Ph. D. Dissertation

Construction of some Generalized Inverses of Operators between Banach Spaces and their Selections, Perturbations and Applications

by

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Contents

Abstract

In this dissertation, continuous homogeneous selections for the set-valued metric generalized inverses T^∂ of linear operators T in Banach spaces are investigated by means of the methods of geometry of Banach spaces. Necessary and sufficient conditions in order that the set-valued metric generalized inverses T^∂ of bounded linear operators T have continuous homogeneous selections are given. The results give an answer to the problem posed by Nashed and Votruba. Secondly, the perturbations of the Moore-Penrose metric generalized inverses for linear operators in Banach spaces are described. Using the notion of metric stable perturbation and the theorem of generalized orthogonal decomposition, under some assumptions we give some error estimates of the single-valued Moore-Penrose metric generalized inverses for bounded linear operators. Moreover, an estimate of the norm of the perturbation of the metric generalized inverse is given. The concepts of generalized regular points and narrow spectrum points of bounded linear operators on Hilbert spaces are introduced. It is proved that some properties of the narrow spectrum are the same as of the spectrum but some other properties are distinguished by these two notions. Finally, it is shown that the well known problem of the existence of invariant subspaces for bounded linear operators on separable Hilbert spaces can be restricted to the problem of the operators with the narrow spectrum only.

Keywords Generalized Inverses; Metric Generalized Inverses; Moore-Penrose Metric Generalized Inverse; Perturbation; Spectrum; Regular Point; Narrow Spectrum; Invariant Subspace

Chapter 1 Introduction

1.1 Generalized Inverses

The observation that generalized inverses are like prose (“Good Heavens! For more than forty years I have been speaking prose without knowing it” - Molière, *Le Bourgois Gentilhomme*) is nowhere truer than in the literature of linear operators. In fact, generalized inverses of integral and differential operators were studied by Fredholm, Hilbert, Schmoidt, Bounitzky, Hurwitz, and others, before E. H. Moore introduced formally the notion of generalized inverses in an algebraic setting, see, e.g., the historic survey by W. T. Reid [84].

The theory of generalized inverses has its genetic roots essentially in the context of so called “ ill-posed ” linear problems. It is well known that if A is a nonsingular (square) matrix, then there exists a unique matrix B, which is called the inverse of A, such that $AB = BA = I$, where I is the identity matrix. If A is a singular or a rectangular (but not square) matrix, no such matrix B exist. Now if A^{-1} exists, then the system of linear equations $Ax = b$ has the unique solution $x = A^{-1}b$ for each b. On the other hand, in many cases, solutions of a system of linear equations exist even when the inverse of the matrix defining these equations does not exist. Also in the case when the equations are inconsistent, there is often interested in a least-squares solutions, i.e., vectors that minimize the sum of the squares of the residuals. These problems, along with many others in numerical linear algebra, optimization and control, statistics, and other areas of analysis and applied mathematics, are readily handled via the concept of a generalized inverse (or pseudo inverse) of a matrix or a linear operator.

In a paper given at the Fourteenth Western Meeting of the American Mathematical Society at the University of Chicago, April, 1920, Professor E. H. Moore first drew attention to a “useful extension of the classical notion of the reciprocal of a nonsingular square matrix” [68]. The definition of the pseudo inverse of a $m \times n$ matrix A, denoted by A^+, originally given by E. H. Moore, has been interpreted by A. Ben-Israel and A. Charnes [17] in the following way: A^+ is the pseudo inverse of A if

$$AA^+ = P_{R(A)}, \ A^+A = P_{R(A^+)}, \tag{1.1}$$

where $P_{R(A)}$ is an orthogonal projection on the range space of A. E. H. Moore established the existence and uniqueness of A^+ for any A, and gave an explicit form for A^+ in terms of the subdeterminants of A and A^*, the conjugate transpose of A. Various properties of A^+ and the relationships among A, A^* and A^+ were incorporated in his General Analysis, and concurrently an algebraic basis and extensions were given by J. von Neumann [77] in his studies on regular rings.

Unaware of Moore’s results, A. Bjerhammar [22, 23] and R. Penrose[79, 80] both gave independent treatments of the pseudo inverse. In 1955 R. Penrose [79] sharpened and extended A. Bjerhammar’s results on linear systems, and showed that E. H. Moore’s inverse for a given matrix A is the unique matrix X satisfying the following four equations:

$$AXA = A, \tag{1.2}$$

$$XAX = X, \tag{1.3}$$

$$(AX)^* = AX, \tag{1.4}$$

$$(XA)^* = XA, \tag{1.5}$$

where A^* is the conjugate transpose of A. These conditions (1.2), (1.3), (1.4), (1.5) are equivalent to Moore’s conditions equation (1.1). The latter discovery has been so important and fruitful that this unique inverse (called by some mathematicans the generalized inverse) is now commonly called the Moore-Penrose inverse.

Since the first publication on this subject by E. H. Moore [68] many other papers appeared. Namely, generalized inverses for matrices were given by C. L. Siegel in [88], and for operators by Y. Y. Tseng [92, 93, 94, 95], F. J. Murray and J. von Neumann [70], F. V. Atkinson in [8, 9], and others. Revival of interest in the subject centered around the least squares properties (not mentioned by E. H. Moore) of certain generalized inverses.

There are several types of generalized inverses such as generalized inverses of matrices, generalized inverses of linear operators, algebraic generalized inverses, metric generalized inverses, Moore-Penrose metric generalized inverses, generalized inverses of nonlinear operators (see [3, 5, 10, 18, 19, 38]), etc. A lot of research on the theory and applications of generalized inverses has been done in the last decades (see [61, 66, 72, 104, 111, 117]), etc. In Chapter 6, two applications are showed, which are extracted from [104] and [117], respectively. One of them is that least extremal solutions of ill-posed Neumann boundary value problem for semilinear elliptic equations in L^p. Another one concerns the structure of the set of extremal solutions of ill-posed operator equation $Tx = y$ with $\text{codim}R(T) = 1$.

It is well known that linear generalized inverses have many important applications, especially in numerical approximation [72], nonlinear analysis [61], and the structural theory of Banach manifolds [66], but generally speaking, other linear inverses than metric generalized inverses were not suitable to construct the extremal solutions, the minimal norm solutions, and the best approximate solutions of an ill-posed linear operator equations in Banach spaces [73]. In order to solve the best approximation problems for ill-posed linear operator equations in Banach spaces, it is necessary to study the metric generalized inverses of linear operators between Banach spaces. This kind of generalized inverses, which are set-valued bounded homogeneous operators, was introduced by M. Z. Nashed and G. F. Votruba in 1974 in [73]. In the same paper they raised the following suggestion: "The problem of obtaining selections with nice properties for the metric generalized inverses is worth studying."

Metric generalized inverses of linear operators between Banach spaces are multivalued and in general nonlinear, so the problem of constructing their selections is natural and important. An important progress in this direction has been made by constructing some selections of metric generalized inverses of linear operators in Banach spaces (see [48, 97, 99]). In 2008, H. Hudzik, Y. W. Wang and W. J. Zheng established bounded homogeneous selections for the set-valued metric generalized inverses of linear operators on Banach spaces [48]. In 2009, C. Wang, S. P. Qu and Y. W. Wang obtained linear continuous selections for metric generalized inverses of bounded linear operators [97]. In Chapter 3 of this dissertation, some continuous homogeneous selections for the set-valued metric generalized inverses of linear operators in Banach spaces are investigated by using the methods of geometry of Banach spaces. Some necessary and sufficient conditions in order that bounded linear operators have continuous homogeneous selections for the set-valued metric generalized inverses are also given. These results are answers to the problem (mentioned above) formulated by Nashed and Votruba in [73].

1.2 Perturbation Theory of Moore-Penrose Metric Generalized Inverses

Throughout this dissertation, “perturbation theory” means “perturbation theory for linear operators”. There are other disciplines in mathematics called perturbation theory, such as the ones in analytical dynamics (celestial mechanics) and in nonlinear oscillation theory. All of them are based on the idea of studying a system deviating slightly from a simple ideal system for which the complete solution of the problem under consideration is known. However the problems they treat and the tools that they use are quite different.

Perturbation theory was created by L. Rayleigh and E. Schrödinger [51, 71], and it occupies an important place in applied mathematics. L. Rayleigh gave a formula for computing the natural frequencies and modes of a vibrating system deviating slightly from a simpler system which admits a complete determination of the frequencies and modes [83]. E. Schrödinger developed a similar method, with higher generality and systematization, for the eigenvalue problems that appear in quantum mechanics [86].

In the last years the group of mathematicians working in the perturbation theory, involving several directions in analytical dynamics and nonlinear oscillation theory etc, increased essentially [15, 26, 27, 28, 39, 51, 54, 108]. There is a wide literature of the results towards linear operators, especially generalized inverses [28, 33, 34, 35, 36, 47, 56, 57, 105, 115], etc. Since its creation, the theory has occupied an important place in applied mathematics. During the last decades it has grown into a mathematical discipline with its own interests and techniques [51].

There are some perturbations theories for generalized inverses such as linear generalized inverses and nonlinear generalized inverses. Although the perturbation of linear generalized inverses have been studied, and numerous results were obtained [55, 85, 90, 91, 107, 109], the problems of nonlinear generalized inverses remaind unsolved except some initiated study of this theory by us in [56, 57].

The Moore-Penrose metric generalized inverses of operators between Banach spaces are bounded homogeneous and nonlinear (in general) operators, which can be applied to of ill-posed boundary value problems concerning some equations. In 1995, Z. W. Li and Y. W. Wang introduced the notion of Moore-Penrose generalized inverses for closed linear operators with dense domain between Banach spaces [103]. In 2003, H. Wang and Y. W. Wang introduced the notion of Moore-Penrose metric generalized inverses of linear operators between Banach spaces [104]. In 2006, some description concerning the solution of the equality $Tx = b$ through the Moore-Penrose metric generalized inverse was obtained in [56]. In 2008, H. F. Ma and Y. W. Wang gave the definition of metric stable perturbation. After that a new method has been developed in [57] to analyze the perturbation problems for Moore-Penrose metric generalized inverses with respect to a special norm. In Chapter 4, the perturbations theory of Moore-Penrose metric generalized inverses for operators between Banach spaces was further studied. By using the continuity of the metric projection operators and the quasi-additivity of metric generalized inverses, we obtain a complete description of Moore-Penrose single-valued metric generalized inverses of operators on Banach spaces.

1.3 Spectrum and Narrow Spectrum

Spectral theory of operators is an important part of functional analysis. Many applications require the spectral theory. This theory has numerous applications in many branches of mathematics and physics including matrix theory, function space theory, complex analysis, differential and integral equations, control theory and quantum physics [32, 46, 53, 81]. In the recent years, spectral theory has witnessed an explosive development. There are many types of spectra for one or several commuting operators, for example, the approximate point spectrum, Taylor spectrum, local spectrum, essential spectrum, etc [24, 37, 41, 69], all of them with important applications. In Chapter 5, we introduce a new type of spectrum, which is called the narrow spectrum for bounded linear operators on Hilbert spaces, by using the concept of locally fine points. Some properties and applications of the narrow spectrum are presented. We show that the narrow spectrum, which form a smaller set than the spectrum, can still keep some important properties of the spectrum. In our studies of the narrow spectrum, the concept of locally fine points plays an important role.

In 1999, J. P. Ma (one of the students of Y. Y. Tseng) introduced the concept of locally fine points for operator valve maps through the concept of generalized inverses, as the notion which guarantee some stabilities of the existence of generalized inverses (see [61, 63]). This concept has been extensively studied in the last years. Such studies appear for example in: a local linearization theorem , a local conjugacy theorem, a generalized preimage theorem in global analysis, a series of the rank theorems for some operators [61, 62, 63, 64, 65, 66, 67].

Let E be a separable infinite-dimensional complex Hilbert space, $B(E)$ be the set of all bounded linear operators from E into itself. The invariant subspace problem can be formulated as: "Does every operator in $B(E)$ have a nontrivial invariant subspace ? " , and it is one of the most important problems in functional analysis. This problem remains still open for non-separable infinite-dimensional complex Hilbert spaces.

It has its origins approximately in 1935 when (according to [6]) J. von Neumann proved (in his unpublished paper) that every compact operator on a separable infinite dimensional complex Hilbert space has a non-trivial invariant subspace (the proof uses the spectral theorem for normal operators [76]). Since then, the invariant subspace problem has motivated enormous literature in operator theory. The books [16, 20, 78, 82], the lecture notes [7] and [44], and the survey papers [1] and [40] are centered around the invariant subspace problem. Related open problems and some conjectures appeared in [2]. The invariant subspaces appear in a natural way in prediction theory (see A. N. Kolmogorov [52], and N. Wiener [114]), and in mathematical physics.

The problem of the existence of nontrivial invariant subspaces for bounded linear operators on separable Hilbert spaces is reformulated in my dissertation as a problem of the narrow spectrum of bounded linear operators on Hilbert spaces . A sufficient condition for this is given in Theorem 5.2.11.

Chapter 2 Preliminaries

Throughout this dissertation, we will denote by $D(T)$, $R(T)$ and $N(T)$ the domain, the range and the null space of an operator T, respectively. Let X and Y be two real Banach spaces. The space of all bounded linear operators from X to Y is denoted by $B(X,Y)$, $B(X,X) =: B(X)$. Write $H(X,Y)$ for the space of all bounded homogenous operators from X to Y, $H(X,X) =: H(X)$. Similarly, write $L(X,Y)$ for the space of all linear operators from X to Y (if $T \in L(X,Y)$, the domain $D(T)$ of T is just a subspace of X). In this dissertation, θ is always a zero vector in vector space. X^* is the conjugate space of X and $x^*(x) =:< x^*, x >$. $\mathbb{F}$ will denote either the real field, $\mathbb{R}$, or the complex field $\mathbb{C}$.

2.1 Some Geometric Properties of Banach Spaces

Definition 2.1.1 [13] The operator $F_X : X \to X^*$ defined by

$$F_X(x) = \left\{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\right\},\ x \in X$$

is called the duality mapping of X.

Remark 2.1.1 There always exists the non-zero duality mapping of X. In fact, for any $x_1, x_2 \in X$, if $x_1 \neq x_2$, then $x_1 - x_2 \neq \theta$. Let us define

$$x_0 = x_1 - x_2,\ X_0 =\ \{\ \lambda x_0 \mid\ \lambda \in \mathbb{C}\},$$

and

$$f_0(\lambda x_0)\ =\ \lambda\|x_0\|\ \ (\forall\lambda \in \mathbb{C}).$$

Then

$$f_0(x_0) \;=\; \|x_0\| \text{ and } \|f_0\|_0 = 1,$$

where $\|f_0\|_0$ is the norm of f_0 on X_0. By the Hahn-Banach Theorem, there exist some $f \in X^*$ such that

$$f(x_0) \;=\; f_0(x_0) \;=\; \|x_0\| \text{ and } \|f\| \;=\; \|f_0\|_0 \;=\; 1.$$

Therefore, there exist some $f \in X^*\backslash\theta$ such that

$$< f, \frac{x_0}{\|x_0\|} > = \; \|\frac{x_0}{\|x_0\|}\|^2 = \; 1 \; = \; \|f\|^2 \; .$$

Proposition 2.1.2 [13] The duality mapping of X has the following properties:

1. it is homogeneous.
2. it is injective or strictly monotone if and only if X is strictly convex.
3. it is surjective if and only if X is reflexive.
4. it is single-valued if and only if X is smooth.
5. it is additive if and only if X is a Hilbert space.

Definition 2.1.2 [89] If $K \subset X$, the set-valued mapping $\mathcal{P}_K : X \to K$ defined by

$$\mathcal{P}_K(x) = \{y \in K : \|x - y\| = d_K(x)\}\,, \quad (x \in X),$$

where $d_K(x) = \inf_{y\in K} \|x - y\|$, is called the metric projection.

1. K is said to be proximinal if $\mathcal{P}_K(x) \neq \emptyset$ for any $x \in X$.
2. K is said to be semi-Chebyshev if $\mathcal{P}_K(x)$ is at most a single point set for each $x \in X$.
3. K is called a Chebyshev set if it is both proximinal and semi-Chebyshev.

When K is a Chebyshev set, we denote $\mathcal{P}_K(x)$ by $\pi_K(x)$ for any $x \in X$.

Remark 2.1.3 Every Chebyshev set is closed and every closed convex set in a rotund reflexive space is Chebyshev. In particular every non-empty closed convex set in Hilbert space is Chebyshev (see [25]).

Lemma 2.1.4 [89] If X is a normed linear space, and L is a subspace of X, then

(*i*) $\pi_L^2(x) = \pi_L(x)$ for all $x \in \mathcal{D}(\pi_L)$, i.e. π_L is idempotent;

(*ii*) $\|x - \pi_L(x)\| \leq \|x\|$ for all $x \in \mathcal{D}(\pi_L)$.

