FROM QUESTIONS TO PROOFS

BETWEEN THE LOGIC OF QUESTIONS AND PROOF THEORY

DOROTA LESZCZYŃSKA-JASION

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Last but not least, all my gratitude and love to my children, to my beloved husband, and my husband's indispensable sense of humour.

Preface

This book gathers most of the results of my work on proof-theoretical tools of the logic of questions known as Inferential Erotetic Logic, especially those focused on the method of Socratic proofs.

Chapter 1 is an overview of the main achievements that belong both to the area of the logic of questions and proof theory. I describe the most important frameworks and indicate the place of my research against this background.

Chapter 2 presents the main characters: erotetic calculi for classical logic, both propositional and first-order, in various variants, and erotetic calculi for the class of the so-called basic modal logics. The method of Socratic proofs was invented by Andrzej Wiśniewski; it is a method of transforming questions concerning logical properties and relations such as validity or derivability, which is at the same time a proof method for an underlying logic. The method certainly lies at the very centre of the intersection of the logic of questions and proof theory. The erotetic calculi presented here are notational variants of the original calculi. I have decided to work on variants of the method with the unified notation consequently extended to the quantifier level, but I present both versions of the calculi. The chapter also contains an analysis of admissibility and derivability of erotetic rules which is basically of my authorship, as well as the algorithm for proof-search in the erotetic calculi for the first-order logic (though the idea is derived from Fitting, 1983).

The part of Chapter 2 devoted to modal logics is based mainly on my monograph (Leszczyńska, 2007), but the erotetic calculi are presented in a new version; the difference concerns the definitions of provisos of applicability of modal rules. I compare the two approaches in detail in Appendix A.

Chapter 3 presents the natural, indispensable environment of the method of Socratic proofs. Erotetic calculi are calculi of questions, and should be analysed as such; therefore, in this chapter, I recall the main ideas of Minimal Erotetic Semantics developed for this purpose by Andrzej Wiśniewski. The version of the notion of admissible partition for the first-order case which is contained in this book is new, while the definition of admissible partitions for modal logics is also of my authorship, but has been already presented in Leszczyńska, 2007. The discussion concerning erotetic implication is based on my research report Leszczyńska-Jasion, 2017. The remaining parts of Chapter 3, concerning the proof-theoretical settlement of the multiple-conclusion entailment relation, are published here for the first time.

Chapter 4 may be viewed as the main part of this monograph. It is focused on exploring relations between various proof systems by defining the so-called "translations" between them. More specifically, the chapter shows how to translate a Socratic proof—that is, a sequence of questions of a certain kind—into a proof of the relevant formula in other proof systems, such as sequent calculus or analytic tableaux. Sections 4.1 and 4.2 are based on Leszczyńska-Jasion et al., 2013. The case of analytic tableaux described in Sections 4.5 and 4.6 was prepared by myself earlier in the form of a research report (Leszczyńska-Jasion, 2015). During work on this report I used the original versions of erotetic calculi by Wiśniewski, and I was planning to modify the results contained in Leszczyńska-Jasion, 2015 *via* the notational differences introduced here. However, I then realised that some specific details of the translation would be lost, namely the games one has to go into with the apparently transparent double-classical-negation, and therefore decided to leave it in the original version. The last subsection 4.7 is partially based on Chlebowski and Leszczyńska-Jasion, 2015. The remaining parts of Chapter 4, namely Sections 4.3 and 4.4, are published for the first time.

The various parts of this book were written over the last, more or less, five years. Another monograph was also written during this period—a doctoral dissertation by Szymon Chlebowski (Chlebowski, 2018); I was the thesis' technical supervisor, hence many parts of the two monographs have been discussed jointly. Therefore there is a certain convergence of topics and techniques used in the two monographs. Although the results are properly separated, this may give the impression that one of the books complements the other. This is indeed the case.

I have strived to write the monograph in a self-contained way. Nonetheless, some basic background in algorithms (like the ability to read pseudocode) may be needed, and basic knowledge in the field of modal logics is necessary to read Chapter 4. If needed, the reader should consult Cormen et al., 2009; Hughes and Cresswell, 1996.

Needless to say, I feel solely responsible for all the mistakes this book contains.

Chapter 1

Between the logic of questions and proof theory. An overview

1.1. Introduction

The relation between questions and proofs is of deep philosophical significance. Both the ability to pose the right questions and the capacity to provide arguments in their strictest and most indisputable form—the form of proofs—are hallmarks of human intelligence and creativity.

Despite the fact that both disciplines: that is the logic of questions and proof theory, have a long-standing tradition and numerous widely recognised results, the idea of working strictly in the two areas seems relatively novel. In this chapter we draw a panorama of research conducted at the junction of the two disciplines and we place there the issues undertaken in this book.

The logic of questions is a branch of philosophical logic that develops formal models of many diverse aspects of the way questions function in natural language. The logic of questions is often called "erotetic logic", from the Greek word "erotema" which means "question". In the early '80s Groenendijk and Stokhof would write:

> There is a vast, and rapidly growing, literature on questions and question-answering. The subject has had the long-standing and almost continuous attention in many areas of study, including linguistics, logic, philosophy of language, computer science, and certainly others besides. (Groenendijk and Stokhof, 1984, p. 211)

During the almost 40 years since then, the subject has not ceased to attract researchers' attention, while the growth in literature has not slowed down, which shows how difficult it is today, if at all possible, to present a complete and comprehensive account of the discipline. Some choices are inevitable. We try to explain those made here.

There are a number of very important issues in the logic of questions that will *not* be discussed in this book: types of question, the answerhood property, types of answer, partial answers, replies and answers, the kind of objects questions are, effectiveness and completeness of question-answer systems, operations on questions, presuppositions of questions, and many others. The reader can find many discussions of these topics elsewhere: (Harrah, 2002), (Belnap and Steel, 1976), (Wiśniewski, 1995), (Wiśniewski, 2015), (Groenendijk and Stokhof, 2011), (Ginzburg, 2011), (Brożek, 2011), there are also well-written articles available in online encyclopaedias:[1] (Harrah, 1998), (Cross and Roelofsen, 2018).

Most logical theories of questions distinguish between *interrogative sentences* and *questions*. The former are expressions of a language and the latter are their meanings, just as *propositions* are assumed to be the meanings of *statements*. Moreover, since the influential paper by Hamblin (Hamblin, 1958) and the book by Belnap and Steel (Belnap and Steel, 1976)[2] the standard has been to explicate the meaning of an interrogative in terms of its set of possible answers. Although this elementary distinction, interrogative/question, plays its role in the philosophical and logical analysis of natural language, we shall not use it here. The main reason for this simplification is that we are not trying to tackle the problem of what questions (the meanings) really are, and neither do we refer to pragmatics, where the distinction between interrogatives and questions is needed. So we will speak of questions as expressions of language.

We will use the notion of *direct answer to a question*. As put by Harrah, a direct answer "gives exactly what the question calls for", it is logically sufficient and immediate (see Harrah, 2002, p. 1).[3] In the formal framework presented here, direct answers are statements; we do not model any other kinds of possible replies to questions. It is fully

[1] The reader should be careful here—the focus of the entry (Cross and Roelofsen, 2018) on the inquisitive semantics approach seems disproportionate with respect to the other frameworks.

[2] It is widely acknowledged that (Belnap and Steel, 1976) was the first monograph developing the logical account of questions, although the book by Tadeusz Kubiński (Kubiński, 1971) was published prior to it. The latter was written in Polish, and before the English-language version (Kubiński, 1980) was published, Belnap's account became a standard.

[3] See also Wiśniewski, 2015, where the author uses the notion of *principal possible answer* in this context.

compatible with the first of the so-called Hamblin's postulates (Hamblin, 1958, p. 162):

Hamblin's postulates

1. An answer to a question is a statement.
2. Knowing what counts as an answer is equivalent to knowing the question.
3. The possible answers to a question are an exhaustive set of mutually exclusive possibilities.

Contrary to some popular and influential frameworks (like Belnap and Steel, 1976, or inquisitive semantics approach—references will be given below), we do not assign a logical value to a question. Answers to questions can be true or false, whereas questions are *sound* or *unsound*. Usually it is assumed that a sound question is one which has a true direct answer, while an unsound question is one which has no true direct answer, such as:

- *Is Ann in the bedroom or in the living room?*

asked in a situation when Ann is in the kitchen. In a formal account the notion of soundness is relativized to some sort of a model, just like the notion of truth is relativized to a model. (Different models may be viewed as different formal descriptions of possible situations.) Moreover, a question which is sound in every possible model is called *safe*, otherwise it is *risky*.

Set-of-answers methodology

In this monograph we shall follow the *syntactic*, or *semi-reductionistic* approach to questions (see Wiśniewski, 2013, Chapter 2, especially Section 2.3). This means that we identify questions with expressions of a formal language of, more or less, the following form:[4]

$$?\{A_1, \ldots, A_n\}$$

where $A_1, \ldots, A_n$ are all the direct answers to the question, among which there are at least two different direct answers. The set-of-answers

[4] In this book we limit ourselves to questions with *finite* sets of direct answers, but, in general, the set-of-answers methodology does not preclude infinite sets of answers.

methodology (*SA-methodology*, for short) is often viewed as a methodological explication of the second postulate by Hamblin. It is neutral with respect to the possible answer to a question "what is a question", since it is compatible with, but does not entail, the entire reduction of questions to sets of answers.

The main advantage of SA-methodology is the giant step towards formalization. It opens the possibility of using the semantic apparatus defined for the statements in defining relations between questions.

Proof theory

Proof theory is a well-established discipline that models the process of argumentation, especially of proving, by means of formal systems in which proofs are constructed.[5] Sometimes it is said that its aim is to examine proofs as formal objects. The author of this book prefers to think that it is the process of proving (constructing a proof) that is being examined and formalized.

To a great extent, this book is a result of reflection on the merits that the logic of questions can gain by using methods specific to proof theory. But it also works the other way round: the content of Chapter 4 of this book shows how proof theory can benefit from using questions.

The aim of this book

The primary aim of this book is to provide a systematic presentation of all the important achievements that the logic of questions has gained while moving towards proof theory. The attention is focused on, but not limited to, proof-theoretical tools of Inferential Erotetic Logic; in this chapter many other developments are discussed. In the remaining chapters the author limits herself exclusively to Inferential Erotetic Logic.

Preliminaries

We use double quotation marks (*i.e.*: " ") as quotation marks and single quotation marks (*i.e.*: ' ') in two roles: to indicate that an expression is mentioned (not used) and as Quinean corners. We do not use them

[5] There are many handbooks and well-written articles about proof theory, we recommend (von Plato, 2016), (Buss, 1998b), (D'Agostino, 1999), (Fitting, 1990), (Priest, 2008), (Indrzejczak, 2013, 2014) (in Polish), (Negri and von Plato, 2001), and for more advanced readers: (Smullyan, 1968), (Troelstra and Schwichtenberg, 2000), (Buss, 1998a), (Indrzejczak, 2010), (Avron, 1993), (Jarmużek, 2013) (in Polish).

whenever there is no risk of a misunderstanding. Generally, the first priority is readability, then preciseness.

We will use '$\Box$' to indicate the end of a proof, the end of a definition, and the end of an example.

Notation: we use '$\varnothing$' for the empty set, '$\subseteq$' for set inclusion and '$\subset$' for proper set inclusion. If X is a set, then the symbol '2^X' is used for the power set of X, that is, the set of all subsets of X. The abbreviation "iff" is used for "if and only if".

1.2. Inferential Erotetic Logic

From the very beginning, Inferential Erotetic Logic (let us use 'IEL' from now on) was developed as an alternative to the "received view", that is, mainly the one developed in Belnap and Steel, 1976. The book created a paradigm in the logic of questions which was vital in the eighties. According to this paradigm *it is not* the aim of the logic of questions to work on inferential systems. There is no deductive system to be discovered (created), since questions (interrogatives) do not serve as premises and conclusions—this is the standpoint of Belnap and Steel which was definitely rejected by Wiśniewski.[6] According to Wiśniewski, there are important cognitive processes with questions involved, and the primary goal of IEL is their logical analysis. More precisely, IEL defines different types of erotetic reasoning, that is, reasoning in which questions play the role of premises and/or conclusion, and explicates criteria of correctness of such reasoning. The main notion introduced for this purpose, which will be defined and used also in this monograph, is that of *erotetic implication*.[7]

There are two conditions defining the notion of erotetic implication. First, if question Q_1 erotetically implies question Q_2, then soundness of the first question, Q_1, warrants soundness of the second question, Q_2. This means that if Q_1 is *well posed* in a given situation, then so is Q_2. Or in other words, if in a given situation it is reasonable to ask Q_1, then it is also reasonable to ask Q_2. For example, *if* question:

[6] For a concise introduction to IEL see Wiśniewski, 2001. The book (Wiśniewski, 1995) presents the results obtained until the early nineties, and (Wiśniewski, 2013) presents the latest account of IEL.

[7] The reader may consult any of: Wiśniewski, 2001; Wiśniewski, 1994 or Wiśniewski, 2013. We give a formal definition of the notion of erotetic implication in Chapter 3.

- *Is Ann in the bedroom or in the living room?*

is sound in a given situation, when we can hear Ann's voice, then so is:

- *Does her voice come from the bedroom or from the living room?*

in the same situation. The second condition defining the notion of erotetic implication amounts to the fact that Q_2 is *asked for a purpose*: every direct answer to Q_2 must bring one closer to answering Q_1.

IEL adopts the SA-methodology. Let us recall that it is in line with the first and second postulate of Hamblin. On the other hand, the only restriction put on the direct answers to a question $Q = ?\{A_1, \ldots, A_n\}$ is that they differ syntactically, that is, any two direct answers among $A_1, \ldots, A_n$ are *different expressions*. There is nothing in this restriction that would yield that $\{A_1, \ldots, A_n\}$ is a set of "mutually exclusive possibilities". Hence IEL does not follow the third postulate by Hamblin, which is, by the way, conceived as the most controversial—see Harrah, 2002, p. 5.

IEL provides one of the predominant contemporary frameworks for systematic logical reflection on questions. The others are the *Interrogative Model of Inquiry* created by Hintikka, the paradigm based on *inquisitive semantics* started by Groenendijk and Stokhof, and an epistemic account of the logic of questions, which was also initiated by Hintikka, and is nowadays represented by many researchers.[8] (Let us stress that we are focusing on *logical* approaches.) The paradigms certainly influence each other. For example, IEL was in many respects inspired by Hintikka's account of the logic of questions as a general theory of reasoning, but on the other hand, it was developed as an alternative to the Interrogative Model of Inquiry (see, for example, Wiśniewski, 2001). The epistemic approach to questions developed by Czech logicians is clearly inspired by IEL (see Peliš, 2016), and there are also less transparent, but still significant, interconnections between the inquisitive and other paradigms (see Wiśniewski and Leszczyńska-Jasion, 2015; Łupkowski, 2015; Hamami, 2015 for the interconnections).

Over the two and a half decades and more of its development, certain important proof systems grounded in IEL have emerged and evolved; the main achievements in this area are the method of Socratic proofs and the method of synthetic tableaux.

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[8] The ideas of the Interrogative Model of Inquiry are best represented in Hintikka, 1999; for the inquisitive semantics paradigm see Ciardelli et al., 2015; Ciardelli and Roelofsen, 2011 and also Ciardelli, 2016. Among the representatives of the epistemic approach one should indicate Peliš, 2016. We give more references in the sequel.

The method of Socratic proofs

The method of Socratic proofs offers a formal explication of the idea of answering questions by pure questioning. In the realm of the very logic of questions it is a formal method of transforming questions by the so-called “erotetic rules”, which analyse the logical structure of a question and, in a way, simplify this structure, while still preserving certain important semantic properties of the transformed question. The erotetic rules form erotetic calculi. In the realm of proof-theory, each erotetic calculus constitutes a proof system for a certain specific logic. Thus the method of Socratic proofs is a proof method. More specifically, it is a proof method which enables answering questions concerning, for example, the validity of a formula by transforming the questions into a form which may be called “rhetoric”; an answer to such a rhetoric question may be considered obvious, because the question concerns some fundamental, well-know properties of validity in a given logic.

The method has been introduced in Wiśniewski, 2004, adjusted to the first-order level in Wiśniewski and Shangin, 2006 and Wiśniewski, 2006, and developed further in: Wiśniewski et al., 2005; Leszczyńska, 2004; Skura, 2005; Leszczyńska, 2007; Leszczyńska-Jasion, 2008, 2009; Chlebowski and Leszczyńska-Jasion, 2015; Chlebowski, 2018; Szczepiński, 2018. Some applications of the method are analysed in Leszczyńska-Jasion et al., 2013; Chlebowski et al., 2017. An exposition together with intuitive background of the method may be found in Wiśniewski, 2013. The next chapter describes the method in detail, and the last one is focused on applications of the method concerned with translations into other proof systems.

It is worth emphasizing that the translations *via* erotetic rules are possible due to their *semantic invertibility*. Generally speaking, semantic invertibility is a property complementing *semantic correctness* (the latter is also called *adequacy* or *soundness* of a rule). If a rule is semantically correct, then one cannot go from a true (or tautological) premise to a false (not tautological) conclusion.[9] If a rule is semantically invertible, then one cannot go from a false premise to a true conclusion. Thus correct rules transform truths only into truths, and invertible rules derive truths only from truths. In the realm of erotetic calculi, semantic correctness of a rule amounts to transmission of soundness of a ques-

[9] There is an important difference between transmitting validity and transmitting truth. The second property, that is, transmitting truth, is stronger. We deliberately simplify the account here.

tion from a premise to a conclusion, and semantic invertibility of a rule amounts to transmission of soundness of a question from a conclusion to a premise.

Due to the properties described above the method of Socratic proofs enables transforming questions in a way which warrants that all semantic information is transmitted from question to question. This property makes backtracking redundant, hence the source of possible merits for implementation. Last but not least, the semantic property of invertibility may be viewed as just another consequence of the fact that erotetic calculi *are calculi of questions* designed to maintain the property of erotetic implication between questions.

The method of synthetic tableaux

From the proof-theoretical perspective, the method of synthetic tableaux belongs to a different family of methods than the method of Socratic proofs. While the method of Socratic proofs is to some extent based on sequent calculi, the method of synthetic tableaux creates a class on its own, staying in close proximity with the standard method of truth-tables, or with resolution. The reader will find the original account of the method in Urbański, 2001a, 2002b. Further developments of the method are described in Urbański, 2002a, 2004, 2005, 2001b, and applications of the method in the domain of abduction may be found in Urbański, 2010; Komosinski et al., 2012; Komosinski et al., 2014.

By and large, a synthetic tableau is created by taking two complementary, mutually exclusive assumptions, like 'p' and 'not-p', and deriving from them conclusions by the so-called synthetic rules, that is, rules that build complex sentences from simpler ones. The aim is to derive the same conclusion from each set of assumptions taken at the start. If a propositional formula (an analogue of a statement, but in a formal language) can be derived from each such set of assumptions, then it must be a law of logic.

Contrary to appearances, the method of synthetic tableaux is not a tableau method in the widely accepted meaning of the term, but we shall not go into detail here, as that would be going beyond the scope of this book. But it is worth explaining here that the method has been invented and developed partially to serve in analysis of *erotetic search scenarios*, which may be viewed as erotetic decision trees. Roughly speaking, an erotetic search scenario shows which questions should be asked, and when they should be asked, in order to solve an initial problem. It starts with a question expressing the initial problem, and each subsequent question is erotetically implied by some previous one.[10] An erotetic search scenario contains questions and declarative sentences, and a synthetic tableau may be viewed as a declarative part of a scenario. Thus, in a way, the method of synthetic tableaux may be considered a tool for generating part of an erotetic search scenario. Nevertheless, the method constitutes an achievement on its own.

1.3. The logics of questions: the main paradigms

Below we briefly describe other important paradigms in the contemporary logic of questions.

Interrogative Model of Inquiry

Let us state here that Jaakko Hintikka was one of the most influential philosophers of the second half of the 20th century. His contributions were crucial to formal and mathematical logic, non-classical logic, epistemology and philosophy of science, and probably many other disciplines. His Interrogative Model of Inquiry (IMI, for short) has been a source of inspiration for a couple of decades. To put it in Hintikka's words:

> Its basic idea is the same as that of the oldest explicit form of reasoning in philosophy, the Socratic method of questioning or *elenchus*. In it, all new information enters into an argument or a line of reasoning in the form of answers to questions that the inquirer addresses to a suitable source of information. (Hintikka, 2007, p. 18)

Hintikka models scientific inquiry as a question-driven process, which may also be conceived as a game between the Inquirer, who asks questions, and the Nature, who answers. The logic of questions, as well as mathematical game theory, are essential tools in developing IMI.

The logic of questions developed by Hintikka is built with the use of epistemic logic. According to the epistemic (or imperative-epistemic) approach, a question is an epistemic request, which is to be satisfied by an answer. Thus, for example, the question:

[10] For more information on erotetic search scenarios see Wiśniewski, 2003; Urbański, 2001b; Łupkowski, 2016—to mention the most important publications concerning the issue.

- *Is Ann in the bedroom or in the living room?*

expresses the following request (or imperative):

- *Bring it about that I know that Ann is in the bedroom or I know that Ann is in the living room.*

or

- *Make it the case that I know that Ann is in the bedroom or that I know that Ann is in the living room.*

In this account, the epistemic state described by the sentence:

- *I know that Ann is in the bedroom or I know that Ann is in the living room.*

is called the *desideratum* of the question. The modal "I know that" is modelled by modal epistemic logic, that is, a modal logic in which the necessity operator '$\Box$' is read as knowledge.

IMI gave rise to interrogative tableaux (described below) and also to imperative-epistemic approach to questions.

Epistemic approaches to questions

This popular trend was actually initiated by Lennart Åqvist, and then developed further by Hintikka. Åqvist formulated a theory of epistemic requests in the sixties; see the summary and bibliography in Harrah, 2002. The approach to questions described above is often called the "Make Me Know" approach, as it takes questions to be synonymous with an imperative of the form "Let it be the case that I know ..." or "Make it the case that I know ..." (see Harrah, 2002, p. 24).

The epistemic analysis, as well as many of Hintikka's other ideas, have influenced also Groenendijk and Stokhof, the authors of the inquisitive semantics, which is referred below. But at the moment, let us very briefly describe "epistemic extensions" of IEL developed, *int.al.*, in Peliš, 2016, Peliš and Majer, 2011; Peliš and Majer, 2010; Švarný et al., 2014. The representatives of this approach borrow certain concepts from IEL and redefine them in an epistemic setting. This makes it possible to model the questioning-answering process taking into account the questioner's knowledge and ignorance. Interestingly, the approach is further extended with the apparatus of public announcement logic, which makes the whole approach a dynamic one.

Among the interesting and still fresh ideas developed in this framework let us mention that of *askability*. Askability is a much more refined version of soundness of a question. A question is *askable* iff three conditions are satisfied (Peliš, 2016, p. 66):

1. **Non-triviality**: It is not reasonable to ask a question if the answer is known.
2. **Admissibility**: Each direct answer is considered as possible.
3. **Context**: At least one of the direct answers must be the right one.

The three conditions are further explicated on the grounds of modal logics, and further *epistemic erotetic implication* is defined in terms of *transmission of askability*; that question Q_1 implies question Q_2 in this sense means that whenever the conditions of askability of Q_1 are satisfied in a given situation, then the conditions of askability of Q_2 are satisfied as well in this situation (*cf.* Peliš, 2016, p. 71). We find this idea appealing.

Inquisitive semantics

The book (Groenendijk and Stokhof, 1984) contains an exposition of many ideas underlying the development of inquisitive semantics. The motivations follow from linguistics and philosophy of language, from theory of meaning and understanding.

Here are the main assumptions underlying this account. The fundamental difference between a statement and a question is in their *content*, not in their form. Every sentence has an *informative content* and an *inquisitive content*. If the first content, that is, the information carried by a sentence, is sufficient to resolve the issue it raises, that is, its inquisitive content, then the sentence is a statement, otherwise it is a question. For example, the sentence:

- *Ann is in the living room.*

raises no issue that is unsettled by its informative content, whereas:

- *Is Ann in the bedroom or in the living room?*

raises an issue it cannot resolve. For this reason the former sentence is a statement, whereas the latter is an interrogative (a question). According to this approach, the sentence:

- *Ann is in the bedroom or in the living room.*

has the same informative and inquisitive content as the aforementioned question, and hence it is also a question.[11]

To sum up, in the inquisitive-semantics approach, on the level of formal language the categories of statements and questions are not defined syntactically.

The central notion of inquisitive semantics is that of *support* of a statement by an information state. Support generalizes the traditional concept of truth. Technically, it is defined on the grounds of semantics of modal logics, where an information state is a set of possible worlds. For example, the statement:

- *Ann is in the bedroom.*

is supported by each information state which consists solely of possible worlds where Ann is actually in the bedroom (one may think of a possible world as a description of how the world could be).

The generalization of truth by the notion of support allows for the creation of a framework in which the semantic properties of declaratives and questions are analysed by the same tools. The representatives of inquisitive semantics take a position regarding ascribing truth/falsity to questions which is the opposite to the position adopted here. They follow Belnap and take questions to be true or false in a given information state. Roughly speaking, the question:

- *Is Ann in the bedroom or in the living room?*

is true in a given state if the state settles the question, that is, supports one of its direct answers: either in each world of the state Ann is in the bedroom or in each world of the state Ann is in the living room.

Interestingly, the relation of entailment defined in this framework unifies the following logical notions: (i) a statement being a logical consequence of another statement, (ii) a statement logically resolving a question, (iii) a question logically presupposing a statement, (iv) a question logically determining another. The last relation is also called *dependency* in this framework; we shall go back to dependency in a moment.

Let us state that it is not the aim of the author of this book to convince the reader that the framework of IEL is the only one relevant

[11] There are numerous differing accounts of the matters among the inquisitive-semantics researchers, we simplify this presentation out of necessity. Our presentation is based mainly on (Ciardelli, 2016).

for the aims that the author has set. It is possible that it is not the case. But it is a fact that the author has worked in this framework, and that the results contained in this monograph have been obtained by using tools from IEL (as far as the logic of questions is concerned). It is also a fact that all the other frameworks described above are appealing.

There are numerous results that come from combining questions and proofs in one framework. In order to provide a systematization of these, we propose to look at the results through the prism of the main goals that the researcher has set. Therefore we propose to distinguish the following goals.

- $\mathbf{G_1}$ Designing problem-solving strategies with reference to the questioning-answering process.
- $\mathbf{G_2}$ Defining relations between questions that may be called *inferential* or *deductive*, but also purely semantic analysis of these relations.
- $\mathbf{G_3}$ Formal study of structures containing questions. Just as proofs, that is structures containing statements, are objects of formal study, so too are structures containing questions.

We elaborate the distinction below.

1.4. Questions and problem-solving strategies

The common feature of the developments described in this section is that they analyse how to use questions in order to get a solution of a problem, or to obtain a proof that something is true. Often a formal model of an information-seeking process, or knowledge-seeking process, is involved, and its description includes both declarative and interrogative steps. We describe some of such developments in chronological order.

Interrogative tableaux

As we have explained above, IMI (*Interrogative Model of Inquiry*) developed by Hintikka aims to model rational inquiry in science, and it does so by, *int.al.*, formulating rules of question-and-answer procedures in the style of tableau systems.

Interrogative tableaux were proposed, *int.al.*, in Hintikka, 1992, developed in Harris, 1994 in terms of game-theoretical semantics, then proposed in a more expanded form in Hintikka et al., 1999. Proof-theoretically, the system is based on Beth's semantic tableaux[12] or on a sequent calculus, but it is extended with the possibility of considering different answers to questions asked during inquiry. Generally speaking, an interrogative tableau indicates the moves of Nature and the Inquirer. The declarative steps are guided by standard tableau rules; for example, if the Inquirer finds a statement of the form '$A\&B$' in her part of the tableau, then she may add both A and B to this part of the tableau. Interrogative moves are guided by the interrogative rules, and they amount to, first, asking questions the answers to which may be included in the appropriate part of a tableau, and, second, considering the results of adding the different answers.

Since (Harris, 1994) it has been known that interrogative tableaux are subject to game-theoretical interpretations. The interpretations have been further developed, *e.g.*, in Genot, 2009, Genot and Jacot, 2010.

Erotetic search scenarios

In a way, erotetic search scenarios (*e-scenarios*, for short) explicate closely related ideas. As we have roughly explained above, an e-scenario for a given problem is a tree with a question expressing the initial problem in the root, a set of declarative premises under the root and some further questions, each of which is erotetically implied by some previous one. Some of them, called *queries*, are answered, and the e-scenario shows what to do next if such-and-such an answer to a query is arrived at. Each branch of the tree ends with a direct answer to the initial problem.

The basic improvement with respect to interrogative tableaux is that asking queries in an e-scenario is, in a way, guided by the relation of erotetic implication, whereas in the case of interrogative tableaux no such guidance is provided.

The idea of e-scenarios was introduced in Wiśniewski, 2003, and examined and/or applied in Urbański, 2001b; Urbański and Łupkowski, 2010; Łupkowski and Leszczyńska-Jasion, 2015; Leszczyńska-Jasion and Łupkowski, 2016; Łupkowski, 2016, 2017; see also Wiśniewski, 2013, Part III.

[12] The system presented by Beth (see Beth, 1969) was a first intuitive, semantically motivated proof method. It is a proof system in which a formula is analysed in a tableau divided into columns with true formulas and false formulas. A formula to be considered is placed in the "false" column, and then one tries to derive a contradiction.

Natural deduction systems for inquisitive logics

Natural deduction systems for inquisitive propositional logic InqB and certain fragments of the first-order system InqBQ are presented in Chapters 3 and 4 of (Ciardelli, 2016). Interestingly, interrogative tableaux, e-scenarios and natural deduction systems for inquisitive logics may be viewed as formalizing the same idea. In Ciardelli's words:

> [Q]uestions have a very important role to play in inferences: they make it possible to formalize arguments involving generic information of a certain type, such as *where Alice lives* [. . .] questions may be used as placeholders for arbitrary information of the corresponding type. By manipulating such placeholders, we may then provide formal proofs [. . .]. (Ciardelli, 2016, p. 77)

Roughly speaking, in the context of interrogative tableaux, "manipulating placeholders" is tantamount to analysing the result of adding different direct answers to a suitable part of a tableau, and in the e-scenarios setting, it is tantamount to choosing a specific set of direct answers to the queries asked in an e-scenario. Actually, "choosing a specific set of answers" is described as an operation of *contraction* performed on an e-scenario.[13]

Both in the (classical) propositional and in the first-order case the system is based on natural deduction, with special rules for the inquisitive connectives characterizing questions in this framework.[14] The rules of the system are expressive enough to generate proofs (or, rather, we would say *derivations*) in which a question is derived as the final conclusion from assumptions among which one may have both statements and questions. The main result concerning this system is the inductive definition of a procedure which, for each such proof deriving a question, and for each choice of answers (called resolutions) to questions occurring in the proof as premises, transforms it into a proof of a specific answer (resolution) to the final question with statements as premises. The whole enterprise leads to an interesting computational interpretation of proofs

[13] See Wiśniewski, 2013, Part III, especially the last chapter titled "E-scenarios and Problem Solving".

[14] Once again, it is a far-going simplification for the purpose of this presentation. Actually, the inquisitive connectives behave like intuitionistic connectives, and there is a clear (and well-examined, see Ciardelli, 2016, Chapter 3) correspondence between inquisitive logic and intuitionistic logic. But anyway, there is more in the inquisitive natural deduction system than that.

containing questions, "reminiscent of the proofs-as-programs interpretation of intuitionistic logic" (Ciardelli, 2016, p. 77).

1.5. Questions and relations between them

Once again, we describe the most important results in this area in chronological order.

Back in the fifties Hamblin has already suggested defining a relation of *containment of questions* (see Harrah, 2002). The idea was that one question contained the other when one could deduce an answer to the second question from every answer to the first. This general idea of resolving one question by answering another is vital and vivid in many other definitions of relations between questions.

Belnap (see Belnap and Steel, 1976 or Harrah, 2002) defines the relation of *erotetical equivalence* between interrogatives. Two interrogatives are in relation to one another iff for every direct answer to one of them there is an equivalent direct answer to the second interrogative, and *vice versa*: for every direct answer to the second interrogative there is a direct answer to the first one which is equivalent to it.

Belnap also defined a *propositional implication* which is a generalization of entailment between statements. Belnap calls an interrogative *true in an interpretation* iff some direct answer to the interrogative is true in the interpretation. Then one can define *propositional implication* between a set X of *quasi-wffs* (that is, expressions which may be propositional formulas or interrogatives) and a quasi-wff A as the transition of truth: X implies A in this sense iff A is true in every interpretation in which every element of X is true.

As mentioned before, Kubiński was probably the first logician who considered systems of questions with the aim of describing diverse relations between questions and possibly defining deductive moves between questions (see Wiśniewski, 2016, 1997). Probably the most important relation defined by Kubiński is that of the *equipollence of questions*. Two questions are equipollent iff there exists a bijection (that is, a function which settles a one-to-one correspondence of elements) that to each direct answer to one question assigns an equivalent direct answer to the second question. Observe that the existence of a bijection makes the relation *stronger* than the relation of erotetical equivalence defined by Belnap. Kubiński also considered, among many others, relations of being *weaker than* and being *stronger than* between questions, and formulated systems of questions to capture these relations.

Wiśniewski, inspired (among others) by Kubiński, defined, *int.al.*, two important relations: evocation of a question by a set of declarative premises and erotetic implication. We have already described the latter so let us turn to the former. The relation of evocation explicates the criteria of validity of inferences in which a question *is posed* on the basis of some declarative premises (statements), like in:

- *The Twin Prime Conjecture*[15] *is one of the still unsettled problems of number theory. It's fascinating because of the simplicity of its expression. Hm. Number* n *is a prime. Is* $n-2$ *or* $n+2$ *prime as well?*

Evocation is defined by two conditions: first, if in a given situation the premises are true, then the question *evoked* by them is sound, but on the other hand, and this is the second condition, no direct answer to the evoked question follows from the premises. This is the case in the above example. First, the question whether $n-2$ or $n+2$ is prime is sound (it is safe, actually), and second, no answer to it follows from the premises. On the other hand, no answer to the question can lead to a solution of the famous number theory problem; but nevertheless, the problem *triggers* the question. As in the case of erotetic implication, the notion of evocation seems to capture a frequently occurring and important cognitive phenomenon.

Finally, the notion of *dependency of questions* is defined in the framework of inquisitive semantics. The relation holds between question Q_1 and Q_2 whenever settling the first question, Q_1, entails settling the second. In this situation we say that Q_2 *depends on* Q_1. For example, question:

- Q_2 = *Is Ann in the bedroom or in the living room?*

depends in this sense on question:

- Q_1 = *Does her voice come from the bedroom or from the living room?*

Let us observe that the relation has, in a sense, *opposite direction* to erotetic implication. Namely, dependency holds between Q_1 and Q_2 in

[15] The Twin Prime Conjecture asserts that there are infinitely many twin primes, that is, primes differing by 2, like 3 and 5, 5 and 7, 11 and 13...

our example, whereas the "natural" direction of erotetic implication is rather from Q_2 to Q_1.[16] Although this is ostensibly but a shallow observation, it actually shows something more significant. In the inquisitive framework the more specific information must come first before the more general information. Observe that, in a way, it is natural for the order of entailment, and the relation of dependency is a special case of the inquisitive generalized notion of entailment. That is to say, in the inquisitive framework the direction of entailment (dependency) between questions is driven by the issue:

- ***inquisitive dependency direction:*** *What question can be settled by information settling Q?*

On the other hand, in the erotetic framework the direction of erotetic implication between questions is driven by the issue:

- ***erotetic implication direction:*** *What question should I ask in order to gain information which will settle Q?*

Further refinements of the notion of dependency are possible, such as the notion of *compliance* which "judges whether a certain dialogue move is coherent with respect to previous moves" (Groenendijk and Roelofsen, 2009). However, bringing the notion closer in would force us to introduce a massive portion of inquisitive semantics, and that would lead us far astray from the topic of this book.

1.6. Questions and systems of questions

The remaining content of this monograph is definitely located in the third area of $\mathbf{G_3}$. It is our aim to study systems (calculi) of questions, especially sequences (structures) of questions generated within such systems, just like formal proofs are studied. We use the concepts, tools and methods specific to proof theory but our interest is in question transformations.

When we go deeper into the proof-theoretical considerations, it turns out that the method of Socratic proofs has some properties that make it suitable for proof translations, *i.e.*, for defining functions that transform

[16] Erotetic implication holds between Q_2 and Q_1 on the basis of the additional declarative premise: that one can hear Ann's voice coming from the bedroom or from the living room.

a Socratic proof of sequent $\vdash A$ into a proof of sequent $\vdash A$, or of formula A, in some other system. As we have already explained, the main feature which makes the method of Socratic proofs suitable for this kind of translation is the invertibility of rules, referring to *whole structures* contained in questions. Typically, the structure is a finite sequence of sequents of some formal language, and each such sequence preserves all of the semantic information contained in a given problem.

However, before we continue, let us mention some other results that we would definitely place in the area of $\mathbf{G_3}$. The very problem of defining systems of questions capturing certain inferential relations between them, as undertaken by Kubiński, is certainly a good example of a $\mathbf{G_3}$-kind of enterprise. The intriguing problems of effectiveness and completeness—analogues (more or less) of similar problems in the realm of declaratives—are also good examples of problems that one may encounter when studying systems of questions in a formal way. For example, Harrah's incompleteness theorem states that if the set of questions of a formal language is effectively enumerable, and also the set of direct answers to a question of the language is effectively enumerable, then *not every* set of sentences of the language can be the set of direct answers to a question.[17]

Obviously, the notion of completeness may be understood in many ways. One may be, for example, tempted to give a *complete* axiomatic account of a system capturing a certain relation involving questions, and one can do so with success: see, for example, Skura and Wiśniewski, 2015 and Wiśniewski, 2016, where axiomatic accounts for evocation are developed. In Chapter 3 we discuss the issue of erotetic implication and observe that this relation deserves axiomatic descriptions but still lacks one. To put it in Harrah's words (Harrah, 2002, p. 10) "[t]his area invites and awaits exploration". It certainly does!

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[17] For a more precise account see Harrah, 1969; Harrah, 2002, Section 3 and Wiśniewski, 1995, p. 98. See also Wiśniewski and Pogonowski, 2010 for a strengthening of Harrah's incompleteness theorem.

Chapter 2

Erotetic calculi for classical logic and for modal logics

In this chapter we describe the most representative erotetic calculi. As we have mentioned in the previous chapter, erotetic calculi are calculi transforming questions of certain formal languages. The aim of the transformation is to solve the problem expressed by the initial question, but at the same time the erotetic rules constitute an inferential relation between questions whose semantic counterpart is a special case of erotetic implication.

We start with the classical case. First, we introduce the formal languages, all the necessary conventions, and three erotetic calculi for classical propositional logic (CPL for short) and classical first-order logic (FOL for short).

Thus $\mathcal{L}_{\mathsf{CPL}}$ stands for the language of CPL which contains the following symbols: countably infinitely many propositional variables: p_1, p_2, p_3 ..., propositional connectives: $\neg, \wedge, \vee, \rightarrow$ and brackets: (,). We will use p, q, r for propositional variables of $\mathcal{L}_{\mathsf{CPL}}$ and p_i as a metavariable. VAR stands for the set of all propositional variables of $\mathcal{L}_{\mathsf{CPL}}$. The notion of formula of language $\mathcal{L}_{\mathsf{CPL}}$ is given by the following definition.

Definition 1 (formula of language $\mathcal{L}_{\mathsf{CPL}}$).

1. *Each element of VAR is a formula of language* $\mathcal{L}_{\mathsf{CPL}}$.
2. *If A and B are formulas of language* $\mathcal{L}_{\mathsf{CPL}}$, *then the expressions of the forms:* '$\neg(A)$', '$(A) \wedge (B)$', '$(A) \vee (B)$', '$(A) \rightarrow (B)$' *are also formulas of language* $\mathcal{L}_{\mathsf{CPL}}$.
3. *Nothing else is a formula of language* $\mathcal{L}_{\mathsf{CPL}}$. □

As usually, we assume that $\neg$ (negation) binds stronger than binary connectives, and that $\wedge$ (conjunction) and $\vee$ (disjunction) bind stronger than $\rightarrow$ (implication). We omit brackets when there is no risk of a confusion.

The notion of a formula of a language is sometimes defined in Backus-Naur form, simply called BNF[1]. The BNF notation involves specifying a syntactic category (*e.g.*, that of a formula) and then giving recursive equations to show how to generate the members of this category. For example, the following:

$$A ::= p \mid \neg A \mid A \wedge A \mid A \vee A \mid A \rightarrow A$$

produces the set of all formulas of language $\mathcal{L}_{\mathsf{CPL}}$[2]. From now on, we will describe formal languages in the BNF notation, but in every other aspect the description will follow (Wiśniewski and Shangin, 2006), (Wiśniewski, 2006) or (Wiśniewski, 2013).

$\mathcal{L}_{\mathsf{FOL}}$ stands for the language of FOL which contains the following symbols: quantifiers $\forall$ ("for all"), $\exists$ ("there is"), countably infinitely many individual variables: $x_1, x_2, x_3, \ldots$, countably infinitely many individual parameters: $a_1, a_2, a_3, \ldots$, countably infinitely many predicates of arbitrary arity $n \in \mathbb{N}$: $P^n_1, P^n_2, P^n_3, \ldots$, comma: , and brackets: (,). The sets of individual variables of $\mathcal{L}_{\mathsf{FOL}}$ and that of individual parameters of $\mathcal{L}_{\mathsf{FOL}}$ are disjoint. We will use x, y, z for individual variables, a, b, c for individual parameters, and P, Q, R for predicates (their arity will be clear from the context).

$\mathcal{L}_{\mathsf{FOL}}$ is not the full language of first-order logic, since there are no function symbols to build complex terms. These could be added, however—see Chlebowski, 2018. Observe that the only closed terms of the language are individual parameters. In the case of richer first-order languages it is usually assumed that individual constants are the object-level symbols of the language, and then parameters are treated as proof-theoretical tools that play their role only in the completeness proof (see, *e.g.*, Fitting, 1990). Here, for simplicity, we follow the account of (Wiśniewski and Shangin, 2006) and (Smullyan, 1968), therefore we have no individual constants, whereas individual parameters are symbols of $\mathcal{L}_{\mathsf{FOL}}$. In fact, in this setting the distinction constant/parameter becomes inessential.

[1] The BNF notation has been introduced in Backus, 1959 and Naur, 1961 (the name established, *int.al.*, in Knuth, 1964) in the context of specifying syntax of the programming language ALGOL; it is, however, commonly used by logicians and linguists. See Goldblatt, 1992, p. 3 for another concise description of the BNF notation.

[2] Strictly speaking, this need not be the same set of expression as that defined in Definition 1, possible difference lies in using brackets.

Terms and *atomic formulas of language* $\mathcal{L}_{\mathsf{FOL}}$ in the BNF grammar, respectively:

$$t ::= x \mid a$$

$$F ::= P^n(t_1, \ldots, t_n)$$

Finally, *formulas of language* $\mathcal{L}_{\mathsf{FOL}}$ are defined by:

$$A ::= F \mid \forall x A \mid \exists x A \mid \neg A \mid A \wedge A \mid A \vee A \mid A \to A$$

Symbols: x_i, a_i, P_i and t_i will be used as the respective metavariables. We use A, B, C, possibly with subscripts, as metavariables for formulas, and X, Y, Z, possibly with subscripts, for sets of formulas; this pertains both to the language of CPL and FOL. In the case of CPL, we will use the notions "formula" and "sentence" interchangeably. In the case of FOL we make the standard distinction between "formula", "sentential function" and "sentence".

For substitutions we use the notation: $A[x/t]$, which refers to the result of substituting a term t for an individual variable x in a formula A. We assume that using this notation implies that x is free in A and that t is free for x in A. If the reader is not familiar with the notion of substitution, she/he may find sufficient background in logic textbooks like (Troelstra and Schwichtenberg, 2000), (Buss, 1998b), (Fitting, 1990), and (Batóg, 2003) in Polish.

Later we will use S, T for finite, possibly empty, sequences of formulas of language $\mathcal{L}_{\mathsf{FOL}}$. If $S = \langle A_1, \ldots, A_{n-1}, A_n \rangle$ is such a sequence of formulas, then by '$\bigvee S$' and '$\bigwedge S$' we refer to the following disjunction/ conjunction of the terms of S, respectively:

$$A_1 \vee (\ldots \vee (A_{n-1} \vee A_n) \ldots)$$

$$A_1 \wedge (\ldots \wedge (A_{n-1} \wedge A_n) \ldots)$$

2.1. Supplementing languages with sequents and questions

Erotetic calculi are worded in languages containing questions. However, the declarative expressions of the languages are, first of all, sequents. As we have explained in the previous chapter, the initial motivation behind the construction of erotetic calculi was to create calculi transforming questions concerning important (semantic or proof-theoretical) properties or relations defined in the underlying logic—like theoremhood, consequence relation, or validity and entailment in semantic terms. Therefore the use of sequents was natural, as a sequent constitutes an important unit of proof-theoretical information.

Roughly speaking, sequents are formal expressions of the form:

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$$

concerning *inferences* or *deductions* from premises $A_1, \ldots, A_n$ to a conclusion which is a single formula among $B_1, \ldots, B_m$. This kind of expressions has been introduced and studied by Paul Hertz[3] in order to examine the formal properties of proofs in axiomatic systems. The first sequent calculi (for the classical and the intuitionistic logic) have been introduced by Gerhard Gentzen (in Genzten, 1935) as technical tools used to prove some properties of the calculus of natural deduction (mainly its completeness and normalization). Gentzen used the arrow '$\Rightarrow$' as the sequent sign, the use of the turnstile symbol '$\vdash$' in this context comes from Curry (see Curry, 1950). Here we follow Wiśniewski's account, therefore we use the turnstile.

Actually, there are various types of erotetic calculi for the classical logic, both propositional and first-order. Here we present three variants depending on the structure of sequents involved: these may be right-sided, both-sided or left-sided; but there are more possibilities that have been examined thoroughly in Chlebowski, 2018. The systematic study of dual erotetic calculi is beyond the scope of this book, so we will only briefly refer to one of the dual calculi for CPL in Section 4.7.

Our description of syntax is based mainly upon (Wiśniewski, 2013, pp. 22-24). Let $\mathcal{L}_{\mathsf{L}}$ stand for the language of L, where L is CPL or FOL. In the case of each logic L, we construct a richer language, labelled '$\mathcal{L}^?_{\vdash\mathsf{L}}$'. The vocabulary of $\mathcal{L}^?_{\vdash\mathsf{L}}$ includes the vocabulary of $\mathcal{L}_{\mathsf{L}}$, the comma ',' (if this is not present in the language), the semicolon ';' and the following signs: ?, $\vdash$, ng ($\mathcal{L}^?_{\vdash\mathsf{L}}$-negation), & ($\mathcal{L}^?_{\vdash\mathsf{L}}$-conjunction). Language $\mathcal{L}^?_{\vdash\mathsf{L}}$ has two categories of well-formed expressions: declarative well-formed

[3] Hertz studied sentences (*sätze*) of the form '$A_1, \ldots, A_n \longrightarrow B$', that is, single succedent sequents. See Legris, 2014, and also Indrzejczak, 2014 in Polish, or historical notes in Indrzejczak, 2010. See also Negri and von Plato, 2001; Troelstra and Schwichtenberg, 2000 for an introduction to structural proof theory. Also the Stanford Encyclopedia of Philosophy contains entries on proof theory of great educational value, see for example "The Development of Proof Theory" by Jan von Plato (von Plato, 2016).

formulas (d-wffs for short) and erotetic well-formed formulas, that is, questions. We start with atomic d-wffs of $\mathcal{L}^?_{\vdash\mathsf{L}}$, that is, sequents.

Both-sided sequents are expressions of the form:

$$S \vdash A \tag{2.1}$$

where S is a finite, possibly empty, sequence of formulas of $\mathcal{L}_\mathsf{L}$ and A is a formula of $\mathcal{L}_\mathsf{L}$. The turnstile symbol '$\vdash$', which is an object-level sign in this context, is supposed to refer to the derivability relation in the underlying logic L. This interpretation justifies the use of the turnstile symbol instead of the arrow '$\Rightarrow$'. Thus the expression (2.1) may be thought of as representing a metalevel statement "formula A is derivable from sequence S in logic L", where the derivability from a sequence is understood as derivability from the set of terms of the sequence.

Right-sided sequents are expressions of the form:

$$\vdash S \tag{2.2}$$

where S is a finite and non-empty sequence of formulas of $\mathcal{L}_\mathsf{L}$. In this context the turnstile symbol may be thought of as referring to the validity of the disjunction $\bigvee S$ in L.

Left-sided sequents are expressions of the form:

$$S \vdash \tag{2.3}$$

where, again, S is a finite and non-empty sequence of formulas of $\mathcal{L}_\mathsf{L}$. Sequents of this form may be interpreted as stating that the set of terms of sequence S is inconsistent in L.

The presentation may be generalized as follows. We will assume that a sequent of language $\mathcal{L}^?_{\vdash\mathsf{L}}$ is an expression of the form:

$$S \vdash T \tag{2.4}$$

where S and T are finite sequences of formulas of $\mathcal{L}_\mathsf{L}$, and either T is a one-term sequence (then we write (2.1) instead of $S \vdash \langle A \rangle$), or S is empty but T is not (then we write (2.2)), or, finally, T is empty and S is not (then we write (2.3)).[4] Traditionally, S is called *antecedent* of sequent (2.4), and T is called *succedent* of sequent (2.4). Sequents allowing one formula in the succedent are called *single succedent* or *single-conclusioned*, and those allowing more formulas are called *multisuccedent*

[4] We do not consider the case with both T and S empty, but it makes sense to allow the empty sequent when both canonical and dual sequents are taken into account—see Chlebowski, 2018.

or *multi-conclusioned*. The usual interpretation of a multisuccedent sequent of the form (2.4), called *denotational* or *material*, is in terms of validity of formula '$\bigwedge S \rightarrow \bigvee T$' (see Negri and von Plato, 2001, p. 47; Shoesmith and Smiley, 1978, p. 33). We will come back to the issue of interpretation of sequents in Section 3.5 of Chapter 3, where we develop a purely syntactic interpretation of multisuccedent sequents.

In structural proof theory it is more common to use multisets[5] instead of sequences as antecedents and succedents of sequents.[6] However, the use of sequences has some merits in the context of possible implementations, as a finite sequence has a natural representation in such simple and common data structures like lists, arrays, tables, and so on.

We will use lower-case Greek letters ϕ, ψ, possibly with subscripts, for sequents of $\mathcal{L}^?_{\vdash\mathsf{L}}$. Now we may say that an *atomic d-wff* of language $\mathcal{L}^?_{\vdash\mathsf{L}}$ is a sequent of the form (2.4), where S and T are suitably restricted. Then *compound d-wffs* of $\mathcal{L}^?_{\vdash\mathsf{L}}$ are defined by the following BNF grammar:

$$\mathfrak{t} ::= \phi \mid ng\mathfrak{t} \mid \mathfrak{t}\&\mathfrak{t}$$

We will not make use of compound d-wffs of language $\mathcal{L}^?_{\vdash\mathsf{L}}$ until the next chapter, when we describe the notion of an answer to a question of $\mathcal{L}^?_{\vdash\mathsf{L}}$. As yet we will deal with single *sequents* and/or their *sequences*, and so we will use upper-case Greek letters Φ, Ψ, with subscripts if needed, for *sequences of sequents* of $\mathcal{L}^?_{\vdash\mathsf{L}}$.

Questions of $\mathcal{L}^?_{\vdash\mathsf{L}}$ are given by:

$$Q ::= ?(\phi_1; \ldots; \phi_n)$$

More concisely, we may represent questions of $\mathcal{L}^?_{\vdash\mathsf{L}}$ as '$?(\Phi)$', where Φ is a finite and non-empty sequence of sequents of $\mathcal{L}^?_{\vdash\mathsf{L}}$. We will omit the angle brackets indicating sequences and will write:

$$?(S_1 \vdash T_1 \ ; \ \ldots \ ; \ S_n \vdash T_n) \tag{2.5}$$

instead of '$?(\langle S_1 \vdash T_1 \ ; \ \ldots \ ; \ S_n \vdash T_n \rangle)$'. We use semicolon to separate the terms of the sequence of sequents. The sequents '$S_1 \vdash T_1 \ ; \ \ldots \ ;$

[5] Formally, by a multiset we mean a set X together with a function $f : X \longrightarrow \mathbb{N}$, that is, a pair (X, f). The function assigns to each member of X the number of its occurrences. To use the nice phrases from Troelstra and Schwichtenberg, 2000, p. 5, a multiset is a "set with multiplicity" or a "sequence *modulo* the ordering".

[6] See Negri and von Plato, 2001; Troelstra and Schwichtenberg, 2000. Other variants, like sets, are also considered. Poggiolesi (2011) contains a transparent classification of all the variants in the context of sequent calculi for modal logics.

$S_n \vdash T_n$' are called *constituents of question* (2.5). We will also say that the question (2.5) is *based on sequents* '$S_1 \vdash T_1$; ... ; $S_n \vdash T_n$'.

In a sense, the semicolon which we use for separating sequents behaves like a metalevel conjunction—a question of $\mathcal{L}^?_{\vdash \mathsf{L}}$ of the form (2.5) asks whether in the case of each i the meta-level statement expressed by the d-wff '$S_i \vdash T_i$' is true. Thus erotetic calculi built in languages using both-sided sequents of the form (2.1) are calculi transforming questions concerning the statements about derivability, or—semantically—entailment. Erotetic calculi using sequents of the form (2.3) are calculi transforming questions concerning the statements about inconsistency—defined syntactically or semantically. Naturally, the following issue arises: if the validity of a disjunction is the semantically defined subject of questions based on right-sided sequents of the form (2.2), then how should we interpret it in syntactic terms? The issue is addressed in Chapter 3, especially in Section 3.5.

2.2. Erotetic rules for the classical case

Below we present the rules of three erotetic calculi for FOL and three erotetic calculi for CPL. Each calculus is syntactically uniform: its rules operate on questions whose constituents are of only one of the three forms: (2.1), (2.2), (2.3).

Calculus $\mathbb{E}^{\mathbf{PQ}}$ transforms questions based on both-sided sequents. The propositional fragment of this calculus has been first presented in Wiśniewski, 2004 and the first-order version in Wiśniewski and Shangin, 2006. Calculus $\mathbb{E}^{\mathbf{RPQ}}$ transforms questions based on right-sided sequents; it has been considered in Wiśniewski, 2006. Calculus $\mathbb{E}^{\mathbf{LPQ}}$ transforms questions based on left-sided sequents. A version of left-sided erotetic calculus for CPL has been considered in Wiśniewski, 2004, pp. 316-318. All the information concerning erotetic calculi for classical logic has been gathered in Chlebowski, 2018, where also an alternative, uniform method of proving completeness has been developed.

In the presentation of the calculi we will use the unified notation introduced by Raymond Smullyan (Smullyan, 1968). As we shall see further in this chapter, we use the unified notation also with respect to modal logics. The unified notation is displayed in Table 2.1. We also use the following notion of the complement, $\overline{A}$, of a formula A:

$$\overline{A} = \begin{cases} B & \text{if } A \text{ is of the form } \neg B \\ \neg A & \text{otherwise} \end{cases}$$

Table 2.1: α-, β-, γ-, δ-formulas

α	α_1	α_2	β	β_1	β_2
$A \wedge B$	A	B	$\neg(A \wedge B)$	$\neg A$	$\neg B$
$\neg(A \vee B)$	$\neg A$	$\neg B$	$A \vee B$	A	B
$\neg(A \to B)$	A	$\neg B$	$A \to B$	$\neg A$	B

γ	$\gamma(t)$	δ	$\delta(t)$
$\forall xA$	$A[x/t]$	$\neg\forall xA$	$\neg A[x/t]$
$\neg\exists xA$	$\neg A[x/t]$	$\exists xA$	$A[x/t]$

The calculi $\mathbf{E^{PQ}}$, $\mathbf{E^{RPQ}}$ and $\mathbf{E^{LPQ}}$ considered by Andrzej Wiśniewski have been formulated with the use of the unified notation with respect to propositional formulas, but not with respect to quantified formulas, which means that the calculi considered here are in fact notational variants of $\mathbf{E^{PQ}}$, $\mathbf{E^{RPQ}}$ and $\mathbf{E^{LPQ}}$ by Wiśniewski. (In Section 2.2.4 we discuss soundness of our variants and sketch the proof of their completeness with respect to the original erotetic calculi.) The use of the operation of complement instead of negation seems more homogeneous with the use of the α-, β-notation. While in standard Gentzen-sequent systems the rules characterize logical operators, in the unified-notation version the rules characterize some more general binary and unary truth-functions. In such a context the negation sign is a device to express the duality of these functions, while the negation function is not characterized on its own.

In the presentation of the rules for quantifiers we will use also the "κ-notation" defined in Table 2.2. The notation was introduced in the paper (Wiśniewski and Shangin, 2006) in order to capture the cases of double negation, quantifiers in the scope of negation and an empty quantification in one schema. Since we use the γ-, δ-notation here, 'κ' refers only to double negation and an empty quantification. In order to prevent some trivial applications of the rules we assume that the γ-, δ-notation does not account for an empty quantification. In other words, when we use, *e.g.*, the notation 'γ', '$\gamma(x/t)$' in one context in order to refer to '$\forall xA$' and '$A[x/t]$', then we assume that x has at least one free occurrence in A.

Table 2.2: κ–notation for $\mathcal{L}_{\mathsf{CPL}}$ and for $\mathcal{L}_{\mathsf{FOL}}$

	κ		κ^*
$\mathcal{L}_{\mathsf{CPL}}$	$\neg\neg A$		A
$\mathcal{L}_{\mathsf{FOL}}$	$\neg\neg A$		A
	$\forall x_i A$, $\exists x_i A$	where x_i is not free in A	A
	$\neg\forall x_i A$, $\neg\exists x_i A$	where x_i is not free in A	$\neg A$

Erotetic calculi (of appropriate "sidedness") for propositional logic are obtained from the first-order version by dropping the rules for quantifiers and by restricting κ to refer only to formulas of the form '$\neg\neg A$'. In this way, in the propositional case rules L_κ and R_κ become the rules for double negation (see Table 2.3).

Let $\mathbb{E}$ stand for one of: $\mathbb{E}^{\mathbf{PQ}}$, $\mathbb{E}^{\mathbf{RPQ}}$, $\mathbb{E}^{\mathbf{LPQ}}$, or the propositional part of one of these. The notion of *proof in* erotetic calculus $\mathbb{E}$ is introduced as a finite sequence, contrary to the derivations-as-trees tradition of structural proof theory. The counterpart of *derivation* is *Socratic transformation*, which is defined as follows (the definition has been introduced for the first time in Wiśniewski, 2004):

Definition 2 (Socratic transformation)**.** *Let Q be a question of language $\mathcal{L}^?_{\vdash\mathsf{L}}$. A* Socratic transformation of question Q via the rules of calculus $\mathbb{E}$ *is a sequence $\langle Q_1, Q_2, \ldots \rangle$ of questions of language $\mathcal{L}^?_{\vdash\mathsf{L}}$ such that $Q = Q_1$, and each Q_i $(i > 1)$ results from Q_{i-1} by one of the rules of $\mathbb{E}$.* □

In the propositional case, Socratic transformations are always finite, but in the first-order case they may be infinite.

