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Every infinite-dimensional non-archimedean Fréchet space has an orthogonal basic sequence

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ABSTRACT

It is proved that any infinite-dimensional non-archimedean metrizable locally convex space has an orthogonal basic sequence.

1. INTRODUCTION

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [3], [4] and [2].

Any infinite-dimensional Banach space of countable type is linearly homeomorphic to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero (with the sup-norm), so it has a Schauder basis ([2], Theorem 3.16). Hence any infinite-dimensional Banach space has a basic sequence.

It is still unknown whether every infinite-dimensional Fréchet space (i.e. a metrizable complete lcs) of countable type has a Schauder basis. Nevertheless any metrizable lcs of finite type has an orthogonal Schauder basis ([1], Theorem 3.5). In [1] it is shown that any lcs in which not every bounded set is a compactoid, has an orthogonal basic sequence.

In this paper we prove that any infinite-dimensional metrizable lcs has an

1991 Mathematics Subject Classification. Primary: 46S10, 46A35 orthogonal basic sequence. This solves the problem stated in [1], whether any infinite-dimensional Fréchet space has a basic sequence.

2. PRELIMINARIES

A sequence (x_n) in a lcs E is a *Schauder basis of* E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $\alpha_n \in \mathbb{K}$ and the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n (n \in \mathbb{N})$ are continuous. A sequence in a lcs E is a *basic sequence in* E if it is a Schauder basis of its closed linear span in E.

The linear span of a subset A of a linear space E is denoted by $\operatorname{lin} A$.

By a *seminorm* on a linear space E we mean a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x+y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\operatorname{Ker} p := \{x \in E : p(x) = 0\} = \{0\}$.

A family P of continuous seminorms on a lcs E is a *base of continuous seminorms on* E if for every continuous seminorm p on E there exists $q \in P$ with $p \leq q$.

Every metrizable lcs E has a non-decreasing sequence of continuous seminorms $\{p_n\}$ which forms a base of continuous seminorms on E.

A metrizable lcs E is *of countable type* if it contains a linearly dense countable set. A lcs E is *of finite type* if for each continuous seminorm p on E the quotient space $E/\operatorname{Ker} p$ is finite-dimensional.

Norms p, q on a linear space E are *equivalent* if there exist positive numbers a, b such that $ap(x) \leq q(x) \leq bp(x)$ for every $x \in E$. Every two norms on a finite-dimensional linear space are equivalent. Every n-dimensional lcs is linearly homeomorphic to the Banach space $\mathbb{K}^n$.

Let $t \in (0, 1]$ and p be a seminorm on a linear space E. An element $x \in E$ is *t-orthogonal to a subspace M of E with respect to p* if $p(\alpha x + y) \geq t \max\{p(\alpha x), p(y)\}$ for all $\alpha \in \mathbb{K}, y \in M$. A sequence $(x_n) \subset E$ is *t-orthogonal with respect to p* if $p(\sum_{i=1}^{n} \alpha_i x_i) \geq t \max\{p(\alpha_i x_i) : 1 \leq i \leq n\}$ for all $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$. A sequence (x_n) in a lcs E is *orthogonal in E* if the family P of all continuous seminorms p on E for which (x_n) is 1-orthogonal with respect to p forms a base of continuous seminorms on E. (In [2] a sequence (x_n) in a normed space $(E, \|\cdot\|)$ is called orthogonal if it is 1-orthogonal with respect to the norm $\|\cdot\|$.)

In [1] it is shown that

(A) A sequence (x_n) in a lcs E is orthogonal in E if and only if there is a base P of continuous seminorms on E and a map $g : P \to (0, 1], p \to t_p$ such that (x_n) is t_p-orthogonal with respect to p for each $p \in P$ ([1], Proposition 2.6).

(B) Every orthogonal sequence of non-zero elements in a lcs E is a basic sequence in E ([1], Proposition 1.4).

3. RESULTS

First, we prove the following

Lemma 1. *Let M be a finite-dimensional subspace of linear space F with dim*

$F = \aleph_0$ *and let* $q_1, \ldots, q_n$ *be norms on* F. *Then for every* $t \in (0,1)$ *there exists* $x \in F\backslash\{0\}$ *which is t-orthogonal to* M *with respect to* q_i *for* $1 \le i \le n$.

