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On closed subspaces with Schauder bases in non-archimedean Fréchet spaces

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ABSTRACT

The main purpose of this paper is to prove that a non-archimedean Fréchet space of countable type is normable (respectively nuclear; reflexive; a Montel space) if and only if any its closed subspace with a Schauder basis is normable (respectively nuclear; reflexive; a Montel space). It is also shown that any Schauder basis in a non-normable non-archimedean Fréchet space has a block basic sequence whose closed linear span is nuclear. It follows that any non-normable non-archimedean Fréchet space contains an infinite-dimensional nuclear closed subspace with a Schauder basis. Moreover, it is proved that a non-archimedean Fréchet space E with a Schauder basis contains an infinite-dimensional complemented nuclear closed subspace with a Schauder basis if and only if any Schauder basis in E has a subsequence whose closed linear span is nuclear.

INTRODUCTION

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [5], [7] and [6]. Schauder and orthogonal bases in locally convex spaces are studied in [1], [2], [3] and [4].

Any infinite-dimensional Banach space E of countable type is isomorphic to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero (with the sup-norm) (see [6], Theorem 3.16), so every closed subspace of E has a Schauder basis.

There exist Fréchet spaces of countable type without a Schauder basis (see [9]). Nevertheless, any infinite-dimensional Fréchet space F of finite type is isomorphic to the Fréchet space $\mathbb{K}^{\mathbb{N}}$ of all sequences in $\mathbb{K}$ with the topology of pointwise convergence (see [3], Theorem 3.5), so every closed subspace of F has a Schauder basis. Moreover, any infinite-dimensional Fréchet space contains an infinite-dimensional closed subspace with a Schauder basis (see [8]). It is also known that any closed subspace of $c_0 \times \mathbb{K}^{\mathbb{N}}$ has a Schauder basis (see [12], Proposition 9). On the other hand any infinite-dimensional Fréchet space which is not isomorphic to any of the following spaces: $c_0, \mathbb{K}^{\mathbb{N}}, c_0 \times \mathbb{K}^{\mathbb{N}}$, contains a closed subspace without a Schauder basis (see [12], Theorem 7).

In this paper we study closed subspaces with Schauder bases in Fréchet spaces.

In Section 1 we investigate normable closed subspaces. First, we show that a Fréchet space is normable if and only if each of its closed subspaces with a Schauder basis is normable (Theorem 1.5). Then we prove that a Fréchet space with a Schauder basis (x_n) contains a closed subspace isomorphic to c_0 if and only if (x_n) has a subsequence (x_{k_n}) whose closed linear span is isomorphic to c_0 (Proposition 1.6). It is known that a Fréchet space contains a closed subspace isomorphic to c_0 if and only if it contains a bounded non-compactoid subset (see [4], Corollary 7.6). It follows that a Fréchet space of countable type is a Montel space (respectively a reflexive space) if and only if each of its closed subspaces with a Schauder basis is a Montel space (respectively a reflexive space) (Corollaries 1.11 and 1.12).

In Section 2 we are interested in nuclear closed subspaces. First, we prove that a Fréchet space of countable type is nuclear if and only if each of its closed subspaces with a Schauder basis is nuclear (Theorem 2.2). Next, we show that any Schauder basis in a non-normable Fréchet space has a block sequence whose closed linear span is nuclear (Theorem 2.3). It follows that any non-normable Fréchet space contains an infinite-dimensional nuclear closed subspace with a Schauder basis (Theorem 2.7). It is of interest to note that there exists a non-normable metrizable lcs E such that any nuclear subspace of E is finite-dimensional (Example 2.8). We also show that a Fréchet space E with a Schauder basis (x_n) contains an infinite-dimensional complemented nuclear closed subspace with a Schauder basis if and only if (x_n) has a subsequence (x_{k_n}) whose closed linear span is nuclear (Proposition 2.6).

PRELIMINARIES

The linear hull of a subset A in a linear space E is denoted by $\operatorname{lin}A$.

Let (y_n) be a sequence in a linear space E. Let $(k_n) \subset \mathbb{N}$ be an increasing sequence and let $(\beta_n) \subset \mathbb{K}$. Put $z_n = \sum_{i=k_n}^{k_{n+1}-1} \beta_i y_i$ for $n \in \mathbb{N}$. The sequence (z_n) is a *block sequence* of (y_n) if $\max_{k_n \leq i < k_{n+1}} |\beta_i| > 0$ for any $n \in \mathbb{N}$.

Let E, F be locally convex spaces. A map $T : E \to F$ is called a *linear homeomorphism* if T is linear, one-to-one, surjective and the maps T, T^{-1} are continuous. E is *isomorphic* to F if there exists a linear homeomorphism $T : E \to F$.

A sequence (x_n) in a lcs E is a *Schauder basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$ and the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n$ ($n \in \mathbb{N}$) are continuous.

By a *seminorm* on a linear space E we mean a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x+y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\ker p := \{x \in E : p(x) = 0\} = \{0\}$.

The set of all continuous seminorms on a metrizable lcs E is denoted by $\mathcal{P}(E)$. A non-decreasing sequence $(p_k) \subset \mathcal{P}(E)$ is a *base* in $\mathcal{P}(E)$ if for every $p \in \mathcal{P}(E)$ there exists $k \in \mathbb{N}$ with $p \leq p_k$. A sequence (p_k) of norms on E is a *base of norms* in $\mathcal{P}(E)$ if it is a base in $\mathcal{P}(E)$.

Any metrizable lcs E possesses a base (p_k) in $\mathcal{P}(E)$ and every metrizable lcs E with a continuous norm has a base of norms (p_k) in $\mathcal{P}(E)$.

A lcs E is *of finite type* if for each continuous seminorm p on E the quotient space $(E/\ker p)$ is finite-dimensional. A metrizable lcs E is *of countable type* if it contains a linearly dense countable subset.

Norms p, q on a linear space E are *equivalent* if there exist positive numbers a, b such that $ap(x) \leq q(x) \leq bp(x)$ for every $x \in E$. Every two norms on a finite-dimensional linear space are equivalent. Every n-dimensional lcs is linearly homeomorphic to the Banach space $\mathbb{K}^n$.