Furthermore, if L is a semi-Chebyshev subspace, then

(*iii*) $\pi_L(\alpha x) = \alpha \pi_L(x)$ for all $x \in X$ and $\alpha \in R$, i.e. π_L is homogeneous;

(*iv*) $\pi_L(x + y) = \pi_L(x) + \pi_L(y) = \pi_L(x) + y$ for all $x \in \mathcal{D}(\pi_L)$ and $y \in L$, i.e. π_L is quasi-additive.

Lemma 2.1.5 If L is a closed subspace of X, then the following statements are equivalent

(*i*) π_L is a linear operator;

(*ii*) $\pi_L^{-1}(\theta)$ is a linear subspace of X;

(*iii*) $\pi_L^{-1}(y)$ is a linear manifold of X for every $y \in L$.

Remark 2.1.6 This result has been obtained in [89] under the assumption that the underlying Banach space X is reflexive and strictly convex, but it is easy to show that the result remains valid under the weaker assumption that L be a Chebyshev subspace of X.

Theorem 2.1.7 [102] (**Generalized Orthogonal Decomposition Theorem**) Let L be a proximinal subspace of X. Then for any $x \in X$, we have the decomposition

$$x = x_1 + x_2,$$

where $x_1 \in L$ and $x_2 \in F_X^{-1}\left(L^{\perp}\right)$. In this case we have $X = L + F_X^{-1}(L^{\perp})$. If L is a *Chebyshev* subspace of X, then the decomposition is unique and

$$x = \mathcal{P}_L(x) + x_2, \quad x_2 \in F_X^{-1}(L^{\perp}).$$

In this case we have $X = \mathcal{P}_L(x) \dot{+} F_X^{-1}(L^{\perp})$, where $\mathcal{P}_L(x) = \{\pi_L x\}$.

Lemma 2.1.8 [48] Let L be a subspace of X, $x \in X \backslash \overline{L}$ and $x_0 \in L$. Then $x_0 \in \mathcal{P}_L(x)$ if and only if

$$F_X(x - x_0) \cap L^{\perp} \neq \emptyset,$$

where F_X is the duality mapping of X and $L^{\perp} = \{x^* \in X^* : \langle x^*, x\rangle = \theta,\ x \in L\}$.

Definition 2.1.3 [50] A nonempty subset C of X is said to be approximately compact, if for any sequence $\{x_n\}$ in C and any $y \in X$ such that $||x_n - y|| \to \operatorname{dist}(y, C) := \inf\{||y - z|| : z \in C\}$, we have that $\{x_n\}$ has a Cauchy subsequence. X is called approximately compact if any nonempty closed and convex subset of X is approximately compact.

Remark 2.1.9 (i) If C is approximatively compact, then $C \neq \emptyset$.
(ii) If C is approximatively compact, then C is a closed and approximinal set.

Lemma 2.1.10 [30] Let C be a semi-Chebyshev closed subset of X. If C is an approximately compact, then C is a Chebyshev subset and the metric projector π_C is continuous.

Definition 2.1.4 [51] Let $T \in B(X, Y)$. The minimum modulus $\gamma(T)$ of T is defined by

$$\gamma(T) = \inf\{\|T(x)\| : \operatorname{dist}(x, N(T)) = 1\}.$$

Thus, from the definition of $\gamma(T)$, we deduce that

$$\|T(x)\| \geq \gamma(T)\operatorname{dist}(x, N(T)), \ \forall x \in X.$$

Lemma 2.1.11 [100] If $T \in H(X, Y)$, the addition and the scalar multiplication are defined as usual in linear structures. If the norm of T is defined as

$$\|T\| = \sup_{\|x\|=1} \|Tx\|, \quad T \in H(X, Y), \tag{2.1}$$

then $(H(X, Y), \|\cdot\|)$ is a Banach space.

Definition 2.1.5 [31] Let $T \in L(X, Y)$. If $D(T)$ is dense in X, T is said to be densely defined.

Definition 2.1.6 [116] Let $T \in L(X, Y)$. If $x \in D(T)$, and $y = Tx$ when $x_n \in D(T)$, $x_n \to x$ and $Tx_n \to Y$, then T is said to closed operator.

2.2 Metric Generalized Inverse

Definition 2.2.1 An operator $T^+ \in B(Y, X)$ is said to be a generalized inverse of an operator $T \in B(X, Y)$ provided

$$TT^+T = T \text{ and } T^+TT^+ = T^+.$$

Remark 2.2.1 This is of course an extension of the notion of the bounded linear inverse T^{-1} of T. It is well known that an operator $T \in B(X, Y)$ has a generalized inverse in $B(Y, X)$ if and only if $N(T)$ and $R(T)$ are both splited, which means that there exist linear subspaces $R^+ \subset X$ and $N^+ \subset Y$ such that the following decompositions of X and Y hold:

$$X = N(T) \oplus R^+, \;\; Y = R(T) \oplus N^+.$$

In this case, R^+ and N^+ are called topological complements of $N(T)$ and $R(T)$, respectively. In this case T is said to be double splited.

For any $T \in L(X, Y)$, an element $x_0 \in X$ is said to be an extremal solution of the equation $Tx = y$, if $x = x_0$ minimizes the functional $||Tx - y||$ on X, that is, $\inf\{\|Tx-y\| : x \in X\} = \|Tx_0-y\|$. Any extremal solution with the minimal norm is called the best approximate solution (b.a.s. for short). In 1974, M. Z. Nashed and G. F. Votruba introduced the concept of the metric generalized inverse for linear operators between Banach spaces, which are set-valued operators in general.

Definition 2.2.2 [73] Let $T \in L(X, Y)$, and consider a $y \in Y$ such that $Tx = y$ has the best approximate solution in X. We define

$$T^{\partial}(y) = \{x \in X : x \text{ is the best approximate solution to } Tx = y\}$$

and call the set-valued mapping $y \to T^{\partial}(y)$ the metric generalized inverse of T. Here

$$D\left(T^{\partial}\right) = \{y \in Y : Tx = y \text{ has a best approximate solution in } X\}.$$

A (in general nonlinear) function $T^{\sigma}(y) \in T^{\partial}(y)$ is called a selection for the metric generalized inverse.

Definition 2.2.3 [106] Let $T \in L(X,Y)$, $N(T)$ and $R(T)$ be Chebyshev subspaces of X and Y, respectively. If there exists a homogeneous operator $T^M : D(T^M) \to D(T)$ such that:

1. $TT^MT = T$ on $D(T)$.
2. $T^MTT^M = T^M$ on $D(T^M)$.
3. $T^MT = I_{D(T)} - \pi_{\overline{N(T)}}$ on $D(T)$.
4. $TT^M = \pi_{\overline{R(T)}}$ on $D(T^M)$,

then T^M is called the Moore-Penrose metric generalized inverse of T, where $I_{D(T)}$ is the identity operator on $D(T)$ and $D(T^M) = R(T) \dotplus F_Y^{-1}(R(T)^\perp)$.

Lemma 2.2.2 Let X and Y be Banach spaces, $T \in L(X,Y)$, $N(T)$ and $R(T)$ be Chebyshev subspaces of X and Y, respectively. If T has a Moore-Penrose metric generalized inverse T^M, then

(1) T^M is unique on $D(T^M)$, and $T^My = (T|_{C(T)})^{-1}\pi_{\overline{R(T)}}y$ when $y \in D(T^M)$, where $D(T^M) = R(T) \dotplus F_Y^{-1}(R(T)^\perp)$;

(2) there exists a linear inner inverse T^- from $R(T)$ to $D(T)$ ($i.e., TT^-T = T$) such that

$$T^My = (I_{D(T)} - \pi_{\overline{N(T)}})T^-\pi_{\overline{R(T)}}y, \tag{2.2}$$

for $y \in D(T^M)$.

Remark 2.2.3 This result has been obtained in [106] by H. Wang and Y. W. Wang under the assumption that the underlying Banach space X and Y are strictly convex, but it is easy to show that the result remains valid under the weaker assumption that $N(T)$ and $R(T)$ be Chebyshev subspaces of X and Y, respectively.

Theorem 2.2.4 [75, 98] Let $T \in B(X,Y)$, $N(T)$ and $R(T)$ be Chebyshev subspaces of X and Y, respectively. Then there exists a unique Moore-Penrose metric generalized inverse T^M of T such that

$$T^M(y) = (T\ |_{C(T)})^{-1}\pi_{\overline{R(T)}}(y)$$

for any $y \in D(T^M)$, where $D(T^M) = R(T) \dotplus F_Y^{-1}(R(T)^\perp)$, $C(T) = D(T) \cap F_X^{-1}(N(T)^\perp)$.

Remark 2.2.5 In Theorem 2.2.4, since $\pi_{\overline{R(T)}}$ and $(T|_{C(T)})^{-1}$ are all bounded homogenous operators, T^M is also bounded homogenous operator. Thus, the norm of T^M is well defined by (2.1) in Lemma 2.1.11.

Chapter 3 Selections of Metric Generalized Inverses

3.1 Criteria for the Metric Generalized Inverses of Linear Operators

To get continuous selections of the metric generalized inverses in a Banach space, we first refine Theorem 3.1 and Theorem 3.2 in [48] , obtaining Theorems 3.1.1 and 3.1.2.

Theorem 3.1.1 Let $T \in L(X,Y)$ and assume that $\overline{R(T)}$ is an approximatively compact subspace of Y and $N(T)$ is a proximinal subspace of X. If $\mathcal{P}_{\overline{R(T)}}(y) \subset R(T)$ for each $y \in R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right)$, then

1. $D\left(T^{\partial}\right) = R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right)$;

2. for all $y \in D\left(T^{\partial}\right)$
$$T^{\partial}(y) = \mathcal{P}\left(T^{-1}\mathcal{P}_{\overline{R(T)}}(y);\theta\right),$$
where $T^{-1}\mathcal{P}_{\overline{R(T)}}(y) = \left\{x \in D(T) : T(x) \in \mathcal{P}_{\overline{R(T)}}(y)\right\}$.

Proof 1. Since $\overline{R(T)}$ is approximatively compact in Y, which is a proximinal subspace. If $Y = \overline{R(T)}$, then
$$D\left(T^{\partial}\right) = R(T) = R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right).$$
If $Y \neq \overline{R(T)}$, it follows from the Hahn-Banach Theorem that there exists a $y^* \in \overline{R(T)}^{\perp} = R(T)^{\perp} \subset Y^*$ such that $||y^*|| = 1$. Hence $R(T)^{\perp} \neq \{\theta\}$.

Take any $y \in Y \backslash \overline{R(T)}$. Since $\overline{R(T)}$ is a proximinal subspace of Y, we have that $\mathcal{P}_{\overline{R(T)}}(y) \neq \emptyset$. Taking any $y_0 \in \mathcal{P}_{\overline{R(T)}}(y)$, by Lemma 2.1.8, we have

$$F_Y(y - y_0) \cap R(T)^{\perp} \neq \emptyset.$$

Hence $\theta \neq y - y_0 \in F_Y^{-1}\left(R(T)^{\perp}\right)$, whence

$$F_Y^{-1}\left(R(T)^{\perp}\right) \neq \{\theta\}.$$

We claim that

$$D\left(T^{\partial}\right) = R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right).$$

Indeed, the operator equation $Tx = y$ has a best approximate solution $x_0 \in D(T)$ for any $y \in D\left(T^{\partial}\right)$. Thus $y_0 = Tx_0 \in R(T)$ satisfies the equalities

$$\begin{aligned} \|y - y_0\| &= \|y - Tx_0\| \\ &= \inf_{x \in D(T)} \|y - Tx\| \\ &= \operatorname{dist}(y, R(T)) \\ &= \operatorname{dist}\left(y, \overline{R(T)}\right). \end{aligned}$$

Hence $y_0 \in \mathcal{P}_{R(T)}(y)$. It follows from Lemma 2.1.8 that

$$F_Y(y - y_0) \cap R(T)^{\perp} \neq \emptyset.$$

Setting $y_1 = y - y_0$, we have $y_1 \in F_Y^{-1}\left(R(T)^{\perp}\right)$, whence

$$y = y_0 + y_1 \in R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right).$$

Therefore

$$D\left(T^{\partial}\right) \subset R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right). \tag{3.1}$$

Conversely, for any $y \in R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right)$, we claim that the operator equation $Tx = y$ has a best approximate solution in $D(T)$, that is,

$$R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right) \subset D\left(T^{\partial}\right). \tag{3.2}$$

We will divide the proof of (3.2) into three steps.

Step 1. For any $y \in R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right)$, there exists $b_y \in \mathcal{P}_{R(T)}(y)$ with

$$\|y - b_y\| = \inf_{z \in R(T)} \|y - z\|.$$

In fact, for any $y \in R(T) \dot{+} F_Y^{-1}\left(R(T)^{\perp}\right)$, there exist $b_y \in R(T)$ and $y_1 \in F_Y^{-1}\left(R(T)^{\perp}\right)$ such that

$$y = b_y + y_1,$$

i.e.

$$y - b_y = y_1 \in F_Y^{-1}\left(R(T)^{\perp}\right),$$

whence

$$F_Y(y - b_y) \cap R(T)^{\perp} \neq \emptyset.$$

Again, by Lemma 2.1.8, we have

$$b_y \in \mathcal{P}_{R(T)}(y).$$

Step 2. For any $b_y \in \mathcal{P}_{R(T)}(y)$, we claim that there exists $\hat{x}_{b_y} \in D(T)$ such that

(i) $\hat{x}_{b_y}$ is a minimal norm solution to the operator equation $Tx = b_y$;

(ii) $\hat{x}_{b_y}$ is an extremal solution to the operator equation $Tx = y$.

Indeed, $b_y \in R(T)$ for any $b_y \in \mathcal{P}_{R(T)}(y)$, whence there exists $x' \in D(T)$ such that $Tx' = b_y$. Since $N(T)$ is a proximinal subspace of X, we may choose $x'' \in \mathcal{P}_{N(T)}(x')$. Defining $\hat{x}_{b_y} = x' - x''$, we have

$$\hat{x}_{b_y} \in \left(I_{D(T)} - \mathcal{P}_{N(T)}\right)(x'),$$

whence

$$T\hat{x}_{b_y} = Tx' = b_y,$$

i.e. $\hat{x}_{b_y} \in T^{-1}b_y$,

Next, we are going to show that $\hat{x}_{b_y} \in \mathcal{P}_{T^{-1}b_y}(\theta)$, i.e. $\hat{x}_{b_y}$ is a minimal norm solution to the operator equation $Tx = b_y$, which will prove (i). For any $v \in N(T)$, setting $w = x'' + v$, we have $w \in N(T)$. Since $x'' \in \mathcal{P}_{N(T)}(x')$, we get

$$\begin{aligned} ||\hat{x}_{b_y} - \theta|| &= ||x' - x''|| \\ &\leq ||x' - w|| \\ &= ||x' - x'' - v|| \\ &= ||\hat{x}_{b_y} - v||, \end{aligned}$$

i.e. $\theta \in \mathcal{P}_{N(T)}\left(\hat{x}_{b_y}\right)$. Now it follows from Lemma 2.1.8 that

$$F_X\left(\hat{x}_{b_y}\right) \cap N(T)^{\perp} \neq \emptyset.$$

Then, we take $\hat{x}^* \in F_X\left(\hat{x}_{b_y}\right) \cap N(T)^{\perp}$, obtaining

$$\langle \hat{x}^*, \hat{x}_{b_y} \rangle = \ ||\hat{x}^*||^2 = ||\hat{x}_{b_y}||^2.$$

For any $x \in T^{-1}b_y$, we have $Tx = b_y = T\hat{x}_{by}$, whence

$$x_0 := x - \hat{x}_{b_y} \in N(T)$$

and

$$\begin{aligned} ||\hat{x}_{b_y}||^2 &= \langle \hat{x}^*, \hat{x}_{b_y} \rangle = \langle \hat{x}^*, \hat{x}_{b_y} + x_0 \rangle \\ &= \langle \hat{x}^*, x \rangle \leq ||\hat{x}^*|| \cdot ||x|| \\ &\leq ||\hat{x}_{b_y}|| \cdot ||x||. \end{aligned}$$

This implies that $\hat{x}_{b_y} \in \mathcal{P}_{T^{-1}b_y}(\theta)$.

Since $b_y \in \mathcal{P}_{R(T)}(y)$ and $b_y = T\hat{x}_{b_y}$, we have

$$\begin{aligned} ||y - T\hat{x}_{b_y}|| &= ||y - b_y|| \\ &= \inf_{z \in R(T)} ||y - z|| \\ &= \inf_{x \in D(T)} ||y - Tx||, \end{aligned}$$

i.e. $\hat{x}_{b_y}$ is an extremal solution to the operator equation $Tx = y$, so (ii) follows.