Proof. Let $1 \le i \le n$ and let $(F_i, \tilde{q}_i)$ be the completion of (F, q_i). Since $(F_i, \tilde{q}_i)$ is an infinite-dimensional Banach space of countable type and M is its closed subspace then there exists a linear continuous projection P_i of F_i onto M of norm less than or equal to t^{-1} ([2], Theorem 3.16). Any $x \in F \cap \text{Ker } P_i$ is t-orthogonal to M with respect to q_i. Indeed, let $\alpha \in \mathbb{K}, y \in M$. Since $y = P_i(\alpha x + y)$ and $\|P_i\| \le t^{-1}$ then $q_i(y) \le t^{-1} q_i(\alpha x + y)$. Hence

$$q_i(\alpha x) = q_i(\alpha x + y - y) \le \max\{q_i(\alpha x + y), q_i(y)\} \le t^{-1} q_i(\alpha x + y).$$

It follows that $t^{-1} q_i(\alpha x + y) \ge \max\{q_i(\alpha x), q_i(y)\}$. Thus x is t-orthogonal to M with respect to q_i. Clearly,

$$\begin{aligned} \dim(F/F \cap \bigcap_{i=1}^{n} \text{Ker } P_i) &\le \sum_{i=1}^{n} \dim(F/F \cap \text{Ker } P_i) \\ &\le \sum_{i=1}^{n} \dim(F_i/\text{Ker } P_i) \\ &= n\dim M < \infty. \end{aligned}$$

Hence $G = F \cap \bigcap_{i=1}^{n} \text{Ker } P_i \ne \{0\}$. It is obvious that any $x \in G$ is t-orthogonal to M with respect to q_i for $1 \le i \le n$. □

Now, we can prove our main result.

Theorem 2. *Any infinite-dimensional metrizable locally convex space E has an orthogonal basic sequence.*

Proof. Let $\{p_n\}$ be a non-decreasing sequence of continuous seminorms on E forming a base of continuous seminorms on E.

If $\dim(E/\text{Ker } p_n) < \infty$ for all $n \in \mathbb{N}$, then E is of finite type and it has an orthogonal Schauder basis ([1], Theorem 3.5).

Now, suppose that there exists $k \in \mathbb{N}$ with $\dim(E/\text{Ker } p_k) = \infty$. We can assume that $k = 1$. Let $\{x_n + \text{Ker } p_1 : n \in \mathbb{N}\}$ be a linearly independent sequence in $E/\text{Ker } p_1$ and put $F = \text{lin}\{x_n : n \in \mathbb{N}\}$. Clearly $\dim F = \aleph_0$ and $q_n = p_n|F$ is a norm on F for each $n \in \mathbb{N}$.

Let $(s_n) \subset (0,1)$ be a sequence with $\prod_{n=1}^{\infty} s_n = s > 0$. By Lemma 1 we can construct inductively a sequence $(y_n) \subset F\backslash\{0\}$ such that for every $n \in \mathbb{N}$ y_{n+1} is s_{n+1}-orthogonal to $\text{lin}\{y_1, \ldots, y_n\}$ with respect to q_i for $1 \le i \le n$. We prove that there exists a sequence $(t_m) \subset (0,1)$ such that the sequence (y_n) is t_m-orthogonal with respect to q_m for any $m \in \mathbb{N}$. Let $m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$. Then,

$$q_1\left(\sum_{i=1}^{m} \alpha_i y_i\right) \ge s_m \max\left\{q_1\left(\sum_{i=1}^{m-1} \alpha_i y_i\right), q_1(\alpha_m y_m)\right\} \ge \ldots \ge$$

$$s_m s_{m-1} \dots s_1 \max\{q_1(\alpha_i y_i) : 1 \le i \le m\} \ge s \max\{q_1(\alpha_i y_i) : 1 \le i \le m\}.$$

Let $E_m = \mathrm{lin}\{y_1, \dots, y_m\}$. Since the norms $q_1|E_m$, $q_m|E_m$ are equivalent then there exists $d_m \in (0, 1)$ such that for arbitrary $\alpha_1, \dots, \alpha_m \in \mathbb{K}$ we have

$$q_m\left(\sum_{i=1}^{m} \alpha_i y_i\right) \ge d_m \max\{q_m(\alpha_i y_i) : 1 \le i \le m\}.$$

Let $k > m$ and $\alpha_1, \dots, \alpha_k \in \mathbb{K}$. Then,

$$\begin{aligned} q_m\left(\sum_{i=1}^{k} \alpha_i y_i\right) &\ge s_k s_{k-1} \dots s_{m+1} \max\left\{q_m\left(\sum_{i=1}^{m} \alpha_i y_i\right), q_m(\alpha_{m+1} y_{m+1}), \dots, q_m(\alpha_k y_k)\right\} \\ &\ge s d_m \max\{q_m(\alpha_i y_i) : 1 \le i \le k\}. \end{aligned}$$

Thus the sequence (y_n) is t_m-orthogonal with respect to q_m for $t_m = sd_m$, $m \in \mathbb{N}$. Hence (y_n) is t_m-orthogonal with respect to p_m for $m \in \mathbb{N}$. Using (A) and (B) we obtain that (y_n) is an orthogonal basic sequence in E. □

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