After (Wiśniewski and Shangin, 2006), by a *pure sentence of* $\mathcal{L}_{\mathsf{FOL}}$ we mean a sentence which contains no occurrences of individual parameters. Then by a *pure sequent* of language $\mathcal{L}^?_{\vdash\mathsf{FOL}}$ we mean a sequent which contains only pure sentences of $\mathcal{L}_{\mathsf{FOL}}$. In the case of $\mathcal{L}_{\mathsf{CPL}}$, "pure sequent" and "sequent" means the same thing.

Concatenation operation and some terminology

We will use two signs for the operation of the concatenation of two finite sequences. The sign: '$'$' will serve as the concatenation sign for sequences of formulas of $\mathcal{L}_{\mathsf{L}}$. Thus '$S\ '\ T$', where $S = \langle A_1, \ldots, A_n \rangle$ and $T = \langle B_1, \ldots, B_k \rangle$, means the sequence:

$$\langle A_1, \ldots, A_n, B_1, \ldots, B_k \rangle$$

where both n and k may equal 0, thus concatenating empty sequences is permitted. For the concatenation of finite sequences of sequents we shall use the semicolon, like in: '$\Phi\ ;\ \Psi$'. The semicolon occurs in two meanings in this work (except for the usual punctuation function): as a separator between terms of a sequence of sequents (like in (2.5) above on page 38), and as a concatenation sign; however, this should not lead to a confusion.

Let us recall that letters S, T refer to finite sequences of formulas of language $\mathcal{L}_{\mathsf{L}}$ ($\mathsf{L} \in \{\mathsf{CPL}, \mathsf{FOL}\}$). By the inscription '$S(A)$' we shall refer to a finite sequence of formulas of $\mathcal{L}_{\mathsf{L}}$ such that A is its term. In other words, we can say that a sequence is of the form '$S(A)$', or that it is of the form:

$$S_1\ '\ \langle A \rangle\ '\ S_2 \tag{2.6}$$

and in both cases we refer to the same class of sequences. Further, we will use both ways of describing sequences, depending on the context.

In order to indicate that both A and B are terms of S we shall write '$S(A)(B)$'; the order of A and B in S is arbitrary, thus '$S(A)(B)$' and '$S(B)(A)$' have the same meaning (*i.e.*, refer to the same class of sequences). By '$S(A/B)$', where B is a formula of $\mathcal{L}_{\mathsf{L}}$, we will mean the result of replacing the distinguished term of sequence S, that is A, with B. More specifically, if '$S(A)$' is written as (2.6), then '$S(A/B)$' is of the form:

$$S_1\ '\ \langle B \rangle\ '\ S_2$$

Consequently, the following: '$S(A_1/B_1)(A_2/B_2)$' refers to the result of the simultaneous replacement of term A_1 with B_1, and term A_2 with B_2, and so on. We will also use the following inscription: '$S(A/B_1, B_2)$', where B_1, B_2 are also formulas of $\mathcal{L}_{\mathsf{L}}$, to refer to the result of replacing the distinguished term of S, that is A, with two terms: B_1, B_2. Generally, if '$S(A)$' is written as (2.6), and $B_1, \ldots, B_k$ are formulas of $\mathcal{L}_{\mathsf{L}}$, then by '$S(A/B_1, \ldots, B_k)$' we mean a sequence of the form:

$$S_1\ '\ \langle B_1, \ldots, B_k \rangle\ '\ S_2$$

Example 1.

$$S = \langle p, \neg(q \wedge p), r, q \wedge p, p \vee \neg p \rangle$$

$$S(q \wedge p/s) = \langle p, \neg(q \wedge p), r, s, p \vee \neg p \rangle$$
$$S(p/\neg r, \neg q) = \langle \neg r, \neg q, \neg(q \wedge p), r, q \wedge p, p \vee \neg p \rangle$$

□

2.2.1. Calculus $\mathbb{E}^{\mathbf{PQ}}$

The rules of erotetic calculus $\mathbb{E}^{\mathbf{PQ}}$ are presented in Table 2.3. In the presentation, instead of "individual parameter of $\mathcal{L}_{\mathsf{FOL}}$" we write simply "parameter". The calculus is worded in language $\mathcal{L}^{?}_{\vdash\mathsf{FOL}}$ with both-sided sequents. We introduce the notion of Socratic proof and give an example.

Table 2.3: Rules of $\mathbb{E}^{\mathbf{PQ}}$

$$\frac{?(\Phi \ ; \ S(\alpha) \vdash C \ ; \ \Psi)}{?(\Phi \ ; \ S(\alpha/\alpha_1, \alpha_2) \vdash C \ ; \ \Psi)} \; L_\alpha$$

$$\frac{?(\Phi \ ; \ S \vdash \alpha \ ; \ \Psi)}{?(\Phi \ ; \ S \vdash \alpha_1 \ ; \ S \vdash \alpha_2 \ ; \ \Psi)} \; R_\alpha$$

$$\frac{?(\Phi \ ; \ S(\beta) \vdash C \ ; \ \Psi)}{?(\Phi \ ; \ S(\beta/\beta_1) \vdash C \ ; \ S(\beta/\beta_2) \vdash C \ ; \ \Psi)} \; L_\beta$$

$$\frac{?(\Phi \ ; \ S \vdash \beta \ ; \ \Psi)}{?(\Phi \ ; \ S\,'\, \overline{\beta_1} \vdash \beta_2 \ ; \ \Psi)} \; R_\beta$$

$$\frac{?(\Phi \ ; \ S(\kappa) \vdash C \ ; \ \Psi)}{?(\Phi \ ; \ S(\kappa/\kappa^*) \vdash C \ ; \ \Psi)} \; L_\kappa \qquad \frac{?(\Phi \ ; \ S \vdash \kappa \ ; \ \Psi)}{?(\Phi \ ; \ S \vdash \kappa^* \ ; \ \Psi)} \; R_\kappa$$

$$\frac{?(\Phi \ ; \ S(\gamma) \vdash C \ ; \ \Psi)}{?(\Phi \ ; \ S(\gamma/\gamma, \gamma(a_i)) \vdash C \ ; \ \Psi)} \; L_\gamma$$

a_i is any parameter

$$\frac{?(\Phi \ ; \ S \vdash \gamma \ ; \ \Psi)}{?(\Phi \ ; \ S \vdash \gamma(a_i) \ ; \ \Psi)} \; R_\gamma$$

a_i is a parameter which does not occur in $S \vdash \gamma$

$$\frac{?(\Phi \ ; \ S(\delta) \vdash C \ ; \ \Psi)}{?(\Phi \ ; \ S(\delta/\delta(a_i)) \vdash C \ ; \ \Psi)} \; L_\delta$$

a_i is a parameter which does not occur in $S(\delta) \vdash C$

$$\frac{?(\Phi \ ; \ S \vdash \delta \ ; \ \Psi)}{?(\Phi \ ; \ S\,'\, \overline{\delta} \vdash \delta(a_i) \ ; \ \Psi)} \; R_\delta$$

a_i is any parameter

Definition 3 (Socratic proof in $\mathbb{E}^{\mathbf{PQ}}$)**.** *Let* $S \vdash A$ *be a pure sequent of language* $\mathcal{L}^{?}_{\vdash \mathsf{L}}$. *A* Socratic proof of $S \vdash A$ in $\mathbb{E}^{\mathbf{PQ}}$ *is a finite Socratic transformation of question* $?(S \vdash A)$ *via the rules of* $\mathbb{E}^{\mathbf{PQ}}$ *such that each constituent of the last question of the transformation is of one of the following forms:*

(*a*) $S(B) \vdash B$, *or*

(*b*) $S(B)(\neg B) \vdash C$.

If there exists a Socratic proof of a sequent in $\mathbb{E}^{\mathbf{PQ}}$, *then we say that the sequent is* provable in $\mathbb{E}^{\mathbf{PQ}}$. □

Here is an example of a Socratic proof in $\mathbb{E}^{\mathbf{PQ}}$. To the right we indicate the rule applied to a question.

Example 2. *Sequent* $\forall x \forall y(P(x,y) \to \neg P(y,x)) \vdash \neg \exists x P(x,x)$ *is provable in* $\mathbb{E}^{\mathbf{PQ}}$. *Below A stands for formula* $\forall x \forall y(P(x,y) \to \neg P(y,x))$.

$$\cfrac{\cfrac{\cfrac{\cfrac{?(\ A \ \vdash \ \neg \exists x P(x,x)\ )}{?(\ A \ \vdash \ \neg P(a,a)\ )}\ R_{\gamma}}{?(\ A, \forall y(P(a,y) \to \neg P(y,a)) \ \vdash \ \neg P(a,a)\ )}\ L_{\gamma}}{?(\ A, \forall y(P(a,y) \to \neg P(y,a)), P(a,a) \to \neg P(a,a) \vdash \neg P(a,a)\ )}\ L_{\gamma}}{?(\ A, \forall y(P(a,y) \to \neg P(y,a)), \neg P(a,a), \neg P(a,a) \vdash \neg P(a,a)\ )}\ L_{\beta}$$

□

Terminology

Suppose that question s_{n+1} of Socratic transformation $\mathbf{s}$ results from question s_n by erotetic rule $\mathbf{r}$ of $\mathbb{E}^{\mathbf{PQ}}$. Then we call s_n the *question-premise* and we call s_{n+1} the *question-conclusion* of this application of $\mathbf{r}$. The constituent of question s_n whose schema is distinguished in the question-premise of $\mathbf{r}$ will be called the *sequent-premise* of question s_n. (It is a sequent-counterpart of what is usually called the *active formula.*) Moreover, we will say that the constituent (or each of the two constituents) of question s_{n+1} whose schema is (are) distinguished in the question-conclusion of $\mathbf{r}$ *results from* the sequent-premise of s_n, or that it is the *sequent-conclusion of question* s_{n+1}. Obviously, each question of $\mathbf{s}$ has exactly one sequent-premise (or, if it is the last question of $\mathbf{s}$— it has none) and at most two sequents-conclusions; observe that these sequents-conclusions are "new" with respect to the previous question. We will also say that sequent ψ *results in* **s** *from* sequent ϕ *by erotetic rule* **r** if for some term s_n of **s**, the term s_{n+1} results from s_n by **r**, ϕ is the sequent-premise of s_n and ψ is the sequent-conclusion (respectively, one of the two sequents-conclusions) of s_{n+1}. We will apply the same terminology to the other erotetic calculi presented in this work.

Before we continue, let us observe that, according to Definition 3, a Socratic proof in calculus $\mathbb{E}^{\mathbf{PQ}}$ starts with a question whose only constituent is a pure sequent, and the rules of the calculus do not allow for introducing a sentential function into a question-conclusion. Thus every formula that occurs in a Socratic proof is a sentence of $\mathcal{L}_{\mathsf{FOL}}$, though not necessarily a pure one. As we shall see, the same pertains to the other calculi presented in this Section.

2.2.2. Calculus $\mathbb{E}^{\mathbf{RPQ}}$

The right-sided calculus $\mathbb{E}^{\mathbf{RPQ}}$ is worded in language $\mathcal{L}^{?}_{\vdash\mathsf{FOL}}$, but with the right-sided sequents only. Table 2.4 presents the rules of the calculus.

Table 2.4: Rules of $\mathbb{E}^{\mathbf{RPQ}}$

$$\frac{?(\Phi\ ;\ \vdash S(\alpha)\ ;\ \Psi)}{?(\Phi\ ;\ \vdash S(\alpha/\alpha_1)\ ;\ \vdash S(\alpha/\alpha_2)\ ;\ \Psi)}\ R_\alpha$$

$\dfrac{?(\Phi\ ;\ \vdash S(\beta)\ ;\ \Psi)}{?(\Phi\ ;\ \vdash S(\beta/\beta_1,\beta_2)\ ;\ \Psi)}\ R_\beta$	$\dfrac{?(\Phi\ ;\ \vdash S(\kappa)\ ;\ \Psi)}{?(\Phi\ ;\ \vdash S(\kappa/\kappa^*)\ ;\ \Psi)}\ R_\kappa$
$\dfrac{?(\Phi\ ;\ \vdash S(\gamma)\ ;\ \Psi)}{?(\Phi\ ;\ \vdash S(\gamma/\gamma(a_i))\ ;\ \Psi)}\ R_\gamma$ a_i is a parameter which does not occur in $\vdash S(\gamma)$	$\dfrac{?(\Phi\ ;\ \vdash S\ (\delta)\ ;\ \Psi)}{?(\Phi\ ;\ \vdash S(\delta/\delta,\delta(a_i))\ ;\ \Psi)}\ R_\delta$ a_i is an arbitrary parameter

As before, we present the definition of the notion of Socratic proof in $\mathbb{E}^{\mathbf{RPQ}}$ and give an example.

Definition 4 (Socratic proof in $\mathbb{E}^{\mathbf{RPQ}}$)**.** *Let* $\vdash A$ *be a pure sequent of* $\mathcal{L}^{?}_{\vdash\mathsf{L}}$*.* A Socratic proof of $\vdash A$ in $\mathbb{E}^{\mathbf{RPQ}}$ *is a finite Socratic transformation of question* $?(\vdash A)$ *via the rules of* $\mathbb{E}^{\mathbf{RPQ}}$ *such that each constituent of the last question of the transformation has the following form:*

(*c*) $\vdash S(B)(\neg B)$.

If there exists a Socratic proof of a sequent in $\mathbb{E}^{\mathbf{RPQ}}$*, then we say that the sequent is* provable in $\mathbb{E}^{\mathbf{RPQ}}$. □

Example 3. *Sequent:* $\vdash \forall x(P(x) \to Q(x)) \to (\forall x P(x) \to \forall x Q(x))$ *is provable in* $\mathbb{E}^{\mathbf{RPQ}}$*. Below* A *stands for '*$\neg\forall x(P(x) \to Q(x))$*', and* B *stands for '*$\neg\forall x P(x)$*'.*

$$\cfrac{?(\vdash\ \forall x(P(x) \to Q(x)) \to (\forall x P(x) \to \forall x Q(x))\ )}{\cfrac{?(\vdash\ A, \forall x P(x) \to \forall x Q(x)\ )}{\cfrac{?(\vdash\ A, B, \forall x Q(x)\ )}{\cfrac{?(\vdash\ A, B, Q(a)\ )}{\cfrac{?(\vdash\ A, \neg(P(a) \to Q(a)), B, Q(a)\ )}{\cfrac{?(\vdash\ A, \neg(P(a) \to Q(a)), B, \neg P(a), Q(a)\ )}{?(\vdash\ A, P(a), B, \neg P(a), Q(a)\ ;\ \vdash\ A, \neg Q(a), B, \neg P(a), Q(a)\ )}\ R_\alpha}\ R_\delta}\ R_\delta}\ R_\gamma}\ R_\beta}\ R_\beta$$

□

2.2.3. Calculus $\mathbb{E}^{\mathbf{LPQ}}$

Similarly as before, calculus $\mathbb{E}^{\mathbf{LPQ}}$ is worded in language $\mathcal{L}^{?}_{\vdash\mathsf{FOL}}$, but this time the left-sided sequents are used. Table 2.5 presents the rules of the calculus.

Table 2.5: Rules of $\mathbb{E}^{\mathbf{LPQ}}$

$\dfrac{?(\Phi\ ;\ S(\alpha) \vdash\ ;\ \Psi)}{?(\Phi\ ;\ S(\alpha/\alpha_1, \alpha_2) \vdash\ ;\ \Psi)}\ L_\alpha$	$\dfrac{?(\Phi\ ;\ S(\kappa) \vdash\ ;\ \Psi)}{?(\Phi\ ;\ S(\kappa/\kappa^*) \vdash\ ;\ \Psi)}\ L_\kappa$
$\dfrac{?(\Phi\ ;\ S(\beta) \vdash\ ;\ \Psi)}{?(\Phi\ ;\ S(\beta/\beta_1) \vdash\ ;\ S(\beta/\beta_2) \vdash\ ;\ \Psi)}\ L_\beta$	
$\dfrac{?(\Phi\ ;\ S(\gamma) \vdash;\ \Psi)}{?(\Phi\ ;\ S(\gamma/\gamma, \gamma(a_i)) \vdash\ ;\ \Psi)}\ L_\gamma$ a_i is an arbitrary parameter	$\dfrac{?(\Phi\ ;\ S(\delta) \vdash\ ;\ \Psi)}{?(\Phi\ ;\ S(\delta/\delta(a_i)) \vdash\ ;\ \Psi)}\ L_\delta$ a_i is a parameter which does not occur in $S(\delta) \vdash$

Definition 5 (Socratic proof in $\mathbb{E}^{\mathbf{LPQ}}$)**.** *Let* $A \vdash$ *be a pure sequent of language* $\mathcal{L}^{?}_{\vdash\mathsf{L}}$*. A* Socratic proof of $A \vdash$ in $\mathbb{E}^{\mathbf{LPQ}}$ *is a finite Socratic*

transformation of $?(A \vdash)$ *via the rules of* $\mathbb{E}^{\mathbf{LPQ}}$ *such that each constituent of the last question of the transformation is of the form:*

(*d*) $S(B)(\neg B) \vdash$

If there exists a Socratic proof of a sequent in $\mathbb{E}^{\mathbf{LPQ}}$*, then we say that the sequent is* provable in $\mathbb{E}^{\mathbf{LPQ}}$. □

Example 4. *Sequent* $\neg\exists x(P(x) \to \forall x P(x)) \vdash$ *is provable in* $\mathbb{E}^{\mathbf{LPQ}}$*. Below* C *stands for* $\neg\exists x(P(x) \to \forall x P(x))$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{?(\, C \vdash)}{?(\, C, \neg(P(a) \to \forall x P(x)) \vdash)}\, L_\gamma}{?(\, C, P(a), \neg\forall x P(x) \vdash)}\, L_\alpha}{?(\, C, P(a), \neg\forall x P(x), \neg P(b) \vdash)}\, L_\delta}{?(\, C, \neg(P(b) \to \forall x P(x)), P(a), \neg\forall x P(x), \neg P(b) \vdash)}\, L_\gamma}{?(\, C, P(b), \neg\forall x P(x), P(a), \neg\forall x P(x), \neg P(b) \vdash)}\, L_\alpha$$

□

As in the case of analytic tableau systems, in erotetic calculus $\mathbb{E}^{\mathbf{LPQ}}$ we prove validity of A by showing *inconsistency* of '$\neg A$'.

2.2.4. Soundness and completeness in the classical case

We start with the familiar semantics of CPL and FOL and then supply the declarative part of $\mathcal{L}^?_{\vdash\mathsf{CPL}}$ and $\mathcal{L}^?_{\vdash\mathsf{FOL}}$ with semantics.

Semantics of CPL and FOL

In the case of CPL we use the standard notion of *Boolean valuation* (*valuation*, for short), which is a function v from the set of all formulas of $\mathcal{L}_{\mathsf{CPL}}$ to the set $\{0, 1\}$ of truth values, and is such that (i) $v(A \wedge B) = 1$ iff $v(A) = v(B) = 1$, (ii) $v(A \vee B) = 0$ iff $v(A) = v(B) = 0$, (iii) $v(A \to B) = 1$ iff $v(A) = 0$ or $v(B) = 1$, (iv) $v(\neg A) = 1$ iff $v(A) = 0$. If $v(A) = 1$, then we say that A is *true under* v, otherwise we say it is *false under* v. A formula of $\mathcal{L}_{\mathsf{CPL}}$ which is true under every valuation is called CPL-*valid*. If X is a set of formulas of $\mathcal{L}_{\mathsf{CPL}}$ such that for each $A \in X$, A is true under a valuation v, then we say that X is *satisfied by* v, or that v is a CPL-*model of* X.

In the case of $\mathcal{L}_{\mathsf{FOL}}$, we use the model-theoretical semantics. The central notion of this is that of an *interpretation* $\mathcal{I}$ which is a pair $\langle U, f\rangle$, where U is a non-empty set called the *universe* or the *domain of the interpretation* and f is a function assigning "the meaning" to the non-logical symbols of the language (that is, in our case, to parameters and predicates). Function f works in the following way: for each parameter a_i, $f(a_i) \in U$, that is, $f(a_i)$ is an object in the universe of the interpretation $\mathcal{I}$; for each predicate P_i of arity n, $f(P_i) \subseteq U^n$, that is, $f(P_i)$ is an n-argument relation in the universe. Then we may say that an atomic formula $P_i(a_1, \ldots, a_n)$ is *true in* $\mathcal{I}$, if $\langle f(a_1), \ldots, f(a_n)\rangle \in f(P_i)$, that is, relation $f(P_i)$ holds between the objects $f(a_1), \ldots, f(a_n)$ assigned to parameters $a_1, \ldots, a_n$. In order to interpret formulas containing individual variables we need an *object assignment in* $\mathcal{I}$ which is a function σ from the set of all individual variables to U. Then we define an interpretation of a term t_i in $\mathcal{I}$ under σ, symbolically $t_i^{\mathcal{I}}[\sigma]$, as follows:

$$t_i^{\mathcal{I}}[\sigma] = \begin{cases} \sigma(t_i) & \text{if } t_i \text{ is an individual variable} \\ f(t_i) & \text{if } t_i \text{ is a parameter} \end{cases}$$

Now we can say that an atomic formula $P_i(t_1, \ldots, t_n)$ is *satisfied by* assignment σ in $\mathcal{I}$, symbolically:

$$\mathcal{I} \models P_i(t_1, \ldots, t_n)\,[\sigma]$$

if relation $f(P_i)$ holds between the objects $t_1^{\mathcal{I}}[\sigma], \ldots, t_n^{\mathcal{I}}[\sigma]$. If σ is an object assignment, then for an object $u \in U$, let $\sigma(x_i/u)$ stand for the object assignment which is identical to σ except that it maps x_i to u. Now we may define inductively:

1. $\mathcal{I} \models \neg A\,[\sigma]$ iff $\mathcal{I} \not\models A\,[\sigma]$.
2. $\mathcal{I} \models A \wedge B\,[\sigma]$ iff $\mathcal{I} \models A\,[\sigma]$ and $\mathcal{I} \models B\,[\sigma]$.
3. $\mathcal{I} \models A \vee B\,[\sigma]$ iff $\mathcal{I} \models A\,[\sigma]$ or $\mathcal{I} \models B\,[\sigma]$.
4. $\mathcal{I} \models A \rightarrow B\,[\sigma]$ iff $\mathcal{I} \not\models A\,[\sigma]$ or $\mathcal{I} \models B\,[\sigma]$.
5. $\mathcal{I} \models \forall x_i A\,[\sigma]$ iff $\mathcal{I} \models A\,[\sigma(x_i/u)]$ for each $u \in U$.
6. $\mathcal{I} \models \exists x_i A\,[\sigma]$ iff $\mathcal{I} \models A\,[\sigma(x_i/u)]$ for some $u \in U$.

A formula of language $\mathcal{L}_{\mathsf{FOL}}$, A, is *true under* $\mathcal{I}$, symbolically:

$$\mathcal{I} \models A$$

if $\mathcal{I} \models A\,[\sigma]$ holds for every object assignment σ in $\mathcal{I}$. Formula A is called FOL-*valid* if it is true under every interpretation $\mathcal{I}$.

If X is a set of formulas of $\mathcal{L}_{\mathsf{FOL}}$ and $\mathcal{I}$ is an interpretation of $\mathcal{L}_{\mathsf{FOL}}$ such that each formula in X is true under $\mathcal{I}$, then we say that $\mathcal{I}$ is a FOL-*model of* X. For simplicity, we introduce a certain "unification" of terminology. We will call Boolean valuations CPL-*interpretations* (of language $\mathcal{L}_{\mathsf{CPL}}$) and model-theoretical interpretations—FOL-*interpretations* (of $\mathcal{L}_{\mathsf{FOL}}$), and we will use v and $\mathcal{I}$ for both CPL- and FOL-interpretations. If an L-interpretation is an L-model of a singleton set $X = \{A\}$, then we call this interpretation an L-*model of formula* A. On the other hand, if an L-interpretation $\mathcal{I}$ is not an L-model of A, then we say that A *is false under* $\mathcal{I}$, or that $\mathcal{I}$ *falsifies* A. We write

$$X \models_{\mathsf{L}} A \tag{2.7}$$

for the *entailment in* L which is a semantic relation understood as *transmission of truth*, formally defined as follows: a set X of formulas of language $\mathcal{L}_{\mathsf{L}}$ *entails in* L a formula, A, of language $\mathcal{L}_{\mathsf{L}}$, symbolically (2.7), if every L-model of set X is an L-model of A. Finally, a set X of formulas of $\mathcal{L}_{\mathsf{L}}$ is called L-*satisfiable* or *semantically consistent in* L, if there exists an L-model of X; otherwise X is called L-*unsatisfiable* or *semantically inconsistent in* L.

Semantics of the declarative part of $\mathcal{L}^{?}_{\vdash\mathsf{L}}$

The semantics of sequents is usually described by means of an intuitive and straightforward generalisation of the semantics of the underlying logic. Thus in the case of CPL, we shall say that a sequent $S \vdash T$ is *correct under valuation* v iff the corresponding formula '$\bigwedge S \to \bigvee T$' is true under v. This approach extends easily to the case of FOL. (Recall the *denotational* or *material* interpretation of sequents mentioned in the previous chapter.)

Thus let $S \vdash T$ be a sequent of $\mathcal{L}^{?}_{\vdash\mathsf{L}}$ and let $\mathcal{I}$ be an L-interpretation of $\mathcal{L}_{\mathsf{L}}$. We will say that $S \vdash T$ is *correct under* $\mathcal{I}$ if either there is a formula in S which is false under $\mathcal{I}$ or there is a formula in T which is true under $\mathcal{I}$. It follows from this definition that:

Corollary 1. *Let S be a finite sequence of sentences of $\mathcal{L}_{\mathsf{L}}$, and let T be a finite sequence of formulas of $\mathcal{L}_{\mathsf{L}}$ (not necessarily sentences in the case of $\mathsf{L} = \mathsf{FOL}$). Let $\mathcal{I}$ be an L-interpretation of $\mathcal{L}_{\mathsf{L}}$.*

1. *If $S \vdash T$ is a both-sided sequent of language $\mathcal{L}^?_{\vdash \mathsf{L}}$, that is, the sequent is of the form $S \vdash B$, then it is correct under $\mathcal{I}$ iff the formula of the form $\bigwedge S \to B$ is true under $\mathcal{I}$.*

2. *If $S \vdash T$ is a left-sided sequent of language $\mathcal{L}^?_{\vdash \mathsf{L}}$, that is, the sequent is of the form $S \vdash$, then it is correct under $\mathcal{I}$ iff the formula of the form $\bigwedge S$ is false under $\mathcal{I}$.*

3. *If $S \vdash T$ is a right-sided sequent of language $\mathcal{L}^?_{\vdash \mathsf{L}}$, that is, the sequent is of the form $\vdash T$, then it is correct under $\mathcal{I}$ iff the formula of the form $\bigvee T$ is true under $\mathcal{I}$.*

The next lemma states that all the rules of erotetic calculi for classical logic, except for R_γ and L_δ of the respective $\mathbb{E}$, preserve joint correctness of all constituents of a question in both directions: from the question-premise of a rule to the question-conclusion of the rule, and also from the question-conclusion to the question-premise. Let us observe that the first direction (premise to conclusion) should correspond to what is usually called the *soundness* or the *correctness* of a rule, and then the converse (conclusion to premise) would correspond to the *invertibility* of a rule. It is worth noticing, however, that as far as in a sequent calculus the proof of its soundness usually hinges on the soundness of the rules, in the case of erotetic calculi it is the other direction—the invertibility of the rules—that is necessary to prove their soundness. This difference between the two types of calculi follows from the fact that they have opposite directions of *proving* as defined by the rules. In fact, it is well-known that in sequent calculi the direction of proving defined by the rules is converse to the direction of actual proof-search. In erotetic calculi the two relations: that of proving defined by the rules, and that of actual proof-search, are unified.

Let $\mathbb{E}$ stand for one of $\mathbb{E}^{\mathbf{PQ}}$, $\mathbb{E}^{\mathbf{RPQ}}$, $\mathbb{E}^{\mathbf{LPQ}}$ or of their propositional parts. In the proofs of the following lemmas we analyse only some of the quantifier cases, more details may be found elsewhere: Chlebowski, 2018; Wiśniewski, 2004; Wiśniewski and Shangin, 2006. Since we aim at the transmission of correctness in *Socratic proofs*, we assume that the constituents of the analysed question cannot contain sentential functions.

Lemma 1 (correctness and invertibility of the rules of $\mathbb{E}$)**.** *Suppose that Q^* results from Q by a rule of erotetic calculus $\mathbb{E}$ other than R_γ and L_δ. Let $\mathcal{I}$ be an L-interpretation of $\mathcal{L}_{\mathsf{L}}$. Then each constituent of Q is correct under $\mathcal{I}$ iff each constituent of Q^* is correct under $\mathcal{I}$.*

Proof. Assume that question $Q^* = ?(\Phi\ ;\ S\ '\ \overline{\delta} \vdash \delta(a_i)\ ;\ \Psi)$ results from $Q = ?(\Phi\ ;\ S \vdash \delta\ ;\ \Psi)$ by rule R_δ of calculus $\mathbb{E}^{\mathbf{PQ}}$, and suppose that there is a constituent of Q^* that is not correct under $\mathcal{I} = \langle U, f \rangle$. If the constituent is not the sequent-conclusion of Q^*, then it is a term of Φ or Ψ, and so it is also present in Q. Thus suppose that '$S\ '\ \overline{\delta} \vdash \delta(a_i)$' is not correct under $\mathcal{I}$. Then each sentence in sequence S, in particular, sentence '$\overline{\delta}$', is true under $\mathcal{I}$. If δ is of the form '$\exists x_j A$', then '$\overline{\delta}$' is '$\neg\exists x_j A$', and hence δ must be false under $\mathcal{I}$; similarly if δ is of the form '$\neg\forall x_j A$'. Thus $S \vdash \delta$ is not correct under $\mathcal{I}$. In this way we have shown that if each constituent of Q is correct under $\mathcal{I}$, then each constituent of Q^* is correct under $\mathcal{I}$. Observe that the presence of '$\overline{\delta}$' on the left side of the sequent is necessary to conduct this argument, since the falsity of $\delta(a_i)$ under $\mathcal{I}$ for one (though arbitrary) a_i would not be sufficient to prove that δ is also false.

Now suppose that sequent $S \vdash \delta$ is not correct under $\mathcal{I}$. Then each sentence of '$S\ '\ \overline{\delta}$' is true under $\mathcal{I}$ and $\mathcal{I} \not\models \delta$. Hence there is at least one object assignment σ under which:

$$\mathcal{I} \not\models \delta\,[\sigma] \tag{2.8}$$

If δ is of the form '$\exists x_j A$', then $\delta(a_i)$ is of the form '$A[x_j/a_i]$' (where a_i is an arbitrary parameter). Then (2.8) holds iff for each object $u \in U$:

$$\mathcal{I} \not\models A\,[\sigma(x_j/u)] \tag{2.9}$$

but for object $u = f(a_i)$, (2.9) yields $\mathcal{I} \not\models A\,[\sigma(x_j/f(a_i))]$, which shows that also:

$$\mathcal{I} \not\models A[x_j/a_i]\,[\sigma(x_j/f(a_i))] \tag{2.10}$$

and this entails that $\mathcal{I} \not\models A[x_j/a_i]$. If δ is of the form '$\neg\forall x_j A$', then the reasoning goes along the same lines, but in (2.9) we have for all $u \in U$:

$$\mathcal{I} \models A\,[\sigma(x_j/u)]$$

in (2.10):

$$\mathcal{I} \models A[x_j/a_i]\,[\sigma(x_j/f(a_i))]$$

which yields:

$$\mathcal{I} \not\models \neg A[x_j/a_i]\,[\sigma(x_j/f(a_i))]$$

In both cases we arrive at the conclusion that sequent '$S\ '\ \overline{\delta} \vdash \delta(a_i)$' is not correct under $\mathcal{I}$. Thus we have shown that if each constituent of question Q^* is correct under $\mathcal{I}$, then each constituent of question Q is correct under $\mathcal{I}$. □

As to rules R_γ and L_δ, the correctness of each constituent of a question-premise warrants the correctness of each constituent of the respective question-conclusion, but not the other way round. For the bottom-up direction we can prove only the transmission of the stronger property: being correct under *every* FOL-interpretation.

Lemma 2 (correctness and invertibility of R_γ, L_δ). *Suppose that Q^* results from Q by rule R_γ or L_δ of $\mathbb{E}$. Then*

1. *if $\mathcal{I}$ is a FOL-interpretation of $\mathcal{L}_{\mathsf{FOL}}$ such that each constituent of Q is correct under $\mathcal{I}$, then also each constituent of Q^* is correct under $\mathcal{I}$; and moreover*

2. *if each constituent of question Q^* is correct under **every** FOL-interpretation of $\mathcal{L}_{\mathsf{FOL}}$, then each constituent of question Q is correct under **every** FOL-interpretation of $\mathcal{L}_{\mathsf{FOL}}$.*

Proof. We shall consider only the case of R_γ with $\gamma = \forall x_j A$, the remaining cases are analogous. We prove both clauses *1* and *2* by transposition.

Thus suppose that:

$$Q = ?(\Phi\ ;\ S \vdash \forall x_j A\ ;\ \Psi)$$

$$Q^* = ?(\Phi\ ;\ S \vdash A[x_j/a_i]\ ;\ \Psi)$$

For the first clause, suppose that $\mathcal{I}$ is an arbitrary FOL-interpretation of $\mathcal{L}_{\mathsf{FOL}}$, and that sequent '$S \vdash A[x_j/a_i]$' of the question-conclusion Q^* is not correct under $\mathcal{I}$. Then every term of sequence S is true under $\mathcal{I}$, but $\mathcal{I} \not\models A[x_j/a_i]$, which yields immediately that $\mathcal{I} \not\models \forall x_j A$. Thus the sequent-premise of the question-premise Q is not correct under $\mathcal{I}$.

For the second clause assume that sequent '$S \vdash \forall x_j A$' is not correct under some interpretation $\mathcal{I} = \langle U, f\rangle$, that is, each term of S is true under $\mathcal{I}$, but '$\forall x_j A$' is not. Then there exists an object assignment σ, and an object $u \in U$ such that $\mathcal{I} \not\models A\,[\sigma(x_j/u)]$. Now we indicate an interpretation that falsifies the sequent-conclusion. Let $\mathcal{I}^* = \langle U, f^*\rangle$ be a FOL-interpretation that differs from $\mathcal{I}$ **at most** with respect to the assignment $f^*(a_i) = u$. Recall that parameter a_i is new with respect to '$S \vdash \forall x_j A$', hence still $\mathcal{I}^*$ makes true each term of S, but at the same time $\mathcal{I}^* \not\models A[x_j/a_i]$, which was to be proved. □

It is also easy to see that:

Corollary 2. *Every sequent of language $\mathcal{L}^?_{\vdash \mathsf{L}}$ of one of the forms: (a)–(d) specified in Definitions 3, 4, 5 is correct under every* L*-interpretation of $\mathcal{L}_\mathsf{L}$.*

The above corollary, together with Lemmas 1 and 2, yields soundness of $\mathbb{E}$ with respect to the correctness of sequents under L-interpretations, and also with respect to the semantics of the underlying logic L.

Theorem 1 (soundness of $\mathbb{E}$)**.** *If there exists a Socratic proof of a pure sequent $S \vdash T$ of language $\mathcal{L}^?_{\vdash \mathsf{L}}$ in $\mathbb{E}$, then the sequent is correct under every* L*-interpretation of $\mathcal{L}_\mathsf{L}$. Moreover,*

1. *<u>the both-sided case and the right-sided case</u>: if there exists a Socratic proof of sequent '$\vdash A$' in $\mathbb{E}$, then A is* L*-valid.*
2. *<u>the left-sided case</u>: if there exists a Socratic proof of sequent '$\overline{A} \vdash$' in $\mathbb{E}$, then A is* L*-valid.*

Proof. Suppose that **p** is a Socratic proof of sequent $S \vdash T$ in $\mathbb{E}$ and assume that **p** has n terms (questions). By the definition of Socratic proof and by the above corollary, each constituent of the last (n-th) term of **p** is correct under every $\mathcal{L}$-interpretation. By the invertibility of the rules of $\mathbb{E}$ (Lemmas 1, 2) also each constituent of the $(n-1)$-st term of **p** is correct under every $\mathcal{L}$-interpretation. The argument is analogous in the inductive step, therefore, by decreasing induction with respect to $i : 1 \leq i \leq n$, sequent $S \vdash T$ is correct under every $\mathcal{L}$-interpretation. We skip the remaining details. □

A Socratic proof *proves* correctness of a sequent under every L-interpretation. However, as we have established, sequents represent meta-level statements concerning properties/relations defined in the underlying logic. Therefore we may say that a Socratic proof proves the truth of the corresponding meta-level statement. We will go back to this issue in the next chapter, where we also present semantics for the erotetic parts of languages with sequents.

Completeness

In Chlebowski, 2018 the author presents the proofs of soundness and completeness of, *int.al.*, all erotetic calculi for classical logic which are considered here. For proving completeness of the calculi the technique of consistency properties is used in Chlebowski, 2018 in a uniform manner.

It is worth noticing that for dual erotetic calculi Chlebowski develops the technique of *dual refutability properties*.

Since the subject is thoroughly examined elsewhere, there is no need for us to show the completeness of erotetic calculi directly and "from scratch". We shall do something different. Below we sketch ("sketch" is a proper word) a proof of Theorem 4, which compares the original calculus $\mathbf{E}^{\mathbf{PQ}}$ by Wiśniewski and calculus $\mathbb{E}^{\mathbf{PQ}}$ considered in this book, and shows completeness of the latter with respect to the former. In the proof of the theorem we actually analyse only one case differing the two erotetic calculi. The rules of $\mathbf{E}^{\mathbf{PQ}}$ are presented below. It is easy to see that the other differences between the two calculi cannot influence completeness.

Erotetic calculus $\mathbf{E}^{\mathbf{PQ}}$

Here are the rules of erotetic calculus $\mathbf{E}^{\mathbf{PQ}}$: $\mathbf{L}_\alpha$, $\mathbf{R}_\alpha$, $\mathbf{L}_\beta$, $\mathbf{R}_\beta$, $\mathbf{L}_\kappa$, $\mathbf{R}_\kappa$, $\mathbf{L}_\forall$, $\mathbf{R}_\forall$, $\mathbf{L}_\exists$ and $\mathbf{R}_\exists$, as presented in Wiśniewski and Shangin, 2006. In the case of $\mathbf{L}_\alpha$, $\mathbf{R}_\alpha$, $\mathbf{L}_\beta$, there is no difference to L_α, R_α, L_β of $\mathbb{E}^{\mathbf{PQ}}$.

$$\frac{?(\Phi\ ;\ S\,'\,\alpha\,'\,T\ \vdash\ C\ ;\ \Psi)}{?(\Phi\ ;\ S\,'\,\alpha_1\,'\,\alpha_2\,'\,T\ \vdash\ C\ ;\ \Psi)}\ \mathbf{L}_\alpha \qquad \frac{?(\Phi\ ;\ S\ \vdash\ \alpha\ ;\ \Psi)}{?(\Phi\ ;\ S\ \vdash\ \alpha_1\ ;\ S\ \vdash\ \alpha_2\ ;\ \Psi)}\ \mathbf{R}_\alpha$$

$$\frac{?(\Phi\ ;\ S\,'\,\beta\,'\,T\ \vdash\ C\ ;\ \Psi)}{?(\Phi\ ;\ S\,'\,\beta_1\,'\,T\ \vdash\ C\ ;\ S\,'\,\beta_2\,'\,T\ \vdash\ C\ ;\ \Psi)}\ \mathbf{L}_\beta$$

Rules $\mathbf{R}_\beta$, $\mathbf{L}_\kappa$, $\mathbf{R}_\kappa$ use the notions 'β_1^*' and κ^{**}; the definition are given below. Let us recall that in the formulation of L_β of $\mathbb{E}^{\mathbf{PQ}}$ (see Table 2.3 on page 43) we have used the complement '$\overline{\beta_1}$' instead of 'β_1^*'. The difference between the two notions comes to light in the case $\beta = A \vee B$, when A is of the form '$\neg C$', since then $\beta_1^* = \neg A = \neg\neg C$, and $\overline{\beta_1} = C$.

$$\frac{?(\Phi\ ;\ S\ \vdash\ \beta\ ;\ \Psi)}{?(\Phi\ ;\ S\,'\,\beta_1^*\ \vdash\ \beta_2\ ;\ \Psi)}\ \mathbf{R}_\beta$$

$$\frac{?(\Phi\ ;\ S\,'\,\kappa\,'\,T\ \vdash\ C\ ;\ \Psi)}{?(\Phi\ ;\ S\,'\,\kappa^{**}\,'\,T\ \vdash\ C\ ;\ \Psi)}\ \mathbf{L}_\kappa \qquad \frac{?(\Phi\ ;\ S\ \vdash\ \kappa\ ;\ \Psi)}{?(\Phi\ ;\ S\ \vdash\ \kappa^{**}\ ;\ \Psi)}\ \mathbf{R}_\kappa$$

β	β_1	β_2	β_1^*
$\neg(A \wedge B)$	$\neg A$	$\neg B$	A
$A \vee B$	A	B	$\neg A$
$A \rightarrow B$	$\neg A$	B	A

The original κ-notation used in Wiśniewski and Shangin, 2006 enhances the two additional cases of negated quantifiers.

κ	κ^{**}
$\neg\neg A$	A
$\forall x_i A$ where x_i is not free in A	A
$\exists x_i A$ where x_i is not free in A	A
$\neg\forall x_i A$	$\exists x_i \neg A$
$\neg\exists x_i A$	$\forall x_i \neg A$

In the case of the quantifier rules $\mathbf{L}_\forall$, $\mathbf{R}_\forall$, $\mathbf{L}_\exists$ and $\mathbf{R}_\exists$ it is assumed that x_i is free in A. In the case of $\mathbf{L}_\forall$ and $\mathbf{R}_\exists$, τ is an arbitrary parameter. In the case of $\mathbf{R}_\forall$ and $\mathbf{L}_\exists$, τ is a parameter which does not occur in, respectively, sequent '$S \vdash \forall x_i A$', sequent '$S\,'\,\exists x_i A\,'\,T \vdash C$'.

$$\frac{?(\Phi\ ;\ S\,'\,\forall x_i A\,'\,T \vdash C\ ;\ \Psi)}{?(\Phi\ ;\ S\,'\,\forall x_i A\,'\,A[x_i/\tau]\,'\,T \vdash C\ ;\ \Psi)}\ \mathbf{L}_\forall$$

$$\frac{?(\Phi\ ;\ S \vdash \forall x_i A\ ;\ \Psi)}{?(\Phi\ ;\ S \vdash A[x_i/\tau]\ ;\ \Psi)}\ \mathbf{R}_\forall \qquad \frac{?(\Phi\ ;\ S\,'\,\exists x_i A\,'\,T \vdash C\ ;\ \Psi)}{?(\Phi\ ;\ S\,'\,A[x_i/\tau]\,'\,T \vdash C\ ;\ \Psi)}\ \mathbf{L}_\exists$$

$$\frac{?(\Phi\ ;\ S \vdash \exists x_i A\ ;\ \Psi)}{?(\Phi\ ;\ S\,'\,\forall x_i \neg A \vdash A[x_i/\tau]\ ;\ \Psi)}\ \mathbf{R}_\exists$$

In Section 4.5 of Chapter 4 we shall go back to the original calculi $\mathbf{E^{PQ}}$, $\mathbf{E^{RPQ}}$ and $\mathbf{E^{LPQ}}$; we present there the rules of $\mathbf{E^{RPQ}}$ and $\mathbf{E^{LPQ}}$.

In Wiśniewski and Shangin, 2006 the reader will find the proofs of the following theorems (formulated in a slightly different way, as we have adjusted terminology here).

Theorem 2 (soundness of $\mathbf{E^{PQ}}$)**.** *Let $S \vdash A$ be a pure sequent. If $S \vdash A$ is provable in $\mathbf{E^{PQ}}$, then $S \vdash A$ is correct under every* FOL-*interpretation.*

Theorem 3 (completeness of $\mathbf{E^{PQ}}$)**.** *Let $S \vdash A$ be a pure sequent. If $S \vdash A$ is correct under every* FOL-*interpretation, then $S \vdash A$ is provable in $\mathbf{E^{PQ}}$.*

We continue with:

Theorem 4. *Every sequent provable in $\mathbf{E^{PQ}}$ has a Socratic proof in $\mathbb{E}^{\mathbf{PQ}}$.*

Proof. Suppose that there exists a Socratic proof of sequent ϕ in calculus $\mathbf{E^{PQ}}$, and let $\mathbf{p}$ stand for the proof. We analyse the construction of $\mathbf{p}$ and indicate how a Socratic proof $\mathbf{p}^*$ of ϕ in $\mathbb{E}^{\mathbf{PQ}}$ can be constructed.

If in the construction of $\mathbf{p}$, rules $\mathbf{L}_\kappa$, $\mathbf{R}_\kappa$ were never applied with respect to formulas of the form '$\neg\exists x_i A$', '$\neg\forall x_i A$', rule $\mathbf{R}_\beta$ has not been applied in the case differing the two accounts, and also rule $\mathbf{R}_\exists$ has not been applied, then $\mathbf{p}$ is a Socratic proof of ϕ in $\mathbb{E}^{\mathbf{PQ}}$. Suppose that up to a certain question, only the remaining rules were applied (those that are the same in both erotetic calculi), and then rule $\mathbf{L}_\kappa$ is applied to a question with respect to sequent of the form: '$S\,'\,\neg\exists x_i A\,'\,T \vdash C$', where the sequent-conclusion is: '$S\ '\ \forall\neg x_i A\ '\ T \vdash C$'. Then the application of $\mathbf{L}_\kappa$ must be omitted in $\mathbf{p}^*$, that is, the question-conclusion of this application is omitted. Every time we have in $\mathbf{p}$ a consecutive occurrence of '$\forall\neg x_i A$', in $\mathbf{p}^*$ we put an occurrence of '$\neg\exists x_i A$' instead. If it happens that rule $\mathbf{L}_\forall$ is applied in $\mathbf{p}$ with respect to an occurrence of '$\forall\neg x_i A$' (in a relevant sequent), we switch it to the application of L_γ with respect to an occurrence of '$\neg\exists x_i A$' (in a relevant sequent). Finally, it may happen, that a constituent, ϕ, of the last question of $\mathbf{p}$ is basic due to:

(a) an occurrence of '$\forall\neg x_i A$' on the left and right side of the turnstile,

(b) an occurrence of '$\forall\neg x_i A$' and '$\neg\forall\neg x_i A$' on the left side of the turnstile.

In case (a), the relevant constituent of the last question of $\mathbf{p}^*$ has an occurrence of '$\neg\exists x_i A$' left of the turnstile and an occurrence of '$\forall\neg x_i A$' on the right side. Then we apply R_γ introducing a parameter, a_i, and then L_γ with respect to a_i. We obtain a question which in the place of ϕ has a constituent with an occurrence of '$\neg A(a_i)$' on the both sides of the sequent. If (b) is the case, then we apply first L_δ, and then L_γ.

All the other cases of the applications of rules $\mathbf{L}_\kappa$ and $\mathbf{R}_\kappa$ may be analysed in the same way. Moreover, if rule $\mathbf{R}_\exists$ is applied in $\mathbf{p}$, then in $\mathbf{p}^*$ we apply R_δ. Now in $\mathbf{p}^*$ we have '$\neg\exists x_i A$', where in $\mathbf{p}$ stands '$\forall\neg x_i A$'. We reason as above. □

The soundness and completeness theorems for erotetic calculus $\mathbf{E^{PQ}}$ may be found in Wiśniewski, 2004—for the propositional part—and in Wiśniewski and Shangin, 2006 for the first-order case. Calculus $\mathbf{E^{RPQ}}$ has been described in Wiśniewski, 2006; its completeness is proved with respect to completeness of $\mathbf{E^{PQ}}$ (but in a much more precise way than we did above). As to the left-sided version, the propositional version called

$\mathbf{L}^{**}$ is presented in Wiśniewski, 2004, but its soundness and completeness are not analysed in this paper.

Completeness of calculi $\mathbb{E}^{\mathbf{RPQ}}$ and $\mathbb{E}^{\mathbf{LPQ}}$ may be analysed with respect to completeness of $\mathbb{E}^{\mathbf{PQ}}$ (more or less in the same way as in Wiśniewski, 2006). Since the detailed proofs are contained in Chlebowski, 2018, we state without proof:

Theorem 5 (completeness of $\mathbb{E}$ wrt the semantics of sequents)**.** *If a pure sequent of the form '$\vdash A$' is correct under every* L*-interpretation of* $\mathcal{L}_{\mathsf{L}}$*, then there exists a Socratic proof of the sequent in* $\mathbb{E}^{\mathbf{PQ}}$*, and there also exists a Socratic proof of the sequent in* $\mathbb{E}^{\mathbf{RPQ}}$*. What is more, if a pure sequent of the form '$\overline{A} \vdash$' is correct under every* L*-interpretation of* $\mathcal{L}_{\mathsf{L}}$*, then there exists a Socratic proof of the sequent in* $\mathbb{E}^{\mathbf{LPQ}}$*.*

Theorem 6 (completeness of $\mathbb{E}$ wrt the semantics of L)**.** *If a formula, A, is* L*-valid, then, first, there exists a Socratic proof of sequent '$\vdash A$' in* $\mathbb{E}^{\mathbf{PQ}}$*, second, there exists a Socratic proof of sequent '$\vdash A$' in* $\mathbb{E}^{\mathbf{RPQ}}$*, and third, there exists a Socratic proof of sequent '$\overline{A} \vdash$' in* $\mathbb{E}^{\mathbf{LPQ}}$*.*

In Chlebowski, 2018 also the following, more general version of Theorem 6, is proved:[7]

Theorem 7 (completeness of $\mathbb{E}$ wrt the semantics of L—a generalisation)**.** *Let ϕ be a pure sequent of language* $\mathcal{L}^{?}_{\vdash\mathsf{FOL}}$ *that is correct under every* L*-interpretation. Then:*

1. *if ϕ is of the form '$S \vdash A$', then ϕ is provable in* $\mathbb{E}^{\mathbf{PQ}}$*,*
2. *if ϕ is of the form '$\vdash T$', then ϕ is provable in* $\mathbb{E}^{\mathbf{RPQ}}$*,*
3. *if ϕ is of the form '$S \vdash$', then ϕ is provable in* $\mathbb{E}^{\mathbf{LPQ}}$*.*

2.2.5. Proof-search in erotetic calculi for FOL

Despite of, or rather due to the undecidability of FOL, any proof system for FOL should go with at least a sketch of a proof-search procedure. We develop such a procedure in this section, using some notions which are known from the literature of the subject and are rather simple. First of all, in order to warrant that the procedure terminates, we use the notion

[7] The three cases are analysed there separately, see Chlebowski, 2018, pp. 72-74. As previously, we slightly modify the quoted theorems, mainly to adjust the terminology.

of *quantifier depth*, "Q-depth" for short, which is a formal parameter bounding the number of applications of the quantifier rules. It is assumed that the value of Q-depth is supplied by the user of the procedure. The notion was used by Melvin Fitting (see Fitting, 1990).

We start with introducing the necessary terminology. Until the end of this section, $\mathbb{E}$ shall stand for any of $\mathbb{E}^{\mathbf{PQ}}$, $\mathbb{E}^{\mathbf{RPQ}}$, $\mathbb{E}^{\mathbf{LPQ}}$. Any sequent of language $\mathcal{L}^{?}_{\vdash\mathsf{FOL}}$ which is of one of the following forms (*cf.* Definitions 3, 4, 5):

(a) $S(B) \vdash B$

(b) $S(B)(\neg B) \vdash C$

(c) $\vdash S(B)(\neg B)$

(d) $S(B)(\neg B) \vdash$

will be called a *basic sequent*. Another useful category is that of a permanently open sequent[8]. A sequent will be called *permanently open* iff it is not a basic sequent and it does not fall under the schema of a sequent-premise of any rule of the respective erotetic calculus $\mathbb{E}$. In other words, a permanently open sequent is an "open" (not basic) sequent with respect to whom no rule is applicable. When a Socratic transformation starts with a pure sequent of language $\mathcal{L}^{?}_{\vdash\mathsf{FOL}}$, an occurrence of a permanently open sequent will be rather rare, but still rare is not impossible.

Further it will occur useful to generalise the notion of Socratic proof in the following way:

Definition 6. *Let Q be a question of language $\mathcal{L}^{?}_{\vdash\mathsf{FOL}}$. A Socratic transformation of question Q via the rules of $\mathbb{E}$ is* successful *if it is finite and each constituent of its last question is a basic sequent.*

Let us observe that a Socratic proof of sequent ϕ is, by definition, a successful Socratic transformation of question $?(\phi)$, but not the other way round—a successful Socratic transformation may start with a question $?(\phi_1, \ldots, \phi_n)$ based on an arbitrary number n of constituents of an arbitrary form.

Let us also quote the following result by Szymon Chlebowski (Chlebowski, 2018, pp. 37-38) for further use:[9]

[8] We have borrowed this term from (Wiśniewski and Shangin, 2006), where the notion of "permanently unsuccessful sequent" was used in the completeness proof of the original $\mathbf{E}^{\mathbf{PQ}}$.

[9] The original version is numbered "**Lemma 5**", we have also slightly altered the formulation of the lemma.

Lemma 3 (Separation lemma). *Let* $\mathbb{E} \in \{\mathbb{E}^{\mathbf{PQ}}, \mathbb{E}^{\mathbf{RPQ}}, \mathbb{E}^{\mathbf{LPQ}}\}$. *Let* $\phi_1, \ldots, \phi_n$ *stand for sequents of* $\mathcal{L}^?_{\vdash\mathsf{FOL}}$. *These two conditions are equivalent:*

1. *there exists a successful Socratic transformation of question* $?(\phi_1, \ldots, \phi_n)$ *in* $\mathbb{E}$*;*
2. *for each* $i : 1 \leq i \leq n$*, there exists a successful Socratic transformation of question* $?(\phi_i)$ *in* $\mathbb{E}$.

Proof. See Chlebowski, 2018, pp. 38-39. □

The applications of the quantifier rules must be subjected to reasonable and intuitive restrictions. First, if there is a choice among application of, to give an example, L_γ and L_δ of $\mathbb{E}^{\mathbf{PQ}}$, then L_δ should go first. Second, the new constant introduced by the application of L_δ should not be chosen completely at random. Third, after the new constants are introduced, rule L_γ should reuse them. Until the end of this section our work aims at formalizing these restrictions into the form of Algorithm 1.

From now on, we assume that the individual parameters of $\mathcal{L}_{\mathsf{FOL}}$ are given in an alphabetical order, *e.g.*, as follows:

$$a_1, a_2, \ldots, a_n, \ldots \tag{2.11}$$

In Table 2.6 we present additional erotetic rules which will be used in Algorithm 1.

Table 2.6: Additional erotetic rules

$$\frac{?(\Phi\, ;\, S(\gamma)\ \vdash C\, ;\, \Psi)}{?(\Phi\, ;\, S(\gamma/\gamma, \gamma(c_n), \ldots, \gamma(c_1), \gamma(c^*))\ \vdash C\, ;\, \Psi)}\ L^*_\gamma$$

$$\frac{?(\Phi\, ;\, S\ \vdash\ \delta\, ;\, \Psi)}{?(\Phi\, ;\, S\,'\,\langle\overline{\delta}, \overline{\delta}(c_n), \ldots, \overline{\delta}(c_1)\rangle\ \vdash\ \delta(c^*)\, ;\, \Psi)}\ R^*_\delta$$

$$\frac{?(\Phi\, ;\vdash S(\delta)\, ;\, \Psi)}{?(\Phi\, ;\vdash S(\delta/\delta, \delta(c_n), \ldots, \delta(c_1), \delta(c^*))\, ;\, \Psi)}\ R^{**}_\delta$$

(∗) $c_1, \ldots, c_n$ are *all (and only)* the ind. parameters which occur in the sequent-premise, and ind. parameter c^* is the first one in the alphabetical order (2.11) which is new to the sequent-premise.

The first rule L^*_γ is to be added to $\mathbb{E}^{\mathbf{PQ}}$. Rule L^*_γ for calculus $\mathbb{E}^{\mathbf{LPQ}}$ is obtained from the presented L^*_γ by dropping 'C' from the succedents of sequents. Rule R^*_δ is designed for calculus $\mathbb{E}^{\mathbf{PQ}}$, and rule R^{**}_δ—for calculus $\mathbb{E}^{\mathbf{RPQ}}$.

We show that each of the additional rules is derivable in the respective calculus $\mathbb{E}$.

Admissibility and derivability of the erotetic rules

In Wiśniewski and Shangin, 2007 some rules admissible in erotetic calculus $\mathbf{E}^{\mathbf{PQ}}$ (the original version of the erotetic system for FOL) are introduced and analysed. Some of them, but not all, are derivable. The authors introduce also a set of admissible structural rules. However, there is no general notion of admissibility and derivability in the erotetic context.

Sara Negri and Jan von Plato define the notion of admissibility as follows (Negri and von Plato, 2001, p. 20):

> "Given a system of rules $\mathbf{G}$, we say that a rule with premisses $S_1, \ldots, S_n$ and conclusion S is *admissible* in $\mathbf{G}$ if, whenever an instance of $S_1, \ldots, S_n$ is derivable in $\mathbf{G}$, the corresponding instance of S is derivable in $\mathbf{G}$."

where derivability of a sequent S_i in $\mathbf{G}$ is defined as the existence of the relevant derivation. In other words, if a rule is admissible, then its addition to a proof system (a system of rules, to follow the quotation) does not influence the derivability relation generated by the system. Any conclusion derivable from a set of premises after the addition must have been derivable before the addition. It means, *int.al.*, that adding an admissible rule to a sound calculus must result in a sound calculus.

In the simplest case, admissibility is met in the form of derivability of a rule[10]. Referring to the quotation above, we would say that a rule with premises $S_1, \ldots, S_n$ and conclusion S is *derivable* in $\mathbf{G}$ if every instance of S is derivable in $\mathbf{G}$ from the respective instances of $S_1, \ldots, S_n$. Usually it means that we are able to develop a schema of derivation leading from premises $S_1, \ldots, S_n$ to conclusion S, in which only the rules of $\mathbf{G}$ are applied; therefore a derived rule is considered a tool to make useful shortcuts in a proof.

[10] See also the definitions of admissibility and derivability by Andrzej Indrzejczak (Indrzejczak, 2010, p. 22).

Let us recall that $\mathbb{E} \in \{\mathbb{E}^{\mathbf{PQ}}, \mathbb{E}^{\mathbf{RPQ}}, \mathbb{E}^{\mathbf{LPQ}}\}$. Adjusting the notion of derivability to the erotetic rules seems fairly simple:

Definition 7 (derivability in $\mathbb{E}$)**.** *Let* **r** *be an erotetic rule of the form:*

$$\frac{Q}{Q^*}\ \mathbf{r}$$

We will say that **r** *is* derivable in *erotetic calculus* $\mathbb{E}$ *iff there exists a finite Socratic transformation* **s** *of question* Q *via the rules of* $\mathbb{E}$ *such that* Q^* *is its last term.* □

Indeed, this is how the notion is understood in Wiśniewski and Shangin, 2007.