Step 3. For any $y \in R(T) \dot{+} F_Y^{-1}\left(R(T)^{\perp}\right)$, we claim that the operator equation $Tx = y$ has a best approximate solution, i.e.

$$R(T) \dot{+} F_Y^{-1}\left(R(T)^{\perp}\right) \subset D\left(T^{\partial}\right). \tag{3.3}$$

Indeed, we define for any $y \in D(T^{\partial}) = R(T) \dot{+} F_Y^{-1}\left(R(T)^{\perp}\right)$:

$$L(y) = \inf_{b_y \in \mathcal{P}_{R(T)}(y)} \left\{ ||\hat{x}_{b_y}|| \ : \ \hat{x}_{b_y} \text{ is a minimal norm solution to } Tx = b_y \right\}.$$

Next, we choose a sequence $\left\{||\hat{x}_{b_y^{(n)}}||\right\}$ such that

$$||\hat{x}_{b_y^{(n)}}|| \ \geq \ ||\hat{x}_{b_y^{(n+1)}}||, \quad Tx_{b_y^{(n)}} = b_y^{(n)} \quad (n = 1, 2, \cdots)$$

and

$$L(y) = \lim_{n\to\infty} ||\hat{x}_{b_y^{(n)}}||.$$

Since $\overline{R(T)}$ is an approximately compact subspace of Y, $\{b_y^{(n)}\} \subset \mathcal{P}_{R(T)}(y) \subset R(T) \subset \overline{R(T)}$, and

$$||y - b_y^{(n)}|| \ = \inf_{z\in \overline{R(T)}} ||y - z||. \tag{3.4}$$

We may assume without loss of generality that $\{b_y^{(n)}\}$ is a Cauchy sequence in Y. By the completeness of Y, there exists $b_y^{(0)} \in \overline{R(T)}$ such that

$$b_y^{(0)} = \lim_{n\to\infty} b_y^{(n)}.$$

From (3.4), we have

$$||y - b_y^{(0)}|| \ = \inf_{z\in \overline{R(T)}} ||y - z||.$$

Hence $b_y^{(0)} \in \mathcal{P}_{\overline{R(T)}}(y)$. Since $y \in R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right)$, then by using the fact that $\mathcal{P}_{\overline{R(T)}}(y) \subset R(T)$ for each $y \in R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right)$, we have

$$b_y^{(0)} \in \mathcal{P}_{\overline{R(T)}}(y) \subset R(T),$$

and hence

$$b_y^{(0)} \in \mathcal{P}_{R(T)}(y).$$

From the Steps 1 and 2, we know that there exists an $\hat{x}_{b_y^{(0)}} \in D(T)$ such that $\hat{x}_{b_y^{(0)}}$ is a minimal norm solution to the operator equation $Tx = b_y^{(0)}$, and a best approximate solution to the operator equation $Tx = y$.

For any extremal solution $\bar{x} \in D(T)$ to the operator equation $Tx = y$, we have $T\bar{x} \in \mathcal{P}_{R(T)}(y)$. Let us set $b_y = T\bar{x} \in \mathcal{P}_{R(T)}(y)$ in Step 2. There exists $\hat{x}_{b_y} \in D(T)$ such that $\hat{x}_{b_y}$is a minimal norm solution of the operator equation $Tx = b_y$, so

$$||\hat{x}_{b_y}|| \ \le \ ||\bar{x}||.$$

By the definition of $\left\{||\hat{x}_{b_y^{(n)}}||\right\}$, let $n \to \infty$ such that

$$||\hat{x}_{b_y^{(0)}}|| \ \le ||\hat{x}_{b_y}|| \ \le \ ||\bar{x}||.$$

Hence $\hat{x}_{b_y^{(0)}}$ is a best approximate solution of the operator equation $Tx = y$ and then (3.3) follows.

Combining (3.1) and (3.3) we obtain

$$D\left(T^{\partial}\right)=R\left(T\right)\dotplus F_{Y}^{-1}\left(R\left(T\right)^{\perp}\right).$$

2. For any $y\in D\left(T^{\partial}\right)$, by the definition of T^{∂}, there exists a best approximate solution $x_0\in D\left(T\right)$, whence

$$Tx_0\in R\left(T\right)\subset\overline{R\left(T\right)}\text{ and }Tx_0\in\mathcal{P}_{\overline{R(T)}}\left(y\right),$$

i.e. $\mathcal{P}_{\overline{R(T)}}\left(y\right)\neq\emptyset$.

It is obvious that $\mathcal{P}_{\overline{R(T)}}\left(y\right)$ is a closed convex subset of Y. Since $y\in D\left(T^{\partial}\right)=R\left(T\right)\dotplus F_{Y}^{-1}\left(R\left(T\right)^{\perp}\right)$, by the condition $\mathcal{P}_{\overline{R(T)}}\left(y\right)\subset R\left(T\right)$ for each $y\in R\left(T\right)\dotplus F_{Y}^{-1}\left(R\left(T\right)^{\perp}\right)$, we see that

$$T^{-1}\mathcal{P}_{\overline{R(T)}}\left(y\right)\neq\emptyset,$$

where $T^{-1}\mathcal{P}_{\overline{R(T)}}\left(y\right)=\left\{x\in D\left(T\right):Tx\in\mathcal{P}_{\overline{R(T)}}\left(y\right)\right\}$, which is a nonempty convex subset of X. For any $y\in D\left(T^{\partial}\right)$ and any $x_0\in T^{\partial}\left(y\right)$, by the definition of $T^{\partial}\left(y\right)$, we see that $x_0\in T^{-1}\mathcal{P}_{\overline{R(T)}}\left(y\right)$ and

$$||x_0||=\inf\left\{||x||:x\in T^{-1}\mathcal{P}_{\overline{R(T)}}\left(y\right)\right\},$$

whence $\mathcal{P}\left(T^{-1}\mathcal{P}_{\overline{R(T)}}\left(y\right);\theta\right)\neq\emptyset$ and

$$T^{\partial}\left(y\right)\subset\mathcal{P}\left(T^{-1}\mathcal{P}_{\overline{R(T)}}\left(y\right);\theta\right).\tag{3.5}$$

Conversely, for any $y\in\mathcal{P}\left(T^{-1}\mathcal{P}_{\overline{R(T)}}\left(y\right);\theta\right)$, by the definition of the set-valued metric projection and the definition of the set-valued metric generalized inverse, we see that $y\in T^{\partial}\left(y\right)$, i.e.

$$\mathcal{P}\left(T^{-1}\mathcal{P}_{\overline{R(T)}}\left(y\right);\theta\right)\subset T^{\partial}\left(y\right).\tag{3.6}$$

Combining (3.5)and (3.6), we obtain

$$T^{\partial}\left(y\right)=\mathcal{P}\left(T^{-1}\mathcal{P}_{\overline{R(T)}}\left(y\right);\theta\right),\quad y\in D\left(T^{\partial}\right).$$

This finishes the proof. □

Theorem 3.1.2 Let $T\in L\left(X,Y\right)$ and $\overline{R\left(T\right)}$ be an approximately compact Chebyshev subspace of Y, and $N\left(T\right)$ be a proximinal subspace of X. Then

1. $D\left(T^{\partial}\right) = R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right)$;

2. for all $y \in D\left(T^{\partial}\right)$, we have

$$T^{\partial}(y) = \left(I_{D(T)} - \mathcal{P}_{N(T)}\right) T^{-1} \pi_{\overline{R(T)}}(y).$$

Proof First we show that

$$\pi_{\overline{R(T)}}(y) \in R(T), \quad \forall y \in R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right). \tag{3.7}$$

Indeed, since $\overline{R(T)}$ is an approximately compact Chebyshev subspace of Y, for any $y \in R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right)$, there exists a unique $\pi_{\overline{R(T)}}(y)$ such that $\mathcal{P}_{\overline{R(T)}}(y) = \left\{\pi_{\overline{R(T)}}(y)\right\}$. On the other hand, there exist $y_0 \in R(T)$ and $y_1 \in F_Y^{-1}\left(R(T)^{\perp}\right)$ such that $y = y_0 + y_1$, whence

$$y - y_0 = y_1 \in F_Y^{-1}\left(\overline{R(T)}^{\perp}\right) = F_Y^{-1}\left(R(T)^{\perp}\right).$$

Therefore

$$F_Y(y - y_0) \cap R(T)^{\perp} \neq \emptyset.$$

From Lemma 2.1.8, we have $y_0 \in \mathcal{P}_{\overline{R(T)}}(y) = \left\{\pi_{\overline{R(T)}}(y)\right\}$, and hence

$$\pi_{\overline{R(T)}}(y) = y_0 \in R(T),$$

which shows that (3.7) holds.

By Theorem 3.1.1, we have

$$D\left(T^{\partial}\right) = R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right),$$

and

$$T^{\partial}(y) = \mathcal{P}\left(T^{-1}\pi_{\overline{R(T)}}(y); \theta\right), \qquad \forall y \in D\left(T^{\partial}\right).$$

In order to finish the proof, we need only to show that

$$\mathcal{P}\left(T^{-1}\pi_{\overline{R(T)}}(y); \theta\right) = \left(I_{D(T)} - \mathcal{P}_{N(T)}\right) T^{-1}\pi_{\overline{R(T)}}(y), \quad \forall y \in D\left(T^{\partial}\right), \tag{3.8}$$

where $\pi_{\overline{R(T)}}(y) \in R(T)$.

Taking arbitrary $y \in D\left(T^{\partial}\right)$ and $x \in \mathcal{P}\left(T^{-1}\pi_{\overline{R(T)}}(y)\,;\theta\right)$, we obtain

$$x \in T^{-1}\mathcal{P}_{\overline{R(T)}}(y)$$

and

$$\|x\| \ = \inf\{\|w\| : w \in T^{-1}\pi_{\overline{R(T)}}(y)\}. \tag{3.9}$$

Theorem 2.1.7 implies that x can be decomposed in the form $x = x_1 + x_2$, where

$$x_1 \in \mathcal{P}_{N(T)(x)}\ ,\ x_2 \in F_X^{-1}\left(N(T)\right)^{\perp}.$$

Hence

$$T(x_2) = T(x - x_1) = T(x) = \pi_{\overline{R(T)}}(y),$$

i.e.

$$x_2 \in T^{-1}\pi_{\overline{R(T)}}(y).$$

For any $v \in N(T)$, we have $x_2 - v \in T^{-1}\pi_{\overline{R(T)}}(y)$, and hence it follows from (3.9) that

$$\|x_2 - (-x_1)\| \ = \ \|x\| \leq \|x_2 - v\|,$$

i.e. $x_1 \in \mathcal{P}_{N(T)}(x_2)$. Therefore

$$\begin{aligned} x &= x_2 - (-x_1) \\ &\in I_{D(T)} - \mathcal{P}_{N(T)}(x_2) \\ &\subset I_{D(T)} - \mathcal{P}_{N(T)}T^{-1}\pi_{\overline{R(T)}}(y), \end{aligned}$$

and consequently

$$\mathcal{P}\left(T^{-1}\pi_{\overline{R(T)}}(y)\,;\theta\right) \subset \left(I_{D(T)} - \mathcal{P}_{N(T)}\right)T^{-1}\pi_{\overline{R(T)}}(y)\,. \tag{3.10}$$

Conversely, taking arbitrary $\tilde{x} \in (I_{D(T)} - \mathcal{P}_{N(T)})T^{-1}\pi_{\overline{R(T)}}(y)$, $y \in D\left(T^{\partial}\right)$, there exists $x' \in T^{-1}\pi_{\overline{R(T)}}(y)$ such that

$$\tilde{x} \in (I_{D(T)} - \mathcal{P}N(T))(\tilde{x}).$$

Hence, there exists $x'' \in \mathcal{P}_{N(T)}(x')$ such that

$$\tilde{x} = x' - x'' \text{ and } x'' \in N(T),$$

consequently

$$T(\tilde{x}) = T(x') = \pi_{\overline{R(T)}}(y).$$

Thus we have $\tilde{x} \in T^{-1}\pi_{\overline{R(T)}}(y)$. Next, we will verify that $\tilde{x} \in \mathcal{P}(T^{-1}\pi_{\overline{R(T)}}(y);\theta)$.

Taking arbitrary $v \in N(T)$ and setting $w = x'' + v$, we get $w \in N(T)$. Noticing that $x'' \in \mathcal{P}_{N(T)}(x')$, for all $v \in N(T)$, we have

$$\begin{aligned} \|\tilde{x} - \theta\| &= \|x' - x''\| \\ &\le \|x' - w\| \\ &= \|x' - x'' - v\| \\ &= \|\tilde{x} - v\|. \end{aligned}$$

Whence $\theta \in \mathcal{P}_{N(T)}(\tilde{x})$. It follows from Theorem 2.1.8 that

$$F_X(\tilde{x}) \cap N(T)^{\perp} \neq \emptyset.$$

Choosing $\tilde{x}^* \in F_X(\tilde{x}) \cap N(T)^{\perp}$ such that

$$\langle \tilde{x}^*, \tilde{x} \rangle \;=\; \|\tilde{x}^*\|^2 \;=\; |\tilde{x}\|^2.$$

For any $x \in T^{-1}\pi_{\overline{R(T)}}(y)$, we have $T(x) = T(\tilde{x}) = \pi_{\overline{R(T)}}(y)$, whence $x - \tilde{x} \in N(T)$.

Let $x_0 = x - \tilde{x}$. Then $x = x_0 + \tilde{x},\ x_0 \in N(T)$. Therefore

$$\begin{aligned} \|\tilde{x}\|^2 &= \langle \tilde{x}^*, \tilde{x} \rangle \\ &= \langle \tilde{x}^*, \tilde{x} + x_0 \rangle \\ &= \langle \tilde{x}^*, x \rangle \\ &\le \|\tilde{x}^*\|\|x\| \\ &= \|\tilde{x}\|\|x\|. \end{aligned}$$

Hence, it follows that $\|\tilde{x}\| \le \|x\|$ for any $x \in T^{-1}\pi_{\overline{R(T)}}(y)$, i.e. $\tilde{x} \in P(T^{-1}\pi_{\overline{R(T)}}(y);\theta)$. Thus

$$\left(I_{D(T)} - \mathcal{P}_{N(T)}\right) T^{-1}\pi_{\overline{R(T)}}(y) \subset \mathcal{P}\left(T^{-1}\pi_{\overline{R(T)}}(y);\theta\right). \tag{3.11}$$

Combining (3.10) and (3.11), we obtain

$$\left(I_{D(T)} - \mathcal{P}_{N(T)}\right) T^{-1}\pi_{\overline{R(T)}}(y) = \mathcal{P}\left(T^{-1}\pi_{\overline{R(T)}}(y);\theta\right),$$

which finished the proof. □

3.2 Continuous Homogeneous Selections of Metric Generalized Inverses of Linear Operators

Theorem 3.2.1 Let $T \in L(X, Y)$ be a densely defined closed linear operator. Suppose that $R(T)$ is an approximately compact Chebyshev subspace of Y and $N(T)$ is a proximinal subspace that is topologically complemented in X. If the set-valued projection $\hat{\mathcal{P}}_{N(T)} : D(T) \to 2^{N(T)}$ has a continuous homogeneous selection $\hat{\pi}_{N(T)} : D(T) \to N(T)$, where $\hat{\mathcal{P}}_{N(T)}$ is the restriction of $\mathcal{P}_{N(T)}$ to $D(T)$ and $\hat{\pi}_{N(T)}$ is the restriction of $\pi_{N(T)}$ to $D(T)$, then the metric generalized inverse $T^{\partial} : Y \to 2^{D(T)}$ has a continuous homogeneous selection $T^{\sigma} : Y \to D(T)$. In this case, we have

$$T^{\sigma} = \left(I_{D(T)} - \hat{\pi}_{N(T)}\right) T_0^{-1} \pi_{R(T)},$$

where $T_0 = T|_{N(T)^c \cap D(T)}$ is the restriction of T to the subspace $N(T)^c \cap D(T)$, and $N(T)^c$ is a topologically complemented subspace of $N(T)$ in X.

Proof Since $R(T)$ is an approximately compact Chebyshev subspace of Y, by Lemma 2.1.4 and Lemma 2.1.10, the metric projection $\pi_{R(T)} : Y \to R(T)$ is a single-valued continuous homogeneous operator.

On the other hand, since $N(T)$ is a topologically complemented subspace of X, there exists a closed subspace $N(T)^c$ of X such that

$$X = N(T) \oplus N(T)^c.$$

Let $T_0 := T|_{N(T)^c \cap D(T)}$ be the restriction of T to the subspace $N(T)^c \cap D(T)$. Then we claim that

$$T_0 : N(T)^c \cap D(T) \to R(T)$$

is one-to-one and onto, whence the converse operator $T_0^{-1} : R(T) \to N(T)^c \cap D(T)$ exists and is a linear operator.

Indeed, if $x, y \in N(T)^c \cap D(T)$ are such that $T_0(x) = T_0(y)$, then

$$x - y \in N(T) \text{ and } x - y \in N(T)^c.$$

Since $N(T) \cap N(T)^c = \{\theta\}$, we see that $x = y$, i.e. T_0 is one-to-one.

On the other hand, for any $y \in R(T)$, there exists an $x \in D(T)$ such that $y = Tx$. Since $x \in D(T) \subset X = N(T) \oplus N(T)^c$, there exist $x_0 \in N(T)$ and $x_1 \in N(T)^c$ such that $x = x_0 + x_1$. Hence

$$x_1 = x - x_0 \in N(T)^c \cap D(T),$$

which satisfies the equalities $Tx_1 = Tx = y$, i.e. T_0 is onto.

