_j is a unique index for which $\alpha_j \cap N_{i_j}$ is non-empty.*

Proof. Take any loop in ω and homotope it to be in general position to $\mathcal{N}$. Since they have a complementary dimensions, their intersection is a finite set. Now, cut the obtained loop into paths α_i as it is required. □

It will cause no confusion if we use the same letter to designate a path $\gamma\colon [0,1] \to W$ and its image.

Lemma 3.8. *Suppose there is a path $\gamma\colon [0,1] \to W$ such that $\gamma \cap \mathcal{N} = \gamma \cap N_j = \{x, y\}$, where $x = \gamma(0) \in X$ and $y = \gamma(1) \in Y$ are in the different connected components X and Y of N_j, and which joins x and y from the same side, i.e. $\gamma \cap P_t(N_j) = \varnothing$ for any $t \in [-1, 0)$ or for any $t \in (0, 1]$. Then there is a system $\mathcal{N}' = (N'_1, \ldots, N'_r)$ such that $N_i = N'_i$ for $i \neq j$, N'_j has a one less connected component than N_j and $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$.*

Proof. First, change $\gamma\colon [0,1] \to W$ to be an embedded arc in $\operatorname{Int} W$ with the same properties as in the statement. Take a subset of small, closed tubular neighbourhood $P(\gamma)$ of γ parametrized by $\gamma \times \mathrm{D}^{n-1}_3$ such that $P(\gamma) \cap \mathcal{N} = P(\gamma) \cap N_j$, where $\mathrm{D}^{n-1}_t = \{x \in \mathbb{R}^{n-1} \ : \ ||x|| \leq t\}$ is a closed disc of radius t. We may assume that $P(\gamma) \cap X = \{x\} \times \mathrm{D}^{n-1}_3$ and $P(\gamma) \cap Y = \{y\} \times \mathrm{D}^{n-1}_3$. Now, perform the connected sum operation of X and Y along γ in W, i.e. we define the new submanifold

$$A = X \#_\gamma Y := \left(X \setminus \{x\} \times \mathrm{D}^{n-1}_2\right) \cup \left(\gamma \times \partial\, \mathrm{D}^{n-1}_2\right) \cup \left(Y \setminus \{y\} \times \mathrm{D}^{n-1}_2\right).$$

Obviously, A is a topological manifold, smoothly embedded outside $\{x, y\} \times \partial\, \mathrm{D}^{n-1}_2$. Thus take an open ε-neighbourhood U of $\{x, y\} \times \partial\, \mathrm{D}^{n-1}_2$ and smooth the corners inside U. Hence we may assume that A is a 2-sided smooth submanifold of W with product neighbourhood $P(A)$ such that

$$\begin{aligned} P(A \setminus U) = \quad & P\left(X \cup Y \setminus \left(\{x, y\} \times \mathrm{D}^{n-1}_2\right) \setminus U\right) \cup \\ & \cup \left(\gamma([\varepsilon, 1-\varepsilon]) \times (\mathrm{D}^{n-1}_3 \setminus \operatorname{Int} \mathrm{D}^{n-1}_1)\right). \end{aligned}$$

Since γ joins X and Y from the same side, the orientations of their normal bundles induces the orientation of $P(A)$, and thus a framing of A.

Let $\mathcal{N}' = (N'_1, \ldots, N'_r)$ be a system of hypersurfaces such that $N_i = N'_i$ for $i \neq j$ and $N'_j = (N_j \setminus (X \cup Y)) \cup A$. We will show that $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$. Let $[\alpha] \in \pi_1(W)$ be any class of loops in W with the basepoint outside $P(\mathcal{N})$ and $P(\gamma)$. We may assume that α does not intersect $\{x, y\} \times \mathrm{D}^{n-1}_2 \cup U$ and it is in general position to $\mathcal{N}'$. Write $\alpha = \alpha_1 \cdot \ldots \cdot \alpha_k$ as in Lemma 3.7 with respect to the system $\mathcal{N}'$, so $\varphi_{\mathcal{N}'}([\alpha]) = a_{i_1}^{\epsilon_1} \ldots a_{i_k}^{\epsilon_k}$. Note that $\varphi_{\mathcal{N}}([\alpha])$ is obtained from $\varphi_{\mathcal{N}'}([\alpha]) = a_{i_1}^{\epsilon_1} \ldots a_{i_k}^{\epsilon_k}$ by removing these $a_{i_j}^{\epsilon_j}$ which correspond to α_j such that $\alpha_j \cap \gamma \times \partial\, \mathrm{D}^{n-1}_2 \neq \varnothing$. However, if α_j intersects $\gamma \times \partial\, \mathrm{D}^{n-1}_2$ and goes inside $\gamma \times \mathrm{D}^{n-1}_2$ (i.e. it has the end point in $\gamma \times \mathrm{D}^{n-1}_2$), then α_{j+1} also intersects $\gamma \times \partial\, \mathrm{D}^{n-1}_2$, since it needs to leave $\gamma \times \mathrm{D}^{n-1}_2$ and does not intersect $\{x, y\} \times \mathrm{D}^{n-1}_2$. Thus $a_{i_j} = a_{i_{j+1}}$ and $\epsilon_{j+1} = -\epsilon_j$. Therefore $\varphi_{\mathcal{N}}([\alpha]) = \varphi_{\mathcal{N}'}([\alpha])$, so $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'}$. □

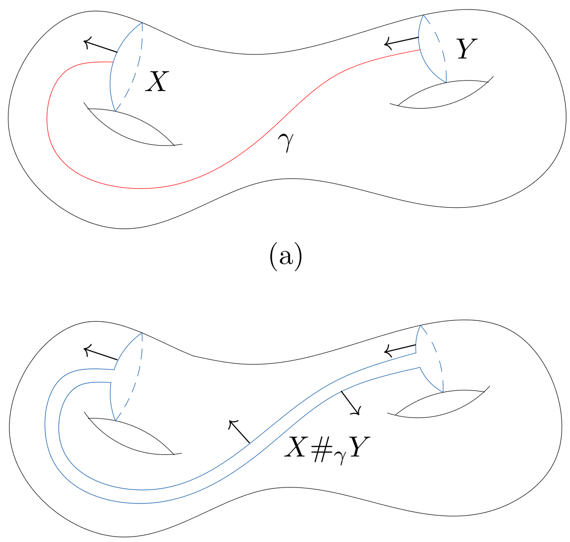


Figure 3.1: Example of connected sum operation of submanifolds X and Y along a curve γ which joins them from the same side. The arrows indicate normal vectors from the framing.

We call the constructed submanifold $X\#_\gamma Y$ the **connected sum** of X and Y **along** γ.

Proposition 3.9. *Let $\mathcal{N} = (N_1, \ldots, N_r)$ be a system of hypersurfaces in W such that $\varphi_{\mathcal{N}}$ is an epimorphism and there are no paths as in the statement of Lemma 3.8. Then there is a unique independent and regular system $\mathcal{A} = (A_1, \ldots, A_r)$ in W such that for each j the submanifold A_j is a component of N_j and there is a loop α_j such that $\alpha_j \cap \mathcal{N} = \alpha_j \cap A_j$ is a single point. In particular, if $\mathcal{N}$ is regular, then it is independent.*

Proof. Since $\varphi_{\mathcal{N}}$ is an epimorphism, for any j there is a loop α_j in W such that $f_{\mathcal{N}} \circ \alpha_j$ represents the generator of $F_r = \pi_1 \left(\bigvee_{i=1}^r \mathrm{S}_i^1\right)$ which corresponds to S_j^1. As in Lemma 3.7 we may consider α_j as the concatenation of paths $\alpha_1^j, \ldots, \alpha_k^j$ such that $a_j = \varphi_{\mathcal{N}}([\alpha_j]) = a_{i_1}^{\epsilon_1} \ldots a_{i_k}^{\epsilon_k}$, where $a_i = [\mathrm{S}_i^1]$. If $k > 1$, then there is some cancellation in the word $a_{i_1}^{\epsilon_1} \ldots a_{i_k}^{\epsilon_k}$, so for some l both α_l^j and α_{l+1}^j intersect the same submanifold N_{i_l}. If they intersect two different components of N_{i_l}, then it leads to a path as in the statement of Lemma 3.8, a contradiction. However, if they intersect N_{i_l} in the same connected component X, then we may assume that the starting point of α_l^j and the endpoint of α_{l+1}^j are in $P_t(X)$ for some $t \in [-2, 2] \setminus [-1, 1]$ by reparameterizing $P(N_{i_l})$. Since X is connected, we may substitute these paths by an arc in $P_t(X)$ joining these endpoints, which provides a loop with reduced the number of paths in the representation from Lemma 3.7. Proceeding inductively we may assume that $\alpha_j \cap \mathcal{N} = \alpha_j \cap N_j$ is a single point.

Note that if there were two components X and Y of N_j with loops α_X and α_Y with the same basepoint intersecting $\mathcal{N}$ only in single points of X and Y, respectively, then they would determine a path joining X and Y as in Lemma 3.8. Thus for any j there is a unique connected component A_j of N_j with this property.

The system $\mathcal{A} = (A_1, \ldots, A_r)$ is regular by definition and independent by the above property of A_j's. The uniqueness of $\mathcal{A}$ follows by the uniqueness of its components.

If $\mathcal{N}$ is regular, then $\mathcal{N} = \mathcal{A}$, so it is independent. □

Remark 3.10. Using the techniques as in the paper of O. Cornea [10] one can show that for a closed manifold M if $\mathcal{N}$ is not regular and $\varphi_{\mathcal{N}}$ is surjective, then there is an independent and regular system $\mathcal{N}' = (N_1', \ldots, N_r')$ in M such that $\mathcal{N}' \subset \mathcal{N}$ (but not necessarily $N_i' \subset N_i$), without the assumption on the existence of paths as in Lemma 3.8.

Theorem 3.11. *Any epimorphism* $\varphi\colon \pi_1(W) \to F_r$ *is induced by a regular and independent system of hypersurfaces.*

Proof. Let $\mathcal{N} = (N_1, \ldots, N_r)$ be a system inducing φ. By Lemma 3.8 we may assume that there is no path as in the statement of the lemma. Thus by Proposition 3.9 we consider a regular and independent system $\mathcal{A} = (A_1, \ldots, A_r)$ such that A_j is a component of N_j and for each j there is a loop α_j such that $\alpha_j \cap \mathcal{N} = \alpha_j \cap A_j$ is a single point. Therefore $\varphi_{\mathcal{N}}([\alpha_j]) = \varphi_{\mathcal{A}}([\alpha_j])$ for each j and $\varphi_{\mathcal{A}}\colon \pi_1(W) \to F_r$ is surjective. We will show that $\ker \varphi_{\mathcal{N}} \subset \ker \varphi_{\mathcal{A}}$.

Let $[\alpha] \in \ker \varphi_{\mathcal{N}}$ and write $\alpha = \alpha_1 \cdot \ldots \cdot \alpha_k$ as in Lemma 3.7 with respect to the system $\mathcal{N}$. We proceed by induction on k, which is even since $\varphi_{\mathcal{N}}([\alpha]) = 1$. If $k = 0$, then $\alpha \cap \mathcal{N} = \varnothing$, so $\alpha \in W|\mathcal{N} \subset W|\mathcal{A}$ and therefore $[\alpha] \in \ker \varphi_{\mathcal{A}}$. Suppose that any element in $\ker \varphi_{\mathcal{N}}$ represented by a loop which can be written as the concatenation of less than k paths as in Lemma 3.7 is also contained in $\ker \varphi_{\mathcal{A}}$. Let $\alpha = \alpha_1 \cdot \ldots \cdot \alpha_k$ for $[\alpha] \in \ker \varphi_{\mathcal{N}}$. Since $1 = \varphi_{\mathcal{N}}([\alpha]) = a_{i_1}^{\epsilon_1} \ldots a_{i_k}^{\epsilon_k}$, there is an index m such that $a_{i_m} = a_{i_{m+1}}$ and $\epsilon_{m+1} = -\epsilon_m$, so $i_m = i_{m+1} =: j$. Thus both the paths α_m and α_{m+1} intersect the same component X of N_j since there are no paths as in Lemma 3.8. Obviously, we may extend slightly the tubular neighbourhood of X and assume that the beginning of the path α_m and the end of α_{m+1} are in $P_t(X)$ for some $t \notin [-1, 1]$. Since X is connected, so also is $P_t(X)$, there is an arc γ in $P_t(X)$ joining these two points. Thus we may define the loop

$$\beta = \alpha_1 \cdot \ldots \cdot \alpha_{m-1} \cdot (\gamma \cdot \alpha_{m+2}) \cdot \alpha_{m+3} \cdot \ldots \cdot \alpha_k$$

which has $k - 2$ paths as in Lemma 3.7. Write $\varphi_{\mathcal{N}}([\alpha]) = \omega \cdot a_j^{\epsilon_m} a_j^{-\epsilon_m} \cdot \omega'$. Evidently, $\varphi_{\mathcal{N}}([\beta]) = \omega\omega' = 1$ and by induction hypothesis $\varphi_{\mathcal{A}}([\beta]) = 1$. It is clear that in both the cases $X = A_j$ or $X \neq A_j$ we get $\varphi_{\mathcal{A}}([\alpha]) = \varphi_{\mathcal{A}}([\beta]) = 1$, so $[\alpha] \in \ker \varphi_{\mathcal{A}}$. By induction $\ker \varphi_{\mathcal{N}} \subset \ker \varphi_{\mathcal{A}}$.

Therefore $\varphi_{\mathcal{A}} = \eta \circ \varphi_{\mathcal{N}}$ for some epimorphism $\eta\colon F_r \to F_r$. Since free groups are Hopfian (see [6]), η is an isomorphism, so $\ker \varphi_{\mathcal{N}} = \ker \varphi_{\mathcal{A}}$. Because $[\alpha_j]$'s generate a subgroup of $\pi_1(W)$ mapped isomorphically onto F_r by $\varphi_{\mathcal{N}}$ and $\varphi_{\mathcal{A}}$ on which they are equal, we obtain $\varphi_{\mathcal{A}} = \varphi_{\mathcal{N}}$ everywhere and the theorem is proved. □

3.3 Corank of a group

Let G be a finitely generated group and $\varphi\colon G \to F_r$ be an epimorphism. The number r is called the **rank** of an epimorphism φ. The **corank** of G is defined as the largest rank of an epimorphism from G onto a free group and it is denoted by $\operatorname{corank}(G)$. Since G is finitely generated it is well-defined and

$$\operatorname{corank}(G) \leq \operatorname{rank}_{\mathbb{Z}} \operatorname{Ab}(G),$$

where $\operatorname{Ab}(G)$ is the abelianization of G. In the case when $G = \pi_1(X)$ the corank of G is also called the first non-commutative Betti number of X (cf. [13]).

Example 3.12. A finitely generated group G has its corank equal to 0 if and only if its abelianization is finite. If $\operatorname{corank}(G) = 1$, then $\operatorname{rank}(\operatorname{Ab}(G)) \geq 1$. In particular, groups with infinite abelianization and which do not have F_2 as a subgroup (e.g. amenable groups) have corank equal to 1.

Moreover, $\operatorname{corank}(F_r) = r$.

For each function $f\colon W \to \mathbb{R}$ with finitely many critical points on (W, W_-, W_+) there is the associated Reeb epimorphism $\varphi_f\colon \pi_1(W) \to \pi_1(\mathcal{R}(f)) \cong F_r$, where $r = \beta_1(\mathcal{R}(f))$, so this implies that

$$\mathcal{R}(W, W_-, W_+) \leq \operatorname{corank}(\pi_1(W)).$$

Remark 3.13. I. Gelbukh [15, Theorem 3.1] showed that the inequality

$$\beta_1(\mathcal{R}(f)) \leq \operatorname{corank}(\pi_1(X))$$

holds for any connected and locally path-connected topological space X and continuous function $f\colon X \to \mathbb{R}$ under the assumption that the Reeb space $\mathcal{R}(f)$ is in fact a finite topological graph (thus a Reeb graph). This is an interesting question which classes of functions satisfies this condition. O. Saeki [58, Theorem 3.1] proved that is suffices to take smooth functions with finitely many critical values on a closed manifold to assure that their Reeb spaces are finite graphs.

Proposition 3.14. *If Σ is a closed surface of Euler characteristic $\chi(\Sigma) = 2 - k$, then*

$$\operatorname{corank}(\pi_1(\Sigma)) = \left\lfloor \frac{k}{2} \right\rfloor = \mathcal{R}(\Sigma),$$

Thus $\operatorname{corank}(\pi_1(\Sigma_g)) = g$ *and* $\operatorname{corank}(\pi_1(S_g)) = \lfloor \frac{g}{2} \rfloor$.

Proof. By Theorem 3.11 we know that $\operatorname{corank}(\pi_1(\Sigma))$ is equal to the maximum size r of a regular and independent system $\mathcal{N}$ of hypersurfaces in Σ. Note that $\mathcal{N}$ consists of r circles, so $\Sigma|\mathcal{N}$ is a compact surface with $2r$ boundary components. Attaching $2r$ discs to them we obtain a closed surface Σ' of Euler characteristic $\chi(\Sigma') = 2 - k + 2r$. Since $\chi(\Sigma') \leq 2$, we have $r \leq \frac{k}{2}$.

It is easy to construct an independent system $\mathcal{N}$ in Σ of size $r = \lfloor \frac{k}{2} \rfloor$. For this, note that if $\Sigma = \Sigma_g$ is orientable of genus g, then $k = 2g$ and $\Sigma_g = \#_{i=1}^{g} T^2$, so we may take one non-separating circle for each torus T^2 and perform the connected sum along discs omitting the circles. Similarly, if $\Sigma = S_g$ is non-orientable of genus g, then $k = g$ and $S_{2m} = \#_{i=1}^{m} K$ for $k = 2m$ even, or $S_{2m+1} = (\#_{i=1}^{m} K) \# \mathbb{R}\mathrm{P}^2$ for $k = 2m+1$ odd, where K is the Klein bottle. In both the cases we have $m = \lfloor \frac{k}{2} \rfloor$ circles non-separating S_g, one from each Klein bottle in the connected sum.

We may also take a Morse function $f\colon \Sigma \to \mathbb{R}$ with $\beta_1(\mathcal{R}(f)) = \lfloor \frac{k}{2} \rfloor = \mathcal{R}(\Sigma)$ (Corollary 2.10) and choose associated regular and independent system of hypersurfaces of size $\beta_1(\mathcal{R}(f))$. ∎

It is known (see [10, 13, 26]) that the corank satisfies the equalities

$$\begin{aligned} \operatorname{corank}(G \times H) &= \max\{\operatorname{corank}(G), \operatorname{corank}(H)\}, \\ \operatorname{corank}(G * H) &= \operatorname{corank}(G) + \operatorname{corank}(H), \end{aligned}$$

where $G * H$ is the free product of groups G and H.

We show in the following proposition that the corank of direct product of two groups is an upper bound for the corank of their arbitrary group extension.

Proposition 3.15. *Let G be an extension of a finitely generated group N by a finitely generated group H, i.e. we have the short exact sequence*

$$1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} H \to 1.$$

Then

$$\operatorname{corank}(H) \leq \operatorname{corank}(G) \leq \max\{\operatorname{corank}(N), \operatorname{corank}(H)\} = \operatorname{corank}(N \times H).$$

Proof. It is known that G is also finitely generated. The lower bound comes easily from the fact that the composition of π with any epimorphism $H \to F_r$ gives an epimorphism $G \to F_r$.

For the upper bound suppose that $\psi\colon G \to F_r$ is an epimorphism. Note that $F_N := \psi(\iota(N))$ is a finitely generated free subgroup of F_r. If $F_N = 1$ is trivial, then $\ker \pi = \iota(N) \subset \ker \psi$ and thus ψ factorizes through π inducing epimorphism $H \to F_r$, so $r \leq \operatorname{corank}(H)$. Assume that $F_N \neq 1$. Then as a finitely generated normal subgroup of a free group it has finite index k which by the Nielsen–Schreier formula [6] satisfies the equality $\operatorname{rank} F_N = (r-1)k+1$. Thus $\operatorname{rank} F_N \geq (r-1)\cdot 1 + 1 = r$. However, $\psi \circ \iota\colon N \to F_N$ is an epimorphism, so $r \leq \operatorname{rank} F_N \leq \operatorname{corank}(N)$. □

Example 3.16. We know that $\operatorname{corank}(F_r) = r$. Let $F_n < F_r$ be a finite index $k > 1$ normal subgroup of rank n, thus F_r is an extension of F_n by F_r/F_n, where $\operatorname{corank}(F_r/F_n) = 0$. However, $n = (r-1)k + 1 > r$ if $r > 1$, so the upper bound from the above proposition is not always attainable.

Example 3.17. Even for a semidirect product the corank can be arbitrary value in the above range. Let $0 \leq k \leq n$ be an integer and define

$$G_k := F_n \rtimes \mathbb{Z}_{/2} = \langle x_1, \ldots, x_n, c \mid c^2 = 1,\ c x_i c = x_i^{-1} \text{ for } i = 1, \ldots, k,\ c x_j c = x_j \text{ for } j = k+1, \ldots, n \rangle.$$

It is obvious that any epimorphism $G_k \to F_r$ factorizes through

$$G_k/(\mathbb{Z}_{/2})^{G_k} = \langle x_1, \ldots, x_n \mid x_i^2 = 1 \text{ for } i = 1, \ldots, k \rangle,$$

where $(\mathbb{Z}_{/2})^{G_k}$ is the normal closure of $\mathbb{Z}_{/2}$ in G_k. This quotient group is clearly isomorphic to

$$F_{n-k} * \underbrace{\mathbb{Z}_{/2} * \ldots * \mathbb{Z}_{/2}}_{k \text{ times}}.$$

Thus $\operatorname{corank}(G_k) = n - k$. For $n = k$ the abelianization $\operatorname{Ab}(G_n) = (\mathbb{Z}_{/2})^{n+1}$ is finite, so $\operatorname{corank}(G_n) = 0$, although G_n has a free nonabelian subgroup.

3.4 Equivalence and strong equivalence of epimorphisms

R. Grigorchuk, P. Kurchanov and H. Zieschang in [19, 21] studied epimorphisms onto free groups from fundamental groups of compact surfaces. As in their papers, we call two homomorphisms $\varphi, \psi\colon G \to H$ **equivalent** if there exist isomorphisms $\nu\colon G \to G$ and $\eta\colon H \to H$ such that $\varphi \circ \nu = \eta \circ \psi$. They are called **strongly equivalent** if one can choose $\eta = \mathrm{id}_H$. In this case we write $\varphi \simeq \psi$. Obviously, strong equivalence implies equivalence of homomorphisms. We are interested in the case $H = F_r$.

We will apply the results of the previous section to the problem of classification of epimorphisms onto free groups up to equivalence and strong equivalence. In particular, we give an alternative proof of the following theorem.

Theorem 3.18 (Grigorchuk–Kurchanov–Zieschang [19–21])**.** *If Σ is a closed surface of Euler characteristic $\chi(\Sigma) = 2 - k$ and $1 \leq r \leq \lfloor \frac{k}{2} \rfloor = \operatorname{corank}(\pi_1(\Sigma))$, then there exist finite numbers p and q of classes of epimorphisms $\pi_1(\Sigma) \to F_r$ with respect to equivalence and strong equivalence, respectively. More precisely,*

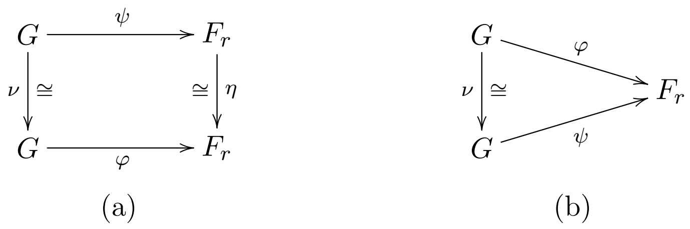


Figure 3.2: Equivalence (a) and strong equivalence (b) of epimorphisms onto free groups.