The notion of admissibility is more problematic. Let us recall the fact that the direction of proving defined by the erotetic rules is converse to the direction of proving as defined by the rules of a standard sequent calculus. We have noticed one consequence of this fact: the proof of soundness of an erotetic calculus depends on the invertibility of the rules, whereas in the case of sequent calculi, the proof of soundness of the calculus relies on the correctness of the rules. The following definition of admissibility reveals another consequence of the above mentioned fact.

Definition 8 (admissibility in $\mathbb{E}$)**.** *Let* **r** *be an erotetic rule of the form:*

$$\frac{Q}{Q^*}\ \mathbf{r}$$

We will say that **r** *is* admissible in *erotetic calculus* $\mathbb{E}$ *iff the following conditions hold:*

1. *if there exists a successful Socratic transformation of question* Q^* *via the rules of* $\mathbb{E}$*, then there exists a successful Socratic transformation of question* Q *via the rules of* $\mathbb{E}$*, and*

2. *if there exists a successful Socratic transformation of question* Q *via the rules of* $\mathbb{E}$*, then there exists a successful Socratic transformation of question* Q^* *via the rules of* $\mathbb{E}$. □

Let us observe that condition *1.* in the definition of admissibility captures the basic intuition behind admissibility, which is also expressed in the cited quotation. On the face of it, it may seem that it is clause *2.* that captures the idea, but the point is that the "final conclusion" of a successful Socratic transformation is expressed by its *first* element, not the last one—as it is in the case of standard proofs.

For example, suppose that we are able to prove the following dependency:

(d) if there exists a successful Socratic transformation via the rules of $\mathbb{E}^{\mathbf{RPQ}}$ of question '$?(\vdash T, A \,;\, \vdash T, \neg A\,)$', then there exists a successful Socratic transformation via the rules of $\mathbb{E}^{\mathbf{RPQ}}$ of question '$?(\vdash T\,)$'.

Taking dependency (d) for granted, when searching for a Socratic proof of sequent '$\vdash\ T$', one can pass from question '$?(\vdash T\,)$' to question '$?(\vdash T, A \,;\, \vdash\ T, \neg A\,)$', since finishing a Socratic transformation of the later question with a success yields that there exists a successful Socratic transformation of the former question. In other words, proving dependency (d) shows the admissibility of the following rule in $\mathbb{E}^{\mathbf{RPQ}}$:

$$\frac{?(\vdash T\,)}{?(\vdash T, A \,;\, \vdash T, \neg A\,)}$$

Let us stress that the dependency expressed by (d) *does not* yield that there is a Socratic transformation "leading" from question '$?(\vdash\ T\,)$' to question '$?(\vdash\ T, A \,;\, \vdash\ T, \neg A\,)$'; in fact, there is no such Socratic transformation—the rule is not derivable in $\mathbb{E}^{\mathbf{RPQ}}$.

However, clause *1.* of Definition 8 does not warrant correctness of a rule. Let us consider, for example, the following rule:

$$\frac{?(\vdash A, \neg A\,)}{?(\vdash A \wedge \neg A\,)}$$

where A is a sentence of $\mathcal{L}_{\mathsf{FOL}}$. Unfortunately, for each calculus $\mathbb{E}$ considered in this book, the rule satisfies condition *1.* of Definition 8, as *there is no* successful Socratic transformation of question '$?(\vdash A \wedge \neg A\,)$' via the rules of $\mathbb{E}$. But the rule is clearly incorrect. Hence we need condition *2.* in Definition 8 to warrant correctness. We may observe that this is another consequence of the fact that a Socratic proof starts with a question concerning the *final conclusion* and proceeds to axioms. In a usual setting, admissibility of a rule of a sequent calculus warrants its correctness, but—as we can see—it is not the case with erotetic calculi.

Finally, the following should be the case:

Corollary 3. *If a rule is derivable in* $\mathbb{E} \in \{\mathbb{E}^{\mathbf{PQ}}, \mathbb{E}^{\mathbf{RPQ}}, \mathbb{E}^{\mathbf{LPQ}}\}$*, then it is admissible in* $\mathbb{E}$.

Proof. Suppose that an erotetic rule $\mathbf{r}$ of the form:

$$\frac{Q}{Q^*}$$

(from now on, we will write "$\mathbf{r} = Q/Q^*$") is derivable in $\mathbb{E}$. Then there is a Socratic transformation $\langle Q_1, \ldots, Q_n \rangle$ via the rules of $\mathbb{E}$ such that $Q_1 = Q$ and $Q_n = Q^*$. Assume that $\mathbf{s} = \langle s_1, \ldots, s_k \rangle$ is a successful Socratic transformation of question Q^* via the rules of $\mathbb{E}$. Then the following sequence of questions: $\langle Q_1, \ldots, Q_n, s_2, \ldots, s_k \rangle$ is a successful Socratic transformation of question Q via the rules of $\mathbb{E}$. This shows that clause *1.* of Definition 8 is satisfied.

The case of clause *2.* is tricky. Assume that $\mathbf{r} = Q/Q^*$ is derivable in $\mathbb{E}$, and that there exists a successful Socratic transformation $\mathbf{s}$ of Q via the rules of $\mathbb{E}$. Suppose that question Q has the form '$?(\phi_1, \ldots, \phi_m)$'. By Lemma 3, for each $i : 1 \leq i \leq m$, there is a successful Socratic transformation of question '$?(\phi_i)$', which, by soundness of $\mathbb{E}$ (Theorem 1), yields that for each i, ϕ_i is a sequent correct under every L-interpretation. Since in the construction of $\mathbf{s}$ only the rules of $\mathbb{E}$ were used, and each of them is correct (see Lemmas 1, 2), also each constituent of question Q^* is a sequent correct under every L-interpretation. By completeness of $\mathbb{E}$ (Theorem 7), each such sequent has a Socratic proof in $\mathbb{E}$. Once again, by Lemma 3, there exists a successful Socratic transformation of question Q^* via the rules of $\mathbb{E}$. Which finishes the proof. □

Corollary 4. *If an erotetic rule* $\mathbf{r}$ *belongs to* $\mathbb{E}$*, then* $\mathbf{r}$ *is both derivable and admissible in* $\mathbb{E}$*.*

Proof. Clearly, if $\mathbf{r} = Q/Q^*$ is a rule of $\mathbb{E}$, then $\langle Q, Q^* \rangle$ is the Socratic transformation which shows that $\mathbf{r}$ is derivable in $\mathbb{E}$. By the previous corollary, $\mathbf{r}$ is also admissible in $\mathbb{E}$. □

Further in this work we will be interested only in derivable erotetic rules. It is clear, however, that the more general notion of admissibility is of great importance.

Lemma 4. *The erotetic rules presented in Table 2.6 are derivable in the respective erotetic calculi, that is:*

1. L^*_γ *is derivable in* $\mathbb{E}^{\mathbf{PQ}}$*,*
2. L^*_γ *with empty-succedent sequents is derivable in* $\mathbb{E}^{\mathbf{LPQ}}$*,*

3. R_δ^* *is derivable in* $\mathbb{E}^{\mathbf{PQ}}$*, and*

4. R_δ^{**} *is derivable in* $\mathbb{E}^{\mathbf{RPQ}}$*.*

Proof. We shall consider only the third case, the remaining ones are analogous. Suppose that Q is a question of $\mathcal{L}^?_{\vdash\mathsf{FOL}}$ of the following form:

$$?(\Phi\ ;\ S\ \vdash\ \delta\ ;\ \Psi)$$

We apply the original rule R_δ of $\mathbb{E}^{\mathbf{PQ}}$ using individual parameter c^* specified in condition $(*)$ and obtain:

$$?(\Phi\ ;\ S\,'\,\overline{\delta} \vdash\ \delta(c^*)\ ;\ \Psi)$$

Let $c_1, \ldots, c_n$ be all (and only) the parameters which occur in sequent '$S \vdash \delta$'. Formula $\overline{\delta}$ is a γ-formula: '$\neg\exists x A(x)$', if δ is of the form '$\exists x A(x)$', and '$\forall x A(x)$', if δ is of the form '$\neg\forall x A(x)$'. We apply rule L_γ n times, starting with individual parameter c_1 up to c_n, and thus arrive at:

$$?(\Phi\ ;\ S\,'\,\overline{\delta}\,'\,\overline{\delta}(c_n)\,'\ldots'\,\overline{\delta}(c_1)\ \vdash\ \delta(c^*)\ ;\ \Psi)$$

which finishes the proof. □

We continue with the terminology needed for Algorithm 1. The rules: R_β of $\mathbb{E}^{\mathbf{RPQ}}$, L_α and R_β of $\mathbb{E}^{\mathbf{PQ}}$, L_α of $\mathbb{E}^{\mathbf{LPQ}}$ and all the R_κ, L_κ rules will be called *simple non-branching rules*, or *SNB-rules* for short. The SNB-rules are non-branching, as they introduce only one sequent-conclusion, and they either perform no operations on quantifiers or deal with an empty quantification. The following rules: R_α of $\mathbb{E}^{\mathbf{RPQ}}$, R_α and L_β of $\mathbb{E}^{\mathbf{PQ}}$ and L_β of $\mathbb{E}^{\mathbf{LPQ}}$ are called *branching rules*, or *B-rules* for short—these are the rules which introduce two sequent-conclusions in the question-conclusion. The remaining rules are non-branching quantifier rules. Among them the rules: R_γ of $\mathbb{E}^{\mathbf{RPQ}}$, R_γ and L_δ of $\mathbb{E}^{\mathbf{PQ}}$, L_δ of $\mathbb{E}^{\mathbf{LPQ}}$ will be called *N-rules*, since they always introduce a new parameter. The remaining quantifier rules are the original R_δ of $\mathbb{E}^{\mathbf{RPQ}}$, R_δ and L_γ of $\mathbb{E}^{\mathbf{PQ}}$, L_γ of $\mathbb{E}^{\mathbf{LPQ}}$, and also the additional admissible rules presented in Table 2.6. They will be called *O-rules*, where "O" is for "old". The basic task of O-rules is to "use" an already introduced parameter or parameters (hence an "old" one); however, it may happen, as we know, that they need to be used with a new one (compare Example 4, p. 47).

We stipulate that when an N-rule is applied, then the new parameter introduced into the sequent-conclusion is not arbitrary, but is exactly the first one in the alphabetical order (2.11) which is new to the sequent-premise. We also assume that the rules: L^*_γ, R^*_δ, R^{**}_δ are used instead of the original O-rules. An application of any of these rules yields an introduction of a new parameter which, again, is the first one in the alphabetical order (2.11) new to the relevant sequent.

We will also use the standard notion of *active formula* of a rule, by which we mean the formula-occurrence whose schema is distinguished in the sequent-premise of the rule.

The algorithm uses four procedures given below: SNB, N, O and B. We start with SNB.

Procedure SNB(question, Q)

Data: question $= ?(\Phi; \phi; \Psi)$; `/* the sequent-premise` ϕ `*/`

1 $Q \leftarrow ?(\Phi; \phi; \Psi)$; `/* is distinguished in 'question' */`
2 $m \leftarrow$ the number of formula occurrences in ϕ (on both sides)
3 $j \leftarrow 1$
4 **while** $j \leq m$ **do**
5   **if** *j-th formula occurrence in ϕ is active in an SNB-rule* **r** **then**
6   apply **r**
7         $\phi \leftarrow$ sequent-conclusion of **r**
8         $Q \leftarrow ?(\Phi; \phi; \Psi)$
9   **else**
10         $j \leftarrow j + 1$
11   **end**
12 **end**

When the condition of if-then-else clause (lines 5–11 of Procedure SNB) is satisfied, the value of j does not change, which has the consequence that all active formulas of SNB-rules are found before the while-loop is exited; hence when the procedure runs to an end, the distinguished constituent of Q has no active formulas of SNB-rules. In the worst case, this means that for each subformula of a formula occurrence in a sequent, either the subformula or its negation is checked for potential application of an SNB-rule.

As we have mentioned before, in order to deal with the undecidability of FOL we borrow an elegant trick from (Fitting, 1990), used by Fitting in a Prolog theorem-prover for FOL. Since we aim at a terminating algorithm, we set at the start the so-called *Q-depth*, which is the number of individual parameters that may be used when a sequent is being "decomposed". Then provability of a sequent is always tested at a given Q-depth.

Procedure N(question, numeral, Q, q)

```
   Data: question = ?(Φ; φ; Ψ), numeral
1  Q ← ?(Φ; φ; Ψ)
2  q ← numeral
3  m ← the number of formula occurrences in φ (on both sides)
4  j ← 1
5  while j ≤ m do
6  |  if j-th formula occurrence in φ is active in an N-rule r then
7  |  |  apply r
8  |  |  φ ← sequent-conclusion of r
9  |  |  Q ← ?(Φ; φ; Ψ)
10 |  |  q ← q + 1
11 |  else
12 |  |  j ← j + 1
13 |  end
14 end
```

Similarly as in the case of Procedure SNB, the construction of if-then-else clause of Procedure N warrants that before the while-loop is exited, there are no formulas-premises of N-rules. In line 10, the value of q, which counts the number of new parameters introduced during the search, is raised.

Procedure O deals with the applications of O-rules (that is, rules $L^*_\gamma, R^*_\delta, R^{**}_\delta$), but it is written in detail only for the case of L^*_γ. The remaining cases may be easily obtained from the one presented here. In the case of this procedure, rising the value of j in line 10 warrants that all the formulas of the "original" sequent are checked for possible applications of L^*_γ, and at the same time it prevents from repetitions of applications of L^*_γ with respect to the same active formula. Each application of L^*_γ is associated with introducing one new parameter, and this fact is taken into account in line 11.

Procedure B deals with the branching rules. A branching rule is applied at most once during executing this procedure, this is warranted by line 13 of the procedure.

Procedure O(question, numeral, Q, q)

Data: question $= ?(\Phi; \phi; \Psi)$, numeral

$Q \leftarrow ?(\Phi; \phi; \Psi)$

$q \leftarrow$ numeral

$m \leftarrow$ the number of formula occurrences in ϕ (on both sides)

$j \leftarrow 1$

while $j \leq m$ **do**

    if *j-th formula occurrence is a γ-formula* **then**

        apply L^*_γ

        $S\,'\,\langle\gamma, \gamma(c_n), \ldots, \gamma(c_1), \gamma(c^*)\rangle\,'\,T \vdash C \leftarrow$ sequent-conclusion of L^*_γ

        $Q \leftarrow ?(\Phi; S\,'\,\langle\gamma, \gamma(c_n), \ldots, \gamma(c_1), \gamma(c^*)\rangle\,'\,T \vdash C; \Psi)$

        $j \leftarrow j + n + 2$

        $q \leftarrow q + 1$

    else

        $j \leftarrow j + 1$

    end

end

Algorithm 1 incorporates the following ideas: (i) economy—branching rules are used only when necessary, after verifying applicability of the other rules, (ii) the applications of quantifier rules are subject to intuitive restrictions mentioned above, (iii) in the case of the negative answer the algorithm differentiates between a situation of actual deciding that no Socratic proof exists (this happens when a permanently open sequent occurs) and a situation when no Socratic proof has been obtained with the use of q distinct individual parameters, where q is the Q-depth proposed by the "user". For example, sequent $\neg\exists x(P(x) \rightarrow \forall x P(x)) \vdash$ (compare Example 4) is not provable in $\mathbb{E}^{\mathbf{LPQ}}$ with Q-depth equal to 1, but it is with a Q-depth grater than 2. Value 2 is not enough, as after applying the N-rule L^*_δ, Procedure O is executed and r gets value 3 before there is a chance to run Procedure SNB again.

Let us also point at line 21 of the algorithm which, if executed, resets the value of r. This happens when a basic sequent is found and we move to another constituent, which yields that the new constants are counted from the beginning[11].

[11] There is a subtlety: it may and it will happen that there are already new parameters at the "branch". *E.g.*, if $q = 10$ and there are already 10 parameters

Procedure B(question, numeral, Q, n)

Data: question $= ?(\Phi; \phi; \Psi)$, numeral

$Q \leftarrow ?(\Phi; \phi; \Psi)$
$n \leftarrow$ numeral
$m \leftarrow$ the number of formula occurrences in ϕ (on both sides)
$j \leftarrow 1$
while $j \leq m$ **do**
    if *j-th formula occurrence in ϕ is active in a B-rule* **r then**
        apply **r**
        $\psi_1 \leftarrow$ the left sequent-conclusion of **r**
        $\psi_2 \leftarrow$ the right sequent-conclusion of **r**
        $\Psi \leftarrow \psi_2 ; \Psi$
        $Q \leftarrow ?(\Phi; \phi; \Psi)$
        $n \leftarrow n + 1$
        $j \leftarrow m + 1$
    else
        $j \leftarrow j + 1$
    end
end

2.3. The modal case

In Leszczyńska, 2007 the author presented erotetic calculi for the class of basic modal propositional logics. By *basic modal logics* we mean K and all of its proper extensions characterized by any combination of the following properties of the accessibility relation: seriality (also called "extendability"), reflexivity, transitivity, symmetry and Euclideaness. There are exactly fifteen different basic modal logics. We use the notation for the names of the logics after Rajeev Goré:[12] K, D, T, KB, DB, B, K4, D4, S4, KB4, S5, K5, D5, K45, D45. In Leszczyńska, 2007 the logics K, D, T, K4, KB, S4 and S5 were discussed in detail, together with soundness and completeness proofs. The author presented also erotetic calculi for the other basic modal logics. In Leszczyńska-Jasion, 2008 another six

introduced before the B-rule has been applied, then with the clean account for r another constituent may have 20 parameters at the end. The point is, however, to restrict the use of new parameters rather than to do it accurately. This could be easily improved if necessary.

[12] See Goré, 1999.

Algorithm 1: Algorithm for proof-search in $\mathbb{E}$

Data: question $?(\phi)$ of language $\mathcal{L}^{?}_{\vdash\mathsf{FOL}}$, where ϕ is a pure sequent of $\mathcal{L}^{?}_{\vdash\mathsf{FOL}}$, Q-depth

Result: finite Socratic transformation of question $?(\phi)$

$n \leftarrow 1$
$i \leftarrow 1$
$r \leftarrow 0$
$q \leftarrow$ Q-depth
$\Phi \leftarrow \varnothing$
$\Psi \leftarrow \varnothing$
while ϕ *is not permanently open* **and** $r \leq q$ **do**
    SNB$(?(\Phi; \phi; \Psi), Q)$
    N$(?(\Phi; \phi; \Psi), r, Q, r)$
    O$(?(\Phi; \phi; \Psi), r, Q, r)$
    B$(?(\Phi; \phi; \Psi), n, Q, n)$
    if ϕ *is a basic sequent* **then**
        if $i = n$ **then**
            return "Sequent ϕ has a proof."
        else
            $\Phi \leftarrow \Phi \,;\, \phi$
            $i \leftarrow i + 1$
            $\phi \leftarrow i$-th constituent of Q
            $\phi \,;\, \Xi \leftarrow \Psi$
            $\Psi \leftarrow \Xi$
            $r \leftarrow 0$
        end
    end
end
if ϕ *is permanently open* **then**
    return "Sequent ϕ is not provable in $\mathbb{E}$."
else
    return "Sequent ϕ is not provable in $\mathbb{E}$ with Q-depth q."
end

modal logics were "erotetised": S4.2, S4.3, S4F, S4R, S4M and G. Until the end of this chapter, unless stated otherwise, L represents the basic modal logics.

2.3.1. Language and semantics

Language $\mathcal{M}$ is the language of modal propositional logics built upon $\mathcal{L}_{\mathsf{CPL}}$ with '$\Box$' (necessity) and '$\Diamond$' (possibility). $FORM_{\mathcal{M}}$ is for the set of formulas of $\mathcal{M}$ given by the following BNF grammar:

$$A ::= p \mid \Box A \mid \Diamond A \mid \neg A \mid A \wedge A \mid A \vee A \mid A \rightarrow A$$

Together with the α-, β-notation for formulas of propositional language we use also the π-, ν-notation introduced in Fitting, 1983. Table 2.7 presents the notation. Once again, we will treat doubly negated formulas as κ-formulas, but since there are no other κ-formulas, we will write explicitly '$\neg\neg A$' in the rules.

Table 2.7: π-, ν-notation

ν	ν_0	π	π_0
$\Box A$	A	$\neg\Box A$	$\neg A$
$\neg\Diamond A$	$\neg A$	$\Diamond A$	A

We shall use the standard Kripke semantics for modal logics as presented for example in Hughes and Cresswell, 1996. Thus by a *modal frame* (or simply *frame*) we mean an ordered pair $\langle W, R\rangle$ such that W is a non-empty set whose elements are called *possible worlds*, and $R \subseteq W \times W$. A *valuation on a frame* $\langle W, R\rangle$ is a function $V : FORM_{\mathcal{M}} \times W \longrightarrow \{0, 1\}$. A triple $\langle W, R, V\rangle$, where $\langle W, R\rangle$ is a frame and V is a valuation on it, is called a *model*.

A formula A of language $\mathcal{M}$ is called *true in a world* w of model $\langle W, R, V\rangle$ iff $V(A, w) = 1$. It is *true in a model* $\langle W, R, V\rangle$ iff it is true in every world of the model. And finally, a formula true in every model is called K-*valid*. Some of the well-know connections between the properties of accessibility relations and validity in logics above K are presented in Table 2.8. (In the table we recall also the definitions of the properties.) A formula A of language $\mathcal{M}$ is called L-valid iff it is true in every model $\langle W, R, V\rangle$ such that R has the L-properties.

Table 2.8: L-properties

logic L	L-properties	definition
K	none	
D	seriality	$\forall x \exists y (xRy)$
T	reflexivity	$\forall x (xRx)$
KB	symmetry	$\forall x, y (xRy \rightarrow yRx)$
DB	seriality and symmetry	
B	reflexivity and symmetry	
K4	transitivity	$\forall x, y, z (xRy \wedge yRz \rightarrow xRz)$
D4	seriality and transitivity	
S4	reflexivity and transitivity	
KB4	symmetry and transitivity	
S5	reflexivity, symmetry and transitivity	
K5	Euclideaness	$\forall x, y, z (xRy \wedge xRz \rightarrow yRz)$
D5	seriality and Euclideaness	
K45	transitivity and Euclideaness	
D45	seriality, transitivity and Euclideaness	

Language $\mathcal{M}^?_\vdash$ of erotetic calculi $\mathbb{E}^{\mathsf{L}}$ for modal logic L is built upon language $\mathcal{M}$. The construction is analogous to that of $\mathcal{L}^?_{\vdash \mathsf{L}}$ built upon $\mathcal{L}_{\mathsf{L}}$ (for $\mathsf{L} \in \{\mathsf{CPL}, \mathsf{FOL}\}$), with the following difference: beyond the vocabulary of $\mathcal{M}$, the comma, the semicolon and the signs: ?, $\vdash$, ng, &, language $\mathcal{M}^?_\vdash$ contains also countably infinitely many numerals: $1, 2, \ldots$ Language $\mathcal{M}^?_\vdash$ has well-formed formulas (*wffs*, for short) of three categories: indexed formulas, declarative wffs and questions. *Indexed formulas of* $\mathcal{M}^?_\vdash$ are expressions of the form:

$$(A)^{i_1,\ldots,i_n} \tag{2.12}$$

where A is a formula of $\mathcal{M}$ and $i_1, \ldots, i_n$ is an n-term ($n \geq 1$) sequence of numerals. The sequence will be called an *index of wff* (2.12). The numerals are meant to indicate possible worlds, and the order in which the numerals occur in an index presents a partial description of the accessibility relation. The last numeral of an index will play a special role, therefore we will use the inscription: 'σ/i' to indicate that the last term of sequence σ is i. Hence an expression of the form:

$$(A)^{\sigma/i}$$

represents an arbitrary indexed formula whose index ends with numeral i. We will use variables $i, j, \ldots$ for numerals and σ, ρ, possibly with subscripts, for indices.[13]

Sequents of language $\mathcal{M}^?_\vdash$ are expressions of the form:

$$\vdash (A_1)^{\sigma_1/i_1}, \ldots, (A_n)^{\sigma_n/i_n} \tag{2.13}$$

where $(A_1)^{\sigma_1/i_1}, \ldots, (A_n)^{\sigma_n/i_n}$ is an n-term $(n \geq 1)$ sequence of indexed formulas of $\mathcal{M}^?_\vdash$. In the sequel we will sometimes omit round brackets and write (2.13) rather as:

$$\vdash A_1^{\sigma_1/i_1}, \ldots, A_n^{\sigma_n/i_n}$$

Generally, it will depend on readability of the expression.

Until the end of this chapter we use ϕ, ψ for sequents of language $\mathcal{M}^?_\vdash$. As before, d-wffs of language $\mathcal{M}^?_\vdash$ are given by the following BNF grammar:

$$\mathsf{t} ::= \phi \mid ng\mathsf{t} \mid \mathsf{t}\&\mathsf{t}$$

and questions of $\mathcal{M}^?_\vdash$ are given by:

$$Q ::= ?(\phi_1; \ldots; \phi_n)$$

where '$\phi_1; \ldots; \phi_n$' is an n-term $(n \geq 1)$ sequence of sequents of $\mathcal{M}^?_\vdash$.

Similarly as before, metavariables $A, B, \ldots$ will be used for formulas of modal language $\mathcal{M}$, metavariables $S, T, \ldots$ for finite (possibly empty) sequences of indexed formulas, Greek upper-case letters Φ, Ψ for finite (possibly empty) sequences of sequents of $\mathcal{M}^?_\vdash$.

2.3.2. Rules of modal erotetic calculi

If ϕ is a sequent of $\mathcal{M}^?_\vdash$ of the form (2.13), then $\mathbf{I}_W\{\phi\}$ is the set of all numerals occurring in the indices of formulas of ϕ, defined as follows:

$$\mathbf{I}_W\{\phi\} = \{j : j \text{ is a term of some } \sigma_k/i_k, 1 \leq k \leq n\}$$

[13] We have used σ before for object assignments in model-theoretical interpretations, but this should not cause any confusion.

and $\mathbf{I}_R\,[\phi]$ is a set of ordered pairs of numerals that may be "gathered" from the indices, formally:

$$\mathbf{I}_R\,[\phi] = \{\langle j, j' \rangle : j \text{ immediately precedes } j' \text{ in some } \sigma/i_k, 1 \leq k \leq n\}$$

Table 2.9 presents the rules of erotetic calculi $\mathbb{E}^{\mathsf{L}}$.

Table 2.9: Rules of modal $\mathbb{E}^{\mathsf{L}}$

$$\frac{?(\Phi\ ;\ \vdash S(\alpha^{\sigma})\ ;\ \Psi)}{?(\Phi\ ;\ \vdash S(\alpha^{\sigma} \,/\, \alpha_1^{\sigma})\ ;\ \vdash S(\alpha^{\sigma} \,/\, \alpha_2^{\sigma})\ ;\ \Psi)}\ R_\alpha$$

$$\frac{?(\Phi\ ;\ \vdash S(\beta^{\sigma})\ ;\ \Psi)}{?(\Phi\ ;\ \vdash S(\beta^{\sigma} \,/\, \beta_1^{\sigma},\ \beta_2^{\sigma})\ ;\ \Psi)}\ R_\beta$$

$$\frac{?(\Phi\ ;\ \vdash S(\neg\neg A^{\sigma})\ ;\ \Psi)}{?(\Phi\ ;\ \vdash S(\neg\neg A^{\sigma} \,/\, A^{\sigma})\ ;\ \Psi)}\ R_{\neg\neg}$$

$$\frac{?(\Phi\ ;\ \vdash S(\nu^{\sigma/i})\ ;\ \Psi)}{?(\Phi\ ;\ \vdash S(\nu^{\sigma/i} \,/\, \nu_0^{\sigma/i,j})\ ;\ \Psi)}\ R_\nu$$

$$\frac{?(\Phi\ ;\ \vdash S(\pi^{\sigma/i})\ ;\ \Psi)}{?(\Phi\ ;\ \vdash S(\pi^{\sigma/i} \,/\, \pi^{\sigma/i},\ \pi_0^{j})\ ;\ \Psi)}\ R_\pi$$

Rule R_ν may be applied provided that j is new with respect to sequent $\vdash S(\nu^{\sigma/i})$. By '$\sigma/i, j$' we mean the concatenation of sequences 'σ/i' and the one-term sequence '$\langle j \rangle$'. For convenience, we assume that:

$$j = max(\mathbf{I}_W\{\vdash S(\nu^{\sigma/i})\}) + 1 \tag{2.14}$$

which guarantees that the new j is exactly the first (in $\leq$) numeral new with respect to the sequent in question.

The proviso of applicability of rule R_π depends on L. In order to present it we define what follows. If R is a binary relation, then by *reflexive closure of* R we mean the smallest binary relation $R^\star$ such that $R \subseteq R^\star$ and $R^\star$ is reflexive. Similarly for *symmetric*, *transitive*, *Euclidean* closures.[14] We will write R^{ref}, R^{sym}, R^{trans}, R^{Eucl} for the

[14] More formally, given a universe U, the P-closure of a relation $R \subseteq U \times U$, where P is one of the properties: reflexivity, symmetry, transitivity, Euclideaness, is the intersection of all relations $R^{\star\star}$, $U \times U \supseteq R^{\star\star} \supseteq R$ that have the property P.

respective closures of R. By *transitive and Euclidean* closure of R we mean the smallest binary relation $R^\star$ such that $R \subseteq R^\star$ and $R^\star$ is both transitive and Euclidean. We shall write $R^{trans,Eucl}$ for such a closure. Similarly for any other combination of properties.[15]

Now let ϕ stand for $\vdash S(\pi^{\sigma/i})$. Table 2.10 presents the provisos of applicability of rule R_π.

Table 2.10: Provisos of applicability of rule R_π

Calculus	Proviso of applicability of rule R_π
$\mathbb{E}^{\mathsf{K}}$, $\mathbb{E}^{\mathsf{D}}$	$(i,j) \in \mathbf{I}_R\,[\phi]$
$\mathbb{E}^{\mathsf{T}}$	$(i,j) \in \mathbf{I}_R\,[\phi]^{ref}$
$\mathbb{E}^{\mathsf{KB}}$, $\mathbb{E}^{\mathsf{DB}}$	$(i,j) \in \mathbf{I}_R\,[\phi]^{sym}$
$\mathbb{E}^{\mathsf{B}}$	$(i,j) \in \mathbf{I}_R\,[\phi]^{ref,sym}$
$\mathbb{E}^{\mathsf{K4}}$, $\mathbb{E}^{\mathsf{D4}}$	$(i,j) \in \mathbf{I}_R\,[\phi]^{trans}$
$\mathbb{E}^{\mathsf{S4}}$	$(i,j) \in \mathbf{I}_R\,[\phi]^{ref,trans}$
$\mathbb{E}^{\mathsf{KB4}}$	$(i,j) \in \mathbf{I}_R\,[\phi]^{sym,trans}$
$\mathbb{E}^{\mathsf{S5}}$	$(i,j) \in \mathbf{I}_R\,[\phi]^{ref,sym,trans}$
$\mathbb{E}^{\mathsf{K5}}$, $\mathbb{E}^{\mathsf{D5}}$	$(i,j) \in \mathbf{I}_R\,[\phi]^{Eucl}$
$\mathbb{E}^{\mathsf{K45}}$, $\mathbb{E}^{\mathsf{D45}}$	$(i,j) \in \mathbf{I}_R\,[\phi]^{trans,Eucl}$

The above summary of the provisos is new. In Appendix A we present the old version, where the provisos are defined without explicit reference to the properties of accessibility relation. We also show that the two accounts are equivalent.

For logics which are serial but not reflexive, *i.e.*, L = D, DB, D4, D5, D45, the corresponding calculus $\mathbb{E}^{\mathsf{L}}$ contains additionally the following rule:

$$\frac{?(\Phi\ ;\ \vdash S(\pi^{\sigma/i})\ ;\ \Psi)}{?(\Phi\ ;\ \vdash S(\pi^{\sigma/i}\,/\,\pi^{\sigma/i},\ \pi_0^{i,j})\ ;\ \Psi)}\ R_{\pi D}$$

with the proviso that j is new with respect to sequent $\vdash S(\pi^{\sigma/i})$ (in practice, we put a restriction on j as in (2.14)).

[15] There is a subtlety. According to this convention $R^{trans,Eucl}$ is the same relation as $R^{Eucl,trans}$. On the other hand $(R^{trans})^{Eucl}$ and $(R^{Eucl})^{trans}$ need not be the same thing. Consider $R = \{(0,1),(2,1),(2,3)\}$. We have $R^{trans} = R$, and $(R^{trans})^{Eucl} = R^{Eucl} = R \cup \{(1,1),\ (1,3),\ (3,1),\ (3,3)\}$. It follows that $(R^{Eucl})^{trans} = R^{Eucl} \cup \{(0,3)\}$, therefore $(R^{trans})^{Eucl} \neq (R^{Eucl})^{trans}$. Observe also that $R^{trans,Eucl} = (R^{Eucl})^{trans}$.

Thus formally we shall say that $R^{property_1,property_2}$ is the intersection of all relations $R^{\star\star}$ such that $U \times U \supseteq R^{\star\star} \supseteq R$ and $R^{\star\star}$ has the two properties *property*$_1$, *property*$_2$.

Socratic transformations are defined as in the classical case: as (finite) sequences of questions regulated by the erotetic rules (see Definition 2 on page 41). A Socratic proof in $\mathbb{E}^{\mathsf{L}}$, the erotetic calculus for modal logic L, starts with a question of the form $?(\vdash (A)^1)$, where the numeral is to represent an arbitrary possible world of an arbitrary Kripke model. Formally,

Definition 9 (Socratic proof in $\mathbb{E}^{\mathsf{L}}$). *Let* $\vdash (A)^1$ *be a sequent of language* $\mathcal{M}^{?}_{\vdash}$. *A* Socratic proof of $\vdash (A)^1$ in $\mathbb{E}^{\mathsf{L}}$ *is a finite Socratic transformation* $\mathbf{s}$ *of the question* $?(\vdash (A)^1)$ *via the rules of* $\mathbb{E}^{\mathsf{L}}$ *such that each constituent of the last question of* $\mathbf{s}$ *has the following form:*

$(e)\ \vdash S((B)^{\sigma_1/i})((\neg B)^{\sigma_2/i})$ □

Let us stress that 'σ_1/i' and 'σ_2/i' may be *different* sequences. Since possible numerals are indicated by single numerals (not by their sequences) it is not necessary that B and $\neg B$ have the same index but that their indices end with the same numeral i.

Here are some examples.

Example 5. *A Socratic proof of sequent* $\vdash (\Box(p \to q) \to (\Box p \to \Box q))^1$ *in* $\mathbb{E}^{\mathsf{K}}$. *Below* A *stands for* '$\neg\Box(p \to q)$' *and* B *for* '$\neg(\Box p)$'.

$$\cfrac{?\,(\vdash (\Box(p \to q) \to (\Box p \to \Box q))^1\,)}{\cfrac{?\,(\vdash A^1, (\Box p \to \Box q)^1\,)}{\cfrac{?\,(\vdash A^1, B^1, (\Box q)^1\,)}{\cfrac{?\,(\vdash A^1, B^1, (q)^{1,2}\,)}{\cfrac{?\,(\vdash A^1, (\neg(p \to q))^2, B^1, (q)^{1,2}\,)}{\cfrac{?\,(\vdash A^1, (\neg(p \to q))^2, B^1, (\neg p)^2, (q)^{1,2}\,)}{?\,(\vdash A^1, (p)^2, B^1, (\neg p)^2, (q)^{1,2}\,;\ \vdash A^1, (\neg q)^2, B^1, (\neg p)^2, (q)^{1,2}\,)}\,R_\alpha}\,R_\pi}\,R_\pi}\,R_\nu}\,R_\beta}\,R_\beta$$

□

Example 6. *A Socratic proof of sequent* $\vdash (\Diamond(\neg\Diamond(p \wedge q) \to \Box\Diamond\neg q \vee \Box\Box\neg p))^1$ *in* $\mathbb{E}^{\mathsf{D45}}$. *Below* A *stands for the whole formula* '$\Diamond(\neg\Diamond(p\wedge q) \to \Box\Diamond\neg q \vee \Box\Box\neg p)$', B *stands for* '$\Diamond(p \wedge q)$', *and* C *stands for* '$\Diamond\neg q$'. *In the constituents of the last question we skip the inner contexts.*

$$
\begin{array}{ll}
?(\vdash (\Diamond(\neg\Diamond(p \wedge q) \rightarrow \Box\Diamond\neg q \vee \Box\Box\neg p))^{1}\ ) & \\
\hline
?(\vdash A^{1}, (\neg\Diamond(p \wedge q) \rightarrow \Box\Diamond\neg q \vee \Box\Box\neg p)^{1,2}\ ) & R_{\pi D} \\
\hline
?(\vdash A^{1}, (\neg\neg\Diamond(p \wedge q))^{1,2}, (\Box\Diamond\neg q \vee \Box\Box\neg p)^{1,2}\ ) & R_{\beta} \\
\hline
?(\vdash A^{1}, B^{1,2}, (\Box\Diamond\neg q \vee \Box\Box\neg p)^{1,2}\ ) & R_{\neg\neg} \\
\hline
?(\vdash A^{1}, B^{1,2}, (\Box\Diamond\neg q)^{1,2}, (\Box\Box\neg p)^{1,2}\ ) & R_{\beta} \\
\hline
?(\vdash A^{1}, B^{1,2}, C^{1,2,3}, (\Box\Box\neg p)^{1,2}\ ) & R_{\nu} \\
\hline
?(\vdash A^{1}, B^{1,2}, C^{1,2,3}, (\Box\neg p)^{1,2,4}\ ) & R_{\nu} \\
\hline
?(\vdash A^{1}, B^{1,2}, C^{1,2,3}, (\neg p)^{1,2,4,5}\ ) & R_{\nu} \\
\hline
?(\vdash A^{1}, B^{1,2}, (p \wedge q)^{5}, C^{1,2,3}, (\neg p)^{1,2,4,5}\ ) & R_{\pi} \\
\hline
?(\vdash A^{1}, B^{1,2}, (p \wedge q)^{5}, C^{1,2,3}, (\neg q)^{5}, (\neg p)^{1,2,4,5}\ ) & R_{\pi} \\
\hline
?(\vdash \ldots, (p)^{5}, \ldots, (\neg p)^{1,2,4,5}\ ;\ \vdash \ldots, (q)^{5}, \ldots, (\neg q)^{5}, \ldots\ ) & R_{\alpha}
\end{array}
$$

□

The diagram on the left illustrates the relation between numerals in the above example. After closing the relation with respect to transitivity and Euclideanes it becomes universal in the set $\{2,3,4,5\}$ (on the right). This picture illustrates the justification of applications of R_{π}.

In terms of chains (see Appendix A) the proviso of applicability of R_{π} is satisfied as:

- the first application of R_{π}: $\langle 2,4,5\rangle$ is a directed $\mathbf{I}_R[\phi]$-chain, where 'ϕ' stands for '$\vdash A^{1}, B^{1,2}, C^{1,2,3}, (\neg p)^{1,2,4,5}$',

- the second application of R_{π}: $\langle 1,2,3\rangle$ and $\langle 1,2,4,5\rangle$ are both directed $\mathbf{I}_R[\psi]$-chains, where 'ψ' stands for '$\vdash A^{1}, B^{1,2}, (p \wedge q)^{5}, C^{1,2,3}, (\neg p)^{1,2,4,5}$'.

Soundness and completeness of erotetic calculi for basic modal logics

In the modal case the denotational interpretation of sequents is not a straightforward one. We can see that the explicit representation of semantic information in the sequents makes proving in the system easier, but complicates the meta-analysis.

For this reason in Leszczyńska, 2007, where the proofs of soundness and completeness with respect to Kripke semantics were presented, the author uses the so-called *interpretation functions* which map the semantic information encoded in a sequent into Kripke frames. By and large, it is not the case that every sequent can be interpreted in every Kripke frame, but a sequent of the form $\vdash (A)^1$ can, and under such an interpretation function it represents the formula A in an arbitrary world of a Kripke frame. The following theorems are arrived at in Leszczyńska, 2007:

Theorem 8 (soundness of $\mathbb{E}^{\mathsf{L}}$ wrt Kripke semantics). *If there exists a Socratic proof of a sequent* $\vdash (A)^1$ *in* $\mathbb{E}^{\mathsf{L}}$*, then the formula* A *is* L*-valid.*

See Theorem 2.3 in Leszczyńska, 2007, p. 48, and:

Theorem 9 (completeness of $\mathbb{E}^{\mathsf{L}}$ wrt Kripke semantics). *If* A *is a formula of* $\mathcal{M}$*, and is* L*-valid, then the sequent* $\vdash (A)^1$ *is provable in* $\mathbb{E}^{\mathsf{L}}$*.*

See Theorem 3.1 in Leszczyńska, 2007, p. 67. Completeness is proved by quite standard counter-model construction on the basis of a *complete* Socratic transformation of question $?(\vdash (A)^1)$ which is not a Socratic proof of the relevant sequent. A complete Socratic transformation is one constructed according to a certain procedure described in Leszczyńska, 2007. However, the procedure was developed for the purpose of completeness proof only, and it needs not terminate. The problem of developing a terminating procedure for the case of transitive modal logics was solved by the author in Leszczyńska-Jasion, 2009.

Erotetic calculi and other proof methods for modal logics: a discussion

After over a dozen years from its birth, it seems like a good time for a recapitulation of the method of Socratic proofs for modal logics.

From the point of view of the main traditional tasks of the proof theory, such as designing proof systems and proving their soundness and completeness, the method of Socratic proofs for modal logics does not distinguish itself among many other similar methods. What are the "similar methods" here?

As we have seen, erotetic calculi for modal logics are expressed in languages with indexed formulas which allow to express semantic information. For this reason, the erotetic calculi may be viewed as a kind of labelled deductive systems.[16] On the other hand, since the constituents of questions transformed by erotetic rules are *sequents*, the proof-theoretical relatives of the method of Socratic proofs seem to be proof methods based on sequent calculi. Among these, there are "ordinary" sequent systems, formulated in a "pure" language, *i.e.*, sequent calculi in which no semantic information is represented directly. Following (Goré, 1999), we may call this kind of systems *implicit*, as opposed to *explicit* systems, where the semantic information is explicitly represented. Implicit sequent systems for modal logics were developed already in the fifties of the XX-th century (*e.g.* Ohnishi and Matsumoto, 1957, 1959; see Poggiolesi, 2011 or Wansing, 2002 for an overview). The characteristic feature of implicit systems is their independence of the underlying semantics. In the margin—in Avron, 1996, the author postulates this feature as one characterising a "good" general proof-theoretical framework.[17]

In summary, among the proof-theoretical relatives mentioned above, the closest must be those based on sequent calculi, but with explicit representation of the semantic information, like in Negri, 2005; Orłowska, 1996; Konikowska, 1999. However, due to the deep affinity between sequent calculi and tableaux systems, one can mention here also explicit tableaux systems, like those in Viganò, 2000; Fitting, 1983; Priest, 2008.

Going back to the traditional tasks of the proof theory, as we have said, it seems like there is nothing in this field that can be obtained by the method of Socratic proofs for modal logics and that could not be obtained by other methods listed in the previous paragraph. However, erotetic calculi have been designed for the purpose of the modelling of what is now called *internal question processing*.[18] Certainly, not every other method could do. On the other hand, in the next chapter, where the issues of the logic of questions are analysed in more detail, it is argued that in the field of processing questions containing modal expressions, even the formalism of the method of Socratic proofs did not fulfil the erotetic task in a fully satisfactory way.

However, the erotetic formalism still has its potential in the proof theory, especially in the light of the new trends and tasks that have emerged

[16] Compare Gabbay, 1996; Viganò, 2000; Russo, 1995; Fitting, 1983.

[17] See also Poggiolesi, 2011 on this feature of proof systems.

[18] See Wiśniewski, 2013, Section 8.2.5.

within this discipline during the last few decades. There is a visible trend connected with the theory of computation, where the research priority occurs to be that of effectiveness of proving, and where the importance of computational issues is still growing.[19] There is also a clear awareness of diversity of various proof methods,[20] a tendency to construct and describe proof systems in an algorithmic manner, as they are described, *e.g.*, in Fitting, 1990, and a common awareness that these "[a]lgorithms are devised not to be «contemplated» but to be implemented".[21]

The potential of the method of Socratic proofs in the computationally oriented proof theory *is not* due to its general complexity. All indicates that it has the general complexity of a standard analytic tableaux methods—and this is not impressive.[22] However, we believe that the research on effectiveness of proof systems can be greatly enhanced by a comparative study of proof methods with diverse computational characteristics, and probably also by designing meta-formats combining such different proof methods. And from this perspective, the method of Socratic proofs is promising due to its quasi-hypersequent format. Transforming sequences of sequents by semantically invertible rules allows to retain all the semantic information in the transformed unit, and thus further the information can be recovered, when needed, in the form of a tree labelled with sequents, or in the form of a tableau. We shall examine these possibilities in Chapter 4.

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[19] See Troelstra and Schwichtenberg, 2000; Negri and von Plato, 2001; Buss, 1998b; Pudlák, 1998; D'Agostino et al., 1999; Indrzejczak, 2010; Pfenning, 2001; Schwichtenberg and Wainer, 2012; Cook and Nguyen, 2010.

[20] Paleo, 2017 is an interesting enterprise.

[21] Marcello D'Agostino in D'Agostino, 1999, p. 45.

[22] That the complexity of the two methods is in the same class is quite obvious when we analyse the method of Socratic proofs in the variant of Socratic trees, which will be described in Chapter 4. The computational drawbacks of tableaux methods are analysed at least since the early '80s, see Boolos, 1984.

In Grzelak and Leszczyńska-Jasion, 2018, the authors describe in detail an algorithmic procedure of generating proofs in an axiomatic system for CPL by means of the method of Socratic proofs for this logic. The result yields that the method of Socratic proofs for CPL is polynomially simulated by the axiomatic system for CPL considered in Grzelak and Leszczyńska-Jasion, 2018. This result is perfectly in line with the results described and/or obtained in D'Agostino, 1990; D'Agostino, 1992; D'Agostino and Mondadori, 1994, and it is clear that the method of Socratic proofs is affected by the same semantic / proof-theoretical / computational defect described by D'Agostino and Mondadori in the case of a standard analytic tableau method. On the other hand, there is an erotetic version of the **KE** system introduced by D'Agostino and Mondadori as a remedy, see Leszczyńska-Jasion et al., 2018, Section 5.

Chapter 3

Erotetic calculi are calculi of questions

The pivotal characteristic of erotetic calculi is that its construction serves both purely proof-theoretical purposes and more philosophical—erotetic—aims. Although this book focuses on proof-theoretical aspects of the method, the picture would certainly not be finished without the analysis of questions, since—we believe—the erotetic purpose was prior for the author of the method.

The idea standing behind the method of Socratic proofs was to model reasoning in which a question concerning derivability in the underlying logic is answered by "pure questioning", without the reference to an external source of information (see, first of all, the introductions to Wiśniewski, 2004 and Wiśniewski and Shangin, 2006). Therefore the construction of "Socratic" calculi consists in a formalization of rules transforming questions concerning derivability in the underlying logic. One could ask: what is meant by "derivability" here, but we shall not make an attempt to answer this question until Section 3.5. At the time being, we assume that some common intuitions concerning the notion are sufficient.

The process of transformation of questions is viewed as a step-by-step simplification of the structure of the initial problem. The transformations were called "Socratic", since the agent asking whether '$S \vdash A$' holds is driven through the consecutive questions to a final question which may be conceived as "rhetoric", as the answer to it is "obvious". What is meant by the "obviousness" is that the last question of a Socratic transformation concerns such basic cases of derivability like derivability of a formula from a set to which the formula belongs, or derivability of a formula from a set containing a formula and its negation. In other words, the last "rhetoric" question of a Socratic transformation concerns reflexivity and/or *ex falso quodlibet* as *obvious* properties of derivability.

Yet the more important issue is that of the legitimacy of the transitions between questions. The construction of erotetic calculi warrants that between a question-premise and a question-conclusion a very important relation holds: it is the relation of erotetic implication, a central notion for Inferential Erotetic Logic[1] (as previously, we will use 'IEL' for short). As we have explained in the first chapter, the notion of erotetic implication has been developed as a semantic criterion of correctness of erotetic reasoning, a tool dedicated to describe *valid erotetic inferences.* Therefore once we establish that the relation of erotetic implication holds between an instance of a question-premise and an instance of a question-conclusion of a rule of our erotetic calculus, we will be guaranteed that every Socratic transformation constructed within the calculus represents a correct erotetic reasoning.

It is the aim of this chapter to provide all the technical details in order to establish this connection. Therefore we, first, introduce the definition of erotetic implication for a simple language and explain the most important ideas on this example. Then, second, we introduce the details needed to define the notion of erotetic implication in the case of more complex languages considered in this book. This provides us with an additional semantic analysis of questions of the languages presented here and allows us to state that the rules of erotetic calculi formalize an inferential relation between questions whose semantic counterpart is the relation of erotetic implication.

Proof-theoretical properties determined by the erotetic goal

However, before we proceed to this task, let us take one more look at some of the properties of the erotetic calculi that are determined exactly by the fact that the method originated as a formalization of erotetic reasoning.

The initial motivation standing behind the method of Socratic proofs yields the use of both-sided, single succedent sequents—that is, sequents with single formula in the succedent—as expressions concerning derivability. Let us observe that it is in line with the so-called *operational interpretation* of sequents,[2] and also in line with thinking of a sequent calculus as a formal theory of derivability relation. However, traditionally, sequent calculi for classical logic are formalized with the use of mul-

[1] See Wiśniewski, 1994; 1995; 2001; 2013. See also the first chapter of this book.

[2] See Negri and von Plato, 2001, especially the beginning of Chapter 3, see also Section 2.1 of Chapter 2 of this book.

tisuccedent sequents, and it is commonly accepted that this is the feature that characterizes classical logic, as opposed to intuitionistic logic, formalized by the use of single succedent sequents. As one can see in Wiśniewski, 2004 and Wiśniewski and Shangin, 2006, as well as in the previous chapter of this book, the method of Socratic proofs for CPL and FOL is formalized with the use of single succedent sequents. However, it is not an argument against the commonly accepted view concerning the proof-theoretical difference between intuitionistic and classical logic, since—as we have seen in the previous chapter—the property of "single-succedentness" is somewhat artificially maintained by shifting a formula from the succedent to the antecedent of a sequent, every time when an intuitionistically problematic rule is applied, that is, exactly in the cases of β-formulas and δ-formulas on the right side of the turnstile. The paradigmatic examples of such cases are '$S \vdash A \vee B$' and '$S \vdash \exists x_j A$', which are transformed into '$S \,{'}\, \overline{A} \vdash B$' and '$S \,{'}\, \neg\exists x_j A \vdash A(a_i)$', respectively. In other words, it is tempting to say that the difference between "the classical" and "the intuitionistic" in this setting comes to the possibility of shifting a formula from right to left, and the possibility is another way of allowing more than one formula in the succedent.

Despite its attractiveness in the context of formalizing erotetic reasoning, the idea of "single-succedentness" has not been extended to non-classical logics. Modal logics (Leszczyńska, 2007), (Leszczyńska-Jasion, 2008), paraconsistent logics (Wiśniewski et al., 2005), (Chlebowski and Leszczyńska-Jasion, 2015) and a relevant logic (Szczepiński, 2018) were formalized by the use of multisuccedent sequents, and only in the case of the relevant logic it is clear that shifting is *really* forbidden. In Section 3.5 we will have something more to say about using multisuccedent sequents in the context of calculi formalizing erotetic reasoning.

Last but not least, also the notion of a basic sequent is, to some extent, related to the erotetic issues by the idea of an *obvious answer* to a question concerning derivability. What about derivability in logics that violate the properties of reflexivity and/or *ex falso*? As we have seen in the previous chapter, in the case of classical logic each basic sequent (the presence of which makes the affirmative answer to a question concerning derivability *obvious*) expresses at least one of the two properties. By and large, the criterion of being *basic*, and cosequently the notion of an *obvious answer*, depends on the underlying logic, and thus must be adapted to the analysed case. For example, in the case of paraconsistent and relevant logics, *ex falso* holds in some moderated form or does not hold at all. Therefore in the case of erotetic calculus for FDE (for "First Degree Entailment"—one of the primary relevance logics) basic sequents are only those referring to reflexivity (and therefore relevance) of derivability. In the paraconsistent case, the logic is augmented in a classical environment, since the language of the erotetic calculus contains, *int.al*, the classical negation except for the paraconsistent one. Therefore the basic sequents can be defined by the same general schemes that pertain to classical logic, where, obviously, in the case of a formula and its negation in the antecedent of a sequent, the negation is classical.

3.1. Erotetic implication: language $\mathcal{L}^?_{\mathsf{CPL}}$

Here we define in a formal way some of the notions introduced in Chapter 1. At the start, we follow (Wiśniewski, 2013).

We construct a language with questions, built upon the language $\mathcal{L}_{\mathsf{CPL}}$ of CPL, in the following way. First, we add the question mark '?', braces '{', '}' and the comma ',' to language $\mathcal{L}_{\mathsf{CPL}}$. By a *d-wff of language* $\mathcal{L}^?_{\mathsf{CPL}}$ we mean a formula of language $\mathcal{L}_{\mathsf{CPL}}$. A *question of language* $\mathcal{L}^?_{\mathsf{CPL}}$ is an expression of the form:

$$?\{A_1, \ldots, A_n\} \tag{3.1}$$

where $n > 1$ and $A_1, \ldots, A_n$ are pairwise syntactically distinct d-wffs of $\mathcal{L}^?_{\mathsf{CPL}}$. D-wffs $A_1, \ldots, A_n$ are called *direct answers* to question (3.1). By 'dQ' we denote the set of direct answers to question Q.

As previously, we use Boolean valuations (valuations, for short) in construing semantics for CPL. However, the following generalization of entailment in CPL will be useful in this chapter (see Shoesmith and Smiley, 1978):

Definition 10 (multiple-conclusion entailment in CPL). *Let X and Y stand for arbitrary sets of formulas of language $\mathcal{L}_{\mathsf{CPL}}$. We define the following relation:* set X multiple-conclusion entails set Y in $\mathcal{L}_{\mathsf{CPL}}$, *symbolically, $X \Vdash_{\mathsf{CPL}} Y$, iff:*

- *for every Boolean valuation v, if every element of X is true under v, then there is at least one element of Y true under v.* □

We quote the following definition from (Wiśniewski, 2013, p. 67):

Definition 11 (erotetic implication in $\mathcal{L}_{\mathsf{CPL}}$). *A* question Q implies a question Q^* on the basis of a set of d-wffs X (*in symbols:* $\mathsf{Im}(Q, X, Q^*)$) *iff:*

1. *for each* $A \in dQ$, $X \cup \{A\} \Vdash_{\mathsf{CPL}} dQ^*$, *and*
2. *for each* $B \in dQ^*$, *there exists a non-empty proper subset* Y *of* dQ *such that* $X \cup \{B\} \Vdash_{\mathsf{CPL}} Y$. □

If $\mathsf{Im}(Q, X, Q^*)$, then Q is often called the *implying question*, or the *initial question* (as Q *initiates* erotetic reasoning), the elements of X are called *declarative premises* (of the reasoning), and Q^* is called the *implied question*, or the *question-conclusion* of the reasoning. Also the abbreviation *e-implication* is sometimes used instead of "erotetic implication".

Clause 1 of Definition 11 expresses the property of the *transmission of soundness/truth into soundness*, which is an analogue of the transmission of truth characterizing entailment relation between declaratives. As we have explained in the first chapter, the counterpart of truth under valuation v is *soundness of a question under valuation* v; question Q is said to be *sound under* v iff at least one of its direct answers is true under v. Hence clause 1 of Definition 11 warrants that if the initial question Q is sound and the declarative premises in X are true, then the question-conclusion Q^* is sound as well. Clause 2 of the definition was designed to express the idea that each answer to the question-conclusion Q^* must be potentially useful in resolving the problem expressed by the initial question Q, that is, it must determine a non-empty proper subset Y of the set of direct answers to Q in which one may look for the true answer to Q, whenever Q^* is sound and the declarative premises in X are all true. The second property is called *open-minded cognitive usefulness* or *goal-directedness* (for more information see, *e.g.*, Wiśniewski, 2013, Chapter 5, Wiśniewski, 2001).

3.2. Erotetic implication in $\mathcal{L}^?_{\mathsf{CPL}}$: a discussion

The second property of erotetic implication amounts to the existence of a proper subset of dQ where the solution of the initial problem lies. However, for those who would say that the *mere existence* of the proper subset of dQ is less than *determining* the solution (whether partial or proper) we have the following suggestion. (Let us recall that the symbol '2^X' is used as the power set of X.)

Definition 12 (functional erotetic implication)**.** *Let* Q *and* Q^* *be questions of* $\mathcal{L}^?_{\mathsf{CPL}}$, *and let* X *stand for a set of d-wffs of* $\mathcal{L}^?_{\mathsf{CPL}}$. Question Q functionally implies question Q^* on the basis of X *iff:*

1. *for each* $A \in dQ$, $X \cup \{A\} \Vdash_{\mathsf{CPL}} dQ^*$, *and*

2. *there is a function* $f : dQ^* \longrightarrow 2^{dQ} \setminus \{dQ, \varnothing\}$ *such that for each* $B \in dQ^*$, $X \cup \{B\} \Vdash_{\mathsf{CPL}} f(B)$. □

Observe that the relation of *functional* erotetic implication is certainly a strengthening of the original notion. Clause 2 of Definition 12 assumes not only the existence of a proper subset for each direct answer to dQ^*, but the existence of a function that assigns the proper subsets to the direct answers. It seems that by putting some additional restrictions on the function (*i.e.*, that it is recursive, or that a procedure of counting its values is explicitly given) one can obtain new interesting—computational, perhaps—versions of the notion of erotetic implication.

On the other hand, sometimes it is argued that the notion of erotetic implication is a very strong concept, perhaps too strong due to the universal quantifier used in clause 2. of Definition 11 (see, for example, Urbański et al., 2016; Grobler, 2006). It is argued that when one looks for instances of reasoning with questions that may be modelled by the notion of erotetic implication, then it is more realistic to use the following weaker concept:

Definition 13 (weak erotetic implication). *Let* Q *and* Q^* *be questions of* $\mathcal{L}^?_{\mathsf{CPL}}$, *and let* X *stand for a set of d-wffs of* $\mathcal{L}^?_{\mathsf{CPL}}$. Question Q w-implies question Q^* on the basis of X (*in symbols:* $\mathsf{Im_w}(Q, X, Q_1)$) *iff:*

1. *for each* $A \in \mathsf{d}Q : X \cup \{A\} \Vdash_{\mathsf{CPL}} \mathsf{d}Q^*$, *and*

2. *for some* $B \in \mathsf{d}Q^*$ *: there exists a non-empty proper subset* Y *of* $\mathsf{d}Q$ *such that* $X \cup \{B\} \Vdash_{\mathsf{CPL}} Y$. □

In the case of the w-implication, it is enough if only one of the answers to the implied question leads us closer to a solution. Yet still, the transmission of soundness and truth into soundness (clause 1.) must be retained.

In fact, there is good empirical evidence (see Urbański et al., 2016) to claim that when a human agent solves a problem expressed by a question by means of asking other (implied) questions, (s)he tends to do it in a way which should be modelled formally by means of w-implication rather than e-implication.

One may also find various reasons to consider the following weakening of the original e-implication. (See, *e.g.*, Grobler, 2006 in the context of the philosophy of science.)

