DESCRIPTIVE TOPOLOGY IN NON-ARCHIMEDEAN FUNCTION SPACES $C_p(X, \mathbb{K})$. PART I

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Abstract. Let $\mathbb{K}$ be a non-archimedean field and let X be an ultraregular space. We study the non-archimedean locally convex space $C_p(X, \mathbb{K})$ of all $\mathbb{K}$-valued continuous functions on X endowed with the pointwise topology. We show that $\mathbb{K}$ is spherically complete if and only if every polar metrizable locally convex space E over $\mathbb{K}$ is weakly angelic. This extends a result of Kiyosawa–Schikhof for polar Banach spaces. For any compact ultraregular space X we prove that $C_p(X, \mathbb{K})$ is Fréchet-Urysohn if and only if X is scattered (a non-archimedean variant of Gerlits–Pytkeev's result). If $\mathbb{K}$ is locally compact we show the following: (1) For any ultraregular space X the space $C_p(X, \mathbb{K})$ is K-analytic if and only if it has a compact resolution (a non-archimedean variant of Tkachuk's theorem); (2) For any ultrametrizable space X the space $C_p(X, \mathbb{K})$ is analytic if and only if X is σ-compact (a non-archimedean variant of Christensen's theorem).

1. Introduction

By a non-archimedean field we mean a non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$ such that $|\alpha+\beta| \leq \max\{|\alpha|, |\beta|\}$ for all $\alpha, \beta \in \mathbb{K}$. Throughout by $\mathbb{K}$ we mean a non-archimedean field.

Let E be a linear space over $\mathbb{K}$. By a *seminorm* on E we mean a function $p : E \to [0, \infty)$ such that $p(x + y) \leq \max\{p(x), p(y)\}$ and $p(\alpha x) = |\alpha|p(x)$ for all $x, y \in E$ and $\alpha \in \mathbb{K}$. A seminorm p on E is a *norm* if $\ker p := \{x \in E : p(x) = 0\} = \{0\}$. A set $A \subset E$ is *absolutely convex*, if for any $\alpha, \beta \in \mathbb{K}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$ and any $x, y \in A$ we have $\alpha x + \beta y \in A$. A topological vector space E over $\mathbb{K}$ is *locally convex space* (lcs) if E admits a basis of neighbourhoods of zero consisting of absolutely convex sets.

All topological spaces and locally convex spaces considered in this paper are Hausdorff.

Let E be an lcs over $\mathbb{K}$. For a seminorm p on E the map $\overline{p} : E_p \to [0, \infty), x+\ker p \mapsto p(x)$, is a norm on $E_p := E/\ker p$. E is of *countable type* if for any continuous seminorm p on E the normed space $(E_p, \overline{p})$ contains a linearly dense countable subset. E is *strictly of countable type* if E contains a linearly dense countable subset. If E is strictly of countable

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type, then it is of countable type; the converse holds for any metrizable lcs. If $\mathbb{K}$ is locally compact, then E is strictly of countable type iff E is separable.
A subset C of E is *compactoid* if for each neighbourhood U of zero in E there is a finite subset A of E such that $C \subset U + \mathrm{co}A$, where $\mathrm{co}A$ is the absolutely convex hull of A.

For non-archimedean notions we refer the reader to [19], [22] and [29].

Let Y be a topological space. A family $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of [compact] subsets of Y covering Y such that $K_\alpha \subset K_\beta$ if $\alpha \leq \beta$ in $\mathbb{N}^\mathbb{N}$ is said to be a [*compact*] *resolution* of Y. Clearly every σ-compact space admits a compact resolution. A resolution $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of an lcs E over $\mathbb{K}$ is *compactoid* if the sets $K_\alpha, \alpha \in \mathbb{N}^\mathbb{N}$, are compactoid in E.

Y is *analytic* if it is a continuous image of $\mathbb{N}^\mathbb{N}$. Y is a *Lindelöf* Σ*-space* if there is an upper semi-continuous (usco) multivalued compact-valued map φ from a non-empty subset $\Omega \subset \mathbb{N}^\mathbb{N}$ into Y with $\bigcup\{\varphi(\alpha) : \alpha \in \Omega\} = Y$ ([2], [15]). If the same holds for $\Omega = \mathbb{N}^\mathbb{N}$, then Y is called *K-analytic*. Every K-analytic space has a compact resolution; the converse fails ([25], [6]). Countable unions and products of K-analytic [analytic] subspaces of a space are K-analytic [analytic]. Closed subspaces of a K-analytic [analytic] space are K-analytic [analytic], see [21], [27].

Y is *web-compact* ([18]) if there exists a nonempty subset $\Sigma \subset \mathbb{N}^\mathbb{N}$ and a family $\{A_\alpha : \alpha \in \Sigma\}$ of subsets of Y with $\overline{\bigcup\{A_\alpha : \alpha \in \Sigma\}} = Y$ such that, if $\alpha = (n_j) \in \Sigma$ and $C_{n_1,\dots,n_k} := \bigcup\{A_\beta : \beta = (m_j) \in \Sigma \text{ and } m_j = n_j \text{ for } j = 1, \dots, k\}$, and $x_k \in C_{n_1,\dots,n_k}$ for all $k \in \mathbb{N}$, then (x_k) has a cluster point in X. Separable spaces, as well as, Lindelöf Σ-spaces are web-compact ([18, Examples (B)]).

Y is *angelic* ([10]) if every relatively countably compact set A in Y is relatively compact and for each $x \in \overline{A}$ there is a sequence in A converging to x. In angelic spaces the countably compactness, compactness, and sequential compactness are equivalent, see [10, Theorem 3.3(1)]. If Y is a web-compact space and Z is a metric space, then $C_p(Y, Z)$ is angelic ([18, Corollary 1.3]). Clearly any metric space is angelic and so, by [10, Theorem 3.3(2)], any regular space with a weaker metric topology is angelic.

Motivated by the above results we proved in [13] a somewhat surprising

Theorem 1. *Assume that* $\mathbb{K}$ *is locally compact. Then a non-archimedean Banach space* E *over* $\mathbb{K}$ *is separable iff* E *is weakly Lindelöf.*

A topological space X is called *ultraregular* if for each $x \in X$ and any closed subset F of X not containing x there exist clopen disjoint subsets U and V in X such that $x \in U$ and $F \subset V$. It is easy to see that X is ultraregular iff X is zero-dimensional, i.e., X has a basis consisting of clopen subsets.

A topological space X is $\mathbb{K}$-replete if it is homeomorphic to a closed subset of the cartesian product $\mathbb{K}^B$ for some set B ([4, p. 98]).

The present paper begins a systematic study of descriptive topology in non-archimedean function spaces $C_p(X,\mathbb{K})$ for ultraregular spaces X.

Non-archimedean variants of deep results of Tkachuk, Gerlits–Pytkeev, and Christensen, respectively, are shown (see the Abstract).

$C_p(X,\mathbb{K})$ denotes the corresponding lcs as well as the underlying vector space of continuous functions.

2. Non-archimedean angelic spaces

The following observation leads to a non-archimedean definition of the analyticity.

