Axiomatic Extension of Risk Measurement

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AXIOMATIC EXTENSION OF RISK MEASUREMENT

Introduction: In the article the author introduce and prove the additional axiom of measure of risk. She checks, by the method of mathematical proving, which from the well-known functions of risk fulfill this additional axiom. This proofs will be conducted for functions such as: Value at Risk, Expected Shortfall, Median, Absolute Median Deviation, Maximum , Maximum Loss, Half Range, and Arithmetic Average. In other words the purpose of the paper is studying which from the above functions fulfill the additional axiom of measure of risk, which can enrich the Arzner's and other axioms. This axiom is not a consequence of the Arzner's and other axioms. Furthermore the author researches mathematically if mentioned functions of risk retain properties after replacing the stochastic order with partial order. At the end the author presents the new measure of risk which fulfill all the axioms of measure of risk and the additional axiom.

Key words: axioms of risk measure, coherence, VaR, ES,

JEL Classification: C01

1. Introduction

The important step in the process of risk management is risk measuring, which let on the control and monitoring of risk. The importance of this step results from the exposure of the trader on the results of random events. In this article the author will concentrate on well- known and popular in practice and science measures of risk. She will takes into consideration also some new measures of risk. She will check the property of monotonicity of this different risk measures for the new definition of random variable order, which is mathematically partial order. This proofs will be conducted for functions such as: Value at Risk, Expected Shortfall, Median, Absolute Median Deviation, Maximum Loss, Half Range, Maximum, and Arithmetical Average. The purpose of the paper is also studying which from the above functions fulfill the additional axiom of measure of risk, which can enrich the Artzner's and other axioms. This survey unable enlargement of risk measurement theory, and its new application.

2.Methods

At the beginning the author will present the definition of measure of risk

Definition (Risk Measure)

Measure of risk is a function which maps the elements of some linear subspace V of some random variables space on (Ω, F, P), which contains the constants in real variables space.

$$\rho : V \to R,$$

It fulfills the following axioms

1) monotonicity

for every $X, Y \in V$, if $X \leq Y$ then , $\rho(X) \leq \rho(Y)$.

It means that if the portfolio X generates losses with a smaller probability then the risk joined with this portfolio is smaller.

2) invariance : For every $a \in R$ and for every $X \in V$

$$\rho(X+a)=\rho(X)+a.$$

This axiom may be interpreted such that when we add some money to the portfolio with value the risk joined with this portfolio is rising, because we may invest more money and lost more. As the values of risk measures are real we can compare them and order them if they fulfill the above axioms [Arzner and others, 1997].

Def.(Coherent measure)

The measure of risk is coherent if it fulfill the conditions:

3) Positive homogeneousness

For every $\lambda \geq 0$ and for every $X \in V$ the truth is that

$$\rho(\lambda X)=\lambda\rho(X).$$

This axiom may denote that multiplying the quantity of investment causes the risk increases proportionally. The example may be the leverage effect in stock market investing.

4) subadditivity

For every $X, Y \in Y$ there exists the relation:

$$\rho(X+Y)\leq\rho(X)+\rho(Y).$$

In well diversified portfolio the total risk of a loss value is not bigger than the risk of its individual loss values. The rules of coherence let on the consequence in risk assessment. [Artzner and others, 1997] , [Uniejewski, 2004]. We will sum up the information about risk measures researched in this article. The Value of Risk is the biggest value that can be lost as a result of investing in portfolio with a given time horizon and with a given tolerance level [Best, 2000]. *VaR* is defined as a loss which can’t be overran or achieved. It is very popular and universal. It is used by banks, investment funds, pension funds, investment homes… There exists some modifications of this measure. There are RiskMetrics, CFaR, EaR. There exists two alternative models of VaR. Jajuga [Jajuga, 2000] defined VaR as a special quantile

$$P(W > W_0 - VaR) = 1 - \alpha$$

W – a market value at the end of the considering period, W_0 - a market value in a given moment α - a tolerance level. In this article we will connect to another written definition of VaR

$$\inf\{x \in R : F(x) \geq \alpha\}.$$

On the base of VaR Expected Shortfall was created, which is also named the conditional value of risk, and denoted as CVaR or TVAR. ES assess the value of risk in classical way focusing on external results. It is clear that the expected loss on the portfolio may be equal or higher than some qaintile. Usually one assumes to the calculations of *ES* - 5% level of confidence. Formally *Expected Shortfall* may be defined as follow:

$$ES_\alpha = \frac{1}{\alpha}\int_0^\alpha VaR_\gamma(X)d\gamma,$$

and in a discrete case as follow:

$$ES_\alpha = -\frac{1}{\alpha}\left(E\left[X1_{\{X \leq x_\alpha\}}\right] + x_\alpha\left(\alpha - P\left[X \leq x_\alpha\right]\right)\right).$$

Expected Shortfall [see Trzpiot G., 2004 i Acerbi C., Tasche D., 2002], may be interpreted as the mean of the worst $(1-\alpha)$ % losses on condition that this losses are bigger then value of risk. Other considered in this article measures are plain and doesn't demand commentary, such us *ML*, Maximum, Median, Median Absolute Deviation in continuous and discrete case. In this article we will take into consideration two alternative definitions of random variables.