Definition 14 (falsificationist erotetic implication)**.** *Let* Q *and* Q^* *be questions of* $\mathcal{L}^?_{\mathsf{CPL}}$*, and let* X *stand for a set of d-wffs of* $\mathcal{L}^?_{\mathsf{CPL}}$. Question Q f-implies question Q^* on the basis of X (*in symbols:* $\mathsf{Im_f}(Q, X, Q^*)$) *iff:*

1. *for each* $A \in \mathsf{d}Q : X \cup \{A\} \Vdash_{\mathsf{CPL}} \mathsf{d}Q^*$*, and*
2. *for some* $B \in \mathsf{d}Q^* : X \cup \{B\}$ *eliminates some direct answer* C *to* Q*, that is, for each valuation* v*, if all the elements of* $X \cup \{B\}$ *are true under* v*, then the direct answer* C *is false under* v*.* □

The reader may find much more comments concerning the notions explicated in Definitions 11, 13, 14, intuitions behind them, and relations between them, in the book (Wiśniewski, 2013). For the purpose of this section it will be enough if we note once again that Definitions 11, 13, 14 do not differ with respect to the first clause.

The remaining part of this section is devoted to showing that whenever the first clause—common to all the four relations defined above in Definitions 11, 12, 13, 14—holds for two question Q, Q^* and a set X, then one can always construct an "interpolant"[3] question that will be erotetically implied by Q on the basis of X, and will imply Q^* on the basis of X enlarged with a *presupposition of* Q. The enlargement seems to be a small price for the result.

The notion of a presupposition of a question comes from Belnap (Belnap and Steel, 1976), and is usually understood as a d-wff which is entailed by each direct answer to the question (see Wiśniewski, 2013, Section 4.3), that is, roughly, a d-wff that must be true if the question is sound. In the present context we are interested in a special type of presupposition, called *prospective.*

Definition 15 (prospective presupposition of a question of $\mathcal{L}^?_{\mathsf{CPL}}$)**.** *Let* $Q = ?\{A_1, \ldots, A_n\}$ *be a question of* $\mathcal{L}^?_{\mathsf{CPL}}$*. By a* prospective presupposition of Q*, symbolically* PPQ*, we mean*

$$PPQ := A_1 \vee \ldots \vee A_n$$

□

The notion of prospective presupposition is usually understood in a more general way (see Wiśniewski, 2013, Section 4.3), but the above definition contains exactly what is needed in the present context.

[3] I owe Szymon Chlebowski the observation of the analogy between Theorem 10 and the Craig Interpolation Theorem.

Suppose that $Q = ?\{A_1, \ldots, A_n\}$ and $Q^* = ?\{B_1, \ldots, B_k\}$ are two questions such that for certain set X of d-wffs, the first clause of Definition 11 (and thus also of Definitions 12, 13, 14) holds. We do not presume anything about the second clause, so we may deal with (functional) erotetic implication, w-implication, f-implication, or no implication at all. Let us introduce the following operation.

$$Q \times Q^* := ?\{A_1 \wedge B_1, A_1 \wedge B_2, \ldots, A_1 \wedge B_k, \\ A_2 \wedge B_1, A_2 \wedge B_2, \ldots, A_2 \wedge B_k, \ldots, \\ A_n \wedge B_1, A_n \wedge B_2, \ldots, A_n \wedge B_k\}$$

That is, operation '$\times$' applied to questions Q and Q^* produces another question whose set $d(Q \times Q^*)$ of all direct answers equals:

$$d(Q \times Q^*) = \{A_i \wedge B_j : 1 \leq i \leq n, 1 \leq j \leq k\}$$

Observe that, by the assumption concerning the first clause of Definitions 11-14, for each $i : 1 \leq i \leq n$:

$$X \cup \{A_i\} \Vert\!\!\models_{\mathsf{CPL}} \{B_1, \ldots, B_k\}$$

It is easy to see that then also (for each i):

$$X \cup \{A_i\} \Vert\!\!\models_{\mathsf{CPL}} \{A_i \wedge B_1, \ldots, A_i \wedge B_k\}$$

and since $\{A_i \wedge B_1, \ldots, A_i \wedge B_k\} \subseteq d(Q \times Q^*)$, the more (for each i):

$$X \cup \{A_i\} \Vert\!\!\models_{\mathsf{CPL}} d(Q \times Q^*) \tag{3.2}$$

Moreover, for each i and for each j $(1 \leq j \leq k)$:

$$X \cup \{A_i \wedge B_j\} \Vert\!\!\models_{\mathsf{CPL}} \{A_i\} \tag{3.3}$$

Since $\{A_i\}$ is a non-empty proper subset of dQ, from (3.2) and (3.3) it follows that:

$$\mathsf{Im}(Q, X, Q \times Q^*)$$

On the other hand, for each i $(1 \leq i \leq n)$, and for each j $(1 \leq j \leq k)$:

$$X \cup \{A_i \wedge B_j\} \Vert\!\!\models_{\mathsf{CPL}} \{B_j\}$$

which warrants that also

$$X \cup \{A_i \wedge B_j\} \Vert\!\!\models_{\mathsf{CPL}} dQ^* \tag{3.4}$$

since $\{B_j\} \subseteq dQ^*$. Let us observe that (3.4) amounts to the first clause of the definition of erotetic implication. For the second one we need the prospective presupposition of question Q. The following holds:

$$X \cup \{PPQ, B_j\} \Vdash_{\mathsf{CPL}} \{A_1 \wedge B_j, \ldots, A_n \wedge B_j\} \tag{3.5}$$

since for every valuation v that makes each d-wff in $X \cup \{PPQ, B_j\}$ true, at least one of $A_1, \ldots, A_n$ is true under v, and hence also at least one of the listed conjunctions $A_1 \wedge B_j, \ldots, A_n \wedge B_j$ is true under v. As $\{A_1 \wedge B_j, \ldots, A_n \wedge B_j\}$ is a non-empty, proper subset of $d(Q \times Q^*)$, (3.4) and (3.5) tell us that:

$$\mathsf{Im}(Q \times Q^*, X, Q^*)$$

To sum up, if the soundness of question Q and the truth of declaratives in X warrant the soundness of Q^*, then $\mathsf{Im}(Q, X, Q \times Q^*)$, and $\mathsf{Im}(Q \times Q^*, X \cup \{PPQ\}, Q^*)$. More formally:

Theorem 10 (interpolation theorem for questions)**.** *Let Q and Q^* be questions of $\mathcal{L}^?_{\mathsf{CPL}}$, and let X stand for a set of d-wffs of $\mathcal{L}^?_{\mathsf{CPL}}$. If questions Q, Q^* and set X are such that for each $A_i \in dQ$:*

$$X \cup \{A_i\} \Vdash_{\mathsf{CPL}} dQ^*$$

then $\mathsf{Im}(Q, X, Q \times Q^*)$ and $\mathsf{Im}(Q \times Q^*, X \cup \{PPQ\}, Q^*)$.

Apart from throwing a new light on the notion of erotetic implication, we strongly believe that this result may have significant consequences for the construction of erotetic search scenarios (mentioned in the first chapter), but this topic definitely goes beyond the scope of this book.

Another observation makes use of the notion of a *safe question of language $\mathcal{L}^?_{\mathsf{CPL}}$*.

Definition 16 (safe and risky question of $\mathcal{L}^?_{\mathsf{CPL}}$)**.** *A question Q of $\mathcal{L}^?_{\mathsf{CPL}}$ is* safe *iff $\varnothing \Vdash_{\mathsf{CPL}} dQ$. A question which is not safe is called* risky. □

Let us observe that in the case of safe questions the first condition of our "implications", that warranting transmission of soundness and truth into soundness, is quite trivially satisfied. Therefore:

Corollary 5. *Let Q and Q^* stand for safe questions of $\mathcal{L}^?_{\mathsf{CPL}}$. Then:*

1. $\mathsf{Im}(Q, \varnothing, Q \times Q^*)$, *and*
2. $\mathsf{Im}(Q \times Q^*, \varnothing, Q^*)$.

Proof. Since Q^* is safe, $\varnothing \Vdash_{\mathsf{CPL}} dQ^*$. Trivially, for each $A \in dQ$, $\varnothing \cup \{A\} \Vdash_{\mathsf{CPL}} dQ^*$, which makes Theorem 10 applicable. Hence it follows that $\mathsf{Im}(Q, \varnothing, Q \times Q^*)$ and $\mathsf{Im}(Q \times Q^*, \{PPQ\}, Q^*)$. Therefore clause 1 is true. Now we show that also $\mathsf{Im}(Q \times Q^*, \varnothing, Q^*)$.

By Definition 16, for every valuation v, the set dQ contains a d-wff true under v. dQ is a finite set and thus PPQ is a well defined CPL-valid formula. Since, by Theorem 10, it holds that $\mathsf{Im}(Q \times Q^*, \{PPQ\}, Q^*)$, by the definition of erotetic implication (the first clause): for each $B \in d(Q \times Q^*)$, $\{PPQ, B\} \Vdash_{\mathsf{CPL}} dQ^*$. But since PPQ is true under every valuation, it yields that $\{B\} \Vdash_{\mathsf{CPL}} dQ^*$, which proves the first clause of the definition of erotetic implication for $\mathsf{Im}(Q \times Q^*, \varnothing, Q^*)$. The second clause holds by an analogous argument. □

Let $\mathcal{Q}$ stand for the set of all safe questions of language $\mathcal{L}^?_{\mathsf{CPL}}$. If we take the set of declarative premises X to be empty, then the relation of erotetic implication $\mathsf{Im}(Q, \varnothing, Q^*)$ between questions on the basis of the empty set of declarative premises becomes a binary relation. Let us make this observation more formal.

Definition 17. *Let Q and Q^* stand for questions of $\mathcal{L}^?_{\mathsf{CPL}}$. We say that* relation Im_{safe} holds between Q and Q^*, *symbolically* $\mathsf{Im}_{safe}(Q, Q^*)$, *iff* $\{Q, Q^*\} \subseteq \mathcal{Q}$ *and* $\mathsf{Im}(Q, \varnothing, Q^*)$. *By* Im^{tr}_{safe} *we mean the transitive closure of* Im_{safe}, *that is,* $\mathsf{Im}^{tr}_{safe}(Q, Q^*)$ *iff there exists a question* Q^{**} *of* $\mathcal{L}^?_{\mathsf{CPL}}$ *such that* $\mathsf{Im}_{safe}(Q, Q^{**})$ *and* $\mathsf{Im}_{safe}(Q^{**}, Q)$. □

Then by Corollary 5:

Corollary 6. $\mathsf{Im}^{tr}_{safe} = \mathcal{Q} \times \mathcal{Q}$.

In other words, Im^{tr}_{safe} is the full relation in the set $\mathcal{Q}$ of all safe questions of language $\mathcal{L}^?_{\mathsf{CPL}}$.

How to interpret this result? It shows that as long as only safe questions are considered, the transitive closure of erotetic implication holds between any two questions. In a way, this result in the field of the logic of questions is an analogue of the fact that as long as only valid declarative formulas are considered, entailment is a “trivial” notion, since the relation holds between any two such formulas. Seen from this perspective, the “trivialization” of the transitive closure of erotetic implication between safe questions is nothing surprising. It only shows that theories of questions which restrict their attention to safe questions may be way too narrow. Any entailment relation between questions defined on the grounds of such a theory may encounter a kind of collapse into a relation which holds between any two questions.

On the other hand, it is worth to stress that even if entailment considered only among valid formulas is not a very interesting notion, a *particular* move from a valid formula to a valid formula conducted within a proof system *is* a matter of interest, exactly because there are other than valid formulas. This shows that erotetic implication needs its proof systems—systems like the erotetic calculi presented in this book.

However, when risky questions come into the picture, the results presented in this section show another interesting thing. According to Theorem 10, if two questions fail to satisfy the second clause of Definition 11, but they satisfy the first, then there is always a way to put a "connector" (an interpolant) between them to retain the e-implication flow.

Originally, the notions of weak and that of falsificationist implication have been introduced because it was thought that the original relation is too strong in the logical sense. Just as deductive reasoning serves as a normative yardstick in the field of reasoning with declaratives—but hence it fails to give a successful account of some other non-normative reasoning, like inductive, probabilistic, and so on—so is erotetic implication a normative yardstick for reasoning with questions, yet incapable of capturing some interesting cases of erotetic reasoning. But the results presented in this section show that w-implication and f-implication do not have more explanative power, as every case of "weaker" implication may be captured by successive e-implications. *As to explanative power, it is rather the other way round!* Reasoning with w-implication or f-implication is like an *enthymematic erotetic argument*: enthymematic, since we reveal a hidden erotetic premise and use the erotetic implication to explain why the reasoning works.

3.3. Erotetic implication: languages $\mathcal{L}^{?}_{\vdash\mathsf{L}}$

In Section 2.1 we have described the syntax of language $\mathcal{L}^{?}_{\vdash\mathsf{L}}$, but we have not presented the notion of an answer to a question. Recall that questions of this language are of the form:

$$?(S_1 \vdash T_1, \ldots, S_n \vdash T_n) \tag{3.6}$$

where $\langle S_1 \vdash T_1, \ldots, S_n \vdash T_n \rangle$ is a finite and non-empty sequence of sequents of $\mathcal{L}^{?}_{\vdash\mathsf{L}}$. By *affirmative answer to question* (3.6) we mean an expression of the form:

$$(S_1 \vdash T_1 \ \& \ldots \& \ (S_{n-1} \vdash T_{n-1} \ \& \ S_n \vdash T_n) \ldots) \tag{3.7}$$

and by *negative answer to question* (3.6) we mean an expression of the form:

$$ng(S_1 \vdash T_1 \ \& \ldots \& \ (S_{n-1} \vdash T_{n-1} \ \& \ S_n \vdash T_n) \ldots) \tag{3.8}$$

We will use afQ and ngQ for the answers of the form (3.7), (3.8), respectively. Thus all questions of $\mathcal{L}^?_{\vdash \mathsf{L}}$ are polar questions whose sets of direct answers are of the forms $\{afQ, ngQ\}$.

Definition 11 is based upon the semantic notion of Boolean valuation; in the case of $\mathcal{L}^?_{\vdash \mathsf{L}}$, however, we need something more general. The notions introduced below are central tools of the so-called *Minimal Erotetic Semantics* (MiES, for short), a very general framework for semantic analysis of both declaratives and questions developed by Andrzej Wiśniewski.[4] The primary notion is that of a partition of a language (the notion comes from Shoesmith and Smiley, 1978).

Definition 18 (partition of language $\mathcal{L}^?_{\vdash \mathsf{L}}$). *Let* $\mathsf{D}_{\mathcal{L}^?_{\vdash \mathsf{L}}}$ *be the set of d-wffs of language* $\mathcal{L}^?_{\vdash \mathsf{L}}$. *By* partition of $\mathsf{D}_{\mathcal{L}^?_{\vdash \mathsf{L}}}$ (*or* partition of language $\mathcal{L}^?_{\vdash \mathsf{L}}$) *we mean an ordered pair* $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ *such that* $\mathsf{T_P} \cup \mathsf{U_P} = \mathsf{D}_{\mathcal{L}^?_{\vdash \mathsf{L}}}$ *and* $\mathsf{T_P} \cap \mathsf{U_P} = \varnothing$. □

Observe that nothing more is assumed about $\mathsf{T_P}$ (the set of "truths" in a partition) and $\mathsf{U_P}$ (the set of "untruths" in a partition) except for the two conditions.

In the case of complex languages with questions, like $\mathcal{L}^?_{\vdash \mathsf{L}}$, the counterpart of the semantic notion of Boolean valuation is that of *admissible partition*. Definition 19 presents the notion for the case of language $\mathcal{L}^?_{\vdash \mathsf{CPL}}$. It is a straightforward generalization of the notion given by Andrzej Wiśniewski (see Wiśniewski, 2013, pp. 28-29), which works for all the three erotetic calculi for CPL presented in this book.

Definition 19 (admissible partition of $\mathcal{L}^?_{\vdash \mathsf{CPL}}$). *Let* $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ *be a partition of language* $\mathcal{L}^?_{\vdash \mathsf{CPL}}$, *and let* $\mathfrak{t}$ *and* $\mathfrak{u}$ *stand for arbitrary d-wffs of this language. We say that* P *is an* admissible partition of language $\mathcal{L}^?_{\vdash \mathsf{CPL}}$ *iff the following conditions hold:*

[4] As Wiśniewski himself admits (see Wiśniewski, 2018, Section 2), some ideas come from Belnap (Belnap and Steel, 1976) and Shoesmith and Smiley (Shoesmith and Smiley, 1978), but MiES goes far beyond them. For the details of MiES see Wiśniewski, 2013 or Wiśniewski, 2015.

1. ‘$\mathfrak{t}\&\mathfrak{u}$’ $\in \mathsf{T_P}$ *iff both* $\mathfrak{t} \in \mathsf{T_P}$ *and* $\mathfrak{u} \in \mathsf{T_P}$;

2. $\mathfrak{t} \in \mathsf{T_P}$ *iff* ‘$ng\mathfrak{t}$’ $\notin \mathsf{T_P}$;

3. ‘$S \vdash T(\alpha)$’ $\in \mathsf{T_P}$ *iff both* ‘$S \vdash T(\alpha/\alpha_1)$’ $\in \mathsf{T_P}$ *and* ‘$S \vdash T(\alpha/\alpha_2)$’ $\in \mathsf{T_P}$;

4. ‘$\vdash T(\beta)$’ $\in \mathsf{T_P}$ *iff* ‘$\vdash T(\beta/\beta_1, \beta_2)$’ $\in \mathsf{T_P}$;

5. ‘$S \vdash \beta$’ $\in \mathsf{T_P}$ *iff* ‘$S\ '\ \overline{\beta_1} \vdash \beta_2$’ $\in \mathsf{T_P}$;

6. ‘$S \vdash T(\kappa)$’ $\in \mathsf{T_P}$ *iff* ‘$S \vdash T(\kappa/\kappa^*)$’ $\in \mathsf{T_P}$;

7. ‘$S(\alpha) \vdash T$’ $\in \mathsf{T_P}$ *iff* ‘$S(\alpha/\alpha_1, \alpha_2) \vdash T$’ $\in \mathsf{T_P}$;

8. ‘$S(\beta) \vdash T$’ $\in \mathsf{T_P}$ *iff both* ‘$S(\beta/\beta_1) \vdash T$’ $\in \mathsf{T_P}$ *and* ‘$S(\beta/\beta_2) \vdash T$’ $\in \mathsf{T_P}$;

9. ‘$S(\kappa) \vdash T$’ $\in \mathsf{T_P}$ *iff* ‘$S(\kappa/\kappa^*) \vdash T$’ $\in \mathsf{T_P}$. □

Let us recall that in the case of CPL, κ is restricted to formulas of the form ‘$\neg\neg A$’. It is also worth recalling that the sequents are presented in a general form but S and T are subject to constraints described in Section 2.1.

The notion of admissible partition for the case of erotetic calculus $\mathbf{E}^{\mathsf{PQ}}$ has been presented in Wiśniewski and Shangin, 2006. The calculus needs both-sided, single succedent sequents, thus the notion presented in Wiśniewski and Shangin, 2006 concerns only sequents of this category. (It is also worth to stress that in Wiśniewski and Shangin, 2006 the construction of admissible partitions stems from intuitions concerning ‘$\vdash$’ as referring to derivability, and therefore it includes a dose of structural properties.) However, language $\mathcal{L}^{?}_{\vdash \mathsf{FOL}}$ defined in this book has also right-sided and left-sided sequents, thus here we present, again, a more general version of the notion of admissible partition.

Definition 20 (admissible partition of $\mathcal{L}^{?}_{\vdash \mathsf{FOL}}$)**.** *Let* $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ *be a partition of language* $\mathcal{L}^{?}_{\vdash \mathsf{FOL}}$. *We say that* P *is an* admissible partition of language $\mathcal{L}^{?}_{\vdash \mathsf{FOL}}$ *iff clauses* 1–9 *from Definition* 19 *are satisfied for* P, *and, moreover, the following conditions hold:*

10. *For each parameter* a_i *which does not occur in sequent* ‘$S \vdash T(\gamma)$’,

    ‘$S \vdash T(\gamma)$’ $\in \mathsf{T_P}$ *iff* ‘$S \vdash T(\gamma/\gamma(a_i))$’ $\in \mathsf{T_P}$.

11. For each parameter a_i which does not occur in sequent '$S(\delta) \vdash T$',
'$S(\delta) \vdash T$' $\in \mathsf{T_P}$ *iff* '$S(\delta/\delta(a_i)) \vdash T$' $\in \mathsf{T_P}$.

12. Finally, for each parameter a_i of language $\mathcal{L}_{\mathsf{FOL}}$,

12.1. '$S(\gamma) \vdash$' $\in \mathsf{T_P}$ *iff* '$S(\gamma/\gamma, \gamma(a_i)) \vdash$' $\in \mathsf{T_P}$,
12.2. '$S(\gamma) \vdash C$' $\in \mathsf{T_P}$ *iff* '$S(\gamma/\gamma, \gamma(a_i)) \vdash C$' $\in \mathsf{T_P}$,
12.3. '$\vdash S(\delta)$' $\in \mathsf{T_P}$ *iff* '$\vdash S(\delta/\delta, \delta(a_i))$' $\in \mathsf{T_P}$,
12.4. '$S \vdash \delta$' $\in \mathsf{T_P}$ *iff* '$S\,'\,\overline{\delta} \vdash \delta(a_i)$' $\in \mathsf{T_P}$. □

Let us observe that clause 10 from Definition 20 is equivalent to the following (metalanguage) conjunction:

- for each parameter a_i which does not occur in sequent '$S \vdash T(\gamma)$', if G, then H, AND
- for each parameter a_i which does not occur in sequent '$S \vdash T(\gamma)$', if H, then G

where G and H stand for, respectively:

$$G : \text{'}S \vdash T(\gamma)\text{'} \in \mathsf{T_P}$$

$$H : \text{'}S \vdash T(\gamma/\gamma(a_i))\text{'} \in \mathsf{T_P}$$

However, by assumption, the quantification "for each parameter a_i ..." is empty over G, therefore, taking some quantifier laws for granted in the metalanguage,[5] clause 10 occurs to be equivalent to the conjunction of the following two clauses:

10.1 If '$S \vdash T(\gamma)$' $\in \mathsf{T_P}$, then for each parameter a_i which does not occur in sequent '$S \vdash T(\gamma)$', '$S \vdash T(\gamma/\gamma(a_i))$' $\in \mathsf{T_P}$.

10.2 If for some parameter a_i which does not occur in sequent '$S \vdash T(\gamma)$', '$S \vdash T(\gamma/\gamma(a_i))$' $\in \mathsf{T_P}$, then '$S \vdash T(\gamma)$' $\in \mathsf{T_P}$.

Clause 11 of Definition 20 can be analysed in the same manner.

Finally, we are in a position to define the notion of entailment between d-wffs of languages $\mathcal{L}^?_{\vdash\mathsf{CPL}}$ and $\mathcal{L}^?_{\vdash\mathsf{FOL}}$, and the notion of erotetic implication between questions of the languages.

[5] If variable x_i does not occur free in A, then '$\forall x_i(A \to B) \leftrightarrow (A \to \forall x_i B)$' and '$\forall x_i(B \to A) \leftrightarrow (\exists x_i B \to A)$' are the laws of FOL.

Definition 21 (entailment in $\mathcal{L}^?_{\vdash\mathsf{L}}$)**.** *Let* $\mathsf{L} \in \{\mathsf{CPL}, \mathsf{FOL}\}$*. Suppose that* $\mathfrak{X}$ *is a set of d-wffs of language* $\mathcal{L}^?_{\vdash\mathsf{L}}$ *and* $\mathfrak{t}$ *is a single d-wff of the language. We say that* set $\mathfrak{X}$ entails formula $\mathfrak{t}$ in language $\mathcal{L}^?_{\vdash\mathsf{L}}$*, symbolically:*

$$\mathfrak{X} \vDash_{\mathcal{L}^?_{\vdash\mathsf{L}}} \mathfrak{t}$$

iff there is no admissible partition P *of language* $\mathcal{L}^?_{\vdash\mathsf{L}}$ *such that* $\mathfrak{X} \subseteq \mathsf{T}_\mathsf{P}$ *and* $\mathfrak{t} \notin \mathsf{T}_\mathsf{P}$*.* □

As before, dQ stands for the set of direct answers to Q.

Definition 22 (erotetic implication in $\mathcal{L}^?_{\vdash\mathsf{L}}$)**.** *Let* $\mathsf{L} \in \{\mathsf{CPL}, \mathsf{FOL}\}$*. Suppose that* Q *and* Q^* *are questions of* $\mathcal{L}^?_{\vdash\mathsf{L}}$ *and that* $\mathfrak{X}$ *is a set of d-wffs of* $\mathcal{L}^?_{\vdash\mathsf{L}}$*. We say that* question Q implies question Q^* on the basis of set $\mathfrak{X}$ of d-wffs (*symbolically,* $\mathsf{Im}_\mathsf{L}(Q, \mathfrak{X}, Q^*)$) *iff for each admissible partition* P *of language* $\mathcal{L}^?_{\vdash\mathsf{L}}$ *the following holds:*

1. *for each* $\mathfrak{t} \in dQ$*: if* $\mathfrak{X} \cup \{\mathfrak{t}\} \subseteq \mathsf{T}_\mathsf{P}$*, then* $dQ^* \cap \mathsf{T}_\mathsf{P} \neq \varnothing$*; and*
2. *for each* $\mathfrak{u} \in dQ^*$*: there is a non-empty proper subset* $\mathfrak{Y}$ *of* dQ *such that if* $\mathfrak{X} \cup \{\mathfrak{u}\} \subseteq \mathsf{T}_\mathsf{P}$*, then* $\mathfrak{Y} \cap \mathsf{T}_\mathsf{P} \neq \varnothing$*.* □

In the case of questions of language $\mathcal{L}^?_{\vdash\mathsf{L}}$, dQ is always a two-element set. For this reason the second clause of Definition 22 may take the following simplified form.

2^*. *For each* $\mathfrak{u} \in dQ^*$*, there is a single answer* $\mathfrak{t} \in dQ$ *such that if* $\mathfrak{X} \cup \{\mathfrak{u}\} \subseteq \mathsf{T}_\mathsf{P}$*, then* $\mathfrak{t} \in \mathsf{T}_\mathsf{P}$*.*

One may think, however, of more sophisticated cases of erotetic implication for which the more general clause is more appropriate (see Wiśniewski, 2013).

We end this section with the following theorem:

Theorem 11. *Let* $\mathbb{E}$ *stand for one of* $\mathbb{E}^{\mathbf{PQ}}$*,* $\mathbb{E}^{\mathbf{RPQ}}$*,* $\mathbb{E}^{\mathbf{LPQ}}$ *or their propositional parts. If question* Q^* *results from question* Q *by a rule of* $\mathbb{E}$*, then* $\mathsf{Im}_\mathsf{L}(Q, \varnothing, Q^*)$*.*

Proof. The proof is by cases and we analyse only some of them, namely R_α of $\mathbb{E}^{\mathbf{PQ}}$, L_γ of $\mathbb{E}^{\mathbf{LPQ}}$ and R_γ of $\mathbb{E}^{\mathbf{RPQ}}$.

In the first case Q and Q^* are of the forms:

$$Q = ?(\Phi\ ;\ S \vdash \alpha\ ;\ \Psi) \qquad Q^* = ?(\Phi\ ;\ S \vdash \alpha_1\ ;\ S \vdash \alpha_2\ ;\ \Psi)$$

Let $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ be an arbitrary admissible partition of $\mathcal{L}^?_{\vdash \mathsf{FOL}}$. The reasoning is straightforward. If $afQ \in \mathsf{T_P}$, then, by clause 1 of Definition 19, also '$S \vdash \alpha$' $\in \mathsf{T_P}$ and, by clause 3 of the same Definition, $\{S \vdash \alpha_1, S \vdash \alpha_2\} \subseteq \mathsf{T_P}$. But then again, by clause 1, $afQ^* \in \mathsf{T_P}$. If $ngQ \in \mathsf{T_P}$, then we make use of clauses 1 and 2 of Definition 19, which yield together that at least one constituent of question Q is not in $\mathsf{T_P}$. If the constituent is '$S \vdash \alpha$', then at least one of '$S \vdash \alpha_1$', '$S \vdash \alpha_2$' is not in $\mathsf{T_P}$, which yields that $ngQ^* \in \mathsf{T_P}$. Altogether this shows that clause 1 of the definition of erotetic implication (see Definition 22) is satisfied. If we recall clause 2^*, then it is easy to see that the rest of the proof comes to showing that "if $afQ^* \in \mathsf{T_P}$, then $afQ \in \mathsf{T_P}$" and "if $ngQ^* \in \mathsf{T_P}$, then $ngQ \in \mathsf{T_P}$", which is analogous to what has been already shown.

Now suppose that Q^* results from Q by L_γ of $\mathbb{E}^{\mathbf{LPQ}}$. Then the questions are of the forms:

$$Q = ?(\Phi \ ; \ S(\gamma) \vdash \ ; \ \Psi) \qquad Q^* = ?(\Phi \ ; \ S(\gamma/\gamma, \gamma(a_i)) \vdash \ ; \ \Psi)$$

where a_i is an arbitrary parameter of $\mathcal{L}_{\mathsf{FOL}}$. By clause 12.1 of Definition 20, for each such parameter '$S(\gamma) \vdash$' $\in \mathsf{T_P}$ iff '$S(\gamma/\gamma, \gamma(a_i)) \vdash$' $\in \mathsf{T_P}$. Together with clause 1 of Definition 19 this proves that $afQ \in \mathsf{T_P}$ iff $afQ^* \in \mathsf{T_P}$, and together with clause 2—that $ngQ \in \mathsf{T_P}$ iff $ngQ^* \in \mathsf{T_P}$, which is exactly what we need for erotetic implication to hold.

For the case like R_γ of $\mathbb{E}^{\mathbf{RPQ}}$ the reasoning is analogous. We omit the part involving clauses 1 and 2 of Definition 19. We use clause 10 of Definition 20 taking S to be empty. The questions are of the forms:

$$Q = ?(\Phi \ ; \ \vdash T(\gamma) \ ; \ \Psi) \qquad Q^* = ?(\Phi \ ; \ \vdash T(\gamma/\gamma(a_i)) \ ; \ \Psi)$$

where a_i is a parameter that does not occur in sequent '$\vdash T(\gamma)$'. Suppose that $afQ \in \mathsf{T_P}$, it follows then that '$\vdash T(\gamma)$' $\in \mathsf{T_P}$. Then also, by clause 10 (see clause 10.1), '$\vdash T(\gamma/\gamma(a_i))$' $\in \mathsf{T_P}$. Now suppose that $afQ^* \in \mathsf{T_P}$. Then it follows that '$\vdash T(\gamma/\gamma(a_i))$' $\in \mathsf{T_P}$, and, since a_i does not occur in '$\vdash T(\gamma)$', clause 10 (see clause 10.2) applies to a_i, thus '$\vdash T(\gamma)$' $\in \mathsf{T_P}$. The reasoning is analogous for the negative answers. Suppose that $ngQ \in \mathsf{T_P}$ and assume that $\vdash T(\gamma) \notin \mathsf{T_P}$. Then also $\vdash T(\gamma/\gamma(a_i)) \notin \mathsf{T_P}$, since clause 10 holds for every parameter a_i that does not occur in the relevant sequent (see the transposition of clause 10.2). Thus also $ngQ^* \in \mathsf{T_P}$. Now assume that $ngQ^* \in \mathsf{T_P}$, and additionally that $\vdash T(\gamma/\gamma(a_i)) \notin \mathsf{T_P}$, where a_i does not occur in $\vdash T(\gamma)$. Since clause 10 holds for every such parameter (transposition of 10.1), it follows that $\vdash T(\gamma) \notin \mathsf{T_P}$, and therefore $ngQ \in \mathsf{T_P}$. □

The above Theorem concerns a special case of erotetic implication—one with empty set of declarative premises. It is called *pure erotetic implication* (see Wiśniewski, 2013 for more information).

But the situation is even more specific. The proof of the above Theorem shows that if question Q^* results from question Q by a rule of $\mathbb{E}$, then the affirmative answer to one of the questions entails, in the sense of Definition 21, the affirmative answer to the second question, and the same holds with respect to the negative answers to the two questions. It is a special case of equipollence of questions defined by Tadeusz Kubiński.

3.4. Erotetic implication: the modal case

As in the classical case, the erotetic part of language $\mathcal{M}^?_\vdash$ may be supplied with a semantics to the effect that the notion of erotetic implication can be defined. This subsection is partially based on Section 2.4 of (Leszczyńska, 2007), but the definitions are different.

In this subsection L represents the 15 basic modal propositional logics (see Section 2.3 of the previous chapter). Until the end of this subsection we also shall use letters $\mathfrak{t}$, $\mathfrak{u}$ for d-wffs of language $\mathcal{M}^?_\vdash$ and $\mathfrak{X}$ for sets of d-wffs of this language. We define the notion of partition of language $\mathcal{M}^?_\vdash$ just as before (*cf.* Definition 18), and we connect the notion of the admissibility of a partition with the underlying logic L. For instance, a K-admissible partition of language $\mathcal{M}^?_\vdash$ is defined as follows:

Definition 23 (K-admissible partition of $\mathcal{M}^?_\vdash$). *A partition* $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P} \rangle$ *of language* $\mathcal{M}^?_\vdash$ *is* K-admissible *iff the following conditions are satisfied:*

1. '$\mathfrak{t}\&\mathfrak{u}$' $\in \mathsf{T_P}$ *iff both* $\mathfrak{t} \in \mathsf{T_P}$ *and* $\mathfrak{u} \in \mathsf{T_P}$;
2. $\mathfrak{t} \in \mathsf{T_P}$ *iff* '$\underline{ng}\mathfrak{t}$' $\notin \mathsf{T_P}$;
3. '$\vdash S(\alpha^{\sigma})$' $\in \mathsf{T_P}$ *iff both* '$\vdash S(\alpha^{\sigma} \,/\, \alpha_1^{\sigma})$' $\in \mathsf{T_P}$ *and* '$\vdash S(\alpha^{\sigma} \,/\, \alpha_2^{\sigma})$' $\in \mathsf{T_P}$;
4. '$\vdash S(\beta^{\sigma})$' $\in \mathsf{T_P}$ *iff* '$\vdash S(\beta^{\sigma} \,/\, \beta_1^{\sigma},\, \beta_2^{\sigma})$' $\in \mathsf{T_P}$;
5. '$\vdash S(\neg\neg A^{\sigma})$' $\in \mathsf{T_P}$ *iff* '$\vdash S(\neg\neg A^{\sigma} \,/\, A^{\sigma})$' $\in \mathsf{T_P}$;
6. *for each numeral j which is not an element of* $\mathbf{I}_W\{\vdash S(\nu^{\sigma/i})\}$,

   '$\vdash S(\nu^{\sigma/i})$' $\in \mathsf{T_P}$ *iff* '$\vdash S(\nu^{\sigma/i} \,/\, \nu_0^{\sigma/i,j})$' $\in \mathsf{T_P}$;

7. for each numeral j such that $\langle i, j\rangle \in \mathbf{I}_R[\vdash S(\pi^{\sigma/i})\,]$,
 '$\vdash S(\pi^{\sigma/i})$' $\in \mathsf{T_P}$ *iff* '$\vdash S(\pi^{\sigma/i} \,/\, \pi^{\sigma/i},\ \pi_0^{j})$' $\in \mathsf{T_P}$. □

For further use we set what follows. By $\mathbf{I}_R[\varphi]^{\mathsf{L}}$ we mean the respective closure of relation $\mathbf{I}_R[\varphi]$ which is specific to L (see Table 2.8).

Definition 24 (L-admissible partition of $\mathcal{M}^?_\vdash$)**.** *A partition* $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P}\rangle$ *of language* $\mathcal{M}^?_\vdash$ *is* L-admissible *iff it satisfies clauses 1-6 of Definition 23, clause 7*:*

7. for each numeral j such that $\langle i, j\rangle \in \mathbf{I}_R[\vdash S(\pi^{\sigma/i})\,]^{\mathsf{L}}$,*
 '$\vdash S(\pi^{\sigma/i})$' $\in \mathsf{T_P}$ *iff* '$\vdash S(\pi^{\sigma/i} \,/\, \pi^{\sigma/i},\ \pi_0^{j})$' $\in \mathsf{T_P}$

and for $\mathsf{L} = \mathsf{D}, \mathsf{DB}, \mathsf{D4}, \mathsf{D5}, \mathsf{D45}$ *also clause D.:*

D. for each j which is not an element of $\mathbf{I}_W\{\vdash S(\pi^{\sigma/i})\,\}$,
 '$\vdash S(\pi^{\sigma/i})$' $\in \mathsf{T_P}$ *iff* '$\vdash S(\pi^{\sigma/i} \,/\, \pi^{\sigma/i},\ \pi_0^{\sigma/i,j})$' $\in \mathsf{T_P}$. □

As in the case of classical logic, the notion of L-admissible partition allows to express both entailment between d-wffs of $\mathcal{M}^?_\vdash$ and erotetic implication between questions of this language. The definitions follow strictly those presented above.

Definition 25 (entailment in $\mathcal{M}^?_\vdash$)**.** *Let $\mathfrak{X}$ be a set of d-wffs of language $\mathcal{M}^?_\vdash$ and let $\mathfrak{t}$ stand for a single d-wff of the language. We say that* set $\mathfrak{X}$ entails formula $\mathfrak{t}$ in language $\mathcal{M}^?_\vdash$, *symbolically:* $\mathfrak{X} \models_{\mathcal{M}^?_\vdash \mathsf{L}} \mathfrak{t}$ *iff there is no L-admissible partition P of language $\mathcal{M}^?_\vdash$ such that $\mathfrak{X} \subseteq \mathsf{T_P}$ and $\mathfrak{t} \notin \mathsf{T_P}$.* □

As previously, the notion of erotetic implication may be defined in a general form, as in Definition 26.

Definition 26 (erotetic implication in $\mathcal{M}^?_\vdash$ relative to L)**.** *Let Q and Q^* be questions of $\mathcal{M}^?_\vdash$ and let $\mathfrak{X}$ be a set of d-wffs of $\mathcal{M}^?_\vdash$. We say that* question Q implies in $\mathcal{M}^?_\vdash$ relatively to L question Q^* on the basis of set $\mathfrak{X}$ of d-wffs (*symbolically:* $\mathbf{Im}_{\mathsf{L}}(Q, \mathfrak{X}, Q^*)$) *iff for each L-admissible partition P of language $\mathcal{M}^?_\vdash$,*

1. *for each $\mathfrak{t} \in dQ$, if $\mathfrak{X} \cup \{\mathfrak{t}\} \subseteq \mathsf{T_P}$, then $dQ^* \cap \mathsf{T_P} \neq \varnothing$; and*

2. *for each $\mathfrak{u} \in dQ^*$, there is a non-empty proper subset $\mathfrak{Y}$ of dQ such that if $\mathfrak{X} \cup \{\mathfrak{u}\} \subseteq \mathsf{T_P}$, then $\mathfrak{Y} \cap \mathsf{T_P} \neq \varnothing$.* □

The following theorem is true.

Theorem 12. *Let* Q *and* Q^* *be questions of* $\mathcal{M}^?_\vdash$. *If* Q^* *results from* Q *by a rule* $\mathbf{r}$ *of* $\mathbb{E}^{\mathsf{L}}$, *then* $\mathbf{Im}_{\mathsf{L}}(Q, \varnothing, Q^*)$.

Proof. Let us consider the case of $\mathbf{r} = R_\pi$. Questions Q and Q^* are of the following forms:

$$Q = ?(\Phi\,;\, \vdash S(\pi^{\sigma/i})\,;\, \Psi) \qquad Q^* = ?(\Phi\,;\, \vdash S(\pi^{\sigma/i}/\pi^{\sigma/i}, \pi_0^j)\,;\, \Psi)$$

Suppose that $\mathsf{P} = \langle \mathsf{T_P}, \mathsf{U_P}\rangle$ is an L-admissible partition of language $\mathcal{M}^?_\vdash$. We show that $afQ \in \mathsf{T_P}$ iff $afQ^* \in \mathsf{T_P}$, and that $ngQ \in \mathsf{T_P}$ iff $ngQ^* \in \mathsf{T_P}$.

Hence assume that $afQ \in \mathsf{T_P}$. Then, in particular, '$\vdash S(\pi^{\sigma/i})$' $\in \mathsf{T_P}$. Since the rule is applied in calculus $\mathbb{E}^{\mathsf{L}}$, the proviso of applicability of R_π must be satisfied, thus $\langle i, j\rangle \in \mathbf{I}_R\left[\vdash S(\pi^{\sigma/i})\right]^{\mathsf{L}}$. By clause *7** of Definition 24, '$\vdash S(\pi^{\sigma/i}/\pi^{\sigma/i}, \pi_0^j)$' $\in \mathsf{T_P}$, hence also $afQ^* \in \mathsf{T_P}$. The converse implication—if $afQ^* \in \mathsf{T_P}$, then $afQ \in \mathsf{T_P}$—follows by the same clause *7**.

We reason analogously for the negative answers. Numeral j is such that $\langle i, j\rangle \in \mathbf{I}_R\left[\vdash S(\pi^{\sigma/i})\right]^{\mathsf{L}}$, therefore, by clause *7**, '$\vdash S(\pi^{\sigma/i})$' $\notin \mathsf{T_P}$ iff '$\vdash S(\pi^{\sigma/i}/\pi^{\sigma/i}, \pi_0^j)$' $\notin \mathsf{T_P}$. This is enough to show that $ngQ \in \mathsf{T_P}$ iff $ngQ^* \in \mathsf{T_P}$, we skip the remaining details.

The reasoning is analogous for the cases of $\mathbf{r} \in \{R_\nu, R_{\pi D}\}$. We make use of clause *6.* of Definition 23 and clause *D.* of Definition 24, respectively, and of the observation that if j equals $max(\mathbf{I}_W\{\phi\}) + 1$ (where ϕ is $\vdash S(\nu^{\sigma/i})$ or $\vdash S(\pi^{\sigma/i})$, respectively), then it is not an element of $\vdash S(\nu^{\sigma/i})$ / $\vdash S(\pi^{\sigma/i})$, respectively. □

Thus, as in the classical case, we can say that the rules of erotetic calculi for modal logics formalize a special case of erotetic implication.

Unfortunately, the above Definition 26 of erotetic implication does not go in line with any natural, straightforward interpretation of sequents of $\mathcal{M}^?_\vdash$. Although we can say that each sequent of the form $\vdash (A)^1$ can be interpreted as a question concerning validity of formula A in a specific class of Kripke frames, this interpretation does not extend easily to other sequents of $\mathcal{M}^?_\vdash$. There are at least two problems to overcome in order to develop a satisfying erotetic theory of questions concerning "derivability" in modal logics. The first one concerns the proper interpretation of "derivability", since, as is well-known, there are at least two relations of entailment in modal logics: the local and the global one, and there are two respective notions of derivability to be studied. Erotetic calculi for modal logics, just as prefixed tableau systems, are amenable to capture the local relation, whereas it is the global relation that we would expect to be represented by '$\vdash$'. The second problem is strictly connected with the first one: it is the use of indices, *i.e.*, explicitness of the system, the same feature which makes proving in erotetic calculi for modal logics easy and intuitive. Apparently, to much semantic details makes it difficult to interpret sequents in terms of derivability. Therefore, the author of this book now believes that the presented account of modal logics in the erotetic setting is not satisfying for the erotetic aims; a simple idea of (Rautenberg, 1983) or hypersequent calculi for modal logics would be better suited for "erotetisation".

3.5. Erotetic calculi and multiple-conclusion relations

Finally, if we study questions concerning derivability, it is natural to ask what kind of derivability is at sake. Usually, derivability is a notion relativized to a specific proof system of a given logic. If one analyses the examples provided by Wiśniewski (*e.g.*, in the introductions to papers (Wiśniewski, 2004), (Wiśniewski and Shangin, 2006)), then it seems correct to interpret the notion as referring to an axiomatic account or a natural deduction system for the underlying logic.

As we have seen in Section 2.1, sequents of language $\mathcal{L}^{?}_{\vdash\mathsf{L}}$ have quite natural interpretations in semantic terms. Both-sided, single succedent sequents may be interpreted as statements concerning semantic entailment in L, left-sided sequents are then to be viewed as statements concerning semantically defined inconsistency in L, and then the right-sided sequents '$\vdash S$' refer—in the simplest terms—to the L-validity of the disjunction of the terms of S. Here comes the time for generalisations.

3.5.1. Semantic interpretation

In the works by Wiśniewski the relation of multiple-conclusion entailment is used in interpreting sequents. We have introduced the notion for the case of CPL (see Section 3.1). Let us now formulate the definition for FOL.

Definition 27 (multiple-conclusion entailment in FOL). *Let X and Y stand for arbitrary sets of formulas of language $\mathcal{L}_{\mathsf{FOL}}$. We say that* set X multiple-conclusion entails set Y in FOL, *symbolically,* $X \Vdash_{\mathsf{FOL}} Y$, *iff:*

- *for every FOL-interpretation $\mathcal{I}$ of $\mathcal{L}_{\mathsf{FOL}}$, if each element of X is true under $\mathcal{I}$, then there is at least one element of Y true under $\mathcal{I}$.* □

Let $\mathsf{L} \in \{\mathsf{CPL}, \mathsf{FOL}\}$. Introducing the notion of multiple-conclusion entailment on the level of the underlying logic L yields the opportunity to interpret every sequent of language $\mathcal{L}^{?}_{\vdash \mathsf{L}}$ as referring to multiple-conclusion entailment in L. At the moment, we do not use partitions, but refer to the underlying semantics (in this respect we do not follow Wiśniewski strictly, *cf.* Wiśniewski, 2013, Section 3.2).

We will need one more notion—that of a *safeset*, introduced in the paper (Wiśniewski, 2017, p. 327).

Definition 28 (safeset in L). *A set X of formulas of $\mathcal{L}_{\mathsf{L}}$ is called a* safeset in L *iff for each L-interpretation $\mathcal{I}$ of $\mathcal{L}_{\mathsf{L}}$ there is a formula in X which is true under $\mathcal{I}$.* □

Example 7. *For any sentence A of $\mathcal{L}_{\mathsf{L}}$, the set $\{A, \neg A\}$ is both semantically inconsistent in L and a safeset in L.* □

The following holds:

Corollary 7. *Let X and Y stand for sets of formulas of language $\mathcal{L}_{\mathsf{L}}$.*

1. *X is semantically incosistent in L, that is, there is no L-model of X, iff $X \Vdash_{\mathsf{L}} \varnothing$.*
2. *X entails a formula A in L iff $X \Vdash_{\mathsf{L}} \{A\}$.*
3. *Y is a safeset in L iff $\varnothing \Vdash_{\mathsf{L}} Y$.*

Thus, as we can see, the relation of multiple-conclusion entailment in L generalizes all the three properties/relations that underlie the semantic interpretation of sequents. Consequently, the rules transforming expressions which involve sequents, as questions of our languages do, can be interpreted as driving as towards a solution of problems concerning the multiple-conclusion entailment relation.

There is, however, an unnatural difference between the self-imposing syntactic interpretation of both-sided and left-sided sequents and hence the rules dealing with them, and no obvious interpretation of right-sided sequents in this aspect.

The next section is devoted to developing a syntactical relation which generalizes both the derivability in the underlying logic and syntactically defined inconsistency, and at the same time it encompasses the cases of right-sided sequents. In other words, we shall define a proof-theoretical counterpart of the semantic relation of multiple-conclusion entailment.

3.5.2. Proof-theoretical interpretation

The claim of this Section is the following. Each sequent of the languages designed for classical logic, and considered in this book, can be interpreted proof-theoretically (that is, syntactically) in terms of a general relation of multiple-conclusion consequence that will be defined in this section. Roughly speaking, the meaning of the relation is the following: Y "follows from" X, iff Y presents a space of possibilities such that whatever can be derived from the whole Y **by cases**, can be "just" derived from X. In other words, Y presents the cases to be considered in proofs by cases from X. In fact, it is a situation that we encounter within natural deduction systems, when we apply the rule for eliminating disjunction.

To a great extent, this section is motivated by the account of consequence relations in the book Shoesmith and Smiley, 1978. Recently, in Wiśniewski, 2017; 2018, Andrzej Wiśniewski also developed many generalizations inspired by similar ideas, but the specific problems undertaken, and the solutions developed in (Wiśniewski, 2017, 2018) and here, are different. As we said before, the intuitive meaning of the turnstile symbol '$\vdash$', as an element of the object-level language $\mathcal{L}^?_{\vdash \mathsf{L}}$, is as referring to *derivability* in L.

Until the end of this chapter, $\mathsf{L} \in \{\mathsf{CPL}, \mathsf{FOL}\}$. Let $FORM_\mathsf{L}$ stand for the set of all formulas of language $\mathcal{L}_\mathsf{L}$. Suppose that the following relation:

$$\vdash_\mathsf{L} \subseteq 2^{FORM_\mathsf{L}} \times FORM_\mathsf{L}$$

is a consequence relation generated by a proof system which is sound and complete with respect to $\models_\mathsf{L}$, that is:

$$X \vdash_\mathsf{L} A \quad \text{iff} \quad X \models_\mathsf{L} A$$

Let us stress that this time '$\vdash_\mathsf{L}$' is not an object-level symbol. We think of '$\vdash_\mathsf{L}$' as referring to proof-theoretical *derivability* or *deducibility*, but do not assume anything more specific about the relation, except for compactness and the following three properties (see Shoesmith and Smiley, 1978, p. 15):

Overlap	If $A \in X$, then $X \vdash_{\mathsf{L}} A$.
Dilution	If $X^* \vdash_{\mathsf{L}} A$ and $X^* \subseteq X$, then $X \vdash_{\mathsf{L}} A$.
Cut for Sets	If $X, Z \vdash_{\mathsf{L}} A$ and $X \vdash_{\mathsf{L}} C$ for each $C \in Z$, then $X \vdash_{\mathsf{L}} A$.

The first property is also called *reflexivity* in the literature, the second one—*monotonicity*, and the third one expresses a form of *transitivity* of the relation. The last property may be expressed in a more familiar way:

Transitivity If $X \vdash_{\mathsf{L}} F$ and $\{F\} \vdash_{\mathsf{L}} A$, then $X \vdash_{\mathsf{L}} A$.

The following holds:

Corollary 8. *If* $\vdash_{\mathsf{L}} \subseteq 2^{FORM_{\mathsf{L}}} \times FORM_{\mathsf{L}}$ *is a relation satisfying* Dilution *and* Cut for Sets*, then it owns* Transitivity *as well.*

Proof. Suppose that $X \vdash_{\mathsf{L}} F$ and $\{F\} \vdash_{\mathsf{L}} A$. By Dilution, $X \cup \{F\} \vdash_{\mathsf{L}} A$. Since $X \vdash_{\mathsf{L}} F$ (it holds for every formula from $\{F\}$), by Cut for Sets also $X \vdash_{\mathsf{L}} A$. □

We will write '$\vdash_{\mathsf{L}} A$' instead of '$\varnothing \vdash_{\mathsf{L}} A$'. A natural way to express the fact that A is a $\vdash_{\mathsf{L}}$-consequence of the empty set is to say that A *is a thesis of* L. As we wrote, we also assume that '$\vdash_{\mathsf{L}}$' is compact:

Compactness $X \vdash_{\mathsf{L}} A$ iff for some finite subset $Y \subseteq X$, $Y \vdash_{\mathsf{L}} A$.

The assumption is quite natural when the consequence relation is thought of as generated by one of, lets say, standard proof systems, where *proof* is usually a finite structure (sequence, tree, *etc.*) of formulas or other objects, like sequents. Assuming that $\vdash_{\mathsf{L}}$ is compact, it is also trivially true that:

Corollary 9. *If X is non-empty, then $X \vdash_{\mathsf{L}} A$ iff for some finite and non-empty $Y \subseteq X$, $Y \vdash_{\mathsf{L}} A$.*

The reason for taking non-emptiness into consideration becomes clear in the sequel. Now let us introduce the following auxiliary concept:

Definition 29 (by-cases-consequence in L). *Let A stand for a formula of language $\mathcal{L}_{\mathsf{L}}$ and let X be a set of formulas of this language. We say that A is a* by-cases-consequence of X in L*, if for every finite and non-empty $Y \subseteq X$, $Y \vdash_{\mathsf{L}} A$. If A is a by-cases-consequence of X, then we write:*

$$X \vdash_{\mathsf{L}}^{bc} A$$

□

Definition 29 states that no matter how we choose the (finite number of) premises from X, we will always arrive at A. As we can see from Corollary 9 and Definition 29, in the case of non-empty sets, the definition generalizes a property which in the case of '$\vdash_{\mathsf{L}}$' is expressed by an existential quantification. The relation of by-cases-consequence in L is stronger than $\vdash_{\mathsf{L}}$ in the following sense:

Corollary 10. *Let $X \neq \varnothing$. If $X \vdash_{\mathsf{L}}^{bc} A$, then $X \vdash_{\mathsf{L}} A$.*

Proof. Since X is non-empty, there exists a finite and non-empty subset Y of X, and, by assumption, $Y \vdash_{\mathsf{L}} A$. Then, by Dilution, $X \vdash_{\mathsf{L}} A$. □

Imagine that X is a search space, or a "knowledge base", however, not necessarily a consistent one. Suppose that we have some doubts regarding X—we have a reason to believe that some of the assumptions are wrong, but we do not know which are the problematic ones. Then we may have reasonable doubts as to a conclusion that follows from this or another choice of premises taken from X. If, however, the conclusion follows from every such (finite) choice of premises, then we can feel much more certain about it. The notion of by-cases-consequence expresses this kind of comfortable situation when we can derive a conclusion with high level of certainty, although the premises are highly uncertain. In the margin, observe that this kind of certainty requires Dilution (*i.e.*, monotonicity).

As a side effect of the universal quantification in Definition 29, we have:

Corollary 11. *$\varnothing \vdash_{\mathsf{L}}^{bc} A$ for any formula A of language $\mathcal{L}_{\mathsf{L}}$.*

which may be interpreted as an expression of the fact that when there are no premises for reasoning by cases, then we are as helpless as in the face of inconsistency. Also the following holds:

Corollary 12. *If A is a thesis of L, then $X \vdash_{\mathsf{L}}^{bc} A$ for every set X.*

Proof. Once again, it is worth to indicate that the result needs Dilution. For if A is a thesis of L, then $\varnothing \vdash_{\mathsf{L}} A$. Assume that X is arbitrary. Then, by the Dilution property, for each finite and non-empty subset Y of X, $Y \vdash_{\mathsf{L}} A$. Thus $X \vdash^{bc}_{\mathsf{L}} A$ for every set X. □

Let us now note that:

Corollary 13. *$X \vdash^{bc}_{\mathsf{L}} A$ iff for each singleton $\{B\} \subseteq X$, $\{B\} \vdash_{\mathsf{L}} A$.*

Proof. The first implication follows from the very Definition 29, and the converse implication requires Dilution. □

Example 8. *Suppose that $\vdash_{\mathsf{CPL}}$ is the consequence relation generated by the axiomatic account of* CPL *by Hilbert and Bernays.*[6] *Thus $X \vdash_{\mathsf{CPL}} A$ iff there exists a derivation of formula A from X governed by the rules of the axiomatic system. We shall not prove this, by it is well-known that the relation satisfies Overlap, Dilution, and Cut for Sets. Formula 'p' of language $\mathcal{L}_{\mathsf{CPL}}$ is a by-cases-consequence of the set: $\{p \wedge q, p, p \vee p\}$, and also of the set: $\{A \wedge B : A = p, B \in VAR\}$, symbolically:*

$$\{p \wedge q, p, p \vee p\} \vdash^{bc}_{\mathsf{CPL}} p$$

$$\{A \wedge B : A = p, B \in VAR\} \vdash^{bc}_{\mathsf{CPL}} p$$

On the other hand, $\{p, q\} \vdash_{\mathsf{CPL}} p$, but $\{p, q\} \nvdash^{bc}_{\mathsf{CPL}} p$, because $\{q\} \nvdash_{\mathsf{CPL}} p$. □

Example 9. *Let $\vdash_{\mathsf{FOL}}$ be the consequence relation generated by the axiomatic system Q_1 used by Smullyan.*[7] *Once again, $X \vdash_{\mathsf{FOL}} A$ iff there exists a derivation of formula A from X governed by the rules of Q_1. As previously—the relation owns the properties of Overlap, Dilution, and Cut for Sets. Then formula '$\exists x P(x)$' is a by-cases consequence in* FOL *of the set $\{P(a_i) : a_i \text{ is a parameter of } \mathcal{L}_{\mathsf{FOL}}\}$.* □

Let us also note that:

Corollary 14. *Suppose that $\vdash_{\mathsf{L}}$ satisfies the three properties defined above and that it is sound and complete with respect to $\models_{\mathsf{L}}$. Let Y be a finite set of formulas of $\mathcal{L}_{\mathsf{L}}$. Then for each non-empty set $Y^* \subseteq Y$, it holds that $Y^* \models_{\mathsf{L}} \bigvee Y$. Hence also $Y^* \vdash_{\mathsf{L}} \bigvee Y$. It follows that $Y \vdash^{bc}_{\mathsf{L}} \bigvee Y$.*

[6] Presented in Hilbert and Bernays, 1968.

[7] See Smullyan, 1968, p. 81. Let us recall that our language $\mathcal{L}_{\mathsf{FOL}}$ is defined as in Smullyan, 1968, p. 43.

Observe that the relation '$\vdash^{bc}_{\mathsf{L}}$' is reflexive in the following sense:

Corollary 15. *For each formula A of language $\mathcal{L}_{\mathsf{L}}$, $\{A\} \vdash^{bc}_{\mathsf{L}} A$.*

But it is not reflexive in the more general sense expressed by the property called "Overlap". Moreover, the relation is non-monotonic:

Example 10. *Let $\vdash_{\mathsf{CPL}}$ be defined as in Example 8. We have: $\{p\} \vdash^{bc}_{\mathsf{CPL}} p$, but $\{p, q\} \nvdash^{bc}_{\mathsf{CPL}} p$.* □

Similarly:

Example 11. *Let $\vdash_{\mathsf{FOL}}$ be defined as in Example 9. We have: $\{\forall x P(x)\} \vdash^{bc}_{\mathsf{FOL}} \forall x P(x)$, but $\{\forall x P(x), \forall x Q(x)\} \nvdash^{bc}_{\mathsf{FOL}} \forall x P(x)$* □

The above examples show that the relation of by-cases-consequence in L *is not* a consequence relation in the sense explicated by Shoesmith and Smiley. The relation is not reflexive in the sense defined by "Overlap" (it is, however, reflexive in the most straightforward sense given by Corollary 15). What is more, the relation does not own the property of monotonicity expressed by "Dilution", but, interestingly, it does own a property which is *dual* to Dilution with respect to set inclusion, namely:

Corollary 16. *If $X \vdash^{bc}_{\mathsf{L}} A$ and $Y \subseteq X$, then $Y \vdash^{bc}_{\mathsf{L}} A$.*[8]

Before we continue, let us prove one more result which also explains the meaning of our by-cases-consequence. Theorem 13 states that *if* the notion of *derivability* explicated by '$\vdash_{\mathsf{L}}$' presupposes that the *premises* of a *derivation* are joined with conjunction, *and* the implication connective is such that the deduction theorem holds, *then* the notion of *derivability by cases* explicated by '$\vdash^{bc}_{\mathsf{L}}$' presupposes *disjunctive* interpretation of the set of premises. Once again, an association with the rule for disjunction elimination present in many natural deduction systems seems inevitable. However, what may seem interesting is also the fact that the theorem and its proof highlight what needs to be assumed about the language and its connectives in order to derive the result. In particular, the assumptions concerning the behaviour of conjunction and disjunction in the context of '$\vdash_{\mathsf{L}}$' coincide with those expressed by Shoesmith and Smiley in their book (Shoesmith and Smiley, 1978).