Proposition 2. *If $\mathbb{K}$ is locally compact, then $\mathbb{K}^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$.*

Proof. By [24, Proposition 6], $\mathbb{K}$ is homeomorphic to the product $\mathbb{N}\times k^{\mathbb{N}}$, where k is the residue class field of $\mathbb{K}$. It follows that $\mathbb{K}^{\mathbb{N}}$ is homeomorphic to $(\mathbb{N}\times k)^{\mathbb{N}}$. The field k is finite since $\mathbb{K}$ is locally compact. Thus $\mathbb{K}^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. □

Hence an lcs E over locally compact $\mathbb{K}$ is analytic iff E is a continuous image of $\mathbb{K}^{\mathbb{N}}$, and E is K-analytic iff there exists an upper-semicontinuous multivalued compact-valued map φ from $\mathbb{K}^{\mathbb{N}}$ into E with $\bigcup\{\varphi(\alpha):\alpha\in\mathbb{K}^{\mathbb{N}}\}=E$.

For Proposition 3 see [25], [6], or [7].

Proposition 3 (Talagrand)**.** *Any angelic space with a compact resolution is K-analytic. Any regular space with a compact resolution admitting a weaker metrizable topology is analytic.*

We need also the following concept (compare to [7]). A polar lcs E over $\mathbb{K}$ is in the class $\mathfrak{G}$ if its $*$-weak topological dual $(E',\sigma(E',E))$ has a resolution $\{A_\alpha:\alpha\in\mathbb{N}^{\mathbb{N}}\}$ such that every countable subset of $A_\alpha, \alpha\in\mathbb{N}^{\mathbb{N}}$, is equicontinuous. In this case $\{A_\alpha:\alpha\in\mathbb{N}^{\mathbb{N}}\}$ is said to be a $\mathfrak{G}$-representation of E.

(1) *Polar (LM)-spaces are in the class $\mathfrak{G}$; in particular, polar metrizable lcs are in the class $\mathfrak{G}$.* Indeed, let $E=(E,\xi)$ be an (LM)-space, i.e. there exists an increasing sequence (called defining for E) $E_n:=(E_n,\xi_n), n\in\mathbb{N}$, of metrizable locally convex spaces with $E=\bigcup_n E_n$ such that $\xi_{n+1}|E_n\le\xi_n$ for each $n\in\mathbb{N}$ and ξ is the finest locally convex topology on E such that $\xi|E_n\le\xi_n$ for all $n\in\mathbb{N}$. For each $n\in\mathbb{N}$ let (U_j^n) be a decreasing countable basis of neighbourhoods of zero in E_n. For $\alpha=(n_k)\in\mathbb{N}^{\mathbb{N}}$ we put $A_\alpha:=\bigcap_{k=1}^\infty(U_{n_k}^k)^\circ$. Then the family $\{A_\alpha:\alpha\in\mathbb{N}^{\mathbb{N}}\}$ is a resolution of E'. Fix an $\alpha\in\mathbb{N}^{\mathbb{N}}$ and a sequence $(f_n)\subset A_\alpha$. Then the set $U:=\bigcup_{k=1}^\infty\sum_{j=1}^k U_{n_j}^j$ is a neighbourhood of zero in E such that $(f_n)\subset U^\circ$. Hence (f_n) is equicontinuous. This proves that E is in the class $\mathfrak{G}$.

(2) *Polar (DF)-spaces are in the class $\mathfrak{G}$.* Let a polar lcs E be a (DF)-space i.e. E admits a fundamental sequence (S_n) of bounded sets, and for every sequence (V_n) of

absolutely convex neighbourhoods of zero in E such that $V = \bigcap_{n=1}^{\infty} V_n$ is bornivorous, the set V is a neighbourhood of zero in E. Take $\beta \in \mathbb{K}$ with $|\beta| > 1$ and $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$. Put $A_\alpha := \bigcap_{k=1}^{\infty} \beta^{n_k} S_k^\circ$. Fix any sequence $(f_n) \subset A_\alpha$. Note that (f_n) is $\beta(E', E)$-bounded. Choose $(t_k) \subset \mathbb{R}$ such that $t_k > \sup_n \sup_{x \in S_k} |f_n(x)|, k \in \mathbb{N}$. Take $(u_k) \subset \mathbb{K}$ with $0 < |u_k| t_k < 1$ for each $k \in \mathbb{N}$. Put $U = \bigcap_{n=1}^{\infty} \{x \in E : |f_n(x)| < 1\}$. Then $u_k S_k \subset U$ for each $k \in \mathbb{N}$. This proves that U is bornivorous in E. Thus U is a neighbourhood of zero in E and the sequence (f_n) is equicontinuous, since $(f_n) \subset U^\circ$.

Although the class $\mathfrak{G}$ is large, we note the following important

Proposition 4. *(a) Let $E \in \mathfrak{G}$ and let $c(E', E)$ be the locally convex topology on E' of the uniform convergence on compactoid subsets of E. Then the lcs $E'_c = (E', c(E', E))$ has a compactoid resolution. Thus E'_c is of countable type and any compactoid subset of E is metrizable.*

(b) Let X be an ultraregular space. Then $C_p(X, \mathbb{K})$ belongs to $\mathfrak{G}$ iff X is countable.

Proof. (a) Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of E. Any countable subset of $A_\alpha, \alpha \in \mathbb{N}^{\mathbb{N}}$, is equicontinuous. Thus using [12, Lemma 9] and [12, Remark 8] (or [19, Theorem 3.8.12]), we infer that $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compactoid resolution of E'_c. By [12, Proposition 5 and Theorem 2] the space E'_c is of countable type and any compactoid subset of E is metrizable.

(b) Assume that $C_p(X, \mathbb{K}) \in \mathfrak{G}$. Since $C_p(X, \mathbb{K})$ is dense in $\mathbb{K}^X$, then (as easily seen) $E := \mathbb{K}^X$ also belongs to the class $\mathfrak{G}$. By [19, Theorems 5.3.11 and 5.4.1] every bounded subset of E is compactoid. Thus $c(E', E) = \beta(E', E)$, where $\beta(E', E)$ is the strong topology on E', so the lcs $\mathbb{K}^{(X)} = \bigoplus_{x \in X} \mathbb{K}$, isomorphic to the strong dual $(E', \beta(E', E))$ of E, is of countable type. Hence, by [19, Remark 4.2.17(a)], the space X is countable. If X is countable, the subspace $C_p(X, \mathbb{K})$ of $\mathbb{K}^X$ is a polar metrizable lcs. Thus $C_p(X, \mathbb{K}) \in \mathfrak{G}$. □

In [14, Corollary 2.2 and Theorem 2.3] Kiyosawa and Schikhof proved that in the weak topology of a non-archimedean Banach space over a spherically complete non-archimedean field the statements in the Eberlein-Šmulian theorem hold. We extend this result. A corresponding variant for spaces over the real and complex numbers has been proved in [7].

Theorem 5. *The following assertions are equivalent:*

(i) The field $\mathbb{K}$ is spherically complete.

(ii) Every lcs E in the class $\mathfrak{G}$ is weakly angelic, i.e. $(E, \sigma(E, E'))$ is angelic.

(iii) Every weakly compact set in any lcs E in the class $\mathfrak{G}$ is weakly metrizable.

Proof. (i) $\Rightarrow$ (ii): Consider two cases.