Let p be a seminorm on a linear space E and $t \in (0, 1]$. An element $x \in E$ is *t-orthogonal to a subspace M of E with respect to p* if $p(\alpha x + y) \geq t \max\{p(\alpha x), p(y)\}$ for all $\alpha \in \mathbb{K}, y \in M$. A sequence $(x_n) \subset E$ is *t-orthogonal with respect to p* if $p(\sum_{i=1}^{n} \alpha_i x_i) \geq t \max_{1 \leq i \leq n} p(\alpha_i x_i)$ for all $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$.

Let $(t_k) \subset (0, 1]$. A sequence (x_n) in a metrizable lcs E is *(t_k)-orthogonal with respect to $(p_k) \subset \mathcal{P}(E)$* if (x_n) is t_k-orthogonal with respect to p_k for every $k \in \mathbb{N}$. (If $t_k = 1$ for $k \in \mathbb{N}$, then we shall write 1-orthogonal instead of (1)-orthogonal.)

A sequence (x_n) in a metrizable lcs E is *orthogonal* if it is 1-orthogonal with respect to some base (p_k) in $\mathcal{P}(E)$. (In [6], a sequence (x_n) in a normed space $(E, \|\cdot\|)$ is called orthogonal if it is 1-orthogonal with respect to the norm $\|\cdot\|$.)

An orthogonal sequence (x_n) of non-zero elements in a metrizable lcs E is a *basic orthogonal sequence* in E. A linearly dense basic orthogonal sequence in a metrizable lcs E is an *orthogonal basis* in E.

Any block sequence of an orthogonal basis in a metrizable lcs E is a basic orthogonal sequence in E.

A sequence (x_n) in a metrizable lcs E is orthogonal in E if and only if it is (t_k)-orthogonal with respect to some base (p_k) in $\mathcal{P}(E)$ for some $(t_k) \subset (0, 1]$ (see [3], Proposition 2.6).

Every orthogonal basis in a metrizable lcs E is a Schauder basis in E (see [3], Proposition 1.4) and every Schauder basis in a Fréchet space E is an orthogonal basis in E (see [3], Proposition 1.7).

A subset A of a lcs E is *compactoid* if for each neighbourhood U of 0 in E there exists a finite subset $B = \{b_1, \ldots, b_n\}$ of E such that $A \subset U + \operatorname{co} B$, where $\operatorname{co} B = \{\sum_{i=1}^{n} \alpha_i b_i : \alpha_1, \ldots, \alpha_n \in \mathbb{K}, |\alpha_1|, \ldots, |\alpha_n| \leq 1\}$ is the *absolutely convex hull* of B.

A bounded subset A in a lcs E is compactoid if and only if any orthogonal sequence $(x_n) \subset A$ tends to 0 in E (see [3], Theorem 2.2).

Let E and F be locally convex spaces. The linear map $T : E \to F$ is *compact* if there exists a neighbourhood U of 0 in E such that $T(U)$ is compactoid in F.

For any seminorm p on a lcs E the map $\overline{p} : (E/\ker p) \to [0, \infty), x + \ker p \to p(x)$ is a norm on $(E/\ker p)$.

A lcs E is *nuclear* if for every continuous seminorm p on E there exists a continuous seminorm q on E with $q \geq p$ such that the canonical map

$$\varphi_{p,q} : ((E/\ker q), \overline{q}) \to ((E/\ker p), \overline{p}), x + \ker q \to x + \ker p$$

is compact. A subspace of a nuclear lcs is nuclear (see [7], Proposition 1.2).

Let E be a Fréchet space with a Schauder basis (x_n) which is 1-orthogonal with respect to a base of norms (p_k) in $\mathcal{P}(E)$. Then E is nuclear if and only if

$$\forall k \in \mathbb{N}\ \exists m > k : \lim_n [p_k(x_n)/p_m(x_n)] = 0$$

(see [2], Propositions 2.4 and 3.5).

Let $B = (b_{k,n})$ be an infinite matrix consisting of positive real numbers such that $b_{k,n} \leq b_{k+1,n}$ for all $k, n \in \mathbb{N}$. The *Köthe space* associated with the matrix B is the space $K(B) = \{(\alpha_n) \subset \mathbb{K} : \lim_n |\alpha_n| b_{k,n} = 0 \text{ for all } k \in \mathbb{N}\}$ with the following standard base of norms (p_k): $p_k((\alpha_n)) = k \max_n |\alpha_n| b_{k,n}, k \in \mathbb{N}$. The space $K(B)$ is a Fréchet space and the sequence (e_n) of coordinate vectors forms the standard Schauder basis in $K(B)$ (see [2], Proposition 2.2). The basis (e_n) is 1-orthogonal with respect to the base (p_k).

1. ON NORMABLE CLOSED SUBSPACES

Using the ideas of the proofs of Lemma 1, [8], Theorem 2, [8], and Proposition 9, [12], we show the following three lemmas.

Lemma 1.1. *Let $n \in \mathbb{N}$ and let $p_1, \ldots, p_n$ be continuous seminorms on a metrizable lcs E of countable type. Let M be a finite-dimensional subspace of E. Then for every $t \in (0, 1)$ there exists a closed subspace L of E with* $\dim(E/L) < \infty$ *such that any $x \in L$ is t-orthogonal to M with respect to p_i for all $1 \leq i \leq n$.*

Proof. Let $1 \leq i \leq n$ and $F_i = E/\ker p_i$. Let $\pi_i : E \to F_i$ be the quotient mapping. Denote by $(G_i, \tilde{p}_i)$ the completion of the normed space $(F_i, \overline{p_i})$ of countable type . Then there exists a linear continuous projection Q_i of G_i onto $\pi_i(M)$ of norm less than or equal to t^{-1} (see [6], Theorem 3.16 and its proof). Let $H_i = F_i \cap \ker Q_i$ and $E_i = \pi_i^{-1}(H_i)$. Any $x \in E_i$ is t-orthogonal to M with respect to p_i. Indeed, let $\alpha \in \mathbb{K}, m \in M, z = \pi_i(m)$ and $y = \pi_i(x)$. Since $z = Q_i(\alpha y + z)$, then $\overline{p_i}(z) \leq t^{-1}\overline{p_i}(\alpha y + z)$. Hence $\overline{p_i}(\alpha y + z) \geq t \max\{\overline{p_i}(\alpha y), \overline{p_i}(z)\}$ (see [6], Lemma 3.2). Thus $p_i(\alpha x + m) \geq t \max\{p_i(\alpha x), p_i(m)\}$.