[8] The author has deliberately omitted the case of "Cut for Sets". If the property is *literally* rewritten for $\vdash^{bc}_{\mathsf{L}}$, then it holds trivially by Corollary 16. The author, however, feels that the definition of the property of Cut for Sets must be somehow *dually* reformulated. The reformulation remains an open problem.

Theorem 13. *Suppose that the consequence relation* $\vdash_{\mathsf{L}}$ *is defined as above, that is, it satisfies, int.al.,* Dilution *and* Cut for Sets. *Suppose also that the behaviour of the connectives* $\wedge, \vee, \rightarrow$ *in the context of* $\vdash_{\mathsf{L}}$ *is such that:*

(*a*) *for any formulas* A, B *of* $\mathcal{L}_{\mathsf{L}}$, $\{A\} \vdash_{\mathsf{L}} A \vee B$ *and* $\{B\} \vdash_{\mathsf{L}} A \vee B$.

(*b*) *for any formulas* A, B, C *of* $\mathcal{L}_{\mathsf{L}}$, *if* $\{A\} \vdash_{\mathsf{L}} C$ *and* $\{B\} \vdash_{\mathsf{L}} C$, *then* $\{A \vee B\} \vdash_{\mathsf{L}} C$.

(*c*) *for arbitrary sentences* $B_1, \ldots, B_n$ *of the language, and an arbitrary formula* $A : \{B_1, \ldots, B_n\} \vdash_{\mathsf{L}} A$ *iff* $\vdash_{\mathsf{L}} (B_1 \wedge \ldots \wedge B_n) \rightarrow A$.

Let X *be a finite, non-empty set of sentences of language* $\mathcal{L}_{\mathsf{L}}$ *and let* A *be a formula of* $\mathcal{L}_{\mathsf{L}}$. *Then:*

$$X \vdash^{bc}_{\mathsf{L}} A \text{ iff } \vdash_{\mathsf{L}} \bigvee X \rightarrow A$$

Proof. Let $X = \{B_1, \ldots, B_n\}$ be a finite, non-empty set of sentences of language $\mathcal{L}_{\mathsf{L}}$ and let A be a formula of $\mathcal{L}_{\mathsf{L}}$.

If $X \vdash^{bc}_{\mathsf{L}} A$, then also $\{B_i\} \vdash_{\mathsf{L}} A$ for each $B_i \in X$, by Corollary 13. By (b) applied $n-1$ times, $\{\bigvee X\} \vdash_{\mathsf{L}} A$. '$\bigvee X$' is a sentence of $\mathcal{L}_{\mathsf{L}}$, and hence by (c), $\vdash_{\mathsf{L}} \bigvee X \rightarrow A$.

For the second implication, assume that $\vdash_{\mathsf{L}} \bigvee X \rightarrow A$. Then, by (c), $\{\bigvee X\} \vdash_{\mathsf{L}} A$. Let $B_i \in X$ be arbitrary. By (a), we have both $\{B_i\} \vdash_{\mathsf{L}} B_i \vee B_{i+1}$ (for $1 \leq i \leq n-1$) and $\{B_i\} \vdash_{\mathsf{L}} B_{i-1} \vee B_i$ (for $2 \leq i \leq n$). Referring to (a) an appropriate number of times, and by Transitivity of $\vdash_{\mathsf{L}}$, we arrive at $\{B_i\} \vdash_{\mathsf{L}} \bigvee X$ (as to Transitivity, by assumption, Dilution and Cut for Sets hold, hence Transitivity holds as well by Corollary 8). Once again, by Transitivity, $\{B_i\} \vdash_{\mathsf{L}} A$. But B_i was an arbitrary element of X, therefore, by Corollary 13, $X \vdash^{bc}_{\mathsf{L}} A$. □

Finally, let us define the following:

Definition 30 (multiple-conclusion consequence)**.** *Let* X *and* Y *be sets of formulas of* $\mathcal{L}_{\mathsf{L}}$, *and let* $\vdash_{\mathsf{L}}$ *stand for a relation defined as above, that is,* $\vdash_{\mathsf{L}} \subseteq 2^{FORM_{\mathsf{L}}} \times FORM_{\mathsf{L}}$ *and* $\vdash_{\mathsf{L}}$ *satisfies* Overlap, Dilution *and* Cut for Sets. *We will say that* Y *is a* multiple-conclusion consequence of X in L, *symbolically,*

$$X \Vdash_{\mathsf{L}} Y$$

iff for every formula A *of* $\mathcal{L}_{\mathsf{L}}$: *if* $Y \vdash^{bc}_{\mathsf{L}} A$, *then* $X \vdash_{\mathsf{L}} A$. □

Let us recall the intuitions we have described at the beginning of this section. We aim at a description of a multiple-conclusion relation between sets of formulas, X and Y, which is such that Y presents an overall space of possibilities for reasoning by cases from X. Definition 30 says that $X \Vdash_{\mathsf{L}} Y$ holds iff every formula derivable from Y by cases is derivable from X. The next theorem states that, as far as finite sets of formulas are considered, if the relations $\vdash_{\mathsf{L}}$ and $\models_{\mathsf{L}}$ coincide, then the relations $\Vdash_{\mathsf{L}}$ and $\|\!\!\models_{\mathsf{L}}$ also coincide.

Theorem 14. *Let X be a finite set of formulas of $\mathcal{L}_{\mathsf{L}}$ and let Y be a finite set of sentences of $\mathcal{L}_{\mathsf{L}}$. Assume also that $\vdash_{\mathsf{L}}$ is sound and complete with respect to $\models_{\mathsf{L}}$. Then $X \Vdash_{\mathsf{L}} Y$ iff $X \|\!\!\models_{\mathsf{L}} Y$.*

Proof. Suppose that $\mathsf{L} = \mathsf{FOL}$. We actually show the proof only for this case, as the one for CPL may be obtained from this. (Let us observe that when $\mathsf{L} = \mathsf{CPL}$, the assumption that Y contains sentences is not essential, as a sentence of $\mathcal{L}_{\mathsf{CPL}}$ is simply a formula of $\mathcal{L}_{\mathsf{CPL}}$.)

Assume that $X \not\|\!\!\models_{\mathsf{L}} Y$. Then there is a FOL-interpretation $\mathcal{I}$ which is a model of X but is not a model of any element of Y. Since the elements are sentences and Y is finite, it then holds that '$\bigvee Y$' is not true under $\mathcal{I}$. For this reason, X does not entail in L formula '$\bigvee Y$', *i.e.* $X \not\models_{\mathsf{L}} \bigvee Y$. Therefore $X \nvdash_{\mathsf{L}} \bigvee Y$. On the other hand, $Y \vdash^{bc}_{\mathsf{L}} \bigvee Y$ by Corollary 14. Sentence '$\bigvee Y$' is thus an example of a formula A that is a by-cases consequence of Y, yet is not a consequence of X. By Definition 30, it follows that $X \nVdash_{\mathsf{L}} Y$.

For the converse direction, let us observe that if the relations '$\vdash_{\mathsf{L}}$' and '$\models_{\mathsf{L}}$' coincide, then '$\vdash_{\mathsf{L}}$' satisfies the assumptions of Theorem 13. Now assume that $X \nVdash_{\mathsf{L}} Y$. Then there is a formula, A, such that $Y \vdash^{bc}_{\mathsf{L}} A$ and $X \nvdash_{\mathsf{L}} A$. Thus also $X \not\models_{\mathsf{L}} A$, that is, there is an L-interpretation $\mathcal{I}$ which is a model of X but is not a model of A. Since $Y \vdash^{bc}_{\mathsf{L}} A$, by Theorem 13, $\vdash_{\mathsf{L}} \bigvee Y \to A$. Thus also $\models_{\mathsf{L}} \bigvee Y \to A$, and since $\mathcal{I}$ is not a model of A, it is neither a model of $\bigvee Y$. This means that $\mathcal{I}$ is not a model of any element of Y, as they are sentences. Hence $X \not\|\!\!\models_{\mathsf{L}} Y$. □

We now establish the potential proof-theoretical interpretation of the three types of sequents considered in this book. For this purpose let us start with a straightforward corollary from Theorem 14 and Corollary 7.

Corollary 17. *Let X be a finite set of formulas of $\mathcal{L}_{\mathsf{L}}$, let Y be a finite set of sentences of $\mathcal{L}_{\mathsf{L}}$, and let A be a single sentence of the language. Then:*

1. $X \Vdash_{\mathsf{L}} \varnothing$ *iff* $X \| \!\models_{\mathsf{L}} \varnothing$ *iff* X *is semantically* L*-inconsistent,*

2. $X \Vdash_{\mathsf{L}} \{A\}$ *iff* $X \| \!\models_{\mathsf{L}} \{A\}$ *iff* $X \models_{\mathsf{L}} A$*,*

3. $\varnothing \Vdash_{\mathsf{L}} Y$ *iff* $\varnothing \| \!\models_{\mathsf{L}} Y$ *iff* Y *is a safeset in* L*.*

Let us establish some more coincidences.

Definition 31 (syntactic inconsistency in L)**.** *Let X be a set of formulas of language $\mathcal{L}_{\mathsf{L}}$. We say that X is* syntactically inconsistent in L *iff $X \vdash_{\mathsf{L}} A$ for every formula A of $\mathcal{L}_{\mathsf{L}}$. We say that a formula B of $\mathcal{L}_{\mathsf{L}}$ is* syntactically inconsistent in L *iff the singleton set $\{B\}$ is syntactically* L*-inconsistent.* □

Corollary 18. *$X \Vdash_{\mathsf{L}} \varnothing$ iff X is syntactically inconsistent in* L*.*

Proof. By Corollary 11, $\varnothing \vdash^{bc}_{\mathsf{L}} A$ for any formula A. Thus if $X \Vdash_{\mathsf{L}} \varnothing$, then by Definition 30, $X \vdash_{\mathsf{L}} A$ for any A, and hence X is syntactically inconsistent in L.

For the converse direction let us observe that if $X \vdash_{\mathsf{L}} A$ for any A, then the implication defining the multiple-conclusion consequence must be satisfied for the case of $X \Vdash_{\mathsf{L}} \varnothing$. □

Corollary 19. *$\varnothing \Vdash_{\mathsf{L}} Y$ iff for every A such that $Y \vdash^{bc}_{\mathsf{L}} A$, A is a thesis of* L*.*

Proof. Directly from Definition 30: $\varnothing \Vdash_{\mathsf{L}} Y$ iff for every formula A of $\mathcal{L}_{\mathsf{L}}$, if $Y \vdash^{bc}_{\mathsf{L}} A$, then $\varnothing \vdash_{\mathsf{L}} A$, that is, A is a thesis of L. □

In other words, Corollary 19 states that $\varnothing \Vdash_{\mathsf{L}} Y$, or, semantically: Y is a safeset in L, iff the only formulas that are by-cases-consequences (in L) of Y are the theses of L.

Corollary 20. *$X \vdash_{\mathsf{L}} A$ iff $X \Vdash_{\mathsf{L}} \{A\}$.*

Proof. Assume that $X \vdash_{\mathsf{L}} A$ and that $\{A\} \vdash^{bc}_{\mathsf{L}} B$. Then $\{A\} \vdash_{\mathsf{L}} B$, by Corollary 10, and also $X \vdash_{\mathsf{L}} B$ by Dilution. Thus if $X \vdash_{\mathsf{L}} A$, then $X \Vdash_{\mathsf{L}} \{A\}$. For the other direction observe that $\{A\} \vdash^{bc}_{\mathsf{L}} A$ (recall Corollary 15). By Definition 30, $X \vdash_{\mathsf{L}} A$. □

To sum up: erotetic calculi, no matter what kind of sequents are used, can be interpreted as transforming questions concerning the relation of multiple-conclusion entailment (semantic version), or the relation of multiple-conclusion consequence (syntactic, *i.e.*, proof-theoretical interpretation).

An open question is to provide a full interpretation of modal logics in these lines.

Chapter 4

Erotetic calculi: proofs

The last chapter of this book concerns issues of construction of proofs in various deductive systems, issues of comparing the systems, and simulating one system in another. Let us summarize what we wish to achieve in this field:

- a deeper insight into the interrelations between different proof methods;
- algorithms of translation written mostly in pseudocode;
- an algorithm of producing proofs in implicit sequent systems for modal logics, which is not based on backtracking: a proof derived in an explicit system is turned into a proof in implicit system.

4.1. Socratic trees

In Chapter 2 we have seen that the structure of a sequent is a rich medium of expression in proof theory. Moreover, the structure of sequences of sequents seems to be very convenient in formulating proof-search algorithms—see Section 2.2.5. However, for the purpose of translations between an erotetic calculus and such deductive systems as a sequent calculus or a tableau system, the structure of a tree is necessary. In this section we "extract" the structure of a tree from the structure of a Socratic transformation, in order to use it further in the translation procedures.

We use a set-theoretic account of the notion of tree. The reader may consult Appendix B, if necessary.

The content of this section is based on Section 2 of (Leszczyńska-Jasion et al., 2013). The aim of this paper was to define a translation of a Socratic transformation into a tree, called *Socratic tree*, whose nodes are *annotated sequents*. Here we present the construction described in Leszczyńska-Jasion et al., 2013, but we generalize it to all erotetic calculi considered in Chapter 2, therefore in this section $\mathbb{E} \in \{\mathbb{E}^{\mathbf{PQ}}, \mathbb{E}^{\mathbf{LPQ}}, \mathbb{E}^{\mathbf{RPQ}}, \mathbb{E}^{\mathsf{K}}, \mathbb{E}^{\mathsf{D}}, \mathbb{E}^{\mathsf{T}}, \mathbb{E}^{\mathsf{KB}}, \mathbb{E}^{\mathsf{DB}}, \mathbb{E}^{\mathsf{B}}, \mathbb{E}^{\mathsf{K4}}, \mathbb{E}^{\mathsf{D4}}, \mathbb{E}^{\mathsf{S4}}, \mathbb{E}^{\mathsf{KB4}}, \mathbb{E}^{\mathsf{K5}}, \mathbb{E}^{\mathsf{D5}}, \mathbb{E}^{\mathsf{K45}}, \mathbb{E}^{\mathsf{D45}}, \mathbb{E}^{\mathsf{S5}}\}$.

First, as in Leszczyńska-Jasion et al., 2013, we introduce the notion of an *annotated sequent*. A sequent is annotated in order to maintain the information about its position in a Socratic transformation. If $\mathbf{s} = \langle s_1, s_2, \ldots \rangle$ is a Socratic transformation, then an ordered triple $\langle n, i, \phi \rangle$ where sequent ϕ is i-th constituent of n-th question s_n of $\mathbf{s}$, is called an *annotated sequent of Socratic transformation* $\mathbf{s}$. Thus $\langle n, i \rangle$ may be thought of as "coordinates" of the sequent in the transformation. The triple should be relativized to $\mathbf{s}$, in order to make this sufficiently precise, but we will omit the relativization, as it should not cause any ambiguity. Observe that a sequent may occur in many places of a Socratic transformation $\mathbf{s}$, but a sequent together with its coordinates in $\mathbf{s}$ is a unique entity with respect to $\mathbf{s}$.

For simplicity, an annotated sequent $\langle n, i, \phi \rangle$ will be sometimes symbolised by ϕ_i^n. We will also use ϱ, σ, τ, possibly with subscripts, for annotated sequents.

Let us recall that we say that sequent ψ *results in* $\mathbf{s}$ *from* sequent ϕ *by erotetic rule* $\mathbf{r}$ if for some term s_n of $\mathbf{s}$, the term s_{n+1} results from s_n by $\mathbf{r}$, ϕ is the sequent-premise of s_n and ψ is the (or one of the two) sequent-conclusion of s_{n+1}. We will also say that *annotated sequent* ϕ_i^n *is active in* s_n if i-th constituent of question s_n is active in this question; and, analogously, we will say that *annotated sequent* ϕ_k^{n+1} *results from annotated sequent* ψ_i^n if k-th constituent of question s_{n+1} results from i-th constituent of question s_n.

The rules of erotetic calculi operate on questions which are based on sequences of sequents. However, a rule acts upon only one constituent of a question leaving the remaining sequents unchanged. From the point of view of the logic of questions this is a desired effect, since an application of an erotetic rule amounts to a reformulation of the problem expressed by the question-premise. What is more, this construction of rules warrants that the rules are semantically invertible, which yields, on the one hand, that the relation of erotetic implication holds between the question-premise and the question-conclusion, and on the other hand, that the erotetic system is *confluent*. In a confluent system, whenever one starts with a provable formula, there are no "bad moves" in the construction of a derivation that would lead to a "dead end", from where no proof can be found.[1] Confluency is a desirable property, favourable for implementation.

However, from the present point of view, the unchanged sequents that are "left" in the resulting question are redundant and thus we will aim at leaving them out of the tree (*cf.* Definition 32). We shall first define a relation, $P_{\mathbf{s}}$, that links two annotated sequents of Socratic transformation $\mathbf{s}$ whenever one of these annotated sequents is active and the other is its sequent-conclusion. Moreover, if after an application of a rule a sequent is rewritten in the next question (that is, a rule has been applied with respect to some other constituent), then the relation $P_{\mathbf{s}}$ links the annotated sequent and its annotated repetition (*cf.* Definition 33). Next, we will consider the transitive closure of relation $P_{\mathbf{s}}$. And finally, in order to get rid of the redundant repetitions of sequents we will restrict the transitive closure of $P_{\mathbf{s}}$ to the set $X_{\mathbf{s}}$ (*cf.* Definition 34).

Suppose that $\mathbf{s}$ is an arbitrary but fixed Socratic transformation of a question of the form '$?(S \vdash T)$' via the rules of $\mathbb{E}$. The form of sequent '$S \vdash T$' is suitably restricted, depending on $\mathbb{E}$. Definition 32 specifies the set of nodes of the "target" Socratic tree.

Definition 32. *By $X_{\mathbf{s}}$ we mean the smallest set of annotated sequents of $\mathbf{s}$ such that $\phi_1^1 \in X_{\mathbf{s}}$ and for each $n \geq 1$, if ψ_i^n is the active annotated sequent in question s_n, then:*

1. *if s_{n+1} results from s_n by a non-branching rule, then $\chi_i^{n+1} \in X_{\mathbf{s}}$,*
2. *if s_{n+1} results from s_n by a branching rule, then $(\omega_1)_i^{n+1} \in X_{\mathbf{s}}$ and $(\omega_2)_{i+1}^{n+1} \in X_{\mathbf{s}}$.* □

According to Definition 32, the set $X_{\mathbf{s}}$ contains the annotated sequent ϕ_1^1, which takes the role of the origin of the tree, and the annotated sequents-conclusions of the questions of $\mathbf{s}$. The following example illustrates the idea (Example 12 also comes from Leszczyńska-Jasion et al., 2013).

Example 12. *Here is a Socratic transformation $\mathbf{s}\star$ of question $?((p \vee p) \vee (q \vee q) \vdash p \vee q)$ via the rules of the propositional part of $\mathbb{E}^{\mathbf{PQ}}$; $\mathbf{s}\star$ is a Socratic proof of sequent $(p \vee p) \vee (q \vee q) \vdash p \vee q$ in $\mathbb{E}^{\mathbf{PQ}}$. The elements of $X_{\mathbf{s}\star}$ are framed.*

[1] See Hählne, 2001, p. 121: "A tableau calculus is *proof confluent*, if from every tableau for an unsatisfiable set of sentences a closed tableau can be constructed."

$$\begin{array}{cl}
?(\boxed{(p \lor p) \;\lor\; (q \lor q) \;\vdash\; p \lor q}) & \\ \cline{1-1}
 & L_\beta \\
?(\boxed{p \lor p \;\vdash\; p \lor q} \;;\; \boxed{q \lor q \;\vdash\; p \lor q}) & \\ \cline{1-1}
 & L_\beta \\
?(\boxed{p \;\vdash\; p \lor q} \;;\; \boxed{p \;\vdash\; p \lor q} \;;\; q \lor q \;\vdash\; p \lor q\;) & \\ \cline{1-1}
 & R_\beta \\
?(\boxed{p, \neg p \;\vdash\; q} \;;\; p \;\vdash\; p \lor q \;;\; q \lor q \;\vdash\; p \lor q\;) & \\ \cline{1-1}
 & R_\beta \\
?(\; p, \neg p \;\vdash\; q \;;\; \boxed{p, \neg p \;\vdash\; q} \;;\; q \lor q \;\vdash\; p \lor q\;) & \\ \cline{1-1}
 & L_\beta \\
?(\; p, \neg p \;\vdash\; q \;;\; p, \neg p \;\vdash\; q \;;\; \boxed{q \;\vdash\; p \lor q} \;;\; \boxed{q \;\vdash\; p \lor q}) & \\ \cline{1-1}
 & R_\beta \\
?(\; p, \neg p \;\vdash\; q \;;\; p, \neg p \;\vdash\; q \;;\; \boxed{q, \neg p \;\vdash\; q} \;;\; q \;\vdash\; p \lor q\;) & \\ \cline{1-1}
 & R_\beta \\
?(\; p, \neg p \;\vdash\; q \;;\; p, \neg p \;\vdash\; q \;;\; q, \neg p \;\vdash\; q \;;\; \boxed{q, \neg p \;\vdash\; q}) &
\end{array}$$

□

In order to define a relation $R_{\mathbf{s}}$ on the set $X_{\mathbf{s}}$ we need a certain auxiliary concept. Recall that $\mathbf{s}$ is an arbitrary but fixed Socratic transformation of a question of the form '$?(S \vdash T)$' via the rules of $\mathbb{E}$.

Definition 33. *Let $Y_{\mathbf{s}}$ be the set of all annotated sequents of Socratic transformation $\mathbf{s}$. By $P_{\mathbf{s}}$ we mean the smallest binary relation on $Y_{\mathbf{s}}$ such that for each $n \geq 1$:*

1. *If $s_{n+1} = ?(\psi_1; \ldots; \psi_j)$ results from $s_n = ?(\phi_1; \ldots; \phi_j)$ by a non-branching rule, then for each $i \leq j$: $\langle (\phi_i)^n_i,\ (\psi_i)^{n+1}_i \rangle \in P_{\mathbf{s}}$.*

2. *If $s_{n+1} = ?(\psi_1; \ldots; \psi_{j+1})$ results from $s_n = ?(\phi_1; \ldots; \phi_j)$ by a branching rule, and k-th constituent of s_n is active in s_n, then:*

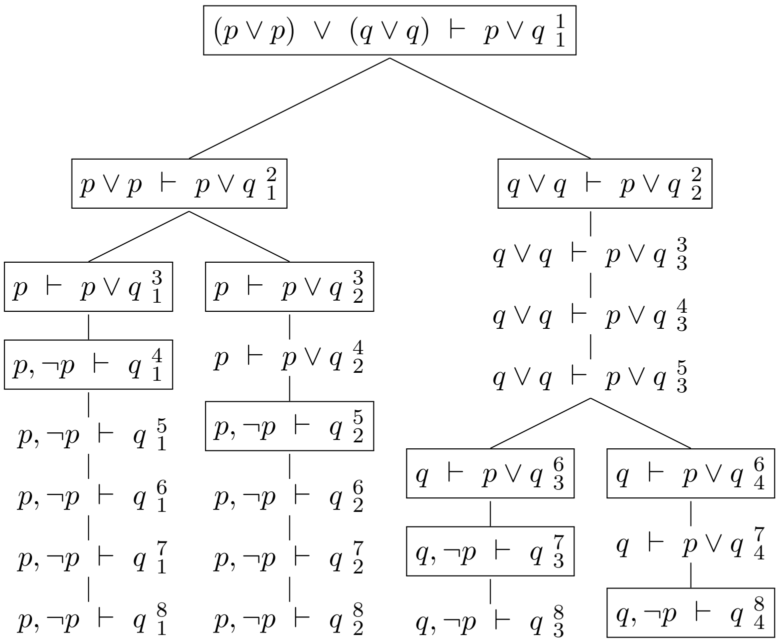
   (a) for each $i \leq k, \langle (\phi_i)^n_i,\ (\psi_i)^{n+1}_i \rangle \in P_{\mathbf{s}}$,

   (b) for each $i : k \leq i \leq j, \langle (\phi_i)^n_i, (\psi_{i+1})^{n+1}_{i+1} \rangle \in P_{\mathbf{s}}$. □

It follows from Definition 33 that if $\langle \varrho, \sigma \rangle \in P_{\mathbf{s}}$, then either ϱ and σ represent the same (differently annotated) sequent or ϱ is active in some question of $\mathbf{s}$ and σ results from ϱ.

Once again, we illustrate the definitions with Socratic transformation $\mathbf{s}\star$ (see Figure 4.1). The lines display the $P_{\mathbf{s}\star}$ relation. The elements of $X_{\mathbf{s}\star}$ are framed.

Figure 4.1: Illustration of Definition 33



□

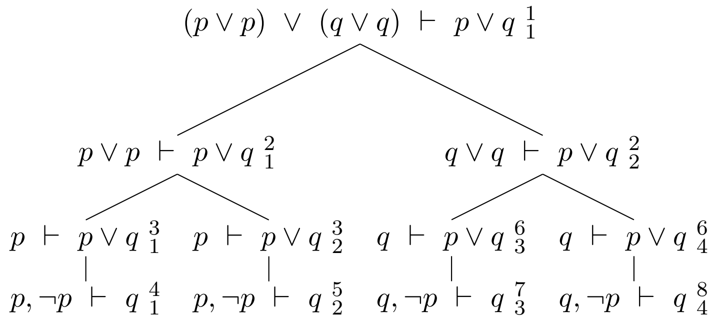
Finally, let us introduce:

Definition 34. *Let* $P^{tr}_{\mathbf{s}}$ *be the transitive closure of* $P_{\mathbf{s}}$. *Then by* $R_{\mathbf{s}}$ *we mean relation* $P^{tr}_{\mathbf{s}}$ *restricted to set* $X_{\mathbf{s}}$, *that is* $\langle \varrho, \sigma \rangle \in R_{\mathbf{s}}$ *iff both* $\langle \varrho, \sigma \rangle \in P^{tr}_{\mathbf{s}}$ *and* $\varrho, \sigma \in X_{\mathbf{s}}$. □

If we apply Definition 34 to our previous example, we will receive the structure presented in Figure 4.2. Only the elements of $X_{\mathbf{s}\star}$ are displayed, the lines represent relation $R_{\mathbf{s}\star}$.

By $\mathbf{Tr}(\mathbf{s})$ we will refer to the following structure: $\langle X_{\mathbf{s}}, R_{\mathbf{s}} \rangle$. It is easy to see that the structure is a tree with origin ϕ^1_1. A proof of this fact for the case of $\mathsf{L} = \mathsf{CPL}, \mathsf{FOL}$ may be found in Leszczyńska-Jasion et al., 2013. After the authors of this paper, we will call the structure $\mathbf{Tr}(\mathbf{s})$ the *Socratic tree determined by a Socratic transformation* $\mathbf{s}$ of a question based on a single sequent. Definitions 32, 33 and 34 give us a recipe how to construct $\mathbf{Tr}(\mathbf{s})$ for a given Socratic transformation $\mathbf{s}$. In Leszczyńska-Jasion et al., 2013 also an algorithm of construction of a Socratic tree is given.

Figure 4.2: Illustration of Definition 34



□

Let us also observe that the relation of immediate $R_{\mathbf{s}}$-successor coincides with the erotetic rules in the following sense (see also Leszczyńska-Jasion et al., 2013, Corollary 2, p.970):

Corollary 21. *Let* $\mathbf{s}$ *be a Socratic transformation via the rules of* $\mathbb{E}$ *and let* $\mathbf{Tr}(\mathbf{s}) = \langle X_{\mathbf{s}}, R_{\mathbf{s}} \rangle$ *be the Socratic tree determined by* $\mathbf{s}$*. If annotated sequent* $\langle n, i, \phi \rangle$ *of* $\mathbf{Tr}(\mathbf{s})$ *is an immediate* $R_{\mathbf{s}}$*-successor of annotated sequent* $\langle m, k, \psi \rangle$ *of* $\mathbf{Tr}(\mathbf{s})$*, then sequent* ϕ *results in Socratic transformation* $\mathbf{s}$ *from sequent* ψ *by a rule of* $\mathbb{E}$*.*

That is to say: Socratic trees do the job they were created for—they retain the skeleton of sequents that were actually modified by the erotetic rules and leave the other sequents.

In Leszczyńska-Jasion et al., 2013 the construction of Socratic tree is described for the case of Socratic transformations obtained by the rules of $\mathbf{E}^*$ (the propositional part of $\mathbf{E}^{\mathbf{PQ}}$) or $\mathbf{E}^{\mathbf{PQ}}$, but there is nothing in the construction that depends on this version of erotetic calculus. More specifically, the algorithm transforming a finite Socratic transformation into a Socratic tree depends on the notion of "conclusion-sequent" and the relation between sequents generated by the erotetic rules, and not on "both-sidedness" of sequents. Therefore an analogous procedure may be used to transform into trees finite Socratic transformations generated in the other erotetic calculi considered here. Moreover, the same pertains to modal logics. Thus we take it for granted that:

Theorem 15. *Any finite Socratic transformation of a question of the form* $?(\phi)$ *via the rules of* $\mathbb{E}$ *may be transformed into a Socratic tree of annotated sequent* $\langle 1, 1, \phi \rangle$*.*

Proof. See Leszczyńska-Jasion et al., 2013. □

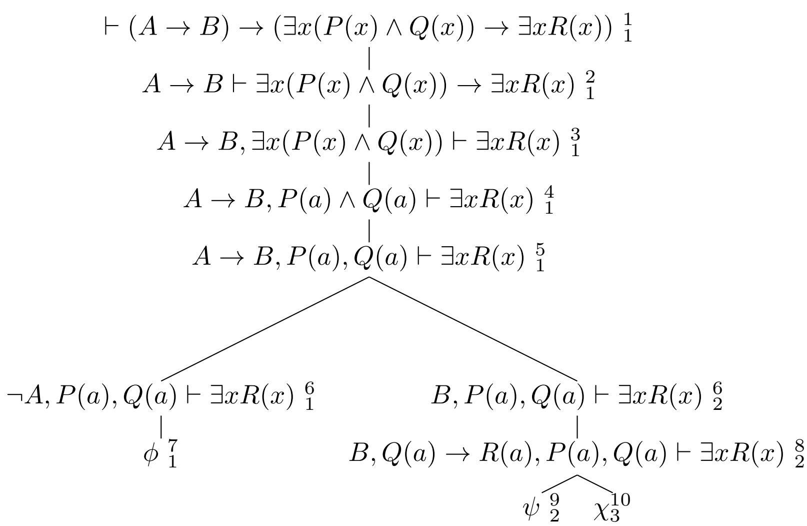
Corollary 21 also holds for all the calculi listed in Theorem 15.

Let us introduce some further examples.

Example 13. *Here is a Socratic proof of sequent:* $\vdash (\exists x P(x) \to \forall x(Q(x) \to R(x))) \to (\exists x(P(x) \wedge Q(x)) \to \exists x R(x))$ *in calculus* $\mathbb{E}^{\mathbf{PQ}}$. *To increase the readability of the example, we put* A *for formula '*$\exists x P(x)$*', and* B *for '*$\forall x(Q(x) \to R(x))$*'. Starting with the* 8^{th} *question we also put* ϕ *for sequent '*$\neg A, \neg P(a), P(a), Q(a) \vdash \exists x R(x)$*', and* ψ *for sequent '*$B, \neg Q(a), P(a), Q(a) \vdash \exists x R(x)$*'.*

$$\begin{array}{cl}
1.\ ?(\ \vdash (A \to B) \to (\exists x(P(x) \wedge Q(x)) \to \exists x R(x))\ ) & \\
\hline
2.\ ?(\ A \to B \vdash \exists x(P(x) \wedge Q(x)) \to \exists x R(x)\ ) & R_\beta \\
\hline
3.\ ?(\ A \to B, \exists x(P(x) \wedge Q(x)) \vdash \exists x R(x)\ ) & R_\beta \\
\hline
4.\ ?(\ A \to B, P(a) \wedge Q(a) \vdash \exists x R(x)\ ) & L_\delta \\
\hline
5.\ ?(\ A \to B, P(a), Q(a) \vdash \exists x R(x)\ ) & L_\alpha \\
\hline
6.\ ?(\ \neg A, P(a), Q(a) \vdash \exists x R(x)\ ;\ B, P(a), Q(a) \vdash \exists x R(x)\ ) & L_\beta \\
\hline
7.\ ?(\ \neg A, \neg P(a), P(a), Q(a) \vdash \exists x R(x)\ ;\ B, P(a), Q(a) \vdash \exists x R(x)\ ) & L_\gamma \\
\hline
8.\ ?(\ \phi\ ;\ B, Q(a) \to R(a), P(a), Q(a) \vdash \exists x R(x)\ ) & L_\gamma \\
\hline
9.\ ?(\ \phi\ ;\ B, \neg Q(a), P(a), Q(a) \vdash \exists x R(x)\ ;\ B, R(a), P(a), Q(a) \vdash \exists x R(x)\ ) & L_\beta \\
\hline
10.\ ?(\ \phi\ ;\ \psi\ ;\ B, R(a), P(a), Q(a), \neg\exists x R(x) \vdash R(a)\ ) & R_\delta
\end{array}$$

And here is the Socratic tree determined by this Socratic proof:



where $\chi = B, R(a), P(a), Q(a), \neg\exists x R(x) \vdash R(a)$ □

Let us observe that the Socratic tree displayed above bears a resemblance to a sequent calculus proof.

Example 14. *Here is a Socratic proof of the following left-sided sequent:*

$$\neg(\forall x(P(x) \wedge Q(x)) \rightarrow \forall x P(x) \wedge \exists x Q(x)) \ \vdash$$

in calculus $\mathbb{E}^{\mathbf{LPQ}}$. Below, A stands for '$\forall x(P(x) \wedge Q(x))$'.

$$\begin{array}{cl}
1.\ ?(\ \neg(\forall x(P(x) \wedge Q(x)) \rightarrow \forall x P(x) \wedge \exists x Q(x))\ \vdash\ ) & \\ \hline
2.\ ?(\ \forall x(P(x) \wedge Q(x)), \neg(\forall x P(x) \wedge \exists x Q(x)) \vdash\ ) & L_\alpha \\ \hline
3.\ ?(\ A, \neg\forall x P(x) \vdash\ ;\ A, \neg\exists x Q(x) \vdash\ ) & L_\beta \\ \hline
4.\ ?(\ A, \neg P(a) \vdash\ ;\ A, \neg\exists x Q(x) \vdash\ ) & L_\delta \\ \hline
5.\ ?(\ A, P(a) \wedge Q(a), \neg P(a) \vdash\ ;\ A, \neg\exists x Q(x) \vdash\ ) & L_\gamma \\ \hline
6.\ ?(\ A, P(a), Q(a), \neg P(a) \vdash\ ;\ A, \neg\exists x Q(x) \vdash\ ) & L_\alpha \\ \hline
7.\ ?(\ A, P(a), Q(a), \neg P(a) \vdash\ ;\ A, P(a) \wedge Q(a), \neg\exists x Q(x) \vdash\ ) & L_\gamma \\ \hline
8.\ ?(\ A, P(a), Q(a), \neg P(a) \vdash\ ;\ A, P(a), Q(a), \neg\exists x Q(x) \vdash\ ) & L_\alpha \\ \hline
9.\ ?(\ A, P(a), Q(a), \neg P(a) \vdash\ ;\ A, P(a), Q(a), \neg\exists x Q(x), \neg Q(a) \vdash\ ) & L_\gamma
\end{array}$$

Here is the Socratic tree determined by this Socratic proof.

$$\neg(\forall x(P(x) \wedge Q(x)) \rightarrow \forall x P(x) \wedge \exists x Q(x)) \ \vdash \ {}^{1}_{1}$$

$$\forall x(P(x) \wedge Q(x)), \neg(\forall x P(x) \wedge \exists x Q(x)) \vdash \ {}^{2}_{1}$$

$A, \neg\forall x P(x) \vdash \ {}^{3}_{1}$	$A, \neg\exists x Q(x) \vdash \ {}^{3}_{2}$
$A, \neg P(a) \vdash \ {}^{4}_{1}$	$A, P(a) \wedge Q(a), \neg\exists x Q(x) \vdash \ 7_2$
$A, P(a) \wedge Q(a), \neg P(a) \vdash \ {}^{5}_{1}$	$A, P(a), Q(a), \neg\exists x Q(x) \vdash \ 8_2$
$A, P(a), Q(a), \neg P(a) \vdash \ {}^{6}_{1}$	$A, P(a), Q(a), \neg\exists x Q(x), \neg Q(a) \vdash \ {}^{9}_{2}$

$\Box$

The above construction resembles a Hintikka-style tableau.

Example 15. *Here is a Socratic proof of sequent $\vdash (\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q))^1$ in calculus $\mathbb{E}^{\mathsf{K}}$ (see also Example 5). Let us recall that this time numeral 1 in the upper index is a part of indexed formulas. A stands for indexed formula '$\neg\Box(p \rightarrow q)$' and B for '$\neg\Box p$'.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{?(\vdash (\Box(p \to q) \to (\Box p \to \Box q))^1)}{?(\vdash (\neg\Box(p \to q))^1, (\Box p \to \Box q)^1)}\,R_\beta}{?(\vdash (\neg\Box(p \to q))^1, (\neg\Box p)^1, (\Box q)^1)}\,R_\beta}{?(\vdash (\neg\Box(p \to q))^1, (\neg\Box p)^1, (q)^{1,2})}\,R_\nu}{?(\vdash A^1, (\neg(p \to q))^2, (\neg\Box p)^1, (q)^{1,2})}\,R_\pi}{?(\vdash A^1, (\neg(p \to q))^2, B^1, (\neg p)^2, (q)^{1,2})}\,R_\pi}{?(\vdash A^1, (p)^2, B^1, (\neg p)^2, (q)^{1,2} \,;\, \vdash A^1, (\neg q)^2, B^1, (\neg p)^2, (q)^{1,2})}\,R_\alpha$$

Here is the Socratic tree determined by this Socratic proof. In order to increase the readability of the example, we do not display the annotations. Let us observe that the Socratic tree bears striking resemblance to a proof in a sequent system for modal logics.

$$\vdash (\Box(p \to q) \to (\Box p \to \Box q))^1$$
$$|$$
$$\vdash (\neg\Box(p \to q))^1, (\Box p \to \Box q)^1$$
$$|$$
$$\vdash (\neg\Box(p \to q))^1, (\neg\Box p)^1, (\Box q)^1$$
$$|$$
$$\vdash (\neg\Box(p \to q))^1, (\neg\Box p)^1, (q)^{1,2}$$
$$|$$
$$\vdash A^1, (\neg(p \to q))^2, (\neg\Box p)^1, (q)^{1,2}$$
$$|$$
$$\vdash A^1, (\neg(p \to q))^2, B^1, (\neg p)^2, (q)^{1,2}$$
$$\swarrow \qquad \searrow$$
$$\vdash A^1, (p)^2, B^1, (\neg p)^2, (q)^{1,2} \qquad \vdash A^1, (\neg q)^2, B^1, (\neg p)^2, (q)^{1,2}$$

□

Example 16. *A Socratic proof of sequent* $?(\vdash (\Box p \to \Box\Box\Diamond p)^1)$ *in calculus* $\mathbb{E}^{\mathsf{D5}}$*. This time we aim at a resemblance with Priest's analytic tableaux for modal logics (see Priest, 2008). When* R_ν *is applied, the information concerning accessibility relation is written in the form of a relational formula "*iRj*" placed after* $(\nu_0)^j$*, instead of extending the index in '*$(\nu_0)^{i,j}$*'; similarly when* $R_{\pi D}$ *is applied.*

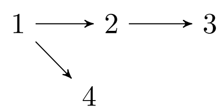
$$\dfrac{\dfrac{?(\vdash (\Box p \to \Box\Box\Diamond p)^1)}{?(\vdash (\neg\Box p)^1, (\Box\Box\Diamond p)^1)}\,R_\beta}{?(\vdash (\neg\Box p)^1, (\Box\Diamond p)^2, 1R2)}\,R_\nu$$

a continuation of Example 15:

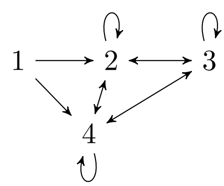
$$\dfrac{\dfrac{\dfrac{\vdots}{?(\ \vdash (\neg\Box p)^1, (\Diamond p)^3, 2R3, 1R2\ )}\ R_\nu}{?(\ \vdash (\neg\Box p)^1, (\neg p)^4, 1R4, (\Diamond p)^3, 2R3, 1R2\ )}\ R_{\pi D}}{?(\ \vdash (\neg\Box p)^1, (\neg p)^4, 1R4, (\Diamond p)^3, (p)^4, 2R3, 1R2\ )}\ R_\pi$$

The last application of R_π is permitted since $\langle 3, 4\rangle$ belongs to the Euclidean closure of $\{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 1, 4\rangle\}$, as the following diagram illustrates.

The diagram of relation $R = \{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 1, 4\rangle\}$:



The diagram of the Euclidean closure of R:



Let us also observe that if the proviso of applicability of R_ν is formulated in terms of chains (see Appendix A), then the rule may be applied since there are the following directed $\mathbf{I}_R[\varphi]$-chains (where φ stands for '$\vdash (\neg\Box p)^1, (\neg p)^4, 1R4, (\Diamond p)^3, 2R3, 1R2$'): directed $\mathbf{I}_R[\varphi]$-$[1, 3]$-chain $\langle 1, 2, 3\rangle$ and directed $\mathbf{I}_R[\varphi]$-$[1, 4]$-chain $\langle 1, 4\rangle$.

Here is the Socratic tree determined by the last Socratic proof.

$$\vdash (\Box p \to \Box\Box\Diamond p)^1$$
$$|$$
$$\vdash (\neg\Box p)^1, (\Box\Box\Diamond p)^1$$
$$|$$
$$\vdash (\neg\Box p)^1, (\Box\Diamond p)^2, 1R2$$
$$|$$
$$\vdots$$

a continuation of the Socratic tree:

$$\vdots$$

$$\vdash (\neg\Box p)^1, (\Diamond p)^3, 2R3, 1R2$$

$$\vdash (\neg\Box p)^1, (\neg p)^4, 1R4, (\Diamond p)^3, 2R3, 1R2$$

$$\vdash (\neg\Box p)^1, (\neg p)^4, 1R4, (\Diamond p)^3, p^4, 2R3, 1R2$$

□

In the next section we exploit some of the indicated resemblances between proof systems. It is worth to stress at this point that displaying and formalizing the resemblances is quite easy when the method of Socratic proofs is used. This is due to the fact that the quasi-hypersequent structures transformed by the erotetic rules—the sequences of sequents on which questions are based—are very rich semantically, as they retain all the semantic information after each application of an erotetic rule. It makes the method of Socratic proofs suitable for proof-search in various proof systems.

4.2. Translation into sequent systems for classical logic

The content of this section is partially based on the paper Leszczyńska-Jasion et al., 2013.

In Leszczyńska-Jasion et al., 2013 the authors have shown how Socratic proofs in $\mathbf{E^{PQ}}$ determine proofs in Gentzen-style sequent calculus called $\mathbf{G^{PQ}}$. Given soundness and completeness of the original $\mathbf{E^{PQ}}$ (proved in Wiśniewski and Shangin, 2006), soundness and completeness of $\mathbf{G^{PQ}}$ is proved with respect to $\mathbf{E^{PQ}}$ *via* the procedure "translating" a Socratic proof into a proof of sequent in $\mathbf{G^{PQ}}$.

Here we adopt the same strategy with respect to the issue of soundness and completeness. Let us recall that the original erotetic calculus $\mathbf{E^{PQ}}$ have rules for quantifiers expressed without the use of γ-, δ-notation. Since we have chosen to work with a variant of $\mathbf{E^{PQ}}$ expressed in the γ-, δ-notation, we need to introduce an analogous variant of sequent system. We will call this system $\mathbb{G}^{\mathbf{PQ}}$ (the one introduced in Leszczyńska-Jasion et al., 2013 is called $\mathbf{G^{PQ}}$). The proof of soundness of the sequent calculus $\mathbb{G}^{\mathbf{PQ}}$ follows the proof of soundness of $\mathbb{E}^{\mathbf{PQ}}$, and completeness of the sequent calculus is entailed by the existence of the translating procedure.

The rules of calculus $\mathbb{G}^{\mathbf{PQ}}$ are displayed in Table 4.1.

Table 4.1: Rules of calculus $\mathbb{G}^{\mathbf{PQ}}$

$\dfrac{S(\alpha_1, \alpha_2) \vdash C}{S(\alpha) \vdash C}\ L^{\mathbb{G}}_{\alpha}$	$\dfrac{S \vdash \alpha_1 \quad S \vdash \alpha_2}{S \vdash \alpha}\ R^{\mathbb{G}}_{\alpha}$
$\dfrac{S(\beta_1) \vdash C \quad S(\beta_2) \vdash C}{S(\beta) \vdash C}\ L^{\mathbb{G}}_{\beta}$	$\dfrac{S\ '\ \overline{\beta_1} \vdash \beta_2}{S \vdash \beta}\ R^{\mathbb{G}}_{\beta}$
$\dfrac{S(\kappa^*) \vdash C}{S(\kappa) \vdash C}\ L^{\mathbb{G}}_{\kappa}$	$\dfrac{S \vdash \kappa^*}{S \vdash \kappa}\ R^{\mathbb{G}}_{\kappa}$
$\dfrac{S(\gamma, \gamma(a_i)) \vdash C}{S(\gamma) \vdash C}\ L^{\mathbb{G}}_{\gamma}$ a_i is any parameter	$\dfrac{S \vdash \gamma(a_i)}{S \vdash \gamma}\ R^{\mathbb{G}}_{\gamma}$ a_i is a parameter which does not occur in $S \vdash \gamma$
$\dfrac{S(\delta(a_i)) \vdash C}{S(\delta) \vdash C}\ L^{\mathbb{G}}_{\delta}$ a_i is a parameter which does not occur in $S(\delta) \vdash C$	$\dfrac{S\ '\ \overline{\delta} \vdash \delta(a_i)}{S \vdash \delta}\ R^{\mathbb{G}}_{\delta}$ a_i is any parameter

Let us recall that the rules of calculus $\mathbb{G}^{\mathbf{PQ}}$ operate on both-sided, single succedent sequents of the form (2.1), that is:

$$S \ \vdash \ A$$

where S is a finite, possibly empty sequence of formulas of language $\mathcal{L}_{\mathsf{FOL}}$. Axioms of the calculus are sequents falling under one of the schemes:

(*a*) $S(B) \ \vdash \ B$

(*b*) $S(B)(\neg B) \ \vdash \ C$

specified in Definition 3. There are no primary structural rules. In particular, there is no weakening, hence the general form of axioms. The rules are invertible and context-shearing (multiplicative). The calculus is confluent.

The proofs of the next three lemmas follow the proofs of Lemma 1 and 2 presented in Section 2.2.4 of Chapter 2; we skip it.

Lemma 5. *Let* $\mathbf{r}$ *stand for one of the following one-premise rules of* $\mathbb{G}^{\mathbf{PQ}}$*:* $L^{\mathbb{G}}_{\alpha}$, $R^{\mathbb{G}}_{\beta}$, $L^{\mathbb{G}}_{\kappa}$, $R^{\mathbb{G}}_{\kappa}$, $L^{\mathbb{G}}_{\gamma}$, $R^{\mathbb{G}}_{\delta}$*; thus* $\mathbf{r}$ *is any of the one-premise rules except for* $R^{\mathbb{G}}_{\gamma}$ *and* $L^{\mathbb{G}}_{\delta}$*. Then for each* FOL*-interpretation* $\mathcal{I}$ *of language* $\mathcal{L}_{\mathsf{FOL}}$*, a sequent falling under the schema of the premise of* $\mathbf{r}$ *is correct under* $\mathcal{I}$ *iff the respective sequent falling under the schema of the conclusion of* $\mathbf{r}$ *is correct under* $\mathcal{I}$*.*

Lemma 6. *Let* $\mathbf{r}$ *be a two-premises rule of* $\mathbb{G}^{\mathbf{PQ}}$*, that is,* $\mathbf{r} \in \{R^{\mathbb{G}}_{\alpha}, L^{\mathbb{G}}_{\beta}\}$*. Then for each* FOL*-interpretation* $\mathcal{I}$ *of language* $\mathcal{L}_{\mathsf{FOL}}$*, sequents falling under the schema of the premises of* $\mathbf{r}$ *are both correct under* $\mathcal{I}$ *iff the respective sequent falling under the schema of the conclusion of* $\mathbf{r}$ *is correct under* $\mathcal{I}$*.*

Lemma 7. *Let* $\mathbf{r} \in \{R^{\mathbb{G}}_{\gamma}, L^{\mathbb{G}}_{\delta}\}$*. Then:*

1. *if* $\mathcal{I}$ *is a* FOL*-interpretation of* $\mathcal{L}_{\mathsf{FOL}}$ *such that a sequent falling under the schema of the premise of* $\mathbf{r}$ *is correct under* $\mathcal{I}$*, then the respective sequent falling under the schema of the conclusion of* $\mathbf{r}$ *is correct under* $\mathcal{I}$ *as well; and moreover*

2. *if a sequent falling under the conclusion schema of* $\mathbf{r}$ *is correct under* ***every*** FOL*-interpretation of* $\mathcal{L}_{\mathsf{FOL}}$*, then the respective sequent-premise of* $\mathbf{r}$ *is correct under* ***every*** FOL*-interpretation of* $\mathcal{L}_{\mathsf{FOL}}$*.*

In accordance with traditional accounts (see, for example, Troelstra and Schwichtenberg, 2000, p. 20), by a *proof of sequent* $S \vdash A$ *in* $\mathbb{G}^{\mathbf{PQ}}$ we mean a finite tree labelled with sequents, regulated by the rules of $\mathbb{G}^{\mathbf{PQ}}$ and such that the origin is labelled with sequent $S \vdash A$ and the leaves are labelled with axioms of the calculus. What we mean by "regulation" is that each node-label is connected with the label(s) of the immediate successor(s) node(s) (if there are any) according to one of the rules of the calculus, that is, the node-label falls under the schema of the conclusion of the rule, whereas the immediate successor(s) label(s) falls (fall) under the schema of its premise(s).

Lemmas 5, 6 and 7 entail:

Theorem 16 (soundness of $\mathbb{G}^{\mathbf{PQ}}$ wrt semantics of sequents)**.** *If a sequent is provable in calculus* $\mathbb{G}^{\mathbf{PQ}}$*, then it is correct under every* FOL*-interpretation.*

Theorem 17 (soundness of $\mathbb{G}^{\mathbf{PQ}}$ wrt semantics of $\mathcal{L}_{\mathsf{FOL}}$). *If sequent* $\vdash A$ *is provable in calculus* $\mathbb{G}^{\mathbf{PQ}}$*, then* A *is* FOL*-valid.*

Now we show that after a certain reformulation of its structure a Socratic tree $\mathbf{Tr}(\mathbf{s}) = \langle X_{\mathbf{s}}, R_{\mathbf{s}} \rangle$ produces a proof in $\mathbb{G}^{\mathbf{PQ}}$. In Leszczyńska-Jasion et al., 2013 the authors have used the structure of annotated sequents and assumed that if a node of a Socratic tree is an object of the form $\langle n, i, \phi \rangle$, then the pair $\langle n, i \rangle$ constitutes a node of the parallel Gentzen-style tree and the sequent ϕ constitutes its label.[2]

Thus suppose that $\langle X_{\mathbf{s}}, R_{\mathbf{s}} \rangle$ is the Socratic tree determined by a Socratic transformation $\mathbf{s}$. Then we put: $X_{\mathbf{sG}}$—the set of pairs of numerals which are parts of the annotated sequents in $X_{\mathbf{s}}$, $R_{\mathbf{sG}}$—the relation extracted from relation $R_{\mathbf{s}}$ in a similar way; formally:

Definition 35. *Let* $\langle X_{\mathbf{s}}, R_{\mathbf{s}} \rangle$ *be the Socratic tree determined by a Socratic transformation* $\mathbf{s}$*. Then set* $X_{\mathbf{sG}}$ *and relation* $R_{\mathbf{sG}}$ *on this set are defined as follows:*

1. $\langle n, i \rangle \in X_{\mathbf{sG}}$ *iff for some sequent* ϕ, $\langle n, i, \phi \rangle \in X_{\mathbf{s}}$,
2. $\langle \langle n_1, i_1 \rangle, \langle n_2, i_2 \rangle \rangle \in R_{\mathbf{sG}}$ *iff for some sequents* ϕ_1, ϕ_2,
   $\langle \langle n_1, i_1, \phi_1 \rangle, \langle n_2, i_2, \phi_2 \rangle \rangle \in R_{\mathbf{s}}$. □

Obviously, the structure $\langle X_{\mathbf{sG}}, R_{\mathbf{sG}} \rangle$ is a tree, since $\langle X_{\mathbf{s}}, R_{\mathbf{s}} \rangle$ is a tree. Moreover, the tree is labelled with labelling function $\eta_{\mathbf{s}}$ such that if $\langle i, j, \phi \rangle \in X_{\mathbf{s}}$, then sequent ϕ is the label $\eta_{\mathbf{s}}(\langle i, j \rangle)$ of node $\langle i, j \rangle$. A triple $\langle X_{\mathbf{sG}}, R_{\mathbf{sG}}, \eta_{\mathbf{s}} \rangle$, that is, the tree $\langle X_{\mathbf{sG}}, R_{\mathbf{sG}} \rangle$ together with its labelling function $\eta_{\mathbf{s}}$, will be called the *labelled tree determined by a Socratic transformation* $\mathbf{s}$.

Further, it is shown in Leszczyńska-Jasion et al., 2013 that the following theorem is true with respect to calculi $\mathbf{E}^{\mathbf{PQ}}$ and $\mathbf{G}^{\mathbf{PQ}}$.

Theorem 18. *If* $\langle X_{\mathbf{s}}, R_{\mathbf{s}} \rangle$ *is the Socratic tree determined by a Socratic proof of sequent* $S \vdash A$ *in* $\mathbf{E}^{\mathbf{PQ}}$*, then the labelled tree* $\langle X_{\mathbf{sG}}, R_{\mathbf{sG}}, \eta_{\mathbf{s}} \rangle$ *is a proof of* $S \vdash A$ *in* $\mathbf{G}^{\mathbf{PQ}}$.

As we have explained above, calculi $\mathbb{E}^{\mathbf{PQ}}$ and $\mathbb{G}^{\mathbf{PQ}}$ considered here are variants of calculi considered in Leszczyńska-Jasion et al., 2013—variants modulo uniform notation with respect to quantifier formulas. There is

[2] In Leszczyńska-Jasion et al., 2013 the authors have also shown how this feature of the construction of a tree may be used in order to simplify the algorithm of tree construction. It is also a good idea to compare the construction of trees used in Leszczyńska-Jasion et al., 2013 with that in Kaye, 2008.

nothing, however, in the proof of the above theorem that depends on this difference between the calculi, therefore we may rely on the proof presented in Leszczyńska-Jasion et al., 2013 and state:

Theorem 19. *If $\langle X_{\mathbf{s}}, R_{\mathbf{s}}\rangle$ is the Socratic tree determined by a Socratic proof of sequent $S \vdash A$ in $\mathbb{E}^{\mathbf{PQ}}$, then the labelled tree $\langle X_{\mathbf{sG}}, R_{\mathbf{sG}}, \eta_{\mathbf{s}}\rangle$ is a proof of $S \vdash A$ in $\mathbb{G}^{\mathbf{PQ}}$.*

Theorem 19, together with Theorems 5 and 6 (completeness of $\mathbb{E}$) entail:

Theorem 20 (completeness of $\mathbb{G}^{\mathbf{PQ}}$)**.** *If sequent $\vdash A$ is correct under every* FOL*-interpretation of $\mathcal{L}_{\mathsf{FOL}}$, then there exists a proof of the sequent in $\mathbb{G}^{\mathbf{PQ}}$. Moreover, if A is a* FOL*-valid formula, then there exists a proof of sequent $\vdash A$ in $\mathbb{G}^{\mathbf{PQ}}$.*

If $\mathbf{s}$ is a Socratic proof of sequent $S \vdash A$ in $\mathbb{E}^{\mathbf{PQ}}$, then the labelled tree $\langle X_{\mathbf{sG}}, R_{\mathbf{sG}}, \eta_{\mathbf{s}}\rangle$ will be called the *proof of $S \vdash A$ in $\mathbb{G}^{\mathbf{PQ}}$ based upon* $\mathbf{s}$. We will denote it by $\mathbf{G}(\mathbf{s})$.

Example 17. *Here is the proof of sequent $(p \vee p) \ \vee \ (q \vee q) \ \vdash \ p \vee q$ based upon Socratic transformation from Example 12 (see page 113).*

$$\dfrac{\dfrac{\dfrac{p, \neg p \ \vdash \ q}{p \ \vdash \ p \vee q} \quad \dfrac{p, \neg p \ \vdash \ q}{p \ \vdash \ p \vee q}}{p \vee p \ \vdash \ p \vee q} \qquad \dfrac{\dfrac{q, \neg p \ \vdash \ q}{q \ \vdash \ p \vee q} \quad \dfrac{q, \neg p \ \vdash \ q}{q \ \vdash \ p \vee q}}{q \vee q \ \vdash \ p \vee q}}{(p \vee p) \ \vee \ (q \vee q) \ \vdash \ p \vee q}$$

□

Example 18. *Here is the proof of sequent*
$\vdash (\exists x P(x) \rightarrow \forall x(Q(x) \rightarrow R(x))) \rightarrow (\exists x(P(x) \wedge Q(x)) \rightarrow \exists x R(x))$
based upon Socratic transformation from Example 13 (see page 117).

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\phi}{\neg A, P(a), Q(a) \vdash \exists x R(x)}\,\mathbf{L}_{\delta} \qquad \cfrac{\cfrac{\psi \quad \cfrac{\chi}{B, R(a), P(a), Q(a) \vdash \exists x R(x)}\,\mathbf{R}_{\delta}}{B, Q(a) \rightarrow R(a), P(a), Q(a) \vdash \exists x R(x)}\,\mathbf{L}_{\beta}}{B, P(a), Q(a) \vdash \exists x R(x)}\,\mathbf{L}_{\gamma}}{A \rightarrow B, P(a), Q(a) \vdash \exists x R(x)}\,\mathbf{L}^{\mathbb{G}}_{\beta}}{A \rightarrow B, P(a) \wedge Q(a) \vdash \exists x R(x)}\,\mathbf{L}^{\mathbb{G}}_{\alpha}}{A \rightarrow B, \exists x(P(x) \wedge Q(x)) \vdash \exists x R(x)}\,\mathbf{L}^{\mathbb{G}}_{\delta}}{A \rightarrow B \vdash \exists x(P(x) \wedge Q(x)) \rightarrow \exists x R(x)}\,\mathbf{R}^{\mathbb{G}}_{\beta}}{\vdash (A \rightarrow B) \rightarrow (\exists x(P(x) \wedge Q(x)) \rightarrow \exists x R(x))}\,\mathbf{R}^{\mathbb{G}}_{\beta}$$

□

4.3. Translation into sequent system for modal logic K

We will derive proofs of sequents with modal formulas in a similar manner for the case of K, D, T, K4, D4, S4, that is, all normal logics which are neither symmetric nor Euclidean. This time we have to deal with semantic information expressed in the form of indices. Probably the most interesting aspect of the translation provided in this section is the relation between the purely syntactic entities, like sequents, in which no additional semantic information is explicitly represented, and sequents enhanced with this kind of additional information, like indices, labels and the like.