Case 1. $\mathbb{K}$ is locally compact. Let $E \in \mathfrak{G}$. Let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of E. Then each A_α is relatively countably compact in $\sigma(E', E)$. Hence $E'_\sigma = (E', \sigma(E', E))$

is web-compact. Indeed, for $\alpha=(n_j)\in\mathbb{N}^{\mathbb{N}}$ and $k\in\mathbb{N}$ we put

$$C_{n_1,\dots,n_k} := \bigcup\{A_\beta : \beta=(m_j)\in\mathbb{N}^{\mathbb{N}} \text{ and } m_j=n_j \text{ for } j=1,\dots,k\}.$$

If $x_k\in C_{n_1,\dots,n_k}$, $k\in\mathbb{N}$, then for every $k\in\mathbb{N}$ there exists $\beta_k=(m_j^k)\in\mathbb{N}^{\mathbb{N}}$ such that $x_k\in A_{\beta_k}$ and $m_j^k=n_j$ for $j=1,2,\dots,k$. Let $\gamma_j=\max\{m_j^k : k\in\mathbb{N}\}$ for $j\in\mathbb{N}$ and $\gamma=(\gamma_j)$. Then $\beta_k\le\gamma$, so $A_{\beta_k}\subset A_\gamma$ for every $k\in\mathbb{N}$. Hence $(x_k)\subset A_\gamma$. The set A_γ is relatively countably compact in E'_σ so (x_k) has a cluster point in E'_σ.

Since E'_σ is web-compact and $\mathbb{K}$ is metrizable, the space $C_p(E'_\sigma,\mathbb{K})$ is angelic, by [18, Corollary 1.3]. Hence $E_\sigma=(E,\sigma(E,E'))$ is angelic, since it is isomorphic to a subspace of $C_p(E'_\sigma,\mathbb{K})$.

Case 2. $\mathbb{K}$ is not locally compact. Let A be a relatively countably compact set in E_σ. It is easy to see that A and its closure $\overline{A}^\sigma$ are precompact in E_σ. By [19, Theorem 5.6.1], $\overline{A}^\sigma$ is precompact in E, so it is compactoid in E. By Proposition 4, $\overline{A}^\sigma$ is metrizable in E. Using [19, Theorem 5.2.12] we infer that $\overline{A}^\sigma$ is metrizable in E_σ. It is easy to see that $\overline{A}^\sigma$ is sequentially compact, so $\overline{A}^\sigma$ is a metric compact subspace of E_σ. It follows that E_σ is angelic.

(ii) $\Rightarrow$ (i): Suppose that $\mathbb{K}$ is not spherically complete. In [14] it was proved that for any set I of non-measurable cardinality with $|I|\ge 2^{\aleph_0}$ (in particular, for $I=[0,1]$) the set $\{\chi_S : S\subset I\}$ of $\mathbb{K}$-valued characteristic functions is weakly compact in the polar Banach space $\ell^\infty(I,\mathbb{K})$ but is not weakly sequentially compact. Thus $\ell^\infty(I,\mathbb{K})$ is not weakly angelic by [10, Theorem 3.3].

(iii) $\Rightarrow$ (i) follows also from the above example from [14].

(i) $\Rightarrow$ (iii): If E belongs to the class $\mathfrak{G}$ and $X\subset E$ is weakly compact, X is compact in E by [19, Theorem 5.6.1]. By Proposition 4 the space X is metrizable in E, hence in $\sigma(E,E')$. □

We showed that if $\mathbb{K}$ is spherically complete and $E\in\mathfrak{G}$, the weak compactness, weak countably compactness and weak sequential compactness for E coincide, [10, Theorem 3.3]. The following corollary provides a non-archimedean variant of the Amir-Lindenstrauss theorem [1].

Corollary 6. *Assume that the field $\mathbb{K}$ is locally compact. Then $E\in\mathfrak{G}$ is separable iff E admits a sequence (K_n) of weakly compact sets such that the linear span of $\bigcup_{n=1}^\infty K_n$ is dense in E. Consequently, every weakly compactly generated Banach space is separable.*

Proposition 7. *A non-zero metrizable lcs E over the field $\mathbb{K}$ is separable iff E has a compactoid resolution and $\mathbb{K}$ is separable.*

Proof. Assume that E is separable. Then $\mathbb{K}$ is separable and E is of countable type. By [12, Theorem 21], E has a compactoid resolution. For the converse, assume that

$\mathbb{K}$ is separable and E is a metrizable lcs with a compactod resolution. We apply [12, Proposition 5] to deduce that E is of countable type. Thus E is separable, since $\mathbb{K}$ is separable. □

Proposition 8. *Assume that the field $\mathbb{K}$ is locally compact. Then $E \in \mathfrak{G}$ is analytic iff E is separable and has a compact resolution.*

Proof. Assume that $E \in \mathfrak{G}$ is separable and has a compact resolution. Let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of E and $U_m = \{f \in E' : m|f(x_n)| < 1 \text{ for } 1 \leq n \leq m\}, m \in \mathbb{N}$. Then (U_m) is a decreasing sequence of absolutely convex neighbourhoods of zero in $E'_\sigma = (E', \sigma(E', E))$ with $\bigcap_{m=1}^{\infty} U_m = \{0\}$. Thus there exists on E' a metrizable locally convex topology weaker than $\sigma(E', E)$. Hence E'_σ is angelic. Using Proposition 4, it is easy to see that E'_σ has a compact resolution. Thus it is separable by Proposition 3. This implies that E admits a weaker metric topology. Applying again Proposition 3 we have that E is analytic. The converse is obvious. □

Corollary 9. *Assume that the field $\mathbb{K}$ is locally compact. Let E be a separable (LM)-space over $\mathbb{K}$ such that every bounded set in E is relatively compact. Then E is analytic.*

Proof. Let $\beta \in \mathbb{K}$ with $|\beta| > 1$. Let (E_n) be an increasing defining sequence of metrizable lcs for E. It is easy to see that there exists a decreasing basis $(U_j^n)_{j=1}^{\infty}$ of absolutely convex neighbourhoods of zero in E_n for $n \in \mathbb{N}$, such that $U_j^n \subset U_j^{n+1}$ for all $j, n \in \mathbb{N}$. For $\alpha = (k_j) \in \mathbb{N}^{\mathbb{N}}$ we put $A_\alpha^n := \bigcap_j \beta^{k_j} U_j^n$ and $K_\alpha := \bigcup_{n \leq k_1} A_\alpha^n$. Clearly every K_α is bounded in E, since every A_α^n is bounded in E_n. The closures in E of the sets $K_\alpha, \alpha \in \mathbb{N}^{\mathbb{N}}$, form a compact resolution in E. Now we apply Proposition 8. □

Recall that the real nonseparable (so non-analytic) space $c_0(I, \mathbb{R})$ is K-analytic in the weak topology of $c_0(I, \mathbb{R})$ ([25]). The following non-archimedean case is somewhat striking.

Proposition 10. *For a metrizable lcs E over locally compact $\mathbb{K}$ the following conditions are equivalent:*

(i) E is analytic.
(ii) $(E, \sigma(E, E'))$ is analytic.
(iii) $(E, \sigma(E, E'))$ is K-analytic.

Proof. (i) ⇒ (ii) ⇒ (iii) are clear. (iii) ⇒ (i): If E is metrizable and $(E, \sigma(E, E'))$ is K-analytic, then by [19, Theorem 5.6.1] we conclude that E has a compact resolution. Then by Proposition 3 the space E is analytic. □

3. A non-archimedean Tkachuk's theorem

In this section we assume that $\mathbb{K}$ is locally compact. We show (Theorem 14) that for an ultraregular space X the space $C_p(X, \mathbb{K})$ is K-analytic iff $C_p(X, \mathbb{K})$ has a compact resolution; we will use Theorem 14 to deduce even a stronger result, see Corollary 15. This provides a non-archimedean counterpart of Tkachuk's Theorem of [26]. We need some preparation work.