Let $L = \bigcap_{i=1}^n E_i$. Any $x \in L$ is t-orthogonal to M with respect to p_i for all $1 \leq i \leq n$. Clearly, L is a closed subspace of E and

$$\dim(E/L) \le \sum_{i=1}^{n} \dim(E/E_i) = \sum_{i=1}^{n} \dim(F_i/H_i)$$
$$\le \sum_{i=1}^{n} \dim(G_i/\ker Q_i) < \infty. \quad \square$$

Lemma 1.2. *Let E be a metrizable lcs with a base (p_k) in $\mathcal{P}(E)$. Assume that $(s_n) \subset (0,1)$ with $s = \prod_{n=1}^{\infty} s_n > 0$. Then any sequence $(y_n) \subset (E \setminus \ker p_1)$ such that y_{n+1} is s_{n+1}-orthogonal to $\operatorname{lin}\{y_1, \dots, y_n\}$ with respect to p_i for all $1 \le i \le n$ and $n \in \mathbb{N}$, is orthogonal in E.*

Proof. It is enough to show that the sequence (y_n) is (t_m)-orthogonal with respect to (p_m) for some $(t_m) \subset (0,1]$ (see [3], Proposition 2.6).

Let $m \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_m \in \mathbb{K}$. Then

$$p_1\left(\sum_{i=1}^{m} \alpha_i y_i\right) \ge \left(\prod_{i=1}^{m} s_i\right) \max_{1 \le i \le m} p_1(\alpha_i y_i).$$

Let $E_m = \operatorname{lin}\{y_1, \dots, y_m\}$. Since the norms $p_1|E_m$, $p_m|E_m$ are equivalent then there exists $d_m \in (0,1)$ such that for arbitrary $\alpha_1, \dots, \alpha_m \in \mathbb{K}$ we have

$$p_m\left(\sum_{i=1}^{m} \alpha_i y_i\right) \ge d_m \max_{1 \le i \le m} p_m(\alpha_i y_i).$$

Let $k > m$ and $\alpha_1, \dots, \alpha_k \in \mathbb{K}$. Then

$$p_m\left(\sum_{i=1}^{k} \alpha_i y_i\right) \ge \left(\prod_{i=m+1}^{k} s_i\right) d_m \max_{1 \le i \le k} p_m(\alpha_i y_i) \ge s d_m \max_{1 \le i \le k} p_m(\alpha_i y_i).$$

Thus the sequence (y_n) is (sd_m)-orthogonal with respect to (p_m). $\square$

Lemma 1.3. *Let E be a Fréchet space with a base of norms (p_k) in $\mathcal{P}(E)$. Assume that for any $k \in \mathbb{N}$ the norms p_1 and p_k are equivalent on some subspace E_k of finite codimension in E. Then E is normable.*

Proof. First, we show that for any $k \in \mathbb{N}$ the norms p_k and p_{k+1} are equivalent on some dense subspace F_k of the normed space (E, p_k). Let $k \in \mathbb{N}$. Denote by G_k the closure of E_{k+1} in (E, p_k). Put $n = \dim(E/G_k)$. Clearly $n < \infty$.

If $n = 0$, then we can take $F_k = E_{k+1}$.

If $n > 0$, then by Lemma 3.14, [6], there exist $e_1, \dots, e_n \in E$ such that $\operatorname{lin}\{e_1, \dots, e_n\} + G_k = E$ and

$$p_k\left(\sum_{i=1}^{n} \alpha_i e_i + x\right) \ge 2^{-n} \max\{\max_{1 \le i \le n} p_k(\alpha_i e_i), p_k(x)\}$$

for all $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $x \in G_k$. Set $F_k = \operatorname{lin}\{e_1, \dots, e_n\} + E_{k+1}$. Of course, F_k is dense in (E, p_k). The norms p_k, p_{k+1} are equivalent on F_k. Indeed, put

$$C = \max\{\max_{1 \le i \le n} [p_{k+1}(e_i)/p_k(e_i)], \max_{x \in E_{k+1}} [p_{k+1}(x)/p_k(x)]\}.$$

Clearly $C < \infty$. Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $x \in E_{k+1}$. Then

$$p_{k+1}\left(\sum_{i=1}^{n} \alpha_i e_i + x\right) \leq \max\{\max_{1\leq i\leq n} p_{k+1}(\alpha_i e_i),\, p_{k+1}(x)\} \leq$$

$$C\max\{\max_{1\leq i\leq n} p_k(\alpha_i e_i),\, p_k(x)\} \leq 2^n C p_k\left(\sum_{i=1}^{n} \alpha_i e_i + x\right).$$