At the end of Chapter 2, we have introduced, after (Goré, 1999), a distinction between implicit and explicit proof systems for modal logics. As we have observed, the method of Socratic proofs is an explicit proof method, as it uses indices to encode the semantic information in the syntax. In this section we show how to transform a Socratic proof of a sequent containing indexed formulas, constructed in an erotetic calculus, into a proof of a corresponding "ordinary" sequent in an implicit sequent calculus, *i.e.*, sequent calculus in which no semantic information is represented directly. Therefore we may say that the algorithms presented in this section describe the relation between an explicit erotetic calculus and an implicit sequent calculus.

Another aspect of explicit and implicit systems which we find worth comparing is the property of *purity* of rules. In Avron, 1993 we find the following definition: a rule of a monotonic Gentzen-type system is called *pure* if, whenever $\Gamma \Rightarrow \Delta$ can be inferred by it from $\Gamma_i \Rightarrow \Delta_i$ $(i = 1, \ldots, n)$, then $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ can also be inferred by it from $\Gamma_i, \Gamma' \Rightarrow \Delta_i, \Delta'$ $(i = 1, \ldots, n$, Γ' and Δ' are arbitrary sets of formulas).[3] The rules of erotetic calculi for modal logics are pure, whereas the modal rules of implicit sequent calculi are not.

Yet there is another important difference between the two types of proof systems, strictly connected with the above discussed, that motivates the study of translation algorithms; it is the property of *confluency* which we have defined at the beginning of this chapter. Let us recall that confluency is a property desirable for implementation, since in a confluent proof system there are no "bad moves" in the construction of a deriva-

[3] See also Poggiolesi, 2011, Chapter 1 on various important properties of the rules of sequent calculi.

tion, and that prevents the necessity of backtracking. Unfortunately, there is no confluent sequent system for modal logics (see Indrzejczak, 2010, subsection 6.2.1). The modal rules of the sequent systems are not invertible, so backtracking may be needed to establish whether a failure in a backward proof-search construction is due to a "bad move", or to unprovability of a sequent. Obviously, we mean the implicit sequent systems here. There are plenty of systems using various kinds of labels that have invertible modal rules, and thus have the property of confluency (we list many of them at the end of Chapter 2). The systems may be useful in automatic proof search. Our result shows that proofs in an implicit sequent system for modal logics may be found by a translation from an explicit proof format which is more amenable to proof search. No backtracking is needed. Obviously, the computational work is done on the level of the explicit proof system, hence using the algorithms presented here together with an inefficient algorithm of proof search in the explicit system brings no computational speedup. We believe, however, that the study of relations between different proof systems may bring us closer to new, efficient algorithms.

Below we present the rules of erotetic calculus $\mathbb{E}^{\mathsf{K}\tau}$ and the corresponding rules of Gentzen-style (implicit) sequent calculus $\mathbb{G}^{\mathsf{K}\tau}$. The second one (the right columns of Tables 4.3, 4.4, 4.5) is the Gentzen-style system. The rules for classical operators are non-controversial; the rules for modal operators are based on these that may be found in Fitting, 1983, Chapter 3 (compare also Wansing, 2002, Section 1). The rules of erotetic calculus $\mathbb{E}^{\mathsf{K}\tau}$ are in a clear way adjusted to the rules of $\mathbb{G}^{\mathsf{K}\tau}$. Thus we obtain a both-sided and multisuccedent version of the method of Socratic proofs for modal logics (the left column). In the case of both calculi the antecedent and the succedent of a sequent may be empty, though not both at the same time.

Conventions: language, concatenation, replacement

We need a separate language of sequents for the implicit systems. The language will be called $\mathcal{M}_{\vdash}^{imp}$. It is built upon $\mathcal{M}$, but contains additionally the turnstile symbol '$\vdash$' and the comma ','. Sequents of $\mathcal{M}_{\vdash}^{imp}$ are expressions of the form:

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_k$$

where $\langle A_1, \ldots, A_n \rangle$ and $\langle B_1, \ldots, B_k \rangle$ represent finite sequences of formulas of $\mathcal{M}$. It may happen that $n = 0$ or $k = 0$, though not both at the same time. Below we will use X, Y for finite, possibly empty sequences of formulas of $\mathcal{M}$. Letters S, T still refer to finite possibly empty sequences of indexed formulas of $\mathcal{M}^{?}_{\vdash}$.

In order to simplify the presentation of the rules, we extend the conventions introduced on page 41. Let us recall that by '$S(A)$' we refer to a sequence of formulas in which A occurs as its term, where the position of the term is arbitrary. Then by '$S(A/\_)$' we mean the result of removing formula A from the place it was before. More formally, if $S(A)$ can be written as:

$$S_1 \,'\, A \,'\, S_2$$

then $S(A/\_)$ refers to:

$$S_1 \,'\, S_2$$

If $S(A)$ occurs in the premise of a rule, then in the conclusion we will write simply '$S(\_)$' instead of '$S(A/\_)$'; this should improve the readability of the rules' schemas. Moreover, if the expression '$S(\_)$' is used first in the premise of a rule, then $S(A)$ used in the conclusion means that A was added at the position indicated by '_', which is an arbitrary position, in fact. Analogous conventions apply to sequences of formulas of language $\mathcal{M}$, that is, to the rules of calculus $\mathbb{G}^{\mathsf{K}\tau}$. Let us explain this in detail on one example. In Table 4.2, the rules defining negation in antecedent of a sequent in calculi $\mathbb{E}^{\mathsf{K}\tau}$ and $\mathbb{G}^{\mathsf{K}\tau}$ are displayed with the use of the concatenation sign '$'$'.

Table 4.2: Left negation rules of $\mathbb{E}^{\mathsf{K}\tau}$ and $\mathbb{G}^{\mathsf{K}\tau}$: concatenation version

$\dfrac{?(\Phi \,;\, S_1 \,'\, \neg A^{\sigma} \,'\, S_2 \,\vdash\, T_1 \,'\, T_2 \,;\, \Psi)}{?(\Phi \,;\, S_1 \,'\, S_2 \,\vdash\, T_1 \,'\, A^{\sigma} \,'\, T_2 \,;\, \Psi)}\; L^{\mathbb{E}}_{\neg}$	$\dfrac{X_1 \,'\, X_2 \,\vdash\, Y_1 \,'\, A \,'\, Y_2}{X_1 \,'\, \neg A \,'\, X_2 \,\vdash\, Y_1 \,'\, Y_2}\; L^{\mathbb{G}}_{\neg}$

None of the calculi contains the structural rule of Exchange, which allows to change the order of formulas in the antecedent or the succedent of a sequent. Therefore the rules are formulated in a way which allows to put the principal formula in the conclusion (that is formula A^{σ} in the case of rule $L^{\mathbb{E}}_{\neg}$, and formula $\neg A$ in the case of rule $L^{\mathbb{G}}_{\neg}$) in an arbitrary place. The same idea may be expressed with the use of our conventions concerning replacement in a way presented in Table 4.3.

The other rules are presented with the "replacement convention": Table 4.4 presents the remaining non-branching rules for the classical

Table 4.3: Left negation rules of $\mathbb{E}^{\mathsf{K}\tau}$ and $\mathbb{G}^{\mathsf{K}\tau}$: replacement version

$$\frac{?(\Phi\,;\,S(\neg A^{\sigma})\ \vdash\ T(\_)\,;\,\Psi)}{?(\Phi\,;\,S(\_)\ \vdash\ T(A^{\sigma})\,;\,\Psi)}\,L^{\mathbb{E}}_{\neg} \qquad \frac{X(\_)\ \vdash\ Y(A)}{X(\neg A)\ \vdash\ Y(\_)}\,L^{\mathbb{G}}_{\neg}$$

connectives, Table 4.5 presents the branching rules for the classical connectives, and Tables 4.6, 4.8 the modal rules. Consequently to the conventions introduced above, if '$S(A)$' occurs in a premise of a rule, then '$S(B)$' in the conclusion (or '$S(B_1, B_2)$', like in rule $L^{\mathbb{E}}_{\wedge}$) should be read as '$S(A/B)$' (or '$S(A/B_1, B_2)$' respectively). Analogous conventions apply to calculus $\mathbb{G}^{\mathsf{K}\tau}$.

Table 4.4: Non-branching rules of $\mathbb{E}^{\mathsf{K}\tau}$ and $\mathbb{G}^{\mathsf{K}\tau}$

$$\frac{?(\Phi\,;\,S(\_)\ \vdash\ T(\neg A^{\sigma})\,;\,\Psi)}{?(\Phi\,;\,S(A^{\sigma})\ \vdash T(\_)\,;\,\Psi)}\,R^{\mathbb{E}}_{\neg} \qquad \frac{X(A)\ \vdash Y(\_)}{X(\_)\ \vdash\ Y(\neg A)}\,R^{\mathbb{G}}_{\neg}$$

$$\frac{?(\Phi\,;\,S((A\wedge B)^{\sigma})\ \vdash T\,;\,\Psi)}{?(\Phi\,;\,S(A^{\sigma}, B^{\sigma})\ \vdash\ T\,;\,\Psi)}\,L^{\mathbb{E}}_{\wedge} \qquad \frac{X(A, B)\ \vdash\ Y}{X(A\wedge B)\ \vdash\ Y}\,L^{\mathbb{G}}_{\wedge}$$

$$\frac{?(\Phi\,;\,S\ \vdash\ T((A\vee B)^{\sigma})\,;\,\Psi)}{?(\Phi\,;\,S\ \vdash\ T(A^{\sigma}, B^{\sigma})\,;\,\Psi)}\,R^{\mathbb{E}}_{\vee} \qquad \frac{X\ \vdash\ Y(A, B)}{X\ \vdash\ Y(A\vee B)}\,R^{\mathbb{G}}_{\vee}$$

$$\frac{?(\Phi\,;\,S(\_)\ \vdash\ T((A\to B)^{\sigma})\,;\,\Psi)}{?(\Phi\,;\,S(A^{\sigma})\ \vdash\ T(B^{\sigma})\,;\,\Psi)}\,R^{\mathbb{E}}_{\to} \qquad \frac{X(A)\ \vdash\ Y(B)}{X(\_)\ \vdash Y(A\to B)}\,R^{\mathbb{G}}_{\to}$$

Observe that in the case of rule $L^{\mathbb{G}}_{\wedge}$ and $R^{\mathbb{G}}_{\vee}$, formula B immediately succeeds formula A in the premise. The branching rules of $\mathbb{G}^{\mathsf{K}\tau}$ have shared contexts, that is, where $Y(A)$ and $Y(B)$ occur in the premises, we assume that formulas A and B occur in the same position.

Finally, the rules for the modal operators are non-branching. In the case of the rules of $\mathbb{E}^{\mathsf{K}\tau}$ "creating a new world", that is $L^{\mathbb{E}}_{\Diamond}$ and $R^{\mathbb{E}}_{\Box}$, j is the smallest numeral which is new with respect to the active constituent of the question-premise. We have used the same restriction previously

Table 4.5: Branching rules of $\mathbb{E}^{\mathsf{K}\tau}$ and $\mathbb{G}^{\mathsf{K}\tau}$

$$\frac{?(\Phi\,;\,S\ \vdash\ T((A\wedge B)^{\sigma})\,;\,\Psi)}{?(\Phi\,;\,S\ \vdash\ T(A^{\sigma})\,;\,S\ \vdash\ T(B^{\sigma})\,;\,\Psi)}\ R^{\mathbb{E}}_{\wedge} \qquad \frac{X\vdash Y(A)\qquad X\vdash Y(B)}{X\vdash Y(A\wedge B)}\ R^{\mathbb{G}}_{\wedge}$$

$$\frac{?(\Phi\,;\,S((A\vee B)^{\sigma})\ \vdash\ T\,;\,\Psi)}{?(\Phi\,;\,S(A^{\sigma})\ \vdash\ T\,;\,S(B^{\sigma})\ \vdash\ T\,;\,\Psi)}\ L^{\mathbb{E}}_{\vee} \qquad \frac{X(A)\vdash Y\qquad X(B)\vdash Y}{X(A\vee B)\ \vdash\ Y}\ L^{\mathbb{G}}_{\vee}$$

$$\frac{?(\Phi\,;\,S((A\to B)^{\sigma})\ \vdash\ T(\_)\,;\,\Psi)}{?(\Phi\,;\,S(\_)\ \vdash\ T(A^{\sigma})\,;\,S(B^{\sigma})\ \vdash\ T(\_)\,;\,\Psi)}\ L^{\mathbb{E}}_{\to}$$

$$\frac{X(\_)\vdash Y(A)\qquad X(B)\vdash Y}{X(A\to B)\vdash Y(\_)}\ L^{\mathbb{G}}_{\to}$$

(see Chapter 2, page 73). Let us also recall that in the case of the rules "settling the worlds", that is, $\mathbf{L}_{\Box}$, $\mathbf{R}_{\Diamond}$, numeral j is already present in the active constituent of the question-premise. The numeral must satisfy the relevant condition of the applicability of the rule. In the case of logic K, it means that j immediately succeeds i in some index of the active sequent.

Table 4.6: Modal rules of $\mathbb{E}^{\mathsf{K}\tau}$

$$\frac{?(\Phi\,;\,S(\Diamond A^{\sigma/i})\ \vdash\ T\,;\,\Psi)}{?(\Phi\,;\,S(A^{\sigma/i,j})\ \vdash\ T\,;\,\Psi)}\ L^{\mathbb{E}}_{\Diamond} \qquad \frac{?(\Phi\,;\,S\ \vdash T(\Diamond A^{\sigma/i})\,;\,\Psi)}{?(\Phi\,;\,S\ \vdash T(\Diamond A^{\sigma/i}, A^{j})\,;\,\Psi)}\ R^{\mathbb{E}}_{\Diamond}$$

$$\frac{?(\Phi\,;\,S(\Box A^{\sigma/i})\ \vdash\ T\,;\,\Psi)}{?(\Phi\,;\,S(\Box A^{\sigma/i}, A^{j})\ \vdash\ T\,;\,\Psi)}\ L^{\mathbb{E}}_{\Box} \qquad \frac{?(\Phi\,;\,S\ \vdash\ T(\Box A^{\sigma/i})\,;\,\Psi)}{?(\Phi\,;\,S\ \vdash\ T(A^{\sigma/i,j})\,;\,\Psi)}\ R^{\mathbb{E}}_{\Box}$$

Analogously to the notion of an active formula we will use that of an *active indexed formula* of a rule $\mathbf{r}$, which is the indexed formula whose schema is distinguished in the sequent-premise of $\mathbf{r}$. If the active indexed formula of $\mathbf{r}$ is of the form $(A)^{\sigma/i}$, then numeral i will be called the *active numeral* of $\mathbf{r}$. Further, next to the notion of constituent-conclusion (sequent-conclusion), we will sometimes use the notion of indexed *formula-conclusion of rule* $\mathbf{r}$, by which we mean the indexed formula (or formulas) whose schema is distinguished in the sequent-conclusion of $\mathbf{r}$. We will also need the notion of *principal numeral*. The principal numeral of rule $\mathbf{r}$ is the last one in the index of the indexed formula-conclusion. In the case of rules $R^{\mathbb{E}}_{\Diamond}$ and $L^{\mathbb{E}}_{\Box}$, there are two formulas-conclusions with different numerals at the end of their indices: $\Diamond A^{\sigma/i}, A^j$, or $\Box A^{\sigma/i}, A^j$, respectively. In this situation we take j to be the principal numeral.[4]

Calculus $\mathbb{G}^{\mathsf{K}\tau}$ used here has been obtained from the sequent calculus considered in Fitting, 1983 by dropping the unified notation, so we first recall the original account by Melvin Fitting before we continue (see Fitting, 1983, p. 90). In Fitting, 1983 the author uses '$\longrightarrow$' for the sequent sign; sequents have finite sets of formulas on the left and right side of the sign. Finally, the ν, π-notation is used, as in Section 2.3 of Chapter 2 of this book.

Table 4.7: Modal rules of sequent calculi by Melvin Fitting

$$\frac{U\sharp, A \longrightarrow V\flat}{U, \Diamond A \longrightarrow V}\,\mathbf{L}_{\Diamond} \qquad \frac{U\sharp \longrightarrow V\flat, A}{U \longrightarrow V, \Box A}\,\mathbf{R}_{\Box}$$

where $U\sharp = \{F : \Box F \in U\}$, $V\flat = \{F : \Diamond F \in V\}$. Though not obvious at the first glance, the above account of the modal rules incorporates weakening, since U, V in the conclusions may contain some extra formulas. Below we illustrate this phenomenon, see Example 21.

Table 4.8 presents the modal rules of calculus $\mathbb{G}^{\mathsf{K}\tau}$. If $X = \langle A_1, \ldots, A_n\rangle$ is a finite sequence of formulas of $\mathcal{M}$, then by '$\Box X$' and $\Diamond X$' we mean:

$$\Box X = \langle \Box A_1, \ldots, \Box A_n\rangle$$

$$\Diamond X = \langle \Diamond A_1, \ldots, \Diamond A_n\rangle$$

We find it much more convenient to present the modal rules of $\mathbb{G}^{\mathsf{K}\tau}$ in the concatenation-convention. Since weakening is not incorporated in

[4] By the way, in the context of sequent calculi, formula-conclusion is usually called *principal formula*. But since we have reversed the rules, using this terminology would be confusing.

the modal rules of $\mathbb{G}^{\mathsf{K}\tau}$, we also state it explicitly. For this purpose we need the notion of subsequence. By *subsequence* of sequence X we mean any sequence that is obtained by deleting some elements of X without changing the order of the other elements. More formally, if finite n-term sequence X is a function from $\{1, \ldots, n\}$ to a certain set (namely, the set of terms of X), then a subsequence of X is any restriction of this function. Accordingly, sequence X^* will be called a *supersequence* of sequence X iff X is a subsequence of X^*. Finally, supersequence X^* of sequence X is called *proper* iff the two are not identical.[5]

Table 4.8: Modal rules and weakening in $\mathbb{G}^{\mathsf{K}\tau}$

$$\frac{X_1\,'\,A\,'\,X_2 \;\vdash\; Y}{\Box X_1\,'\,\Diamond A\,'\,\Box X_2 \;\vdash\; \Diamond Y}\; L^{\mathbb{G}}_{\Diamond} \qquad \frac{X \;\vdash\; Y_1\,'\,A\,'\,Y_2}{\Box X \;\vdash\; \Diamond Y_1\,'\,\Box A\,'\,\Diamond Y_2}\; R^{\mathbb{G}}_{\Box}$$

$$\frac{X \;\vdash\; Y}{X^* \;\vdash\; Y}\; L^{\mathbb{G}}_{W} \qquad \frac{X \;\vdash\; Y}{X \;\vdash\; Y^*}\; R^{\mathbb{G}}_{W}$$

where X^* is a proper supersequence of X — where Y^* is a proper supersequence of Y

On the other hand, the axioms of calculus $\mathbb{G}^{\mathsf{K}\tau}$ are sequents of the form '$B \;\vdash\; B$', exclusively, where B is a formula of language $\mathcal{M}$. It is common to formulate the axioms of a sequent calculus as a no-premise rule Ax, as below. Let us also observe that the no-premise rule Ax_{+W} (to the right) is derivable in $\mathbb{G}^{\mathsf{K}\tau}$ by Ax, $L^{\mathbb{G}}_{W}$ and $R^{\mathbb{G}}_{W}$.

$$\frac{}{B \;\vdash\; B}\; Ax \qquad \frac{}{X(B) \;\vdash Y(B)}\; Ax_{+W}$$

Calculus $\mathbb{G}^{\mathsf{K}\tau}$ is the set of rules:

$$\{L^{\mathbb{G}}_{\neg}, R^{\mathbb{G}}_{\neg}, L^{\mathbb{G}}_{\wedge}, R^{\mathbb{G}}_{\vee}, R^{\mathbb{G}}_{\to}, R^{\mathbb{G}}_{\wedge}, L^{\mathbb{G}}_{\vee}, L^{\mathbb{G}}_{\to}, L^{\mathbb{G}}_{\Diamond}, R^{\mathbb{G}}_{\Box}, L^{\mathbb{G}}_{W}, R^{\mathbb{G}}_{W}, Ax\}.$$

The notion of proof in calculus $\mathbb{G}^{\mathsf{K}\tau}$ is defined in a standard manner:

Definition 36. *Let A be a formula of $\mathcal{M}$. A* proof of sequent $\vdash A$ in calculus $\mathbb{G}^{\mathsf{K}\tau}$ *is a finite tree labelled with sequents of language $\mathcal{M}^{imp}_{\vdash}$,*

[5] The notion of restriction of a function does not preclude identity, therefore we need to distinguish between the proper and improper case.

regulated by the rules of calculus $\mathbb{G}^{\mathsf{K}\tau}$*, with the root labelled with sequent* $\vdash A$*, and the leaves labelled with axioms of the calculus.* □

On the other hand, calculus $\mathbb{E}^{\mathsf{K}\tau}$ is the following set of rules:

$$\{L_{\neg}^{\mathbb{E}}, R_{\neg}^{\mathbb{E}}, L_{\wedge}^{\mathbb{E}}, R_{\vee}^{\mathbb{E}}, R_{\rightarrow}^{\mathbb{E}}, R_{\wedge}^{\mathbb{E}}, L_{\vee}^{\mathbb{E}}, L_{\rightarrow}^{\mathbb{E}}, L_{\Diamond}^{\mathbb{E}}, R_{\Diamond}^{\mathbb{E}}, L_{\Box}^{\mathbb{E}}, R_{\Box}^{\mathbb{E}}\}.$$

Now we may introduce the following:

Definition 37. *A* Socratic transformation of a question Q via the rules of $\mathbb{E}^{\mathsf{K}\tau}$ *is a sequence* $\mathbf{s} = \langle Q_1, Q_2, \dots \rangle$ *of questions of language* $\mathcal{M}_{\vdash}^{?}$ *such that* $Q_1 = Q$ *and for each* $n > 1$*, question* Q_n *results from question* Q_{n-1} *by an application of one of the rules of* $\mathbb{E}^{\mathsf{K}\tau}$*.* □

Definition 38. *Let* $\vdash (A)^1$ *be a sequent of language* $\mathcal{M}_{\vdash}^{?}$*. A* Socratic proof of $\vdash (A)^1$ in $\mathbb{E}^{\mathsf{K}\tau}$ *is a finite Socratic transformation* $\mathbf{s}$ *of the question* $?(\vdash (A)^1)$ *via the rules of* $\mathbb{E}^{\mathsf{K}\tau}$ *such that each constituent of the last question of* $\mathbf{s}$ *is of the following form:*

(*i*) $S(B^{\sigma_1/i}) \vdash T(B^{\sigma_2/i})$ □

Here is an example of a Socratic proof in $\mathbb{E}^{\mathsf{K}\tau}$. A stands for '$\Box(p \rightarrow q)$'. (The numbers of lines are for further reference.)

Example 19.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1.\ ?(\vdash (\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q))^1)}{2.\ ?((A)^1 \vdash (\Box p \rightarrow \Box q)^1)}\ R_{\rightarrow}^{\mathbb{E}}}{3.\ ?((A)^1, (\Box p)^1 \vdash (\Box q)^1)}\ R_{\rightarrow}^{\mathbb{E}}}{4.\ ?((A)^1, (\Box p)^1 \vdash (q)^{1,2})}\ R_{\Box}^{\mathbb{E}}}{5.\ ?((A)^1, (\Box p)^1, (p)^2 \vdash (q)^{1,2})}\ L_{\Box}^{\mathbb{E}}}{6.\ ?((A)^1, (p \rightarrow q)^2, (\Box p)^1, (p)^2 \vdash (q)^{1,2})}\ L_{\Box}^{\mathbb{E}}}{7.\ ?((A)^1, (\Box p)^1, (p)^2 \vdash (q)^{1,2}, (p)^2\ ;\ (A)^1, (q)^2, (\Box p)^1, (p)^2 \vdash (q)^{1,2})}\ R_{\rightarrow}^{\mathbb{E}}$$

□

And here is a proof of sequent $\vdash \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ in $\mathbb{G}^{\mathsf{K}\tau}$.

Example 20.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{p \vdash q, p}\ R_W^{\mathbb{G}} \qquad \dfrac{q \vdash q}{q, p \vdash q}\ L_W^{\mathbb{G}}}{p \rightarrow q, p \vdash q}\ L_{\rightarrow}^{\mathbb{G}}}{\Box(p \rightarrow q), \Box p \vdash \Box q}\ R_{\Box}^{\mathbb{G}}}{\Box(p \rightarrow q) \vdash \Box p \rightarrow \Box q}\ R_{\rightarrow}^{\mathbb{G}}}{\vdash \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)}\ R_{\rightarrow}^{\mathbb{G}}$$

□

In the above example, one application of the modal rule $R^{\mathbb{G}}_{\Box}$ of $\mathbb{G}^{\mathsf{K}\tau}$ corresponds to a sequence of applications of the modal rules of $\mathbb{E}^{\mathsf{K}\tau}$ in lines 4–6 of Example 19. Semantically speaking, one application of $R^{\mathbb{G}}_{\Box}$ of $\mathbb{G}^{\mathsf{K}\tau}$ describes the process of creating a new world and settling the world by means of the available information. In the case of calculus $\mathbb{E}^{\mathsf{K}\tau}$, the same process is described by the three applications of modal rules. However, in order to see the correspondence clearly, the modal rules of $\mathbb{E}^{\mathsf{K}\tau}$ must be applied in an appropriate order. Therefore in the sequel we introduce into calculus $\mathbb{E}^{\mathsf{K}\tau}$ a derived rule which gathers a whole sequence of applications of the modal rules.

Another observation is that since indexed sequents carry more semantic information than "ordinary" sequents, we will need to "skip" part of this information—namely, we will skip these indexed formulas which are rewritten in Socratic transformations in order to retain semantic invertibility of the rules.

Another example presents proofs of sequent '$\vdash \Box(p \to p) \lor q$' in calculus $\mathbb{G}^{\mathsf{K}\tau}$ (the leftmost column), in Fitting's sequent calculus (the middle), and a proof of sequent '$\vdash (\Box(p \to p) \lor q)^1$' in calculus $\mathbb{E}^{\mathsf{K}\tau}$ (the right column). The example illustrates the need of weakening in calculus $\mathbb{G}^{\mathsf{K}\tau}$, where the rule is used to add information—encoded in formula 'q'—which is unnecessary for the derivation to become a proof. The remaining two proofs show how the same problem is undertaken in the other calculi.

Example 21.

calculus $\mathbb{G}^{\mathsf{K}\tau}$

$$\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\vdash p \to p}\,R^{\mathbb{G}}_{\to}}{\vdash \Box(p \to p)}\,R^{\mathbb{G}}_{\Box}}{\vdash \Box(p \to p), q}\,R^{\mathbb{G}}_{W}}{\vdash \Box(p \to p) \lor q}\,R^{\mathbb{G}}_{\lor}$$

Fitting's account

$$\dfrac{\dfrac{\dfrac{p \longrightarrow p}{\longrightarrow p \supset p}}{\longrightarrow \Box(p \supset p), q}\,\mathbf{R}_{\Box}}{\longrightarrow \Box(p \supset p) \lor q}$$

calculus $\mathbb{E}^{\mathsf{K}\tau}$

$$\begin{array}{ll} ?(\vdash (\Box(p \to p) \lor q)^1) & \\ \hline ?(\vdash (\Box(p \to p))^1, (q)^1) & R^{\mathbb{E}}_{\lor} \\ \hline ?(\vdash (p \to p)^{1,2}, (q)^1) & R^{\mathbb{E}}_{\Box} \\ \hline ?((p)^{1,2} \vdash (p)^{1,2}, (q)^1) & R^{\mathbb{E}}_{\to} \end{array}$$

□

In Fitting's account '$\longrightarrow$' is used for the sequent sign and '$\supset$' for implication. In the case of the application of $\mathbf{R}_{\Box}$, for $V = \{\Box(p \supset p), q\}$ we have $V\flat = \{p \supset p\}$, thus the application of $\mathbf{R}_{\Box}$ hides weakening.

It seems like weakening is *the* structural rule characterising modal logics. It is present in the implicit formulations of sequent systems for modal logics, even though sometimes it is hidden, like we have seen.

There are cut-free calculi for modal logics, but not weakening-free; reflexivity can be weakened to atoms (*i.e.* for example axioms only of the form $p_i \vdash p_i$), but weakening cannot. (See Hacking, 1994; Wansing, 2002.)

Analogously to the first-order case, we introduce a generalization of the modal rules. The rules presented in Table 4.9 are derivable rules in $\mathbb{E}^{\mathsf{K}\tau}$. The idea is that when a new world accessible from i is created (numeral j) all the properly modalized formulas that live in i are activated so they can send their suitable subformulas to settle the new world.

If an indexed formula of the form $(A)^{\sigma/i}$ occurs in a sequence S, then we will say that *formula A lives in i in S*.

Let S be a sequence of indexed formulas. We define the result of operation $\sharp^i_j$ performed on S which will be written as '$S\sharp^i_j$'. Let $F_1, \ldots, F_m$ be all formulas such that '$\Box F_k$' lives in i in S (for $k = 1, \ldots, m$; we take $m = 0$ if there are no such formulas). Then we perform the operation according to Algorithm 2.

Algorithm 2: operation $\sharp^i_j$

Data: finite sequence S of indexed formulas, and all formulas $F_1, \ldots, F_m$ such that '$\Box F_k$' lives in i in S **or** $m = 0$ if there are no such formulas

Result: finite sequence $S\sharp^i_j$ of indexed formulas

for $k = 1$ **to** m **do**
  $S \leftarrow S((\Box F_k)^{\sigma_k/i}/(\Box F_k)^{\sigma_k/i}, (F_k)^j)$
end
$S\sharp^i_j \leftarrow S$
return $S\sharp^i_j$

Similarly for the right side of the turnstile. Let T be a sequence of indexed formulas. We define the result of operation $\flat^i_j$ performed on T which will be written as '$T\flat^i_j$'. Let $F_1, \ldots, F_m$ be all formulas such that '$\Diamond F_k$' lives in i in T (for $k = 1, \ldots, m$; and we take $m = 0$ if there are no such formulas). Then we perform the operation according to Algorithm 3. (On the margin, the complexity of Algorithm 2 and Algorithm 3 depends linearly on the length, that is, the number of terms, of sequence S / sequence T, respectively.)

Rule $L^{\mathbb{E}}_{\Diamond\Box}$ states that if '$(\Diamond A)^{\sigma/i}$' occurs in S, that is, antecedent of a constituent of the question-premise, then the occurrence of '$(\Diamond A)^{\sigma/i}$'

Algorithm 3: operation $\flat^i_j$

Data: finite sequence T of indexed formulas, and all formulas $F_1, \ldots, F_m$ such that '$\Diamond F_k$' lives in i in T **or** $m = 0$ if there are no such formulas

Result: finite sequence $T\flat^i_j$ of indexed formulas

```
for k = 1 to m do
  | T ← T((◇F_k)^{σ_k/i} / (◇F_k)^{σ_k/i}, (F_k)^j)
end
T♭^i_j ← T
return T♭^i_j
```

is replaced with the occurrence of $A^{\sigma/i,j}$, where j is new, and *after that* operation $\sharp^i_j$ is performed on sequence $S(A^{\sigma/i,j})$ in the antecedent, and operation $\flat^i_j$ is performed on sequence T in the succedent. Analogously for rule $R^{\mathbb{E}}_{\Box\Diamond}$.

Table 4.9: General modal (erotetic) rules

$$\frac{?(\Phi\,;\ S((\Diamond A)^{\sigma/i})\ \vdash\ T\,;\ \Psi)}{?(\Phi\,;\ S(A^{\sigma/i,j})\sharp^i_j\ \vdash\ T\flat^i_j\,;\ \Psi)}\ L^{\mathbb{E}}_{\Diamond\Box}$$

where $j = max(\mathbf{I}_W\{S((\Diamond A)^{\sigma/i})\ \vdash\ T\}) + 1$

$$\frac{?(\Phi\,;\ S\ \vdash T((\Box A)^{\sigma/i})\,;\ \Psi)}{?(\Phi\,;\ S\sharp^i_j\ \vdash T(A^{\sigma/i,j})\flat^i_j\,;\ \Psi)}\ R^{\mathbb{E}}_{\Box\Diamond}$$

where $j = max(\mathbf{I}_W\{S\ \vdash T((\Box A)^{\sigma/i})\}) + 1$

By "$\mathbb{E}^{\mathsf{K}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$" we will refer to the erotetic calculus resulting by adding the above rules $L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}$ to $\mathbb{E}^{\mathsf{K}\tau}$. The fact that the rules are derivable in $\mathbb{E}^{\mathsf{K}\tau}$ can be proved as in the first-order case (see Lemma 4 in Chapter 2).

Corollary 22. *Rules $L^{\mathbb{E}}_{\Diamond\Box}$ and $R^{\mathbb{E}}_{\Box\Diamond}$ are derivable in $\mathbb{E}^{\mathsf{K}\tau}$.*

Proof. In order to simulate the application of $L^{\mathbb{E}}_{\Diamond\Box}$ to a question with respect to sequent $S \vdash T$, the application of rule $L^{\mathbb{E}}_{\Diamond}$ is followed by

m_1 applications of $L^{\mathbb{E}}_{\Box}$ for each k such that '$\Box F_k$' lives in i in S, $k = 1, \ldots, m_1$, and then also by m_2 applications of $R^{\mathbb{E}}_{\Diamond}$ for each k such that $\Diamond F_k$ lives in i in T, $k = 1, \ldots, m_2$.

Analogously for the other rule. $\Box$

If we now go back to Example 19, we can modify the proof by applying one of the general modal rules as follows (as before, 'A' stands for '$\Box(p \to q)$'):

Example 22.

$$\dfrac{\dfrac{\dfrac{\dfrac{?(\vdash (\Box(p \to q) \to (\Box p \to \Box q))^1)}{?((A)^1 \vdash (\Box p \to \Box q)^1)}\, R^{\mathbb{E}}_{\to}}{?((A)^1, (\Box p)^1 \vdash (\Box q)^1)}\, R^{\mathbb{E}}_{\to}}{?((A)^1, (p \to q)^2, (\Box p)^1, (p)^2 \vdash (q)^{1,2})}\, R^{\mathbb{E}}_{\Box\Diamond}}{?((A)^1, (\Box p)^1, (p)^2 \vdash (q)^{1,2}, (p)^2 \;;\; (A)^1, (q)^2, (\Box p)^1, (p)^2 \vdash (q)^{1,2})}\, R^{\mathbb{E}}_{\to}$$

$\Box$

We proceed to the description of the translation procedure.

Stage 1: regular Socratic transformation

An indexed formula occurring in sequent $S \vdash T$ will be called *atomic in* $S \vdash T$, if it is either an indexed propositional variable, an indexed formula of the form $(\Box F)^\sigma$ occurring in S, or an indexed formula of the form $(\Diamond F)^\sigma$ occurring in T. We will say that sequent $S \vdash T$ is *atomic* if every member of S and T is atomic in it. Moreover, we will use the notion of a basic sequent, as we did in Section 2.2.5; thus by a *basic sequent* we will mean a sequent satisfying condition (i) specified in Definition 38.

All Socratic transformations considered until the end of Section 4.3 will be constructed according to Algorithm 4. We will call them *regular Socratic transformations*. When a regular Socratic transformation is constructed, the rules are applied always with respect to the leftmost indexed formula of the appropriate form that occurs in a sequent. The non-branching rules for the classical connectives are prior to the branching rules (see Procedure CLASSICAL).

The algorithm uses three procedures defined below: MOVE-RIGHT, CLASSICAL and MODAL. The procedures are defined separately in order to make the algorithm easy to comprehend. The Procedure MOVE-RIGHT is called when the current sequent is a basic one but it is not the last constituent of the current question.

Algorithm 4: regular Socratic transformation via the rules of $\mathbb{E}^{\mathsf{K}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$

Data: question $?(\vdash (A)^1)$
Result: regular Socratic transformation of question $?(\vdash (A)^1)$

$n \leftarrow 1$; `/* n counts the number of constituents */`
; `/* of the last question */`
$i \leftarrow 1$; `/* i indicates the current constituent */`
$\mathbf{s} \leftarrow \langle ?(\vdash (A)^1) \rangle$; `/* s stores the Socratic */`
; `/* transformation */`
$\Phi \leftarrow \varnothing$; `/* the constituents */`
; `/* left to the current one */`
$\Psi \leftarrow \varnothing$; `/* the constituents */`
; `/* right to the current one */`
$\phi \leftarrow \vdash (A)^1$; `/* the current constituent */`
while *ϕ is not atomic* **do**
    CLASSICAL$(?(\Phi; \phi; \Psi), n, \mathbf{s}, Q, n, \mathbf{s})$
    if *ϕ is not a basic sequent* **then**
        MODAL$(?(\Phi; \phi; \Psi), \mathbf{s}, Q, \mathbf{s})$
    end
    while *ϕ is a basic sequent* **do**
        if $n = i$ **then**
            return s *is a Socratic proof of the sequent in* $\mathbb{E}^{\mathsf{K}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$.
        else
            MOVE-RIGHT$(\Phi, \Psi, \phi, i, \Phi, \Psi, \phi, i)$
        end
    end
end
return *The sequent is not provable in* $\mathbb{E}^{\mathsf{K}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$.

Procedure MOVE-RIGHT(left-context, right-context, sequent, numeral, Φ, Ψ, ϕ, i)

Data: left-context $=\Phi$, right-context $=\Psi$, sequent $=\phi$, numeral $=i$

$\Phi \leftarrow \Phi; \phi$

$i \leftarrow i+1$

$\phi \leftarrow i$-th constituent of Q

$\phi; \Xi \leftarrow \Psi$

$\Psi \leftarrow \Xi$

If we take $|A|$ to be the length of formula A understood as the number of occurrences of connectives in A, then we may set the length of indexed formulas, finite sequences of such formulas and sequents as follows:

$$|A^{\sigma}| := |A|$$

$$|A_1^{\sigma_1}, \ldots, A_n^{\sigma_n}| := \sum_{i=1}^{n} |A_i^{\sigma_i}|$$

$$|S \vdash T| := |S| + |T|$$

Using these measures one can say that the cost of executing procedure CLASSICAL on an active sequent ϕ is $\mathcal{O}(n \log n)$, where $n = |\phi|$, provided that the applications of rules are counted as basic operations.[6] Under the same assumptions the cost of calling procedure MODAL is linear with respect to n.

In Procedure CLASSICAL, the assignment in line 8 and in line 14 increases the value of m, that is, the number of indexed formulas in the current sequent. After going through Procedure CLASSICAL in Algorithm 4, the "current" sequent of the last question contains only indexed variables and/or indexed modal formulas. After that a check of the closing condition (basic sequent, line 9 of Algorithm 4) is performed. Procedure MODAL is called provided that the active sequent is not a basic one already. The check in line 9 of Algorithm 4 warrants that if the closing condition is met in a world i (that is, there is a formula and its negation with indices ending with numeral i), then this fact is

[6] This is a very rough analysis and it says nothing about complexity of the method with respect to the number of different variables used in the problem to be solved, that is, sequent. It also does not take into account the cost of searching for the active formula. The estimation $\mathcal{O}(n \log n)$ is derived under the assumption that all rules applied are branching, and each application divides "problem" to two equal "subproblems", that is the number of occurrences of connectives is divided by 2.

Procedure CLASSICAL(question,numeral,transformation,$Q, n, \mathbf{s}$)

```
Data: question = ?(Φ; A_1, ..., A_k ⊢ A_{k+1}, ..., A_m; Ψ), numeral,
      transformation
Q ←?(Φ; A_1, ..., A_k ⊢ A_{k+1}, ..., A_m; Ψ)
n ← numeral ;                    /* the number of constituents */
;                                  /* of the current question */
s ← transformation
j ← 1
while j ≤ m do
    if A_j is a formula-premise of a classical non-branching rule r
     then
        apply r
        A_1, ..., A_k ⊢ A_{k+1}, ..., A_m ← sequent-conclusion of r
        Q ← ?(Φ; A_1, ..., A_k ⊢ A_{k+1}, ..., A_m; Ψ)
        s ← s; Q
    else
        if A_j is a formula-premise of a classical branching rule r then
            apply r
            A_1, ..., A_k ⊢ A_{k+1}, ..., A_m ← the left sequent-conclusion
             of r
            ψ ← the right sequent-conclusion of r
            Ψ ← ψ; Ψ
            Q ← ?(Φ; A_1, ..., A_k ⊢ A_{k+1}, ..., A_m; Ψ)
            s ← s; Q
            n ← n + 1
        else
            j ← j + 1
        end
    end
end
```

recognised before another world can be created. In turn, if after going through Procedure CLASSICAL the current sequent is a basic one, and it is the last constituent, then $\mathbf{s}$ is a Socratic proof. If the sequent is a basic one, but not the last one, then we move to the next constituent to the right (line 16 of Algorithm 4). This time the check condition is in the while-loop, in order to find the right-most constituent of the last question which is not basic.

Observe that, in the case of Procedure MODAL, the general modal rule $\mathbf{r} \in \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$ is applied only once, the assignment in line no. 9 of the procedure ends the execution of the while-loop.

Procedure MODAL(question,transformation,Q, s)

Data: question $= ?(\Phi; A_1, \ldots, A_k \vdash A_{k+1}, \ldots, A_m; \Psi)$, transformation
$s \leftarrow$ transformation
$Q \leftarrow ?(\Phi; A_1, \ldots, A_k \vdash A_{k+1}, \ldots, A_m; \Psi)$
$j \leftarrow 1$
while $j \leq m$ **do**
    if A_j *is a formula-premise of* $\mathbf{r} \in \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$ **then**
        apply $\mathbf{r}$
        $Q \leftarrow$ question-conclusion
        $s \leftarrow s; Q$
        $j \leftarrow m + 1$; `/* ends the execution of the while-loop */`
    else
        $j \leftarrow j + 1$
    end
end

The notions of *Socratic transformation via the rules of calculus* $\mathbb{E}^{\mathsf{K}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$ and *Socratic proof in calculus* $\mathbb{E}^{\mathsf{K}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$ are understood as in the case of calculus $\mathbb{E}^{\mathsf{K}\tau}$. A Socratic proof of a sequent which is a regular Socratic transformation of the relevant question is also called *regular Socratic proof*. Finally, observe that since a regular Socratic transformation of a question is given by a deterministic algorithm, it is a unique object.

Stage 2: simplified Socratic trees

Regular Socratic transformations obtained at the previous stage may be further transformed into Socratic trees according to the pattern described previously in Section 4.1. As we have already mentioned, in Leszczyńska-Jasion et al., 2013 the transformation is described in the form of an algorithm, and we could estimate its complexity. For our present purpose it would be redundant, however. Instead of generating a Socratic transformation and then turning it into a tree, we will rather assume that the Socratic tree is constructed together with a regular Socratic transformation, by marking the sequents-conclusions. In fact, the algorithm of construction of regular Socratic transformations can be easily transformed into an algorithm of construction of the Socratic trees determined by the regular Socratic transformation.

Before we continue, let us note some important properties of a Socratic tree determined by a regular Socratic transformation of a question of the form $?(\vdash (A)^1)$. First of all, when the rules are applied according to Algorithm 4, the applications follow certain pattern:

1. Classical rules are applied with respect to formulas living in world 1 (as we start with $?(\vdash (A)^1)$) in the antecedent or the succedent of a sequent, until there are no classical connectives to be eliminated (see the construction of the while-loop in lines 5-24 of Procedure CLASSICAL). If a branching rule is applied, the right sequent-conclusion is left to be analysed later (lines 15-16 of Procedure CLASSICAL).

2. A general modal rule is applied once, if possible (line 10 of Algorithm 4 and line 9 of Procedure MODAL): world 2 accessible to 1 is created and settled.

3. Once again, classical connectives are eliminated in formulas living in world 2 (on the left or the right side of the turnstile), and only in 2, since those in 1 have been analysed already.

$\vdots$

n. A general modal rule is applied once: world j, accessible to i, is created and settled (if possible).

$n+1$. Classical connectives are eliminated in formulas living in world j (left or right of the turnstile), and only j, since the classical connectives of formulas living in the other worlds have been already eliminated.

$\vdots$

The above pattern warrants that the following corollary is true.

Corollary 23. *If* $\mathbf{s}$ *is a regular Socratic transformation and* ϕ *is a constituent-conclusion of a question of* $\mathbf{s}$*, then the principal numeral of* ϕ *is the largest one in this sequent. Moreover, if* ϕ *is a constituent-premise of a classical rule, then the numeral active in* ϕ *is the largest in* ϕ*.*

Let us emphasize that this follows from the fact that we do not analyse symmetric and/or Euclidean logics—when a new world is created, settled, and the “settlers” are analysed by the classical rules, we know that the world is settled once for good.

On the other hand, the applications of modal rules need not be "linear" with respect to the numerals, since there may be more than two modal formulas in a world, say i. It may happen that the leftmost such formula is analysed and it gives birth to a chain of worlds. But then the transformation does not end with a success, so we analyse another modal formula in i. In other words, when ψ results from ϕ by a modal rule, the active numeral in ϕ is not necessarily the largest one in the sequent, but the principal numeral of ψ is the largest one in ψ.

Before we continue, let us recall that together with the notion of a Socratic tree we have introduced the symbol '$R_{\mathbf{s}}$', which refers to the relation in the Socratic tree $\mathbf{Tr}(\mathbf{s})$ determined by $\mathbf{s}$. If we consider Socratic trees determined by regular Socratic transformations, then we may observe what follows:

Corollary 24. *Let* $\mathbf{Tr}(\mathbf{s})$ *stand for a Socratic tree determined by a regular Socratic transformation* $\mathbf{s}$ *of a questions of the form* $?(\vdash (A)^1)$. *If a node of* $\mathbf{Tr}(\mathbf{s})$ *labelled with* ψ *results in* $\mathbf{s}$ *by a classical rule, and its immediate* $R_{\mathbf{s}}$*-successor labelled with* ϕ *results in* $\mathbf{s}$ *by a classical rule, then the principal numeral in* ψ *and* ϕ *is the same numeral.*

Proof. As we have emphasized in the above pattern of the applications of rules in a regular Socratic transformation: when we start with a formula living in 1, we apply the rules for classical connectives to the consecutive sequents as long as they are applicable, and once we are finished with 1, and a new world is introduced, we never come back to 1. The situation is analogous when one of the general modal rules is applied, and a new world is created, settled, and the classical rules are applied with respect to the formulas living in the new world. □

In other words, in Socratic trees determined by regular Socratic transformations, if two consecutive (linked by $R_{\mathbf{s}}$) nodes result in $\mathbf{s}$ by classical rules, then they have the same principal numeral. By the same arguments also:

Corollary 25. *Let* $\mathbf{Tr}(\mathbf{s})$ *stand for a Socratic tree determined by a regular Socratic transformation* $\mathbf{s}$ *of a questions of the form* $?(\vdash (A)^1)$. *If a node of* $\mathbf{Tr}(\mathbf{s})$ *labelled with* ψ *results in* $\mathbf{s}$ *by a modal rule, where the principal numeral in* ψ *is* j*, and its immediate* $R_{\mathbf{s}}$*-successor labelled with* ϕ *results in* $\mathbf{s}$ *by a classical rule, then the principal numeral in* ϕ *is also* j*.*

As we shall see, however, the transition from regular Socratic transformations to Socratic trees is not enough for our present purpose, since

Socratic transformations may contain way too much information that must be now precisely cut off. Before we present the solution to this problem, let us first illustrate the point with examples.

If we compare the Socratic proof in Example 22 on page 137 with the target proof in calculus $\mathbb{G}^{\mathsf{K}\tau}$ (see Example 20 on page 133), then we can see that each of the "ordinary" sequents of language $\mathcal{M}^{imp}_{\vdash}$ represents the settlers of exactly one world, whereas the sequents of language $\mathcal{M}^{?}_{\vdash}$ keep all the information about the various worlds (there are two worlds in this example) together with the information concerning the relation between the worlds. Therefore the next step should result in cleaning away the redundant information. This will be done by a function τ which takes a sequence of indexed formulas and simplifies the sequence by leaving only these formulas that live in a specified world. Formally:

Definition 39. *Let $i \in \mathbb{N}$.*
$\tau(i, \varnothing) = \varnothing$

$$\tau(i, \langle A_1^{\sigma_1}, \ldots, A_k^{\sigma_k} \rangle) = \begin{cases} \langle A_1 \rangle \,' \, \tau(i, \langle A_2^{\sigma_2}, \ldots, A_k^{\sigma_k} \rangle) & \textit{if } \sigma_1 \textit{ ends with } i \\ \tau(i, \langle A_2^{\sigma_2}, \ldots, A_k^{\sigma_k} \rangle) & \textit{otherwise} \end{cases}$$

□

And we extend the definition of τ to act on sequents.

Definition 40. *Let $i \in \mathbb{N}$ and let $S \vdash T$ be a sequent of language $\mathcal{M}^{?}_{\vdash}$. Then:*

$$\tau(i, S \vdash T) = \tau(i, S) \vdash \tau(i, T)$$

□

Let us observe that the result $\tau(i, S \vdash T)$ is either a sequent of language $\mathcal{M}^{imp}_{\vdash}$, or—if there are no formulas living in i—the sequent with empty antecedent and empty succedent. The empty sequent is not a properly built expression of language $\mathcal{M}^{?}_{\vdash}$ (nor of language $\mathcal{M}^{imp}_{\vdash}$), but it will never occur in the derivations considered below. We need it only for the purpose of this definition.

We use function τ to modify sequents in Example 22. First, we generate the Socratic tree on the basis of this example; A stands for $\Box(p \to q)$:

$$
\begin{array}{c}
\vdash (\Box(p \to q) \to (\Box p \to \Box q))^1 \\
| \\
(A)^1 \vdash (\Box p \to \Box q)^1 \\
| \\
(A)^1, (\Box p)^1 \vdash (\Box q)^1 \\
| \\
(A)^1, (p \to q)^2, (\Box p)^1, (p)^2 \vdash (q)^{1,2} \\
\swarrow \qquad\qquad \searrow \\
(A)^1, (\Box p)^1, (p)^2 \vdash (q)^{1,2}, (p)^2 \qquad (A)^1, (q)^2, (\Box p)^1, (p)^2 \vdash (q)^{1,2}
\end{array}
$$

next, we treat the sequents with function τ, and turn the tree upside down (or rather turn it back into the right position):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\tau(2, (A)^1, (\Box p)^1, (p)^2 \vdash (q)^{1,2}, (p)^2) \quad \tau(2, (A)^1, (q)^2, (\Box p)^1, (p)^2 \vdash (q)^{1,2})}{\tau(2, (A)^1, (p \to q)^2, (\Box p)^1, (p)^2 \vdash (q)^{1,2})}}{\tau(1, (A)^1, (\Box p)^1 \vdash (\Box q)^1)}}{\tau(1, (A)^1 \vdash (\Box p \to \Box q)^1)}}{\tau(1, \vdash (\Box(p \to q) \to (\Box p \to \Box q))^1)}
$$

arriving at the tree below, which is almost the proof presented in Example 20, but without the initial applications of weakening to axioms:

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \vdash q, p \quad q, p \vdash q}{p \to q, p \vdash q}\; L^{\mathbb{G}}_{\to}}{\Box(p \to q), \Box p \vdash \Box q}\; R^{\mathbb{G}}_{\Box}}{\Box(p \to q) \vdash \Box p \to \Box q}\; R^{\mathbb{G}}_{\to}}{\vdash \Box(p \to q) \to (\Box p \to \Box q)}\; R^{\mathbb{G}}_{\to}
$$

As we can see, the change of numeral 1 to numeral 2 is driven by the application of the modal rule, and we can feel intuitively that this is the right step here. This observation regarding the choice of numeral—the first argument of τ—will be generalised and stated formally as the use of another function called $\tau_{\mathbf{s}}$, where $\mathbf{s}$ indicates the whole Socratic transformation (see Definition 41). Before we introduce the definition, let us recall that according to Corollary 21, if sequent ϕ is an immediate $R_{\mathbf{s}}$-successor of sequent ψ in a Socratic tree, then ψ is the active sequent of an erotetic rule and ϕ is its sequent-conclusion (possibly one of the two). Thus with regard to every sequent of a Socratic tree, except for the one in the root, we can say that the sequent is a sequent-conclusion of an erotetic rule, and there is always exactly one rule satisfying this claim.

The next definition introduces a *relabelling function*, since it acts on the sequents labelling the nodes of $\mathbf{Tr}(\mathbf{s})$ and assigns them sequents of language $\mathcal{M}_{\vdash}^{imp}$; that is to say, the function replaces one label of a node with another. The relabelling function works as follows: for a numeral i and a sequent ϕ (with indexed formulas), $\tau(i, \phi)$ is a sequent obtained from ϕ by (1) deleting each indexed formula whose index ends with a numeral other than i, and (2) deleting all the indices of the other formulas.

Definition 41. *Let* $\mathbf{s}$ *be a regular Socratic transformation of a question of the form '*$?(\vdash (A)^1)$*' and let* $\mathbf{Tr}(\mathbf{s})$ *be the Socratic tree determined by* $\mathbf{s}$*. Then* $\tau_{\mathbf{s}}$ *is a relabelling function defined recursively as follows:*

1. *If* ϕ *labels the root of* $\mathbf{Tr}(\mathbf{s})$*, then* $\tau_{\mathbf{s}}(\phi) = \tau(1, \phi)$*.*
2. *If* ϕ *labels a node,* ψ *labels the immediate* $R_{\mathbf{s}}$*-predecessor of the node,* ϕ *results from* ψ *in* $\mathbf{s}$ *by a classical rule, and* i *is the numeral such that* $\tau_{\mathbf{s}}(\psi) = \tau(i, \psi)$*, then* $\tau_{\mathbf{s}}(\phi) = \tau(i, \phi)$*.* ***(conservative extension)***
3. *If* ϕ *is a label of a node that results in* $\mathbf{s}$ *from its immediate* $R_{\mathbf{s}}$*-predecessor by a modal rule, then* $\tau_{\mathbf{s}}(\phi) = \tau(j, \phi)$*, where* j *is the principal numeral in* ϕ*.* ***(radical change)*** □

If $\mathbf{Tr}(\mathbf{s}) = \langle X_{\mathbf{s}}, R_{\mathbf{s}} \rangle$ is a Socratic tree, then the structure $\langle X_{\mathbf{s}}, R_{\mathbf{s}}, \tau_{\mathbf{s}} \rangle$ is a labelled tree. $\tau_{\mathbf{s}}$ is a two-argument function; if the second argument, ϕ, is given, then the first argument will be called $\tau_{\mathbf{s}}$*-numeral for* ϕ. The choice of $\tau_{\mathbf{s}}$-numerals by function $\tau_{\mathbf{s}}$ has the following nice property:

Corollary 26. *Let* $\mathbf{s}$ *be a regular Socratic transformation of a question of the form* $?(\vdash (A)^1)$ *and let* $\tau_{\mathbf{s}}$ *be defined as in Definition 41. Then:*

1. *if* ϕ *is a sequent-conclusion of a classical rule, and it is the only immediate* $R_{\mathbf{s}}$*-successor of* ψ*, then*

$$\frac{\tau_{\mathbf{s}}(\phi)}{\tau_{\mathbf{s}}(\psi)}$$

   is an instance of one of the linear classical rules of $\mathbb{G}^{\mathsf{K}\tau}$*.*

2. *if* ϕ_1 *is one of the two immediate* $R_{\mathbf{s}}$*-successors of* ψ*, and the other is* ϕ_2*, then at least one of the following schemes:*

$$\frac{\tau_{\mathbf{s}}(\phi_1) \quad \tau_{\mathbf{s}}(\phi_2)}{\tau_{\mathbf{s}}(\psi)} \qquad or \qquad \frac{\tau_{\mathbf{s}}(\phi_2) \quad \tau_{\mathbf{s}}(\phi_1)}{\tau_{\mathbf{s}}(\psi)}$$

is an instance of one of the branching classical rules of $\mathbb{G}^{\mathsf{K}\tau}$.

Proof. We consider only one rule as an example.

Suppose that sequent ψ occurs in the tree (more precisely, that there is $(m, k, \psi) \in X_{\mathbf{s}}$) and it has exactly two immediate $R_{\mathbf{s}}$-successors ϕ_1 and ϕ_2 (more precisely, (m_1, k_1, ϕ_1) and (m_2, k_2, ϕ_2) for some m_1, k_1, m_2, k_2). Since there are no modal branching rules in $\mathbb{E}^{\mathsf{K}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$, ψ is the active sequent of a classical rule $\mathbf{r}$ of $\mathbb{E}^{\mathsf{K}\tau}$ and ϕ_1, ϕ_2 are its sequents-conclusions. Suppose further that the rule is $\mathbf{r} = L^{\mathbb{E}}_{\vee}$. We have

$$\psi = S_1 \, ' \, (A \vee B)^{\sigma/i} \, ' \, S_2 \vdash T$$

and one of the two:

$$\phi_1 = S_1 \, ' \, (A)^{\sigma/i} \, ' \, S_2 \vdash T \qquad \phi_2 = S_1 \, ' \, (B)^{\sigma/i} \, ' \, S_2 \vdash T$$

or

$$\phi_2 = S_1 \, ' \, (A)^{\sigma/i} \, ' \, S_2 \vdash T \qquad \phi_1 = S_1 \, ' \, (B)^{\sigma/i} \, ' \, S_2 \vdash T$$

(displayed with the use of the concatenation sign). We assume the first possibility, the second one is analogous.

Obviously, the form of $\tau_{\mathbf{s}}(\psi)$, $\tau_{\mathbf{s}}(\phi_1)$ and $\tau_{\mathbf{s}}(\phi_2)$ depends on their $\tau_{\mathbf{s}}$-numerals. Definition 41 warrants that the $\tau_{\mathbf{s}}$-numeral for ϕ_1 and for ϕ_2 is identical to the $\tau_{\mathbf{s}}$-numeral for ψ (see ***conservative extension***), but we do not know the $\tau_{\mathbf{s}}$-numeral for ψ. In order to make the proof more transparent, let us make an assumption which we will justify below, namely that for ψ we have:

$$\tau_{\mathbf{s}}(\psi) = \tau(i, \psi) \tag{4.1}$$

that is, we assume that the $\tau_{\mathbf{s}}$-numeral for ψ is the same numeral i that is *the active numeral* of the application of $L^{\mathbb{E}}_{\vee}$. *If* this is the case, then

$$\tau_{\mathbf{s}}(\psi) = \tau(i, S_1) \, ' \, A \vee B \, ' \, \tau(i, S_2) \vdash \tau(i, T)$$

where $\tau(i, S_1)$, $\tau(i, S_2)$ and $\tau(i, T)$ are possibly empty sequences of formulas of $\mathcal{M}$. For simplicity, let $\tau(i, S_1) = X_1$, $\tau(i, S_2) = X_2$ and $\tau(i, T) = Y$. Then under the assumption (4.1) also:

$$\tau_{\mathbf{s}}(\phi_1) = X_1 \, ' \, A \, ' \, X_2 \vdash Y \qquad \tau_{\mathbf{s}}(\phi_2) = X_1 \, ' \, B \, ' \, X_2 \vdash Y$$

which was to be proved, since then:

$$\frac{\tau_{\mathbf{s}}(\phi_1) \quad \tau_{\mathbf{s}}(\phi_2)}{\tau_{\mathbf{s}}(\psi)}$$

is an instance of $L_{\vee}^{\mathbb{G}}$.