Next, we will prove that $T_0^{-1} : R(T) \to N(T)^c \cap D(T)$ is a closed linear operator. Let $\{x_n\} \subset N(T)^c \cap D(T)$ be such that

$$x_n \to x_0 \text{ and } T_0(x_n) \ = \ Tx_n \to y_0 \text{ as } n \to \infty.$$

Since T is a closed linear operator, we have that $x_0 \in D(T)$ and $y_0 = Tx_0$. On the other hand, $N(T)^c$ is a closed linear subspace, we see that

$$x_0 \in N(T)^c \cap D(T), \ y_0 = T_0(x_0).$$

Therefore T_0 is a closed linear operator, which converse operator

$$T_0^{-1} : R(T) \to N(T)^c \cap D(T)$$

is also a closed linear operator. Since $R(T)$ is a closed linear subspace of Y, whence $R(T)$ is complete, it follows by the Closed Graph Theorem that

$$T_0^{-1} : R(T) \to N(T)^c \cap D(T)$$

is a continuous linear operator.

Since $\hat{\pi}_{N(T)} : D(T) \to N(T)$ is a single-valued continuous homogeneous selection for the set-valued projection $\hat{\mathcal{P}}_{N(T)} : D(T) \to 2^{N(T)}$, we get

$$I_{D(T)} - \hat{\pi}_{N(T)} : D(T) \to N(T)$$

is also a single-valued continuous homogeneous selection for the set-valued mapping $I_{D(T)} - \hat{\mathcal{P}}_{N(T)} : D(T) \to 2^{N(T)}$.

We define $T^\sigma : Y \to D(T)$ by the formula

$$T^\sigma(y) = \left(I_{D(T)} - \hat{\pi}_{N(T)}\right) T_0^{-1} \pi_{R(T)}(y), \qquad y \in Y.$$

Then, by Theorem 3.1.2, we have

$$T^\sigma(y) \ \in \ \left(I_{D(T)} - \hat{\mathcal{P}}_{N(T)}\right) T_0^{-1} \pi_{R(T)}(y)$$

$$\begin{aligned} &\subset \left(I_{D(T)} - \mathcal{P}_{N(T)}\right) T^{-1} \pi_{R(T)} (y) \\ &= T^{\partial} (y) \end{aligned}$$

for any $y \in D\left(T^{\partial}\right)$.

Since $R(T)$ is an approximately compact Chebyshev subspace of Y, by Theorem 3.1.2 and Theorem 2.1.7, we obtain that

$$D\left(T^{\partial}\right) = R(T) \dotplus F_Y^{-1}\left(R(T)^{\perp}\right) = Y$$

and

$$T^{\sigma} = \left(I_{D(T)} - \hat{\pi}_{N(T)}\right) T_0^{-1} \pi_{R(T)}$$

is a continuous homogeneous selection for the metric generalized inverse $T^{\partial} : Y \to 2^{D(T)}$. □

Theorem 3.2.2 Suppose $\mathcal{L}(X, Y)$ be the space of all bounded linear operators T from X to Y with closed range $R(T)$ (the domain $D(T)$ of T is just a subspace of X). Let $T \in \mathcal{L}(X, Y)$, $R(T)$ be an approximately compact Chebyshev subspace of Y, $N(T)$ be a proximinal subspace and is topologically complemented in X. Then the metric generalized inverse $T^{\partial} : Y \to 2^X$ has a continuous homogeneous selection $T^{\sigma} : Y \to X$ if and only if the set-valued projection $\mathcal{P}_{N(T)} : X \to 2^{N(T)}$ has a continuous homogeneous selection $\pi_{N(T)} : X \to N(T)$. In this case, we have

$$T^{\sigma} = \left(I_{D(T)} - \pi_{N(T)}\right) T_0^{-1} \pi_{R(T)},$$

where $T_0 = T|_{N(T)^c}$ is the restriction of T to the subspace $N(T)^c$, and $N(T)^c$ is a topologically complemented subspace of $N(T)$ in X.

Proof Necessity. If the set-valued metric generalized inverse $T^{\partial} : Y \to 2^X$ of T has a continuous homogeneous selection $T^{\sigma} : Y \to X$, defining

$$\pi_{N(T)}(x) := x - T^{\sigma} T x,$$

then $\pi_{N(T)} : X \to N(T)$ is a continuous homogeneous operator. By the definition of T^{∂} and T^{σ}, we see that

$$T\left(\pi_{N(T)}(x)\right) = Tx - TT^{\sigma}Tx = \theta,$$

i.e. $\pi_{N(T)}(x) \in N(T)$, and for any $y \in T^{-1}Tx = x + N(T)$:

$$||T^{\sigma}Tx|| \leq ||y||. \tag{3.12}$$

Let $y = x - z$ for any $z \in N(T)$. Then $y \in x + N(T) = T^{-1}Tx$. From (3.12) we have

$$\begin{aligned} \left|\left|x - \pi_{N(T)}(x)\right|\right| &= ||T^{\sigma}Tx|| \\ &\leq ||y|| \\ &\leq ||x - z|| \end{aligned}$$

for any $z \in N(T)$. Hence,

$$\pi_{N(T)}(x) \in \mathcal{P}_{N(T)}(x), \qquad (x \in X),$$

i.e. $\pi_{N(T)} : X \to N(T)$ is a continuous homogeneous selection for the set-valued projection

$$\mathcal{P}_{N(T)} : X \to 2^{N(T)}.$$

Sufficiency. Since $T \in \mathcal{L}(X, Y)$ is a bounded linear operator defined on X, the fact that T is a densely defined closed linear operator with $D(T) = X$ follows from Theorem 3.2.1. □

Chapter 4 Perturbations of Moore-Penrose Metric Generalized Inverses of Linear Operators

4.1 Perturbation of the Solution of the Operator Equation $Tx = b$

Let $T \in B(X, Y)$. Throughout this section, let $\delta T \in B(X, Y)$, $\overline{T} = T + \delta T$, $b \in R(T)$ and $b \neq \theta$. Let us define $S(T, b) = \{x \in X \mid Tx = b\}$, $S(\overline{T}, b) = \{ x \in X \mid \overline{T}x = b\}$.

Lemma 4.1.1 Let $T \in B(X, Y)$. If $N(T)$ and $R(T)$ are Chebyshev subspaces of X and Y, respectively. Then there exists the Moore-Penrose metric generalized inverse T^M of T such that

$$\frac{1}{\|T^M\|} \ \leq \ \gamma(T) \ \leq \ \frac{\|T^M T\| \|TT^M\|}{\|T^M\|}.$$

Proof By Theorem 2.2.4, there exists a unique Moore-Penrose metric generalized inverse T^M of T such that

$$T^M(y) = (T\mid_{C(T)})^{-1}\pi_{\overline{R(T)}}(y), \ y \in D(T^M).$$

It follows from Remark 2.2.5 that the norm of T^M is well defined by (2.1). For any $x \in X$ and $y \in N(T)$, we have

$$\|T^M T(x-y)\| \ = \ \|T^M Tx\| \ \leq \ \|T^M T\| \ \|x-y\| \tag{4.1}$$

and

$$\text{dist}(x, N(T)) \;\leq\; \|x - (I - T^M T)x\| \;=\; \|T^M T x\|.$$

It follows from (4.1) that

$$\|x - y\| \;\geq \|T^M T x\| \|T^M T\|^{-1},$$

or equivalently

$$\text{dist}(x, N(T)) \geq \|T^M T x\| \|T^M T\|^{-1}.$$

Therefore

$$\|T^M\| \; \|Tx\| \;\geq\; \|T^M T x\| \;\geq\; \text{dist}(x, N(T)) \;\geq\; \frac{\|T^M T x\|}{\|T^M T\|}. \tag{4.2}$$

By the definition of $\gamma(T)$, inequality (4.2) implies that

$$\gamma(T) \;\geq\; \frac{1}{\|T^M\|}$$

and

$$\|Tx\| \;\geq\; \gamma(T)\text{dist}(x, N(T)) \;\geq\; \gamma(T)\; \frac{\|T^M T x\|}{\|T^M T\|}. \tag{4.3}$$

For any $z \in Y$, if we substitute $T^M z$ for x in inequality (4.3), we get

$$\begin{aligned} \|T\; T^M z\| \;&\geq\; \gamma(T)\; \frac{\|T^M T T^M z\|}{\|T^M T\|} \\ &=\; \gamma(T)\; \frac{\|T^M z\|}{\|T^M T\|}. \end{aligned}$$

Therefore

$$\gamma(T) \;\leq\; \frac{\|T^M T\| \; \|T T^M\|}{\|T^M\|}.$$

The proof is completed. □

Lemma 4.1.2 Let $T \in B(X, Y)$. If $N(T)$ and $R(T)$ are Chebyshev subspaces of X and Y, respectively, then

$$\|T\|^{-1} \; \|\delta T \overline{x}\| \;\leq\; \text{dist}(\overline{x},\; S(T, b)) \;\leq\; \|T^M\| \; \|\delta T\| \|\overline{x}\|$$

for every $\overline{x} \in S(\overline{T}, b)$

Proof By Definition 2.1.4, it is easy to see that

$$\|Tx\| \ \geq \ \gamma(T)\ \mathrm{dist}(x, N(T)),\ x \in X.$$

Therefore, for any $\overline{x} \in S(\overline{T}, b)$

$$\mathrm{dist}(\overline{x}, S(T,b)) = \ \mathrm{dist}(\overline{x} - T^M b, N(T)) \leq \ \gamma(T)^{-1}\ \|T(\overline{x} - T^M b)\|. \tag{4.4}$$

It follows from Lemma 4.1.1 that

$$\gamma(T)^{-1} \ \leq \ \|T^M\|. \tag{4.5}$$

This inequality together with (4.4) gives

$$\begin{aligned} \mathrm{dist}(\overline{x}, S(T,b)) \leq & \|T^M\|\ \|T(\overline{x} - T^M b)\| \\ = & \|T^M\|\ \|T\overline{x} - TT^M b\|. \end{aligned}$$

The definition of T^M shows that $TT^M \ = \ \pi_{\overline{R(T)}}$. Thus $TT^M b \ = \ b$. Moreover,

$$\begin{aligned} \mathrm{dist}(\overline{x}, S(T,b)) \quad &\leq \quad \|T^M\|\ \|T\overline{x} - b\| \\ &= \quad \|T^M\|\ \|(\overline{T} - \delta T)\overline{x} - b\| \\ &= \quad \|T^M\|\ \|\overline{T}\overline{x} - b - \delta T\overline{x}\| \\ &= \quad \|T^M\|\ \|\delta T\overline{x}\|. \end{aligned}$$

On the other hand, it is easy to see that

$$\begin{aligned} \|T(\overline{x} \ - \ T^M b)\| \quad &= \quad \|T(\overline{x} \ - T^M b \ - \ y)\| \\ &\leq \quad \|T\|\ \|\overline{x} \ - (T^M b \ + \ y)\| \end{aligned}$$

for any $y \in N(T)$. Therefore

$$\begin{aligned} \mathrm{dist}(\overline{x}, S(T,b)) \quad &\geq \quad \|T\|^{-1}\ \|T(\overline{x} - T^M b)\| \\ &= \quad \|T\|^{-1}\ \|(\overline{T} - \delta T)\overline{x} - TT^M b\| \\ &= \quad \|T\|^{-1}\ \|\overline{T}\overline{x} - \delta T\overline{x} - b\| \\ &= \quad \|T\|^{-1}\ \|\delta T\overline{x}\|. \end{aligned}$$

The proof is completed. □

Theorem 4.1.3 Let $T \in B(X,Y)$, $N(T)$ and $R(T)$ be Chebyshev subspaces of X and Y, respectively. If $\|T^M\|\ \|\delta T\| < 1$, then there exists a unique $x \in S(T,b)$ such that

$$\frac{\|\overline{x}-x\|}{\|x\|} \leq \frac{\|T^M\|\ \|\delta T\|}{1-\|T^M\|\ \|\delta T\|}, \quad \overline{x} \in S(\overline{T},b).$$

Proof For any $x \in S(T,b)$, it follows from $b \neq \theta$ that $\|x\| \neq \theta$. Since $S(T,b) = T^M b + N(T)$ and $N(T)$ is a Chebyshev subspace of X, $S(T,b)$ is a Chebyshev linear manifold. Thus, there exists a unique $x \in S(T,b)$ such that

$$\|\overline{x}-x\| = \operatorname{dist}(\overline{x}, S(T,b)),$$

for any $\overline{x} \in S(\overline{T},b) \subset X$. By Lemma 4.1.2, we have

$$\|\overline{x}-x\| \leq \|T^M\|\ \|\delta T\|\ \|\overline{x}\|,$$

or equivalently

$$\frac{\|\overline{x}-x\|}{\|x\|} \leq \|T^M\|\ \|\delta T\| \frac{\|\overline{x}\|}{\|x\|}. \tag{4.6}$$

Moreover

$$\frac{\|\overline{x}\|}{\|x\|} \leq \frac{\|\overline{x}-x\| + \|x\|}{\|x\|},$$

and

$$\frac{\|\overline{x}\|}{\|x\|} \leq 1 + \frac{\|\overline{x}-x\|}{\|x\|}. \tag{4.7}$$

This means by inequality (4.6) that

$$\frac{\|\overline{x}-x\|}{\|x\|} \leq \frac{\|T^M\|\ \|\delta T\|}{1-\|T^M\|\ \|\delta T\|},$$

which finishes the proof. □

In the following, let $\delta b \in R(T)$, $\overline{b} = b + \delta b \in R(\overline{T})$ and $\overline{b} \neq \theta$. Suppose $S(\overline{T},\overline{b}) = \{x \in X \mid \overline{T}x = \overline{b}\}$.

Lemma 4.1.4 Let $T \in B(X,Y)$, $N(T)$ and $R(T)$ be Chebyshev subspaces of X and Y, respectively. Then

$$\|T\|^{-1}\|\delta T\overline{x} - \delta b\| \leq \operatorname{dist}(\overline{x}, S(T,b)) \leq \|T^M\|\|\delta T\overline{x} - \delta b\|$$

for every $\overline{x} \in S(\overline{T}, \bar{b})$.

Proof It follows from the definition of $\gamma(T)$ that

$$\begin{aligned} \|T(\overline{x} - T^M b)\| &\geq \gamma(T) \operatorname{dist}(\overline{x}, S(T,b)) \\ &= \gamma(T) \operatorname{dist}(\overline{x} - T^M b, N(T)) \end{aligned} \tag{4.8}$$

for all $\bar{x} \in S(\overline{T}, b)$. Lemma 4.1.1 means that

$$\gamma(T)^{-1} \leq \|T^M\|. \tag{4.9}$$

In addition, (4.8) implies that

$$\operatorname{dist}(\overline{x}, S(T,b)) \leq \|T^M\|\|T(\overline{x} - T^M b)\| = \|T^M\|\|T\overline{x} - TT^M b\|.$$

By the definition of T^M, $TT^M = \pi_{\overline{R(T)}}$. Thus, $TT^M b = b$ and

$$\begin{aligned} \operatorname{dist}(\overline{x}, S(T,b)) &\leq \|T^M\|\|T\overline{x} - b\| \\ &= \|T^M\|\|(\overline{T} - \delta T)\overline{x} - b\| \\ &= \|T^M\|\|T\overline{x} - b - \delta T\overline{x}\| \\ &= \|T^M\|\|\delta T\overline{x} - \delta b\|. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|T(\overline{x} - T^M b)\| &= \|T(\overline{x} - T^M b - y)\| \\ &\leq \|T\|\|\overline{x} - (T^M b + y)\| \end{aligned}$$

for any $y \in N(T)$. Hence

$$\begin{aligned} \operatorname{dist}(\overline{x}, S(T,b)) &\geq \|T\|^{-1}\|T(\overline{x} - T^M b)\| \\ &= \|T\|^{-1}\|\overline{T}\overline{x} - TT^M b\| \\ &= \|T\|^{-1}\|(\overline{T} - \delta T)\overline{x} - b\| \end{aligned}$$

$$
\begin{aligned}
&= \|T\|^{-1}\|\overline{T}\overline{x} - \delta T\overline{x} - b\| \\
&= \|T\|^{-1}\|\overline{b} - b - \delta T\overline{x}\| \\
&= \|T\|^{-1}\|\delta b - \delta T\overline{x}\|.
\end{aligned}
$$

The proof is completed. □

Theorem 4.1.5 Let $T \in B(X, Y)$, $N(T)$ and $R(T)$ be Chebyshev subspaces of X and Y, respectively. If $\|T^M\|\ \|\delta T\| < 1$, then there exists a unique $x \in S(T, b)$ such that

$$\frac{\|\overline{x} - x\|}{\|x\|} \le \frac{\|T^M\|(\|\delta T\| + \|T\|\|\delta b\|\|b\|^{-1})}{1 - \|T^M\|\|\delta T\|}$$

for every $\overline{x} \in S(\overline{T}, \overline{b})$.