We shall prove that the normable space (E, p_1) is complete. Let (f_k^1) be a Cauchy sequence in (E, p_1) and $k_n^0 = n, n \in \mathbb{N}$. Then there exists a subsequence (k_n^1) of (k_n^0) such that $p_1(f^1_{k^1_n} - f^1_{k^1_{n+1}}) < n^{-1}, n \in \mathbb{N}$. Since F_1 is a dense subspace of (E, p_1), we can choose a sequence $(f^2_{k^1_n}) \subset F_1$ with $p_1(f^1_{k^1_n} - f^2_{k^1_n}) < n^{-1}$, $n \in \mathbb{N}$. Clearly, $(f^2_{k^1_n})$ is a Cauchy sequence in (E, p_1). Since the norms p_1, p_2 are equivalent on F_1 and $(f^2_{k^1_n}) \subset F_1$, then $(f^2_{k^1_n})$ is a Cauchy sequence in (E, p_2). In this way we can choose in turn for every $v \in \mathbb{N}$ a subsequence (k_n^v) of (k_n^{v-1}) with $p_v(f^v_{k^v_n} - f^v_{k^v_{n+1}}) < n^{-1}, n \in \mathbb{N}$, and a sequence $(f^{v+1}_{k^v_n}) \subset F_v$ with $p_v(f^v_{k^v_n} - f^{v+1}_{k^v_n}) < n^{-1}$, $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$ there exists $s \in \mathbb{N}$ with $s > n$ such that $k^{n+1}_{n+1} = k^n_s$. Since $p_n(f^n_{k^n_i} - f^n_{k^n_{i+1}}) < i^{-1}, n \leq i \leq s-1$, then $p_n(f^n_{k^n_n} - f^n_{k^{n+1}_{n+1}}) < n^{-1}$. Moreover,

$$p_n(f^n_{k^{n+1}_{n+1}} - f^{n+1}_{k^{n+1}_{n+1}}) = p_n(f^n_{k^n_s} - f^{n+1}_{k^n_s}) < s^{-1} < n^{-1}.$$

Hence $p_n(f^n_{k^n_n} - f^{n+1}_{k^{n+1}_{n+1}}) < n^{-1}, n \in \mathbb{N}$. This follows that $p_i(f^n_{k^n_n} - f^{n+1}_{k^{n+1}_{n+1}}) \to 0$ for any $i \in \mathbb{N}$. Thus $(f^n_{k^n_n})$ is a Cauchy sequence in E. Let f be the limit of $(f^n_{k^n_n})$ in E. Since

$$\begin{aligned} p_1(f^1_{k^n_n} - f^n_{k^n_n}) &\leq \max_{1\leq i\leq n-1} p_1(f^i_{k^n_n} - f^{i+1}_{k^n_n}) \\ &\leq \max_{1\leq i\leq n-1} p_i(f^i_{k^n_n} - f^{i+1}_{k^n_n}) < n^{-1},\ n \in \mathbb{N}, \end{aligned}$$

and $p_1(f^n_{k^n_n} - f) \to 0$, then $p_1(f^1_{k^n_n} - f) \to 0$. Hence $p_1(f^1_k - f) \to 0$, because (f^1_k) is a Cauchy sequence in (E, p_1). Thus we have proved that the normable space (E, p_1) is complete. By the open mapping theorem the Fréchet space E is normable. □

Immediately by Lemma 3 we obtain the following.

Proposition 1.4. *Let E be a non-normable Fréchet space with a base of norms (p_k) in $\mathcal{P}(E)$. Then there exists a subsequence (p_{n_k}) of (p_k) such that for any $k \in \mathbb{N}$ the norms p_{n_k} and $p_{n_{k+1}}$ are non-equivalent on any subspace of finite codimension in E.*

Now we can prove our first theorem.

Theorem 1.5. *A Fréchet space is normable if and only if each of its closed subspaces with a Schauder basis is normable.*

Proof. It is enough to show that any non-normable Fréchet space E contains a non-normable closed subspace G with a Schauder basis. Consider two cases.

Case 1. E has a continuous norm. It is easy to see that E contains a non-normable closed subspace F of countable type. Let (N_k) be a sequence of pairwise disjoint infinite sets with $\bigcup_{k=1}^{\infty} N_k = \mathbb{N}$ and let $(s_n) \subset (0,1)$ with $\prod_{n=1}^{\infty} s_n > 0$. By Proposition 1.4 there exists a base of norms (p_k) in $\mathcal{P}(F)$ such that for any $k \in \mathbb{N}$ the norms p_k and p_{k+1} are non-equivalent on any subspace of finite codimension in F. Then using Lemma 1.1, we can construct inductively a sequence $(x_n) \subset F$ such that x_{n+1} is s_{n+1}-orthogonal to $\mathrm{lin}\{x_1, \ldots, x_n\}$ with respect to p_i for $1 \le i \le n, n \in \mathbb{N}$, and $p_k(x_n) < n^{-1} p_{k+1}(x_n)$ for all $n \in N_k$, $k \in \mathbb{N}$. By Lemma 1.2, (x_n) is orthogonal in F. Clearly, $\inf_{n \in \mathbb{N}}[p_k(x_n)/p_{k+1}(x_n)] = 0$ for any $k \in \mathbb{N}$. Hence for every $k \in \mathbb{N}$ the norms p_k and p_{k+1} are non-equivalent on the closed linear span G of (x_n). Thus G is a non-normable closed subspace with a Schauder basis in E.

Case 2. E has no continuous norm. Then E contains a closed subspace G isomorphic to $\mathbb{K}^{\mathbb{N}}$ (see [2], Proposition 2.6), so it has a non-normable closed subspace with a Schauder basis. □

Our next result states when a Fréchet space with a Schauder basis possesses an infinite-dimensional normable closed subspace with a Schauder basis.

Proposition 1.6. *A Fréchet space E with a Schauder basis (x_n) contains a subspace isomorphic to c_0 if and only if (x_n) has a subsequence (x_{k_n}) whose closed linear span is isomorphic to c_0.*

Proof. Assume that (x_n) is 1-orthogonal with respect to a base (p_j) in $\mathcal{P}(E)$ and F is a subspace of E isomorphic to c_0. Then there is $k \in \mathbb{N}$ such that $p_k|F$ is a norm on F and

$$(*) \qquad \forall j \ge k \ \exists s_j > 0 \ \forall y \in F : p_k(y) \ge s_j p_j(y).$$

Put $N_j = \{n \in \mathbb{N} : p_k(x_n) \ge s_j p_j(x_n)\}$ for $j \ge k$. It is easy to check that there exists a sequence $(y_n) \subset (F \setminus \{0\})$ such that $y_n = \sum_{i=n}^{\infty} \alpha_{n,i} x_i, n \in \mathbb{N}$, for some $(\alpha_{n,i})_{i=n}^{\infty} \subset \mathbb{K}$.