Now we go back to the justification of (4.1). We prove it by induction with respect to the structure of $\mathbf{Tr}(\mathbf{s})$ which is—let us recall—determined by a *regular* Socratic transformation. More specifically, the induction is with respect to the level of node ψ in $\mathbf{Tr}(\mathbf{s})$. (If necessary, see Appendix B for the notion of a level of a node in a tree.)

Base step: suppose that ψ is at level 0, then ψ is in the root and, by Definition 41, $\tau_{\mathbf{s}}(\psi) = \tau(1, \psi)$. On the other hand, 1 must be the active numeral in ψ, as there are no other numerals in ψ.

Inductive step: we assume that the induction hypothesis is satisfied for the nodes above ψ and that ϕ results from ψ by a classical rule (linear or branching). Then we need to consider the previous application of a rule, that is, the application of a rule $\mathbf{r}*$ that ψ resulted from. If the rule was classical, then by the inductive hypothesis and by the fact that $\mathbf{s}$ is regular, it follows that the two applications of classical rules are in the same active world (see Corollary 24). If $\mathbf{r}*$ was modal, then the $\tau_{\mathbf{s}}$-numeral for ψ is the new one in ψ, say j, and by Definition 41, $\tau_{\mathbf{s}}(\psi) = \tau(j, \psi)$. Then, by Corollary 25, the classical rule applied immediately after that must act upon a formula living in j.

The reasoning is the same for the other rules. □

The above Corollary states that in the case of the applications of the classical rules the choice of the numeral made by function $\tau_{\mathbf{s}}$ is the proper one, that is, the result of $\tau_{\mathbf{s}}$-translation of a sequent-premise and its sequent(s)-conclusion(s) is an instance of a rule of $\mathbb{G}^{\mathsf{K}\tau}$. Unfortunately, in the general case the choice of the numeral by the use of function $\tau_{\mathbf{s}}$ does not guarantee that the resulting tree is regulated by the rules of $\mathbb{G}^{\mathsf{K}\tau}$. The following example illustrates the difficulty.

Example 23. *The regular Socratic proof of sequent* $\vdash ((\Box p \vee \Box q) \vee \Diamond\neg q)^1$ *in calculus* $\mathbb{E}^{\mathsf{K}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$:

$$\cfrac{?(\vdash ((\Box p \vee \Box q) \vee \Diamond\neg q)^1)}{\cfrac{?(\vdash (\Box p \vee \Box q)^1, (\Diamond\neg q)^1)}{\underline{?(\vdash (\Box p)^1, (\Box q)^1, (\Diamond\neg q)^1)}}\; R^{\mathbb{E}}_{\vee}}\; R^{\mathbb{E}}_{\vee}$$

$$\vdots$$

a continuation of Example 23:

$$
\begin{array}{c}
\vdots \\ \hline
?(\vdash (p)^{1,2}, (\Box q)^{1}, (\Diamond\neg q)^{1}, (\neg q)^{2}) \\ \hline
?(\,(q)^{2} \vdash (p)^{1,2}, (\Box q)^{1}, (\Diamond\neg q)^{1}) \\ \hline
?(\,(q)^{2} \vdash (p)^{1,2}, (q)^{1,3}, (\Diamond\neg q)^{1}, (\neg q)^{3}) \\ \hline
?(\,(q)^{2}, (q)^{3} \vdash (p)^{1,2}, (q)^{1,3}, (\Diamond\neg q)^{1})
\end{array}
\begin{array}{l}
R^{\mathbb{E}}_{\Box\Diamond} \\
R^{\mathbb{R}}_{\neg} \\
R^{\mathbb{E}}_{\Box\Diamond} \\
R^{\mathbb{R}}_{\neg}
\end{array}
$$

□

Suppose that we turn the Socratic proof into a Socratic tree, turn it upside down and act on the sequents with function τ, changing the numeral (the first argument of the function) every time a modal rule is applied:

$$
\begin{array}{c}
\tau(3, (q)^{2}, (q)^{3} \vdash (p)^{1,2}, (q)^{1,3}, (\Diamond\neg q)^{1}) \\ \hline
\tau(3, (q)^{2} \vdash (p)^{1,2}, (q)^{1,3}, (\Diamond\neg q)^{1}, (\neg q)^{3}) \\ \hline
\tau(2, (q)^{2} \vdash (p)^{1,2}, (\Box q)^{1}, (\Diamond\neg q)^{1}) \\ \hline
\tau(2, \vdash (p)^{1,2}, (\Box q)^{1}, (\Diamond\neg q)^{1}, (\neg q)^{2}) \\ \hline
\tau(1, \vdash (\Box p)^{1}, (\Box q)^{1}, (\Diamond\neg q)^{1}) \\ \hline
\tau(1, \vdash (\Box p \vee \Box q)^{1}, (\Diamond\neg q)^{1}) \\ \hline
\tau(1, \vdash ((\Box p \vee \Box q) \vee \Diamond\neg q)^{1})
\end{array}
\qquad
\begin{array}{c}
q \vdash q \\ \hline
\vdash q, \neg q \\ \hline
q \vdash p \\ \hline
\vdash p, \neg q \\ \hline
\vdash \Box p, \Box q, \Diamond\neg q \\ \hline
\vdash \Box p \vee \Box q, \Diamond\neg q \\ \hline
\vdash (\Box p \vee \Box q) \vee \Diamond\neg q
\end{array}
$$

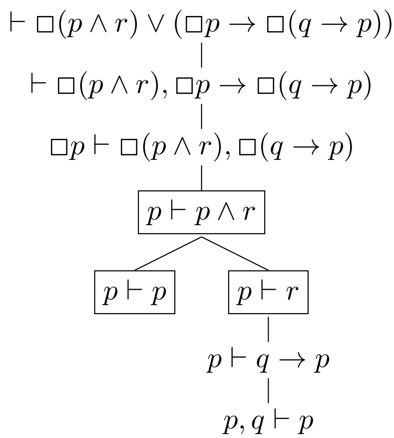
The resulting tree (to the right) is not a proof in $\mathbb{G}^{\mathsf{K}\tau}$. If the second sequent (from the top) was followed by the fifth one, we could just add an application of the proper rule of weakening and it would be fine, but the two sequents in the middle spoil this picture. We can see the reason of this failure. The choice of formula $\Box p$ to be acted upon (in the third question of Example 23) was wrong. The mistake is corrected in the Socratic transformation on the further stage of its construction; it is possible because all the information is still here, in every question of the transformation. But in the ordinary sequent calculus this mistake can be corrected only by means of backtracking, which leads to a new proof—one, in which the mistake is not committed. (Let us recall that ordinary, *i.e.* implicit, sequent calculi for modal logics are not confluent.)

The situation gets even more complicated if the wrong choice accidentally leads to a partial success, as in the example below.

Example 24. *Here is the regular Socratic proof of sequent* '$\vdash (\Box(p \wedge r) \vee (\Box p \to \Box(q \to p)))^{1}$' *in calculus* $\mathbb{E}^{\mathsf{K}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$, '$\chi$' *stands for the sequent* '$(\Box p)^{1}, (p)^{2} \vdash (p)^{1,2}, (\Box(q \to p))^{1}$'.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{?(\vdash (\Box(p \wedge r) \vee (\Box p \to \Box(q \to p)))^1)}{?(\vdash (\Box(p \wedge r))^1, (\Box p \to \Box(q \to p))^1)}\, R^{\mathbb{E}}_{\vee}}{?(\,(\Box p)^1 \vdash (\Box(p \wedge r))^1, (\Box(q \to p))^1)}\, R^{\mathbb{E}}_{\to}}{?(\,(\Box p)^1, (p)^2 \vdash (p \wedge r)^{1,2}, (\Box(q \to p))^1)}\, R^{\mathbb{E}}_{\Box\Diamond}}{?(\chi\ ;\ (\Box p)^1, (p)^2 \vdash (r)^{1,2}, (\Box(q \to p))^1)}\, R^{\mathbb{E}}_{\wedge}}{?(\chi\ ;\ (\Box p)^1, (p)^3, (p)^2 \vdash (r)^{1,2}, (q \to p)^{1,3})}\, R^{\mathbb{E}}_{\Box\Diamond}}{?(\chi\ ;\ (\Box p)^1, (p)^3, (p)^2, (q)^{1,3} \vdash (r)^{1,2}, (p)^{1,3})}\, R^{\mathbb{E}}_{\to}
$$

Below we present the Socratic tree determined by this Socratic proof treated with function τ.



Turning this tree into a proof in calculus $\mathbb{G}^{\mathsf{K}\tau}$ requires (apart from weakening in the leaf) cutting off the three boxed nodes: that is, the whole left branch and a part of the right branch. The challenge now is to settle precisely what must be cut off and how to do it. The idea is the following.

As long as we consider non-symmetric, non-Euclidean normal modal logics, if a formula is a tautology of the logic, then in the construction of the counter-model a chain (linear order) of worlds suffices to prove it: it is the chain leading from the world in which a given formula was assumed to be false to the world in which a contradiction appears. In other words, when constructing a potential counter-model for these logics the semantic information is always carried from a world to its R-successor, never to its R-predecessor. For this reason, linear samples of counter-models are sufficient to show that a formula has no counter-model. A sequence of numerals representing such a "linear sample" (a chain of worlds) in a Socratic transformation will be called a *world line*.[7]

More specifically, a *world line of a branch* of a Socratic tree is a sequence of numerals leading from 1 to a numeral i such that there is a formula that lives in i both in the antecedent and in the succedent of the sequent labelling the leaf of the branch. Let us observe that if a Socratic tree is determined by a regular Socratic transformation, then each branch of the tree has exactly one world line; it follows from the fact that a regular Socratic transformation ends as soon as a contradiction appears—compare our remarks on page 140 concerning the check of the closing condition. Formally,

Definition 42. *Let* $\mathbf{T}(\mathbf{s})$ *stand for a Socratic tree determined by a regular Socratic transformation* $\mathbf{s}$ *of a question of the form* $?(\vdash (A)^1)$*, such that* $\mathbf{s}$ *is a Socratic proof of* $\vdash (A)^1$*. Let* ϕ *be the label of the leaf of a branch* $\mathcal{B}$ *of* $\mathbf{T}(\mathbf{s})$*. The* world line of branch $\mathcal{B}$ *is a finite sequence* $\langle i_1, \ldots, i_n \rangle$ *of numerals such that:*

1. *the last numeral,* i_n*, is such that for some formula* A*,* $(A)^{\sigma_1/i_n}$ *occurs on the left side and* $(A)^{\sigma_2/i_n}$ *occurs on the right side of the turnstile,*

2. *for each* i_k*, where* $k > 1$*,* i_{k-1} *is the numeral that immediately precedes* i_k *in an index of some indexed formula in* ϕ*.* □

Let us note that due to the properties of regular Socratic transformations, we could have defined i_n in *1.* as the principal numeral in the leaf. Let us also note that the condition expressed in *2.* is unambiguous, as long as ϕ occurs in a Socratic transformation—in a Socratic transformation a numeral occurring in an index of a formula in ϕ cannot have two different immediate predecessors in some indices in ϕ; it follows form the fact that indices are extended only by *new* numerals.

Let us recall the following notion. If R is a relation defined on X and $Y \subseteq X$, then by $R\,|_Y$ we mean the restriction of R to Y (and similarly for functions).

---

[7] This way of thinking of countermodel constructions is present from the very beginning of the so-called Kripke semantics: "the semantical theory would lose no generality if only tree models were admitted" (Kripke, 1963). As Kripke points out, after Hintikka, in the cited article, there are formulas not valid in S4, that have no finite *tree* countermodels. But still, we consider provable sequents, containing provable formulas, and for provable formulas it holds true that a linear construction is sufficient.

We define the following operation of deleting a node from a tree.[8]

Definition 43. *If* $\langle X, R\rangle$ *is a tree and* $\mathbf{n} \in X$ *is a node of this tree, then* deleting $\mathbf{n}$ from $\langle X, R\rangle$, *symbolically* $\langle X, R\rangle \setminus \mathbf{n}$, *is an operation performed on the tree that results in the following structure:* $\langle X \setminus \{\mathbf{n}\}, R\,|_{X\setminus\{\mathbf{n}\}}\rangle$. *Thus*

$$\langle X, R\rangle \setminus \mathbf{n} \quad :=^{df} \quad \langle X \setminus \{\mathbf{n}\}, R\,|_{X\setminus\{\mathbf{n}\}}\rangle$$

□

It is easy to see that:

Corollary 27. *If* $\langle X, R\rangle$ *is a tree and* $\mathbf{n} \in X$ *is not a root of the tree, then the structure* $\langle X, R\rangle \setminus \mathbf{n}$ *is also a tree.*

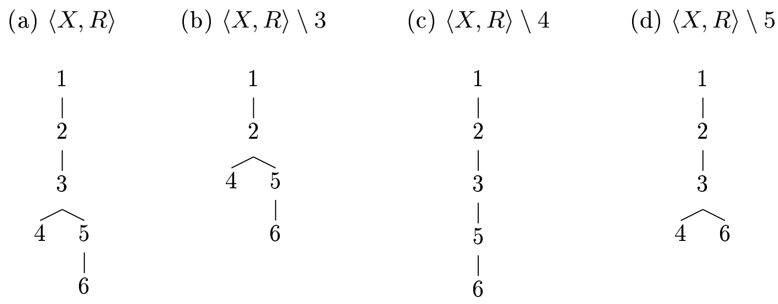
Proof. It is enough to observe that the operation of restriction does not spoil such properties as reflexivity, transitivity, antisymmetry and linearity. We consider only one case: that of transitivity. Thus let $\langle X, R\rangle$ be a tree and suppose that x, y, z are elements of $\langle X, R\rangle \setminus \mathbf{n}$ such that relation $R\,|_{X\setminus\{\mathbf{n}\}}$ holds between x and y, between y and z, but not between x and z. Then also xRy and yRz, and as R is transitive, xRz. If the pair $\langle x, z\rangle$ belongs to R but not to $R\,|_{X\setminus\{\mathbf{n}\}}$, then one of x, z must be the deleted node $\mathbf{n}$. However, if $x = \mathbf{n}$, then the pair $\langle x, y\rangle$ is not in $R\,|_{X\setminus\{\mathbf{n}\}}$, and if $z = \mathbf{n}$, then the pair $\langle y, z\rangle$ is not in $R\,|_{X\setminus\{\mathbf{n}\}}$. A contradiction.

If the deleted node is not the root, then the result must be a tree. Observe, however, that if the root of a tree has exactly one immediate successor, then the result of deleting the root is also a tree. Then the immediate successor becomes the root of the new tree. □

If $\mathbf{Tr} = \langle X, R, \eta\rangle$ is a labelled tree, then by $\mathbf{Tr} \setminus \mathbf{n}$ we mean the tree $\langle X, R\rangle \setminus \mathbf{n}$ together with the restricted labelling function $\eta\,|_{X\setminus\{\mathbf{n}\}}$.

Example 25. $\langle X, R\rangle$ *is such that* $X = \{1, 2, 3, 4, 5, 6\}$ *and* R *is the transitive closure of* $\{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 4\rangle, \langle 3, 5\rangle, \langle 5, 6\rangle\}$. *Below we present the tree (a), and the result of deleting nodes* 3–5 *from* $\langle X, R\rangle$ *((b)–(d)).*

[8] There are well-known definitions of the operation of deletion of a node from a tree which did not, however, suit our needs, therefore we have introduced our own definition. For example, the operation of deletion of a node from a binary search tree produces different result than our operation when a branching point is deleted. Compare Cormen et al., 2009, pp. 295-296.



□

Stage 3: cutting off the "dry branches"

The idea is the following. We take the rightmost branch of a Socratic tree and find its world line. Starting with the leaf, we cut off the nodes labelled with sequents whose principal numeral is from outside the world line. If a deleted node is a branching point, then the whole (left or right) branch must be cut off. After reaching the root, we go to the branch on the left. The idea is stated formally in the form of Procedure PRUNE. The Socratic tree introduced as data must be determined by a regular Socratic *proof*, otherwise the assignment in line 6 may fail, since an open branch has no world line.

When the procedure PRUNE is executed, it calls the recursive procedure REMOVE which deletes a whole subtree of the tree, if necessary. This happens when we arrive at a node whose principal numeral is not in the appropriate world line, thus the node should be removed, but the node has a sibling, which means that it has resulted from the previous node by an application of a branching rule. The branching point will be analysed and removed, if necessary, during another iteration of the while-loop, but the whole subtree starting with the sibling must be removed first, otherwise we will loose the information about its redundancy.

Now let us go back to Examples 23 and 24.

Example 26. *The result of acting with Procedure PRUNE on the Socratic tree determined by the regular Socratic proof of sequent* $\vdash ((\Box p \vee \Box q) \vee \Diamond\neg q)^1$ *(see Example 23).*

The tree has one branch and the world line of its leaf is $\langle 1, 3\rangle$, hence the procedure PRUNE removes the nodes with the principal numeral 2. Further, the function $\tau_{\mathbf{s}}$ acts on the nodes, and the final result is displayed in the left down corner.

Procedure PRUNE(Socratic tree,**s**)

Data: Socratic tree determined by a regular Socratic proof of a sequent of the form $\vdash (A)^1$

Result: Socratic tree **s** pruned

```
s ← Socratic tree
b* ← the rightmost branch of s
repeat
    b ← b*
    n ← the leaf of b
    wl ← the world line of b
    while n is not the root of s do
        if the principal numeral of n is in wl then
            n ← the immediate predecessor of n
        else
            if n has a sibling then
                k ← the sibling of n
                REMOVE(s,k,s)
                m ← the immediate predecessor of n
                s ← s \ n
                n ← m
            else
                m ← the immediate predecessor of n
                s ← s \ n
                n ← m
            end
        end
    end
    if b is not the leftmost branch of s then
        b* ← the branch which is next to the left of b
    end
until b is the leftmost branch of s;
```

Procedure REMOVE(tree,node,**s**)

```
Data: tree, node
Result: tree with the whole subtree starting with node removed
s ← tree
n ← node
if n is a leaf of s then
    s ← s \ n
    return s
else
    if n is a branching point then
        m1 ← the left immediate successor
        m2 ← the right immediate successor
        s ← s \ n
        REMOVE(s,m1,s)
        REMOVE(s,m2,s)
    else
        m ← the immediate successor
        s ← s \ n
        REMOVE(s,m,s)
    end
end
```

a continuation of Example 26:

Socratic tree $\mathbf{Tr}(\mathbf{s})$:

$$(q)^2, (q)^3 \vdash (p)^{1,2}, (q)^{1,3}, (\Diamond\neg q)^1$$
$$|$$
$$(q)^2 \vdash (p)^{1,2}, (q)^{1,3}, (\Diamond\neg q)^1, (\neg q)^3$$
$$|$$
$$(q)^2 \vdash (p)^{1,2}, (\Box q)^1, (\Diamond\neg q)^1$$
$$|$$
$$\vdash (p)^{1,2}, (\Box q)^1, (\Diamond\neg q)^1, (\neg q)^2$$
$$|$$
$$\vdash (\Box p)^1, (\Box q)^1, (\Diamond\neg q)^1$$
$$|$$
$$\vdash (\Box p \vee \Box q)^1, (\Diamond\neg q)^1$$
$$|$$
$$\vdash ((\Box p \vee \Box q) \vee \Diamond\neg q)^1$$

$$\Longrightarrow$$

$\mathbf{Tr}(\mathbf{s})$ pruned

$$(q)^2, (q)^3 \vdash (p)^{1,2}, (q)^{1,3}, (\Diamond\neg q)^1$$
$$|$$
$$(q)^2 \vdash (p)^{1,2}, (q)^{1,3}, (\Diamond\neg q)^1, (\neg q)^3$$
$$|$$
$$\vdash (\Box p)^1, (\Box q)^1, (\Diamond\neg q)^1$$
$$|$$
$$\vdash (\Box p \vee \Box q)^1, (\Diamond\neg q)^1$$
$$|$$
$$\vdash ((\Box p \vee \Box q) \vee \Diamond\neg q)^1$$

$$\Downarrow$$

a continuation of Example 26:

the final tree with the boxed node added		$\tau_{\mathbf{s}}$ acting on sequents:
$q \vdash q$	$\Longleftarrow$	$\tau(3, (q)^2, (q)^3 \vdash (p)^{1,2}, (q)^{1,3}, (\Diamond\neg q)^1)$
$\vdash q, \neg q$		$\tau(3, (q)^2 \vdash (p)^{1,2}, (q)^{1,3}, (\Diamond\neg q)^1, (\neg q)^3)$
$\boxed{\vdash \Box q, \Diamond\neg q}$		$\tau(1, \vdash (\Box p)^1, (\Box q)^1, (\Diamond\neg q)^1)$
$\vdash \Box p, \Box q, \Diamond\neg q$		$\tau(1, \vdash (\Box p \vee \Box q)^1, (\Diamond\neg q)^1)$
$\vdash \Box p \vee \Box q, \Diamond\neg q$		$\tau(1, \vdash ((\Box p \vee \Box q) \vee \Diamond\neg q)^1)$
$\vdash (\Box p \vee \Box q) \vee \Diamond\neg q$		

□

As we have seen in Example 24 above, even after pruning we may need to modify the leaves by applications of $L_W^{\mathbb{G}}$ and/or $R_W^{\mathbb{G}}$. From the above example we can see another case when an application of weakening may be needed—between the premise and the conclusion of the general modal rule. The second sequent (from the top) of the pruned tree is:

$$(q)^2 \vdash (p)^{1,2}, (q)^{1,3}, (\Diamond\neg q)^1, (\neg q)^3$$

its principal numeral is 3, and it is translated by function $\tau_{\mathbf{s}}$ into '$\vdash q, \neg q$'. The next node in the pruned Socratic tree is labelled with:

$$\vdash (\Box p)^1, (\Box q)^1, (\Diamond\neg q)^1$$

and its principal numeral is 1. The result after translation by $\tau_{\mathbf{s}}$ is '$\vdash \Box p, \Box q, \Diamond\neg q$'. Unfortunately, the following

$$\frac{\vdash q, \neg q}{\vdash \Box p, \Box q, \Diamond\neg q}$$

is not an instance of a rule of $\mathbb{G}^{\mathsf{K}\tau}$, but the conclusion can be obtained from the premise by an application of $R_\Box^{\mathbb{G}}$ *and* $R_W^{\mathbb{G}}$. The boxed sequent is the missing one to make the tree a proof in $\mathbb{G}^{\mathsf{K}\tau}$.

In order to add applications of weakening, every case of application of a modal rule needs to be analysed in the following way. Let ϕ stand for the sequent-conclusion and ψ for the sequent-premise of an application of erotetic rule $L_{\Diamond\Box}^{\mathbb{E}}$ or $R_{\Box\Diamond}^{\mathbb{E}}$. Suppose that $\tau_{\mathbf{s}}(\phi) = X' \vdash Y'$ and $\tau_{\mathbf{s}}(\psi) =$

$X \vdash Y$ are the respective nodes treated with function $\tau_{\mathbf{s}}$. Find the leftmost formula of the form '$\Diamond A$' in X such that A occurs in X'. If there is no such formula, then find the leftmost in Y formula of the form '$\Box A$' such that A is in Y'. Let $X^\star \vdash Y^\star$ stand for the result (conclusion) of application of, respectively, $L^{\mathbb{G}}_{\Diamond}$ or $R^{\mathbb{G}}_{\Box}$ with respect to A. If $X' \vdash Y'$ differs from $X^\star \vdash Y^\star$, then add a node labelled with $X^\star \vdash Y^\star$ between these labelled with $X \vdash Y$ and $X' \vdash Y'$. If both: X' differs from $X^\star$ and Y' differs from $Y^\star$, then add also a node labelled with $X' \vdash Y^\star$ between these labelled with $X^\star \vdash Y^\star$ and $X' \vdash Y'$. Then the result is:

$$\cfrac{\cfrac{\cfrac{X \vdash Y}{X^\star \vdash Y^\star}\ L^{\mathbb{G}}_{\Diamond}/R^{\mathbb{G}}_{\Box}}{X' \vdash Y^\star}\ L^{\mathbb{G}}_{W}}{X' \vdash Y'}\ R^{\mathbb{G}}_{W}$$

It will be convenient to shorten the above derivation into an application of a derived rule which will be called $L^{\mathbb{G}}_{\Diamond+W}$ or $R^{\mathbb{G}}_{\Box+W}$, depending on the initial use of $L^{\mathbb{G}}_{\Diamond}$ or $R^{\mathbb{G}}_{\Box}$. This operation reminds "hiding" weakening, but let us emphasize that we have introduced weakening as a primary rule, and $L^{\mathbb{G}}_{\Diamond+W}, R^{\mathbb{G}}_{\Box+W}$ are derived ones.

$$\frac{X_1 \ ' \ A \ ' \ X_2 \vdash Y}{X_1^* \ ' \ \Diamond A \ ' \ X_2^* \vdash Y^*}\ L^{\mathbb{G}}_{\Diamond+W} \qquad \frac{X \vdash Y_1 \ ' \ A \ ' \ Y_2}{X^* \vdash Y_1^* \ ' \ \Box A \ ' \ Y_2^*}\ R^{\mathbb{G}}_{\Box+W}$$

where 'X_1^*' ('X_2^*', 'X^*') is a supersequence, not necessarily proper, of '$\Box X_1$' ('$\Box X_2$', '$\Box X$', respectively), and 'Y^*' ('Y_1^*', 'Y_2^*') is a supersequence, not necessarily proper, of '$\Diamond Y$' ('$\Diamond Y_1$', '$\Diamond Y_2$', respectively).

We "weaken" the leaves in a similar manner. Let us recall that each leaf results from a basic sequent ϕ with indexed formulas, which is such that the $\tau_{\mathbf{s}}$-numeral of the sequent is the one in which the closing condition was met. Therefore there is a formula occurring both on the left and on the right side of the turnstile in $\tau_{\mathbf{s}}(\phi)$. If there is more than one formula on the right, then we add an application of $R^{\mathbb{G}}_{W}$, and if there is more than one formula on the left, then we add an application of $L^{\mathbb{G}}_{W}$. However, if we are not very fond of "clean" axioms, we may feel satisfied with sequents falling under the scheme of Ax_{+W}.

Example 27. *The Socratic tree determined by the regular Socratic proof of sequent* $\vdash (\Box(p \wedge r) \vee (\Box p \rightarrow \Box(q \rightarrow p)))^1$ *in calculus* $\mathbb{E}^{\mathsf{K}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$ *(see Example 24).*

$$\vdash (\Box(p \wedge r) \vee (\Box p \to \Box(q \to p)))^1$$
$$|$$
$$\vdash (\Box(p \wedge r))^1, (\Box p \to \Box(q \to p))^1$$
$$|$$
$$\vdash (\Box(p \wedge r))^1, (\Box p \to \Box(q \to p))^1$$
$$|$$
$$(\Box p)^1 \vdash (\Box(p \wedge r))^1, (\Box(q \to p))^1$$
$$|$$
$$(\Box p)^1, (p)^2 \vdash (p \wedge r)^{1,2}, (\Box(q \to p))^1$$

$$\begin{array}{cc} (\Box p)^1, (p)^2 \vdash (p)^{1,2}, (\Box(q \to p))^1 & (\Box p)^1, (p)^2 \vdash (r)^{1,2}, (\Box(q \to p))^1 \\ & | \\ & (\Box p)^1, (p)^3, (p)^2 \vdash (r)^{1,2}, (q \to p)^{1,3} \\ & | \\ & (\Box p)^1, (p)^3, (p)^2, (q)^{1,3} \vdash (r)^{1,2}, (p)^{1,3} \end{array}$$

The world line of the left branch is $\langle 1, 2\rangle$, and that of the right branch is $\langle 1, 3\rangle$. The analysis by procedure PRUNE starts with the right branch and removes the nodes with 2 as the principal numeral, leaving the following tree (to the left), transformed by function $\tau_{\mathbf{s}}$ (to the right), which further, supplemented with the applications of weakening, becomes a proof in $\mathbb{G}^{\mathsf{K}\tau}$ (below).

$$\begin{array}{ccc} (\Box p)^1, (p)^3, (p)^2, (q)^{1,3} \vdash (r)^{1,2}, (p)^{1,3} & \Longrightarrow & p, q \vdash p \\ | & & | \\ (\Box p)^1, (p)^3, (p)^2 \vdash (r)^{1,2}, (q \to p)^{1,3} & & p \vdash q \to p \\ | & & | \\ (\Box p)^1 \vdash (\Box(p \wedge r))^1, (\Box(q \to p))^1 & & \Box p \vdash \Box(p \wedge r), \Box(q \to p) \\ | & & | \\ \vdash (\Box(p \wedge r))^1, (\Box p \to \Box(q \to p))^1 & & \vdash \Box(p \wedge r), \Box p \to \Box(q \to p) \\ | & & | \\ \vdash (\Box(p \wedge r))^1, (\Box p \to \Box(q \to p))^1 & & \vdash \Box(p \wedge r), \Box p \to \Box(q \to p) \\ | & & | \\ \vdash (\Box(p \wedge r) \vee (\Box p \to \Box(q \to p)))^1 & & \vdash \Box(p \wedge r) \vee (\Box p \to \Box(q \to p)) \end{array}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{p \vdash p}{p, q \vdash p}\, L_W^{\mathbb{G}}}{p \vdash q \to p}\, R_{\to}^{\mathbb{G}}}{\Box p \vdash \Box(q \to p)}\, R_{\Box}^{\mathbb{G}}$$
$$\overline{}$$
$$\vdots$$

a continuation of Example 27:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Box p \vdash \Box(p \wedge r), \Box(q \to p)}\, R_W^{\mathbb{G}}}{\vdash \Box(p \wedge r), \Box p \to \Box(q \to p)}\, R_{\to}^{\mathbb{G}}}{\vdash \Box(p \wedge r) \vee (\Box p \to \Box(q \to p))}\, R_{\vee}^{\mathbb{G}}$$

□

Let us observe that the fact that the procedure starts with the right branch is inessential for the final result. If we started with the left branch, the world line would be $\langle 1, 2\rangle$, no nodes would be removed as the left branch has no sequents with principle numerals from outside the world line. After reaching the root, we would move to the right branch, change the world line to $\langle 1, 3\rangle$, and cut off the nodes with 2 as active, which would lead, however, to cutting off the left branch.

The last task in this section is to prove the following theorem:

Theorem 21. *Let* **s** *be a regular Socratic proof of sequent '*$\vdash (A)^1$*' in calculus* $\mathbb{E}^{\mathsf{K}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$*. If* $\langle X, R\rangle$ *is a tree obtained from* **s** *by the procedure PRUNE, then* $\langle X, R, \tau_{\mathbf{s}} \mid_X\rangle$ *is a proof of '*$\vdash A$*' in calculus* $\mathbb{G}^{\mathsf{K}\tau} \cup \{Ax_{+W},\ L^{\mathbb{G}}_{\Diamond+W},\ R^{\mathbb{G}}_{\Box+W}\}$*.*

First, we prove the following lemma from which it follows immediately that a tree $\langle X, R, \tau_s \mid_X\rangle$ obtained as above is regulated by the rules of $\mathbb{G}^{\mathsf{K}\tau} \cup \{L^{\mathbb{G}}_{\Diamond+W}, R^{\mathbb{G}}_{\Box+W}\}$. Then we go back to the above theorem.

Lemma 8. *Let* **s** *stand for a regular Socratic proof of sequent '*$\vdash (A)^1$*' in calculus* $\mathbb{E}^{\mathsf{K}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$*, and let* $\mathcal{T} = \langle X, R, \tau_{\mathbf{s}} \mid_X\rangle$ *be the tree obtained from* **Tr**(**s**) *by procedure PRUNE.*

1. *If sequent* ψ *labels a node of* $\mathcal{T}$ *that immediately precedes exactly one node, which is labelled by* ϕ*, and the two sequents have the same principal numeral, then the following:*

$$\frac{\tau_{\mathbf{s}}(\phi)}{\tau_{\mathbf{s}}(\psi)}$$

is an instance of a classical linear rule of $\mathbb{G}^{\mathsf{K}\tau}$*.*

2. *If sequent ψ labels a node of $\mathcal{T}$ that immediately precedes two nodes labelled by ϕ_1 and ϕ_2, then the three sequents have the same principal numeral, and the following:*

$$\frac{\tau_{\mathbf{s}}(\phi_1) \quad \tau_{\mathbf{s}}(\phi_2)}{\tau_{\mathbf{s}}(\psi)}$$

is an instance of a classical branching rule of $\mathbb{G}^{\mathsf{K}\tau}$.

3. *If sequent ψ labels a node of $\mathcal{T}$ that immediately precedes exactly one node, which is labelled by ϕ, and the two sequents have different principal numerals, then the following:*

$$\frac{\tau_{\mathbf{s}}(\phi)}{\tau_{\mathbf{s}}(\psi)}$$

is an instance of one of the rules: $L^{\mathbb{G}}_{\Diamond}$, $R^{\mathbb{G}}_{\Box}$, $L^{\mathbb{G}}_{\Diamond+W}$, $R^{\mathbb{G}}_{\Box+W}$.

Proof. The classical cases *1* and *2* seem to be a repetition of the content of Corollary 26. They *seem to be*, but now we deal with a tree after pruning by Procedure PRUNE, so we need to show that the pruning mechanism does not spoil the property expressed by Corollary 26.

Fortunately, it is easy to see that it does not. Let us recall Corollary 23; it states that the principal numeral in a sequent (which is a constituent-conclusion in a question of a regular Socratic transformation, and then becomes a label of a node of a Socratic tree determined by the transformation) is always the largest in the sequent. This entails that no nodes between these labelled by ψ and ϕ could have been deleted during pruning. The same pertains to case *2*.

The situation in *3* is different, since whole subtrees between ψ and ϕ could have been deleted. First, we argue that:

- **Proposition:** Sequent ϕ results from its predecessor in $\mathbf{Tr}(\mathbf{s})$ by one of the general modal rules.

As we shall see, this is the case exactly because $\mathbf{s}$ is regular, and because our tree is after pruning. Sequent ψ is a predecessor of sequent ϕ in $\mathbf{s}$. Suppose that ψ is an *immediate* predecessor of ϕ in $\mathbf{s}$, let i and j stand for the principal numerals of sequents ψ and ϕ, respectively. By assumption, $i \neq j$. If j was not present in ψ, then ϕ resulted from ψ by a modal rule, as required. Suppose that j was present in ψ. The principal numeral in ψ is i, not j, and $\mathbf{s}$ is regular. This yields (see the description on page 142) that the world j has been already set once for good before ψ was derived, therefore no classical rules are applicable with respect to formulas living in j. But then j cannot be principal in ϕ—a contradiction. Therefore if ψ is an immediate predecessor of ϕ in **s**, then ϕ results from ψ by a modal rule. Suppose that ψ is not *immediate* predecessor of ϕ in **s**, that is, some nodes between them have been deleted during pruning. Now consider the principal numeral of the immediate predecessor of ϕ in **s**. If it was j, as in ϕ, then the immediate predecessor of ϕ would not have been deleted during pruning. It is neither possible that j was present in the immediate predecessor and became active by an application of a classical rule—as in the case just considered above. This means that, again, ϕ results from its immediate predecessor by a modal rule, and j is new in the sequent. This ends the justification of **Proposition**.

Suppose that the modal rule is $L^{\mathbb{E}}_{\Diamond\Box}$, the case with $R^{\mathbb{E}}_{\Box\Diamond}$ is analogous. Now we will make an additional assumption to be proved later. Let i stand for the principal numeral of ψ.

- **Additional Assumption**: The formula-premise of the application of $L^{\mathbb{E}}_{\Diamond\Box}$ is a formula, $\Diamond A$, living in i, that is, the principal numeral of ψ.

What happens then is that the sequents are of the forms:

$$\psi = S((\Diamond A)^{\sigma/i}) \vdash T$$

$$\phi = U((A)^{\rho/i,j})\sharp^i_j \vdash V\flat^i_j$$

Let us establish what we know about the relations between the contexts on the right: T and $V\flat^i_j$. Recall that '$V\flat^i_j$' is V "enlarged" with "settlers" that come from i to j. Every indexed formula of the form '$(\Diamond B)^{\sigma^*/i}$' occurring in T (if there was any) is present also in $V\flat^i_j$, since these formulas are not eliminated in the course of a Socratic transformation. What is more, the converse is true: every indexed formula of this form that occurs in $V\flat^i_j$ is also in T. Once again, this follows from the fact that our tree has been pruned: the nodes between ψ and ϕ (if there were any) had their principal numerals outside the world line of the branch, whereas numerals i, j belong to the world line. Hence no rules applied between ψ and ϕ could result in introducing new formulas living in i. The same pertains to the context of formula '$(\Diamond A)^{\sigma/i}$' in '$S((\Diamond A)^{\sigma/i})$' and the context of '$(A)^{\rho/i,j}$' in '$U((A)^{\rho/i,j})\sharp^i_j$', that is, for every indexed formula of the form '$(\Box B)^{\sigma^*/i}$': the formula occurs in '$S((\Diamond A)^{\sigma/i})$' iff it occurs in

'$U((A)^{\rho/i,j})\sharp^i_j$'. On the other hand, the following is also true for the same reason: for every indexed formula of the form '$(\Box B)^{\sigma^*/i}$', the formula occurs in T (on the right side) iff it occurs in '$V\flat^i_j$'. And for the left side: for every indexed formula of the form '$(\Diamond B)^{\sigma^*/i}$' *except for* '$(\Diamond A)^{\sigma/i}$': the formula occurs in '$S((\Diamond A)^{\sigma/i})$' iff it occurs in '$U((A)^{\rho/i,j})\sharp^i_j$'. The formulas of the form '$\Box B$', that were present at the right side, together with the additional (that is, except for '$\Diamond A$') formulas of the form '$\Diamond B$', that were on the left side, constitute the "extra" context that must be added in calculus $\mathbb{G}^{\mathsf{K}\tau}$ by weakening.

Now we are almost done. Let us calculate $\tau_{\mathbf{s}}(\phi)$. The $\tau_{\mathbf{s}}$-numeral for ϕ is j, thus $\tau_{\mathbf{s}}(\phi)$ is a sequent of language $\mathcal{M}^{imp}_{\vdash}$ (no indices) with formulas that lived in j. Let us represent it as:

$$\tau_{\mathbf{s}}(\phi) \quad = \quad X_1\,'\,A\,'\,X_2 \vdash Y$$

where $X_1{}'A'X_2$ is the sequence of all formulas that have been introduced to world j by operation $\sharp^i_j$ (among which A occurs), and Y is the sequence of all formulas introduced to world j by operation $\flat^i_j$ (see Algorithms 2 and 3). On the other hand, $\tau_{\mathbf{s}}(\psi) = \tau(i,\psi)$, and the formulas were introduced to $\tau_{\mathbf{s}}(\phi)$ only by the elimination of suitable modal operators. If there were no additional indexed modal formulas of the form '$(\Box B)^{\sigma^*/i}$' in T nor '$(\Diamond B)^{\sigma^*/i}$' in '$S((\Diamond A)^{\sigma/i})$', then we would have that:

$$\tau_{\mathbf{s}}(\psi) \quad = \quad \Box X_1\,'\,\Diamond A\,'\,\Box X_2 \vdash \Diamond Y$$

and then $\tau_{\mathbf{s}}(\phi)/\tau_{\mathbf{s}}(\psi)$ would be an instance of $L^{\mathbb{G}}_{\Diamond}$. Otherwise, additional $\Diamond$-formula(s) occur on the left side of the turnstile and/or additional $\Box$-formula(s) occur on the right side of the turnstile. Then after the application of $L^{\mathbb{G}}_{\Diamond}$, another application of $L^{\mathbb{G}}_{W}$ and/or $R^{\mathbb{G}}_{W}$ produces sequent $\tau_{\mathbf{s}}(\psi)$, which was to be proved.

Now we go back to the additional assumption which assumes a nice coincidence consisting in the fact that the principal numeral i of sequent ψ must be at the same time the active numeral for sequent ϕ, even if ϕ results from some other sequent-premise, deleted in the course of pruning. The nice coincidence follows from the fact that the world line is a sequence of numerals that are linearly ordered by $<$. We justify this statement.

Let k stand for the numeral which was active in the application of $L^{\mathbb{E}}_{\Diamond\Box}$ that ϕ resulted from. Numeral k must be in the world line, for otherwise j would be outside the world line (see Definition 42). The sequents between ψ and ϕ had their principal numerals outside the world line, therefore k must be present in ψ. Suppose that $k < i$. Then the numerals in ϕ form a tree that has at least two leaves: j is one of the leaves, but i is another one. Since world lines are linearly ordered, the numerals cannot belong to the world line of the branch at the same time. Therefore $k \not< i$, which means that $k = i$, since By Corollary 23 numeral i is the largest in ψ. □

Now we may proceed to the proof of Theorem 21.

Proof of Theorem 21. Let $\mathbf{s}$ be a regular Socratic proof of sequent '$\vdash (A)^1$' in calculus $\mathbb{E}^{\mathsf{K}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$, and assume that $\langle X, R, \tau_{\mathbf{s}}|_X\rangle$ is a tree obtained from $\mathbf{s}$ by the procedure PRUNE, together with the labelling function $\tau_{\mathbf{s}}$ restricted to X. Then

- $\langle X, R, \tau_{\mathbf{s}}|_X\rangle$ is a finite tree labelled with sequents of language $\mathcal{M}^{imp}_{\vdash}$ (by the definition of $\tau_{\mathbf{s}}$),
- by Lemma 8, the tree is regulated by the rules of calculus $\mathbb{G}^{\mathsf{K}\tau} \cup \{L^{\mathbb{G}}_{\Diamond+W}, R^{\mathbb{G}}_{\Box+W}\}$,
- the root of the tree is labelled with sequent $\tau_{\mathbf{s}}(\vdash (A)^1) = \tau(1, \vdash (A)^1)$, which is '$\vdash A$',
- each leaf of $\mathbf{Tr}(\mathbf{s})$ is labelled by a basic sequent, and for each such label ϕ, its $\tau_{\mathbf{s}}$-numeral is the one in which there is a formula on the two sides of the turnstile, therefore $\tau_{\mathbf{s}}(\phi)$ is an axiom Ax or a general axiom Ax_{+W}.

All this entails that $\langle X, R, \tau_{\mathbf{s}}|_X\rangle$ is a proof of sequent $\vdash A$ in calculus $\mathbb{G}^{\mathsf{K}\tau} \cup \{L^{\mathbb{G}}_{\Diamond+W}, R^{\mathbb{G}}_{\Box+W}, Ax_{+W}\}$. □

4.4. Translations for non-symmetric, non-Euclidean extensions of K

The erotetic calculi for the extensions of K inherit the rules of $\mathbb{E}^{\mathsf{K}\tau}$ and possibly obtain some additional rules indicated below. The algorithm for regular Socratic transformations constructed via the rules of an erotetic calculus will be based on Algorithm 12, but the Procedures CLASSICAL and/or MODAL may be altered. Moreover, the definition of relabelling function $\tau_{\mathbf{s}}$ will be modified in some cases.

The Gentzen-style sequent calculi for the extensions of K are based on the calculi presented in Fitting, 1983. As before, they all have been obtained by dropping the unified notation. Interestingly, we have found the calculi presented here convergent with implicit Hintikka-style tableau systems presented in Goré, 1999; their convergence follows from the fact that the tableau systems examined by Rajeev Goré where also inspired by (Fitting, 1983) (and also by (Hintikka, 1955) and (Rautenberg, 1983), as the author admitted). Soundness and completeness of the sequent calculi presented here may be proved by using the techniques used in Goré, 1999, or by reference to arguments used in Fitting, 1983, and we will not examine these issues here.

Calculus $\mathbb{G}^{\mathsf{D}\tau}$ consists of the primary and the derived rules of $\mathbb{G}^{\mathsf{K}\tau}$ and the rule:

$$\frac{X \ \vdash \ Y}{\Box X \ \vdash \ \Diamond Y} \ R_D^{\mathbb{G}}$$

where, let us recall, $\Box X$ is a sequence of formulas such that $\Box A$ is i-th term of $\Box X$ iff A is i-th term of sequence X; in turn, $\Diamond A$ is i-th term of $\Diamond Y$ iff A is i-th term of Y. As previously, it is convenient to add a derived rule summarising the possible combination of applications of $R_D^{\mathbb{G}}$, $L_W^{\mathbb{G}}$ and $R_W^{\mathbb{G}}$. The rule is called '$R_{D+W}^{\mathbb{G}}$' and has the following scheme:

$$\frac{X \ \vdash \ Y}{X^* \ \vdash \ Y^*} \ R_{D+W}^{\mathbb{G}}$$

where X^* is a proper supersequence of $\Box X$ and Y^* is a proper supersequence of $\Diamond Y$.

Calculus $\mathbb{G}^{\mathsf{T}\tau}$ is the set of rules $\mathbb{G}^{\mathsf{K}\tau}$ (the primary and the derived ones) and the following:

$$\frac{X(A) \ \vdash \ Y}{X(\Box A) \ \vdash \ Y} \ L_{refl}^{\mathbb{G}} \qquad \frac{X \ \vdash \ Y(A)}{X \ \vdash \ Y(\Diamond A)} \ R_{refl}^{\mathbb{G}}$$

It is the only modal rule added to obtain an extension of K that does not need to be "spoiled" with weakening.

Calculus $\mathbb{G}^{\mathsf{K4}\tau}$ contains the primary and the derived rules of $\mathbb{G}^{\mathsf{K}\tau}$ except for the modal rules $L_{\Diamond}^{\mathbb{G}}$, $R_{\Box}^{\mathbb{G}}$ which must be replaced with $L_{trans}^{\mathbb{G}}$, $R_{trans}^{\mathbb{G}}$. The use of operations $\sharp$ and $\flat$ is inspired by Fitting's account but the operations' definitions are more complicated due to the fact that we deal with sequences, not sets (see Algorithms 5 and 6). Moreover, this account does not hide weakening as Fitting's account does: we still define the rule separately. The use of '$\Box X_i$' and '$\Diamond Y_i$' in the conclusions of rules $L^{\mathbb{G}}_{trans}$, $R^{\mathbb{G}}_{trans}$ indicates that all formulas in the conclusion are properly "modalized".

Algorithm 5: operation $\sharp$

Data: n-term sequence $\Box X = \langle \Box A_1, \ldots, \Box A_n \rangle$ of formulas of $\mathcal{M}$, $n \geq 0$

Result: $2n$-term sequence $(\Box X)\sharp$ of formulas of $\mathcal{M}$

$(\Box X)\sharp \leftarrow \varnothing$
for $k = 1$ **to** n **do**
  $(\Box X)\sharp \leftarrow (\Box X)\sharp \,'\, \langle \Box A_k, A_k \rangle$
end
return $(\Box X)\sharp$

Algorithm 6: operation $\flat$

Data: n-th term sequence $\Diamond Y = \langle \Diamond A_1, \ldots, \Diamond A_n \rangle$ of formulas of $\mathcal{M}$, $n \geq 0$

Result: $2n$ term sequence $(\Diamond Y)\flat$ of formulas of $\mathcal{M}$

$(\Diamond Y)\flat \leftarrow \varnothing$
for $k = 1$ **to** n **do**
  $(\Diamond Y)\flat \leftarrow (\Diamond Y)\flat \,'\, \langle \Diamond A_k, A_k \rangle$
end
return $(\Diamond Y)\flat$

$$\frac{(\Box X_1)\sharp \,'\, A \,'\, (\Box X_2)\sharp \ \vdash\ (\Diamond Y)\flat}{\Box X_1 \,'\, \Diamond A \,'\, \Box X_2 \ \vdash\ \Diamond Y} \ L^{\mathbb{G}}_{trans}$$

$$\frac{(\Box X)\sharp \ \vdash\ (\Diamond Y_1)\flat \,'\, A \,'\, (\Diamond Y_2)\flat}{\Box X \ \vdash\ \Diamond Y_1 \,'\, \Box A \,'\, \Diamond Y_2} \ R^{\mathbb{G}}_{trans}$$

Calculus $\mathbb{G}^{\mathsf{D4}\tau}$ contains the rules of $\mathbb{G}^{\mathsf{K}\tau}$ for the classical connectives and for weakening, rules $L^{\mathbb{G}}_{trans}$, $R^{\mathbb{G}}_{trans}$ (for transitivity) and rule $R^{\mathbb{G}}_{Dtrans}$ for seriality. We need a new one, since $R^{\mathbb{G}}_{D}$ does not account for the transition of information which is specific to the transitive logics:

$$\frac{(\Box X)\sharp \ \vdash\ (\Diamond Y)\flat}{\Box X \ \vdash\ \Diamond Y} \ R^{\mathbb{G}}_{Dtrans}$$

Finally, calculus $\mathbb{G}^{\mathsf{S4}\tau}$ consists of the rules of $\mathbb{G}^{\mathsf{K}\tau}$ for the classical connectives and for weakening, rules $L^{\mathbb{G}}_{trans}$, $R^{\mathbb{G}}_{trans}$ for transitivity, and $L^{\mathbb{G}}_{refl2}$ and $R^{\mathbb{G}}_{refl2}$, which express reflexivity in the presence of transitivity.

$$\frac{X(\Box A, A) \ \vdash \ Y}{X(\Box A) \ \vdash \ Y}\ L^{\mathbb{G}}_{refl2} \qquad\qquad \frac{X \ \vdash \ Y(\Diamond A, A)}{X \ \vdash \ Y(\Diamond A)}\ R^{\mathbb{G}}_{refl2}$$

In $L^{\mathbb{G}}_{refl2}$ two occurrences of formulas: '$\Box A, A$' in the premise are replaced with one occurrence: '$\Box A$' in the conclusion, similarly for $R^{\mathbb{G}}_{refl2}$.

All rules of our Gentzen-style calculi for modal logics are gathered in Table 4.10. Below we continue with the description of erotetic calculi modified for the purpose of this chapter. The table summarising the descriptions is quite large, so we have placed it in Appendix C.

Table 4.10: calculi $\mathbb{G}^{\mathsf{L}\tau}$

calculus	primary rules	derived rules
$\mathbb{G}^{\mathsf{K}\tau}$	$L^{\mathbb{G}}_{\neg}$, $R^{\mathbb{G}}_{\neg}$, $L^{\mathbb{G}}_{\wedge}$, $R^{\mathbb{G}}_{\vee}$, $R^{\mathbb{G}}_{\to}$, $R^{\mathbb{G}}_{\wedge}$, $L^{\mathbb{G}}_{\vee}$, $L^{\mathbb{G}}_{\to}$, $L^{\mathbb{G}}_{\Diamond}$, $R^{\mathbb{G}}_{\Box}$, $L^{\mathbb{G}}_{W}$, $R^{\mathbb{G}}_{W}$, Ax	Ax_{+W},$L^{\mathbb{G}}_{\Diamond+W}$, $R^{\mathbb{G}}_{\Box+W}$
$\mathbb{G}^{\mathsf{D}\tau}$	all above and $R^{\mathbb{G}}_{D}$	all above and rule $R^{\mathbb{G}}_{D+W}$
$\mathbb{G}^{\mathsf{T}\tau}$	all primary rules of $\mathbb{G}^{\mathsf{K}\tau}$ and $L^{\mathbb{G}}_{refl}$, $R^{\mathbb{G}}_{refl}$	as in $\mathbb{G}^{\mathsf{K}\tau}$
$\mathbb{G}^{\mathsf{K4}\tau}$	$L^{\mathbb{G}}_{\neg}$, $R^{\mathbb{G}}_{\neg}$, $L^{\mathbb{G}}_{\wedge}$, $R^{\mathbb{G}}_{\vee}$, $R^{\mathbb{G}}_{\to}$, $R^{\mathbb{G}}_{\wedge}$, $L^{\mathbb{G}}_{\vee}$, $L^{\mathbb{G}}_{\to}$, $L^{\mathbb{G}}_{trans}$, $R^{\mathbb{G}}_{trans}$, $L^{\mathbb{G}}_{W}$, $R^{\mathbb{G}}_{W}$, Ax	Ax_{+W}, $L^{\mathbb{G}}_{trans+W}$, $R^{\mathbb{G}}_{trans+W}$
$\mathbb{G}^{\mathsf{D4}\tau}$	$L^{\mathbb{G}}_{\neg}$, $R^{\mathbb{G}}_{\neg}$, $L^{\mathbb{G}}_{\wedge}$, $R^{\mathbb{G}}_{\vee}$, $R^{\mathbb{G}}_{\to}$, $R^{\mathbb{G}}_{\wedge}$, $L^{\mathbb{G}}_{\vee}$, $L^{\mathbb{G}}_{\to}$, $R^{\mathbb{G}}_{Dtrans}$, $L^{\mathbb{G}}_{trans}$, $R^{\mathbb{G}}_{trans}$, $L^{\mathbb{G}}_{W}$, $R^{\mathbb{G}}_{W}$, Ax	Ax_{+W}, $L^{\mathbb{G}}_{trans+W}$, $R^{\mathbb{G}}_{trans+W}$, $R^{\mathbb{G}}_{Dtrans+W}$
$\mathbb{G}^{\mathsf{S4}\tau}$	$L^{\mathbb{G}}_{\neg}$, $R^{\mathbb{G}}_{\neg}$, $L^{\mathbb{G}}_{\wedge}$, $R^{\mathbb{G}}_{\vee}$, $R^{\mathbb{G}}_{\to}$, $R^{\mathbb{G}}_{\wedge}$, $L^{\mathbb{G}}_{\vee}$, $L^{\mathbb{G}}_{\to}$, $L^{\mathbb{G}}_{refl2}$, $R^{\mathbb{G}}_{refl2}$, $L^{\mathbb{G}}_{trans}$, $R^{\mathbb{G}}_{trans}$, $L^{\mathbb{G}}_{W}$, $R^{\mathbb{G}}_{W}$, Ax	Ax_{+W}, $L^{\mathbb{G}}_{trans+W}$, $R^{\mathbb{G}}_{trans+W}$

Translation procedure for D

We start with logic D. In the original erotetic account (Leszczyńska, 2007), the serial logics are formalized by the addition of $R_{\pi D}$ rule. We first add analogues of it in the standard notation (rules $L^{\mathbb{E}}_{D}$, $R^{\mathbb{E}}_{D}$ below) and then derive a generalisation (as in the case of K).

$$\frac{?(\Phi\,;\ S((\Box A)^{\sigma/i}) \vdash T\,;\ \Psi)}{?(\Phi\,;\ S((\Box A)^{\sigma/i}, A^{i,j}) \vdash T\,;\ \Psi)}\ L^{\mathbb{E}}_{D}$$

$$\frac{?(\Phi\,;\ S \vdash T((\Diamond A)^{\sigma/i})\,;\ \Psi)}{?(\Phi\,;\ S \vdash T((\Diamond A)^{\sigma/i}, A^{i,j})\,;\ \Psi)}\ R^{\mathbb{E}}_{D}$$

Proviso: j must be new with respect to the sequent-premise. This is enough for the rule to be sound, but as usually we will put more restrictions on applicability of the rule. First, j is not arbitrary but must represent the smallest number not present in the sequent-premise. Second, the rule will be applied only when there is no other way to create the new world accessible from i.

Calculus $\mathbb{E}^{\mathsf{D}\tau}$ contains the rules of $\mathbb{E}^{\mathsf{K}\tau}$ together with $L^{\mathbb{E}}_{\Diamond\Box}$, $R^{\mathbb{E}}_{\Box\Diamond}$ and $L^{\mathbb{E}}_{D}$, $R^{\mathbb{E}}_{D}$. The generalisation of the erotetic rules for seriality is as simple as that (j is new with respect to the active sequent):

$$\frac{?(\Phi\,;\ S \vdash T\,;\ \Psi)}{?(\Phi\,;\ S\sharp D^{i}_{j} \vdash T\flat D^{i}_{j}\,;\ \Psi)}\ R^{\mathbb{E}}_{Dgen}$$

where $\sharp D^{i}_{j}$ is a slightly modified version of $\sharp^{i}_{j}$, and $\flat D^{i}_{j}$ is a slightly modified version of $\flat^{i}_{j}$ (see Algorithms 7 and 8). Numeral i is *active* in the sequent-conclusion of rule $R^{\mathbb{E}}_{Dgen}$.

Algorithm 7: operation $\sharp D^{i}_{j}$

Data: finite sequence S of indexed formulas, and all formulas $F_1, \ldots, F_m$ such that '$\Box F_k$' lives in i in S **or** $m = 0$ if there are no such formulas

Result: finite sequence $S\sharp D^{i}_{j}$ of indexed formulas

1 **for** $k = 1$ **to** m **do**

2 | $S \leftarrow S((\Box F_k)^{\sigma_k/i} / (\Box F_k)^{\sigma_k/i}, (F_k)^{i,j})$

3 **end**

4 $S\sharp D^{i}_{j} \leftarrow S$

5 **return** $S\sharp D^{i}_{j}$

The only difference between Algorithms 2 and 7 is that in the assignment in line 2 there is '$(F_k)^{j}$' in the former and '$(F_k)^{i,j}$' in the latter.

The point is to make sure that we have added the information that i sees j in the resulting sequent. The definition of $\flat^i_j$ is modified in the same way (compare Algorithms 3 and 8).

Algorithm 8: operation $\flat D^i_j$

Data: finite sequence T of indexed formulas, and all formulas $F_1, \ldots, F_m$ such that '$\Diamond F_k$' lives in i in T **or** $m = 0$ if there are no such formulas

Result: finite sequence $S\flat D^i_j$ of indexed formulas

for $k = 1$ **to** m **do**
  $T \leftarrow T((\Diamond F_k)^{\sigma_k/i}/(\Diamond F_k)^{\sigma_k/i}, (F_k)^{i,j})$
end
$T\flat D^i_j \leftarrow T$
return $T\flat D^i_j$

Let us also observe that there is no need to apply the rule if there is an indexed formula of the form '$(\Diamond A)^{\sigma/i}$' on the left, or of the form '$(\Box A)^{\sigma/i}$' on the right, side of the turnstile. And there is no sense to apply the rule if there is no formula of the form '$(\Box A)^{\sigma/i}$' on the left and no formula of the form '$(\Diamond A)^{\sigma/i}$' on the right. If in the sequent-premise of an application of $R^{\mathbb{E}}_{Dgen}$ there is at least one expression '$(\Box A)^{\sigma/i}$' on the left side or '$(\Diamond A)^{\sigma/i}$' on the right side, then we call this application of the rule *non-trivial*.