(1) if Σ is orientable, then $p = q = 1$,

(2) if $\Sigma = S_k$ is non-orientable, then we have:

(a) $p = q = 1$ if the genus $k = 2m + 1$ is odd,

(b) $p = 2$ and $q = 2^r$ if the genus $k = 2m$ is even and $r < m$,

(c) $p = 1$ and $q = 2^r - 1$ if the genus $k = 2m$ is even and $r = m$.

Theorem 3.19 ([20]). *For $m \geq r$ there exists only one class of epimorphisms $F_m \to F_r$ up to strong equivalence.*

It should be noted that the Poincaré conjecture is equivalent to the classification of some pairs of epimorphisms onto free groups, which shows the importance of their studies.

Theorem 3.20 (Stallings–Jaco–Waldhausen–Hempel, [23, 25]). *The Poincaré conjecture holds if and only if for each $g \geq 2$ any two epimorphisms $\pi_1(\Sigma_g) \to F_g \times F_g$ are equivalent.*

The problem of calculating the number of equivalence or strong-equivalence classes of epimorphisms onto free groups seems to be very hard in general. Thus we propose to distinguish the following property of epimorphisms which is related with equivalence relation.

An epimorphism $\varphi\colon G \to F_r$ is **maximal** if it is not factorized by an epimorphism onto a free group of higher rank. We are interested in the question if there exists maximal epimorphisms of rank smaller than corank.

Denote by $\mathrm{Epi}(G, H)$ the set of all epimorphisms $G \to H$.

Lemma 3.21. *Assume that $\varphi, \varphi' \in \mathrm{Epi}(G, F_r)$ are equivalent and $\psi \in \mathrm{Epi}(G, F_{r'})$ factorizes φ. Then there exists $\psi' \in \mathrm{Epi}(G, F_{r'})$ strong equivalent to ψ which factorizes φ'. Therefore two equivalent epimorphisms are both maximal or not maximal.*

Proof. Let $\nu\colon G \to G$ and $\eta\colon F_r \to F_r$ be isomorphisms such that $\varphi' \circ \nu = \eta \circ \varphi$ and let $\alpha\colon F_{r'} \to F_r$ be an epimorphism such that $\varphi = \alpha \circ \psi$ (see Figure 3.3). It suffices to take $\psi' = \psi \circ \nu^{-1}$. □

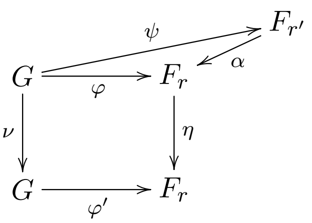


Figure 3.3: Two equivalent epimorphisms are both maximal or not maximal.

Corollary 3.22. *If there is a maximal epimorphism $\varphi\colon G \to F_r$, where $r < \operatorname{corank}(G)$, then there are two non-equivalent epimorphisms $G \to F_r$ (e.g. φ and any non-maximal epimorphism of the same rank).* □

Example 3.23. If $\operatorname{corank}(G) \leq 1$, then trivially any maximal epimorphism has rank equal to $\operatorname{corank}(G)$. In particular, if G is abelian, then all its epimorphisms $G \to F_1 = \mathbb{Z}$ are equivalent by a linear algebra argument. However, we will see in Example 3.26 that not all corank 1 groups have all equivalent epimorphisms of the same rank.

Example 3.24. By Theorems 3.18 and 3.19 groups $\pi_1(\Sigma_g)$, $\pi_1(S_{2g+1})$, $\pi_1(S_2)$ and F_r have all epimorphisms onto free group of the same rank equivalent, thus any their maximal epimorphism is of rank equal to corank. However, although $\pi_1(S_{2g})$ for $g \geq 2$ has two non-equivalent epimorphisms, its maximal epimorphisms have rank equal to corank (see Corollary 4.27).

Proposition 3.25. *Let G and H be groups with coranks n and k, respectively. If $n \geq k \geq 1$ and $n \geq 2$, then $G \times H$ has a maximal epimorphism of rank smaller than $\operatorname{corank}(G \times H) = n$.*

Proof. Take epimorphisms $\eta\colon G \to \mathbb{Z}$ and $\nu\colon H \to \mathbb{Z}$. Then the epimorphism $\varphi\colon G \times H \to \mathbb{Z}$ defined by $\varphi(g,h) = \eta(g) + \nu(h)$ is maximal since, by [13, Lemma 3.2], for $r \geq 2$ any epimorphism $G \times H \to F_r$ factorizes through the projection $\pi_G\colon G \times H \to G$ or $\pi_H\colon G \times H \to H$.

Moreover, if $n > k \geq 1$, any epimorphism $\psi \circ \pi_H$, where $\psi\colon H \to F_k$, is maximal. □

Example 3.26. Recall that $H(3,\mathbb{Z}) = \langle x, y \mid xz = zx,\ yz = zy\rangle$ is the discrete Heisenberg group, where $z = [x,y] = xyx^{-1}y^{-1}$ is the commutator of x and y. The group $G = H(3,\mathbb{Z}) \times \mathbb{Z}$ has $\operatorname{corank}(G) = 1$, but it has two non-equivalent epimorphisms $G \to \mathbb{Z}$.

The element z generates the center $Z(H(3,\mathbb{Z})) \cong \mathbb{Z}$. The group $H(3,\mathbb{Z})$ is nilpotent, so $\operatorname{corank}(H(3,\mathbb{Z})) = 1$ since $\operatorname{Ab}(H(3,\mathbb{Z})) = \mathbb{Z}^2$ is infinite, generated by images of x and y. Thus $\operatorname{corank}(G) = 1$.

Let $\operatorname{pr}_1\colon G \to H(3,\mathbb{Z})$ and $\operatorname{pr}_2\colon G \to \mathbb{Z}$ be the projections onto the first and second factor, respectively, and $\varphi\colon H(3,\mathbb{Z}) \to \mathbb{Z}$ be defined by $\varphi(x) = 1$, $\varphi(y) = 0$. Then pr_2 and $\varphi \circ \operatorname{pr}_1$ are not equivalent. If they were, there would be an isomorphism $\eta\colon G \to G$ such that $\varphi \circ \operatorname{pr}_1 \circ \eta\colon G \to \mathbb{Z}$ maps the generator of $\mathbb{Z}$, the second factor of G, onto generator. However, an isomorphism of groups maps the center of a group to the center. Since $Z(G) = Z(H(3,\mathbb{Z})) \times \mathbb{Z}$ is in the kernel of $\varphi \circ \operatorname{pr}_1$, we obtain a contradiction.

Proposition 3.27. *Let G and H be finitely generated groups. Then the free product $G * H$ have the property that all its maximal epimorphisms have rank equal to its corank if and only if G and H have this property.*

Proof. Let $\operatorname{corank}(G) = n$ and $\operatorname{corank}(H) = k$. Suppose there is a maximal epimorphism $\varphi\colon G * H \to F_r$, where $r \leq n + k = \operatorname{corank}(G * H)$. There are two obvious epimorphisms $\varphi_1\colon G*H \to \varphi(G)*\varphi(H)$ and $\varphi_2\colon \varphi(G)*\varphi(H) \to F_r$ such that $\varphi = \varphi_2 \circ \varphi_1$. By assumption, there are factorizations $\psi_G\colon G \to F_n$ of $\varphi|_G$ and $\psi_H\colon H \to F_k$ of $\varphi|_H$. This leads to the factorization $\psi\colon G * H \to F_n * F_k \cong F_{n+k}$ of φ_1. Since φ_1 factorizes φ and φ is maximal, $r = n + k$.

Conversely, take a maximal epimorphism $\eta\colon G \to F_r$, $r \leq n$ and any epimorphism $\nu\colon H \to F_k$. They define the epimorphism $\varphi\colon G * H \to F_r * F_k \cong F_{r+k}$. If $G * H$ has the property from the statement, there are epimorphisms $\psi\colon G * H \to F_{n+k}$ and $\alpha\colon F_{n+k} \to F_r * F_k$ such that $\varphi = \alpha \circ \psi$. Restricting to H we obtain $\nu = \alpha|_{\psi(H)} \circ \psi|_H$. However, ν is maximal, so $\psi(H)$ has rank k and so $\psi(G)$ has rank equal to n. Since $\eta = \alpha|_{\psi(G)} \circ \psi|_G$ is maximal, $r = n$. Analogously we show the property for the group H. □

Remark 3.28. In view of the above proposition, the study of maximal epimorphisms from fundamental groups of 3-manifolds can be reduced to fundamental groups of prime 3-manifolds. For example, by Proposition 3.25 the fundamental groups $\pi_1(M) = \pi_1(\Sigma_g) \times \mathbb{Z}$ of a manifold $M = \Sigma_g \times \mathrm{S}^1$ for $g \geq 2$ have maximal epimorphisms of rank smaller than $\operatorname{corank}(\pi_1(M)) = g$, so have non-equivalent epimorphisms. Similarly for $\pi_1(S_g \times \mathrm{S}^1)$, where $g \geq 4$.

3.5 Systems of hypersurfaces up to framed cobordism and diffeomorphism

Let us denote by $\mathcal{H}_r(M)$ the set of all independent and regular systems of hypersurfaces in M of size r which omit the basepoint, and by $\mathcal{H}_r^{fr}(M)$ the set of framed cobordism classes of elements of $\mathcal{H}_r(M)$. On each of these sets there is a natural action of $\mathrm{Diff}_\bullet(M)$, the set of self-diffeomorphisms of M which preserve the basepoint, so we may form the orbit space $\mathcal{H}_r^{fr}(M)/_{\mathrm{Diff}_\bullet(M)}$. Note that if $h \in \mathrm{Diff}_\bullet(M)$, then a system $\mathcal{N} = (N_1, \ldots, N_r)$ and its image $h(\mathcal{N}) = (h(N_1), \ldots, h(N_r))$ induce strongly equivalent homomorphisms.

We have the natural map $\Theta\colon \mathcal{H}_r(M) \to \mathrm{Epi}(\pi_1(M), F_r)$ which sends a system $\mathcal{N}$ into the induced epimorphism $\varphi_{\mathcal{N}}$. By Proposition 3.5 it factorizes through the injective map $\overline{\Theta}\colon \mathcal{H}_r^{fr}(M) \to \mathrm{Epi}(\pi_1(M), F_r)$. The Theorem 3.11 states that both these mappings are also surjective.

Corollary 3.29. *The map $\overline{\Theta}\colon \mathcal{H}_r^{fr}(M) \to \mathrm{Epi}(\pi_1(M), F_r)$ is a bijection between the set of all framed cobordism classes of regular and independent systems of hypersurfaces of size r in M and the set of all epimorphisms from $\pi_1(M)$ onto the free group of rank r.* □

Now, let us consider the strong equivalence relation $\simeq$ on $\mathrm{Epi}(G, F_r)$. The composition

$$\mathcal{H}_r^{fr}(M) \to \mathrm{Epi}(\pi_1(M), F_r) \to \mathrm{Epi}(\pi_1(M), F_r)/_{\simeq}$$

is still surjective and it factorizes through the map $\overline{\overline{\Theta}}\colon \mathcal{H}_r^{fr}(M)/_{\mathrm{Diff}_\bullet(M)} \to \mathrm{Epi}(\pi_1(M), F_r)/_{\simeq}$.

Corollary 3.30. *The number of strong equivalence classes of epimorphisms $\pi_1(M) \to F_r$ is not greater than the cardinality of $\mathcal{H}_r^{fr}(M)/_{\mathrm{Diff}_\bullet(M)}$.*

The question is when the latter set is finite. It is for example the case for the surface groups.

Proposition 3.31. *For a closed surface Σ the map*

$$\overline{\overline{\Theta}}\colon \mathcal{H}^{fr}(\Sigma)/_{\mathrm{Diff}_\bullet(\Sigma)} \to \mathrm{Epi}(\pi_1(\Sigma), F_r)/_{\simeq}$$

is a bijection.

Proof. We know that it is surjective. For injectivity it suffices to note that by Dehn–Nielsen Theorem (see [9, Theorem 3.4.6.]) any automorphism of $\pi_1(\Sigma)$ can be represented by a self-diffeomorphism of Σ. If $\varphi_{\mathcal{N}}$ and $\varphi_{\mathcal{N}'}$ are strongly equivalent by $\eta = h_\#$ induced by $h \in \mathrm{Diff}_\bullet(\Sigma)$, then $\varphi_{\mathcal{N}} = \varphi_{\mathcal{N}'} \circ \eta = (f_{\mathcal{N}'} \circ h)_\# = (f_{h^{-1}(\mathcal{N}')})_\# = \varphi_{h^{-1}(\mathcal{N}')}$, so $\mathcal{N}$ and $h^{-1}(\mathcal{N}')$ are framed cobordant. □

Remark 3.32. The same fact is true for any manifold M for which any automorphism of $\pi_1(M)$ is induced by some element of $\mathrm{Diff}_\bullet(M)$. By Mostow Rigidity Theorem it is the case for hyperbolic manifolds of dimension at least 3.

Now, our aim is to calculate $\mathcal{H}_r^{fr}(\Sigma)/_{\mathrm{Diff}_\bullet(\Sigma)}$. We need the following series of three lemmas.

Lemma 3.33. *Let Σ be a non-orientable compact surface with $\partial\Sigma \neq \varnothing$ and $S \subset \partial\Sigma$ be a connected component. Then there exist $h \in \mathrm{Diff}_\bullet(\Sigma)$ such that $h(S) = S$, $h|_S$ is orientation-reversing and $h|_{\partial\Sigma\setminus S} = \mathrm{id}_{\partial\Sigma\setminus S}$.*

Proof. First, assume that Σ is the projective plane $\mathbb{RP}^2$ with one disc B removed, i.e. $\Sigma = \mathbb{RP}^2 \setminus \mathrm{Int}\, B$ and $S = \partial B$. Let $D \subset \mathrm{Int}\,\Sigma$ be another disc and $\Sigma' = \Sigma \setminus D \cup B$ be a Möbius band. Fix a parametrization $\Sigma' \cong [-1,1] \times [0,1]/(t,0) \sim (-t,1)$ for $t \in [-1,1]$ such that $S \subset \mathrm{Int}\,\Sigma'$ is symmetric with respect to the core $\{0\} \times [0,1]$, i.e. if $(t,x) \in S$, then $(-t,x)$ is also in S. Then $h'\colon \Sigma' \to \Sigma'$ defined by $h'(t,x) = (-t,x)$ is a self-diffeomorphism such that $h'|_S\colon S \to S$ has degree -1, so it is orientation-reversing, but on $\partial\Sigma'$ it is orientation-preserving, so isotopic to the identity. Thus we can extend h' to $h\colon \mathbb{RP}^2 \to \mathbb{RP}^2$ such that $h(B) = B$ and $h|_S$ has degree -1, and take $h|_\Sigma$.

In general case, glue a disc B and Σ along S and take a diffeomorphism $\Sigma \cup_S B \to \Sigma'' \# \mathbb{RP}^2$ such that $B \subset \Sigma' \subset \mathbb{RP}^2$ as before. The lemma follows from the first case. □

Lemma 3.34. *Let $\mathcal{N} = (N_1 \cup N_2)$ be a system of size 1 in a manifold M such that N_1 and N_2 are connected. If $M \setminus N_1$ and $M \setminus N_2$ are connected, but $M|\mathcal{N}$ is disconnected, then $\varphi_\mathcal{N}\colon \pi_1(M) \to \mathbb{Z}$ is not surjective.*

Proof. Assume that $\varphi_\mathcal{N}$ is an epimorphism. Then there is a loop α in a general position to $\mathcal{N}$ such that $\varphi_\mathcal{N}([\alpha]) = \pm 1$. As in Lemma 3.7 write $\alpha = \alpha_1 \ldots \alpha_k$ as a concatenation of paths α_i, each of which intersects $\mathcal{N}$ in a single point. Therefore

$$1 \equiv \pm 1 = \varphi_\mathcal{N}([\alpha]) \equiv k \mod 2,$$

so k is odd. By the assumption $M|\mathcal{N}$ has exactly two components. Since N_1, N_2, $M \setminus N_1$ and $M \setminus N_2$ are connected, each α_i joins both the components of $M|\mathcal{N}$. Thus k is even, because α is a loop, so it starts and ends at the same point. It gives a contradiction, so $\varphi_\mathcal{N}$ is not surjective. □

Remark 3.35. While we know that independent systems induce surjective homomorphisms, non-independent systems can induce both surjective or not surjective homomorphisms. The above lemma shows when $\varphi_\mathcal{N}$ is not an epimorphisms and it can be generalized for other similar situations.

Lemma 3.36. *For an independent system $\mathcal{N} = (N_1 \cup \ldots \cup N_r)$ of size 1 in a manifold M there exists a regular and independent system $\mathcal{N}' = (N')$ which is framed cobordant to $\mathcal{N}$, so $\varphi_\mathcal{N} = \varphi_{\mathcal{N}'}$. Moreover,*

(1) The complement $M|\mathcal{N}'$ can be non-orientable if $M|\mathcal{N}$ is non-orientable.

(2) The complement $M|\mathcal{N}'$ is orientable if $M|\mathcal{N}$ is orientable and $M|\mathcal{N} \cup P(N_i)$ is non-orientable for each i.

Proof. The construction of $\mathcal{N}'$ is performed as in the proof of Lemma 3.8 by using arcs γ connecting components of $\mathcal{N}$. They can be found since $\mathcal{N}$ is independent.

Consider a two-sheeted orientation cover $\pi\colon \widetilde{M} \to M$, where

$$\widetilde{M} := \{\mu_x \,|\, x \in M \text{ and } \mu_x \in H_n(M, M \setminus \{x\}) \text{ is a local orientation of } M\}.$$

For (1), if $M|\mathcal{N}$ is non-orientable, then there is a loop α in $M|\mathcal{N}$ which reverse the orientation, which means that it lifts to a path in $\widetilde{M}$ which joins two different local orientations at the basepoint. Since $M|\mathcal{N} \setminus \operatorname{Im}\alpha$ is connected, we may perform the construction of $\mathcal{N}'$ in this space. Then α is also contained in $M|\mathcal{N}'$, so it is non-orientable.

Now, assume that $M|\mathcal{N}$ is orientable, but $M|\mathcal{N} \cup P(N_i)$ is non-orientable for each i. To obtain a contradiction, suppose that $M|\mathcal{N}'$ is non-orientable, so there is a loop α in $M|\mathcal{N}'$ which reverse the orientation and we may assume that it is in general position to $\mathcal{N}$. Using Lemma 3.7 write $\alpha = \alpha_1 \dots \alpha_k$ as a concatenation of paths α_i, each of which intersects $\mathcal{N}$ in a single point. Note that since α is in $M|\mathcal{N}'$, it intersects $\mathcal{N}$ only when it goes into or leaves a tubular neighbourhood $P(\gamma)$ of some arc γ as mentioned in the beginning of the proof. Therefore, as in the proof of Lemma 3.8, if α intersects $\mathcal{N}$ going inside $P(\gamma)$, then it needs to leave $P(\gamma)$ again intersecting $\mathcal{N}$. Thus k is even.

For any i consider α_i which intersects N_j in a point x and take a small closed disc D in M around x such that the cover π is trivial over D and $\partial D \cap \operatorname{Im}\alpha_i = \{x_1, x_2\}$, where x_1 and x_2 lie on the different sides of N_j such that α_i goes from x_1 to x_2. By the assumption, there is a reversing-orientation loop β_i in $M|\mathcal{N} \cup P(N_j)$ intersecting $\mathcal{N}$ only once at x and we may assume that its image agrees with the image of α_i on D. We take a loop α' which differs from α only on the segment of α_i between x_1 and x_2, where it goes as β_i outside D. Note that the local orientations in x_2 assigning by lifts of α and α' are opposed. Repeating this for each α_i we obtain a loop α'' which omits $\mathcal{N}$ and which is still orientation-reversing since we changed the local orientations by β_i an even number of times. This contradicts the fact that $M|\mathcal{N}$ is orientable and proves (2). □

Remark 3.37. In fact, in (1) the complement $M|\mathcal{N}'$ is always non-orientable if $M|\mathcal{N}$ is non-orientable. For this, if $\mathcal{N}''$ is any other regular and independent system framed cobordant to $\mathcal{N}$ such that $M|\mathcal{N}''$ is orientable, then it is also framed cobordant to $\mathcal{N}'$, but it is a contradiction by the next proposition.

Proposition 3.38. *Let M be a non-orientable manifold and let $\mathcal{N}$ and $\mathcal{N}'$ be two regular and independent systems of hypersurfaces in M of the same size r such that $M|\mathcal{N}$ is orientable, but $M|\mathcal{N}'$ is non-orientable. Then $\mathcal{N}$ and $\mathcal{N}'$ are not framed cobordant.*

Proof. Let $\mathcal{N} = (N_1, \dots, N_r)$ and $\mathcal{N}' = (N_1', \dots, N_r')$. We may assume that $\mathcal{N}$ satisfies the conditions in Lemma 3.36 (2) since a framed cobordism between $\mathcal{N}$ and $\mathcal{N}'$ implies a framed cobordism between the systems $\mathcal{N}_* = (N_{i_1}, \dots, N_{i_k})$ and $\mathcal{N}_*' = (N_{i_1}', \dots, N_{i_k}')$, where $i_1 < \dots < i_k$ are all indices such that $M|\mathcal{N} \cup P(N_{i_j})$ is non-orientable. We will show that $\mathcal{N}_*$ and $\mathcal{N}_*'$ are not framed cobordant even as submanifolds, not as systems of hypersurfaces. For this we may use Lemma 3.36 for $(N_{i_1} \cup \dots \cup N_{i_k})$ and $(N_{i_1}' \cup \dots \cup N_{i_k}')$ to assume that $r = 1$.

So now, each of $\mathcal{N}$ and $\mathcal{N}'$ is just a non-separating connected 2-sided submanifold in M, $M|\mathcal{N}$ is orientable and $M|\mathcal{N}'$ is non-orientable. Suppose that $W \subset M \times [0,1]$ is a framed cobordism between $\mathcal{N}$ and $\mathcal{N}'$. Take the orientation cover $\pi\colon \widetilde{M} \to M$ and take the lifts $\widetilde{\mathcal{N}} := \pi^{-1}(\mathcal{N})$ and $\widetilde{\mathcal{N}'} := \pi^{-1}(\mathcal{N}')$. Moreover, by the property of π the complement $\widetilde{M}|\widetilde{\mathcal{N}}$ has two connected components since $M|\mathcal{N}$ is orientable, and $\widetilde{M}|\widetilde{\mathcal{N}'}$ is connected since $M|\mathcal{N}'$ is non-orientable. The cobordism W is lifted to the framed cobordism $\widetilde{W} := (\pi \times \mathrm{id}_{[0,1]})^{-1}(W) \subset \widetilde{M} \times [0,1]$ between $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}'}$. Therefore $\varphi_{\widetilde{\mathcal{N}}} = \varphi_{\widetilde{\mathcal{N}'}}\colon \pi_1(\widetilde{M}) \to \mathbb{Z}$ and $\varphi_{\widetilde{\mathcal{N}'}}$ is surjective, because $\widetilde{\mathcal{N}'}$ is independent. However, $\varphi_{\widetilde{\mathcal{N}}}$ is not surjective, which gives a contradiction.