Proof By the proof of Theorem 4.1.3, there exists a unique $x \in S(T, b)$ such that $x \neq \theta$ and

$$\|\overline{x} - x\| = \operatorname{dist}(\overline{x}, S(T, b)), \quad \forall \overline{x} \in S(\overline{T}, \overline{b}).$$

By Lemma 4.1.4, we have

$$\|\overline{x} - x\| \le \|T^M\|\ \|\delta T\overline{x} - \delta b\|.$$

for every $\overline{x} \in S(\overline{T}, \overline{b})$. Therefore

$$
\begin{aligned}
\frac{\|\overline{x} - x\|}{\|x\|} &\le \frac{\|T^M\|\ \|\delta T\overline{x} - \delta b\|}{\|x\|} \\
&\le \frac{\|T^M\|\ (\|\delta T\|\ \|\overline{x}\| + \|\delta b\|)}{\|x\|} \\
&= \|T^M\|\ \|\delta T\|\frac{\|\overline{x}\|}{\|x\|} + \frac{\|\delta b\|}{\|x\|}\|T^M\|. \qquad (4.10)
\end{aligned}
$$

Furthermore, it follows from $\|Tx\| = \|b\| \le \|T\|\ \|x\|$ that

$$\frac{1}{\|x\|} \le \frac{\|T\|}{\|b\|}.$$

This implies by (4.10) that

$$\frac{\|\overline{x} - x\|}{\|x\|} \le \|T^M\|\ \|\delta T\|\frac{\|\overline{x}\|}{\|x\|} + \frac{\|T\|}{\|b\|}\|\delta b\|\ \|T^M\|. \qquad (4.11)$$

On the other hand, it is easy to see that

$$\frac{\|\overline{x}\|}{\|x\|} \leq \frac{\|\overline{x}-x\| + \|x\|}{\|x\|} = 1+\frac{\|\overline{x}-x\|}{\|x\|}. \tag{4.12}$$

By (4.12) and (4.11), we get

$$\frac{\|\overline{x}-x\|}{\|x\|} \leq \frac{\|T^M\|(\|\delta T\|+\|T\|\|\delta b\|\|b\|^{-1})}{1-\|T^M\|\|\delta T\|}.$$

The proof is completed. □

4.2 Perturbation of Moore-Penrose Metric Generalized Inverse

Definition 4.2.1 [57] Let $T \in B(X,Y)$, $N(T)$ and $R(T)$ be Chebyshev subspaces of X and Y, respectively. Suppose that $\delta T \in B(X,Y)$, $\overline{T} = T+\delta T$, and $N(\overline{T})$ and $R(\overline{T})$ are Chebyshev subspaces of X and Y, respectively. Then δT is called the metric stable perturbation of T. In addition,

$$\Delta(T) = \{\delta T \in B(X,Y) : \delta T \text{ is the metricly stable perturbation of T}\}$$

is called the metric stable perturbation set of T .

Remark 4.2.1 For any $\delta T \in \Delta T$, it follows from Theorem 2.2.4 that there exist the Moore-Penrose metric generalized inverse $\overline{T}^M$ and T^M of $\overline{T}$ and T, respectively, such that

$$\begin{aligned} D(T^M) &= R(T) \dotplus F_Y^{-1}(R(T)^{\perp}), \\ D(\overline{T}^M) &= R(\overline{T}) \dotplus F_Y^{-1}(R(\overline{T})^{\perp}). \end{aligned}$$

Since $R(T)$ and $R(\overline{T})$ are all Chebyshev subspaces of Y. Then Theorem 2.1.7 means $D(T^M) = D(\overline{T}^M) = Y$. Let us define

$$Y_{\delta T}(T) := \{b \in Y \ : \ F_X(\overline{T}^M b - T^M b) \cap N(T)^{\perp} \neq \emptyset\}.$$

It is obvious that $\theta \in Y_{\delta T}(T)$. Therefore,

1. $Y_{\delta T}(T) \neq \emptyset$,

2. $Y_{\delta T}(T) \subset Y$ is a homogenous set.

Remark 4.2.2 Let $T \in B(X,Y)$. If $N(T)$ and $R(T)$ are Chebyshev subspaces of X and Y, respectively, then $\Delta(T)\backslash\{\theta\} \neq \emptyset$.

Indeed, if $\delta T = \tau T$ for any $\tau \in \mathbb{R}$, then $\overline{T} = T + \delta T = (1+\tau)T$, $N(\overline{T}) = N(T)$, $R(\overline{T}) = R(T)$. Thus $\delta T \in \Delta(T)$.

Theorem 4.2.3 Let $T \in B(X,Y)$, $N(T)$ and $R(T)$ be Chebyshev subspaces of X and Y, respectively. If $\delta T \in \Delta(T)$, $\|T^M\|\,\|\delta T\| < 1$, then

$$\begin{aligned} \frac{\|\overline{T}^M - T^M\|_0}{\|\overline{T}^M\|} &\leq \|T^M\|\,\|\delta T\| \\ &\leq \frac{\|T^M\|\,\|\delta T\|}{1 - \|T^M\|\,\|\delta T\|}, \end{aligned} \tag{4.13}$$

where both $\|\overline{T}^M\|$ and $\|T^M\|$ are the norms for bounded homogenous operators, and

$$\|\overline{T}^M - T^M\|_0 = \sup_{\substack{b\neq\theta \\ b\in Y_{\delta T}(T)}} \frac{\|(\overline{T}^M - T^M)(b)\|}{\|b\|}.$$

Here $\|\overline{T}^M - T^M\|_0 = \theta$ when $Y_{\delta T}(T) = \{\theta\}$.

Proof By Theorem 2.2.4, there exist the Moore-Penrose metric generalized inverse T^M and $\overline{T}^M$ of T and $\overline{T}$, respectively, such that

$$\begin{aligned} D(T^M) &= R(T) \dotplus F_Y^{-1}(R(T)^{\perp}), \\ D(\overline{T}^M) &= R(\overline{T}) \dotplus F_Y^{-1}(R(\overline{T})^{\perp}). \end{aligned}$$

Noting that $R(T)$ and $R(\overline{T})$ are both Chebyshev subspaces of Y, Theorem 2.1.7 shows that

$$D(\overline{T}^M) = D(T^M) = Y.$$

If $Y_{\delta T}(T) = \{\theta\}$, then inequality (4.13) is trivial. Otherwise, we have $b \neq \theta$ and $\overline{x} = \overline{T}^M b \in S(\overline{T}, b)$ for all $b \in Y_{\delta T}(T)$. By Lemma 4.1.2, we have

$$\operatorname{dist}(\overline{x}, S(T,b)) \leq \|T^M\|\,\|\delta T\|\,\|\overline{x}\|. \tag{4.14}$$

Since $S(T,b) = T^M b + N(T)$ is a closed linear manifold, $N(T)$ is Chebyshev subspace of X, then $S(T,b)$ is a Chebyshev linear manifold. Hence

$$\begin{aligned} \operatorname{dist}(\overline{x}, S(T,b)) &= \operatorname{dist}(\overline{T}^M b, T^M b + N(T)) \\ &= \|\overline{T}^M b - T^M b - \pi_{N(T)}(\overline{T}^M b - T^M b)\|, \end{aligned} \tag{4.15}$$

where $\pi_{N(T)}$ is a metric projector operator from X to $N(T)$.

Let us denote $x_M^b := (\overline{T}^M - T^M)b$. Since $b \in Y_{\delta T}(T)$ and $b \neq 0$, we assume $x_M^b \neq \theta$ such that

$$F_X(x_M^b) \cap N(T)^\perp \neq \emptyset. \tag{4.16}$$

Take any $x^* \in F_X(x_M^b) \cap N(T)^\perp$ such that

$$\begin{aligned}\|x_M^b\|^2 &= \langle x^*, x_M^b\rangle \\ &= \langle x^*, x_M^b - x\rangle \\ &\leq \|x^*\|\ \|x_M^b - x\|\end{aligned}$$

for any $x \in N(T)$. Since $\|x_M^b\| = \|x^*\| \ \neq \theta$, we get

$$\|x_M^b - \theta\| = \inf_{x\in N(T)} \|x_M^b - x\|.$$

It follows from the assumption that $N(T)$ is a Chebyshev subspace such that

$$\pi_{N(T)}(x_M^b) = \theta. \tag{4.17}$$

Next from equality (4.15), we obtain

$$\operatorname{dist}(\overline{x}, S(T,b)) = \|\overline{T}^M b - T^M b\|.$$

In addition, by inequality (4.14) we obtain

$$\|(\overline{T}^M - T^M)b\| \leq \|T^M\|\ \|\delta T\|\ \|\overline{T}^M\|\ \|b\|,$$

whence

$$\frac{\|(\overline{T}^M - T^M)b\|}{\|b\|} \leq \|T^M\|\ \|\delta T\|\ \|\overline{T}^M\|. \tag{4.18}$$

Taking the supremum of the left hand side over b in (4.18), we get

$$\frac{\|\overline{T}^M - T^M\|_0}{\|\overline{T}^M\|} < \|T^M\|\ \|\delta T\|.$$

By the assumption that $\|T^M\|\ \|\delta T\| < 1$, we have

$$\begin{aligned}\frac{\|\overline{T}^M - T^M\|_0}{\|\overline{T}^M\|} &\leq \|T^M\|\ \|\delta T\|, \\ &\leq \frac{\|T^M\|\ \|\delta T\|}{1 - \|T^M\|\ \|\delta T\|},\end{aligned}$$

and the proof is completed. □

In general, the metric generalized inverse is a bounded homogeneous nonlinear operator, which suggests the discussion of nonlinear generalized inverse. In these circumstances, we are going to change the nonlinear operator into the product of the linear operator and quasi-linear operator, and then partially draw on the discussing of the perturbation of the linear generalized inverses. To obtain the perturbation of Moore-Penrose metric generalized inverse T^M of $T \in B(X, Y)$. At first, we discuss the quasi-additivity of T^M.

Theorem 4.2.4 Let $T \in B(X, Y)$ and assume that $N(T)$ and $R(T)$ are Chebyshev subspaces of X and Y, respectively, and that $\pi_{N(T)}^{-1}(\theta)$ is a linear subspace of X. Then:

(1) there exists a unique Moore-Penrose metric generalized inverse T^M of T, and

$$T^M y = (I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)} y, \quad y \in Y \tag{4.19}$$

where T^- is a linear inner inverses of T;

(2) T^M is quasi-additive (i.e, T^M is quasi-additive on $R(T)$) and

$$T^M(x + y) = T^M x + T^M y$$

for all $x \in Y$, $y \in R(T)$.

Proof (1) Since $N(T)$ and $R(T)$ are Chebyshev subspaces of X and Y, respectively, by Lemma 2.2.2, there exists a unique Moore-Penrose meric generalized inverse T^M of T such that

$$T^M y = (I_{D(T)} - \pi_{N(T)})T^- \pi_{\overline{R(T)}} y, \quad y \in D(T^M),$$

where $D(T^M) = R(T) \dot{+} F_Y^{-1}(R(T)^{\perp})$, and T^- is a linear inner inverse of T. Since $R(T)$ is a Chebyshev subspace of Y, so $D(T^M) = Y$, by Theorem 2.1.7. Therefore equality (4.19) is valid.

(2) Noticing that $\pi_{N(T)}^{-1}(\theta)$ is a linear subspace of X, Lemma 2.1.5 implies that $\pi_{N(T)}$ is a linear operator. Thus $I_{D(T)} - \pi_{N(T)}$ is a linear operator. By Lemma 2.2.2, there exists a linear inner T^- of T. Moreover, $\pi_{\overline{R(T)}} = \pi_{R(T)}$ is bounded quasi-linear

(quasi-additive) metric projector, which shows that T^M is a bounded homogeneous operator. Thus for each $x \in Y$, $y \in R(T)$, we have

$$\begin{aligned}
T^M(x+y) &= (I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)}(x+y) \\
&= (I_{D(T)} - \pi_{N(T)})T^-[\pi_{R(T)}x + y] \\
&= (I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)}x + (I_{D(T)} - \pi_{N(T)})T^- y \\
&= T^M x + (I_{D(T)} - \pi_{N(T)})T^- y \\
&= T^M x + T^M y.
\end{aligned}$$ □

Corollary 4.2.5 Let $T \in B(X,Y)$, $\delta T \in B(X,Y)$, $N(T)$ and $R(T)$ be Chebyshev subspaces of X and Y, respectively. If we assume that $\pi_{N(T)}^{-1}(\theta)$ is a linear subspace of X and $R(\delta T) \subset R(T)$, then $T^M \delta T$ is a linear operator.

Proof By Theorem 4.2.4, there exists a unique Moore-Penrose metric generalized inverse T^M of T such that

$$T^M y = (I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)} y, \quad (y \in Y).$$

By $R(\delta T) \subset R(T)$, it is easy to see that

$$\begin{aligned}
T^M \delta T &= (I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)} \delta T \\
&= (I_{D(T)} - \pi_{N(T)})T^- \delta T.
\end{aligned}$$

Therefore, $T^M \delta T$ is also a linear operator because $(I_{D(T)} - \pi_{N(T)})T^- \delta T$ is a linear operator. The proof is completed. □

In order to prove Theorem 4.2.8, we need the following result.

Lemma 4.2.6 Let $T \in H(X)$. If T is quasi-additive on $R(T)$ and $\|T\| < 1$, then the operator $(I-T)^{-1}$ exists and

(1) $(I-T)^{-1} \in H(X)$;

(2) $(I-T)^{-1} = \sum_{k=0}^{\infty} T^k$;

(3) $\|(I-T)^{-1}\| \le \frac{1}{1-\|T\|}$;

(4) $\|(I-T)^{-1}-I\| \leq \frac{\|T\|}{1-\|T\|}$.

Proof Let $A_n = \sum_{k=0}^{n} T^k$ for all nonnegative integers n. Then A_n are bounded homogenous operators. For all $n > m$, we have

$$\|A_n - A_m\| = \|\sum_{k=m}^{n} T^k\| \leq \sum_{k=m}^{n} \|T\|^k \to 0$$

as $m,\ n \to \infty$. By the completeness of $H(X)$, there exists a unique operator $A \in H(X)$ such that

$$A = \lim_{n\to\infty} A_n = \sum_{k=0}^{\infty} T^k.$$

Since T is quasi-additive on $R(T)$, we have

$$T(I + T + T^2 + \cdots + T^n) = T + T^2 + \cdots + T^{n+1}.$$

Hence

$$(I-T)A_n = (I-T)(I + T + T^2 + \cdots + T^n) = I - T^{n+1}$$

and

$$A_n(I-T) = I - T^{n+1},$$

for each $n \geq 1$. Let $n \to \infty$, we obtain that $A = (I-T)^{-1}$. Therefore

$$\|(I-T)^{-1}\| = \|A\| \leq \frac{1}{1-\|T\|},$$
$$\|(I-T)^{-1} - I\| = \|A - I\| \leq \frac{\|T\|}{1-\|T\|}.$$

This finishes the proof. □

Lemma 4.2.7 Let $T \in B(X,Y)$, $\delta T \in B(X,Y)$ and $\overline{T} = T + \delta T$. Assume that $\pi_{N(T)}^{-1}(\theta)$ is a linear subspace of X, $N(T)$ and $R(T)$ are Chebyshev subspaces of X and Y, respectively. If $\delta T \in \triangle(T)$, $\|T^M\|\|\delta T\| < 1$, $R(\delta T) \subset R(T)$ and $N(T) \subset N(\delta T)$, then

$$R(T) = R(\overline{T}), \qquad N(T) = N(\overline{T}).$$

Proof By Theorem 2.2.4, there exists a unique Moore-Penrose Metric Generalized inverse T^M of T, which is a bounded homogenous operator (see Remark 2.2.5). Since $TT^M = \pi_{R(T)}$, we have

$$\overline{T} = T + \delta T = T(I + T^M \delta T).$$

By the assumption that $\pi_{N(T)}^{-1}(\theta)$ is a linear subspace of X, it follows from Theorem 4.2.4 that T^M is quasi-additive on $R(T) \subset Y$. Moreover, $R(\delta T) \subset R(T)$, therefore $T^M \delta T$ is quasi-additive on $R(T^M \delta T)$. Noticing that

$$\|T^M \delta T\| \leq \|T^M\| \|\delta T\| < 1 \text{ and } -T^M \delta T \in H(X),$$

by Lemma 4.2.6, the operator $(I - (-T^M \delta T))^{-1}$ exists and $(I + T^M \delta T)^{-1} \in H(X)$. Hence

$$T = \overline{T}(I + T^M \delta T)^{-1},$$

which means that $R(T) \subset R(\overline{T})$. It is obvious that $R(\overline{T}) \subset R(T)$. Therefore,

$$R(T) = R(\overline{T}).$$

By the assumption that $N(T) \subset N(\delta T)$ we easily deduce that $N(T) \subset N(\overline{T})$. Noticing that

$$\|\delta T T^M\| \leq \|T^M\| \|\delta T\| < 1 \text{ and } -\delta T T^M \in H(X),$$

by Lemma 4.2.6, the operator $(I - (-\delta T T^M))^{-1}$ exists and $(I + \delta T T^M)^{-1} \in H(X)$. By $T^M T = I - \pi_{N(T)}$, we get

$$\overline{T} = T + \delta T = (I + \delta T T^M)T.$$

Hence

$$T = (I + \delta T T^M)^{-1} \overline{T}.$$

On the other hand, $(I + \delta T T^M)^{-1}$ is a homogenous operator, so for any $x \in N(\overline{T})$, we have

$$Tx = (I + \delta T T^M)^{-1} \overline{T} x = (I + \delta T T^M)^{-1} \theta = \theta,$$

which means that $x \in N(T)$. Therefore

$$N(T) = N(\overline{T}).$$

This finishes the proof. □

Now we are ready to state our result concerning the perturbation of Moore-Penrose metric generalized inverse T^M of T.