Let X be an ultraregular space. The Banaschewski's compactification $\beta_0 X$ of X is an ultranormal compactification of X. Clopen subsets of X have clopen closures in $\beta_0 X$. For each ultraregular space Y and each $f \in C(X,Y)$ there is a unique continuous function $\beta_0 f : \beta_0 X \to \beta_0 Y$ extending f. If X ultranormal then $\beta_0 X = \beta X$, where βX is the Čech-Stone compactification of X, see [4, Theorem 1].

Denote by $\upsilon_0 X$ the set of all elements $x \in \beta_0 X$ such that for every sequence (V_n) of neighbourhoods of x in $\beta_0 X$ we have $\bigcap_{n=1}^{\infty} V_n \cap X \neq \emptyset$. Using [4, Theorem 7] we note $\upsilon_0 X = \{x \in \beta_0 X : (\beta_0 f)(x) \in \mathbb{K} \text{ for every } f \in C(X,\mathbb{K})\}$. It is clear that any continuous function $f : X \to \mathbb{K}$ has a unique continuous extension $\upsilon_0 f : \upsilon_0 X \to \mathbb{K}$ such that $\upsilon_0 f = \beta_0 f|_{\upsilon_0 X}$.

Let X be a topological space. Let $\mathcal{A}$ and $\mathcal{N}$ be families of subsets of X. We say that $\mathcal{N}$ is a network modulo $\mathcal{A}$, if for any $A \in \mathcal{A}$ and any open subset U of X with $A \subset U$ there exists $N \in \mathcal{N}$ such that $A \subset N \subset U$. Let $\mathcal{N}$ be a network modulo $\mathcal{A}$. We say that a function $f \in \mathbb{K}^X$ is $\mathcal{N}$-bounded at a point $x \in X$, if $f(N)$ is bounded in $\mathbb{K}$ for some $N \in \mathcal{N}$ with $x \in N$, and $\mathcal{N}$-bounded if it is bounded at any point of X.

It is known that a topological space X is a Lindelöf Σ-space iff it has a countable network modulo some compact cover of X, see [15]. By a simple modification of the proof of [2, Proposition IV.9.3] we note the following

Proposition 11. *Let $\mathcal{N}$ be a countable network modulo a countably compact cover $\mathcal{A}$ of an ultraregular space X. Then any function $f \in C_p(X,\mathbb{K})$ is $\mathcal{N}$-bounded and the set H of all $\mathcal{N}$-bounded functions $f \in \mathbb{K}^X$ is a Lindelöf Σ-space.*

Proof. Let $\mathcal{N} = \{N_k : k \in \mathbb{N}\}$ and $\mathcal{A} = \{A_\gamma : \gamma \in \Gamma\}$. Let $f \in C_p(X,\mathbb{K})$. Take an $x \in X$. Then $x \in A_\gamma$ for some $\gamma \in \Gamma$. Clearly, f is bounded on A_γ, so on some open subset U of X with $A_\gamma \subset U$. Thus there is a $k \in \mathbb{N}$ with $A_\gamma \subset N_k \subset U$; of course $x \in N_k$ and f is bounded on N_k. So f is $\mathcal{N}$-bounded. Now we shall prove that H is a Lindelöf Σ-space.

Put $M := \mathbb{N}\times\mathbb{N}$. Let $L := \beta_0\mathbb{K}$. Clearly, H is a subspace of the compact space $G := L^X$. For $m = (k,n) \in M$ we set $B_n := \{\lambda \in \mathbb{K} : |\lambda| \leq n\}$ and $G_m := \{f \in G : f(N_k) \subset B_n\}$. Note that B_n is a compact set, so G_m is closed in G.

Let $h \in H$. Put $S_h := \{m \in M : h \in G_m\}$ and $D_h = \bigcap\{G_m : m \in S_h\}$. Then $h \in D_h$ and $D_h \subset H$. Indeed, suppose that there is a $g \in (D_h \setminus H)$. Then g is not $\mathcal{N}$-bounded at some point $x_0 \in X$. Clearly, h is $\mathcal{N}$-bounded at x_0; so for some $m = (k,n) \in M$ we have $x_0 \in N_k$ and $h \in G_m$. Hence $m \in S_h$, so $g \in D_h \subset G_m$; a contradiction, since g is not $\mathcal{N}$-bounded at x_0. Thus $\mathcal{D} = \{D_h : h \in H\}$ is a compact cover of H.

The family

$$\mathcal{M} = \{ \bigcap_{m\in W} G_m \cap H : W \subset M \text{ is finite and non-empty}\}$$

is countable. Let $h \in H$ and let U be an open subset of G with $D_h \subset U \cap H$. Then $\bigcap_{m\in W} G_m \cap (G\backslash U) = \emptyset$ for some finite subset W of S_h. Hence $D_h \subset \bigcap_{m\in W} G_m \cap H \subset U \cap H$. Thus the countable family $\mathcal{M}$ is a network modulo the compact cover $\mathcal{D}$ of H; so H is a Lindelöf Σ-space. □

Applying [3, Theorem 24] we note the following

Corollary 12. *Let Y be a Lindelöf Σ-space. Then a subspace M of $C_p(Y, \mathbb{K})$ is a Lindelöf space iff any closed discrete subset of M is countable.*

We need also the following

Proposition 13. *Let X be an ultraregular space.*

(a) The map $\Phi : C_p(X, \mathbb{K}) \to C_p(\upsilon_0 X, \mathbb{K})$ defined by $f \to \upsilon_0 f$ is linear, injective and surjective. The converse map Φ^{-1} is continuous.

(b) If $(f_n) \subset C_p(X, \mathbb{K})$ and $y \in \upsilon_0 X$, then there exists $x \in X$ with $(\upsilon_0 f_n)(y) = f_n(x)$ for every $n \in \mathbb{N}$.

(c) If a subset A of $C_p(X, \mathbb{K})$ is countable then the map $\Phi|_A$ is continuous.

(d) If $\upsilon_0 X$ is a Lindelöf Σ-space and a subset B of $C_p(X, \mathbb{K})$ is compact, then the map $\Phi|_B$ is continuous.

Proof. (a) is clear.

(b) Let $z_n = (\beta_0 f_n)(y), n \in \mathbb{N}$. Clearly $(z_n) \subset \mathbb{K}$. For each $n \in \mathbb{N}$ set

$$K_{n,m} = \{z \in \mathbb{K} : |z - z_n| < m^{-1}\}$$

for $m \in \mathbb{N}$. By $W_{n,m}$ denote the closure of $K_{n,m}$ in $\beta_0\mathbb{K}$ for each $m \in \mathbb{N}$. It is easy to see that $W_{n,m}, m \in \mathbb{N}$, form a base of neighbourhoods of z_n. Thus $V_{n,m} = (\beta_0 f_n)^{-1}(W_{n,m})$, $m \in \mathbb{N}$, are neighbourhoods of y in $\beta_0 X$ and

$$\bigcap_{m=1}^{\infty} V_{n,m} = (\beta_0 f_n)^{-1}(z_n).$$

Since $y \in \upsilon_0 X$, there exists an

$$x \in \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} V_{n,m} \cap X.$$

Then

$$f_n(x) = (\beta_0 f_n)(x) = (\beta_0 f_n)(y) = (\upsilon_0 f_n)(y)$$

for every $n \in \mathbb{N}$.

(c) Assume $(f_\gamma) \subset A, f_0 \in A$ and $f_\gamma \to f_0$ in $C_p(X,\mathbb{K})$. Let $y \in \upsilon_0 X$. By (b) there is $x \in X$ with $f(x) = (\upsilon_0 f)(y)$ for every $f \in A$. Then $f_\gamma(x) \to f_0(x)$. Hence $(\upsilon_0 f_\gamma)(y) \to (\upsilon_0 f_0)(y)$. Thus $\upsilon_0 f_\gamma \to \upsilon_0 f_0$ in $C_p(\upsilon_0 X,\mathbb{K})$, so the map $\Phi|_A$ is continuous.