To see this, note that $\widetilde{\mathcal{N}}$ can have one or two components. If $\widetilde{\mathcal{N}}$ is connected, then $\varphi_{\widetilde{\mathcal{N}}}$ is evidently not surjective, since $\widetilde{M}|\widetilde{\mathcal{N}}$ is not connected. In the second case when $\widetilde{\mathcal{N}}$ has two components, we use Lemma 3.34.

Thus $\mathcal{N}$ and $\mathcal{N}'$ are not framed cobordant. □

Remark 3.39. The above proposition is easily not true for not regular systems of hypersurfaces.

Now, we may start the calculation of $\mathcal{H}_r^{fr}(\Sigma)/_{\mathrm{Diff}_\bullet(\Sigma)}$. First, let us make a short preparation.

Let $\mathcal{N} = (N_1, \ldots, N_r)$ and $\mathcal{N}' = (N_1', \ldots, N_r')$ be two arbitrary regular and independent systems of hypersurfaces in a closed surface Σ. Thus all N_i, N_i' are circles. Assume that $\Sigma|\mathcal{N}$ and $\Sigma|\mathcal{N}'$ are diffeomorphic. By homogeneity of manifolds, take a diffeomorphism $h' \colon \Sigma|\mathcal{N} \to \Sigma|\mathcal{N}'$ which sends $P_{\pm 1}(N_i)$ onto $P_{\pm 1}(N_i')$. Glue all tubes $P(N_i) \cong [-1,1] \times N_i$ to Σ along $\{-1\} \times N_i$ to obtain a surface $\overline{\Sigma}$ with $2r$ boundary components $\{1\} \times N_i$ and $P_1(N_i)$, $i = 1, \ldots, r$. Let

$$\xi_i \colon P_1(N_i) \to \{1\} \times N_i$$

be a gluing map which leads to Σ. Analogously, we define $\overline{\Sigma'}$ and take ξ_i' for $\mathcal{N}'$. Extend h' to a diffeomorphism $\bar{h} \colon \overline{\Sigma} \to \overline{\Sigma'}$ using $P(N_i) \cong [-1,1] \times \mathrm{S}^1 \cong P(N_i')$, so $\bar{h}(\mathcal{N}) = \mathcal{N}'$. It follows easily that $\bar{h}$ induces $h \in \mathrm{Diff}_\bullet(\Sigma)$ after performing gluing operations via ξ_i and ξ_i' if and only if $\bar{h}^{-1} \circ \xi_i' \circ \bar{h}|_{P_1(N_i)}$ is isotopic to ξ_i for each $i = 1, \ldots, r$. If it is the case, then $h(\mathcal{N}) = \mathcal{N}'$, so $\mathcal{N}$ and $\mathcal{N}'$ are the same elements in $\mathcal{H}_r^{fr}(\Sigma)/_{\mathrm{Diff}_\bullet(\Sigma)}$.

Theorem 3.40. *Let Σ be a closed surface, let r be an integer such that $1 \leq r \leq \operatorname{corank}(\pi_1(\Sigma))$ and set $q = \left|\mathcal{H}_r^{fr}(\Sigma)/_{\mathrm{Diff}_\bullet(\Sigma)}\right|$.*

(1) If Σ is orientable or non-orientable of odd genus, then $q = 1$.

(2) If $\Sigma = S_{2m}$ is non-orientable of genus $2m$, then

- *if $r < m$, then $q = 2^r$,*
- *if $r = m$, then $q = 2^r - 1$.*

As a consequence, q is the number of strong equivalence classes of epimorphisms $\pi_1(\Sigma) \to F_r$ as in Theorem 3.18.

Proof. We use the above notation. If Σ is orientable, then $\Sigma|\mathcal{N}$ and $\Sigma|\mathcal{N}'$ are diffeomorphic surfaces and we may assume that the diffeomorphism h' is orientation-preserving. Since Σ is orientable, all maps ξ_i and ξ_i' are also orientation-preserving, so we obtain $h \in \mathrm{Diff}_\bullet(\Sigma)$ such that $h(N_i) = N_i'$. Therefore $q = 1$.

Now assume that Σ is non-orientable of odd genus. Then $\Sigma|\mathcal{N}$ and $\Sigma|\mathcal{N}'$ are compact surfaces with $2r$ boundary components and of the same odd Euler characteristic, so they are also non-orientable. Using Lemma 3.33 we may change h', by the composition with another diffeomorphism, so that $\bar{h}^{-1} \circ \xi_i' \circ \bar{h}|_{P_1(N_i)}$ and ξ_i are isotopic. As before, it implies that $q = 1$.

Finally, let $\Sigma = S_{2m}$ be non-orientable of even genus $2m$. For any non-empty subset $I \subset \{1, \ldots, r\}$ it is easy to construct a system $\mathcal{N}_I$ such that $\Sigma|\mathcal{N}_I$ is orientable and gluing maps ξ_i^I (defined as before) are orientation-reversing only for $i \in I$. We omit the case when $I = \varnothing$ since then Σ would be orientable. Moreover, for $r < m$ we denote by $\mathcal{N}_0$ a system for which $\Sigma|\mathcal{N}_0$ is non-orientable. Note that if $r = m$, then $\Sigma|\mathcal{N}$ is always the sphere with $2r$ open discs removed, so it is orientable.

By the previous considerations it is clear that the systems $\mathcal{N}_I$ for $\varnothing \neq I \subset \{1, \ldots, r\}$ and $\mathcal{N}_0$ for $r < m$ represent all elements of $\mathcal{H}_r^{fr}(\Sigma)/_{\mathrm{Diff}_\bullet(\Sigma)}$ (for the case when $\Sigma|\mathcal{N}$ is non-orientable we use Lemma 3.33 as before). Thus $q \leq 2^r$ for $r < m$ and $q \leq 2^r - 1$ for $r = m$. We will show that they are different elements of $\mathcal{H}_r^{fr}(\Sigma)/_{\mathrm{Diff}_\bullet(\Sigma)}$. It will be done if we show that the systems are not framed cobordant to each other.

By Proposition 3.38 we known that $\mathcal{N}_0$ is not cobordant to any $\mathcal{N}_I$. If we have two systems $\mathcal{N}_I = (N_1^I, \dots, N_r^I)$ and $\mathcal{N}_J = (N_1^J, \dots, N_r^J)$ for $I \neq J$, then we may assume that there is an index $1 \leq j \leq r$ such that $j \notin I$, but $j \in J$, so ξ_j^I is orientation-preserving, but ξ_j^J is orientation-reversing. If $I = \{i_1, \dots, i_k\}$, form the systems $\mathcal{N}_I^* = (N_{i_1}^I, \dots, N_{i_k}^I)$ and $\mathcal{N}_J^* = (N_{i_1}^J, \dots, N_{i_k}^J)$. By the construction, $\Sigma|\mathcal{N}_I^*$ is orientable, but $\Sigma|\mathcal{N}_J^*$ is non-orientable. Again by Proposition 3.38 we get that $\mathcal{N}_I^*$ and $\mathcal{N}_J^*$ are not framed cobordant, so also $\mathcal{N}_I$ and $\mathcal{N}_J$ cannot be framed cobordant and the proof is complete.

The last statement follows by Proposition 3.31. □

Corollary 3.41. *With the above notation,*

$$\mathcal{H}_r^{fr}(S_{2m})/_{\mathrm{Diff}_\bullet(S_{2m})} = \begin{cases} \{[\mathcal{N}_0], [\mathcal{N}_I] \,:\, \varnothing \neq I \subset \{1, \dots, r\}\} & \text{for } r < m, \\ \{[\mathcal{N}_I] \,:\, \varnothing \neq I \subset \{1, \dots, r\}\} & \text{for } r = m. \end{cases}$$

□

3.6 Analogue of Nielsen transformations for systems of hypersurfaces

We have found out that strong equivalence classes of epimorphisms $\pi_1(M) \to F_r$ can be described by elements of $\mathcal{H}_r^{fr}(M)/_{\mathrm{Diff}_\bullet(M)}$. In this section we show how to get equivalence classes from them.

It is known that the automorphism group $\mathrm{Aut}(F_r)$ of a finitely generated free group F_r is generated by **elementary Nielsen transformations** (see e.g. [6]). On a given ordered basis $(a_1, \dots, a_r)$ we define them as follows:

(T1) $n_\sigma \colon (a_1, \dots, a_r) \mapsto (a_{\sigma(1)}, \dots, a_{\sigma(r)})$ for some permutation $\sigma \in S_r$;

(T2) $n_i \colon (a_1, \dots, a_r) \mapsto (a_1, \dots, a_{i-1}, a_i^{-1}, a_{i+1}, \dots, a_r)$ for some index i;

(T3) $n_{ij} \colon (a_1, \dots, a_r) \mapsto (a_1, \dots, a_{i-1}, a_i a_j, a_{i+1}, \dots, a_r)$ which replaces a_i by $a_i a_j$ for some $i \neq j$.

Note that the transformation (T1) can be obtained from the other two transformations, but it is convenient to use. Thus we have three types of automorphisms: $n_\sigma, n_i, n_{ij} \in \mathrm{Aut}(F_r)$.

Definition 3.42. Let $\mathcal{N} = (N_1, \dots, N_r)$ be an independent and regular system of hypersurfaces in a closed manifold M. We define analogous operations on $\mathcal{H}_r(M)$:

(H1) $\mathcal{N} \mapsto \mathcal{N}^\sigma := (N_{\sigma(1)}, \dots, N_{\sigma(r)})$ for some permutation $\sigma \in S_r$;

(H2) $\mathcal{N} \mapsto \mathcal{N}^i$ is obtained by changing the framing of the submanifold N_i to the one with opposite orientation;

(H3) $\mathcal{N} \mapsto \mathcal{N}^{ij}$ is obtained for $i \neq j$ by replacing N_j by $N_j \#_\gamma P_1(N_i)$, where γ is an arc as in Lemma 3.8 which intersects $\mathcal{N}$ only in two points and joins N_j and $P_1(N_i)$ from the same side.

An arc γ in (H3) always exists since $\mathcal{N}$ is an independent system. Then for the obtained system $\mathcal{N}^{ij}$ we take smaller tubular neighbourhoods to be disjoint, e.g. $P_{[-1,\frac{1}{2}]}(N_i) \cong N_i \times [-1, \frac{1}{2}]$. By Lemma 3.8 the homomorphism $\varphi_{\mathcal{N}^{ij}}$ is the same as induced by the system $(N_1, \ldots, N_i, \ldots, N_j \cup P_1(N_i), \ldots, N_r)$, so it is clear by the definition that $\varphi_{\mathcal{N}^{ij}} = n_{ij} \circ \varphi_{\mathcal{N}}$. Therefore $\varphi_{\mathcal{N}^{ij}}$ is surjective and since obviously $\mathcal{N}^{ij}$ is regular, by Proposition 3.9 it is also independent, so the operation (H3) on $\mathcal{H}_r(M)$ is well defined. It does not depend on the choice of γ up to framed cobordism.

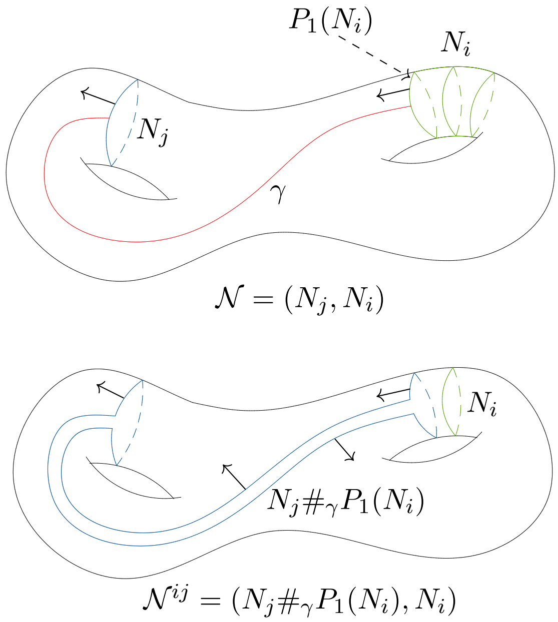


Figure 3.4: The operation (H3) which transforms the system $\mathcal{N} = (N_j, N_i)$ into $\mathcal{N}^{ij} = (N_j \#_\gamma P_1(N_i), N_i)$.

In the same way operations (H1) and (H2) are analogues of (T1) and (T2):

$$\varphi_{\mathcal{N}^\sigma} = n_\sigma \circ \varphi_{\mathcal{N}} \quad \text{and} \quad \varphi_{\mathcal{N}^i} = n_i \circ \varphi_{\mathcal{N}}.$$

Since elementary Nielsen transformations generate $\operatorname{Aut}(F_r)$, we have the following straightforward conclusion.

Proposition 3.43. *Two epimorphisms $\varphi_{\mathcal{N}}$ and $\varphi_{\mathcal{N}'}$ induced by $\mathcal{N}, \mathcal{N}' \in \mathcal{H}_r(M)$ are equivalent if and only if $\mathcal{N}'$ can be transformed by using a finite number of operations (H1) – (H3) to a system $\mathcal{N}''$ such that $\varphi_{\mathcal{N}}$ and $\varphi_{\mathcal{N}''}$ are strongly equivalent.* □

In particular, if M is a manifold for which $\overline{\overline{\Theta}}\colon \mathcal{H}_r^{fr}(M)/_{\mathrm{Diff}_\bullet(M)} \to \operatorname{Epi}(\pi_1(M), F_r)/_{\simeq}$ is a bijection, then $\varphi_{\mathcal{N}}$ and $\varphi_{\mathcal{N}'}$ are equivalent if and only if $\mathcal{N}''$ can be obtained to represent the same element of $\mathcal{H}_r^{fr}(M)/_{\mathrm{Diff}_\bullet(M)}$ as $\mathcal{N}$.

Lemma 3.44. *The operations (H1)–(H3) on $\mathcal{N}$ do not change the orientability of $M|\mathcal{N}$.*

Proof. It is clear for (H1) and (H2). For (H3) if α is an orientation-reversing loop in $M|\mathcal{N}$, then $M|\mathcal{N} \setminus \operatorname{Im} \alpha$ is also connected and a path γ between N_j and $P_1(N_i)$ can be taken to be disjoint from α, so $M|\mathcal{N}^{ij}$ is also non-orientable by Proposition 3.38. If $M|\mathcal{N}$ is orientable, but α is an orientation-reversing loop in $M|\mathcal{N}^{ij}$, then it intersects N_j and $P_1(N_i)$ in $P(\gamma)$, the tubular neighbourhood of γ. When α intersects N_j and goes into $P(\gamma)$, it may pass through $P(\gamma)$ and $P_{[0,1]}(N_i) \cong [0,1] \times N_i$, and it needs to intersect N_j again. Note that $P_{[0,1]}(N_i)$ is orientable, because $P_1(N_i)$ is orientable as a submanifold of orientable manifold $M|\mathcal{N}$. Thus α may be changed to another orientation-reversing loop lying outside $\mathcal{N}$, a contradiction. Therefore $M|\mathcal{N}^{ij}$ is also orientable. □

Theorem 3.45. *Let Σ be a closed surface and $1 \leq r \leq \operatorname{corank}(\pi_1(\Sigma))$ be an integer. Denote by p the number of equivalence classes of epimorphisms $\pi_1(\Sigma) \to F_r$. Then*

(1) If Σ is orientable or non-orientable of odd genus, then $p = 1$.

(2) if $\Sigma = S_{2m}$ is non-orientable of genus $2m$, then

- *if $r < m$, then $p = 2$,*
- *if $r = m$, then $p = 1$.*

Proof. For the first part note that $1 \leq p \leq q$, where q is the number of strong equivalence classes of epimorphisms $\pi_1(\Sigma) \to F_r$, and $q = 1$ if Σ is orientable or non-orientable of odd genus. If Σ is non-orientable of genus $2m$, then by Theorem 3.40 and Proposition 3.43 we need to investigate the operations (H1)–(H3) on the systems $\mathcal{N}_0$ and $\mathcal{N}_I$ for $\varnothing \neq I \subset \{1,\dots,r\}$. Since by the above lemma the operations do not change the orientability of complements of systems, $\varphi_{\mathcal{N}_0}$ and $\varphi_{\mathcal{N}_I}$ cannot be equivalent for any I, so $p \geq 2$ if $r < m$. We will show that all $\mathcal{N}_I$ induce equivalent epimorphisms.

Use the operation (H3) on $\mathcal{N}_J = \mathcal{N} = (N_1,\dots,N_r)$, for $i \notin J$ and $j \in J$, obtaining the system $\mathcal{N}^{ij}$ which represents the same element in $\mathcal{H}_r^{fr}(\Sigma)/_{\mathrm{Diff}_\bullet(\Sigma)}$ as $\mathcal{N}_I$ for some I. We will show that $I = J \cup \{i\}$.

First, note that $l \in J$ if and only if $\Sigma|\mathcal{N} \cup P(N_l)$ is non-orientable. Let us divide the proof into four steps:

- $j \in I$: It follows from the fact that $\Sigma|\mathcal{N}^{ij} \cup P(N_j \#_\gamma P_1(N_i)) = \Sigma|\mathcal{N} \cup P(N_j)$ is non-orientable, since $j \in J$.
- $J \setminus \{j\} \subset I$: Let $l \in J \setminus \{j\}$. Thus there is an orientation-reversing loop α in $\Sigma|\mathcal{N} \cup P(N_l)$ which intersects N_l in a one point. Since a tubular neighbourhood of α is a Möbious band, $\Sigma|N \setminus \operatorname{Im}\alpha$ is also connected and γ using in (H3) can be taken to be disjoint from α. Thus $\operatorname{Im}\alpha \subset \Sigma|\mathcal{N}^{ij} \cup P(N_l)$, so the latter subspace is non-orientable. Therefore $l \in I$ since (H3) does not depend on the choice of γ up to framed cobordism.
- $i \in I$: Take an orientation-reversing loop α in $\Sigma|\mathcal{N} \cup P(N_j)$ intersecting N_j in a one point x, which is a starting point of an arc γ joining N_j with $P_1(N_i)$, and intersecting $P_1(\gamma)$ in a one point y. Thus we may write $\alpha = \alpha_1 \cdot \alpha_2$, where α_1 is a path outside $P(\gamma)$ joining y with x. Let the endpoint of γ in $P_1(N_i)$ correspond to $(1,z) \in \{1\} \times N_i$ and take a path $\tau\colon [-1,1] \to P(N_i) \cong [-1,1] \times N_i$ defined by $\tau(t) = (-t, z)$. Moreover, take a path β from $\tau(1) \in P_{-1}(N_i)$ to y, which is contained in $\Sigma|\mathcal{N} \setminus \operatorname{Im}\gamma$ (such a path exists since γ does not disconnect $\Sigma|\mathcal{N}$). Now, form a loop $\alpha' = \alpha_1 \cdot \gamma \cdot \tau \cdot \beta$, which is contained in $\Sigma|\mathcal{N}^{ij} \cup P_{[-1,\frac{1}{2}]}(N_i)$ by taking a smaller tubular neighbourhood of γ used in the connected sum $N_j \#_\gamma P_1(N_i)$. This loop is orientation-reversing since it is homotopic to $\alpha \cdot \overline{\alpha_2} \cdot \gamma \cdot \tau \cdot \beta$, where $\overline{\alpha_2}$ is the inverse path for α_2, and $\overline{\alpha_2} \cdot \gamma \cdot \tau \cdot \beta$ is orientation-preserving as it can be homotoped to lie in $\Sigma|\mathcal{N} \cup P(N_i)$, which is orientable. Therefore $i \in I$.

- $l \notin I$ if $l \notin J \cup \{i\}$: if $\Sigma|\mathcal{N}^{ij} \cup P(N_l)$ contains an orientation-reversing loop, then as in Lemma 3.44 it leads to an orientation-reversing loop in $\Sigma|N \cup P(N_l)$, a contradiction.

Thus using (H3) we may transform any $\mathcal{N}_J$ to be the same element of $\mathcal{H}_r^{fr}(\Sigma)/_{\mathrm{Diff}_\bullet(\Sigma)}$ as $\mathcal{N}_{\{1,\ldots,r\}}$, so they all induce equivalent epimorphisms. □

Remark 3.46. We do not know any other non-trivial calculations of (strong) equivalence classes of epimorphisms $G \to F_r$. One of the reasons is that the structure of $\hom(G, F_r)$ can be alternatively described in terms of Makanin–Razborov diagrams (see [4, 39, 55, 59]). However, it leads to different structure than (strong) equivalence relation in general. The latter should also be of interest, e.g. due to connections with conjugacy of function (see Section 5.4). Surjectivity of $\overline{\overline{\Theta}}\colon \mathcal{H}_r^{fr}(M)/_{\mathrm{Diff}_\bullet(M)} \to \mathrm{Epi}(\pi_1(M), F_r)/_{\simeq}$. which is a bijection e.g. for hyperbolic manifolds, provides a geometric tool for its studies. A particular question is about finiteness of both these sets.

Chapter 4

Cycle rank of Reeb graphs and Reeb number

In this chapter we are focused on description of cycle rank of Reeb graph of Morse function on a manifold of dimension $n \geq 3$, since we already reached some results in the case of surfaces. During the considerations, we show the realization of the so-called initial graph as the Reeb graph of Morse function closely related with a given epimorphism onto free group. It is the first step in the proof of realization theorem in the next chapter.

Section 4.1 presents a class of Morse functions whose Reeb graphs are trees. In Section 4.2 we develop a very useful technique of combinatorial modifications of Reeb graphs of simple Morse functions. Section 4.3 is devoted to the construction of function, whose Reeb graph is the initial graph, such that a given independent and regular system of hypersurfaces is a part of its level set. Next, in Section 4.4 main characterizations of Reeb number are stated and proved. Moreover, we establish a correspondence between Reeb epimorphisms and regular and independent systems of hypersurfaces without boundary. Section 4.5 discusses the problem of extendability of independent systems of hypersufaces.

4.1 Minimizing the cycle rank – ordered Morse functions

We know that simple Morse functions maximize the cycle rank of Reeb graphs of smooth functions with finitely many critical points. On the other hand, Reeb graphs of self-indexing Morse functions are always trees. In general, ordered Morse functions have also Reeb graphs with no cycles if a manifold is of dimension at least three. These facts will be used in the realization of initial graph as a Reeb graph (Theorem 4.11).

For simplicity, for $\mathcal{R}$-simple Morse functions we define the **index of vertex** v of $\mathcal{R}(f)$ to be the index of the corresponding critical point. We also extend this definition for arbitrary graphs with good orientation — the index of a vertex v of degree 3 is 1 if $\deg_{in}(v) = 2$ and $n-1$ if $\deg_{out}(v) = 2$, when n is known from the context.

Proposition 4.1. *Let $f\colon W \to \mathbb{R}$ be an ordered Morse function on a smooth triad (W, W_-, W_+) of dimension $n \geq 3$. Then $\mathcal{R}(f)$ is a tree. In particular, the Reeb graph of a self-indexing Morse function is a tree (even for $n = 2$).*

Proof. Every critical level of f contains critical points of the same index. Therefore by Lemma 2.7

and Remark 2.8 performing a small perturbation of f we can obtain a simple and still ordered Morse function $g\colon W \to \mathbb{R}$ with the same critical points and of the same index as f such that $\beta_1(\mathcal{R}(f)) \leq \beta_1(\mathcal{R}(g))$.