5 Definition (Standard definition of stochastic order of n degree)
If the variable X dominates stochastically the variable Y, what can be written $X \leq Y$, then

For n=0

$$F_y(y) \leq F_x(x)$$

$$\int_{-\infty}^{y} F_y^{\,n-1}(t)dt \leq \int_{-\infty}^{x} F_x^{\,n-1}(t)dt.$$

It means that with variable Y is joined bigger risk then the risk with the variable X. As some measures of risk don't include probability we can define the order relations without considering probability.

Weak order partial order is the relation reflexive, transitive and anti-simetric. In this way we will define the relation of the order on stochastic variables.

Definition (Partial order on random variables)

$$X \leq Y \Leftrightarrow \underset{x_i, y_i}{\forall} x_i \leq y_i .$$

In the paper the author will analyze mathematically which from the functions of measure are monotonic with this definition of order on random variables. The author will check which from the risk measures fulfill the additional axiom from the paper not published yet of the author. X and Y are risk variables. They may have two different interpretations as the value of the portfolio and also the value of its part. The axiom is the following:

Axiom

From any risk variables named X and Y

$$X \le Y \Rightarrow \mu(Y / X) = \mu(Y) - \mu(X)$$

This axiom may be interpreter that if the portfolio variable X is smaller than bigger portfolio variable Y then the risk joined in the variable Y after removing the variable X is equal the risk of Y minus the risk of X. The author claims that the above axiom is not an obvious consequence of axioms of risk measure and coherence. It is a modification of the one of the basic characteristics of measure resulting from the axioms of mathematical measure. The author will check which from the functions of risk fulfill the additional axiom. Thanks to this new axiom is possible to accurately calculate the risk of a difference of two random variables. This axiom is stronger than the axiom of subbaditivity and is not an obvious consequence of the axioms of subbaditivity and homogeneousness. The risk of difference may be helpful when we reduce the portfolio. For example a bank or an insurance company in the portfolio of loans or insurance policies eliminates the risk through securitization or reinsurance. These companies are interested in risk assessment of the investment portfolio after such reduction.

This axiom has one meaning. The left side may be only partial order. For stochastic order the author suggests the following axiom

$$Y \le X \Rightarrow \mu(Y / X) = \mu(Y) - \mu(X)$$

If the probability of crossing the loss limit is higher for portfolio variables Y then for variables X, then the risk of the portfolio after reduction should be calculated as risk Y minus risk of X.

In the counterexamples bellow she will use the following formula for probability of a difference:

$$P(X - Y) = P(X) - P(X \cap Y).$$

1. Mathematical proofs

At first the author analyze the most popular function of risk.

$VaR = \inf\{x : F(x) \ge \alpha\}$

Counterexample

Let's define the probability distribution of random variable. Suppose that $X \leq Y$

Table 1. The distribution of random variable

y_i	**4**	**7**	x_i	2	3
p_{y_i}	**1/2**	**1/2**	$p_{x\,i}$	1/2	1/2
$y_i - x_j$	*1*	*2*	*4*	*5*	
$p_{i\,j}$	*1/4*	*1/4*	*1/4*	*1/4*	

Source: Own example

$$\mu(Y / X) = \inf \{y_i - x_j : F(y_i - x_j) \geq 0{,}5\} = 4$$

$$\inf \{y_i : F(y_i) \geq 0{,}5\} - \inf \{x_i : F(x_i) \geq 0{,}5\} = 4 - 2 = 2.$$

So $\rho(Y / X) \neq \rho(Y) - \rho(X)$.

For second the author considers expected value

- Expected Value

  Counterexample

  We notice that $X \leq Y$

$$\sum_{i=1}^{n}(x_i - y_j)p_{ij} = \sum_{i=1}^{n} x_i p_{ij} - \sum_{i=1}^{n} y_j p_{ij}, \ \rho(Y/X) = \rho(Y) - \rho(X).$$

So the expected value fulfill the additional axiom.