(d) Let $C := \Phi(B)$. Clearly, any countable subset of B has a cluster point in B. Using (c) we infer that C is countably compact. Hence any closed discrete subset of C is finite, and therefore C is a Lindelöf space. It follows that C is compact. Since $\Phi^{-1}|_C$ is continuous, $\Phi|_B$ is continuous. □

Now, using some ideas of [26] we show that for any ultraregular space X the space $C_p(X,\mathbb{K})$ is K-analytic iff it has a compact resolution.

Theorem 14. *For any ultraregular space X the following conditions are equivalent.*

(a) $C_p(X,\mathbb{K})$ has a compact resolution.

(b) $C_p(\upsilon_0 X,\mathbb{K})$ has a compact resolution.

(c) $C_p(\upsilon_0 X,\mathbb{K})$ is K-analytic.

(d) $C_p(X,\mathbb{K})$ is K-analytic.

Proof. (a) ⇒ (b): Let $\mathcal{A} = \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a compact resolution on the ultraregular space $S = C_p(X,\mathbb{K})$. For $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$ set

$$A_{\alpha|n} = \bigcup \{A_\beta : \beta = (m_k) \in \mathbb{N}^\mathbb{N}, m_k = n_k,\ 1 \le k \le n\} \text{ and } A'_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha|n}.$$

Then $\mathcal{A}' = \{A'_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a countably compact resolution of S by [26, Proposition 2.4], and the family $\mathcal{N} := \{A_{\alpha|n} : \alpha \in \mathbb{N}^\mathbb{N}, n \in \mathbb{N}\}$ is a network modulo $\mathcal{A}'$, by applying [26, Corollary 2.5]. The network $\mathcal{N}$ is countable, since $A_{\alpha|n} = A_{\beta|n}$ whenever $\alpha = (n_k), \beta = (m_k) \in \mathbb{N}^\mathbb{N}$, $n \in \mathbb{N}$, and $n_k = m_k$ for $1 \le k \le n$.

Using Proposition 11 we obtain a Lindelöf Σ-space H with

$$C_p(S,\mathbb{K}) \subset H \subset \mathbb{K}^S.$$

The space X is homeomorphic to the subspace $F := \delta(X)$ of $C_p(S,\mathbb{K})$, where $\delta : X \to C_p(S,\mathbb{K})$ is the evaluation map. Clearly δ is continuous. Denote by G the closure of F in H. Clearly G is a Lindelöf Σ-space, and so G is an ultraregular Lindelöf space. Using [4, Theorem 8], we note $\upsilon_0 G = G$. Note that F is C-embedded in $\mathbb{K}^S$, i.e. any continuous function $g : F \to \mathbb{K}$ can be extended to a continuous function $\hat{g} : \mathbb{K}^S \to \mathbb{K}$. Indeed, let $g \in C(F,\mathbb{K})$. Then $g \circ \delta \in S$, so the function $\hat{g} : \mathbb{K}^S \to \mathbb{K}, h \to h(g \circ \delta)$ is well defined, continuous and $\hat{g}|_F = g$. Hence F is C-embedded in G. It follows that

$$\beta_0 F = \beta_0 G,\ \upsilon_0 F = \upsilon_0 G.$$

Thus $\upsilon_0 F = G$; clearly $\upsilon_0 X$ is homeomorphic to $\upsilon_0 F$. This implies that $\upsilon_0 X$ is a Lindelöf Σ-space. Using Proposition 13(d) we conclude that $\{\Phi(A_\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution of $C_p(\upsilon_0 X,\mathbb{K})$.

(b) $\Rightarrow$ (c) Using Proposition 13(a) we infer that $C_p(X,\mathbb{K})$ has a compact resolution. Thus by the proof of the implication (a) $\Rightarrow$ (b) we conclude that $\upsilon_0 X$ is a Lindelöf Σ-space.

If an uncountable space W has a resolution $\{W_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, then W_α is infinite for some $\alpha \in \mathbb{N}^\mathbb{N}$, see [16, Lemma 2.1]. It follows that any closed discrete subset of $T = C_p(\upsilon_0 X,\mathbb{K})$ is countable. Using Corollary 12 one obtains that T is an ultraregular Lindelöf space, so it is a $\mathbb{K}$-replete by [4, Theorem 8]. Thus T is homeomorphic to a closed subset of the product $\mathbb{K}^B$ for some set B. It follows that T is K-analytic, see [7].

(c) $\Rightarrow$ (d) $C_p(X,\mathbb{K}) = \Phi^{-1}(C_p(\upsilon_0 X,\mathbb{K}))$, and the map Φ^{-1} is continuous, so $C_p(X,\mathbb{K})$ is K-analytic. The implication (d) $\Rightarrow$ (a) is obvious. □

Next Corollary 15 extends Theorem 14 (a) $\Leftrightarrow$ (d).

Corollary 15. *Assume that X is an ultraregular space and $C_p(X,\mathbb{K})$ has a compact resolution. Then $C_p(X,\mathbb{K})$ is angelic. Moreover, if ξ is a regular topology on $C(X,\mathbb{K})$ that is finer than the pointwise one of $C_p(X,\mathbb{K})$, then $(C(X,\mathbb{K}),\xi)$ is K-analytic if and only if it has a compact resolution.*

Proof. ¿From the first part of the proof of Theorem 14 it follows that $\upsilon_0 X$ is a Lindelöf Σ-space, so $\upsilon_0 X$ is web-compact. Then by [18, Corollary 1.3] the space $C_p(\upsilon_0 X,\mathbb{K})$ is angelic. It follows that $C_p(X,\mathbb{K})$ is angelic. Indeed, let $A \subset C_p(X,\mathbb{K})$ be a relatively countably compact set. Let Φ be the map from Proposition 13. Then $\Phi(A)$ is relatively countably compact in $C_p(\upsilon_0 X,\mathbb{K})$, since for every countable subset $B \subset C_p(X,\mathbb{K})$ the map $\Phi|B$ is an homeomorphism. Therefore $\overline{\Phi(A)}$ is compact, since $C_p(\upsilon_0 X,\mathbb{K})$ is angelic. By the continuity of Φ^{-1} the set $\Phi^{-1}(\overline{\Phi(A)})$ is compact in $C_p(X,\mathbb{K})$. Thus $\overline{A} = \overline{\Phi^{-1}(\Phi(A))} \subset \Phi^{-1}(\overline{\Phi(A)})$, so $\overline{A}$ is compact in $C_p(X,\mathbb{K})$ and $\Phi(\overline{A}) \subset \overline{\Phi(A)}$. Let $f \in \overline{A}$. Then $\Phi(f) \in \overline{\Phi(A)}$, so there is a sequence $(f_n) \subset A$ with $\lim_n \Phi(f_n) = \Phi(f)$ in $C_p(\upsilon_0 X,\mathbb{K})$. Hence $\lim_n f_n = f$ in $C_p(X,\mathbb{K})$. Thus $C_p(X,\mathbb{K})$ is angelic.

Let ξ be a regular topology on $C(X,\mathbb{K})$ that is finer than the pointwise one of $C_p(X,\mathbb{K})$.