Suppose that the Reeb graph $\mathcal{R}(g)$ has a cycle. By index-degree correspondence (Proposition 1.19) there are two vertices of degree 3 and of index 1 and $n-1$ which are the highest and the lowest vertices in this cycle, respectively. This is a contradiction, since g is ordered. Hence $\mathcal{R}(g)$ is a tree, so $\mathcal{R}(f)$ also.

The case of a self-indexing Morse function on a surface follows as in Example 2.2. □

Lemma 4.2. *For any ordered Morse function $f\colon W \to \mathbb{R}$ on a smooth triad (W, W_-, W_+), $\dim W \geq 3$, there exist a regular value c such that W_c is non-empty and connected.*

Proof. We may assume that f is simple by changing it on arbitrary small neighbourhoods of critical points. Suppose that f has critical points of indices 1 and $n-1$. Then by Proposition 1.19 a subgraph of $\mathcal{R}(f)$ between the highest vertex of index 1 and the lowest vertex of index $n-1$ is homeomorphic to the interval and c can be taken from levels corresponding to this subgraph. If there is no vertex of index 1 (for $n-1$ we proceed analogously), then $W_- = \varnothing$ and f has one minimum or $W_- \neq \varnothing$ is connected and f does not have any critical point being a minimum. In both the cases $\mathcal{R}(f)$ has a unique vertex v with indegree 0. Thus we may take any regular value through which the edge incident to v passes. □

4.2 Combinatorial modifications of Reeb graphs

This section presents the combinatorial modifications of Reeb graphs which are effective tool in their study. We show that any graph can be transformed using a finite number of modifications to the so-called canonical form. As a conclusion, any number between 0 and $\mathcal{R}(M)$ can be realized as the cycle rank of Reeb graph of Morse function on M.

Lemma 4.3. *Let $f\colon W \to \mathbb{R}$ be a simple Morse function with exactly two critical points p and p' on a smooth triad (W, W_-, W_+), where $n = \dim W \geq 3$. Let $f(p) > f(p')$ and assume that $\mathcal{R}(f)$ is isomorphic to the graph on the left side of the case (i) in Figure 4.1. If $\operatorname{ind}(p) \leq \operatorname{ind}(p')$, then there exists a simple Morse function $g\colon W \to \mathbb{R}$ with the same critical points and of the same index as f, such that $g(p) < g(p')$ and $\mathcal{R}(g)$ is isomorphic to the graph on the right side of the case (i) in Figure 4.1 (in the cases (4) and (5) we require the order of the vertices corresponding to the components of $W_\pm = V_1^\pm \sqcup V_2^\pm \sqcup V_3^\pm$ determined by a permutation $\sigma\colon \{1,2,3\} \to \{1,2,3\}$).*

The above lemma provides a technique of **combinatorial modifications of Reeb graphs** by modifications of simple Morse functions. To be more precise, let $f\colon W \to \mathbb{R}$ be a simple Morse function on a smooth triad (W, W_-, W_+), and let v and w be adjacent vertices of $\mathcal{R}(f)$ which correspond to critical points p and p' of f, respectively, such that $\operatorname{ind}(p) \leq \operatorname{ind}(p')$ and $f(p) > f(p')$. Assume that W', the connected component of $W^{[f(p')-\varepsilon, f(p)+\varepsilon]}$ containing p and p', contains no other critical points of f. Then f can be modified on W' to a simple Morse function g such that the Reeb graphs $\mathcal{R}(f|_{W'})$ and $\mathcal{R}(g|_{W'})$ are isomorphic to the graphs on the left and on the right side of a suitable case in Figure 4.1, respectively.

In fact, except the case (6), if vertices v and w are adjacent, we can always assume that p and p' are two consecutive critical points by rescaling f on the triad corresponding to a small neighbourhood of the edge joining the vertices.

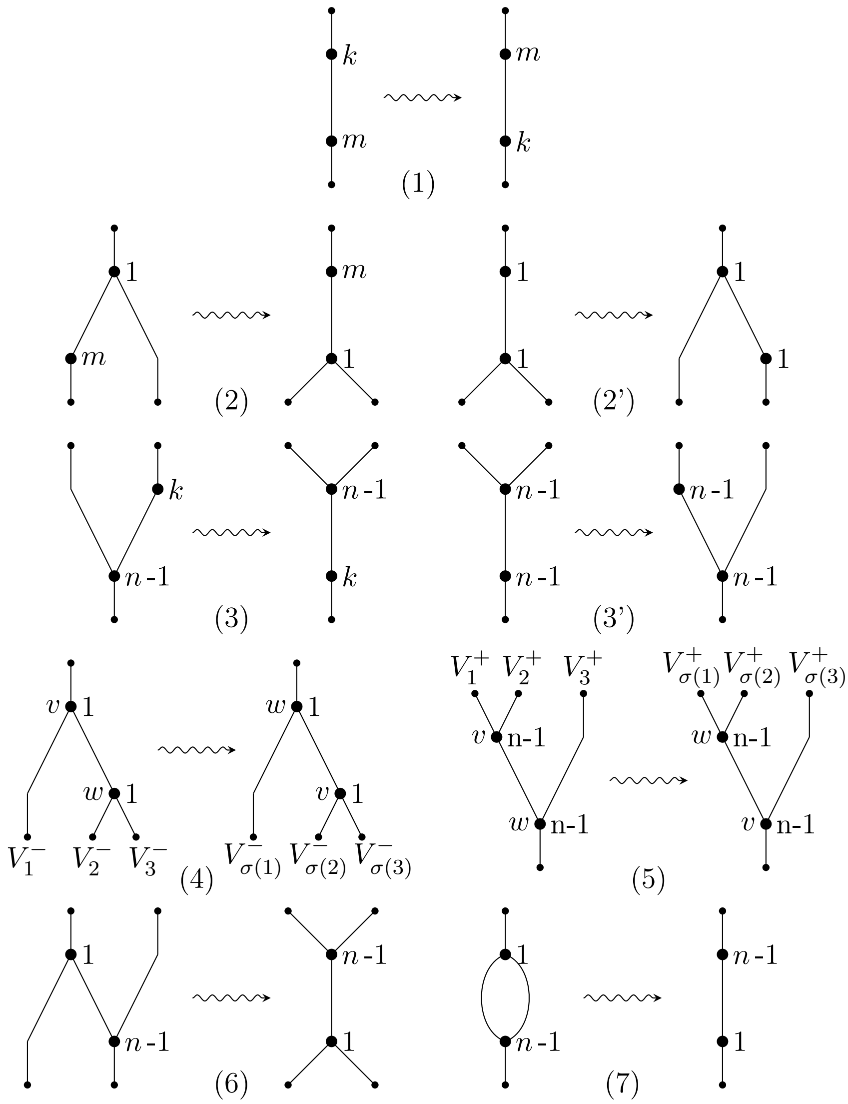


Figure 4.1: Combinatorial modifications of Reeb graphs related to change of the order of two consecutive critical points. On the right sides of vertices are their indices, where $1 \le k \le m \le n-1$.

It is easily seen that for vertices v and w of degree 2 or 3 there are no cases other than those presented in Figure 4.1.

Proof. By Theorem 1.14 there exists a simple Morse function $g\colon W \to \mathbb{R}$ on (W, W_-, W_+) such that $g(p) < g(p')$ and with the same critical points and indices as f. We need only to show changes in the Reeb graph. Let $q_g\colon W \to \mathcal{R}(g)$ be the quotient map, $k := \operatorname{ind}(p) \le \operatorname{ind}(p') =: m$ and let $v := q_g(p)$, $w := q_g(p')$ be the vertices of $\mathcal{R}(g)$. The main properties, that we need, are connectedness of $\mathcal{R}(g)$, the number of connected components of $W_\pm$ and the correspondence from Proposition 1.19.

Case (1). Here W_- and W_+ are connected. Since $W_{g(p)-\varepsilon}$ has the same number of connected components as W_-, $\deg_{in}(v) = 1$. Similarly, $\deg_{out}(w) = 1$. If $\deg_{out}(v) = 2$, there would be $\deg_{in}(w) = 2$, so $k = n-1$ and $m = 1$, a contradiction. Therefore $\deg_{out}(v) = 1 = \deg_{in}(w)$ and so $\mathcal{R}(g)$ is isomorphic to the graph on the right side of (1).

Case (2). Assume that $m \neq 1$. From the same reason as above $\deg_{in}(v) = 2$, and therefore

$\deg_{out}(v) = 1$. Thus $\deg_{in}(w) = \deg_{out}(w) = 1$ and $\mathcal{R}(g)$ is as desired. In the same way we show the other cases when $k \neq m$.

An additional argument is needed when $k = m \in \{1, n-1\}$. Suppose that $k = m = 1$ (for $n-1$ the proof is by duality). Consider a handle decomposition of W corresponding to f. It consists of two 1-handles attached along embeddings of $\mathrm{S}^0 \times \mathrm{D}^{n-1} = \mathrm{D}^{n-1} \sqcup \mathrm{D}^{n-1}$. By handle theory we can isotopically separate the images of these embeddings. Therefore the handles are attached to $W_- \times [0,1]$ along embeddings $\mathrm{D}^{n-1} \sqcup \mathrm{D}^{n-1} \to W_- \times \{1\}$. Since we can attach the handles in any order and because closed connected manifolds are homogeneous (i.e. for any two disjoint copies of D^{n-1} there is a self-diffeomorphism isotopic to the identity which maps the one disc to the another), we can change the embeddings arbitrarily. Thus the handles can be attached to a required components of $W_- \times \{1\}$. $\square$

The above lemma provides modifications which rely on Rearrangement theorem. The Cancellation of handles (Proposition 1.15) gives us two additional modifications of Reeb graphs presented in Figure 4.2. In fact, they work in both ways. We can always assume that the critical points corresponding to vertices in (8) or (9) are consecutive by rescalling the function.

Figure 4.2: Combinatorial modifications of Reeb graphs related to Cancellation of Handles.

Remark 4.4. Note that similar operations were introduced by B. Di Fabio and C. Landi [12] for Reeb graphs of simple Morse functions on closed orientable surfaces. They called them elementary deformations. For manifolds of dimension at least three indices of critical points play important role in the modifications (1) – (7) which can be used only if the index of the upper vertex is not greater than the index of the lower vertex. This causes that many of them do not work the other way around, contrary to the case of surfaces and elementary deformations, where the index of a critical point not being extremum is always equal to 1. Another difference between combinatorial modifications and elementary deformations is in the occurrence of vertices of degree 2 for the former operations. Reeb graphs of simple Morse functions on orientable surfaces have only vertices of degrees 1 and 3 (see Proposition 1.19). Their cycle rank is always equal to the genus of a surface, so the modification (7) also does not occur for them.

Fabio and Landi used the elementary deformations to define an edit distance between Reeb graphs. For this purpose they added labels of vertices of Reeb graph with function values.

Let Γ be a graph with good orientation. A vertex v of degree 3 and of index 1 (respectively of index $n-1$) in Γ is **branching**, if there exist two decreasing (respectively increasing) paths $\gamma, \delta \colon [0,1] \to \Gamma$ such that $\gamma(0) = \delta(0) = v$, their images are disjoint outside v and both $\gamma(1)$ and $\delta(1)$ are vertices of degree 1 in Γ.

Remark 4.5. Note that defined modifcations, except (7), do not change the cycle rank of Reeb graph. One can see it directly or by Lemma 2.5. Moreover, except (4), (5), (8) and (9) they also do not change the property of being branching for a vertex of degree 3. It is easy to check that the same is true for (4) and (5) if one of the vertices is branching and the other not.

Recall that a vertex of degree 1 with an outgoing edge (incoming edge) is called a minimum (maximum) .

Lemma 4.6. *Each simple Morse function $f\colon M \to \mathbb{R}$ on a closed n-manifold M, $n \geq 3$, can be modified using a finite number of combinatorial modifications to a simple Morse function with exactly one minimum and maximum and with the same cycle rank of the Reeb graph.*

Proof. We take the lowest branching vertex v of degree 3 and index 1 and we move it down using the modifications (2), (4) and (6) so that it is adjacent to two minima. Since v is branching, the modification (6) can be used and we do not have to use (7). Also v will still be branching after (4), since it is the lowest vertex with this property (see Remark 4.5). Then we use (8) to remove v and a one minimum and we repeat this procedure for each branching vertex of index 1.

It is an easy exercise to show that obtained Reeb graph has exactly one minimum. The proof for maxima is analogous. □

By the above lemma $\mathcal{R}(M)$ can be attained by simple Morse functions with one minimum and maximum. By Lemma 2.5 for such a function $f\colon M \to \mathbb{R}$ we have $\beta_1(\mathcal{R}(f)) = \frac{\Delta_3(\mathcal{R}(f))}{2}$. Therefore

$$\mathcal{R}(M) = \max\left\{ \frac{\Delta_3(\mathcal{R}(f))}{2} \;\middle|\; \begin{array}{c} f\colon M \to \mathbb{R} \text{ - simple Morse function} \\ \text{with one minimum and maximum} \end{array} \right\}.$$

Definition 4.7 (cf. [12, Definition 1.3], [35])**.** The graph shown in Figure 4.3 (a) is called **the canonical graph** (with a given cycle rank). A graph is in a **canonical form** if it is homeomorphic to the canonical graph and the homeomorphism adds vertices of degree 2 only on non-cyclic edges.

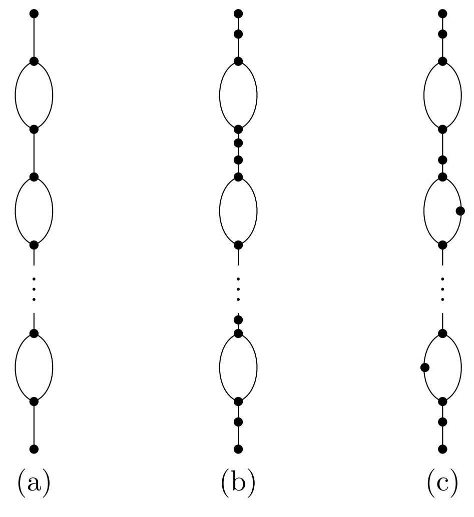


Figure 4.3: (a) the canonical graph; (b) graph in a canonical from; (c) graph not in a canonical form.

The canonical graph with cycle rank equal to g is the Reeb graph of a height function on a closed orientable surface of genus g. A tree is in a canonical form if it is a path.

The following proposition is similar to the ones for orientable surfaces which were shown by E. Kudryavtseva [35, Theorem 1] and Fabio–Landi [12, Lemma 2.6].

Proposition 4.8. *Let $f\colon M \to \mathbb{R}$ be a simple Morse function on a closed manifold M of dimension $n \geq 3$. Then f can be modified using a finite number of combinatorial modifications to a simple Morse function whose Reeb graph has the same cycle rank and is in a canonical form.*

Proof. By Lemma 4.6 we may assume that f has exactly one minimum and maximum. If $\mathcal{R}(f)$ is a tree, then it is in a canonical form, so assume that $\beta_1(\mathcal{R}(f)) \geq 1$.

First, we move down (move up) all vertices of degree 2 and of index 1 (of index $n-1$) in $\mathcal{R}(f)$ using the modifications (1), (2′) and (3) ((1), (2) and (3′)) so that below the highest vertex of degree 2 and of index 1 (above the lowest vertex of degree 2 and of index $n-1$) there will be only other such vertices and the minimum (the maximum).

Let v be the lowest vertex of degree 3 and of index 1 and let w be the highest vertex of degree 3 and of index $n-1$ which meets two different decreasing paths γ and δ starting from v. On paths γ and δ there are vertices of indices $2, \dots, n-1$. We move all of them above v using the modifications (2) and (6) (we do not use (4) and (7)). We obtain a graph with a neighbourhood of v and w as on the left side of (7).

On the path from w to the minimum there may be other vertices of degree 3 and of index $n-1$. Let u be the highest such a vertex. Using (3) we move it up just below w. Now, the situation is as in Figure 4.4 (*i*) and we perform the modifications (5) and (6) as in the figure. We repeat this procedure for all such vertices u. Then below w there are only vertices of degree 2 and the minimum.

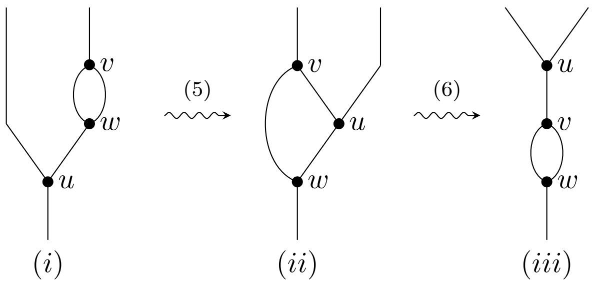


Figure 4.4: Moving u up above the cycle created by v and w.

Performing the entire above procedure for each vertex of degree 3 and of index 1 we obtain a graph in a canonical form. The cycle rank is unchanged. □

As a conclusion we get the main theorem of this section.

Theorem 4.9. *Let M be a closed manifold of dimension $n \geq 2$. For any number $0 \leq k \leq \mathcal{R}(M)$ there exists a Morse function $g\colon M \to \mathbb{R}$ such that $\beta_1(\mathcal{R}(g)) = k$ and it can be simple if M is not an orientable surface.*

Proof. For $n \geq 3$ let $f\colon M \to \mathbb{R}$ be a simple Morse function such that $\beta_1(\mathcal{R}(f)) = \mathcal{R}(M)$ and $\mathcal{R}(f)$ is in a canonical form. By $(\mathcal{R}(M) - k)$-fold use of the modification (7) we get a simple Morse function g such that $\mathcal{R}(g)$ has cycle rank equal to k.

If $n = 2$, then the statement follows by the direct constructions of functions (Corollary 2.22 and Theorem 2.24). The exception for orientable surfaces comes from the fact that by Lemma 2.1 the Reeb graph of a simple Morse function on a closed orientable surface of genus g has always cycle rank equal to g . □

Remark 4.10. The above results concerning canonical form can be derived also for smooth triads by defining the canonical graph admissible for (W, W_-, W_+). Combinatorially, it would be of the same difficulty, but it would complicate the notation. An analogues results will follow from Theorem 4.18 and Theorem 5.6.

4.3 The initial graph

We begin to study the relations between Reeb epimorphisms, systems of hypersurfaces and Reeb graphs. We start by showing a realization of a particular graph as Reeb graphs under several additional conditions, what will later be the base for the realization theorem.

The graph presented in Figure 4.5 (a) is called the **initial graph** (with a given cycle rank r). We distinguish a spanning tree in the initial graph coloured red in the figure. Moreover, we order the edges $e_1, \ldots, e_r$ outside the tree. We distinguish also the version of initial graph admissible for a triad (W, W_-, W_+), see Figure 4.5 (b).

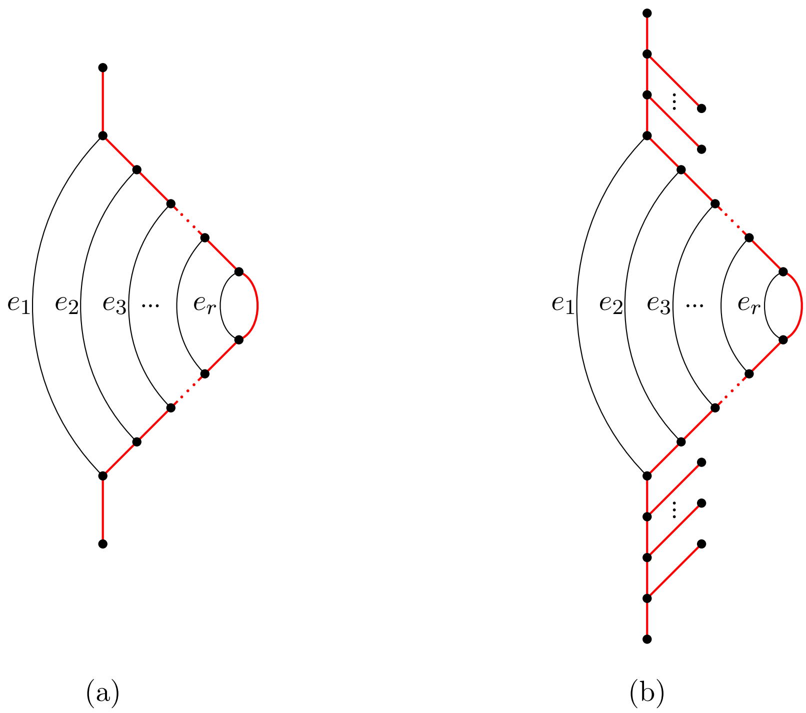


Figure 4.5: (a) the initial graph with distinguished tree and ordered edges outside it; (b) initial graph admissible for (W, W_-, W_+).

The initial graph with cycle rank equal to g occurs easily as the Reeb graph of a height function on an orientable surface of genus g. In fact, by Theorem 2.23 it can be the Reeb graph of a Morse function on any closed surface with the Reeb number at least g.

Theorem 4.11. *Let $\partial W = W_- \sqcup W_+$ and $\mathcal{N} = (N_1, \ldots, N_r)$ be a system of hypersurfaces without boundary in W. Then the induced homomorphism $\varphi_{\mathcal{N}}$ is factorized by the Reeb epimorphism of a simple Morse function $f \colon W \to \mathbb{R}$ on the smooth triad (W, W_-, W_+) such that the connected components of $\mathcal{N}$ are components of some regular level $f^{-1}(c) = \mathcal{N} \sqcup V$, where V is a non-empty submanifold. Moreover, if we allow f to be not necessarily simple for* $\dim W = 2$, *then*

$$\beta_1(\mathcal{R}(f)) = |\pi_0(\mathcal{N})| - |\pi_0(W|\mathcal{N})| + 1.$$

Furthermore, if $\mathcal{N}$ is regular and independent, then $\mathcal{R}(f)$ can be taken to be homeomorphic to the initial graph admissible for (W, W_-, W_+), with cycle rank equal to r and such that N_i corresponds to the edge e_i for each i as in Figure 4.5.

Proof. For the smooth triad $(W|\mathcal{N}, P_{-1}(\mathcal{N}) \sqcup W_+, P_1(\mathcal{N}) \sqcup W_-)$ take an ordered Morse function $h \colon W|\mathcal{N} \to \mathbb{R}$ and a regular value d from Lemma 4.2 such that $V := h^{-1}(d)$ has the same number of connected components as $W|\mathcal{N}$. Let

$$P(V) := h^{-1}([d-\varepsilon, d+\varepsilon]) \cong V \times [-1,1], \; Q_- := h^{-1}((-\infty, d-\varepsilon]) \; \text{ and } \; Q_+ := h^{-1}([d+\varepsilon, \infty)).$$

Then

$$\partial Q_- = P_{-1}(V) \sqcup P_{-1}(\mathcal{N}) \sqcup W_+ \quad \text{and} \quad \partial Q_+ = P_1(V) \sqcup P_1(\mathcal{N}) \sqcup W_-,$$

Thus V, Q_+ and Q_- have the same number of connected components. Now, take simple and ordered Morse functions

$$g_- \colon Q_- \to [-2,-1] \text{ on the triad } (Q_-, \emptyset, \partial Q_-),$$

$$g_+ \colon Q_+ \to [1,2] \text{ on the triad } (Q_+, \partial Q_+, \emptyset).$$

Let us glue them together with suitable projections $P(\mathcal{N} \sqcup V) \to [-1,1]$ obtaining the simple Morse function $f \colon W \to \mathbb{R}$ with regular value 0 such that $f^{-1}(0) = \mathcal{N} \sqcup V$.