- Absolute Median Deviation

  Counterexample

Table 3. The distribution of random variable

y_i	**4**	**5**	x_i	2	3
p_{y_i}	**1/2**	**1/2**	p_{x_i}	1/2	1/2
$y_i - x_i$			*1*	*2*	*3*
p_i			*1/4*	*2/4*	*1/4*

Source: Own example

$$Med(|Y - X - Med(Y - X)|) = Med(|\{5-3, 5-2, 4-3, 4-2\} - 2|)$$
$$= Med(|\{1,2,3\} - 2|) = 0,5$$

$$Med(Y - Med(Y)) = Med|\{5,4\} - 4,5| = Med(\{|0,5, -0,5|\}) = 0,5$$
$$Med(X - Med(X)) = Med(|\{2,3\} - 2,5|) = 0,5$$
$$0,5 > 0 \Rightarrow \rho(Y/X) \neq \rho(Y) - \rho(X).$$

In the paper titled “About the fundamentals of measures of risk” the author proved that Median is a coherent measure of risk, when we define a sum of random variables in a particular way. That’s way it is present in this survey.

- Median

  Counterexample

Table 4. The distribution of random variable

y_i	**4**	**6**		x_i	2	3	3,5
p_{y_i}	**1/2**	**1/2**		p_{x_i}	1/3	1/3	1/3
$y_i - x_j$		*0,5*	*1*	*2*	*2,5*	*3*	*4*

Source: Own example

$Med(Y-X) = 2{,}25 \;\; Med(Y) = 5 \;\; Med(X) = 3$. So

$\rho(Y/X) \neq \rho(Y) - \rho(X)$.

The median doesn't fulfill the additional axiom.

- Maximum

  Counterexample

Table 5. The distribution of random variable

y_i	**4**	**5**	x_i	2	3
p_{y_i}	**1/2**	**1/2**	p_{x_i}	1/2	1/2

Source: Own example

$$\max(5-3{,}5-2{,}4-3{,}4-2) = \max(1{,}2{,}3) = 3$$
$$\max(5{,}4) = 5 \;\; , \max(3{,}2) = 3$$
$$5-3 = 2 \Rightarrow \rho(Y/X) \neq \rho(Y) - \rho(X).$$

We will analyze ML as a well- known measure of risk, presented for example in the paper of Czerniak from 2003.

- $ML = \max_i p_i x_i$

  Counterexample *

Table 6. The distribution of random variable

y_i	**4**	**7**	x_i	2	3
p_{y_i}	**1/2**	**1/2**	p_{x_i}	1/2	1/2
$y_i - x_j$		*1*	*2*	*4*	*5*
$p_{i\,j}$		*1/4*	*1/4*	*1/4*	*1/4*

Source: Own example

$$\rho(Y - X) = \max_{i\,j} p_{i\,j}(y_i - x_j) = \frac{5}{4}.$$
$$\rho(Y) - \rho(X) = \max_i p_{y_i} y_i - \max p_{x\,j} x_j = \frac{7}{2} - \frac{3}{2} = 2.$$

So $\rho(Y - X) \neq \rho(Y) - \rho(X)$

Alternatively from the condition of independence of random variables:

$$\rho(Y - X) = \max_{ij} p_{ij}(y_i - x_j) = \max_{ij} p_i p_j (y_i - x_j) = \max_{ij}(p_i p_j y_i - p_i p_j x_j) \leq \max_{ij}(p_i y_i - p_j x_j) = \max_i p_i y_i - \max p_j x_j = \rho(Y) - \rho(X).$$

- Half Range

Counterexample . Let’s take into account the counterexample *

$$\rho(Y-X)=0{,}5(5\text{-}1)=2\ ,\quad \rho(Y)-\rho(X)=0{,}5(7-4)-0{,}5(3-2)=\frac{3}{2}-\frac{1}{2}=1\,.$$

So

$$\rho(Y-X)\neq\rho(Y)-\rho(X).$$

At the end the author will check which of the above functions of risk are still monotonic after changing stochastic order on partial order

- $VaR=\inf\{x:F(x)\geq\alpha\}$

Let's assume that $X\leq Y\Rightarrow\underset{x}{\forall}x_i\leq y_i$

So

$$\inf\{x:F(x)\geq\alpha\}\leq\inf\{y:F(y)\geq\alpha\}\Rightarrow\rho(X)\leq\rho(Y).$$

- Expected Value

Let's assume that $X\leq Y\Rightarrow\underset{x}{\forall}x_i\leq y_i$.

Counterexample

Table 7. The distribution of random variable

x_i	**1**	**2**	**3**	y_i	1	4	4,5
p_{xi}	**1/3**	**1/3**	**1/3**	p_{y_i}	0,8	0,1	0,1

Source: Own example

$$\sum_i x_i p_{xi}=2\ ,\sum_i y_i p_{y_i}=1{,}65\Rightarrow\rho(X)>\rho(Y).$$

- Absolute Median Deviation

Let's assume that

$$X \le Y \Rightarrow \underset{x}{\forall} x_i \le y_i$$

Counterexample

Table 8. The distribution of random variable

y_i	**1**	**1,2**	**1,3**	x_i	0	0,5	1
p_{yi}	**1/3**	**1/3**	**1/3**	p_{xi}	1/3	1/3	1/3

Source: Own example

$Med\left|X - Med(X)\right| = 0,5$, $Med\left|Y - Med(Y)\right| = 0,1$.