Assume that $Y = (C(X,\mathbb{K}),\xi)$ has a compact resolution. Since $C_p(X,\mathbb{K})$ is angelic, also $(C(X,\mathbb{K}),\xi)$ is angelic, by [10, Theorem 3.3]. Applying Proposition 3 we infer that Y is K-analytic. The converse is obvious. □

4. A non-archimedean Gertlits–Pytkeev's theorem

We prove the following non-archimedean counterpart of Gerlits–Pytkeev's result, see [2, Theorem III.1.2] for the archimedean case. Recall that a topological space X is Fréchet-Urysohn if for each $A \subset X$ and $x \in \overline{A}$ there exists a sequence in A converging to x. Clearly, any subspace of a Fréchet-Urysohn space is Fréchet-Urysohn.

A topological space X is said to be *scattered* if for every closed subset C of X , the set of isolated points of C is dense in C.

Theorem 16. *Let X be an ultraregular compact space. Then the space $C_p(X,\mathbb{K})$ is Fréchet-Urysohn iff X is scattered.*

Proof. Assume that X is scattered. By [2, Theorem III.1.2] the space $C_p(X,\mathbb{R})$ is Fréchet-Urysohn. Take a set $A \subset C_p(X,\mathbb{K})$ and an $f \in \overline{A}$. Clearly, the map

$$T : C_p(X,\mathbb{K}) \to C_p(X,\mathbb{R}), T(g) := |f - g|$$

is continuous. Thus $Tf \in T(\overline{A}) \subset \overline{T(A)}$. Since $C_p(X,\mathbb{R})$ is Fréchet-Urysohn, there exists a sequence $(f_n) \subset A$ such that $Tf_n \to Tf$ in $C_p(X,\mathbb{R})$. It follows that $f_n \to f$ in $C_p(X,\mathbb{K})$. We proved that $C_p(X,\mathbb{K})$ is Fréchet-Urysohn.

Now assume that $C_p(X,\mathbb{K})$ is Fréchet-Urysohn, and assume that X is not scattered. By D denote the Cantor set $\{0,1\}^{\mathbb{N}}$. By [23, Theorem 8.5.4] there exists a continuous surjection ϕ from X onto D. It follows that $C_p(X,\mathbb{K})$ contains a closed subset homeomorphic to $C_p(D,\mathbb{K})$. We prove that $C_p(D,\mathbb{K})$ is even not sequential, i.e. there exists in $C_p(D,\mathbb{K})$ a sequentially closed set which is not closed. Denote by μ the product measure $\nu^{\mathbb{N}}$, where ν is the measure on $\{0,1\}$ with $\nu(\{0\}) = \nu(\{1\}) = 2^{-1}$. Set

$$V(s_1,\dots,s_k) = \{(x_i) \in D : x_i = s_i \text{ for } 1 \le i \le k\}$$

for $s_i \in \{0,1\}, 1 \le i \le k, k \in \mathbb{N}$. Denote by $\mathcal{W}$ the family of all sets of the form

$$\bigcup_{i=1}^{t} V(s_{i,1},\dots,s_{i,k_i})$$

with $\sum_{i=1}^{t} 2^{-k_i} \le 2^{-2}$, where $s_{i,1},\dots,s_{i,k_i} \in \{0,1\}, 1 \le i \le t, t \in \mathbb{N}$. Clearly $\mathcal{W}$ is countable, and any element W of $\mathcal{W}$ is a clopen subset of D with $\mu(W) \le 2^{-2}$. It is easy to see that for every finite subset Z of D there exists $W \in \mathcal{W}$ with $Z \subset W$. Let $\{w_k : k \in \mathbb{N}\}$ be a dense subset of D and let $U_n \in \mathcal{W}$ with $\{w_1,\dots,w_n\} \subset U_n, n \in \mathbb{N}$.

Set

$$\mathcal{W} = \{W_n : n \in \mathbb{N}\},\ V_n = W_n \cup U_n,\ n \in \mathbb{N}.$$

Denote by f_n the characteristic function of the set $(D \setminus V_n)$ for $n \in \mathbb{N}$. Then $f_m(w_n) = 0$ for all $m,n \in \mathbb{N}$ with $m \ge n$. Put $F = \{f_n : n \in \mathbb{N}\}$. Clearly $F \subset C_p(D,\mathbb{K})$. Denote by G the closure of F in $C_p(D,\mathbb{K})$. Let $f_0 : D \to \mathbb{K},\ f_0(x) = 0$. Note that $f_0 \in G$. Indeed, let S be a neighbourhood of f_0 in $C_p(D,\mathbb{K})$. Then there is a finite subset Z of D and $\varepsilon > 0$ such that

$$S_Z := \{f \in C(D,\mathbb{K}) : |f(x)| < \varepsilon \ \text{ for } x \in Z\} \subset S.$$

Since for some $n \in \mathbb{N}$ we have $Z \subset W_n \subset V_n$, then $f_n|Z = 0$. Hence $f_n \in S$.

We prove that $G = F \cup \{f_0\}$. Let $g \in (G \setminus F)$ and $(f_{k_\alpha}) \subset F$ with $\lim f_{k_\alpha} = g$. Let $n \in \mathbb{N}$. Then there exists α_n such that $k_\alpha \ge n$ for all $\alpha \ge \alpha_n$. Hence $f_{k_\alpha}(w_n) = 0$ for $\alpha \ge \alpha_n$, so $g(w_n) = 0$. Thus $g(w_n) = 0$ for all $n \in \mathbb{N}$, so $g = f_0$.

Suppose that for some strictly increasing sequence $(n_k) \subset \mathbb{N}$ we have $\lim f_{n_k} = f_0$. Let $x \in D$. Then $x \in \bigcap_{k=k_0}^{\infty} U_{n_k}$ for some $k_0 \in \mathbb{N}$. Thus

$$\bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} U_{n_k} = D.$$

On the other hand,

$$\mu(\bigcap_{k=m}^{\infty} U_{n_k}) \leq \mu(U_{n_m}) \leq 2^{-2}$$

for any $n \in \mathbb{N}$. Consequently

$$\mu(\bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} U_{n_k}) \leq 2^{-2},$$

a contradiction. It follows that no sequence in F is convergent to f_0 in $C_p(D, \mathbb{K})$. Thus $C_p(D, \mathbb{K})$ is not sequential, hence not a Fréchet-Urysohn space. This implies that $C_p(X, \mathbb{K})$ is not a Fréchet-Urysohn space, a contradiction. □

Problem 17. *Does there exist an ultraregular space X such that the space $C_p(X, \mathbb{K})$ is Fréchet-Urysohn but $C_p(X, \mathbb{R})$ is not Fréchet-Urysohn?*

In [24, Corollary 8] (and in [28, Theorem 3.8]) it was proved that every locally compact $\mathbb{K}$ is homeomorphic to $\mathbb{N} \times D$, where D is the Cantor set. This implies the following

Proposition 18. *(i) If $\mathbb{L}$ and $\mathbb{M}$ are locally compact non-archimedean fields, then for any ultraregular space X the spaces $C_p(X, \mathbb{L})$ and $C_p(X, \mathbb{M})$ are linearly homeomorphic.*

(ii) If $\mathbb{K}$ is locally compact, then $C_p(\mathbb{K}, \mathbb{K})$ is linearly homeomorphic to the space $C_p(D, \mathbb{K})^{\mathbb{N}}$. Hence $C_p(Z, \mathbb{K})$ is not Fréchet-Urysohn for any separable Fréchet space Z.

Proof. (i) Let $\phi : \mathbb{M} \to \mathbb{L}$ be an homeomorphism. Then the map $S : C_p(X, \mathbb{L}) \to C_p(X, \mathbb{M})$ defined by $Sf = \phi^{-1} \circ f$ is a linear homeomorphism.