Let $q_f \colon W \to \mathcal{R}(f)$ and $g \colon \mathcal{R}(f) \to \mathcal{R}(f)/q(W|\mathcal{N}) = \bigvee_{i=1}^r \mathrm{S}^1_i$ be the quotient maps. The map g sends $q_f(W|\mathcal{N})$ to the basepoint and $q_f(P(N_i)) \cong [-1,1]$ linearly and orientation-preserving onto S^1_i. It is clear that $\varphi_{\mathcal{N}} = (f_{\mathcal{N}})_\# = (g \circ q_f)_\# = g_\# \circ (q_f)_\#$, so $\varphi_{\mathcal{N}}$ is factorized by the Reeb epimorphism $\varphi_f = (q_f)_\#$ of f.

Now, let us compute $\beta_1(\mathcal{R}(f))$. The subset $q_f(W|\mathcal{N})$ of $\mathcal{R}(f)$ is homeomorphic to the Reeb graph $\mathcal{R}(f|_{W|\mathcal{N}})$, so it has $|\pi_0(W|\mathcal{N})|$ connected components. If $\dim W \geq 3$, then the components of $\mathcal{R}(f|_{W|\mathcal{N}})$ are trees, because components of $\mathcal{R}(f|_{Q_\pm})$ are trees by Proposition 4.1 (since the Morse functions on $Q_\pm$ are ordered) and they are gluing through $\mathcal{R}(f|_{P(V)}) \cong [-1,1]$, one component of $\mathcal{R}(f|_{Q_\pm})$ with only one component of $\mathcal{R}(f|_{P(V)})$. In the case of surfaces by the same fact we may define Morse functions (self-indexing, not simple) on components of $Q_\pm$ whose Reeb graphs are trees. In both the cases $q_f(W|\mathcal{N})$ has $|\pi_0(W|\mathcal{N})|$ components which are trees, and so they are contractible.

Thus the quotient $\mathcal{R}(f)/q_f(W|\mathcal{N})$ can be obtained from $\mathcal{R}(f)$ by first the contraction of components of $q_f(W|\mathcal{N})$, and then by gluing them to the one point. The first operation does not change the first Betti number, but the second increases it by one for each gluing of two points. Hence $\mathcal{R}(f)/q_f(W|\mathcal{N})$ has cycle rank equal to $\beta_1(\mathcal{R}(f)) + |\pi_0(W|\mathcal{N})| - 1$. On the other hand, it is clear that $\mathcal{R}(f)/q_f(W|\mathcal{N})$ is homeomorphic to the wedge product of $|\pi_0(\mathcal{N})|$ circles. Therefore

$$|\pi_0(\mathcal{N})| = \beta_1(\mathcal{R}(f)) + |\pi_0(W|N)| - 1.$$

Now, let $\mathcal{N}$ be a regular and independent system of hypersurfaces. Let $\dim W \geq 3$. Since $W|\mathcal{N}$ and V are connected, the manifolds $Q_\pm$ are also connected, so we may assume that $g_\pm$ has only one critical point being extremum. Then by Proposition 4.1 the Reeb graph $\mathcal{R}(g_-)$ (resp. $\mathcal{R}(g_+)$) is a tree with one minimum (maximum) and so all vertices of degree 3 have indegree 1 (outdegree 1). By the proved formula on cycle rank we have $\beta_1(\mathcal{R}(f)) = r$. By means of the combinatorial modifications we move up (move down) all vertices of degree 2 in $\mathcal{R}(g_-)$ (in $\mathcal{R}(g_+)$ respectively). Then by using the modification (4) on $f|_{Q_-} = g_-$ and (5) on $f|_{Q_+} = g_+$ we can obtain a simple Morse function on W whose Reeb graph is the initial graph admissible for (W, W_-, W_+) with the desired correspondence between N_i and its edges e_i.

The last statement in the case of surfaces can be similarly obtained with some additional effort or it follows by Theorem 5.4 (Theorem 4.11 is not used in the proof of Theorem 5.4 for surfaces). □

Remark 4.12. For $\dim W = 2$ note that the components of $Q_\pm$ can be either orientable or non-orientable, even if W is non-orientable. Since on an orientable component a simple Morse function has Reeb graph with the maximum cycle rank, it is a reason why simplicity of a Morse function in the theorem is excluded also if W is non-orientable surface.

Using the above theorem we may easily prove the last part of Proposition 3.9. If $\mathcal{N}$ is regular and $\varphi_{\mathcal{N}}$ is surjective, then by Theorem 4.11 the epimorphism $\varphi_{\mathcal{N}}$ is factorized by a Reeb epimorphism of rank $r' = r - |\pi_0(W|\mathcal{N})| + 1$. Since $r \leq r'$, it implies that $|\pi_0(W|\mathcal{N})| \leq 1$, so $\mathcal{N}$ is independent.

This theorem can be also used to easily prove the following known fact for orientable surfaces.

Corollary 4.13. *Any epimorphism $\varphi\colon \pi_1(\Sigma_g) \to F_r$ is factorized through an epimorphism $\pi_1(\Sigma_g) \to F_g$.*

Proof. By Theorem 4.11 φ is factorized by the Reeb epimorphism of a simple Morse function $f\colon \Sigma_g \to \mathbb{R}$, whose rank is equal to $\beta_1(\mathcal{R}(f))$. By Lemma 2.1 the Reeb graph of a simple Morse function on Σ_g has always cycle rank equal to g, so $\beta_1(\mathcal{R}(f)) = g$. □

In fact, since any two epimorphisms $\pi_1(\Sigma_g) \to F_g$ are strongly equivalent by Theorem 3.18, for a fixed epimorphism $\psi\colon \pi_1(\Sigma_g) \to F_g$ any $\varphi\colon \pi_1(\Sigma_g) \to F_r$ is factorized through $\psi \circ \eta$ for some $\eta \in \mathrm{Aut}(\pi_1(\Sigma_g))$.

4.4 Reeb number and corank of fundamental group

In this section we are focused on describing the Reeb number of manifold, algebraically and geometrically.

For a compact manifold W, possibly with boundary, we define (following O. Cornea [10]) the number $C(W)$ to be the maximum number of connected components in a proper, 2-sided submanifold N of W such that $W \setminus N$ is connected. In other words, it is the maximum size of an independent and regular system in W.

O. Cornea showed the equality $\mathcal{R}(M) = C(M)$ for a closed manifold M. The following theorem was proven by W. Jaco [26] for combinatorial manifolds. Cornea announced only inequality $C(W) \geq \mathrm{corank}(\pi_1(W)) - |\pi_0(\partial W)| + 1$ if $\partial W \neq \varnothing$, but the theorem holds also in the smooth category.

Theorem 4.14. *$C(W) = \mathrm{corank}(\pi_1(W))$.*

Proof. If there is an independent and regular system $\mathcal{N}$ of size $k = C(W)$, then the induced homomorphism $\varphi_{\mathcal{N}}$ is onto F_k, so $C(W) \leq \mathrm{corank}(\pi_1(W))$. From the other side, any epimorphism onto the free group of rank equal to $\mathrm{corank}(\pi_1(W))$ is by Theorem 3.11 induced by a regular and independent system, so $C(W) = \mathrm{corank}(\pi_1(W))$. □

Proposition 4.15. *$\mathcal{R}(W, W_-, W_+)$ is equal to the maximum size of an independent and regular system without boundary in W. Thus it does not depend on the partition $\partial W = W_- \sqcup W_+$.*

Proof. By Theorem 4.11 if $\mathcal{N}$ is a regular and independent system without boundary in W of size r, then $\varphi_{\mathcal{N}}$ is factorized by the Reeb epimorphism of rank $|\pi_0(\mathcal{N})| - |\pi_0(W|\mathcal{N})| + 1 = r$, which gives inequality in a one way. However, if f is a simple Morse function on (W, W_-, W_+) such that $\mathcal{R}(f)$ has cycle rank equal to $\mathcal{R}(W, W_-, W_+)$, then components of level sets of f corresponding to edges of $\mathcal{R}(f)$ outside some spanning tree form a regular and independent system of hypersurfaces in W of size $\mathcal{R}(W, W_-, W_+)$ (cf. Section 1.4). $\square$

Therefore, from now on, we will write $\mathcal{R}(W)$ instead of $\mathcal{R}(W, W_-, W_+)$. It is obvious that $\mathcal{R}(W) \leq C(W)$.

Remark 4.16. Note that $\mathcal{R}(W)$ can be defined as the maximum cycle rank among Reeb graphs of Morse functions on W which are constant on connected components of ∂W. We use triads for simplifying considerations.

Let $\operatorname{Cone}(X) := X \times [0,1]/X \times \{1\}$ denote the cone over a space X. The point corresponding to $X \times \{1\}$ is called the vertex of the cone.

For a compact manifold W with boundary $\partial W = A \sqcup B$ define

$$\operatorname{Cone}_{\partial A}(W) := W \cup_A \bigcup_{i=1}^{k} \operatorname{Cone}(A_i),$$

which is obtained by gluing cones $\operatorname{Cone}(A_i)$ and W along A, where $A_1, \ldots, A_k$ are all connected components of A. Let v_i be the vertex of $\operatorname{Cone}(A_i)$. Clearly, we may identify

$$\operatorname{Cone}_{\partial A}(W) \setminus \{v_1, \ldots, v_k\} \cong W \setminus A.$$

Hereafter, we denote by $\langle \pi_1(A) \rangle^{\pi_1(W)}$ the normal subgroup of $\pi_1(W)$ generated by all images of $\pi_1(A_i)$ in $\pi_1(W)$ of the homomorphisms induced by inclusions $A_i \subset W$. By Seifert–van Kampen theorem

$$\pi_1(\operatorname{Cone}_{\partial A}(W)) \cong \pi_1(W)/\langle \pi_1(A) \rangle^{\pi_1(W)}.$$

It is clear that up to isomorphism this group is well-defined without referencing to the basepoint.

Proposition 4.17. *Let W be a compact manifold and $\partial W = A \sqcup B$. Then an epimorphism $\varphi\colon \pi_1(W) \to F_r$ is factorized through the quotient $\pi_1(W)/\langle \pi_1(A) \rangle^{\pi_1(W)}$ if and only if it is induced by an independent and regular system $\mathcal{N}$ such that $\mathcal{N} \cap A = \varnothing$.*

Proof. Set $H := \langle \pi_1(A) \rangle^{\pi_1(W)}$. If $\mathcal{N}$ is an independent and regular system such that $\mathcal{N} \cap A = \varnothing$, then clearly the images in $\pi_1(W)$ of loops in A are contained in the kernel of $\varphi_{\mathcal{N}}$, so $\varphi_{\mathcal{N}}$ is factorized through $\pi_1(W)/H$.

Conversely, assume that $\varphi = \psi \circ \eta$, where $\eta\colon \pi_1(W) \to \pi_1(W)/H$ and $\psi\colon \pi_1(W)/H \to F_r$. We proceed as in the proof of Proposition 3.3. Let ψ be induced by $f\colon \operatorname{Cone}_{\partial A}(W) \to \bigvee_{i=1}^{r} S^1_i$ which is a smooth map outside $\{v_1, \ldots, v_k\}$ and the inverse image of the basepoint. Take regular values $a_i \in S^1_i$ and define

$$N_i = f^{-1}(a_i) \subset \operatorname{Cone}_{\partial A}(W) \setminus \{v_1, \ldots, v_k\} \cong W \setminus A.$$

Thus $\mathcal{N} = (N_1, \ldots, N_r)$ is a system in $\mathrm{Cone}_{\partial A}(W)$ which induces ψ such that $\mathcal{N} \cap A = \varnothing$. Clearly, as a system in W it induces φ. It is easy to check that the procedures in proofs of Lemma 3.8 and Theorem 3.11 give an independent and regular system $\mathcal{N}'$ inducing φ which also satisfies $\mathcal{N}' \cap A = \varnothing$. □

Theorem 4.18. *For an epimorphism $\varphi\colon \pi_1(W) \to F_r$ the following are equivalent:*

(1) $\varphi = \varphi_{\mathcal{N}}$ for an independent and regular system $\mathcal{N}$ without boundary;

(2) φ is factorized through $\pi_1(W)/\langle\pi_1(\partial W)\rangle^{\pi_1(W)}$;

(3) there is a Morse function f (simple if $\dim W \geq 3$) on any triad (W, W_-, W_+) and a spanning tree T in $\mathcal{R}(f)$ such that $\varphi = (p_T \circ q_f)_\#$, where $q_f\colon W \to \mathcal{R}(f)$ and $p_T\colon \mathcal{R}(f) \to \mathcal{R}(f)/T = \bigvee_{i=1}^{r} \mathrm{S}^1$ are quotient maps.

Thus

$$\mathcal{R}(W) = \mathrm{corank}\left(\pi_1(W)/\langle\pi_1(\partial W)\rangle^{\pi_1(W)}\right).$$

Proof. The equivalence of (1) and (2) follows from the above proposition for $A = \partial W$. If $\varphi = \varphi_{\mathcal{N}}$ for an independent and regular system $\mathcal{N} = (N_1, \ldots, N_r)$ without boundary, then by Theorem 4.11 there is a Morse function f (simple if $\dim W \geq 3$) on (W, W_-, W_+) whose Reeb graph has cycle rank equal to r and components of $\mathcal{N}$ correspond to edges outside some spanning tree T of $\mathcal{R}(f)$. Thus $(p_T \circ q)_\# = \varphi_{\mathcal{N}}$. This proves that (1) implies (3), and the converse is clear.

By Proposition 4.15 we get $\mathcal{R}(W) = \mathrm{corank}\left(\pi_1(W)/\langle\pi_1(\partial W)\rangle^{\pi_1(W)}\right)$. □

Corollary 4.19. *Let M be a closed manifold of dimension $n \geq 2$ and let $r \geq 0$ be an integer. The following are equivalent:*

(a) There exists a Morse function $g\colon M \to \mathbb{R}$ such that $\beta_1(\mathcal{R}(g)) = r$.

(b) There exists an epimorphism $\pi_1(M) \to F_r$.

(c) There exist a regular and independent system of hypersurfaces in M of size r.

Moreover, if M is not an orientable surface, then the function g in the case (a) can be taken to be simple. □

A straightforward conclusion is the following equality which has been also proven by I. Gelbukh [14, Theorem 13] for orientable manifolds by means of foliation theory.

Corollary 4.20. *If M is a closed manifold, then $\mathcal{R}(M) = \mathrm{corank}(\pi_1(M))$.* □

The properties of corank imply analogues results for Reeb number.

Corollary 4.21. *Let M and N be closed manifolds of dimension $n \geq 2$. Then*

(a) $\mathcal{R}(M \times N) = \max\{\mathcal{R}(M), \mathcal{R}(N)\}$,

(b) $\mathcal{R}(M \# N) = \mathcal{R}(M) + \mathcal{R}(N)$ if $n \geq 3$.

□

Remark 4.22. The above equation for connected sum is also true if one of the surfaces is orientable, but it does not hold for non-orientable surfaces. Let $K = \mathbb{R}P^2 \# \mathbb{R}P^2$ be the Klein bottle. Then $\mathcal{R}(K) = 1$, but $\mathcal{R}(\mathbb{R}\mathrm{P}^2) = 0$.

Example 4.23. $\mathcal{R}(\#_{i=1}^{g} \mathrm{S}^1 \times \mathrm{S}^{n-1}) = g$ and $\mathcal{R}(\#_{i=1}^{g} \mathrm{S}^1 \times \mathbb{R}P^{n-1}) = g$ for $n \geq 2$.

4.5 Extendability of independent systems of hypersurfaces

Let $\mathcal{N} = (N_1, \ldots, N_r)$ be an independent and regular system of hypersurfaces in W. We say that $\mathcal{N}$ is **extended** by a system $\mathcal{N}'$ if $\mathcal{N}'$ is also a regular and independent system such that $\mathcal{N} \subset \mathcal{N}'$ and their framings determine the same orientation of the normal bundle of $\mathcal{N}$ in W.

Proposition 4.24. *Let $\mathcal{N}$ be an independent and regular system without boundary in W of size r. Then*

$$\operatorname{corank}\left(\pi_1(W)/\langle\pi_1(\mathcal{N})\rangle^{\pi_1(W)}\right) = \operatorname{corank}\left(\pi_1(W|\mathcal{N})/\langle\pi_1(\partial P(\mathcal{N}))\rangle^{\pi_1(W|\mathcal{N})}\right) + r$$

and it is the maximum size of an independent and regular system without boundary in W which extends $\mathcal{N}$. In particular, for a closed manifold M we get

$$\mathcal{R}(M|\mathcal{N}) = \operatorname{corank}\left(\pi_1(M)/\langle\pi_1(\mathcal{N})\rangle^{\pi_1(M)}\right) - r.$$

Proof. Suppose we have a 2-sided connected submanifold $N = N_1$ without boundary with product neighbourhood $P(N)$ in a compact manifold W such that $W|N$ is connected. Thus W is obtained from $W|N$ be gluing the components of boundary $\partial(W|N) = P_{-1}(N) \sqcup P_1(N) \sqcup \partial W$ using a diffeomorphism $h\colon P_{-1}(N) \to P_1(N)$. It is known that $\pi_1(W)$ is the HNN extension of $\pi_1(W|N)$ relative to $h_\#\colon H_{-1} \to H_1$, where $H_t = \pi_1(P_t(N)) < \pi_1(W|N)$. In other words, $\pi_1(W)$ is the free product $\pi_1(W|N) * \mathbb{Z}$ divided by the normal closure K of $\{u\omega u^{-1}h_\#(\omega)^{-1} : \omega \in H_{-1}\}$, where u is the stable letter which generates $\mathbb{Z}$. The group $\pi_1(W|N)$ is a subgroup of $\pi_1(W)$ and the groups H_{-1} and H_1 are conjugated in $\pi_1(W)$. In fact, the normal subgroup $\pi_1(N)^{\pi_1(W)}$ in $\pi_1(W)$ is equal to $\langle H_{-1}\rangle^{\pi_1(W)} = \langle H_1\rangle^{\pi_1(W)} = \langle H_{-1}, H_1\rangle^{\pi_1(W|N)}$. Therefore $\pi_1(W)/\pi_1(N)^{\pi_1(W)}$ is isomorphic to

$$\pi_1(W)/\langle H_{-1}, H_1\rangle^{\pi_1(W|N)} \cong \pi_1(W|N)/\langle H_{-1}, H_1\rangle^{\pi_1(W|N)} * \mathbb{Z}.$$

It gives the first part of the proposition for $r = 1$ since $\operatorname{corank}(G * H) = \operatorname{corank}(G) + \operatorname{corank}(H)$. The general case follows by considering all submanifolds N_i simultaneously and HNN extension with r stable letters.

The description of the number on both sides of this equality follows by Proposition 4.17. □

Example 4.25. The Reeb number of a compact manifold W with non-empty boundary can be smaller than $C(W)$. For example, let $M = \Sigma \times S^1$, where Σ is a closed surface of the Euler characteristic $\chi(\Sigma) = 2 - k \leq 0$. Then

$$\mathcal{R}(M) = \operatorname{corank}(\pi_1(\Sigma) \times \mathbb{Z}) = \max\left(\left\lfloor \frac{k}{2} \right\rfloor, 1\right) = \left\lfloor \frac{k}{2} \right\rfloor \geq 1.$$

Let $\mathcal{N} = (\Sigma \times \{1\})$ and $W := M|\mathcal{N} = \Sigma \times [0,1]$. Then $C(W) = \operatorname{corank}(\pi_1(\Sigma)) = \lfloor \frac{k}{2} \rfloor$. However, $\mathcal{R}(W) = \operatorname{corank}\left((\pi_1(\Sigma) \times \mathbb{Z})/\pi_1(\Sigma)\right) - 1 = 0$ by Proposition 4.24.

Example 4.26. Let $\Sigma_{g,h}$ and $S_{g,h}$ denote, respectively, an orientable and non-orientable surface of genus g with $h \geq 1$ open discs removed. Then

- $\mathcal{R}(\Sigma_{g,h}) = g$ and $C(\Sigma_{g,h}) = 2g + h - 1$,
- $\mathcal{R}(S_{g,h}) = \lfloor \frac{g}{2} \rfloor$ and $C(S_{g,h}) = g + h - 1$.

We have already seen calculations of Reeb numbers in Corollary 2.18. The values of $C(\Sigma)$ follows by Theorem 4.14 and from the fact that $\pi_1(\Sigma_{g,h}) = F_{2g+h-1}$ and $\pi_1(S_{g,h}) = F_{g+h-1}$.

Corollary 4.27. *Any independent, regular and without boundary system $\mathcal{N}$ of hypersurfaces in a compact surface Σ can be extended to that system of size $\mathcal{R}(\Sigma)$.*

Proof. Let r be the size of $\mathcal{N}$. Since Σ is two-dimensional, $\mathcal{N}$ consists of circles in Σ. It is easily seen by the classification theorem of compact surfaces that if $\Sigma = \Sigma_{g,h}$ then $\Sigma|\mathcal{N} \cong \Sigma_{g-r,h+2r}$, and if $\Sigma = S_{g,h}$ then $\Sigma|N \cong S_{g-2r,h+2r}$. By the above example in both the cases $\mathcal{R}(\Sigma|\mathcal{N}) = \mathcal{R}(\Sigma) - r$, so $\mathcal{N}$ can be extended to the size $\mathcal{R}(\Sigma)$. □

The problem of extendability of an independent and regular system $\mathcal{N}$ of hypersurfaces requires further studies. For a moment, let us denote by $E(\mathcal{N})$ the maximum size of an independent and regular system which extends $\mathcal{N}$ and by $F(\varphi_{\mathcal{N}})$ the maximum rank of a free group onto which there is an epimorphism which factorizes $\varphi_{\mathcal{N}}$ (cf. Section 3.4). It is clear that $E(\mathcal{N}) \leq F(\varphi_{\mathcal{N}})$. We would like to know for which closed manifolds M it is the equality for any independent system of hypersurfaces in M. Using Theorem 3.11 it can be shown that $\mathcal{N}$ is framed cobordant to a system which can be extended to size $F(\varphi_{\mathcal{N}})$, however we do not know whether $\mathcal{N}$ can be extended itself. Since $E(\mathcal{N}) = \mathcal{R}(M|N) + r$ by Proposition 4.24, where r is the size of $\mathcal{N}$, this problem is related to the computability of corank and Reeb number.

Chapter 5

Realization problem for Reeb graphs

In this chapter we settle various realization problems for Reeb graphs. Section 5.1 provides a realization up to orientation-preserving homeomorphism of a graph as the Reeb graph of Morse function on a given closed manifold M, provided that the graph has a good orientation and its cycle rank is not greater than the Reeb number of M. Next, in Section 5.2 we show a realization of a graph as the Reeb graph of function with prescribed some components of level sets by system of hypersurfaces in a compact manifold W. Moreover, in Section 5.3 we discuss a realization of epimorphism as Reeb epimorphism. Section 5.4 is devoted to the application of Reeb epimorphisms to the problem of conjugacy of Morse functions. Finally, in Section 5.5 we indicate the role of degree 2 vertices in Reeb graphs of simple Morse functions.