Let $n, j \in \mathbb{N}$ with $j \ge k$. Then $p_k(y_n) = \max_{i \ge n} p_k(\alpha_{n,i} x_i) = p_k(\alpha_{n,i_n} x_{i_n})$ for some $i_n \ge n$, and $p_j(y_n) = \max_{i \ge n} p_j(\alpha_{n,i} x_i) \ge p_j(\alpha_{n,i_n} x_{i_n})$. By $(*)$ we get $p_k(\alpha_{n,i_n} x_{i_n}) = p_k(y_n) \ge s_j p_j(y_n) \ge s_j p_j(\alpha_{n,i_n} x_{i_n})$. Hence $p_k(x_{i_n}) \ge s_j p_j(x_{i_n})$, so $i_n \in N_j$. Thus $\{i_n : n \in \mathbb{N}\} \subset \bigcap_{j=k}^{\infty} N_j$. Since $i_n \ge n$ for any $n \in \mathbb{N}$, the set $N_0 = \bigcap_{j=k}^{\infty} N_j$ is infinite. Denote by G the closed linear span of $\{x_n : n \in N_0\}$. Clearly,

$$(**) \qquad \forall_{n \in N_0} \forall j \ge k : p_k(x_n) \ge s_j p_j(x_n) \ge s_j p_k(x_n).$$

Since $\forall n \in N_0 \ \exists j \ge k : p_j(x_n) > 0$, then $\forall n \in N_0 : p_k(x_n) > 0$. Hence $p_k|G$ is a norm on G. Moreover, $\forall j \ge k \ \forall x \in G : p_k(x) \ge s_j p_j(x)$. Indeed, let $j \ge k$ and $x \in G$. Then $x = \sum_{n \in N_0} \alpha_n x_n$ for some $(\alpha_n)_{n \in N_0} \subset \mathbb{K}$ and by $(**)$ we have

$$p_k(x) = \max_{n \in N_0} p_k(\alpha_n x_n) \ge s_j \max_{n \in N_0} p_j(\alpha_n x_n) = s_j p_j(x).$$

This follows that for any $j \ge k$ the norms $p_j|G$ and $p_k|G$ are equivalent. Thus G is normable, so it is isomorphic to c_0. □

Corollary 1.7. *A metrizable lcs E with an orthogonal basis (x_n) contains an infinite-dimensional normable subspace if and only if (x_n) has a subsequence (x_{k_n}) whose closed linear span is normable.*

By the proof of Proposition 1.6 we obtain

Remark 1.8. *Let (x_n) be a Schauder basis in a Fréchet space E. Assume that (x_n) is 1-orthogonal with respect to a base (p_k) in $\mathcal{P}(E)$. Then (x_n) has a subsequence (x_{k_n}) whose closed linear span is isomorphic to c_0 if and only if there exist an infinite subset M of $\mathbb{N}$, a sequence $(d_k) \subset (0, 1)$ and $k_0 \in \mathbb{N}$ such that $p_k(x_n) \geq d_{k+1}p_{k+1}(x_n) > 0$ for all $k \geq k_0$ and $n \in M$.*

Clearly, any Fréchet space which contains a closed subspace isomorphic to c_0 is non-nuclear. The following example shows that the converse is not true.

Example 1.9. Let (N_i) be a sequence of pairwise disjoint infinite sets with $\bigcup_{i=1}^{\infty} N_i = \mathbb{N}$. For $i \in \mathbb{N}$ and $n \in N_i$ we put $b_{k,n} = k^i$ if $k \leq i$, and $b_{k,n} = k^{in}$ if $k > i$. Clearly, $0 < b_{k,n} \leq b_{k+1,n}$ for all $k, n \in \mathbb{N}$. Let $B = (b_{k,n})$ and $E = K(B)$.

The Köthe space E is non-nuclear and has no subspace isomorphic to c_0.

Indeed, let (e_n) be the standard basis in E and let (p_k) be the standard base in $\mathcal{P}(E)$. Since $[p_1(e_n)/p_i(e_n)] = i^{-i}$ for $i \in \mathbb{N}$ and $n \in N_i$, then $\lim_n[p_1(e_n)/p_i(e_n)] = 0$ for none of $i \in \mathbb{N}$. Thus E is non-nuclear.

Let N_0 be an infinite subset of $\mathbb{N}$. If the set $M_i = N_0 \cap N_i$ is infinite for some $i \in \mathbb{N}$, then $\lim_{n \in M_i}[p_k(e_n)/p_{k+1}(e_n)] = \lim_{n \in M_i}[k/(k+1)]^{ni} = 0$ for any $k > i$; so the closed linear span X_0 of $\{e_n : n \in N_0\}$ is non-normable. If the set M_i is finite for any $i \in \mathbb{N}$, then there exist two increasing sequences $(n_i), (m_i) \subset \mathbb{N}$ such that $n_i \in M_{m_i}$ for any $i \in \mathbb{N}$. Thus $\lim_i[p_k(e_{n_i})/p_{k+1}(e_{n_i})] = \lim_i[k/(k+1)]^{m_i} = 0$ for any $k \in \mathbb{N}$; so X_0 is non-normable, too. By Proposition 1.6 we infer that E has no subspace isomorphic to c_0.

Since a Fréchet space of countable type is a Montel space if and only if it has no subspace isomorphic to c_0 (see [4], Corollary 7.6), then we get

Corollary 1.10. *A Fréchet space F with a Schauder basis (x_n) is a Montel space if and only if (x_n) has no subsequence (x_{k_n}) whose closed linear span is isomorphic to c_0.*

Corollary 1.11. *A Fréchet space E of countable type is a Montel space if and only if each of its closed subspaces with a Schauder basis is a Montel space.*

Using [7], Corollary 9.9, Theorem 10.3 and Theorem 10.4 we obtain

Corollary 1.12. *A Fréchet space of countable type is reflexive if and only if each of its closed subspaces with a Schauder basis is reflexive.*

First, we show the following lemma.