(ii) Let $\psi : \mathbb{N} \times D \to \mathbb{K}$ be an homeomorphism. Let $T_n : C_p(\mathbb{K}, \mathbb{K}) \to C_p(D, \mathbb{K})$ be defined by $(T_n f)(x) := f(\psi(n, x))$ for $x \in D$ and $n \in \mathbb{N}$. Then the map

$$T : C_p(\mathbb{K}, \mathbb{K}) \to C_p(D, \mathbb{K})^{\mathbb{N}}, Tf = (T_n f)$$

is a linear homeomorphism. By Theorem 16, $C_p(D, \mathbb{K})$ is not Fréchet-Urysohn, since D is not scattered. It follows that $C_p(\mathbb{K}, \mathbb{K})$ is not Fréchet-Urysohn, since it is homeomorphic to $C_p(D, \mathbb{K})^{\mathbb{N}}$. The map $P : C_p(\mathbb{K}, \mathbb{K}) \to C_p(\mathbb{K}^{\mathbb{N}}, \mathbb{K})$ defined by $(Pf)(x_1, x_2, \ldots) = f(x_1)$ for any $(x_1, x_2, \ldots) \in \mathbb{K}^{\mathbb{N}}$ is an homeomorphism onto its range. Thus $C_p(\mathbb{K}^{\mathbb{N}}, \mathbb{K})$ is not Fréchet-Urysohn.

Let Z be a separable Fréchet space. If Z is finite-dimensional, then it is homeomorphic to $\mathbb{K}$ ([24, Proposition 6]). If Z is infinite-dimensional, then it is homeomorphic to $\mathbb{K}^{\mathbb{N}}$ ([24, Corollary 5]). It follows that $C_p(Z, \mathbb{K})$ is not Fréchet-Urysohn. □

$C_p(\mathbb{R},\mathbb{R})$ is not Fréchet-Urysohn, otherwise $\mathbb{R}$ would be zero-dimensional by [11]. Moreover, $C_p(\mathbb{K},\mathbb{R})$ is not Fréchet-Urysohn, if $\mathbb{K}$ is locally compact. Indeed, in the opposite case, by the same argument as in the first part of the proof of Theorem 16, the space $C_p(\mathbb{K},\mathbb{K})$ is Fréchet-Urysohn, contrary to Proposition 18(ii).

5. A non-archimedean variant of Christensen's theorem

By Christensen [8], if X is a metric separable space, then $C_p(X,\mathbb{R})$ is analytic iff X is σ-compact. Calbrix ([5]) proved that if X is a completely regular space and $C_p(X,\mathbb{R})$ is analytic, then X is σ-compact. We shall prove (Corollary 22) that $C_p(\mathbb{K}^{\mathbb{N}},\mathbb{K})$ is not analytic, if $\mathbb{K}$ is locally compact; clearly the Fréchet space $\mathbb{K}^{\mathbb{N}}$ is not σ-compact. The following variant (Theorem 20) of a deep theorem of Christensen's [8, Theorem 3.7] describes this relationship in a general case.

We need the following non-archimedean result.

Proposition 19. *Let X be a paracompact ultraregular space. Then there exists a continuous surjection from $C_p(X,\mathbb{K}^{\mathbb{N}})$ onto $C_p(X,\mathbb{R})$.*

Proof. By [21, Theorem 1.2.14] there exists an open continuous map φ from $\mathbb{K}^{\mathbb{N}}$ onto $\mathbb{R}$. The continuous map $S_\varphi : C_p(X,\mathbb{K}^{\mathbb{N}}) \to C_p(X,\mathbb{R}), g \to \varphi \circ g$ is surjective. Indeed, let $f \in C(X,\mathbb{R})$. The multivalued function $F : X \to \mathbb{K}^{\mathbb{N}}, F(x) = \varphi^{-1}(f(x))$ has non-empty closed values. It is easy to see that F is a lower semicontinuous map i.e. for every open subset V of $\mathbb{K}^{\mathbb{N}}$ the set of all $x \in X$ with $F(x) \cap V \neq \emptyset$ is open in X. Using the Zero-dimensional Selection Theorem [20, Theorem 2.4], we get a function $g \in C_p(X,\mathbb{K}^{\mathbb{N}})$ such that $g(x) \in F(x)$ for every $x \in X$. Then $\varphi \circ g = f$. Thus S_φ is surjective. □

Theorem 20. *Assume that $\mathbb{K}$ is locally compact and X is an ultrametrizable space. Then $C_p(X,\mathbb{K})$ is analytic if and only if X is σ-compact.*

Proof. Assume that $C_p(X,\mathbb{K})$ is analytic. Then $C_p(X,\mathbb{K}^{\mathbb{N}})$ is analytic, since it is homeomorphic to $C_p(X,\mathbb{K})^{\mathbb{N}}$ and countable products of analytic spaces are analytic. Thus $C_p(X,\mathbb{R})$ is analytic, as a continuous image of $C_p(X,\mathbb{K}^{\mathbb{N}})$ (Proposition 19). Then by [5, Theorem 2.3.1] the space X is σ-compact.

The proof of the converse uses some ideas from [17, Theorem 2.1]. Assume that (X_l) is an increasing sequence of compact subsets of X covering X. Let d be a metric on X compatible with the topology of X. Denote by B the unit closed ball of $\mathbb{K}$. For $l, n, k \in \mathbb{N}$ we put

$$X_{l,n} = \{(x,y) \in X \times X : X_l \cap K(x,n^{-1}) \cap K(y,n^{-1}) \neq \emptyset\}$$

and

$$F_{k,l,n} = \{f \in B^X : |f(x) - f(y)| \leq k^{-1} \text{ for all } (x,y) \in X_{l,n}\},$$

where $K(a,n^{-1}) = \{z \in X : d(z,a) < n^{-1}\}$ for $a \in X$.

The sets $F_{k,l,n}$ for $k,l,n \in \mathbb{N}$ are compact. Indeed, let $k,l,n \in \mathbb{N}$. For every $(x,y) \in X \times X$ the set

$$V_k(x,y) = \{f \in B^X : |f(x) - f(y)| > k^{-1}\}$$

is open in B^X, since the map $\delta : B^X \to \mathbb{R}$ defined by $f \to |f(x) - f(y)|$ is continuous. Clearly,

$$F_{k,l,n} = B^X \setminus \bigcup\{V_k(x,y) : (x,y) \in X_{l,n}\},$$

and the product space B^X is compact. Thus the set $F_{k,l,n}$ is compact.

Now we prove that $C_p(X,B)$ is equal to

$$F = \bigcap_{k=1}^{\infty}\bigcap_{l=1}^{\infty}\bigcup_{n=1}^{\infty} F_{k,l,n}.$$

Let $f \in F$. Then for all $k,l \in \mathbb{N}$ there exists $n = n(k,l) \in \mathbb{N}$ with $f \in F_{k,l,n}$. Let $z \in X$. For some $l \in \mathbb{N}$ we have $z \in X_l$. Let $k \in \mathbb{N}, n = n(k,l)$ and $x \in X$ with $d(x,z) < n^{-1}$. Then $(x,z) \in X_{l,n}$, so $|f(x) - f(z)| \leq k^{-1}$. It follows that f is continuous at z. Thus $F \subset C_p(X,B)$.