Theorem 4.2.8 Let $T \in B(X,Y)$, $\delta T \in B(X,Y)$ and $\overline{T} = T + \delta T$. Assume that $N(T)$ and $R(T)$ are Chebyshev subspaces of X and Y, respectively. If $\delta T \in \triangle(T)$, $\|T^M\|\|\delta T\| < 1$, $R(\delta T) \subset R(T)$, $N(T) \subset N(\delta T)$, and $\pi_{N(T)}^{-1}(\theta)$ is a linear subspace of X, then T^M and $\overline{T}^M$ exist. Moreover, we have

$$\frac{\|\overline{T}^M - T^M\|}{\|\overline{T}^M\|} \le \|T^M\|\|\delta T\| \le \frac{\|T^M\|\|\delta T\|}{1 - \|T^M\|\|\delta T\|},$$

$$\|\overline{T}^M\| \le \frac{\|T^M\|}{1 - \|T^M\|\|\delta T\|},$$

where $\|T^M\|$ is the bounded homogenous operator norm of T^M.

Proof By Theorem 2.2.4, T^M and $\overline{T}^M$ exist and

$$D(T^M) = R(T) \dotplus F_Y^{-1}(R(T)^\perp),$$
$$D(\overline{T}^M) = R(\overline{T}) \dotplus F_Y^{-1}(R(\overline{T})^\perp),$$

where $F_Y : Y \rightrightarrows Y^*$ is the duality mapping of Y .

Since $R(T)$ and $R(\overline{T})$ are Chebyshev subspaces of Y, by Theorem 2.1.7,

$$D(\overline{T}^M) = D(T^M) = Y.$$

Since $R(\delta T) \subset R(T)$, by the Lemma 4.2.7, we deduce that $R(\overline{T}) = R(T)$. For all $b \in R(\overline{T}) = R(T)$, $x = T^M b \in S(T,b)$, $\overline{x} = \overline{T}^M b \in S(\overline{T},b)$. Lemma 4.1.2 implies that

$$\operatorname{dist}(\overline{x}, S(T,b)) \le \|T^M\|\|\delta T\|\|\overline{x}\|. \tag{4.20}$$

Noticing that

$$S(T,b) = T^M b + N(T),$$

and furthermore that $N(T)$ is a Chebyshev subspace of X, we obtain that $S(T, b)$ is a Chebyshev linear manifold in X. Therefore

$$\begin{aligned}\text{dist}(\overline{x}, S(T, b)) &= \text{dist}(\overline{T}^M b, T^M b + N(T)) \\ &= \|\overline{T}^M b - T^M b - \pi_{N(T)}(\overline{T}^M b - T^M b)\|, \qquad (4.21)\end{aligned}$$

where $\pi_{N(T)}$ is a metric project operator from X into $N(T)$. Since $N(T)$ is a Chebyshev subspace of X, by Theorem 2.1.7, we obtain that

$$X = \pi_{N(T)}(x_) \dotplus C(T), \quad \forall x \in X,$$

where $C(T) = F_X^{-1}(N(T)^\perp)$. This implies that for all $x_1 \in C(T)$, we have $\pi_{N(T)}(x_1) = \theta$. Indeed, the following relation is clearly true

$$x_1 = \theta + x_1, \quad \theta \in N(T), \ (x_1 \in C(T)).$$

By Theorem 2.1.7, we have

$$x_1 = \pi_{N(T)}(x_1) + x_2, \ (x_2 \in C(T)).$$

Moreover, since the decomposition is a unique, we have

$$\pi_{N(T)}(x_1) = \theta.$$

Since $N(T) \subset N(\delta T)$, by Lemma 4.2.7, we obtain that $N(T) = N(\overline{T})$. Hence $C(\overline{T}) = C(T)$. Since $\pi_{N(T)}^{-1}(\theta)$ is a linear subspace of X, by Lemma 2.1.5, $\pi_{N(T)}$ is also a linear operator. Since $\overline{T}^M b \in C(\overline{T}) = C(T)$ and $T^M b \in C(T)$, we have

$$\pi_{N(T)}(\overline{T}^M b - T^M b) = \pi_{N(T)}(\overline{T}^M b) - \pi_{N(T)}(T^M b) = \theta.$$

By (4.21), there holds

$$\text{dist}(\overline{x}, S(T, b)) = \|\overline{T}^M b - T^M b\|.$$

By (4.20) and $\|\overline{x}\| = \|\overline{T}^M b\|$, we have

$$\|(\overline{T}^M - T^M)b\| \leq \|T^M\| \|\delta T\| \|\overline{T}^M b\|.$$

For all $y \in Y \setminus \{\theta\}$, there exists a unique $b \in R(\overline{T}) = R(T)$ such that

$$b = \pi_{R(T)}(y) = \pi_{R(\overline{T})}(y).$$

It follows from Theorem 2.2.4 that

$$\begin{aligned} \overline{T}^M \pi_{R(\overline{T})}(y) = \overline{T}^M(y), \\ T^M \pi_{R(T)}(y) = T^M(y). \end{aligned}$$

Hence

$$\begin{aligned} \|(\overline{T}^M - T^M)y\| &= \|\overline{T}^M(y) - T^M(y)\| \\ &= \|\overline{T}^M \pi_{R(\overline{T})}(y) - T^M \pi_{R(T)}(y)\| \\ &= \|(\overline{T}^M - T^M)b\| \\ &\leq \|T^M\|\|\delta T\|\|\overline{T}^M b\| \\ &= \|T^M\|\|\delta T\|\|\overline{T}^M \pi_{R(\overline{T})}(y)\| \\ &= \|T^M\|\|\delta T\|\|\overline{T}^M(y)\| \\ &\leq \|T^M\|\|\delta T\|\|\overline{T}^M\|\|(y)\|. \end{aligned}$$

Therefore

$$\sup_{\|y\| \neq \theta} \frac{\|(\overline{T}^M - T^M)y\|}{\|y\|} \leq \|T^M\|\|\delta T\|\|\overline{T}^M\|,$$

and

$$\frac{\|\overline{T}^M - T^M\|}{\|\overline{T}^M\|} \leq \|T^M\|\|\delta T\|.$$

Since $\|T^M\|\|\delta T\| < 1$, we have $\theta < 1 - \|T^M\|\|\delta T\| < 1$ and

$$\frac{\|\overline{T}^M - T^M\|}{\|\overline{T}^M\|} \leq \frac{\|T^M\|\|\delta T\|}{1 - \|T^M\|\|\delta T\|}.$$

Moreover,

$$\begin{aligned} \|\overline{T}^M y\| &\leq \|\overline{T}^M y - T^M y\| + \|T^M y\| \\ &= \|(\overline{T}^M - T^M)y\| + \|T^M y\| \\ &\leq \|T^M\|\|\delta T\|\|\overline{T}^M y\| + \|T^M y\|. \end{aligned}$$

Therefore

$$(1 - \|T^M\|\|\delta T\|)\|\overline{T}^M y\| \leq \|T^M y\|,$$

which implies that

$$\|\overline{T}^M y\| \leq \frac{\|T^M\|\|y\|}{1 - \|T^M\|\|\delta T\|}$$

or equivalently

$$\frac{\|\overline{T}^M y\|}{\|y\|} \leq \frac{\|T^M\|}{1 - \|T^M\|\|\delta T\|}.$$

Taking the supremum over $y \in Y \setminus \{\theta\}$, we have

$$\|\overline{T}^M\| \leq \frac{\|T^M\|}{1 - \|T^M\|\|\delta T\|},$$

and the proof is completed. □

If X and Y are Hilbert spaces, then the Moore-Penrose metric generalized inverses of linear operators between Banach spaces coincide with Moore-Penrose generalized inverses under usual sense since the metric projector is linear orthogonal projector. It is easy to deduce the following well-known perturbation result from our above result.

Corollary 4.2.9 Let X and Y be Hilbert spaces, $T \in B(X,Y)$ be with $\overline{D(T)} = \overline{D(\overline{T})} = X$, $R(T)$ be a closed subspace of Y. Then there exists the Moore-Penrose generalized inverse T^+ of T. If $\delta T \in B(X,Y)$, $\|T^+\|\|\delta T\| < 1$ and $R(\overline{T}) \cap N(T^+) = \{\theta\}$, then the Moore-Penrose generalized inverse $\overline{T}^+$ of $\overline{T}$ exists and

$$\|\overline{T}^+\| \leq \frac{\|T^+\|}{1 - \|T^+\|\|\delta T\|},$$

$$\frac{\|\overline{T}^+ - T^+\|}{\|\overline{T}^+\|} \leq \frac{\|T^+\|\|\delta T\|}{1 - \|T^+\|\|\delta T\|}.$$

Proof Since $T \in B(X,Y)$ and $R(T)$ is closed, the Moore-Penrose metric generalized inverse T^+ of T exists. Since $\|T^+\|\|\delta T\| < 1$ and $R(\overline{T}) \cap N(T^+) = \{\theta\}$, there exists the Moore-Penrose generalized inverse $\overline{T}^+$ of $\overline{T}$ (see [112]), which implies that the condition of Theorem 4.2.8 is satisfied. Take $T^M = T^+$ and $\overline{T}^M = \overline{T}^+$. Then it follows from Theorem 4.2.8 that

$$\|\overline{T}^+\| \le \frac{\|T^+\|}{1 - \|T^+\|\|\delta T\|}$$

and

$$\frac{\|\overline{T}^+ - T^+\|}{\|\overline{T}^+\|} \le \frac{\|T^+\|\|\delta T\|}{1 - \|T^+\|\|\delta T\|},$$

which finishes the proof. □

4.3 The Error Bound Estimate of Perturbation for Moore-Penrose Metric Generalized Inverse

Theorem 4.3.1 Let $T \in B(X,Y)$, $N(T)$ and $R(T)$ be Chebyshev subspaces of X and Y, respectively. If $\pi_{N(T)}^{-1}(\theta)$ is a linear subspace of X and $R(T)$ is approximatively compact, then T has a unique and continuous Moore-Penrose metric generalized inverse T^M.

Proof By Theorem 4.2.4, there exists a unique Moore-Penrose metric generalized inverse T^M of T such that

$$T^M y = (I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)} y, \quad y \in Y.$$

Since $R(T)$ is a approximatively compact Chebyshev subspace of Y, it follows from Lemma 2.1.10 that $\pi_{R(T)}$ is continuous. Since $I_{D(T)} - \pi_{N(T)}$ and T^- are bounded linear operators, the operator $(I_{D(T)} - \pi_{N(T)})T^- \pi_{R(T)}$ is bounded and continuous. Thus, there exists a unique and continuous Moore-Penrose metric generalized inverse T^M of T. □

Lemma 4.3.2 Let $T, \delta T \in B(X,Y)$, $N(T)$ and $R(T)$ be Chebyshev subspaces of X and Y, respectively. Assume that $\|T^M\|\|\delta T\| < 1$, $N(T) \subset N(\delta T)$ and $R(\delta T) \subset R(T)$. If $\pi_{N(T)}^{-1}(\theta)$ is a linear subspace of X, and $R(T)$ is approximatively compact, then the following results are true:

(1) $(I+\delta TT^M):Y\to Y$ is bounded, invertible and

$$(I+\delta TT^M)^{-1}=\sum_{k=0}^{\infty}(-1)^k(\delta TT^M)^k, \tag{4.22}$$

where $(I+\delta TT^M)^{-1}\in H(Y)$.

(2) $\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^kT^M$ is convergent in $H(Y,X)$ and

$$\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^kT^M \;=\; T^M(I+\delta TT^M)^{-1}. \tag{4.23}$$

(3) $(I+T^M\delta T):X\to F_X^{-1}(N(T)^{\perp})$ is bounded, invertible and

$$(I+T^M\delta T)^{-1}=\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k, \tag{4.24}$$

where$(I+T^M\delta T)^{-1}\in B(X,X)$.

(4)

$$T^M(I+\delta TT^M)^{-1}=(I+T^M\delta T)^{-1}T^M. \tag{4.25}$$

Proof (1) Since $N(T)$ and $R(T)$ are Chebyshev subspaces of X and Y, respectively, there exists a unique Moore-Penrose metric generalized inverse $T^M\in H(Y,X)$ of T, where $R(T)$ is a closed set, $D(T^M)=Y$ and $R(T^M)=F_X^{-1}(N(T)^{\perp})$. Since $\|T^M\|\|\delta T\|\le r<1$, δTT^M is quasi-additive on $R(\delta TT^M)\subset R(T)$, it follows from Lemma 4.2.6 that $(I+\delta TT^M)$ is invertible and

$$(I+\delta TT^M)^{-1}=\sum_{k=0}^{\infty}(-1)^k(\delta TT^M)^k,$$

where $(I+\delta TT^M)^{-1}\in H(Y)$.

(2) Since $\|T^M\|\|\delta T\|\le r<1$, by Corollary 4.2.5, we have $T^M\delta T\in L(X)$ and

$$\begin{aligned}\|(-1)^k(T^M\delta T)^kT^M\| &= \|(-1)^kT^M(\delta TT^M)^k\|\\ &\le \|T^M\|\|\delta TT^M\|^k\\ &\le \|T^M\|\;r^k\end{aligned}$$

for all $k=0,1,2\cdots$. Hence the series $\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^kT^M$ is absolutely convergent in $H(Y,X)$. Since $\pi_{N(T)}^{-1}(\theta)$ is a linear subspace of X and $R(T)$ is approximatively compact, it follows from Theorem 4.3.1 that T^M is continuous. By Theorem

4.2.4, T^M is quasi-additive on $R(T)$. Hence, by $R(\delta T) \subset R(T)$, we deduce that

$$\begin{aligned} T^M(I+\delta T T^M)^{-1} &= T^M \sum_{k=0}^{\infty}(-1)^k(\delta T T^M)^k \\ &= \sum_{k=0}^{\infty} T^M(-1)^k(\delta T T^M)^k \\ &= \lim_{k\to\infty}[T^M - T^M\delta T T^M + ... + (-1)^k T^M(\delta T T^M)^k] \\ &= \sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k T^M. \end{aligned}$$

(3) It is obvious that $\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k$ is a bounded operator acting from X to $F_X^{-1}(N(T)^{\perp})$. We claim that

$$(I+T^M\delta T)^{-1} = \sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k.$$

Indeed, taking arbitrary $x \in X$, we have

$$x = (I - T^M T)x + T^M T x.$$

Since $N(T) \subset N(\delta T)$, thus $\delta T(I - T^M T) = \theta$. It follows from Corollary 4.2.5 that $T^M\delta T$ is a bounded linear operator. Hence, by equalities (4.22), (4.23) and the inclusion $N(T) \subset N(\delta T)$, we obtain

$$\begin{aligned} &\left[\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k\right](I+T^M\delta T)x \\ &= \sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k(I-T^MT)x + \left[\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k\right](I+T^M\delta T)T^MTx \\ &= (I-T^MT)x + \left[\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k\right]T^M(I+\delta T T^M)Tx \\ &= (I-T^MT)x + T^M\left[\sum_{k=0}^{\infty}(-1)^k(\delta T T^M)^k\right](I+\delta T T^M)Tx \\ &= (I-T^MT)x + T^M(I+\delta T T^M)^{-1}(I+\delta T T^M)Tx \\ &= x. \end{aligned}$$

It is easy to see that

$$x = (I - T^M T)x + T^M T x, \quad x \in F_X^{-1}(N(T)^{\perp}) = R(T^M).$$

Since $\delta T(I - T^M T) = \theta$, T^M is continuous and quasi-additive on $R(T)$, so

$$\begin{aligned}
& (I + T^M\delta T)\left[\sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k\right]x \\
= \ & (I + T^M\delta T)\left[x + \sum_{k=1}^{\infty}(-1)^k(T^M\delta T)^k T^M T x\right] \\
= \ & (I + T^M\delta T)\left[x + T^M\sum_{k=1}^{\infty}(-1)^k(\delta T T^M)^k T x\right] \\
= \ & (I + T^M\delta T)\left[x + T^M((I + \delta T T^M)^{-1}Tx - Tx)\right] \\
= \ & (I + T^M\delta T)\left[x + T^M(I + \delta T T^M)^{-1}Tx - T^M T x\right] \\
= \ & (I + T^M\delta T)x + (I + T^M\delta T)T^M(I + \delta T T^M)^{-1}Tx - (I + T^M\delta T)T^M T x \\
= \ & (I + T^M\delta T)x + T^M(I + \delta T T^M)(I + \delta T T^M)^{-1}Tx - (I + T^M\delta T)T^M T x \\
= \ & (I + T^M\delta T)x + T^M T x - (I + T^M\delta T)T^M T x \\
= \ & x + T^M\delta T(I - T^M T)x \\
= \ & x.
\end{aligned}$$

by (4.22) and (4.23). Therefore

$$(I + T^M\delta T)^{-1} = \sum_{k=0}^{\infty}(-1)^k(T^M\delta T)^k \in B(X),$$

where $(I + T^M\delta T)^{-1} \in B(X, X)$.