5.1 Realization theorem

Here we present the proof from [46] of realization theorem for a closed manifold. It is quite long and it needs new combinatorial modifications of Reeb graphs. In fact, having these modifications defined, the rest of the proof is purely combinatorial in nature. Although it deals only with closed manifolds, since the case of manifolds with boundary would complicate considerations to much, it has the advantage that it provides the combinatorial transformation of the initial graph to a given graph (see Corollary 5.3). This is in addition to the results on transformation of Reeb graph to the canonical form (Kudryavtseva [35] and Fabio–Landi [12]). In the next section we show realization theorem with additional setting of systems of hypersurfaces for a manifold W, possibly with boundary (Theorem 5.4).

For the purpose of this section we adopt the following definition of realizability of a graph as the Reeb graph of function. We say that a function $f\colon M \to \mathbb{R}$ **realizes** a graph Γ with good orientation if $\mathcal{R}(f)$ is orientation-preserving homeomorphic to Γ. If it is the case, Γ is called **realizable** on M by f.

Remark 5.1. From now on, we will use combinatorial modifications for arbitrary graphs with good orientations, not only for Reeb graphs. Let us note that if a graph Γ is realizable on M by a simple Morse function, then any graph Γ' obtained from Γ by using combinatorial modifications is also realizable on M. Conversely, if we want to show a realization of Γ' and we know that Γ is obtained from Γ' by using the reverse combinatorial modifications, it is sufficient to show a realizability of Γ by a simple Morse function. Recall that the modifications (4), (5), (8) and (9) described in Section 4.2 are two-sided.

Theorem 5.2. *Let M be a closed n-dimensional manifold, $n \geq 2$, and Γ be a finite oriented graph. There exists a Morse function $f\colon M \to \mathbb{R}$ such that $\mathcal{R}(f)$ is orientation-preserving homeomorphic to Γ if and only if Γ has a good orientation and $\beta_1(\Gamma) \leq \mathcal{R}(M)$. Moreover, if M is not an orientable surface and $\Delta_3(\Gamma) \leq 3$, then f can be taken to be simple.*

Proof. The case of surfaces is provided by Corollary 2.22 and Theorem 2.24.

Let us assume that $n \geq 3$. Throughout the proof, we will define three additional combinatorial modifications of Reeb graphs ((10), (11) and (12)). The sketch of the proof is as follows. The first step is to reduce the considerations to graphs with vertices of degrees 1, 2 and 3 (see Figure 5.1). Then we will only work with simple Morse functions. Next we reduce the proof to the case of graphs whose all vertices of indegree 2 are above vertices of outdegree 2, as it is for the initial graph. In the last step we proceed by the induction on the number of vertices of degree 1. The crucial part of the proof is to show the induction step. In a graph Γ we consider possible neighbourhoods of an edge e incident to vertex w of degree 3 and to a vertex v of degree 1 (see Figure 5.2). The problematic case is when e is the only edge incoming to w or outgoing from w. For this case if w separates Γ into three connected components, then we provide another modification of Reeb graphs which increases by 1 the number of vertices of degree 1 (see Figure 5.3). It remains the case when $\Gamma \setminus \{w\}$ has two components for any vertices v and w as before. Assume that v is a maximum vertex, w is adjacent to v, v' is a minimum joined with v by a monotonic path and w' is incident to v'. We show that it suffices to consider the case when all increasing paths from v' to v omit an edge e incident to w and an edge e' incident to w', where the edges e and e' are the same for all these paths. Next we provide a construction of specific function inducing the same good orientation as the original one for which the situation looks like in Figure 5.5 (a), i.e. the edge e is so long that it ends below the chosen point x on e'. Then we introduce the last modification of Reeb graphs which increases by 2 the number of vertices of degree 1 (see Figure 5.5), and it completes the proof.

Step 1. We first reduce the problem to graphs whose maximum degree is not greater than 3 by introducing combinatorial modification number (10). Let Γ' be a graph which is obtained from Γ by substituting a small neighbourhood of each vertex v in Γ such that $\deg(v) \geq 4$ into a suitable one denoted by $S(v)$ as in Figure 5.1. Then Γ' is a graph such that $\Delta_3(\Gamma') \leq 3$. If there exists a simple Morse function f on M which realizes Γ', then identifying $S(v)$ with a subset of $\mathcal{R}(f)$ there are $\deg_{out}(v) - 1$ vertices of index $n-1$ below $\deg_{in}(v) - 1$ vertices of index 1 in $S(v)$, all of degree 3. Any vertex of degree 2 in $S(v)$ can be moved outside $S(v)$ by using modifications of Reeb graps. By Theorem 1.14 we can rearrange the corresponding critical points to a single critical level of a new Morse function. Then the vertices in $S(v)$ collapse to a single vertex with neighbourhood homeomorphic to the neighbourhood of v in Γ. If we perform this for all $S(v)$, then the obtained Morse function will realize Γ.

Therefore we may assume that Γ has no vertices of degrees other than 1, 2 and 3. We will show that Γ can be realized by a simple Morse function on M. We may ignore vertices of degree 2 since we are interesting in a homeomorphism type of graphs and we can always move them in a suitable way using combinatorial modifications. Thus assume that Γ has only vertices of degrees 1 or 3.

If there is no a vertex with indegree 2 in Γ below a vertex with outdegree 2, then Γ is called **primitive**. For example, the initial graph is primitive.

Step 2. We use the reverse modification (6) and modifications (4) and (5) (which are two-sided) on Γ to make it primitive. Thus we reduced the problem to primitive graphs (see Remark 5.1).

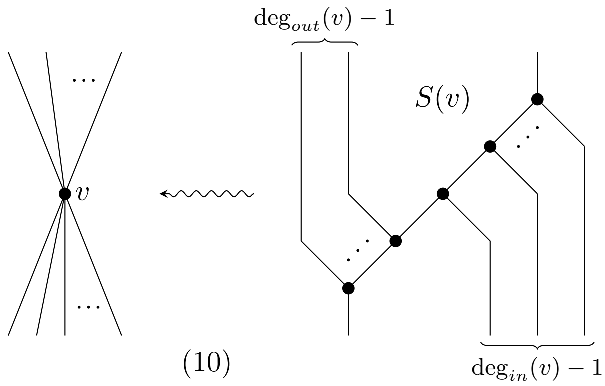


Figure 5.1: The combinatorial modification (10). If the Reeb graph of a simple Morse function has $S(v)$ as a subspace, then it can be modified to the Reeb graph of a Morse function in which $S(v)$ corresponds to a neighbourhood of v in Γ.

Step 3. Assume Γ to be primitive. We proceed by induction on the number of vertices of degree 1. For the base case, suppose that Γ has only one minimum and maximum. By Theorem 4.11 there exists a simple Morse function g on M whose Reeb graph is the initial graph and has cycle rank equal to $\beta_1(\Gamma)$. By Proposition 1.19 and Lemma 2.5 both Γ and $\mathcal{R}(g)$ has $\beta_1(\Gamma)$ vertices of outdegree 2 and $\beta_1(\Gamma)$ vertices of indegree 2. It is easily seen that by using the modifications (4) and (5) on $\mathcal{R}(g)$ we can reorder them to produce a simple Morse function which realizes Γ as the Reeb graph.

Now, let v be a vertex of degree 1 in Γ, e be the edge incident to v and let w be the second vertex incident to e. If w has degree 1, then Γ is the tree on two vertices and this case is provided by the base case. Hence we may assume that $\deg(w) = 3$. We distinguish the following cases for vertices of degree 1 in Γ:

(a) e is not the only edge which incomes to (or outgoes from) w,

(b) e is the only edge incoming to (or outgoing from) w and:

- (b1) $\Gamma \setminus \{w\}$ has three connected components,
- (b2) $\Gamma \setminus \{w\}$ has two components.

In the case (a) by Proposition 1.15 and using the modifications (8) and (9) we can reduce Γ to a graph without v.

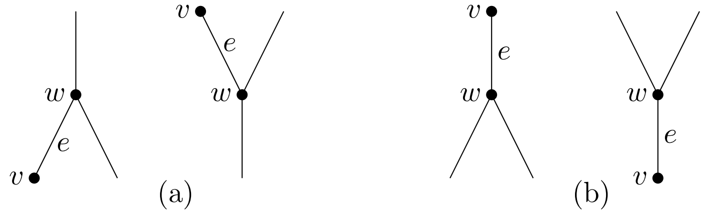


Figure 5.2: Possible configurations of the vertices v and w in Γ.

For the case (b1) suppose that e is an edge incoming to w (the second case when e outgoes from w is analogous) Let u_1 and u_2 be other adjacent vertices of w and let Γ_1, Γ_2 be connected components (except $\{v\}$) of a graph obtained from Γ by removing w and incident edges. Define Γ' as an oriented graph obtained from $-\Gamma_1$ (i.e. Γ_1 with reverse orientation) and from Γ_2 by joining u_1 and u_2 by an edge e'. Figure 5.3 shows the situation schematically. It has one less vertex of degree 1 than Γ, so by the induction hypothesis and Step 2. (since Γ' may not be primitive) there exists a simple Morse function f' on M which realizes Γ'.

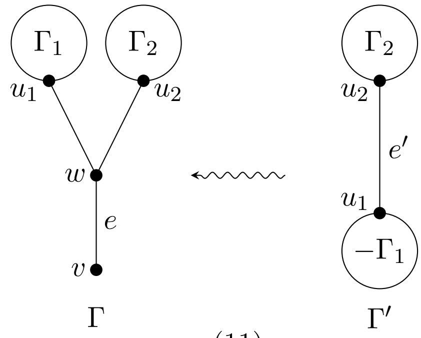


Figure 5.3: Construction of Γ' from Γ. Realization of Γ' as the Reeb graph of a simple Morse function implies realization of Γ, which leads to combinatorial modification number (11).

Let $[a, b]$ be a small interval contained in e' in $\mathcal{R}(f')$ and let G_1, G_2 be the two components of $\mathcal{R}(f')\backslash(a, b)$ such that G_i corresponds to Γ_i. Divide M into three submanifolds $Q_i = q^{-1}(G_i)$, $i = 1, 2$, and $W = q^{-1}([a, b])$ (where $q := q_{f'} : M \to \mathcal{R}(f')$ is the quotient map). The functions $f'|_{Q_1}$, $f'|_{Q_2}$ and $f'|_W$ are funtions on the smooth triads $(Q_1, \varnothing, W_-)$, $(Q_2, W_+, \varnothing)$ and (W, W_-, W_+) respectively, where $W_- = q^{-1}(a)$ and $W_+ = q^{-1}(b)$ are connected submanifolds. Denote by c and d the levels of f' for W_- and W_+, respectively. Let $g : Q_1 \to \mathbb{R}$ be defined by $g(x) = -f'|_{Q_1}(x)+c+d$, which is a function on the triad $(Q_1, W_-, \varnothing)$ such that $g(W_-) = \{d\}$. We also need an ordered simple Morse function $h : W \to [d - \varepsilon, d]$ on the triad $(W, \varnothing, W_- \sqcup W_+)$ with only one critical point being extremum (here it is a minimum). By Propositions 1.19 and 4.1 the Reeb graph $\mathcal{R}(h)$ is homeomorphic to a neighbourhood of w in Γ. Now, define a Morse function f on M which is the piecewise extension of g, $f'|_{Q_2}$ and h. It follows from the construction that f realizes Γ. Since each component of a level set of f contains at most one critical point, f can be taken to be simple. It is the combinatorial modification number (11).

Now, suppose that the only vertices of degree 1 in Γ are from the case (b2). Suppose that Γ has at least two maxima (a proof for minima is analogous).

Let v be a maximum vertex, w vertex with indegree 2 which is adjacent to v and let v' be a minimum joined with v by a monotonic path τ. Since Γ has no vertices from the case (a), v' is adjacent to a vertex w' with outdegree 2. Using the modifications (4) and (5) one can move out all vertices on τ between w and w'. Let x and y be points on the edges incident to w' and w, respectively, as in Figure 5.4 (a).

Suppose that there exists an increasing path γ from x to y. Since Γ has more than two vertices of degree 1, there exists a vertex of degree 3 on γ. There are the following two cases:

(b2-I) there are both the types of vertices of degree 3 on γ,

(b2-II) there is no vertex with outdegree 2 or a vertex with indegree 2 on γ.

For the case (b2-I) let z and z' be vertices on γ adjacent to w and w', respectively. Since Γ is primitive, $\deg_{in}(z) = 2 = \deg_{out}(z')$. Use the modifications (4) and (5) to move out all vertices on γ leaving only z and z', as in Figure 5.4 (b). Now, let us again use (4) to move z on the second edge incident to w and (5) to move z' on the second edge incident to w', as in Figure 5.4 (c). Thus we reduced the number of increasing paths from x to y.

For the case (b2-II) assume that there is no vertex with indegree 2 on γ (the second case is analogous). Let z be the vertex adjacent to w' with outdegree 2. As in the previous case, we move out all vertices on γ other than z (all of them have outdegree 2) and now z is adjacent to w and w'. Use (5) to move z on the second edge incident to w'. Figure 5.4 (d) shows the situation.

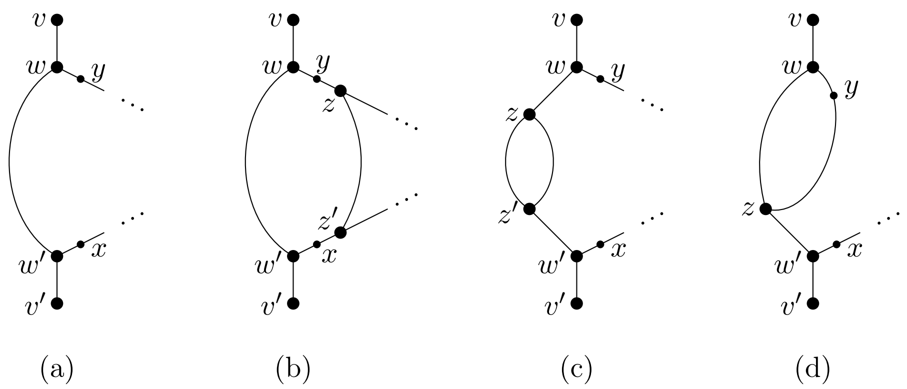


Figure 5.4: Reducing the number of increasing paths from x to y.

For two points p and p' in Γ denote by $\mathrm{IP}(p,p')$ ($\mathrm{DP}(p,p')$ respectively) the subset of Γ consisting of images of all increasing (decreasing) paths from p to p'. Similarly, denote by $\mathrm{IP}(p)$ ($\mathrm{DP}(p)$) the subset consisting of images of all increasing (decreasing) paths starting at p.

Performing the above procedures for each increasing path from x to y we obtain a situation such that $\mathrm{IP}(x,y)=\varnothing$ (equivalently, $x\notin \mathrm{DP}(y)$) and that $\mathrm{IP}(w',w)\setminus\{w',w\}$ is a connected component of $\Gamma\setminus\{w',w\}$. It is clear that if the case (b2-II) was occured, then v' and w' (or v and w) are from the case (b1), so it gives a realization of Γ. Therefore we may assume that we only used a procedure from (b2-I).

Let g be a continuous function on Γ which induces its good orientation and let $b=g(x)$ and $g(\mathrm{DP}(y))=[c,d]$, where $g(y)=d$. Let y' be a point near y on the same edge such that $g(y')=d'<d$. We want to construct a new function g' on Γ such that $g'(y')<g'(x)$ and which induces the same orientation. Let $\varepsilon>0$, $a<b$ and $h\colon [c,d]\to[a-\varepsilon,d]$ be an orientation-preserving homeomorphism of intervals such that $[d',d]$ is mapped to $[a,d]$. Denote by E_y the set of edges e in Γ whose closure $\mathrm{cl}(e)$ intersects $\mathrm{DP}(y)$ in an only one end. If $e\in E_y$, then e is incident to a unique vertex $z\in \mathrm{DP}(y)$. Since $e\not\subset \mathrm{DP}(y)$, e outgoes from z. Let h_e be an orientation-preserving homeomorphism of $[g(z),d_e]$ onto $[h(g(z)),d_e]$, where $g(\mathrm{cl}(e))=[g(z),d_e]$. We define a continuous function $g'\colon\Gamma\to\mathbb{R}$ by

$$g'(s)=\begin{cases} h(g(s)) & \text{if } s\in \mathrm{DP}(y),\\ h_e(g(s)) & \text{if } s\in e\in E_y,\\ g(s) & \text{in other cases.}\end{cases}$$

It is clear that the orientation induced by g' is the same as g and that $g'(y')=a<b=g'(x)$.

Now, let z' and z be points on the edges contained in $\mathrm{IP}(w',w)$ and incident to w' and w, respectively (see Figure 5.5 (a)). Define an oriented graph Γ' obtained from Γ by:

1. removing an open neighbourhood of an edge incident to w and v (to w' and v') with z and y' (z' and x) as boundary points,
2. taking $A = \mathrm{IP}(z', z)$ with the reverse orientation,
3. joining by a segment y' with z and z' with x.

Figure 5.5 (b) shows this construction schematically. It is evident that Γ' has a good orientation and has two less vertices of degree 1 than Γ. Thus Step 2. and induction hypothesis give us a realization of Γ' by a simple Morse function f'.

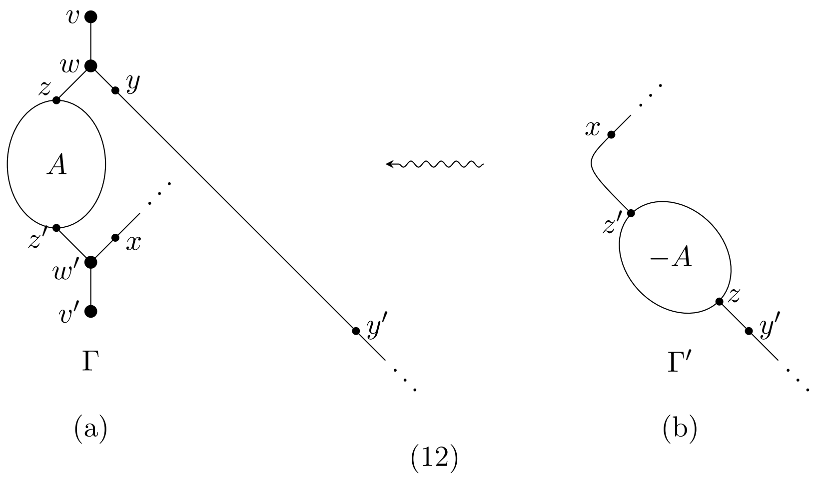


Figure 5.5: Construction of Γ' from Γ. Realization of Γ' by a simple Morse function implies realization of Γ, which leads to combinatorial modification number (12).

Let $q\colon M \to \mathcal{R}(f')$ be the quotient map, $\overline{f'}\colon \mathcal{R}(f') \to \mathbb{R}$ be the induced function and let $a = \overline{f'}(x)$. Let p be a point between y' and z in $\mathcal{R}(f')$. By $[z', x]$, $[y', p]$ and $[p, z]$ we denote the segments joining appropriate points in $\mathcal{R}(f')$. We will construct a simple Morse function f on M which realizes Γ. First, take an orientation-preserving diffeomorphism h of $[\overline{f'}(y'), \overline{f'}(p)]$ onto $[\overline{f'}(y'), a+\varepsilon]$. Next, let $W = q^{-1}([p, z])$ and $W' = q^{-1}([z', x])$. Take ordered and simple Morse functions

$$g\colon W \to [a+\varepsilon, a+2\varepsilon] \text{ on the triad } \big(W, q^{-1}(p) \sqcup q^{-1}(z), \varnothing\big),$$
$$g'\colon W' \to [a-\varepsilon, a] \text{ on the triad } \big(W', \varnothing, q^{-1}(z') \sqcup q^{-1}(x)\big),$$

with exactly one critical point being extremum (maximum and minimum, respectively). By Propositions 1.19 and 4.1 the Reeb graphs $\mathcal{R}(g)$ and $\mathcal{R}(g')$ are homeomorphic to a small neighbourhoods of vertices w and w' in Γ, respectively. Take a submanifold $Q = q^{-1}(-A)$ with boundary $\partial Q = q^{-1}(z) \sqcup q^{-1}(z')$ and let h_Q be an orientation-reversing diffeomorphism of the interval $f'(Q) = \overline{f'}(-A) = [\overline{f'}(z), \overline{f'}(z')]$ onto $[a, a+\varepsilon]$. Now, we define a Morse function f on M by

$$f(s) = \begin{cases} h(f'(s)) & \text{if } s \in q^{-1}([y', p]), \\ g(s) & \text{if } s \in W, \\ g'(s) & \text{if } s \in W', \\ h_Q(f'(s)) & \text{if } s \in Q, \\ f'(s) & \text{in other cases.} \end{cases}$$

It is easily seen that f realizes Γ and can be changed to be simple since connected components of level sets contains at most one critical point. □

In fact, during the proof of the above theorem we showed the following fact.

Corollary 5.3. *For any graph Γ with good orientation there is a finite sequence of combinatorial modifications (1) – (12) which transform the initial graph to Γ up to vertices of degree 2.* □

5.2 Realization of system of hypersurfaces as components of level sets of function

Now we are concerned with adding the condition that components of level sets of constructed function, which correspond to edges outside a chosen spanning tree in desired Reeb graph, form a given independent and regular system $\mathcal{N}$ of hypersurfaces without boundary in a manifold W. The idea of this construction for $\dim W \geq 3$ is simple — we cut a graph along these edges obtaining a tree, and we want to realize it as the Reeb graph of function on $W|\mathcal{N}$ with suitable correspondence of degree 1 vertices and components of $\partial(W|\mathcal{N})$. Thus on the one hand the task is easier than in the previous section, since the considered graph has no cycles, but on the other hand we need to preserve this correspondence for degree 1 vertices.

Theorem 5.4. *Let (W, W_-, W_+) be a smooth triad, $W_\pm = W_1^\pm \sqcup \ldots \sqcup W^\pm_{|\pi_0(W_\pm)|}$ be a decomposition into connected components and $\mathcal{N} = (N_1, \ldots, N_r)$ be a regular and independent system of hypersurfaces without boundary in W. Let Γ be a finite connected graph with good orientation, whose cycle rank is equal to r and which is admissible for (W, W_-, W_+). Distinguish vertices $a_1^\pm, \ldots, a^\pm_{|\pi_0(W_\pm)|}$ of degree 1 in Γ, where all a_i^- have indegree 0 and all a_i^+ have outdegree 0. Moreover, take a spanning tree T of Γ and order the edges outside T as $e_1, \ldots, e_r$. Then there is a Morse function $f\colon W \to \mathbb{R}$ on the triad (W, W_-, W_+), such that $\mathcal{R}(f)$ is orientation-preserving homeomorphic to Γ, each N_i is a component of level set of f which corresponds to the edge e_i and each $W_i^\pm$ corresponds to $a_i^\pm$. Moreover, if $\dim W \geq 3$ and $\Delta(\Gamma) \leq 3$, then f can be taken to be simple.*

Proof. Let Γ' be a tree obtained from Γ by cutting along all edges e_i as it is presented in Figure 5.6. Denote by c_i^- and c_i^+ the vertices of Γ' of outdegree 0 and indegree 0, respectively, obtained by cutting Γ along the edge e_i.