So

$Med\left|X - Med(X)\right| > Med\left|Y - Med(Y)\right|$ So $\rho(X) > \rho(Y)$.

- Median(X)

$X \le Y \Rightarrow \underset{i}{\wedge} x_i \le y_i \Rightarrow Med(X) \le Med(Y)$. So $\rho(X) \le \rho(Y)$.

- Maximum

$X \le Y \Rightarrow \underset{i}{\wedge} x_i \le y_i \Rightarrow Max(X) \le Max(Y)$. So $\rho(X) \le \rho(Y)$.

- $ML = \max_i p_i x_i$

Counterexample

$X \le Y \Rightarrow \underset{i}{\wedge} x_i \le y_i \Rightarrow \max_i p_{x_i} x_i = 1$, $\max_i p_{y_i} y_i = 0,8$.

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Table 9. The distribution of random variable

x_i	**1**	**2**	**3**	y_i	1	4	4,5
$p_{x\,i}$	**1/3**	**1/3**	**1/3**	$p_{y\,i}$	0,8	0,1	0,1

Source: Own example

So $\rho(Y) < \rho(X)$.

The author conducts the proof for Half Range which is not a coherent measure of risk.

- Half Range

Counterexample

Table 10. The distribution of random variable

x_i	**1**	**2**	**3**	y_i	3	4	4,5
$p_{x\,i}$	**1/3**	**1/3**	**1/3**	p_{y_i}	0,8	0,1	0,1

Source: Own example

$X \le Y \Rightarrow \wedge_i x_i \le y_i \Rightarrow 0{,}5(X_{\max} - X_{\min}) = 1, \qquad 0{,}5\,(Y_{\max} - Y_{\min}) = 0{,}75.$ So $\rho(Y) < \rho(X)$.

- Arithmetic Average

$$X \le Y \Rightarrow \wedge_i x_i \le y_i \Rightarrow \frac{1}{n}\sum_i x_i \le \frac{1}{m}\sum_j y_j \Rightarrow \rho(X) \le \rho(Y)$$

The author will prove that Arithmetic Average fulfills axioms of coherent measure of risk and the additional axiom

- Arithmetic Mean

$$\rho(Y-X)=\frac{1}{nm}\sum_i\sum_j(y_i-x_j)==\frac{1}{n}\frac{1}{m}\sum_i\sum_j y_i-\frac{1}{n}\frac{1}{m}\sum_i\sum_j x_j=\frac{1}{n}\sum_i y_i-\frac{1}{m}\sum_j x_j=\rho(Y)-\rho(X).$$

The author will prove that the Arithmetic Mean is a coherent measure of risk

- monotonicity

$$X\leq Y\Rightarrow F_1(y)\leq F_2(x).$$

So

$$\frac{1}{n}\sum_i x_i\leq\frac{1}{n}\sum_j y_j\Rightarrow\rho(X)\leq\rho(Y).$$

- homogenousness

From the properties of number series the author conclude:

$$\rho(\lambda X)=\frac{1}{n}\sum_i\lambda x_i=\lambda\frac{1}{n}\sum_i x_i=\lambda\rho(X).$$

- strong subadditivity

From the properties of number series the author conclude:

$$\rho(X+Y)=\frac{1}{n}\frac{1}{m}\sum_i\sum_j(y_i+x_j)=\frac{1}{n}\frac{1}{m}\sum_i\sum_j y_i+\frac{1}{n}\frac{1}{m}\sum_i\sum_j x_i=\frac{1}{n}\sum_i y_i+\frac{1}{m}\sum_j x_i=\rho(Y)+\rho(X).$$

- invariance

From the properties of number series the author conclude:

$$\rho(X+a)=\frac{1}{n}\sum_{i=1}^{n}(x_i+a)=\frac{1}{n}\sum_{i=1}^{n}x_i+\frac{1}{n}\sum_{i=1}^{n}a=\frac{1}{n}\sum_{i=1}^{n}x_i+a=\rho(X)+a$$

Conclusions

In the survey the author proved that functions of risk such like VaR, , Median Absolute Deviation, Median, Maximum , Maximum Loss and Half Range don't fulfill the additional axiom of measure of risk. When the author takes into consideration the partial order as the order on random variables it occurs that VaR, Median, Maximum and Arithmetic Mean are monotonic and E(X), Absolute Median Deviation, Maximum Loss and Half Range are not monotonic. The example of coherent measures of risk which is monotonic with partial order and fulfills the additional axiom also, is the Arithmetic Average. Unfortunately it doesn't include probability.

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