Lemma 2.1. *Let E be a metrizable lcs with a base (p_k) in $\mathcal{P}(E)$. Assume that*

$\forall k \in \mathbb{N}\ \exists m(k) > k\ \forall \epsilon > 0\ \exists F < E : \dim(E/F) < \infty\ \forall x \in F : p_k(x) \le \epsilon p_{m(k)}(x).$

Then E is nuclear.

Proof. Let $k \in \mathbb{N}, m = m(k)$ and $E_i = (E/\ker p_i)$ for $i \in \mathbb{N}$. We shall prove that the canonical map

$$\varphi : (E_m, \overline{p_m}) \to (E_k, \overline{p_k}), x + \ker p_m \to x + \ker p_k$$

is compact. Let $\epsilon > 0$. Then there exists a subspace F of E with $\dim(E/F) < \infty$ such that $p_k(x) \le 2^{-1}\epsilon p_m(x)$ for any $x \in F$. Without loss of generality we can assume that $F \supset \ker p_m$ and $G_m = (F/\ker p_m)$ is a closed subspace of the normed space $(E_m, \overline{p_m})$. Put $B_m = \{z \in E_m : \overline{p_m}(z) \le 1\}, B_k = \{z \in E_k : \overline{p_k}(z) \le \epsilon\}$, and $n = \dim(E/F)$. Clearly, $\overline{p_k}(\varphi(y)) \le 2^{-1}\epsilon\overline{p_m}(y)$ for $y \in G_m$.

If $n = 0$, then $\varphi(B_m) \subset B_k$.

If $n \ge 1$, then by Lemma 3.14, [6], there exist $z_1, \dots, z_n \in E_m$ such that $\text{lin}\{z_i : 1 \le i \le n\} + G_m = E_m$ and

$$(*) \qquad \overline{p_m}\left(\sum_{i=1}^{n} \alpha_i z_i + y\right) \ge [2^{-(1/n)}]^n \max\{\max_{1 \le i \le n} \overline{p_m}(\alpha_i z_i), \overline{p_m}(y)\}$$

for all $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $y \in G_m$. Clearly, we can assume that $\overline{p_m}(z_i) \ge 2$ for all $1 \le i \le n$. Then

$$\varphi(B_m) \subset \text{co}\{\varphi(z_i) : 1 \le i \le n\} + B_k$$

Indeed, let $z \in B_m$. Then there exist $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $y \in G_m$ such that $z = \sum_{i=1}^{n} \alpha_i z_i + y$. By $(*)$ we get $1 \ge \overline{p_m}(z) \ge \max\{\max_{1 \le i \le n} |\alpha_i|, 2^{-1}\overline{p_m}(y)\}$. Hence $\max_{1 \le i \le n} |\alpha_i| \le 1$ and $\overline{p_k}(\varphi(y)) \le 2^{-1}\epsilon\overline{p_m}(y) \le \epsilon$. Since $\varphi(z) = \sum_{i=1}^{n} \alpha_i\varphi(z_i) + \varphi(y)$, then $\varphi(z) \in \text{co}\{\varphi(z_i) : 1 \le i \le n\} + B_k$. This follows that $\varphi(B_m)$ is compactoid in $(E_k, \overline{p_k})$. Thus φ is compact. Hence E is nuclear. □

Theorem 2.2. *A metrizable lcs E of countable type is nuclear if and only if each of its closed subspaces with an orthogonal basis is nuclear.*

In particular, a Fréchet space of countable type is nuclear if and only if each of its closed subspaces with a Schauder basis is nuclear.

Proof. It is enough to show that any non-nuclear metrizable lcs E of countable type contains a non-nuclear closed subspace with an orthogonal basis. Let (p_k) be a base in $\mathcal{P}(E)$. By Lemma 2.1 we get

$$(*) \qquad \exists k_0 \in \mathbb{N} \forall m \ge k_0 \exists \epsilon_m > 0 \forall F < E : \dim(E/F) < \infty \quad \exists x \in F : p_{k_0}(x) > \epsilon_m p_m(x);$$

clearly, we can assume that $k_0 = 1$. Let (N_m) be a sequence of pairwise disjoint infinite sets with $\bigcup_{m=1}^{\infty} N_m = \mathbb{N}$ and let $(s_n) \subset (0,1)$ with $\prod_{n=1}^{\infty} s_n > 0$. By $(*)$ and Lemma 1.1 we can construct inductively a sequence $(y_n) \subset E$ such that y_{n+1} is s_{n+1}-orthogonal to $\text{lin}\{y_1, \ldots, y_n\}$ with respect to p_i for $1 \leq i \leq n, n \in \mathbb{N}$, and $p_1(y_n) > \epsilon_m p_m(y_n)$ for all $n \in N_m, m \in \mathbb{N}$. Hence $p_1(y_n) > 0$ for any $n \in \mathbb{N}$. By Lemma 1.2, (y_n) is orthogonal in E, so it is 1-orthogonal with respect to some base (q_k) in $\mathcal{P}(E)$. Of course, we can assume that $q_1 \geq p_1$. Let $k \in \mathbb{N}$ and $m \in \mathbb{N}$ with $p_m \geq q_k$. Then $[q_1(y_n)/q_k(y_n)] \geq [p_1(y_n)/p_m(y_n)] > \epsilon_m$ for any $n \in N_m$. Thus $\lim_n [q_1(y_n)/q_k(y_n)] = 0$ for none of $k \in \mathbb{N}$. Therefore the closed linear span of (y_n) is non-nuclear. □

Now we show that any non-normable Fréchet space with a Schauder basis contains an infinite-dimensional nuclear closed subspace with a Schauder basis.

Theorem 2.3. *Let E be a non-normable Fréchet space with a Schauder basis (x_n). Then (x_n) has a block sequence (y_n) whose closed linear span is nuclear.*

Proof. Consider two cases.