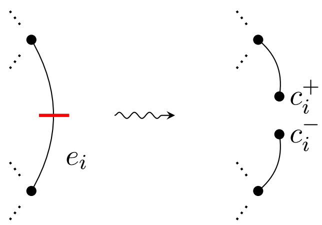
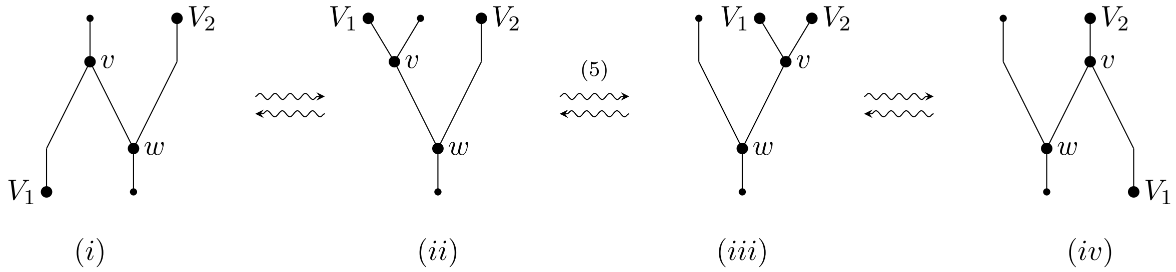


Figure 5.6: Cutting along edge.

For the case $\dim W = 2$ use Corollary 2.22 to obtain a Morse function $f'\colon W|\mathcal{N} \to \mathbb{R}$ on $(W|\mathcal{N}, W_- \sqcup P_{+1}(\mathcal{N}), W_+ \sqcup P_{-1}(\mathcal{N}))$ whose Reeb graph is orientation-preserving homeomorphic to Γ'. By homogeneity of manifolds there is a self-diffeomorphism $h\colon W|\mathcal{N} \to W|\mathcal{N}$ isotopic to identity and permuting the components of $\partial(W|\mathcal{N})$ such that $W_i^\pm$ corresponds to $a_i^\pm$, and $P_{\pm 1}(N_i)$ corresponds to $c_i^\pm$ in $\mathcal{R}(f' \circ h)$, which is orientation-preserving homeomorphic to Γ'. Obviously, we can rescale $f' \circ h$ along the edges incident to $c_i^\pm$ to assume that its value on $P_{-1}(N_i)$ is smaller than its value on $P_{+1}(N_i)$ for each i. Thus we may extend $f' \circ h$ on $P(\mathcal{N})$ to a Morse function on W whose Reeb graph, by the construction, is orientation-preserving homeomorphic to Γ, and N_i corresponds to the edge e_i for each i.

Now, let $\dim W \geq 3$. We proceed analogously as in the proof of Theorem 5.2 with the difference that the manifold has a boundary. However, we deal with the simplest case when graph is a tree. Steps 1 and 2 of the mentioned proof reduce the problem to the case when Γ' has only vertices of degrees 1 and 3 and is primitive, i.e. there is no an increasing path from a vertex with indegree 2 to a vertex with outdegree 2. By Theorem 4.11 for the empty system of hypersurfaces in $W|\mathcal{N}$ there is a simple Morse function $g\colon W|\mathcal{N} \to \mathbb{R}$ whose Reeb graph is the initial graph admissible for the triad $(W|\mathcal{N}, W_- \sqcup P_1(\mathcal{N}), W_+ \sqcup P_{-1}(\mathcal{N}))$. We may increase (or decrease if necessary) the number of vertices of degree 1 by using the modifications (8) and (9), so that $\mathcal{R}(g)$ is the initial graph with the same numbers of minima and maxima vertices as Γ'. Moreover, since Γ' and $\mathcal{R}(g)$ are primitive trees, it forces the same number of vertices of indegree 2 and of outdegree 2. Thus it suffices to appropriately rearrange vertices of degree 3 to produce Γ' from $\mathcal{R}(g)$.

For this purpose, we introduce the combinatorial modification number (13) presented in Figure 5.7, which allows us to transfer a vertex v of indegree 2 onto the second outgoing edge from a vertex w of outdegee 2 adjacent to v. The analogous modification for graphs with opposite orientations is numbered as (14). Since the graphs Γ' and $\mathcal{R}(g)$ are primitive, small neighbourhoods of two adjacent vertices of degree 3 look like in the modifications (4), (5), (13) or (14). Note that these modifications are two-sided, i.e. they work in both directions. We will show that Γ' can be transformed to the initial graph by using them, and so $\mathcal{R}(g)$ can be transformed to Γ' obtaining a simple Morse function $f'\colon W|\mathcal{N} \to \mathbb{R}$ whose Reeb graph is orientation-preserving homeomorphic to Γ'. Moreover, previously rearranging vertices in the initial graph $\mathcal{R}(g)$ using the modifications (4) and (5) we may ensure that the distinguished vertices of degree 1 in $\Gamma' \cong \mathcal{R}(f')$ correspond to appropriate components of $\partial(W|\mathcal{N})$. Again, as in the case for surfaces, we can rescale f' to assume that its value on $P_{-1}(N_i)$ is smaller than its value on $P_{+1}(N_i)$ for each i and extend f' on $P(\mathcal{N})$ to a Morse function f on W whose Reeb graph, by the construction, is orientation-preserving homeomorphic to Γ, and which satisfy all desired conditions.



(13)

Figure 5.7: The combinatorial modification number (13) of Reeb graph of simple Morse function. It transfers a vertex v of degree 3 and indegree 2 onto the second outgoing edge from a vertex w of degree 3 and outdegree 2. The situation with opposite directions leads to modification (14).

Thus, take a monotonic path τ in Γ' between vertices of degree 1 and with maximum number of vertices of degree 3. Assume that there is a vertex v of degree 3 outside τ, which is adjacent to a vertex w of degree 3 on τ. Without loss of generality assume that w has outdegree 2. If v also has outdegree 2, then use the modification (5) to move v on the path τ. For the second case when v has indegree 2, first use (5) to move w up along τ as long as it is possible, preserving adjacency with v. Then use the modification (13) for v and w to move v on τ. The obtained graph is still primitive and has increased number of vertices of degree 3 on τ. Repeating this procedure as long as there is a vertex of degree 3 outside τ we obtain the initial graph.

It only remains to provide the construction of the modification (13). Suppose $g\colon Q \to \mathbb{R}$ is a simple Morse function on a triad (Q, Q_-, Q_+) such that $\mathcal{R}(g)$ is isomorphic to the graph in Figure 5.7 (*i*). Let v and w be adjacent vertices of $\mathcal{R}(g)$ as in the figure and let v_1 be a vertex of degree 1 in $\mathcal{R}(g)$ adjacent to v and corresponding to the submanifold V_1 of Q, so V_1 is a single point (index 0 critical point of g) or component of Q_-. If it is a point, than use the modification (8) and then (9) to obtain the graph from Figure 5.7 (*ii*). Therefore assume V_1 is a component of Q_-. First, rescale the function along the edge between v and v_1 so that the value on v_1 is greater than the value on w. Take a neighbourhood U of this edge containing no other vertices than v and v_1 such that the corresponding submanifold S of Q forms a triad $(S, V_1 \sqcup S_1, S_2)$. Change a function g on S by defining a new simple and ordered Morse function on $(S, S_1, V_1 \sqcup S_2)$ without critical points being extremum. By Proposition 4.1 this produces a function with Reeb graph as in (*ii*). The modification (5) leads to the case (*iii*), and the analogous argument as before allows us to pass to (*iv*). Finally, the modification (14) for a simple Morse function g can be obtained from (13) for the function $-g$. □

Remark 5.5. Note that the constructed function f in the proof of the above theorem in the case of surfaces can be simple if $W|\mathcal{N}$ is non-orientable or has genus 0 since then f' can be taken to be simple by Theorem 2.24. It is the case for example if W has odd genus.

5.3 Realization of epimorphism onto free group as Reeb epimorphism

The previous theorem allows us to provide rigorous realization of epimorphism as the Reeb epimorphism of a Morse function on (W, W_-, W_+). Note that by Theorem 4.18 we need to assume that it factorizes through $\pi_1(W)/\langle \pi_1(\partial W)\rangle^{\pi_1(W)}$. We also deal with classification of Reeb epimorphisms of simple Morse functions on surfaces.

Theorem 5.6. *Let Γ be a finite connected graph with good orientation, (W, W_-, W_+) be a smooth triad and $\varphi\colon \pi_1(W) \to \pi_1(\Gamma)$ be an epimorphism factorized through $\pi_1(W)/\langle \pi_1(\partial W)\rangle^{\pi_1(W)}$. Then there is a Morse function $f\colon W \to \mathbb{R}$ on (W, W_-, W_+) such that $\mathcal{R}(f)$ is orientation-preserving homeomorphic to Γ and under this identification the Reeb epimorphism of f is equal to φ. Moreover, if W is not a surface and $\Delta(\Gamma) \leq 3$, then f can be taken to be simple.*

Proof. Take a spanning tree T of Γ and order the edges outside T by $e_1, \ldots, e_r$. Take the quotient map $p_T\colon \Gamma \to \Gamma/T = \bigvee_{i=1}^r S^1$ which maps e_i onto i-th circle. By Theorem 4.18 the epimorphism $(p_T)_\# \circ \varphi$ is induced by an independent and regular system $\mathcal{N} = (N_1, \ldots, N_r)$ of hypersurfaces without boundary in W of size r. By Theorem 5.4 there is a Morse function $f\colon W \to \mathbb{R}$ whose Reeb graph can be identified with Γ up to vertices of degree 2 and N_i corresponds to e_i. If $q_f\colon W \to \mathcal{R}(f)$ is the quotient map, then by the construction $(p_T)_\# \circ (q_f)_\# = \varphi_\mathcal{N} = (p_T)_\# \circ \varphi$. Since p_T is an isomorphism, $\varphi = (q_f)_\# = \varphi_f$ is the Reeb epimorphism of f. □

Note that the condition $\beta_1(\Gamma) \leq \mathcal{R}(W)$ is fulfilled by the existence of epimorphism φ.

Corollary 5.7. *Let Γ be a finite connected graph with good orientation and $\varphi\colon \pi_1(M) \to \pi_1(\Gamma)$ be an epimorphism, where M is a closed manifold. Then there is a Morse function $f\colon M \to \mathbb{R}$ such that $\mathcal{R}(f)$ is orientation-preserving homeomorphic to Γ and under this identification the Reeb epimorphism of f is equal to φ.* □

Remark 5.8. The Reeb epimorphism of f does not represent a unique epimorphism $\pi_1(W) \to \pi_1(\Gamma)$ in general, because it depends on the homeomorphism between Γ and $\mathcal{R}(f)$. It is unique for example for the initial graph or more generally for graphs such that the identity map is the only orientation-preserving automorphism. Otherwise, Theorem 5.4 provides more rigorous representation of Reeb epimorphism if some additional data is given. For instance, assume that there are distinguished edges $e_1, \ldots, e_r$ outside a spanning tree T of Γ and a regular and independent system $\mathcal{N}$ of hypersurfaces inducing $(p_T)_\# \circ \varphi$. Then the condition that each N_i corresponds to e_i implies the uniqueness of an epimorphisms $\pi_1(W) \to \pi_1(\Gamma)$ represented by the Reeb epimorphism of f.

Remark 5.9. Independently, O. Saeki [58] has proven a similar result for a closed manifold M, which provides a representation of an epimorphism $\varphi\colon \pi_1(M) \to \pi_1(\Gamma)$ as the Reeb epimorphism of a smooth function with finitely many critical values for a finite graph Γ without loops. In addition, since the constructed function has degenerate critical points, it can realize Γ as the Reeb graph up to isomorphism of graphs. However, the number of vertices of degree 2 in the Reeb graph of Morse function cannot be arbitrary (see Theorem 2.23 and Section 5.5), thus we need to ignore them in our construction of Morse function. Moreover, the above theorem together with Theorem 5.4 provide more rigorous representation of φ as the Reeb epimorphism, since we control the components of level sets of the function. It can be crucial in applications of Reeb epimorphisms, e.g. in topological conjugacy of Morse functions (see Section 5.4). Finally, our result deals also with manifolds with boundary.

Therefore for manifolds of dimension at least 3 any epimorphism $\pi_1(M) \to \pi_1(\Gamma)$ is represented as the Reeb epimorphism of a simple Morse function provided that Γ satisfies necessary conditions: it has a good orientation and the maximum degree of its vertices is not greater than 3. Now, let us investigate when in Theorem 5.6 one can take a simple Morse function in the case of surfaces.

We know that simple Morse functions on a closed orientable surface of genus g have a unique property that their Reeb graphs have cycle rank equal to g. For a non-orientable surface Σ there is a simple Morse function with Reeb graph having arbitrary cycle rank between 0 and $\mathcal{R}(\Sigma)$ (Theorem 2.24). However, it turns out that simple Morse functions have also some unique property for non-orientable surfaces of even genus, which can be described in terms of Reeb epimorphisms.

Proposition 5.10. *Let Γ be a graph with good orientation such that $\beta_1(\Gamma) < g$. Then there is a unique strong equivalence class Ξ of epimorphisms $\pi_1(S_{2g}) \to \pi_1(\Gamma)$ such that for any simple Morse function $f\colon S_{2g} \to \mathbb{R}$ having $\mathcal{R}(f) = \Gamma$ its Reeb epimorphisms belongs to Ξ.*

Proof. Let $r := \beta_1(\mathcal{R}(f))$ and $\mathcal{N} = (N_1, \ldots, N_r)$ be an independent and regular system of hypersurfaces in S_{2g} which are connected components of level sets of f and which correspond to edges outside some spanning tree of $\mathcal{R}(f)$. We claim that $S_{2g}|\mathcal{N}$ is non-orientable. Thus assume that $S_{2g}|\mathcal{N}$ is orientable. First, note that $f|_{S_{2g}|\mathcal{N}}$ is a simple Morse function and its Reeb graph $\mathcal{R}(f|_{S_{2g}|\mathcal{N}})$ is a tree. Therefore $S_{2g}|\mathcal{N}$ has a genus 0 as a surface with boundary, i.e. it is a sphere with discs removed. This implies that $r = g$, a contradiction.

Therefore as in the proof of Theorem 3.40 any two Reeb epimorphisms of simple Morse functions $S_{2g} \to \mathbb{R}$ with Γ as Reeb graphs are strongly equivalent and Ξ is represented by systems of hypersurfaces whose complement is non-orientable. □

The following classification follows by Theorem 2.3, Remark 5.5, Theorem 5.6 and Proposition 5.10.

Corollary 5.11. *Let Γ be a graph with good orientation such that $\Delta(\Gamma) \leq 3$ and let $\mathcal{R}_{epi}(\Sigma)$ be the set of all Reeb epimorphisms of simple Morse functions on a closed surface Σ. Take an epimorphism $\psi\colon \pi_1(\Sigma) \to \pi_1(\Gamma)$.*

- *If Σ is orientable of genus g, then $\psi \in \mathcal{R}_{epi}(\Sigma)$ if and only if $\beta_1(\Gamma) = g$.*
- *If Σ is non-orientable of odd genus, then any $\psi \in \mathcal{R}_{epi}(\Sigma)$.*
- *If Σ is non-orientable of even genus $2g$, then $\psi \in \mathcal{R}_{epi}(\Sigma)$ if and only if $\beta_1(\Gamma) = g$, or $\beta_1(\Gamma) < g$ and ψ belongs to a unique strong equivalence class Ξ of epimorphisms $\pi_1(\Sigma) \to \pi_1(\Gamma)$ represented by systems of hypersurfaces whose complement is non-orientable.*

5.4 Topological conjugacy of Morse functions

Now, we are focused on relations between Reeb epimorphisms and Morse functions. The main trouble is that in general different Reeb epimorphisms have different codomains. Although fundamental groups of Reeb graphs with the same cycle ranks are isomorphic, they are not isomorphic in the canonical way. However, this ambiguity can be omitted for oriented graphs for which the identity map is the only orientation-preserving automorphisms (e.g. the initial graph, see Remark 5.8).

Let us restrict our attention to the case of a closed manifold M. The functions f_1 and f_2 on M are called **topologically conjugate** if there are a self-homeomorphism $h\colon M \to M$ and an orientation-preserving homeomorphism $\eta\colon \mathbb{R} \to \mathbb{R}$ such that $f_1 = \eta \circ f_2 \circ h$. In this case h induces the unique homeomorphism $\overline{h}\colon \mathcal{R}(f_1) \to \mathcal{R}(f_2)$ such that $\overline{h} \circ q_1 = q_2 \circ h$ and $\overline{f_1} = \eta \circ \overline{f_2} \circ \overline{h}$, where $q_i := q_{f_i}\colon M \to \mathcal{R}(f_i)$, see Figure 5.8.

Lemma 5.12. *If f_1 and f_2 are simple Morse functions topologically conjugate by h, then their Reeb graphs are isomorphic through $\overline{h}$.*

Proof. If could be a vertex with degree 2 in $\mathcal{R}(f_1)$ mapped by $\overline{h}$ to a point in the interior of edge in $\mathcal{R}(f_2)$, then some smooth product triad would be mapped by h^{-1} homeomorphically onto a smooth triad with exactly one non-degenerate critical point. It is a contradiction by the comparison of Euler characteristics (cf. Theorem 1.12). □

$$\begin{array}{ccccc} M & \xrightarrow{q_1} & \mathcal{R}(f_1) & \xrightarrow{\overline{f_1}} & \mathbb{R} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \overline{h}} & & \downarrow{\scriptstyle \eta} \\ M & \xrightarrow{q_2} & \mathcal{R}(f_2) & \xrightarrow{\overline{f_2}} & \mathbb{R} \end{array}$$

Figure 5.8: Topologically conjugate simple Morse functions have isomorphic Reeb graphs.

We cannot say directly about equivalence and strong equivalence of Reeb epimorphisms since they have distinct codomains, even if Reeb graphs are isomorphic. By the above diagram we see that if f_1 and f_2 are topologically conjugate by h, then $\overline{h}_\# \circ (q_1)_\#$ and $(q_2)_\#$ are strongly equivalent. Thus we say that two Reeb epimorphisms $\varphi_i := \varphi_{f_i}\colon \pi_1(M) \to \pi_1(\mathcal{R}(f_i))$ are **strongly equivalent** if they are strongly equivalent with respect to an isomorphism $k\colon \mathcal{R}(f_1) \to \mathcal{R}(f_2)$, i.e. $k_\# \circ \varphi_1$ and φ_2 are strongly equivalent. In general, it may depend on the chosen isomorphism.

The following theorem due to E. Kulinich [36] and V. Sharko [60] is a classical result in the theory of Morse functions spaces.

Theorem 5.13 ([36, 60], cf. [12])**.** *Two simple Morse functions on a closed orientable surface* Σ *are topologically conjugate by* $h\colon \Sigma \to \Sigma$ *if and only if their Reeb graphs are isomorphic as oriented graphs through* $\overline{h}$.

This theorem allows us to give another proof of a part of Theorem 3.18 for orientable surfaces which uses Reeb graphs. First, for any two epimorphisms $\pi_1(\Sigma_g) \to F_r$ we need to take systems which induce them. Then we extend them to systems of maximum size $\mathcal{R}(\Sigma_g) = g$ by Corollary 4.27 and now we can represent induce epimorphisms $\pi_1(\Sigma_g) \to F_g$ by Reeb epimorphisms of simple Morse functions whose Reeb graphs are the initial graphs. Since their Reeb graphs are isomorphic in a canonical way, by Theorem 5.13 is induced by a self-homeomorphism of Σ_g that maps one system to the another and gives a strong equivalence.

Remark 5.14. Theorem 3.18 for non-orientable surfaces of even genus shows that the analogue of Theorem 5.13 does not hold for them in general. In fact, we may construct two simple Morse functions on S_{2g} whose Reeb graphs are isomorphic, but their Reeb epimorphisms are not strongly equivalent. Thus we must endow Reeb graphs in additional information.

D. Lychak and A. Prishlyak in their work [38] equipped Reeb graphs of a simple Morse function on non-orientable surface with signs $+$ or $-$ near vertices of degree 3, which come from the compatibility of orientations during attaching handles in corresponding critical levels. To be precise, each sign is assigned to a pair of incident edges at a vertex v of degree 3, one of which is incoming to v and the second one is outgoing from v. For the procedure of the assignment of signs we refer the reader to [38]. Two **Reeb graphs with signs** are called **equivalent** if they are isomorphic and it is possible to obtain, under the isomorphism, identical signs by the following operation: for a given edge reverse all signs assigned to it.

Theorem 5.15 (Lychak–Prishlyak [38])**.** *Two simple Morse functions on a closed nonorientable surface are topologically conjugate if and only if their Reeb graphs with signs are equivalent.*

Lemma 5.16. *Let* Γ *be a graph with good orientation whose vertices have degrees* 1 *or* 3*. Then there are exactly* 2^r *equivalence classes of graphs with signs, where* $r = \beta_1(\Gamma)$.

Proof. First, look at the case of the canonical graph. It is an easy exercise that any such graph with signs is equivalent to a configuration of the form showed in Figure 5.9 (a), where in the r places of "?" we can put arbitrary signs. Moreover, all such 2^r configurations are non-equivalent. The same can be shown for the initial graph with configurations of signs as in Figure 5.9 (b).

Now, note that by [12] and Proposition 4.8 there is a sequence of modifications of Reeb graphs which transform Γ to the canonical graph. It is left to the reader to check that these modifications for graphs with vertices of degree 1 or 3 do not change the number of non-equivalent configurations of signs. □

Remark 5.17. For a graph with vertices of degree 2 the number of non-equivalent configurations of signs may vary depending on the position of these vertices in the graph. Moreover, the Reeb graph of a simple Morse function on a non-orientable surface of odd genus has always a vertex of degree 2. The same is true if a surface is non-orientable of even genus $2g$ and the Reeb graph has a cycle rank smaller than g. In the view of Theorem 2.3 it is reasonable to consider the case of simple Morse functions on a non-orientable surface of genus $2g$ whose Reeb graphs have cycle rank equal to g.

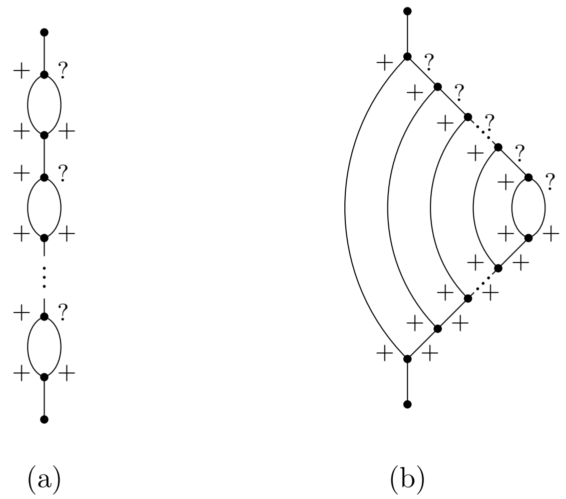


Figure 5.9: The canonical and initial graphs with signs.