The last statement (4) follows easily from (4.22), (4.23) and (4.24), and proof is completed. □

Theorem 4.3.3 Let $T \in B(X, Y)$, $\delta T \in B(X, Y)$, and $\overline{T} = T + \delta T$. Assume that $N(T)$ and $R(T)$ are Chebyshev subspaces of X and Y, respectively, $\delta T \in \triangle(T)$, $\|T^M\|\|\delta T\| < 1$, $N(T) \subset N(\delta T)$ and $R(\delta T) \subset R(T)$. If $\pi^{-1}_{N(T)}(\theta)$ is a linear subspace of X and $R(T)$ is approximatively compact, then:

(1) $N(T) = N(\overline{T})$, $R(T) = R(\overline{T})$;

(2) $\overline{T}^M = T^M(I + \delta T T^M)^{-1} = (I + T^M\delta T)^{-1}T^M$;

(3) $\|\overline{T}^M\| \leq \frac{\|T^M\|}{1 - \|\delta T T^M\|}$;

(4) $\|\overline{T}^M - T^M\| \leq \frac{\|T^M\|\|\delta T T^M\|}{1 - \|\delta T T^M\|}$.

Proof (1) By Lemma 4.2.7, we have $N(T) = N(\overline{T})$, $R(T) = R(\overline{T})$.

(2) Since $N(T)$ and $N(\overline{T})$ are Chebyshev subspaces of X, $R(T)$ and $R(\overline{T})$ are Chebyshev subspaces of Y, thus T^M and $\overline{T}^M$ exist. It follows from Lemma 4.3.2 that the operator $(I + \delta TT^M)$ is invertible and

$$(I + \delta TT^M)^{-1} = \sum_{k=0}^{\infty}(-1)^k(\delta TT^M)^k,$$

where $(I + \delta TT^M)^{-1} \in H(Y)$. Denoting $T^\# := T^M(I + \delta TT^M)^{-1} \in H(Y, X)$, we claim that $T^\# = T^M(I + \delta TT^M)^{-1}$ is the Moore-Penrose metric generalized inverse of $\overline{T}$ and

$$\overline{T}^M = T^M(I + \delta TT^M)^{-1} = (I + T^M\delta T)^{-1}T^M.$$

Indeed,

(i) Since $N(T) \subset N(\delta T)$, so $\delta T(I - T^M T) = \theta$. Hence

$$\begin{aligned}
&\overline{T} - \overline{T}T^\#\overline{T}\\
&= [I - \overline{T}T^M(I + \delta TT^M)^{-1}]\overline{T}\\
&= [I - (T + \delta T)T^M(I + \delta TT^M)^{-1}](T + \delta T)\\
&= [(I + \delta TT^M) - (T + \delta T)T^M](I + \delta TT^M)^{-1}(T + \delta T)\\
&= (I - TT^M)(I + \delta TT^M)^{-1}(T + \delta T)\\
&= (I - TT^M)(I + \delta TT^M)^{-1}(T + \delta TT^M T + \delta T - \delta TT^M T)\\
&= (I - TT^M)(I + \delta TT^M)^{-1}[(I + \delta TT^M)T + \delta T(I - T^M T)]\\
&= (I - TT^M)(I + \delta TT^M)^{-1}(I + \delta TT^M)T = \theta,
\end{aligned}$$

i.e.

$$\overline{T} = \overline{T}T^\#\overline{T}, \text{ on } X.$$

(ii) It follows from (4.25) that

$$T^M(I + \delta TT^M)^{-1} = (I + T^M\delta T)^{-1}T^M.$$

T^M is quasi-additive on $R(T)$, which implies that $T^M(TT^M - I) = \theta$ and

$$\begin{aligned}
&T^\#\overline{T}T^\# - T^\#\\
&= T^M(I + \delta TT^M)^{-1}\overline{T}T^M(I + \delta TT^M)^{-1} - T^M(I + \delta TT^M)^{-1}
\end{aligned}$$

$$= (I + T^M \delta T)^{-1} T^M \overline{T} T^M (I + \delta T T^M)^{-1} - (I + T^M \delta T)^{-1} T^M.$$

Furthermore, $R(\delta T) \subset R(T)$. Thus, $(I + T^M \delta T)$ is a linear operator such that

$$\begin{aligned}
& T^\# \overline{T} T^\# - T^\# \\
&= (I + T^M \delta T)^{-1} T^M [(T + \delta T) T^M (I + \delta T T^M)^{-1} - I] \\
&= (I + T^M \delta T)^{-1} T^M [T T^M + \delta T T^M - I - \delta T T^M](I + \delta T T^M)^{-1} \\
&= (I + T^M \delta T)^{-1} T^M (T T^M - I)(I + \delta T T^M)^{-1} \\
&= \theta,
\end{aligned}$$

which means that $T^\# \overline{T} T^\# = T^\#$ on Y.

(iii) Noting that $N(T) \subset N(\delta T)$, then $N(T) = N(\overline{T})$ and $\delta T = \delta T T^M T$. Since $T^M T = I - \pi_{N(T)}$, we deduce that

$$\overline{T} = T + \delta T = (I + \delta T T^M) T.$$

Hence

$$\begin{aligned}
T^\# \overline{T} &= T^M (I + \delta T T^M)^{-1} (I + \delta T T^M) T \\
&= T^M T \\
&= I - \pi_{N(T)} \\
&= I - \pi_{N(\overline{T})}.
\end{aligned}$$

(iv) It follows from the inclusion $R(\delta T) \subset R(T)$ that $R(\overline{T}) = R(T)$. Hence $\delta T = T T^M \delta T$. Since $T T^M = \pi_{R(T)}$, we have

$$\overline{T} = T + \delta T = T(I + T^M \delta T),$$

and

$$\begin{aligned}
\overline{T} T^\# &= T(I + T^M \delta T)(I + T^M \delta T)^{-1} T^M \\
&= T T^M \ = \pi_{R(T)} \ = \pi_{R(\overline{T})}.
\end{aligned}$$

Therefore, $T^\# = T^M (I + \delta T T^M)^{-1}$ is the Moore-Penrose metric generalized inverse of $\overline{T}$, and

$$\overline{T}^M = T^M (I + \delta T T^M)^{-1} = (I + T^M \delta T)^{-1} T^M.$$

Therefore, we have shown that (2) is valid.

(3) Lemma 4.2.6 shows that

$$\begin{aligned}\|\overline{T}^M\| &= \|T^M(I+\delta T T^M)^{-1}\| \\ &\le \|T^M\|\|(I+\delta T T^M)^{-1}\| \\ &\le \frac{\|T^M\|}{1-\|\delta T T^M\|}.\end{aligned}$$

(4) Lemma 4.2.6 assures that

$$\begin{aligned}\|\overline{T}^M - T^M\| &= \|(I+T^M\delta T)^{-1}T^M - T^M\| \\ &= \|((I+T^M\delta T)^{-1} - I)T^M\| \\ &\le \|(I+T^M\delta T)^{-1} - I\|\|T^M\| \\ &\le \frac{\|T^M\delta T\|\|T^M\|}{1-\|T^M\delta T\|}.\end{aligned}$$

By Theorem 4.3.3 , we assert that

$$\overline{T}^M = (I+T^M\delta T)^{-1}T^M = \left(I - T^M T + T^M(I+\delta T T^M)^{-1}T\right)T^M. \tag{4.26}$$

Indeed

$$\begin{aligned}&\left(I - T^M T + T^M(I+\delta T T^M)^{-1}T\right)T^M \\ &= T^M - T^M T T^M + T^M(I+\delta T T^M)^{-1}T T^M \\ &= (I+T^M\delta T)^{-1}T^M.\end{aligned}$$

□

Theorem 4.3.4 Let T and δT belong to $B(X,Y)$ and $\overline{T} = T + \delta T$. Assume that $N(T)$ and $R(T)$ are Chebyshev subspaces of X and Y, respectively, $\delta T \in \triangle(T)$, $\|T^M\|\|\delta T\| < 1$, $N(T) \subset N(\delta T)$ and $R(\delta T) \subset R(T)$. If $\pi^{-1}_{N(T)}(\theta)$ is a linear subspace of X, $R(T)$ is approximatively compact and $\overline{y} := y + \delta y \in R(T)$ for all $y \in R(T)$, then

$$\frac{\|\overline{x} - x\|}{\|x\|} \le \frac{\kappa}{1-\kappa\varepsilon_T}\left(\varepsilon_y\frac{\|y\|}{\|T\|\|x\|} + \varepsilon_T\right),$$

where $\kappa = \|T\|\|T^M\|$, $\varepsilon_T = \|\delta T\|/\|T\|$, $\varepsilon_y = \|\delta y\|/\|y\|$, $\overline{x} = \overline{T}^M\overline{y}$ and $x = T^M y$.

Proof Noticing that $\overline{T}^M$ is linear on $R(\overline{T}) = R(T)$, it follows from Theorem 4.3.3 and (4.26) that

$$\begin{aligned}
\|\overline{x} - x\| &= \|\overline{T}^M \overline{y} - T^M y\| \\
&= \|\overline{T}^M \delta y + (\overline{T}^M - T^M) y\| \\
&= \|\overline{T}^M \delta y + \left[(I + T^M \delta T)^{-1} - I\right] T^M y\| \\
&\leq \|\overline{T}^M\| \|\delta y\| + \|(I + T^M \delta T)^{-1} - I\| \|T^M y\| \\
&\leq \frac{\|T^M\|}{1 - \|T^M\|\|\delta T\|} \|\delta y\| + \frac{\|T^M\|\|\delta T\|\|x\|}{1 - \|T^M\|\|\delta T\|} \\
&= \frac{\|T^M\|\|T\|}{1 - \|T^M\|\|\delta T\|} \frac{\|\delta y\|}{\|y\|} \frac{\|y\|}{\|T\|} + \frac{\|T^M\|\|T\|\|x\|}{1 - \|T^M\|\|\delta T\|} \frac{\|\delta T\|}{\|T\|} \\
&= \frac{\kappa}{1 - \kappa \varepsilon_T} \left(\varepsilon_y \frac{\|y\|}{\|T\|} + \varepsilon_T \|x\| \right),
\end{aligned}$$

which finished the proof. □

Corollary 4.3.5 If T satisfies the assumptions of Theorem 4.3.4 and T is surjective, then

$$\frac{\|\overline{x} - x\|}{\|x\|} \leq \frac{\kappa}{1 - \kappa \varepsilon_T} \left(\varepsilon_y + \varepsilon_T, \right).$$

where $\kappa = \|T\|\|T^M\|$, $\varepsilon_T = \|\delta T\|/\|T\|$, $\varepsilon_y = \|\delta y\|/\|y\|$.

Proof Since T is surjective, for any $y \in Y$, there exists x such that $Tx = y$, i.e., $y - Tx = \theta$, and $\|y\| \leq \|T\|\|x\|$. Thus by the proof of Theorem 4.3.4, we have

$$\begin{aligned}
\|\overline{x} - x\| &\leq \frac{\kappa}{1 - \kappa \varepsilon_T} \left(\varepsilon_y \frac{\|y\|}{\|T\|} + \varepsilon_T \|x\| \right) \\
&\leq \frac{\kappa}{1 - \kappa \varepsilon_T} \left(\varepsilon_y + \varepsilon_T \right) \|x\|,
\end{aligned}$$

which finished the proof. □

Chapter 5 Narrow Spectrum

5.1 Locally Fine Points

In recent years, Professor Jipu Ma introduced the concept of locally fine points for operator valve maps(see [61, 63]). This concept has been extensively studied in the last years (see [61, 62, 63, 64, 65, 67]). In this Chapter, two new concepts that generalize the notions of regular points and narrow spectrum points for bounded linear operators on Hilbert spaces are studied by using the concept of locally fine points. At first, some definition and theories of locally fine points will be shown in the followings.

Definition 5.1.1 [61] Let E be a topological space, $T_x : E \to B(X,Y)$ be an operator-valued map continuous at $x_0 \in E$, T_{x_0} be double splited and T_0^+ be a generalized inverse of T_{x_0}. Then x_0 is said to be a locally fine point for T_x provided there exists a neighborhood $U \subset X$ of x_0 such that for any $x \in U$, the following equality holds

$$R(T_x) \cap R(I_Y - T_0T_0^+) = \{\theta\}.$$

Remark 5.1.1 The concept of locally fine point involves formally a generalized inverse of T_{x_0}. However, it is independent of the choice of generalized inverses of T_{x_0} by Theorem 1.6 in [63]. Thus, it presents a behavior just for T_x near x_0 in the case when is double splited T_{x_0}.

Lemma 5.1.2 [63] Let $T_x : E \to B(X,Y)$ be continuous at x_0 and $T_{x_0} := T_0$ be double splited. Assume that T_0 has a generalized inverse $T_0^+ \in B(Y,X)$. Then there exists a neighborhood V_0 of x_0 such that the followings hold true:

(i) for each $x \in V_0$, T_x has a generalized inverse $T_x^+ \in B(E,F)$;

(*ii*) the equality $\lim_{x\to x_0} T_x^+ = T_0^+$ is satisfied if and only if x_0 is a locally fine point with respect to T_0^+ .

Lemma 5.1.3 [63] Assume that $T_0 \in B(X,Y)$ has a generalized inverse $T_0^+ \in B(X,Y)$. If $T \in B(Y,X)$ satisfies the inequality $\|T_0^+\|\|T - T_0\| < 1$, then the following conditions are equivalent:

(1) $B = [I_X + T_0^+(T - T_0)]^{-1}T_0^+ = T_0^+[I_Y + (T - T_0T_0^+)]^{-1}$ is a generalized inverse of T;

(2) $R(T) \cap N(T_0^+) = \{\theta\}$;

(3) $(I_E - T_0^+T_0)N(T) = N(T_0)$.

5.2 Generalized regular points and narrow spectrum points

In this chapter, let H be a Hilbert space, $B(H)$ be the set of all bounded linear operators from H into itself and $A \in B(H)$. Let $\sigma(A)$, $\rho(A)$ denote the sets of spectrum points and of generalized regular points of $A \in B(H)$, respectively. For any $A \in B(H)$, let us define $A_\lambda := A - \lambda$, $\forall \lambda \in \mathbb{C}$ ($\mathbb{C}$ denotes the field of complex numbers).

Definition 5.2.1 If $\mu \in \mathbb{C}$ is a locally fine point of A_λ, then μ is said to be a generalized regular point of $A \in B(H)$, if the range $R(A_\mu)$ of A_μ is closed and there exists a positive $\delta >$ such that

$$R(A_\lambda) \cap N^\perp = \{\theta\}$$

whenever $\mid \mu - \lambda \mid < \delta$, where $N^\perp$ denotes the orthogonal complement of $R(A_\mu)$. The set of generalized regular points of A is denoted by $\rho_g(A)$ in the sequel.

Definition 5.2.2 Any point from $\mathbb{C} \setminus \rho_g(A)$ is said to be a narrow spectrum point and the set of all narrow spectrum points of A is denoted in the sequel by $\sigma_N(A)$.

Proposition 5.2.1 For any $A \in B(H)$, there holds the inclusion $\rho_g(A) \supset \rho(A)$.

Proof Since $A-\lambda I$ for any $\lambda \in \rho(A)$ is a regular operator, $R(A_\lambda) = H$ and $R(A_\lambda)$ is closed, so that $R(A_\lambda)^\perp = \{\theta\}$. Then

$$R(A_\mu) \cap R(A_\lambda)^\perp = R(A_\mu) \cap \{\theta\} = \{\theta\}, \ \ \forall \mu \in \mathbb{C}.$$

Hereby, we can conclude that $\lambda \in \rho_g(A)$. □

Proposition 5.2.2 The narrow spectrum $\sigma_N(A)$ is a closed set for any $A \in B(H)$.

Proof Since it is proved in [96] that $\rho_g(A)$ is an open set, the proposition follows immediately. □

The following theorem gives some relationships between $\sigma_N(A)$, $\sigma(A)$ and $\rho_g(A)$.

Theorem 5.2.3 For each $A \in B(H)$, there holds the equality $\sigma(A) \setminus \sigma_N(A) = \rho_g(A) \cap \sigma(A)$.