Case 1. E has a continuous norm. Assume that (x_n) is 1-orthogonal with respect to a base of norms (p_k) in $\mathcal{P}(E)$. Without loss of generality we can assume that for any $k \in \mathbb{N}$ the norms p_k and p_{k+1} are non-equivalent. Then $\inf_n [p_k(x_n)/p_{k+1}(x_n)] = 0$ for $k \in \mathbb{N}$. Let $m, n \in \mathbb{N}$. We can construct a finite sequence $(\alpha_{i_1}, \ldots, \alpha_{i_{n+1}}) \subset (\mathbb{K} \setminus \{0\})$ with $m = i_1 < \cdots < i_{n+1}$ such that $p_1(\alpha_{i_1} x_{i_1}) \geq 1$, $p_k(\alpha_{i_{k+1}} x_{i_{k+1}}) \leq 1$ and $p_{k+1}(\alpha_{i_{k+1}} x_{i_{k+1}}) \geq n \max_{1 \leq j \leq k} p_k(\alpha_{i_j} x_{i_j})$ for any $1 \leq k \leq n$. Let $y_n = \sum_{j=1}^{n+1} \alpha_{i_j} x_{i_j}$ and $1 \leq k \leq n$. Then $p_1(y_n) = \max_{1 \leq j \leq n+1} p_1(\alpha_{i_j} x_{i_j}) \geq 1$ and

$$\max_{k < j \leq n+1} p_k(\alpha_{i_j} x_{i_j}) \leq \max_{k < j \leq n+1} p_{j-1}(\alpha_{i_j} x_{i_j}) \leq 1 \leq p_1(\alpha_{i_1} x_{i_1}) \\ \leq \max_{1 \leq j \leq k} p_k(\alpha_{i_j} x_{i_j}).$$

Hence

$$\frac{p_k(y_n)}{p_{k+1}(y_n)} = \frac{\max_{1 \leq j \leq n+1} p_k(\alpha_{i_j} x_{i_j})}{\max_{1 \leq j \leq n+1} p_{k+1}(\alpha_{i_j} x_{i_j})} \leq \frac{\max_{1 \leq j \leq k} p_k(\alpha_{i_j} x_{i_j})}{p_{k+1}(\alpha_{i_{k+1}} x_{i_{k+1}})} \leq n^{-1}.$$

Thus we can construct inductively a block sequence (y_n) of (x_n) such that we have $[p_k(y_n)/p_{k+1}(y_n)] \leq n^{-1}$ for all $k, n \in \mathbb{N}$ with $k \leq n$. Clearly, (y_n) is 1-orthogonal with respect to (p_k) and $\lim_n [p_k(y_n)/p_{k+1}(y_n)] = 0, k \in \mathbb{N}$. Hence the closed linear span of (y_n) is nuclear.

Case 2. E has no continuous norm. Assume that (x_n) is 1-orthogonal with respect to a base (p_k) in $\mathcal{P}(E)$. It is easy to see that there exist two increasing sequences $(k_n), (m_n) \subset \mathbb{N}$ such that $x_{k_n} \in (\ker p_{m_n} \setminus \ker p_{m_{n+1}}), n \in \mathbb{N}$. Then the closed linear span F of (x_{k_n}) is isomorphic to $\mathbb{K}^{\mathbb{N}}$. Indeed, for any $(\alpha_n) \subset \mathbb{K}$ the sequence $(\alpha_n x_{k_n})$ is convergent to 0 in E. Hence, by the closed graph theorem the linear map $T : \mathbb{K}^{\mathbb{N}} \to F, (\alpha_n) \to \sum_{n=1}^{\infty} \alpha_n x_{k_n}$, is an isomorphism. Clearly, (x_{k_n}) is a block sequence of (x_n) and F is nuclear. □

Corollary 2.4. *Let E be a non-normable metrizable lcs E with an orthogonal basis (x_n). Then (x_n) has a block sequence (y_n) whose closed linear span is nuclear.*

The following example shows that there exists a non-normable Fréchet space H with a Schauder basis (x_n) such that for any subsequence (x_{k_n}) of (x_n) the closed linear span of (x_{k_n}) is non-nuclear.

Example 2.5. Let (N_i) be a sequence of pairwise disjoint infinite sets with $\bigcup_{i=1}^{\infty} N_i = \mathbb{N}$. For $i \in \mathbb{N}$ and $n \in N_i$ we put $b_{k,n} = 1$ if $k < i$, and $b_{k,n} = n$ if $k \geq i$. Clearly, $0 < b_{k,n} \leq b_{k+1,n}$ for all $k, n \in \mathbb{N}$. Let $B = (b_{k,n})$ and $H = K(B)$. Let (e_n) be the standard basis in H and let (p_k) be the standard base in $\mathcal{P}(H)$. Since $[p_k(e_n)/p_{k+1}(e_n)] = n^{-1}$ for any $k \in \mathbb{N}$ and $n \in N_{k+1}$, then H is non-normable.

Let N_0 be an infinite subset of $\mathbb{N}$. If the set $M_i = N_0 \cap N_i$ is infinite for some $i \in \mathbb{N}$, then the closed linear span of $\{e_n : n \in M_i\}$ is isomorphic to c_0, since $p_k(e_n) = p_i(e_n)$ for any $k \geq i$ and $n \in M_i$. If the set M_i is finite for any $i \in \mathbb{N}$, then there exist two increasing sequences $(n_i), (m_i) \subset \mathbb{N}$ such that $n_i \in M_{m_i}$ for any $i \in \mathbb{N}$. Thus $p_k(e_{n_i}) = p_{k+1}(e_{n_i})$ for all $i, k \in \mathbb{N}$ with $i > k + 1$; so the closed linear span of $\{e_{n_i} : i \in \mathbb{N}\}$ is isomorphic to c_0.

This shows that for any infinite subset N_0 of $\mathbb{N}$ the closed linear span X_0 of $\{e_n : n \in N_0\}$ contains a subspace isomorphic to c_0; so X_0 is non-nuclear.

In fact, the space H has not any infinite-dimensional complemented nuclear closed subspace with a Schauder basis. This follows from our next result.

Proposition 2.6. *Let E be a Fréchet space with a Schauder basis (x_n) and F its infinite-dimensional complemented closed subspace with a Schauder basis (y_n). If F is nuclear (respectively a Montel space), then (x_n) has a subsequence (x_{k_n}) whose closed linear span is nuclear (respectively a Montel space).*

Proof. Consider two cases.