To show the inverse inclusion, fix $f \in C_p(X,B)$. Assume that $f \notin F$. Then for some $k,l \in \mathbb{N}$ we note $f \notin \bigcup_{n=1}^{\infty} F_{k,l,n}$. Then $|f(x_n) - f(y_n)| > k^{-1}$ for some $(x_n,y_n) \in X_{l,n}, n \in \mathbb{N}$. Let

$$z_n \in X_l \cap K(x_n, n^{-1}) \cap K(y_n, n^{-1}), n \in \mathbb{N}.$$

Since X_l is a compact metric space, the sequence (z_n) has a subsequence (z_{t_n}) convergent to some $z_0 \in X_l$. Then the sequences $(x_{t_n}), (y_{t_n})$ are convergent in X to z_0. Hence $\lim_n |f(x_{t_n}) - f(y_{t_n})| = 0$, a contradiction. Thus $C_p(X,B) \subset F$. It follows that $C_p(X,B)$ is a $K_{\sigma\delta}$-space i.e. it is of the form $\bigcap_{i=1}^{\infty}\bigcup_{j=1}^{\infty} K_{i,j}$, where $K_{i,j}$ for $i,j \in \mathbb{N}$ are compact sets.

Note that $P = (B \setminus \{0\})$ is homeomorphic to $\mathbb{K}$. Indeed, $\mathbb{K}$ is locally compact, so there exists $\gamma \in \mathbb{K}$ with $|\gamma| = \max\{|\alpha| : \alpha \in \mathbb{K}$ and $|\alpha| < 1\}$ and the residue class field $k = B/\gamma B$ is finite. Put $m = |k|, r = |\gamma|$ and $k = \{\alpha_i + \gamma B : 1 \leq i \leq m\}$; clearly we can assume that $\alpha_1 + \gamma B = \gamma B$. Then the closed balls $B_i = B(\alpha_i, r), 1 \leq i \leq m$, are pairwise disjoint and $S(0,1) = \bigcup_{i=2}^{m} B_i$, where $B(\alpha,t) = \{\beta \in \mathbb{K} : |\beta| \leq t\}$ and $S(0,t) = \{\beta \in \mathbb{K} : |\beta| = t\}$ for $\alpha \in \mathbb{K}, t > 0$. Since $S(0,r^l) = \gamma^l S(0,1)$ for $l \in \mathbb{Z}$, we get $P = \bigcup_{l=0}^{\infty} \gamma^l S(0,1) = \bigcup_{l=0}^{\infty}\bigcup_{i=2}^{m} \gamma^l B_i$ and $\mathbb{K} = B \cup \bigcup_{l=1}^{\infty} \gamma^{-l} S(0,1) = \gamma^{-1} B_1 \cup \bigcup_{l=1}^{\infty}\bigcup_{i=2}^{m} \gamma^{-l} B_i$. Thus P and $\mathbb{K}$ are the sums of infinite countable families of pairwise disjoint balls in $\mathbb{K}$. Clearly, these balls are clopen in $\mathbb{K}$ and pairwise homeomorphic. It shows that P and $\mathbb{K}$ are homeomorphic. Thus B is a one-point compactification of $\mathbb{K}$.

Let $\overline{\mathbb{K}}$ be a compact space homeomorphic to B such that $\mathbb{K} \subset \overline{\mathbb{K}}$. Clearly, $C_p(X, \overline{\mathbb{K}})$ is homeomorphic to $C_p(X,B)$, so it is a $K_{\sigma\delta}$-space. Let $\alpha \in (\mathbb{K} \setminus B)$. It is easy to see that the subset

$$G = \bigcap_{l=1}^{\infty} \bigcup_{n=1}^{\infty} \{f \in \overline{\mathbb{K}}^X : f(X_l) \subset \alpha^n B\}$$

of the compact space $\overline{\mathbb{K}}^X$ is a $K_{\sigma\delta}$-space. Moreover $C_p(X,\mathbb{K}) \subset G \subset \overline{\mathbb{K}}^X$, so $C_p(X,\mathbb{K}) = G \cap C_p(X,\overline{\mathbb{K}})$. Thus $C_p(X,\mathbb{K})$ is a $K_{\sigma\delta}$-space, so it is K-analytic.

X is ultrametrizable and σ-compact, so it is separable. Let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of X. Then (U_m), where $U_m = \{f \in C(X,\mathbb{K}) : m|f(x_n)| < 1 \text{ for } 1 \leq n \leq m\}, m \in \mathbb{N}$, is a decreasing sequence of neighbourhoods of zero in $C_p(X,\mathbb{K})$ with $\bigcap_{m=1}^{\infty} U_m = \{0\}$. Thus there exists a weaker metric topology on $C_p(X,\mathbb{K})$. This, together with Proposition 3, implies that $C_p(X,\mathbb{K})$ is analytic. □

Corollary 21. *Assume that $\mathbb{K}$ is locally compact and X is an ultrametrizable space. Then $C_p(X,\mathbb{K})$ is analytic if and only if $C_p(X,\mathbb{R})$ is analytic.*

Proof. If $C_p(X,\mathbb{K})$ is analytic, then $C_p(X,\mathbb{R})$ is analytic - see the first part of the proof of Theorem 20. Conversely, assume that $C_p(X,\mathbb{R})$ is analytic. By [5], X is σ-compact metric space. Using Theorem 20 we infer that $C_p(X,\mathbb{K})$ is analytic. □

The following corollary extends Christensen's [8, Theorem 5.7.1].

Corollary 22. *Assume that $\mathbb{K}$ is locally compact. Then the space $C_p(\mathbb{K}^{\mathbb{N}},\mathbb{K})$ does not have a compact resolution, so it is not K-analytic.*

Proof. Suppose by contrary that $C_p(\mathbb{K}^{\mathbb{N}},\mathbb{K})$ has a compact resolution. Since $\mathbb{K}^{\mathbb{N}}$ is separable, the regular space $C_p(\mathbb{K}^{\mathbb{N}},\mathbb{K})$ admits a weaker metric topology (see the proof of Theorem 20). This and Proposition 3 yield the analyticity of $C_p(\mathbb{K}^{\mathbb{N}},\mathbb{K})$. Since $\mathbb{K}^{\mathbb{N}}$ is ultrametrizable, we apply Theorem 20 to get that $\mathbb{K}^{\mathbb{N}}$ is σ-compact, a contradiction. □

Corollary 23. *$C_p([0,\omega_1],\mathbb{K}^{\mathbb{N}})$ is not Lindelöf and $C_p([0,\omega_1],\mathbb{K})$ is not K-analytic.*

Proof. Put $X := [0,\omega_1]$. Clearly, the closed unit ball B of the Banach space $C(X,\mathbb{R})$ (with the supremum norm) is closed in $C_p(X,\mathbb{R})$. By [9, Theorem 12.40] the space $C(X,\mathbb{R})$ endowed with the weak topology σ is not Lindelöf. It follows that $(B,\sigma|B)$ is not Lindelöf, since $C(X,\mathbb{R}) = \bigcup_{n=1}^{\infty} nB$. X is a scattered compact space, so the topology of $C_p(X,\mathbb{R})$ coincides with σ on B, see [23, Corollary 19.7.7]. Hence $C_p(X,\mathbb{R})$ is not Lindelöf. Using Proposition 19 we infer that the space $C_p(X,\mathbb{K}^{\mathbb{N}})$ is not Lindelöf.

Since $C_p(X,\mathbb{K}^{\mathbb{N}})$ is homeomorphic to $C_p(X,\mathbb{K})^{\mathbb{N}}$ and countable products of K-analytic spaces are K-analytic (so Lindelöf), the space $C_p(X,\mathbb{K})$ is not K-analytic. □

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