Theorem 5.18. *Let $f_1, f_2 \colon S_{2g} \to \mathbb{R}$ be simple Morse functions on a closed non-orientable surface of genus $2g$ such that $\beta_1(\mathcal{R}(f_1)) = g = \beta_1(\mathcal{R}(f_2))$. Then they are topologically conjugate if and only if their Reeb graphs are isomorphic and their Reeb epimorphisms are strongly equivalent.*

Proof. Denote by $\mathcal{M}(M, \Gamma)$ the set of all simple Morse functions f on M whose Reeb graphs are isomorphic to a graph Γ and by $\mathcal{M}(M, \Gamma)/_{\text{t.c.}}$ the set of their conjugacy classes. Moreover, let Signs(Γ) be the set of equivalence classes of configurations of signs in Γ. By Theorem 5.15 a natural map $\mathcal{M}(S_{2g}, \Gamma)/_{\text{t.c.}} \to \text{Signs}(\Gamma)$ associating with a function its configuration of signs in its Reeb graphs as in [38] is injective. Now, let Γ has vertices of degrees 1 and 3 and $\beta_1(\Gamma) = g$. Then for any configuration of signs in Γ except the one with only pluses we can produce a simple Morse function on S_{2g} which realizes it (see Theorem 2.13 for a procedure). The configuration of signs with only pluses leads to a function on an orientable surface. By Lemma 5.16 the set Signs(Γ) has 2^r elements, so $\mathcal{M}(S_{2g}, \Gamma)/_{\text{t.c.}}$ has $2^r - 1$ elements.

Now, by Theorem 5.6 and Corollary 5.11 the map $\mathcal{M}(S_{2g}, \Gamma)/_{\text{t.c.}}$ to the set of strong equivalence classes of Reeb epimorphisms of functions in $\mathcal{M}(S_{2g}, \Gamma)$ is surjective. Since the latter set has also $2^r - 1$ elements by Theorem 3.18, it is a bijection and the theorem is proved. □

5.5 Bounds on the number of degree two vertices

The results obtained above, concerned with the realization of graphs as Reeb graphs, hold up to homeomorphism for manifolds of dimension at least 3. As we discussed in Remark 5.9, controlled distribution of vertices of degree 2 in Reeb graph is possible in general only if we allow degenerate critical points of functions. In the case of surfaces, we have seen it in Theorems 2.20 and 2.23. In this section, we indicate basic properties of the number of degree 2 vertices in Reeb graphs of Morse functions, which is a topic for further studies. For simplicity, let us restrict our considerations to simple Morse functions on closed manifolds.

Definition 5.19. By $\Delta_2(M)$ we denote the minimal number of vertices of degree 2 in Reeb graphs of simple Morse functions on a manifold M.

Reeb graphs of simple Morse functions on orientable surface have no vertices of degree 2, so $\Delta_2(\Sigma_g) = 0$. In non-orientable case we have the following characterization from Theorem 2.24.

Corollary 5.20. *Let $f\colon S_g \to \mathbb{R}$ be a simple Morse function.*

(1) If $g = 2k$ is even, then $\Delta_2(\mathcal{R}(f))$ is even and $\Delta_2(S_{2k}) = 0$.

(2) If $g = 2k+1$ is odd, then $\Delta_2(\mathcal{R}(f))$ is odd and $\Delta_2(S_{2k+1}) = 1$. □

Note the following trivial observation which follows directly from the definition of $\Delta_2(M)$.

Corollary 5.21. *Let Γ be a finite graph with good orientation and $\Delta(\Gamma) \leq 3$. If $\Delta_2(\Gamma) < \Delta_2(M)$, then there is no simple Morse function $f\colon M \to \mathbb{R}$ whose Reeb graph is isomorphic to Γ.*

Recall the formula for a simple Morse function $f\colon M \to \mathbb{R}$ (Lemma 2.5):

$$\beta_1(\mathcal{R}(f)) = -\frac{k_0 + k_n}{2} + \frac{\Delta_3(\mathcal{R}(f))}{2} + 1,$$

where k_i is the number of critical points of index i. In particular, if f has only two extrema, then $k_0 = k_n = 1$ and so $\beta_1(\mathcal{R}(f)) = \frac{\Delta_3(\mathcal{R}(f))}{2}$.

The **rank** of a finitely generated group G is the smallest cardinality of its generating set. Note that $k_1 \geq \operatorname{rank} \pi_1(M)$, since a Morse function f leads to a CW-decomposition with k_1 cells of dimension 1 (Theorem 1.12). The same argument for $-f$ gives us $k_{n-1} \geq \operatorname{rank} \pi_1(M)$.

By $\operatorname{cat}(X)$ we denote the **Lusternik–Schnirelmann category** of a space X, the minimum number of open sets covering X which are contractible in X. Thus $\operatorname{cat}(X) = 1$ if X is contractible.

Moreover, for a finitely generated R-module M over a principal ideal domain R its rank $\operatorname{rank}_R M$ is defined as the rank over free R-module $M/Tor(M)$, where $Tor(M)$ is a torsion submodule of M.

Proposition 5.22. *For a simple Morse function $f\colon M \to \mathbb{R}$ the number $\Delta_2(\mathcal{R}(f))$ of vertices of degree 2 in $\mathcal{R}(f)$ satisfy the inequalities*

$$\Delta_2(\mathcal{R}(f)) \geq 2(\operatorname{rank}(\pi_1(M)) - \operatorname{corank}(\pi_1(M))), \tag{1}$$

$$\Delta_2(\mathcal{R}(f)) \geq \sum_{i=1}^{n-1} \operatorname{rank}_R H_i(M, R) - 2\operatorname{corank}(\pi_1(M)), \tag{2}$$

$$\Delta_2(\mathcal{R}(f)) \geq \operatorname{cat}(M) - 2\operatorname{corank}(\pi_1(M)) - 2. \tag{3}$$

Moreover, if M is an orientable 3-manifold with Heegaard genus $g(M)$, then $\Delta_2(\mathcal{R}(f))$ is even and

$$\Delta_2(\mathcal{R}(f)) \geq 2(g(M) - \operatorname{corank}(\pi_1(M))). \tag{4}$$

Proof. By Lemma 4.6 we may assume that f has only two extrema without changing $\Delta_2(\mathcal{R}(f))$. Therefore $\operatorname{corank}(\pi_1(M)) \geq \beta_1(\mathcal{R}(f)) = \frac{\Delta_3(\mathcal{R}(f))}{2}$. Thus

$$\begin{aligned}\Delta_2(\mathcal{R}(f)) = k_1 + \ldots + k_{n-1} - \Delta_3(\mathcal{R}(f)) \quad &\geq \quad 2\left(\frac{k_1 + k_{n-1}}{2} - \frac{\Delta_3(\mathcal{R}(f))}{2}\right) \\ &\geq \quad 2\left(\operatorname{rank}(\pi_1(M))\right) - \operatorname{corank}(\pi_1(M))\,.\end{aligned}$$

From Morse inequalities $k_i \geq \operatorname{rank}_R H_i(M,R)$ we obtain (2). It is also known (see [65]) that $\operatorname{cat}(M)$ bounds from below the number of critical points of function on M. Thus $\sum_{i=0}^{n} k_i \geq \operatorname{cat}(M)$ what gives (3).

Finally, for $n = 3$ since f has exactly two extrema and $\chi(M) = 0$, $k_1 = k_2 \geq g(M)$, so

$$\Delta_2(\mathcal{R}(f)) = k_1 + k_2 - \Delta_3(\mathcal{R}(f)) = 2\left(\frac{k_1 + k_2}{2} - \frac{\Delta_3(\mathcal{R}(f))}{2}\right) = 2\left(k_1 - \beta_1(\mathcal{R}(f))\right),$$

what implies the desired inequality. □

Example 5.23. Since $\operatorname{cat}(M) \leq \dim M + 1$ and $\operatorname{cat}(M)$ is not an easy invariant to compute, the inequality (3) is of less utility. However, for $M = \mathbb{R}\mathrm{P}^n$ the bound (3) provides $\Delta_2(\mathbb{R}\mathrm{P}^n) \geq n - 1$ since $\operatorname{cat}(\mathbb{R}\mathrm{P}^n) = n + 1$, while (1) gives only $\Delta_2(\mathbb{R}\mathrm{P}^n) \geq 2$.

More generally, if M is simply connected, then (1) is trivial, but (3) yields $\Delta_2(M) \geq \operatorname{cat}(M) - 2$ and the right-hand side is positive if M is not a sphere. For example, $\operatorname{cat}(\mathbb{C}\mathrm{P}^n) = n+1$, so $\Delta_2(\mathbb{C}\mathrm{P}^n) \geq n - 1$.

Example 5.24. In some cases the bound (1) can be better than (2) and (3).

Take $M = L_p \# L_q$, the connected sum of two 3-dimensional lens spaces such that $\gcd(p, q) = 1$, where $\pi_1(L_k) = \mathbb{Z}/k\mathbb{Z}$. Then $\pi_1(M) = \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$, $\operatorname{rank}(\pi_1(M)) = 2$ and $\operatorname{corank}(\pi_1(M)) = 0$, so (1) gives $\Delta_2(M) \geq 4$. Since $\operatorname{cat}(M) \leq 4$, the bound (3) yields at most $\Delta_2(M) \geq 2$. Moreover, $H_0(M) = H_3(M) = \mathbb{Z}$, $H_1(M) = \mathbb{Z}/pq\mathbb{Z}$ and $H_2(M) = 0$, so $\operatorname{rank}_R H_1(M,R) \leq 1$ and $\operatorname{rank}_R H_2(M,R) \leq 1$ for any principal ideal domain R. Thus from (2) we also obtain at most $\Delta_2(M) \geq 2$

Example 5.25. Similarly, there are examples where the bound (2) is better than (1) and (3).

For n-dimensional torus T^n one can show that $\operatorname{cat}(T^n) = n+1$, so the inequality (3) implies $\Delta_2(T^n) \geq n - 3$, while (1) gives $\Delta_2(T^n) \geq 2(n-1) = 2n - 2$. However, since $\operatorname{rank}_{\mathbb{Z}} H_k(T^n) = \binom{n}{k}$, the formula (2) provides $\Delta_2(M) \geq 2^n - 4$.

Example 5.26. In the case of orientable 3-manifold the bound (4) using Heegaard genus $g(M)$ is sharper than the other three, because $g(M) \geq \operatorname{rank}(\pi_1(M))$, $g(M) \geq \operatorname{rank}_R H_i(M,R)$ for $i = 1, 2$ and $2g(M) \geq \operatorname{cat}(M) - 2$ (the last inequality follows since $\operatorname{cat}(M) - 2 \leq 2$, so it suffices to $g(M) \geq 1$, and for $g(M) = 0$, M is the 3-sphere, so $\operatorname{cat}(M) = 2$).

Let us look at the following easy examples.

- If $M = H(3,\mathbb{R})/H(3,\mathbb{Z})$ is the Heisenberg manifold, then $\operatorname{corank}(\pi_1(M)) = 1$ and $\operatorname{rank}(\pi_1(M)) = g(M) = 2$, so $\Delta_2(M) \geq 2$.
- If M is the Poincare homology sphere, then $\operatorname{corank}(\pi_1(M)) = 0$ and $\operatorname{rank}(\pi_1(M)) = g(M) = 2$, so $\Delta_2(M) \geq 4$.
- If $M = T^3$ is a 3-torus, then $\operatorname{corank}(\pi_1(M)) = 1$ and $\operatorname{rank}(\pi_1(M)) = g(M) = 3$, so $\Delta_2(M) \geq 4$.

Lemma 5.27. *Let $f_i \colon M_i \to \mathbb{R}$ be (simple) Morse functions on n-dimensional manifold M with k_j^i critical points of index j and $\beta_1(\mathcal{R}(f_i)) = r_i$. Then there is a (simple) Morse function $f \colon M \to \mathbb{R}$ on $M = M_1 \# M_2$ such that $\beta_1(\mathcal{R}(f)) = r_1 + r_2$ and which has $k_j^1 + k_j^2$ critical points of index $j \neq 0, n$, and $k_j^1 + k_j^2 - 1$ for $j = 0, n$. Moreover,*

$$\Delta_2(M_1 \# M_2) \leq \Delta_2(M_1) + \Delta_2(M_2).$$

Proof. We may assume that the maximum value of f_1 is the minimum value of f_2, which is denoted by c. If $f_i(p_i) = c$, take n-handles D_i^n corresponding to critical points p_i, so the boundary ∂D_i^n is the component of level set of f_i. Perform the connected sum $M = M_1 \# M_2$ along D_i^n. Then the functions f_i paste together to a Morse function on M with the desired properties. □

It is easy to characterize orientable 3-manifolds with $\Delta_2(M) = 0$.

Proposition 5.28. *Let M be an orientable closed 3-manifold and $r = \operatorname{corank}(\pi_1(M))$. The following are equivalent:*

(1) $M \cong \mathbb{S}_r := \#_{i=1}^r \mathrm{S}^2 \times \mathrm{S}^1$, *where* $\mathbb{S}_0 = S^3$;

(2) $\Delta_2(M) = 0$;

(3) $g(M) = r$;

(4) $\operatorname{rank}(\pi_1(M)) = r$;

(5) $\pi_1(M) \cong F_r$.

Proof. The manifold $\mathrm{S}^2 \times \mathrm{S}^1$ has a Heegaard diagram (H, α) such that α is a co-core C of the 1-handle of H. Therefore the 2-handle of $\mathrm{S}^2 \times \mathrm{S}^1$ is attached to H along $\varphi\colon \mathrm{S}^1 \times \mathrm{D}^1 \to \partial H$ such that $\varphi|_{\mathrm{S}^1 \times \{0\}} = \alpha$, so we can isotopically change φ to move its image outside C. Thus these handles can be attached in any order, so the Morse function f, corresponding to this handle decomposition, has a unique critical point of index 2 below a unique critical point of index 1. Hence f is simple and since the 2-handle is attached to the boundary of the ball D^3, it splits the level set of f into two parts. Therefore $\beta_1(\mathcal{R}(f)) = 1$ and $\mathcal{R}(f)$ has no vertices of degree 2. Thus (1) implies (2) by Lemma 5.27.

Suppose that $\Delta_2(M) = 0$ and take a simple Morse function f on M whose Reeb graph has no vertices of degree 2. We may assume that f has exactly two extrema. Then $k_1 = \beta_1(\mathcal{R}(f))$ is the number of critical points of index 1. Since

$$r = \operatorname{corank}(\pi_1(M)) \le \operatorname{rank}(\pi_1(M)) \le g(M) \le k_1 = \beta_1(\mathcal{R}(f)) \le r,$$

we obtain $g(M) = r$.

The implication (3) to (4) follows also by $\operatorname{corank}(\pi_1(M)) \le \operatorname{rank}(\pi_1(M) \le g(M)$.

Now, assume (4) and take an epimorphism $\varphi\colon \pi_1(M) \to F_r$. If $\{a_1, \ldots, a_r\}$ generates $\pi_1(M)$, then $S = \{\varphi(a_1), \ldots, \varphi(a_r)\}$ generates F_r. This generating set is equivalent, under Nielsen transformations, to free generating set of F_r with also r elements, so S is also free generating set. Thus a homomorphism $\phi\colon F_r \to \pi_1(M)$ defined by $\phi(b_i) = a_i$ is an inverse for φ, so they are isomorphisms.

Finally, for (5) implies (1) assume that $\pi_1(M) \cong F_r$. The case $r = 0$ is the Poincaré conjecture, so suppose that $r \ge 1$. If M is a prime manifold and $M \not\cong \mathrm{S}^2 \times \mathrm{S}^1$, then M is also irreducible, and so aspherical since $\pi_1(M)$ is infinte. Thus M is an Eilenberg-MacLane space $\mathrm{K}(F_r, 1)$, so M has a homotopy type of $\bigvee^r \mathrm{S}^1$, which contradicts the fact that $H_2(M) = \mathbb{Z}^r$.

If $r = 1$, then M is a prime manifold by Grushko theorem (see [6]) and Poincaré conjecture, so as we showed $M \cong \mathrm{S}^2 \times \mathrm{S}^1$. Inductively, if $M = M_1 \# M_2$ is non-trivial decomposition, then by Grushko theorem $\pi_1(M_i)$ is a free group of rank r_i smaller than r, so by induction $M_i \cong \mathbb{S}_{r_i}$, and so $M \cong \mathbb{S}_r$. □

We state without proof the following result, which is a part of forthcoming paper. It relies on the form of generators and relations in the presentation of 3-manifold group.

Theorem 5.29. *If M is orientable 3-manifold with $\pi_1(M)$ torsion-free, then $\Delta_2(M) \geq 4$.*

Example 5.30. The discrete Heisenberg group $H(3, \mathbb{Z})$ is torsion-free (it can be see algebraically or by noting that the Heisenberg manifold $M = H(3, \mathbb{R})/H(3, \mathbb{Z})$ is aspherical), so $\Delta_2(M) \geq 4$, and so the inequality (4) can be improved.

This is a list of open problems we are working on. Let M and M' be closed orientable 3-manifolds.

(a) The invariant $\Delta_2(M)$ is additive with respect to connected sum operation, i.e.

$$\Delta_2(M\#M') = \Delta_2(M) + \Delta_2(M').$$

(b) Assuming (a), it can be show that $\Delta_2(M) = 2$ if and only if $M = \mathbb{S}_r\#L$, where $r = \operatorname{corank}(\pi_1(M))$ and L is a lens space. In this case the inequality (4) is equality.

(c) More generally, our conjecture is that $\Delta_2(M) = 2(g(M) - \operatorname{corank}(\pi_1(M)))$ if and only if $g(M) \neq \operatorname{corank}(\pi_1(M)) + 1$.

(d) If $g(M) = \operatorname{corank}(\pi_1(M)) + 1 \geq 2$, then $\Delta_2(M) = 2(g(M) - \operatorname{corank}(\pi_1(M)) + 1) = 4$.

At the end, let us see on the following examples.

Example 5.31. If $M = \Sigma_g \times \mathrm{S}^1$, then $g(M) = 2g + 1$ and $\operatorname{corank}(\pi_1(M)) = g$. By an explicit construction it can be shown that $\Delta_2(M) = 2g + 2 = 2(g(M) - \operatorname{corank}(\pi_1(M)))$.

In particular, if $M = T^3$ for $g = 1$ we have $\Delta_2(M) = 4 = 2^3 - 4$ as in a bound from Example 5.25.

Example 5.32. Further generalization of above manifolds are orientable S^1-bundles over a closed surface Σ_g. They are classified by elements of $H^2(\Sigma_g, \mathbb{Z}) = \mathbb{Z}$, so any $r \in \mathbb{Z}$ corresponds to a bundle M_r and conversely, any circle bundle over Σ_g is isomorphic to M_r for some $r \in \mathbb{Z}$.

First, $M_0 = \Sigma_g \times \mathrm{S}^1$ is the trivial boundle and it is covered by Example 5.31.

For $r \notin \{0, \pm 1\}$ we have $\operatorname{rank}(\pi_1(M_r)) = 2g+1$ and $\operatorname{corank}(\pi_1(M_r)) = g$. It can be shown that $g(M_r) = 2g + 1$ and $\Delta_2(M) = 2(g(M_r) - \operatorname{corank}(\pi_1(M_r))) = 2g + 2$.

For $g = 1$ we have circle bundles over torus and $M_1 = H(3, \mathbb{R})/H(3, \mathbb{Z})$ is the Heisenberg manifold with $\Delta_2(M_1) = 4$, while $2(g(M_1) - \operatorname{corank}(\pi_1(M_1))) = 2$.

Example 5.33. Another class of 3-manifolds are surface bundles over S^1. Such a manifold is a mapping tori $M(\varphi)$ of a diffeomorphism $\varphi\colon \Sigma \to \Sigma$, where Σ is a closed surface.

Take $\Sigma = T^2$. Then $\pi_1(M(\varphi))$ is an extension of $\pi_1(T^2) = \mathbb{Z}^2$ by $\pi_1(\mathrm{S}^1) = \mathbb{Z}$, so $\operatorname{corank}(\pi_1(M(\varphi))) = 1$ by Proposition 3.15. We can construct a simple Morse function f on $M(\varphi)$ with $\Delta_2(\mathcal{R}(f)) = 4$, $\beta_1(\mathcal{R}(f)) = \operatorname{corank}(\pi_1(M(\varphi))) = 1$ and $k_1 = 3$.

It can be checked that the rank of $\pi_1(M(\varphi))$ is not greater than 3. If $\operatorname{rank}(\pi_1(M(\varphi))) = 3$, then $g(M(\varphi)) = 3$ and so $\Delta_2(M(\varphi)) = 4$.

However, for $\operatorname{rank}(\pi_1(M(\varphi))) = 2$ the inequality (4) can be improved. Again, the Heisenberg manifold provides an example.

Notation

- M – a closed, smooth and connected manifold of dimension $n \geq 2$,
- W – a compact, smooth and connected manifold of dimension $n \geq 2$, possibly with boundary,
- ∂W – the boundary of a manifold W,
- (W, W_-, W_+) – a smooth triad, i.e. $\partial W = W_- \sqcup W_+$,
- Σ_g – the closed orientable surface of genus g,
- S_g – the closed non-orientable surface of genus g,
- D^n – the n-dimensional disc,
- S^n – the n-dimensional sphere, the boundary of D^{n+1}; $\mathrm{S}^{-1} = \varnothing$,
- $\chi(X)$ – the Euler characteristic of a space X,
- $\beta_1(X)$ – the first Betti number of a space X,
- $\operatorname{Im} \varphi$ – the image of a map φ,
- $\pi_0(X)$ – the set of path components of a space X,
- $|S|$ – the cardinality of the set S,
- $\lfloor x \rfloor$ – the floor of a real number x,
- $\operatorname{Int}(X)$ – the interior of a topological space X,
- $X \sqcup Y$ – the disjoint union of spaces X and Y,
- $W_c := f^{-1}(c)$,
- $W^c := f^{-1}((-\infty, c])$,
- $W^I := f^{-1}(I)$ for an interval $I \subset \mathbb{R}$,
- $\operatorname{ind}(p)$ – the index of a non-degenerate critical point p,
- $\mathcal{R}(f)$ – the Reeb graph/Reeb space of a function $f\colon X \to \mathbb{R}$,
- q_f – the quotient map $X \to \mathcal{R}(f)$,
- φ_f – the Reeb epimorphism $(q_f)_\#\colon \pi_1(W) \to \pi_1(\mathcal{R}(f))$ of function $f\colon W \to \mathbb{R}$,
- $\mathcal{R}(W)$ – the Reeb number of a manifold W, see Definition 2.6,

- $f_{\mathcal{N}}$ – a map $W \to \bigvee_{i=1}^{r} \mathrm{S}_i^1$ induced by a system $\mathcal{N}$ of hypersurfaces in a manifold W,
- $\varphi_{\mathcal{N}}$ – a homomorphism $(f_{\mathcal{N}})_{\#} \colon \pi_1(W) \to F_r$ induced by a system $\mathcal{N}$ of hypersurfaces in a manifold W,
- $P(N)$ – a product neighbourhood of 2-sided submanifold N in W,
- $P_t(N)$ – the submanifold corresponding to $N \times \{t\}$ under fixed parametrization $P(N) \cong N \times [-1, 1]$,
- $\mathcal{H}_r(M)$ – the set of all independent and regular systems of hypersurfaces in M of size r ommiting the basepoint,
- $\mathcal{H}_r^{fr}(M)$ – the set of framed cobordism classes of elements of $\mathcal{H}_r(M)$,
- $\mathrm{Diff}_\bullet(M)$ – the set of all self-diffeomorphisms of M which preserve the basepoint,
- $g(M)$ – the Heegaard genus of a closed orientable 3-manifold M,
- F_r – a free group of rank r,
- $\mathrm{Ab}(G)$ – the abelianization of a group G,
- K_2 – the complete graph on two vertices,
- $\Delta(\Gamma)$ – the maximum degee of a vertex in a finite graph Γ,
- $\Delta_k(\Gamma)$ – the number of vertices of degree k in Γ.

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