Case 1. E has a continuous norm. Denote by P a linear continuous projection from E onto F. Let (f_n) and (h_n) be the sequences of coefficient functionals associated with the bases (x_n) and (y_n), respectively. Put $g_n(x) = h_n(Px)$ for $n \in \mathbb{N}$ and $x \in E$. Since

$$1 = |g_n(y_n)| = |g_n(\sum_{k=1}^{\infty} f_k(y_n)x_k)|$$
$$= |\sum_{k=1}^{\infty} f_k(y_n)g_n(x_k)| \leq \max_k |f_k(y_n)g_n(x_k)|, \; n \in \mathbb{N},$$

then for any $n \in \mathbb{N}$ there exists $t_n \in \mathbb{N}$ with $|f_{t_n}(y_n)g_n(x_{t_n})| \geq 1$.

Assume that (x_n) is 1-orthogonal with respect to a base of norms (p_k) in $\mathcal{P}(E)$. For any $k \in \mathbb{N}$ there exist $q_k \in \mathcal{P}(E)$, $s_k \in \mathbb{N}$ with $p_k \leq q_k \leq p_{s_k}$ and

$q_k \circ P \leq p_{s_k}$ such that (y_n) is 1-orthogonal with respect to q_k. For all $n, k \in \mathbb{N}$ we obtain

$$\begin{aligned} p_k(f_{t_n}(y_n)x_{t_n}) &\leq \max_m p_k(f_m(y_n)x_m) = p_k(y_n) \\ &\leq |g_n(x_{t_n})|^{-1} \max_m q_k(g_m(x_{t_n})y_m) = |g_n(x_{t_n})|^{-1} q_k(Px_{t_n}) \\ &\leq p_{s_k}(f_{t_n}(y_n)x_{t_n}). \end{aligned}$$

Hence

$$(*) \qquad p_k(f_{t_n}(y_n)x_{t_n}) \leq p_k(y_n) \leq p_{s_k}(f_{t_n}(y_n)x_{t_n}) \text{ for all } k,\ n \in \mathbb{N}.$$

Put $r_k(y) = \max_n |h_n(y)| p_k(f_{t_n}(y_n)x_{t_n}), k \in \mathbb{N},\ y \in F$.

By $(*)$, we get $r_k(y) \leq \max_n |h_n(y)| q_k(y_n) = q_k(y) \leq p_{s_k}(y)$, and $p_k(y) \leq \max_n |h_n(y)| p_k(y_n) \leq \max_n |h_n(y)| p_{s_k}(f_{t_n}(y_n)x_{t_n}) = r_{s_k}(y)$.

Thus (r_k) is a base of norms in $\mathcal{P}(F)$. Clearly, (y_n) is 1-orthogonal with respect to (r_k) and

$$(**) \qquad \frac{r_k(y_n)}{r_{k+1}(y_n)} = \frac{p_k(f_{t_n}(y_n)x_{t_n})}{p_{k+1}(f_{t_n}(y_n)x_{t_n})} = \frac{p_k(x_{t_n})}{p_{k+1}(x_{t_n})} \quad \text{for all } k,\ n \in \mathbb{N}.$$

If F is nuclear, then for any $k \in \mathbb{N}$ there is $m_k \in \mathbb{N}$ with $\lim_n [r_k(y_n)/r_{m_k}(y_n)] = 0$. Hence $\lim_n [p_k(x_{t_n})/p_{m_k}(x_{t_n})] = 0$ for any $k \in \mathbb{N}$. Thus the set $\{t_n : n \in \mathbb{N}\}$ is infinite and the closed linear span of (x_{t_n}) is nuclear.

If F is a Montel space, then by Corollary 1.10, Remark 1.8 and $(**)$, the set $\{t_n : n \in \mathbb{N}\}$ is infinite and the closed linear span of (x_{t_n}) is a Montel space.

Case 2. E has no continuous norm. As in the proof of Theorem 2.3 one can prove that (x_n) has a subsequence (x_{k_n}) whose closed linear span is isomorphic to $\mathbb{K}^{\mathbb{N}}$. Clearly, $\mathbb{K}^{\mathbb{N}}$ is nuclear, so it is a Montel space, too. □

Lemma 5, [11], states that any non-normable Fréchet space E of countable type which is not isomorphic to $c_0 \times \mathbb{K}^{\mathbb{N}}$ or $\mathbb{K}^{\mathbb{N}}$ contains a non-normable closed subspace with a continuous norm. It is obvious by its proof that any Fréchet space which is not isomorphic to the product of a Banach space and $\mathbb{K}^{\mathbb{N}}$ contains a non-normable closed subspace with a continuous norm. Hence, using Theorems 1.5 and 2.3, we get the following.

Theorem 2.7. *Any non-normable Fréchet space E contains an infinite-dimensional nuclear closed subspace F with a Schauder basis. If E is not isomorphic to the product of a Banach space and $\mathbb{K}^{\mathbb{N}}$, we can claim additionally that F has a continuous norm.*

Next example shows that there is a non-normable metrizable lcs E such that:

(i) any subspace of E with an orthogonal basis is normable (compare with Theorem 1.5);

(ii) any nuclear subspace of E is finite-dimensional (compare with Theorem 2.7).

Example 2.8. Let E be a dense subspace of $c_0 \times \mathbb{K}^{\mathbb{N}}$ with a continuous norm (see [10], Proposition 8 and its proof). Clearly, E is non-normable.

Let G be a subspace with an orthogonal basis in E. It is easy to check that the closure F of G in $c_0 \times \mathbb{K}^{\mathbb{N}}$ has a continuous norm (see [10], Proposition 8). But any closed subspace of $c_0 \times \mathbb{K}^{\mathbb{N}}$ with a continuous norm is normable (see [12], Proposition 9), so G is normable.

Let X be an infinite-dimensional subspace of E. Then X contains a subspace with an orthogonal basis (x_n) (see Lemmas 1.1 and 1.2 or [8], Theorem 2). Thus X contains an infinite-dimensional normable subspace. Hence X is non-nuclear.

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