Proof $\lambda \in \sigma(A) \setminus \sigma_N(A)$ implies that $\lambda \in \sigma(A)$ and $\lambda \notin \sigma_N(A)$, so that $\lambda \in \rho_g(A) \cap \sigma(A)$, which means that

$$\sigma(A) \setminus \sigma_N(A) \subset \rho_g(A) \cap \sigma(A).$$

On the other hand, if $\lambda \in \rho_g(A) \cap \sigma(A)$, then $\lambda \notin \sigma_N(A)$ and $\lambda \in \sigma(A)$, that is, $\lambda \in \sigma(A) \setminus \sigma_N(A)$, which means that

$$\sigma(A) \setminus \sigma_N(A) \supset \rho_g(A) \cap \sigma(A).$$

Thus we conclude that $\sigma(A) \setminus \sigma_N(A) = \rho_g(A) \cap \sigma(A)$. □

Corollary 5.2.4 For any $A \in B(H)$, we have $\lambda \in \sigma(A) \setminus \sigma_N(A)$ if and only if the following conditions are satisfied:

1. $\dim N(A_\lambda) + \dim N(A_\lambda^+) \neq \theta$;
2. $\lambda \in \rho_g(A)$.

Proof It follows from Theorem 5.2.3 that $\lambda \in \sigma(A) \setminus \sigma_N(A) \Longleftrightarrow \lambda \in \rho_g(A) \cap \sigma(A)$ $\Longleftrightarrow \lambda \in \rho_g(A)$ and $\dim N(A_\lambda) + \dim N(A_\lambda^+) \neq \theta$. □

Proposition 5.2.5 The equality $\rho_g(A) = \rho(A) \cup (\sigma(A) \setminus \sigma_N(A))$ holds true for any $A \in B(H)$.

Proof Obviously, $\rho_g(A) = \rho(A) \cup (\rho_g(A) \setminus \rho(A))$. Meanwhile, it is not difficult to verify that

$$\rho_g(A) \setminus \rho(A) = \sigma(A) \setminus \sigma_N(A).$$

In fact,

$$\begin{aligned}\lambda \in \rho_g(A) \setminus \rho(A) &\Longleftrightarrow \lambda \in \sigma(A) \text{ and } \lambda \notin \sigma_N(A) \\ &\Longleftrightarrow \lambda \in \sigma(A) \setminus \sigma_N(A).\end{aligned}$$ □

Proposition 5.2.6 Let $A \in B(H)$. The set $\sigma(A) \setminus \sigma_N(A)$ is an open set in $\mathbb{C}$.

Proof Theorem 5.2.3 means that $\lambda \in \sigma(A) \setminus \sigma_N(A)$, i.e., $\lambda \in \rho_g(A)$. By Proposition 5.2.2, it is easy to see that there exist $\delta_0 > 0$ such that

$$\{\mu \mid |\mu - \lambda| < \delta_0\} \subset \rho_g(A).$$

It follows from Lemma 5.1.3 that there exist $\delta_1 > 0$ such that

$$P_{N(A_\lambda)} N(A_\mu) = N(A_\lambda)$$

whenever $|\mu - \lambda| < \delta_1$. Meanwhile we can conclude that there exist a positive number δ_2 such that

$$P_{N(A_\lambda^+)} N(A_\mu^+) = N(A_\lambda^+)$$

whenever $\mid \mu - \lambda \mid < \delta_2$.

In fact, by Lemma 5.1.2, there exist $\delta_2 > 0$ such that A_μ^+ exists for any μ with $\mid \mu - \lambda \mid < \delta_2$ and $\lim_{\mu \to \lambda} A_\mu^+ = A_\lambda^+$, whence

$$I - A_\mu A_\mu^+ \to I - A_\lambda A_\lambda^+, \ \ \mu \to \lambda,$$

i.e.,

$$P_{N(A_\mu^+)} \to P_{N(A_\lambda^+)}, \ \ \mu \to \lambda.$$

Thus, one can conclude that there exists a positive number δ_3 such that

$$\|P_{N(A_\mu^+)} - P_{N(A_\lambda^+)}\| < 1$$

for any μ with $\mid \mu - \lambda \mid < \delta_3$. It follows from [51] that

$$P_{N(A_\lambda^+)} N(A_\mu^+) = N(A_\lambda^+)$$

whenever $|\mu - \lambda| < \delta_3$.

Defining $\delta := \min\{\delta_0, \delta_1, \delta_2, \delta_3\}$, we have

$$P_{N(A_\lambda)}N(A_\mu) = N(A_\lambda), \ \ P_{N(A_\lambda^+)}N(A_\mu^+) = N(A_\lambda^+)$$

and

$$C(\lambda, \delta) = \{\mu : |\mu - \lambda| < \delta\} \subset \rho_g(A)$$

whenever $|\mu - \lambda| < \delta$. By the assumption that $\lambda \in \rho_g(A) \cap \sigma(A) = \sigma(A) \setminus \sigma_N(A)$ and Corollary 5.2.4, this implies that

$$\dim N(A_\mu) + \dim N(A_\mu^+) \neq \theta$$

and

$$C(\lambda, \delta) \subset \rho_g(A), \ \ \mu \in C(\lambda, \delta).$$

So $C(\lambda, \delta) \subset \sigma(A) \setminus \sigma_N(A)$. □

The translation invariance of the spectrum is an important property in the operator theory. The next proposition shows that the narrow spectrum has also this property.

Proposition 5.2.7 If a is a fixed constant and A is an operator from $B(H)$, then for any $\lambda \in \sigma_N(A)$, we have

$$\lambda - a \in \sigma_N(A - aI).$$

Proof If $\lambda - a \notin \sigma_N(A - aI)$, then $\lambda - a \in \rho_g(A - aI)$. By Definition 5.1.1, there is a neighborhood $U_{\lambda - a}$ of $\lambda - a$ such that

$$R((A - aI) - \mu I) \cap R((A - aI) - (\lambda - a)I)^\perp = \{\theta\}, \ \forall \mu \in U_{\lambda - a}.$$

Furthermore, for the neighborhood $U_\lambda = U_{\lambda - a} + a$ of λ, we have

$$R((A - aI) - (\mu - a)I) \cap R((A - aI) - (\lambda - a)I)^\perp = \{\theta\}, \ \forall \mu \in U_\lambda,$$

i.e.,

$$R(A - \mu I) \cap R(A - \lambda I)^\perp = \{\theta\}, \ \forall \mu \in U_\lambda.$$

By Definition 5.1.1, $\lambda \in \rho_g(A)$, which contradicts the assumption that $\lambda \in \sigma_N(A)$. □

It is well known that if A is a bounded linear operator on a Hilbert space, then $\sigma(A)$ is a nonempty set. We will show that $\sigma_N(A)$ is also a nonempty set in this situation.

Theorem 5.2.8 The set $\sigma_N(A)$ is nonempty for any $A \in B(H)$.

Proof Assume that $\sigma_N(A) = \emptyset$. Then by Proposition 5.2.6, $\sigma(A)$ is an open set in $\mathbb{C}$. This means the $\sigma(A)$ is open and closed set simultaneously. In addition, since $\sigma(A)$ is a nonempty set, $\sigma(A) = \mathbb{C}$, which means that $\rho(A) = \emptyset$, which is a contradiction. □

Proposition 5.2.9 Let $A \in B(H)$ and $\partial\sigma(A)$ be the boundary of $\sigma(A)$. Then

$$\partial\sigma(A) \subset \sigma_N(A).$$

Proof Assume that $\lambda \in \partial\sigma(A)$ and $\lambda \notin \sigma_N(A)$. Noting that $\sigma(A)$ is a closed set and $\lambda \in \sigma(A) \setminus \sigma_N(A)$. By Proposition 5.2.6, one can assert that $\sigma(A) \setminus \sigma_N(A)$ is an open set. This yields that λ is the interior point of $\sigma(A)$, which contradicts the assumption that $\lambda \in \partial\sigma(A)$. So $\partial\sigma(A) \subset \sigma_N(A)$. □

Let $\gamma_{\sigma(A)}$ and $\gamma_{\sigma_N(A)}$ be the spectrum radius of $\sigma(A)$ and $\sigma_N(A)$, respectively. In what just follows, we will show that $\gamma_{\sigma(A)} = \gamma_{\sigma_N(A)}$.

Proposition 5.2.10 For any $A \in B(H)$ there holds the equality $\gamma_{\sigma(A)} = \gamma_{\sigma_N(A)}$.

Proof Obviously, $\gamma_{\sigma(A)} \geq \gamma_{\sigma_N(A)}$. Moreover, since $\sigma(A)$ is a bounded closed set, it follows that there exist $\lambda \in \sigma(A)$ such that $\gamma_{\sigma(A)} = |\lambda|$ and $\lambda \in \partial\sigma(A) \subset \sigma_N(A)$. Hereby, one can concludes that $\gamma_{\sigma(A)} = \gamma_{\sigma_N(A)}$. □

In what follows, we give one condition of the extension of non-trivial invariant subspace of linear bounded operators on Hilbert space.

Theorem 5.2.11 If $A \in B(H)$ and $\sigma(A) \setminus \sigma_N(A) \neq \emptyset$, there exists a non-trivial invariant subspace of A.

Proof By Theorem 5.2.3, we have the equality $\sigma(A) \setminus \sigma_N(A) = \rho_g(A) \cap \sigma(A)$. So, there exists one point $\lambda_0 \in \sigma(A)$ such that $R(\lambda_0)$ is closed. It is easy to check that $\dim N(A_{\lambda_0}) + \dim N(A^*_{\lambda_0}) \neq \theta$. Indeed, if both of them are zero, then $\lambda_0 \in \rho(A)$. This yields a contradiction to $\lambda_0 \in \sigma(A)$.

Let $H_0 = N(A_{\lambda_0})$. If $\dim N(A_{\lambda_0}) > 0$, then

$$Ah = \lambda_0 h, \quad \forall h \in H_0.$$

In the case of $H = H_0$, we have $A = \lambda_0 I$. There exists obviously a non-trivial invariant subspace of A. Hence we can assume that $\{\theta\} \subsetneq H_0 \subsetneq H$. Then we have $\{\theta\} \subsetneq H_0^{\perp} \subsetneq H$. Hereby it follows that H_0 is the non-trivial invariant subspace of A. In fact,

$$Ah = \lambda_0 h \in H_0, \ h \in H_0$$

in this case. Let $H_0 = N(A^*_{\lambda_0})$, if $\dim N(A^*_{\lambda_0}) > \theta$. Then in the case of $H = H_0$, we have

$$A^*_{\lambda_0} h = \theta, \quad h \in H,$$

that is,

$$A^* = \lambda_0 I.$$

So, there is non-trivial invariant subspace of A in this case. Hence, we can assume that $\{\theta\} \subsetneq H_0^{\perp} \subsetneq H$. Obviously,

$$\theta = \langle H_0^{\perp}, A^*_{\lambda_0} H_0 \rangle = \langle A_{\lambda_0} H_0^{\perp}, H_0 \rangle,$$

so that

$$g = A_{\lambda_0} h = Ah - \lambda_0 h \in H_0^{\perp}, \ h \in H_0^{\perp},$$

i.e.,

$$Ah = g + \lambda_0 h \in H_0^{\perp}, \ h \in H_0^{\perp}.$$

This shows that $H_0^{\perp}$ is a non-trivial invariant subspace of A. □

The theorem that we just proved shows that every operator A satisfying $\sigma(A) \setminus \sigma_N(A) \neq \emptyset$ has a non-trivial invariant subspace. Therefore, because of Theorem 5.2.11, the non-trivial invariant subspace problem is reduced to the problem concerning operators with pure narrow spectrum.

Chapter 6 Some Applications of Generalized Inverses

By the methods of metric generalized inverse in Banach space and Schauder fixed point theorem, X. L. Wang, H. Wang, G. Q. Liu and Y. W. Wang proved the existence of the least extremal solution of an ill-posed Neumann boundary value problem for semilinear elliptic equations in L^p $(1 < p < \frac{2n}{n-2})$ and gave a necessary and sufficient condition for a function to be the least extremal solution of the ill-posed Neumann boundary value problem. In order to present some theorems applications of generalized inverses, we need first to give some definitions that will be used in the first two theorems.

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded domain, and its boundary Γ be a C^2-manifold. We consider the Neumann boundary value problem

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u(x)}{\partial x_j}) = f(x, u(x)), & x \in \Omega, \\ \frac{\partial u}{\partial \nu_A}(x) = 0, & x \in \Gamma, \end{cases} \tag{6.1}$$

where $a_{ij} = a_{ji} \in C^{\infty}(\Omega)$ $(i, j = 1, 2, \ldots, n)$, and the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory conditions (see Chapter 2 in [118]), i.e.,

1. $x \to f(x, u)$ is measurable for all $u \in \mathbb{R}$;

2. $u \to f(x, u)$ is continuous for almost every $x \in \Omega$;

and the inequality

$$| f(x, u) | \leq a \mid u \mid^{\frac{p}{q}} + b(x), \quad \text{for a.e. } x \in \Omega, \ \forall u \in \mathbb{R}$$

where $a > 0,\ b \in L^q(\Omega),\ \frac{1}{p} + \frac{1}{q} = 1,$ and $\frac{2n}{n+2} \le q < \infty$. Then

$$\frac{\partial u(x)}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{i,j} \frac{\partial u(x)}{\partial x_j} \cos(\widehat{\nu, x_i}),$$

where ν is the unit outer normal vector on Γ and $(\widehat{\nu, x_i})$ is the angle between the vector ν and the direction x_i.

In general, the Neumann boundary value problem (6.1) is ill-posed. We should study the extremal solution or least extremal solution of the boundary problem. Now let

$$D(A) = \{u \in H^1(\Omega) \subset L^p(\Omega) \ :\ Au \in L^q(\Omega),\ \frac{\partial u}{\partial \nu_A} |_\Gamma = 0\},$$

where

$$\begin{cases} Au &= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij} \frac{\partial u(x)}{\partial x_j}), \\ \frac{\partial u}{\partial \nu_A} &= \sum_{i,j=1}^{n} a_{ij} \frac{\partial u(x)}{\partial x_j} \cos \widehat{(\nu, x_i)},\ u \in H^1(\Omega). \end{cases}$$

In this case, the Neumann boundary value problem (6.1) is equivalent to the following semilinear operator equation:

$$Au = F(u), \quad u \in D(A).$$

Theorem 6.1.1 [104] If $1 < p < \frac{2n}{n-2} (n \ge 3),\ \frac{1}{p} + \frac{1}{q} = 1,\ p < q < \infty$ and the conditions of the introduction in this section are satisfied, then there exists a least extremal solution of the Neumann boundary value problem for the semilinear elliptic equation (6.1).

Theorem 6.1.2 [104] Assume that the hypotheses in Theorem 6.1.1 hold. Then a function $u \in D(A)$ is the least extremal solution of the Neumann boundary value problem (6.1) if and only if u satisfies

$$\int_\Omega |u(x)|^{p-1} sgnu(x) dx = 0$$

and is the weak solution of the equation

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij} \frac{\partial u(x)}{\partial x_j}) = f(x, u(x)) - \frac{1}{|\Omega|} \int_\Omega f(x, u(x)) dx, & x \in \Omega, \\ \frac{\partial u}{\partial \nu_A}(x) = 0, & x \in \Gamma. \end{cases}$$

In 2009, Y. H. Zhao and Y. W. Wang studied the problem of the ill-posed operator equation $Tx = y$ with codim$R(T) = 1$ in normed linear spaces. The structure of the set of extremal solutions of the equation has been obtained by the maximal elements in $N(T^*)$ and the generalized inverse T^+ of T. Furthermore, the representation of the set of the extremal solutions of the equation is given formally.

Theorem 6.1.3[117] Let X, Y be Banach spaces, and let $T : D(T) \subset X \to Y$ be a closed densely defined linear operator. Suppose that codim$R(T) = 1$, $R(T)$ is closed, and $N(T)$ is topologically complemented in X. Then for any $y \in Y \backslash R(T)$, there exists an extremal solution x_0 of the ill-posed linear operator equation $Tx = y$ if and only if for any $y^* \in N(T^*) \subset Y^*$, y^* achieves its norm on $S(Y)$, i.e., there exists a $y_0 \in S(Y)$ such that

$$\langle y^*, y_0 \rangle = \|y^*\|,$$

where $S(Y) = \{y \in Y : \|y\| = 1\}$ is the unit sphere in Y.

Theorem 6.1.4[117] Let X, Y be normed linear spaces, and let T be a closed densely defined linear operator from X to Y. Suppose that codim$R(T) = 1$, $R(T)$ is closed, and $N(T)$ is topologically complemented in X . Let us define for any $y_0^* \in N(T^*) \backslash \{0\}$, let

$$S_0 = \{y_0 \in S(Y) : y_0 \ \text{ is the maximal element of } y_0^*, i.e., \langle y_0^*, y_0 \rangle = \|y_0^*\|\}.$$

Then for any $y \in Y \backslash R(T)$, we have:

1. $Tx = y$ has an extremal solution if and only if $S_0 \neq \emptyset$;

2. $Tx = y$ has at most one extremal solution in M $\cap$ $D(T)$ if and only if S_0 has at most one element, where M is a closed subspace of X such that

$$X = N(T) \oplus M;$$

3. If $S_0 \neq \emptyset$, then the set $Extr_y(T)$ of extremal solutions of the ill-posed linear operator equation $Tx = y$ is represented as

$$Extr_y(T) = \bigcup_{y_0 \in S_0} T^+[y - \frac{\langle y_0^*, y_0 \rangle}{\|y_0^*\|} y_0] + N(T),$$

where T^+ is the generalized inverse of T.

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