Corollary 28. *Every non-trivial application of $R^{\mathbb{E}}_{Dgen}$ is derivable in $\mathbb{E}^{\mathsf{D}\tau}$.*

Proof. As in the previous similar cases: we apply m_1 times rule $L^{\mathbb{E}}_D$ for every '$\Box F_k$', $1 \leq k \leq m_1$ such that '$\Box F_k$' lives in i in S, and then we apply rule $R^{\mathbb{E}}_D$ m_2 times for every '$\Diamond F_k$', $1 \leq k \leq m_2$ such that '$\Diamond F_k$' lives in i in T. □

Now we need to modify the procedure of applications of the modal rules, since we want the rule for seriality to be applied when the general modal rules are not applicable. If numeral i has no immediate successor in the indices of indexed formulas of a sequent ϕ, then we say that i is a *dead end in* ϕ. If an indexed formula of the form '$A^{\sigma/i}$' occurs in ϕ, then we also say that *A is in a dead end in ϕ*. See Procedure MODAL-D.

The next example illustrates the result of applying the tools defined for logic D.

Procedure MODAL-D(question,transformation,Q,$\mathbf{s}$)

Data: question $= ?(\Phi; A_1, \ldots, A_k \vdash A_{k+1}, \ldots, A_m; \Psi)$, transformation

$\mathbf{s} \leftarrow$ transformation
$Q \leftarrow ?(\Phi; A_1, \ldots, A_k \vdash A_{k+1}, \ldots, A_m; \Psi)$
$j \leftarrow 1$
$i \leftarrow 0$; `/* used to decide whether to use seriality or not */`
while $j \leq m$ **do**
  if *A_j is a formula-premise of* $\mathbf{r} \in \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$ **then**
    apply $\mathbf{r}$
    $Q \leftarrow$ question-conclusion
    $\mathbf{s} \leftarrow \mathbf{s}; Q$
    $i \leftarrow 1$
    $j \leftarrow m + 1$; `/* ends the execution of the while-loop */`
  else
    $j \leftarrow j + 1$
  end
end
if $i = 0$ **then**
  if *there is a formula of the form '$\Box F$' in a dead end on the left side, or a formula of the form '$\Diamond F$' in a dead end on the right side* **then**
    apply $R^{\mathbb{E}}_{Dgen}$ with the numeral representing the dead end as active
    $Q \leftarrow$ question-conclusion
    $\mathbf{s} \leftarrow \mathbf{s}; Q$
  end
end

Example 28. *A regular Socratic proof of sequent '$\vdash (\Box p \to \Diamond p)^1$' in calculus $\mathbb{E}^{\mathsf{D}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}, R^{\mathbb{E}}_{Dgen}\}$ translated into a proof of sequent '$\vdash \Box p \to \Diamond p$' in calculus $\mathbb{G}^{\mathsf{D}\tau} \cup \{Ax_{+W}, L^{\mathbb{G}}_{\Diamond+W}, R^{\mathbb{G}}_{\Box+W}, R^{\mathbb{G}}_{D+W}\}$. (In fact, in the case of this example the translation goes from $\mathbb{E}^{\mathsf{D}\tau} \cup \{R^{\mathbb{E}}_{Dgen}\}$ to $\mathbb{G}^{\mathsf{D}\tau}$, but, in general, all the derived rules may be needed—see Theorem 22.) The Socratic tree has one branch with world line $\langle 1, 2\rangle$.*

$$\cfrac{\cfrac{?(\vdash (\Box p \to \Diamond p)^1)}{?((\Box p)^1 \vdash (\Diamond p)^1)}\, R^{\mathbb{E}}_{\to}}{?((\Box p)^1, p^{1,2} \vdash (\Diamond p)^1, p^{1,2})}\, R^{\mathbb{E}}_{Dgen}$$

$$\Downarrow$$

a continuation of Example 28:

$$\dfrac{\dfrac{\tau(2,(\Box p)^1,p^{1,2}\vdash(\Diamond p)^1,p^{1,2})}{\tau(1,(\Box p)^1\vdash(\Diamond p)^1)}}{\tau(1,\vdash(\Box p\rightarrow\Diamond p)^1)} \qquad \Rightarrow \qquad \dfrac{\dfrac{p\vdash p}{\Box p\vdash\Diamond p}\,R^{\mathbb{G}}_{D}}{\vdash\Box p\rightarrow\Diamond p}\,R^{\mathbb{G}}_{\rightarrow}$$

□

The following theorem is true:

Theorem 22. *Let* **s** *be a regular Socratic proof of sequent '$\vdash (A)^1$' in calculus $\mathbb{E}^{\mathsf{D}\tau}\cup\{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}, R^{\mathbb{E}}_{Dgen}\}$. If $\langle X,R\rangle$ is a tree obtained from* **s** *by the procedure PRUNE, then $\langle X,R,\tau_{\mathbf{s}}\mid_X\rangle$ is a proof of '$\vdash A$' in calculus $\mathbb{G}^{\mathsf{D}\tau}\cup\{Ax_{+W}, L^{\mathbb{G}}_{\Diamond+W}, R^{\mathbb{G}}_{\Box+W}, R^{\mathbb{G}}_{D+W}\}$.*

For this purpose we need to analyse the content of Lemma 8 and adjust it to the present case. It is easy to see that nothing changes in the analysis of applications of the classical rules, nor in the justification of **Additional Assumption**. What is specific to D is the change in the situation described in clause *3* of Lemma 8:

Lemma 9. *Let* **s** *stand for a regular Socratic proof of sequent '$\vdash (A)^1$' in calculus $\mathbb{E}^{\mathsf{D}\tau}\cup\{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}, R^{\mathbb{E}}_{Dgen}\}$, and let $\mathcal{T}=\langle X,R,\tau_{\mathbf{s}}\mid_X\rangle$ be the tree obtained from* **Tr(s)** *by procedure PRUNE.*

1. and 2. as in Lemma 8.

3. If sequent ψ labels a node of $\mathcal{T}$ that immediately precedes exactly one node, which is labelled by ϕ, and the two sequents have different principal numerals, then the following:

$$\frac{\tau_{\mathbf{s}}(\phi)}{\tau_{\mathbf{s}}(\psi)}$$

is an instance of one of the rules: $L^{\mathbb{G}}_{\Diamond}$, $R^{\mathbb{G}}_{\Box}$, $L^{\mathbb{G}}_{\Diamond+W}$, $R^{\mathbb{G}}_{\Box+W}$, $R^{\mathbb{G}}_{D}$ or $R^{\mathbb{G}}_{D+W}$.

Proof. We reason as in the proof of Lemma 8: the fact that **s** is regular and that $\mathcal{T}$ has been pruned yields that ϕ results in **Tr(s)** from its predecessor by one of the modal rules: $L^{\mathbb{E}}_{\Diamond\Box}$, $R^{\mathbb{E}}_{\Box\Diamond}$, $R^{\mathbb{E}}_{Dgen}$. If it is $L^{\mathbb{E}}_{\Diamond\Box}$ or $R^{\mathbb{E}}_{\Box\Diamond}$, then we show that $\tau_{\mathbf{s}}(\phi)/\tau_{\mathbf{s}}(\psi)$ is an instance of one of $L^{\mathbb{G}}_{\Diamond}$, $R^{\mathbb{G}}_{\Box}$, $L^{\mathbb{G}}_{\Diamond+W}$, $R^{\mathbb{G}}_{\Box+W}$. If it is $R^{\mathbb{E}}_{Dgen}$, then in an analogous way we show that $\tau_{\mathbf{s}}(\phi)/\tau_{\mathbf{s}}(\psi)$ is an instance of $R^{\mathbb{G}}_{D}$ or $R^{\mathbb{G}}_{D+W}$. □

Translation procedure for T

As we shall see, reflexive logics may require labelling some formulas with additional information; this will happen during the generation of a regular Socratic transformation. The additional information will be placed in the bottom index of formulas. We assume that when the construction of a regular Socratic transformation starts, there are no labels in the bottom indices. If this is added to a formula A, then we say that this occurrence of formula in a Socratic transformation *has a bottom label*. Then the occurrence takes the following form: 'A_{refl}'. However, the bottom labels are not parts of the deductive machinery, that is, they are not defined in the rules.

Calculus $\mathbb{E}^{\mathsf{T}\tau}$ contains the rules of $\mathbb{E}^{\mathsf{K}\tau}$ and the two rules displayed below, which capture reflexivity. The derived rules of $\mathbb{E}^{\mathsf{K}\tau}$: $L^{\mathbb{E}}_{\Diamond\Box}$ and $R^{\mathbb{E}}_{\Box\Diamond}$, are also derived rules of $\mathbb{E}^{\mathsf{T}\tau}$.

$$\frac{?(\Phi\ ;\ S((\Box A)^{\sigma/i})\ \vdash\ T\ ,\ \Psi)}{?(\Phi\ ;\ S((\Box A)^{\sigma/i}, A^{i})\ \vdash\ T\ ;\ \Psi)}\ L^{\mathbb{E}}_{refl}$$

$$\frac{?(\Phi\ ;\ S\ \vdash\ T((\Diamond A)^{\sigma/i})\ ;\ \Psi)}{?(\Phi\ ;\ S\ \vdash\ T((\Diamond A)^{\sigma/i}, A^{i})\ ;\ \Psi)}\ R^{\mathbb{E}}_{refl}$$

In the original account (Leszczyńska, 2007) there was one scheme to capture the rules corresponding to $L^{\mathbb{E}}_{\Box}$, $R^{\mathbb{E}}_{\Diamond}$ and to the above two, with an appropriate provisos of its applicability. But in the present context it is advisable to set apart the cases of $i \neq j$ and $i = j$.

When generating regular Socratic transformations via the rules of $\mathbb{E}^{\mathsf{T}\tau}$ Procedure CLASSICAL is replaced with SETTLE-T. The latter includes the applications of $L^{\mathbb{E}}_{refl}$ and $R^{\mathbb{E}}_{refl}$.

Let us emphasize again that the information concerning the bottom index is not defined in the rule, but it is used in the algorithm applying the rules. The respective assignment is present in line 23 of Procedure SETTLE-T. The bottom labels play two roles. They prevent redundant repetitions of applications of the reflexivity-rules $L^{\mathbb{E}}_{refl}$, $R^{\mathbb{E}}_{refl}$, but they are also used on the level of "decoding" indexed transformations into purely syntactic ones. The bottom label is a hint that the sequents after translation must fit the rules $L^{\mathbb{G}}_{refl}$, $R^{\mathbb{G}}_{refl}$ of $\mathbb{G}^{\mathsf{T}\tau}$.

Function τ will be replaced with τ_{T}. Formulas of the form '$\Box F$', even if in the appropriate world, will be neglected by the function if they have the bottom label (see Definition 44).

Procedure SETTLE-T(question,numeral,transformation,$Q, n, \mathbf{s}$)

Data: question $= ?(\Phi; A_1, \ldots, A_k \vdash A_{k+1}, \ldots, A_m; \Psi)$, numeral, transformation

$Q \leftarrow ?(\Phi; A_1, \ldots, A_k \vdash A_{k+1}, \ldots, A_m; \Psi)$
$n \leftarrow$ numeral ; `/* counts constituents */`
; `/* of the current question */`
$\mathbf{s} \leftarrow$ transformation
$j \leftarrow 1$
while $j \leq m$ **do**
  if *A_j is a formula-premise of a classical non-branching rule* $\mathbf{r}$ **then**
    apply $\mathbf{r}$
    $A_1, \ldots, A_k \vdash A_{k+1}, \ldots, A_m \leftarrow$ sequent-conclusion of $\mathbf{r}$
    $Q \leftarrow ?(\Phi; A_1, \ldots, A_k \vdash A_{k+1}, \ldots, A_m; \Psi)$
    $\mathbf{s} \leftarrow \mathbf{s}; Q$
  else
    if *A_j is a formula-premise of a classical branching rule* $\mathbf{r}$ **then**
      apply $\mathbf{r}$
      $A_1, \ldots, A_k \vdash A_{k+1}, \ldots, A_m \leftarrow$ the left sequent-conclusion of $\mathbf{r}$
      $\psi \leftarrow$ the right sequent-conclusion of $\mathbf{r}$
      $\Psi \leftarrow \psi; \Psi$
      $Q \leftarrow ?(\Phi; A_1, \ldots, A_k \vdash A_{k+1}, \ldots, A_m; \Psi)$
      $\mathbf{s} \leftarrow \mathbf{s}; Q$
      $n \leftarrow n + 1$
    else
      if *A_j is of the form $\circledast F$ ($\circledast = \Box$ in the antecedent, $\circledast = \Diamond$ in the succedent), and it has no bottom label* **then**
        apply rule $\mathbf{r} \in \{L^{\mathbb{E}}_{refl}, R^{\mathbb{E}}_{refl}\}$
        $\circledast F \leftarrow (\circledast F)_{refl}$; `/* bottom label added */`
        $A_1, \ldots, A_k \vdash A_{k+1}, \ldots, A_m \leftarrow$ sequent-conclusion of $\mathbf{r}$
        $Q \leftarrow ?(\Phi; A_1, \ldots, A_k \vdash A_{k+1}, \ldots, A_m; \Psi)$
        $\mathbf{s} \leftarrow \mathbf{s}; Q$
        $j \leftarrow j + 1$
      else
        $j \leftarrow j + 1$
      end
    end
  end
end

Definition 44. *Let* $i \in \mathbb{N}$.

$\tau_{\mathsf{T}}(i, \varnothing) = \varnothing$

$$\tau_{\mathsf{T}}(i, \langle A_1^{\sigma_1}, \ldots, A_k^{\sigma_k} \rangle) = \begin{cases} \tau_{\mathsf{T}}(i, \langle A_2^{\sigma_2}, \ldots, A_k^{\sigma_k} \rangle) & \textit{if } \sigma_1 \textit{ does not end with } i, \textit{ or } A_1^{\sigma_1} \textit{ has a bottom label} \\ \langle A_1 \rangle' \tau_{\mathsf{T}}(i, \langle A_2^{\sigma_2}, \ldots, A_k^{\sigma_k} \rangle) & \textit{otherwise} \end{cases}$$

□

Similarly, definition 41 must be altered—we will use function $\tau_{\mathsf{T}\mathbf{s}}$ instead of $\tau_{\mathbf{s}}$.

Definition 45. *Let* **s** *be a regular Socratic transformation of a question of the form '*$?(\vdash (A)^1)$*' via the rules of* $\mathbb{E}^{\mathsf{T}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$, *and let* $\mathbf{Tr}(\mathbf{s})$ *be the Socratic tree determined by* **s**. *Then* $\tau_{\mathsf{T}\mathbf{s}}$ *is a relabelling function defined recursively as follows:*

1. *If* ϕ *labels the root of* $\mathbf{Tr}(\mathbf{s})$, *then* $\tau_{\mathsf{T}\mathbf{s}}(\phi) = \tau_{\mathsf{T}}(1, \phi)$.

2. *If* ϕ *labels a node,* ψ *labels the immediate* $R_{\mathbf{s}}$*-predecessor of the node,* ϕ *results from* ψ *in* **s** *by a classical rule, and* i *is the numeral such that* $\tau_{\mathsf{T}\mathbf{s}}(\psi) = \tau_{\mathsf{T}}(i, \psi)$, *then* $\tau_{\mathsf{T}\mathbf{s}}(\phi) = \tau_{\mathsf{T}}(i, \phi)$. ***(conservative extension)***

3. *If* ϕ *labels a node,* ψ *labels the immediate* $R_{\mathbf{s}}$*-predecessor of the node,* ϕ *results from* ψ *in* **s** *by* $L^{\mathbb{E}}_{refl}$ *or* $R^{\mathbb{E}}_{refl}$, *and* i *is the numeral such that* $\tau_{\mathsf{T}\mathbf{s}}(\psi) = \tau_{\mathsf{T}}(i, \psi)$, *then* $\tau_{\mathsf{T}\mathbf{s}}(\phi) = \tau_{\mathsf{T}}(i, \phi)$. ***(conservative extension reflectively continued)***

4. *If* ϕ *labels a node and* ϕ *results in* **s** *from its immediate* $R_{\mathbf{s}}$*-predecessor by a modal rule other than* $L^{\mathbb{E}}_{refl}$, $R^{\mathbb{E}}_{refl}$, *then* $\tau_{\mathsf{T}\mathbf{s}}(\phi) = \tau_{\mathsf{T}}(j, \phi)$, *where* j *is the principal numeral in* ϕ. ***(radical change)***

□

As a matter of fact, clause 3, **(conservative extension reflectively continued)** is a special case of 4, **(radical change)** (believe me, it happens!), as the principal numeral in the case of application of $L^{\mathbb{E}}_{refl}$, $R^{\mathbb{E}}_{refl}$ is i.

The following example shows how the regular Socratic proof of sequent '$\vdash (\Box p \to p)^1$' (the specific axiom of T) in $\mathbb{E}^{\mathsf{T}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$ is translated into a proof of '$\vdash \Box p \to p$' in $\mathbb{G}^{\mathsf{T}\tau} \cup \{Ax_{+W}, L^{\mathbb{G}}_{\Diamond+W}, R^{\mathbb{G}}_{\Box+W}\}$. The resulting Socratic proof has only one branch, and the world line has only one element: 1. We can see how, by the use of the bottom label, one occurrence of '$\Box p$' is saved in this transformation and the other lost, as required.

Example 29.

$$\cfrac{\cfrac{?(\vdash (\Box p \to p)^1)}{?((\Box p)^1 \vdash p^1)}\ R^{\mathbb{E}}_{\to}}{?((\Box p)^1_{refl}, p^1 \vdash p^1)}\ L^{\mathbb{E}}_{refl}$$

$$\Downarrow$$

$$\cfrac{\cfrac{\tau_{\mathsf{T}}(1, (\Box p)^1_{refl}, p^1 \vdash p^1)}{\tau_{\mathsf{T}}(1, (\Box p)^1 \vdash p^1)}}{\tau_{\mathsf{T}}(1, \vdash (\Box p \to p)^1)} \quad \Rightarrow \quad \cfrac{\cfrac{p \vdash p}{\Box p \vdash p}\ L^{\mathbb{G}}_{refl}}{\vdash \Box p \to p}\ R^{\mathbb{G}}_{\to}$$

□

As previously, we establish the following:

Theorem 23. *Let* **s** *be a regular Socratic proof of sequent '$\vdash (A)^1$' in calculus $\mathbb{E}^{\mathsf{T}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$. If $\langle X, R\rangle$ is a tree obtained from* **s** *by the procedure PRUNE, then $\langle X, R, \tau_{\mathsf{Ts}} |_X\rangle$ is a proof of '$\vdash A$' in calculus $\mathbb{G}^{\mathsf{T}\tau} \cup \{Ax_{+W}, L^{\mathbb{G}}_{\Diamond+W}, R^{\mathbb{G}}_{\Box+W}\}$.*

In the present case, clause *3* of Lemma 8 remains the same, but we need to modify clause *1*.

Lemma 10. *Let* **s** *stand for a regular Socratic proof of sequent '$\vdash (A)^1$' in calculus $\mathbb{E}^{\mathsf{T}\tau} \cup \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$, and let $\mathcal{T} = \langle X, R, \tau_{\mathsf{Ts}} |_X\rangle$ be the tree obtained from* **Tr**(**s**) *by procedure PRUNE.*

1. *If sequent ψ labels a node of $\mathcal{T}$ that immediately precedes exactly one node, which is labelled by ϕ, and the two sequents have the same principal numeral, then the following:*

$$\frac{\tau_{\mathsf{Ts}}(\phi)}{\tau_{\mathsf{Ts}}(\psi)}$$

is either an instance of a classical linear rule of $\mathbb{G}^{\mathsf{T}\tau}$ or an instance of one of $L^{\mathbb{G}}_{refl}$, $R^{\mathbb{G}}_{refl}$.

2. *and 3. as in Lemma 8.*

Proof. If ϕ results in **Tr**(**s**) from its predecessor by one of the modal rules: $L^{\mathbb{E}}_{refl}$, $R^{\mathbb{E}}_{refl}$, then the active occurrence of modal formula '$(\Box A)^{\sigma}$' (respectively, '$(\Diamond A)^{\sigma}$') has no bottom label yet; otherwise the rule would not have been applied—see line 21 of Procedure SETTLE-T. Therefore the occurrence of '$\Box A$' (of '$\Diamond A$') is not lost under τ_{T} in sequent $\tau_{\mathsf{Ts}}(\psi)$, but it is lost in $\tau_{\mathsf{Ts}}(\phi)$, where it already has the bottom index. For this reason $\tau_{\mathsf{Ts}}(\phi)/\tau_{\mathsf{Ts}}(\psi)$ is an instance of $L^{\mathbb{G}}_{refl}$ ($R^{\mathbb{G}}_{refl}$, respectively). □

Translation for K4

It is time to deal with the transitive logics. In the original erotetic calculi the difference between K and K4 is in the proviso of applicability of R_{π}. Let us recall: A^{j} may be introduced (from world i) for any pair $\langle i, j\rangle$ that belongs to the transitive closure of the set of pairs of numerals in the indices. The rules of $\mathbb{E}^{\mathsf{K4}\tau}$ are the classical rules of $\mathbb{E}^{\mathsf{K}\tau}$, modal rules $L^{\mathbb{E}}_{\Diamond}$ and $R^{\mathbb{E}}_{\Box}$ (as in the case of $\mathbb{E}^{\mathsf{K}\tau}$), and rules $L^{\mathbb{E}}_{\Box}$, $R^{\mathbb{E}}_{\Diamond}$ with the suitable proviso. In order to generalise the modal rules we need to modify (again) the definitions of $\sharp$ and $\flat$. See Algorithm 9, $\flat_j$ is defined analogously.

Algorithm 9: operation $\sharp_j$ for transitive logics

Data: finite sequence S of indexed formulas,
set R of pairs of numerals $\langle n, m\rangle$ such that n immediately precedes m in an index of a formula in S
all formulas $F_1, \ldots, F_m$ such that $\Box F_k$ lives in i_k in S where $\langle i_k, j\rangle$ belongs to the transitive closure of R **or** $m = 0$ if there are no such formulas
Result: finite sequence $S\sharp_j$ of indexed formulas
for $k = 1$ **to** m **do**
  $S \leftarrow S((\Box F_k)^{\sigma_k/i_k}/(\Box F_k)^{\sigma_k/i_k}, (F_k)^{j})$
end
$S\sharp_j \leftarrow S$
return $S\sharp_j$

Here is the generalisation of modal rules for the case of K4. Let us remind that '$S(A^{\sigma/i,j})\sharp_j$' and '$T(A^{\sigma/i,j})\flat_j$' are understood in such a way that, first, the occurrence of '$(\Diamond A)^{\sigma/i}$' (respectively, '$(\Box A)^{\sigma/i}$') is replaced with '$A^{\sigma/i,j}$', and then, second, the operation ($\sharp_j$, $\flat_j$) is performed on the resulting sequence.

$$\frac{?(\Phi\ ;\ S((\Diamond A)^{\sigma/i})\ \vdash\ T\ ;\ \Psi)}{?(\Phi\ ;\ S(A^{\sigma/i,j})\sharp_j\ \vdash\ T\flat_j\ ;\ \Psi)}\ L^{\mathbb{E}}_{trans}$$

$$\frac{?(\Phi\ ;\ S\ \vdash T((\Box A)^{\sigma/i})\ ;\ \Psi)}{?(\Phi\ ;\ S\sharp_j\ \vdash T(A^{\sigma/i,j})\flat_j\ ;\ \Psi)}\ R^{\mathbb{E}}_{trans}$$

This time we state without proof:

Corollary 29. *Rules $L^{\mathbb{E}}_{trans}$ and $R^{\mathbb{E}}_{trans}$ are derivable in $\mathbb{E}^{\mathsf{K4}\tau}$.*

For the sake of defining regular Socratic transformations via the rules of $\mathbb{E}^{\mathsf{K4}\tau}$ we modify only Procedure MODAL: in line 5 we use $\mathbf{r} \in \{L^{\mathbb{E}}_{trans}, R^{\mathbb{E}}_{trans}\}$ instead of $\mathbf{r} \in \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$—see page 141. This modification will be called "MODAL-K4".

The main difference between the explicit and implicit (purely syntactical) proof methods for transitive modal logics is that in the latter accounts the information about transitivity is transferred by rewriting formulas of the form '$\Box A$' on the left ('$\Diamond A$' on the right). In explicit formats the same information is encoded in the additional indices/labels. We need to modify function τ to the effect of expressing the information about transitivity in the syntactical way. Unfortunately, the function must act in a different way on the left and right side (see Definition 46).

For simplicity, we will say that *numeral i can see numeral j in a sequent ϕ* iff the pair $\langle i, j\rangle$ belongs to the transitive closure of the set of pairs $\mathbf{I}_R[\phi]$.

Definition 46. *Let $i \in \mathbb{N}$ and let ϕ be a sequent of $\mathcal{M}^{?}_{\vdash}$. We define functions τ_{4L} and τ_{4R} as follows:*

$\tau_{4L}(\phi, i, \varnothing) = \varnothing$

$\tau_{4L}(\phi, i, \langle A_1^{\sigma_1}, \ldots, A_k^{\sigma_k}\rangle) =$

$$= \begin{cases} \langle A_1\rangle' \tau_{4L}(\phi, i, \langle A_2^{\sigma_2}, \ldots, A_k^{\sigma_k}\rangle) & \textit{if } \sigma_1 \textit{ ends with } i, \textbf{\textit{or}}\ A_1 = \Box F \\ & \textit{and } \sigma_1 \textit{ ends with a numeral} \\ & \textit{that can see } i \textit{ in } \phi \\ \tau_{4L}(\phi, i, \langle A_2^{\sigma_2}, \ldots, A_k^{\sigma_k}\rangle) & \textit{otherwise} \end{cases}$$

and

$\tau_{4R}(i, \varnothing) = \varnothing$

$\tau_{4R}(\phi, i, \langle A_1^{\sigma_1}, \ldots, A_k^{\sigma_k}\rangle) =$

$$= \begin{cases} \langle A_1 \rangle' \tau_{4R}(\phi, i, \langle A_2^{\sigma_2}, \ldots, A_k^{\sigma_k} \rangle) & \textit{if } \sigma_1 \textit{ ends with } i, \textbf{ or } A_1 = \Diamond F \\ & \textit{and } \sigma_1 \textit{ ends with a numeral} \\ & \textit{that can see } i \textit{ in } \phi \\ \tau_{4R}(\phi, i, \langle A_2^{\sigma_2}, \ldots, A_k^{\sigma_k} \rangle) & \textit{otherwise} \end{cases}$$

□

We join the two functions in the following definition:

Definition 47. *Let $i \in \mathbb{N}$ and let $\phi = S \vdash T$ be a sequent of $\mathcal{M}^?_\vdash$. Then:*

$$\tau_4(i, S \vdash T) = \tau_{4L}(\phi, i, S) \vdash \tau_{4R}(\phi, i, T)$$

□

Let us present:

Example 30. *A regular Socratic proof of '$\vdash (\Box p \rightarrow \Box\Box p)^1$' in $\mathbb{E}^{\mathsf{K4}\tau} \cup \{L^{\mathbb{E}}_{trans}, R^{\mathbb{E}}_{trans}\}$ translated into a proof of '$\vdash \Box p \rightarrow \Box\Box p$' in calculus $\mathbb{G}^{\mathsf{K4}\tau} \cup \{Ax_{+W}, L^{\mathbb{G}}_{trans+W}, R^{\mathbb{G}}_{trans+W}\}$. The Socratic tree has one branch with world line $\langle 1, 2, 3 \rangle$.*

$$\dfrac{\dfrac{\dfrac{?(\vdash (\Box p \rightarrow \Box\Box p)^1)}{?((\Box p)^1 \vdash (\Box\Box p)^1)}\, R^{\mathbb{E}}_{\rightarrow}}{?((\Box p)^1, p^2 \vdash (\Box p)^{1,2})}\, R^{\mathbb{E}}_{trans}}{?((\Box p)^1, p^3, p^2 \vdash p^{1,2,3})}\, R^{\mathbb{E}}_{trans} \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{\tau_4(3, (\Box p)^1, p^{1,3}, p^{1,2} \vdash p^{1,2,3})}{\tau_4(2, (\Box p)^1, p^{1,2} \vdash (\Box p)^{1,2})}}{\tau_4(1, (\Box p)^1 \vdash (\Box\Box p)^1)}}{\tau_4(1, \vdash (\Box p \rightarrow \Box\Box p)^1)}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{\Box p, p \vdash p}{\Box p, p \vdash \Box p}}{\Box p \vdash \Box\Box p}}{\vdash \Box p \rightarrow \Box\Box p}$$

□

Similarly as before, a modification *modulo* weakening is needed in order to obtain a proof in $\mathbb{G}^{\mathsf{K4}\tau}$, though here the situation is less transparent. The transition:

$$\dfrac{\Box p, p \vdash \Box p}{\Box p \vdash \Box\Box p}$$

(in the middle) is an instance of $R^{\mathbb{G}}_{trans}$:

$$\frac{(\Box X)\sharp \ \vdash \ (\Diamond Y_1)\flat \,{}'\, A \,{}'\, (\Diamond Y_2)\flat}{\Box X \ \vdash \ \Diamond Y_1 \,{}'\, \Box A \,{}'\, \Diamond Y_2} \ R^{\mathbb{G}}_{trans}$$

with $A = \Box p$, $\Box A = \Box\Box p$, $\Box X = \langle \Box p\rangle$ in the conclusion and $(\Box X)\sharp = \langle \Box p, p\rangle$ in the premise. The top transition:

$$\frac{\Box p, p \vdash p}{\Box p, p \vdash \Box p}$$

has weakening hidden, since from $\Box p, p \vdash p$ we can derive $\Box p \vdash \Box p$ by $R^{\mathbb{G}}_{trans}$ ($A = p$, $\Box A = \Box p$, $(\Box X)\sharp = \langle \Box p, p\rangle$, $\Box X = \langle \Box p\rangle$), and then we can move to $\Box p, p \vdash \Box p$ by weakening. The whole proof in $\mathbb{G}^{\mathsf{K4}\tau}$ should go as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{p \vdash p}{\Box p, p \vdash p}\ L^{\mathbb{G}}_{W}}{\Box p \vdash \Box p}\ R^{\mathbb{G}}_{trans}}{\Box p, p \vdash \Box p}\ L^{\mathbb{G}}_{W}}{\Box p \vdash \Box\Box p}\ R^{\mathbb{G}}_{trans}}{\vdash \Box p \rightarrow \Box\Box p}\ R^{\mathbb{G}}_{\rightarrow}$$

We can see clearly that the first steps are redundant (we could start with $\Box p \vdash \Box p$) but redundancy seems inevitable in such translations. To sum up, the above example indicates that, as previously, we need to add weakening to the modal rules.

$$\frac{(\Box X_1)\sharp \,{}'\, A \,{}'\, (\Box X_2)\sharp \ \vdash \ (\Diamond Y)\flat}{X_1^* \,{}'\, \Diamond A \,{}'\, X_2^* \ \vdash \ Y^*} \ L^{\mathbb{G}}_{trans+W}$$

$$\frac{(\Box X)\sharp \ \vdash \ (\Diamond Y_1)\flat \,{}'\, A \,{}'\, (\Diamond Y_2)\flat}{X^* \ \vdash \ Y_1^* \,{}'\, \Box A \,{}'\, Y_2^*} \ R^{\mathbb{G}}_{trans+W}$$

where in $L^{\mathbb{G}}_{trans+W}$: 'X_1^*' is a supersequence of '$\Box X_1$', 'X_2^*' is a supersequence of '$\Box X_2$', and 'Y^*' is a supersequence of '$\Diamond Y$', and at least in one case the supersequence is proper; similarly, in $R^{\mathbb{G}}_{trans+W}$: 'X^*' is a supersequence of '$\Box X$', 'Y_1^*' is a supersequence of '$\Diamond Y_1$', and 'Y_2^*' is a supersequence of '$\Diamond Y_2$', and at least in one case the supersequence is proper.

Function τ_{4s} is defined as function τ_{s} in Definition 41 but with τ_4 playing the role of τ.

Definition 48. *Let* $\mathbf{s}$ *be a regular Socratic transformation of a question of the form '*$?(\vdash (A)^1)$*' via the rules of* $\mathbb{E}^{\mathsf{K4}\tau} \cup \{L^{\mathbb{E}}_{trans}, R^{\mathbb{E}}_{trans}\}$*, and let* $\mathbf{Tr}(\mathbf{s})$ *be the Socratic tree determined by* $\mathbf{s}$*. Then* τ_{4s} *is a relabelling function defined recursively as follows:*

1. *If* ϕ *labels the root of* $\mathbf{Tr}(\mathbf{s})$*, then* $\tau_{\mathsf{4s}}(\phi) = \tau_4(1, \phi)$*.*
2. *If* ϕ *labels a node,* ψ *labels the immediate* $R_{\mathbf{s}}$*-predecessor of the node,* ϕ *results from* ψ *in* $\mathbf{s}$ *by a classical rule, and* i *is the numeral such that* $\tau_{\mathsf{4s}}(\psi) = \tau_4(i, \psi)$*, then* $\tau_{\mathsf{4s}}(\phi) = \tau_4(i, \phi)$*.* ***(conservative extension)***
3. *If* ϕ *is a label of a node that results in* $\mathbf{s}$ *from its immediate* $R_{\mathbf{s}}$*-predecessor by a modal rule, then* $\tau_{\mathsf{4s}}(\phi) = \tau_4(j, \phi)$*, where* j *is the principal numeral in* ϕ*.* ***(radical change)*** □

Lemma 8 is modified in clause *3* as follows.

Lemma 11. *Let* $\mathbf{s}$ *stand for a regular Socratic proof of '*$\vdash (A)^1$*' in calculus* $\mathbb{E}^{\mathsf{K4}\tau} \cup \{L^{\mathbb{E}}_{trans}, R^{\mathbb{E}}_{trans}\}$*, and let* $\mathcal{T} = \langle X, R, \tau_{\mathsf{4s}} |_X \rangle$ *be the tree obtained from* $\mathbf{Tr}(\mathbf{s})$ *by procedure PRUNE.*

1. *and 2. as in Lemma 8.*
3. *If sequent* ψ *labels a node of* $\mathcal{T}$ *that immediately precedes exactly one node, which is labelled by* ϕ*, and the two sequents have different principal numerals, then the following:*

$$\frac{\tau_{\mathsf{4s}}(\phi)}{\tau_{\mathsf{4s}}(\psi)}$$

is an instance of one of the rules: $L^{\mathbb{G}}_{trans}$*,* $R^{\mathbb{G}}_{trans}$*,* $L^{\mathbb{G}}_{trans+W}$*, or* $R^{\mathbb{G}}_{trans+W}$*.*

Proof. As previously, the proof is almost a repetition of the proof of Lemma 8. We observe that $\mathbf{s}$ is regular and that $\mathcal{T}$ has been pruned, and for this reason the two sequents: ϕ and ψ share the same modalized formulas (modalized with '$\Box$' on the left side of the turnstile, and with '$\Diamond$' on the right side) that live in worlds which can see the principal j. If there are any other formulas in $\tau_{\mathsf{4s}}(\psi)$ that do not fit the scheme of conclusion, then $\tau_{\mathsf{4s}}(\phi)/\tau_{\mathsf{4s}}(\psi)$ is an instance of $L^{\mathbb{G}}_{trans+W}$ or $R^{\mathbb{G}}_{trans+W}$. Otherwise it is an instance of $L^{\mathbb{G}}_{trans}$ or $R^{\mathbb{G}}_{trans}$. □

Lemma 11 yields that the following theorem is true.

Theorem 24. *Let* **s** *be a regular Socratic proof of sequent '$\vdash (A)^1$' in calculus* $\mathbb{E}^{\mathsf{K4}\tau} \cup \{L^{\mathbb{E}}_{trans}, R^{\mathbb{E}}_{trans}\}$. *If* $\langle X, R\rangle$ *is a tree obtained from* **s** *by the procedure PRUNE, then* $\langle X, R, \tau_{\mathsf{4s}} |_X\rangle$ *is a proof of A in calculus* $\mathbb{G}^{\mathsf{K4}\tau} \cup \{Ax_{+W}, L^{\mathbb{G}}_{trans+W}, R^{\mathbb{G}}_{trans+W}\}$.

Translation for D4

Erotetic calculus $\mathbb{E}^{\mathsf{D4}\tau}$ for logic D4 contains the rules for the classical connectives, modal rules $L^{\mathbb{E}}_{\Diamond}$, $R^{\mathbb{E}}_{\Box}$ of calculus $\mathbb{E}^{\mathsf{K}\tau}$, modal rules $L^{\mathbb{E}}_{D}$ and $R^{\mathbb{E}}_{D}$ for seriality, and rules $L^{\mathbb{E}}_{\Box}$, $R^{\mathbb{E}}_{\Diamond}$ with the proviso of applicability for transitive logics. As previously, we add the generalised versions. In the following rules, that capture seriality, numeral j is new with respect to the active sequent:

$$\frac{?(\Phi\ ;\ S((\Box A)^{\sigma/i}) \vdash T\ ;\ \Psi)}{?(\Phi\ ;\ S((\Box A)^{\sigma/i}, A^{i,j})\sharp_j \vdash T\flat_j\ ;\ \Psi)}\ L^{\mathbb{E}}_{Dgen}$$

$$\frac{?(\Phi\ ;\ S \vdash T((\Diamond A)^{\sigma/i})\ ;\ \Psi)}{?(\Phi\ ;\ S\sharp_j \vdash T((\Diamond A)^{\sigma/i}, A^{i,j})\flat_j\ ;\ \Psi)}\ R^{\mathbb{E}}_{Dgen2}$$

where $\sharp_j$ and $\flat_j$ are defined as for the transitive logics (see Algorithm 9), so they gather all the information from the worlds that can see the new j. (Let us also recall that we have already introduced rule $R^{\mathbb{E}}_{Dgen}$ for logic D, for this reason the above rule is called "$R^{\mathbb{E}}_{Dgen2}$".)

As previously, we state without proof:

Corollary 30. *Rules* $L^{\mathbb{E}}_{Dgen}$, $R^{\mathbb{E}}_{Dgen2}$ *are derivable in* $\mathbb{E}^{\mathsf{D4}\tau}$.

Corollary 31. *Rules* $L^{\mathbb{E}}_{trans}$ *and* $R^{\mathbb{E}}_{trans}$ *are derivable in* $\mathbb{E}^{\mathsf{D4}\tau}$.

For the case of defining regular Socratic transformations constructed *via* the rules of $\mathbb{E}^{\mathsf{D4}\tau}$ we use Procedure MODAL-D, see page 169, with the following modifications:

- line 6 of the procedure: "$\mathbf{r} \in \{L^{\mathbb{E}}_{trans}, R^{\mathbb{E}}_{trans}\}$" instead of "$\mathbf{r} \in \{L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}\}$",
- line 18 of the procedure: "$R^{\mathbb{E}}_{Dgen2}$" instead of "$R^{\mathbb{E}}_{Dgen}$".

This modification of Procedure MODAL-D will be called "MODAL-D4". For translating sequent we use the relabelling function τ_{4s} as defined for K4 (see Definition 48).

Example 31. *The regular Socratic proof of '$\vdash (\Box p \to \Diamond\Diamond p)^1$' in $\mathbb{E}^{\mathsf{D4}\tau} \cup \{L^{\mathbb{E}}_{trans}, R^{\mathbb{E}}_{trans}, L^{\mathbb{E}}_{Dgen}, R^{\mathbb{E}}_{Dgen2}\}$ translated into a proof of '$\vdash \Box p \to \Diamond\Diamond p$' in $\mathbb{G}^{\mathsf{D4}\tau} \cup \{Ax_{+W}, L^{\mathbb{G}}_{trans+W}, R^{\mathbb{G}}_{trans+W}, R^{\mathbb{G}}_{Dtrans+W}\}$.*

$$\cfrac{\cfrac{\cfrac{?(\vdash (\Box p \to \Diamond\Diamond p)^1)}{?((\Box p)^1 \vdash (\Diamond\Diamond p)^1)}\ R^{\mathbb{E}}_{\to}}{?((\Box p)^1, p^{1,2} \vdash (\Diamond\Diamond p)^1, (\Diamond p)^2)}\ L^{\mathbb{E}}_{Dgen}}{?((\Box p)^1, p^3, p^{1,2} \vdash (\Diamond\Diamond p)^1, (\Diamond p)^3, (\Diamond p)^2, p^{2,3})}\ R^{\mathbb{E}}_{Dgen2}$$

$$\Downarrow$$

$$\cfrac{\cfrac{\cfrac{\tau_4(3, (\Box p)^1, p^3, p^{1,2} \vdash (\Diamond\Diamond p)^1, (\Diamond p)^3, (\Diamond p)^2, p^{2,3})}{\tau_4(2, (\Box p)^1, p^{1,2} \vdash (\Diamond\Diamond p)^1, (\Diamond p)^2)}}{\tau_4(1, (\Box p)^1 \vdash (\Diamond\Diamond p)^1)}}{\tau_4(1, \vdash (\Box p \to \Diamond\Diamond p)^1)}$$

$$\Downarrow$$

$$\cfrac{\cfrac{\cfrac{\Box p, p \vdash \Diamond\Diamond p, \Diamond p, \Diamond p, p}{\Box p, p \vdash \Diamond\Diamond p, \Diamond p}}{\Box p \vdash \Diamond\Diamond p}}{\vdash \Box p \to \Diamond\Diamond p}$$

□

We present some calculations to explain the above example. We indicate the first argument of functions τ_{4L}, τ_{4R} by 'ϕ', as the sequent is clear from the context. We have:

$\tau_{4L}(\phi, 2, \langle(\Box p)^1, p^{1,2}\rangle) = \langle\Box p, p\rangle$, because 1 can see 2 in the sequent, similarly: $\tau_{4R}(\phi, 2, \langle(\Diamond\Diamond p)^1, (\Diamond p)^2\rangle) = \langle\Diamond\Diamond p, \Diamond p\rangle$

$\tau_{4L}(\phi, 3, \langle(\Box p)^1, p^3, p^{1,2}\rangle) = \langle\Box p, p\rangle$, as 1 can see 3 in the sequent. The occurrence of 'p' in 3 is left, and the other is lost.

$\tau_{4R}(\phi, 3, \langle(\Diamond\Diamond p)^1, (\Diamond p)^3, (\Diamond p)^2, p^{2,3}\rangle) = \langle\Diamond\Diamond p, \Diamond p, \Diamond p, p\rangle$, since 1 and 2 can see 3.

The final tree is not a proof in $\mathbb{G}^{\mathsf{D4}\tau}$, but we can see how to "fix it" by weakening.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\Box p, p \vdash p}\, L_W^{\mathbb{G}}}{\Box p, p \vdash \Diamond\Diamond p, \Diamond p, \Diamond p, p}\, R_W^{\mathbb{G}}}{\Box p \vdash \Diamond\Diamond p, \Diamond p}\, R_{Dtrans}^{\mathbb{G}}}{\Box p, p \vdash \Diamond\Diamond p, \Diamond p}\, L_W^{\mathbb{G}}}{\Box p \vdash \Diamond\Diamond p}\, R_{Dtrans}^{\mathbb{G}}}{\vdash \Box p \to \Diamond\Diamond p}\, R_{\to}^{\mathbb{G}}$$

Therefore the only missing element in the picture is the following derived rule of $\mathbb{G}^{\mathsf{D4}\tau}$:

$$\frac{(\Box X)\sharp \ \vdash \ (\Diamond Y)\flat}{X^* \ \vdash \ Y^*}\, R_{Dtrans+W}^{\mathbb{G}}$$

where 'X^*' is a supersequence of '$\Box X$', 'Y^*' is a supersequence of '$\Diamond Y$', and at least one of the supersequences is proper.

We are ready to conclude:

Theorem 25. *Let* **s** *be a regular Socratic proof of sequent '$\vdash (A)^1$' in calculus $\mathbb{E}^{\mathsf{D4}\tau} \cup \{L_{trans}^{\mathbb{E}}, R_{trans}^{\mathbb{E}}, L_{Dgen}^{\mathbb{E}}, R_{Dgen2}^{\mathbb{E}}\}$. If $\langle X, R \rangle$ is a tree obtained from* **s** *by the procedure PRUNE, then $\langle X, R, \tau_{\mathsf{4s}}|_X \rangle$ is a proof of '$\vdash A$' in calculus $\mathbb{G}^{\mathsf{D4}\tau} \cup \{Ax_{+W}, L_{trans+W}^{\mathbb{G}}, R_{trans+W}^{\mathbb{G}}, R_{Dtrans+W}^{\mathbb{G}}\}$.*

Lemma 8 is modified in clause *3* as follows.

Lemma 12. *Let* **s** *stand for a regular Socratic transformation via the rules of calculus $\mathbb{E}^{\mathsf{D4}\tau} \cup \{L_{trans}^{\mathbb{E}}, R_{trans}^{\mathbb{E}}, L_{Dgen}^{\mathbb{E}}, R_{Dgen2}^{\mathbb{E}}\}$, and let $\mathcal{T} = \langle X, R, \tau_{\mathsf{4s}}|_X \rangle$ be the tree obtained from* **Tr**(**s**) *by procedure PRUNE.*

1. and 2. as in Lemma 8.

3. If sequent ψ labels a node of $\mathcal{T}$ that immediately precedes exactly one node, which is labelled by ϕ, and the two sequents have different principal numerals, then the following:

$$\frac{\tau_{\mathsf{4s}}(\phi)}{\tau_{\mathsf{4s}}(\psi)}$$

is an instance of one of: $L_{trans}^{\mathbb{G}}$, $R_{trans}^{\mathbb{G}}$, $L_{trans+W}^{\mathbb{G}}$, $R_{trans+W}^{\mathbb{G}}$, $R_{Dtrans}^{\mathbb{G}}$, $R_{Dtrans+W}^{\mathbb{G}}$.

Proof. As before, the classical cases listed in Lemma 8, the justification of the fact that a modal rule has been applied "between" ψ and ϕ, and the reasons for **Additional Assumption**—they all run as in Lemma 8. Further, if the modal rule was, *e.g.*, $L^{\mathbb{E}}_{trans}$, then $\psi = S(A^{\sigma/i,j})\sharp_j \vdash T\flat_j$, and either $\phi = S((\Diamond A)^{\sigma/i}) \vdash T$ or it has a proper supersequence of $S((\Diamond A)^{\sigma/i})$ on the left and/or a proper supersequence of T on the right. We compare the definitions of $\sharp_j$ and $\sharp$ (and so on for $\flat$) and arrive at the conclusion that $\tau_{\mathsf{4s}}(\phi)/\tau_{\mathsf{4s}}(\psi)$ is an instance of $L^{\mathbb{G}}_{trans}$ or $L^{\mathbb{G}}_{trans+W}$. Similarly for the other cases. □

Translation for S4

Calculus $\mathbb{E}^{\mathsf{S4}\tau}$ for S4 contains the rules for the classical connectives and $L^{\mathbb{E}}_{\Diamond}$, $R^{\mathbb{E}}_{\Box}$ of calculus $\mathbb{E}^{\mathsf{K}\tau}$, rules $L^{\mathbb{E}}_{refl}$ and $R^{\mathbb{E}}_{refl}$ for reflexivity and rules $L^{\mathbb{E}}_{\Box}$, $R^{\mathbb{E}}_{\Diamond}$ with the proviso of applicability for transitive logics. Next, we add—as derived rules—the generalised versions of transitivity rules, that is, $L^{\mathbb{E}}_{trans}$ and $R^{\mathbb{E}}_{trans}$. The proof of their derivability in $\mathbb{E}^{\mathsf{S4}\tau}$ is as before, we skip it.

Corollary 32. *Rules $L^{\mathbb{E}}_{trans}$ and $R^{\mathbb{E}}_{trans}$ are derivable in $\mathbb{E}^{\mathsf{S4}\tau}$.*

The notion of regular Socratic transformation is defined with reference to Procedures SETTLE-T (instead of CLASSICAL) and MODAL-K4. Let us recall that Procedure SETTLE-T makes use of bottom labels, marking a modal formula after an application of the rules for reflexivity. In the case of logic T, the bottom label played two roles: preventing from redundant applications of the reflexivity rules (see the condition expressed in line 21 of Procedure SETTLE-T), and distinguishing between the occurrences of modal formulas that must be saved, and these that must be lost under the translation function (see Definition 44). The situation is different in the case of S4. We use Procedure SETTLE-T, and we want the bottom labels to play their preventing role. On the other hand, the relevant occurrences of modal formulas will not disappear during the translation, since now they are needed to encode the property of transitivity in the syntactic manner.

For this reason, we assume that after generating a regular Socratic proof by Algorithm 4 using Procedure SETTLE-T instead of CLASSICAL, and Procedure MODAL-K4 instead of MODAL, all bottom labels (if there are any) are removed from the Socratic proof. The translation function to work for S4 will be τ_4, thus we assume that it takes as the second argument a sequent that contains no indexed formulas with bottom labels.

The final relabelling function, called τ_{S4s}, is defined as in the case of T (function τ_{Ts}, Definition 45), but when the numerals are settled, then function τ_{4} is used (Definition 47). Here is the whole definition:

Definition 49. *Let* $\mathbf{s}$ *be a regular Socratic transformation of a question of the form '*$?(\vdash (A)^1)$*' via the rules of* $\mathbb{E}^{\mathsf{S4}\tau} \cup \{L^{\mathbb{E}}_{trans}, R^{\mathbb{E}}_{trans}\}$ *and let* $\mathbf{Tr}(\mathbf{s})$ *be the Socratic tree determined by* $\mathbf{s}$*. Then* τ_{S4s} *is a relabelling function defined recursively as follows:*

1. *If* ϕ *labels the root of* $\mathbf{Tr}(\mathbf{s})$*, then* $\tau_{\mathsf{S4s}}(\phi) = \tau_{\mathsf{4}}(1, \phi)$*.*
2. *If* ϕ *labels a node,* ψ *labels the immediate* $R_{\mathbf{s}}$*-predecessor of the node,* ϕ *results from* ψ *in* $\mathbf{s}$ *by a classical rule or by one of* $L^{\mathbb{E}}_{refl2}$*,* $R^{\mathbb{E}}_{refl2}$*, and* i *is the numeral such that* $\tau_{\mathsf{S4s}}(\psi) = \tau_{\mathsf{4}}(i, \psi)$*, then* $\tau_{\mathsf{S4s}}(\phi) = \tau_{\mathsf{4}}(i, \phi)$*.*
3. *If* ϕ *labels a node and* ϕ *results in* $\mathbf{s}$ *from its immediate* $R_{\mathbf{s}}$*-predecessor by a modal rule other than* $L^{\mathbb{E}}_{refl}$*,* $R^{\mathbb{E}}_{refl}$*, then* $\tau_{\mathsf{S4s}}(\phi) = \tau_{\mathsf{4}}(j, \phi)$*, where* j *is the principal numeral in* ϕ*.*

□

Example 32.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{?(\vdash (\Diamond\Box\Diamond p \to \Diamond p)^1)}{?((\Diamond\Box\Diamond p)^1 \vdash (\Diamond p)^1)}\ R^{\mathbb{E}}_{\to}}{?((\Diamond\Box\Diamond p)^1 \vdash (\Diamond p)^1_{refl}, p^1)}\ R^{\mathbb{E}}_{refl}}{?((\Box\Diamond p)^{1,2} \vdash (\Diamond p)^1_{refl}, p^2, p^1)}\ L^{\mathbb{E}}_{trans}}{?((\Box\Diamond p)^{1,2}_{refl}, (\Diamond p)^2 \vdash (\Diamond p)^1_{refl}, p^2, p^1)}\ L^{\mathbb{E}}_{refl}}{?((\Box\Diamond p)^{1,2}_{refl}, (\Diamond p)^3, p^{2,3} \vdash (\Diamond p)^1_{refl}, p^3, p^2, p^1)}\ L^{\mathbb{E}}_{trans}$$

Function $\tau_{\mathsf{S4}}\mathbf{s}$ assigns numerals to the nodes of the above regular Socratic transformation $\mathbf{s}$ as follows (the bottom labels disappear):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\tau_{\mathsf{4}}(3, (\Box\Diamond p)^{1,2}, (\Diamond p)^3, p^{2,3} \vdash (\Diamond p)^1, p^3, p^2, p^1)}{\tau_{\mathsf{4}}(2, (\Box\Diamond p)^{1,2}, (\Diamond p)^2 \vdash (\Diamond p)^1, p^2, p^1)}}{\tau_{\mathsf{4}}(2, (\Box\Diamond p)^{1,2} \vdash (\Diamond p)^1, p^2, p^1)}}{\tau_{\mathsf{4}}(1, (\Diamond\Box\Diamond p)^1 \vdash (\Diamond p)^1, p^1)}}{\tau_{\mathsf{4}}(1, (\Diamond\Box\Diamond p)^1 \vdash (\Diamond p)^1)}}{\tau_{\mathsf{4}}(1, \vdash (\Diamond\Box\Diamond p \to \Diamond p)^1)}$$

Below to the left we have the resulting tree, to the right the tree supplemented with the missing intermediate nodes. We can see that the "gaps" may be filled with weakening, so the tree (to the right) is constructed with the use of more general derived rules $L^{\mathbb{G}}_{trans+W}$, $R^{\mathbb{G}}_{trans+W}$.

$$
\begin{array}{c}
\Box\Diamond p, \Diamond p, p \vdash \Diamond p, p \\ \hline
\Box\Diamond p, \Diamond p \vdash \Diamond p, p \\ \hline
\Box\Diamond p \vdash \Diamond p, p \\ \hline
\Diamond\Box\Diamond p \vdash \Diamond p, p \\ \hline
\Diamond\Box\Diamond p \vdash \Diamond p \\ \hline
\vdash \Diamond\Box\Diamond p \rightarrow \Diamond p
\end{array}
\quad \Rightarrow \quad
\begin{array}{cl}
\Box\Diamond p, \Diamond p, p \vdash \Diamond p, p & \\ \hline
\boxed{\Box\Diamond p, \Diamond p \vdash \Diamond p} & L^{\mathbb{G}}_{trans} \\ \hline
\Box\Diamond p, \Diamond p \vdash \Diamond p, p & R^{\mathbb{G}}_{W} \\ \hline
\Box\Diamond p \vdash \Diamond p, p & L^{\mathbb{G}}_{refl2} \\ \hline
\boxed{\Diamond\Box\Diamond p \vdash \Diamond p} & L^{\mathbb{G}}_{trans} \\ \hline
\Diamond\Box\Diamond p \vdash \Diamond p, p & R^{\mathbb{G}}_{W} \\ \hline
\Diamond\Box\Diamond p \vdash \Diamond p & R^{\mathbb{G}}_{refl2} \\ \hline
\vdash \Diamond\Box\Diamond p \rightarrow \Diamond p & R^{\mathbb{G}}_{\rightarrow}
\end{array}
$$

In the transition from the topmost sequent '$\Box\Diamond p, \Diamond p, p \vdash \Diamond p, p$' to '$\Box\Diamond p, \Diamond p \vdash \Diamond p$', in the sequent-premise we have: $(\Box X_1)\flat = \Box\Diamond p, \Diamond p$, $A = p$, $(\Diamond Y)\flat = \Diamond p, p$, and in the sequent-conclusion we have: $\Box X_1 = \Box\Diamond p$, $\Diamond A = \Diamond p$, and $\Diamond Y = \Diamond p$. In the transition from '$\Box\Diamond p \vdash \Diamond p, p$' to '$\Diamond\Box\Diamond p \vdash \Diamond p$' (the middle application of $L^{\mathbb{G}}_{trans}$), we have: $A = \Box\Diamond p$ and $\Diamond A = \Diamond\Box\Diamond p$, and the succedent as in the topmost sequent and its successor. □

For the last time, we prove the S4-counterpart of Lemma 8:

Lemma 13. *Let* **s** *stand for a regular Socratic proof of sequent '*$\vdash (A)^1$*' in calculus* $\mathbb{E}^{\mathsf{S4}\tau} \cup \{L^{\mathbb{E}}_{trans}, R^{\mathbb{E}}_{trans}\}$*, and let* $\mathcal{T} = \langle X, R, \tau_{\mathsf{S4s}} |_X \rangle$ *be the tree obtained from* $\mathbf{Tr}(\mathbf{s})$ *by procedure PRUNE.*

1. *If sequent* ψ *labels a node of* $\mathcal{T}$ *that immediately precedes exactly one node, which is labelled by* ϕ*, and the two sequents have the same principal numeral, then the following:*

$$\frac{\tau_{\mathsf{S4s}}(\phi)}{\tau_{\mathsf{S4s}}(\psi)}$$

is either an instance of a classical linear rule of $\mathbb{G}^{\mathsf{T}\tau}$ *or an instance of one of* $L^{\mathbb{G}}_{refl2}$, $R^{\mathbb{G}}_{refl2}$.

2. *as in Lemma 8.*

3. *If sequent ψ labels a node of $\mathcal{T}$ that immediately precedes exactly one node, which is labelled by ϕ, and the two sequents have different principal numerals, then the following:*

$$\frac{\tau_{\mathsf{S4s}}(\phi)}{\tau_{\mathsf{S4s}}(\psi)}$$

is an instance of one of the rules: $L^{\mathbb{G}}_{trans}$, $R^{\mathbb{G}}_{trans}$, $L^{\mathbb{G}}_{trans+W}$, or $R^{\mathbb{G}}_{trans+W}$.

Proof. As in the previous cases. □

Theorem 26. *Let* **s** *be a regular Socratic proof of sequent '$\vdash (A)^1$' in calculus $\mathbb{E}^{\mathsf{S4}\tau} \cup \{L^{\mathbb{E}}_{trans}, R^{\mathbb{E}}_{trans}\}$. If $\langle X, R\rangle$ is a tree obtained from* **s** *by the procedure PRUNE, then $\langle X, R, \tau_{\mathsf{S4s}} |_X\rangle$ is a proof of '$\vdash A$' in calculus $\mathbb{G}^{\mathsf{S4}\tau} \cup \{Ax_{+W}, L^{\mathbb{G}}_{trans+W}, R^{\mathbb{G}}_{trans+W}\}$.*

4.5. The case of analytic tableaux for classical logic

In this section we go back to $\mathsf{L} \in \{\mathsf{CPL}, \mathsf{FOL}\}$. We define a procedure "translating" a Socratic proof of a sequent '$\vdash A$' into a closed analytic tableau for formula '$\neg A$', that is, into a proof of A in a system of analytic tableaux. We also present the analytic tableau system and introduce certain additional rules to it. Then we analyse the notions of admissibility and derivability of the rules.

We shall consider here the original account of the erotetic calculi for classical logic, that is, $\mathbf{E^{PQ}}$, but also the right-sided and left-sided counterparts of $\mathbf{E^{PQ}}$. We will call them $\mathbf{E^{RPQ}}$ and $\mathbf{E^{LPQ}}$, respectively. Let us recall that calculus $\mathbf{E^{RPQ}}$ has been examined in Wiśniewski, 2006, the propositional part of the left-sided version has been introduced in Wiśniewski, 2004, but this particular formulation of the first-order case has not been considered elsewhere. The rules of calculus $\mathbf{E^{PQ}}$ has been presented in Section 2.2.4 of Chapter 2. The remaining two calculi have the rules for α-, β-formulas defined exactly as in $\mathbb{E}^{\mathbf{RPQ}}$, $\mathbb{E}^{\mathbf{LPQ}}$ (respectively), and also the rules presented in Table 4.11.

The considerations presented in this section apply also to the propositional parts of $\mathbf{E^{PQ}}$, $\mathbf{E^{RPQ}}$, $\mathbf{E^{LPQ}}$. We will use $\mathbb{E}$ as a metavariable referring to the six calculi: $\mathbf{E^{PQ}}$, $\mathbf{E^{RPQ}}$, $\mathbf{E^{LPQ}}$ and their propositional parts.

Table 4.11: Rules of $\mathbf{E^{RPQ}}$ and $\mathbf{E^{LPQ}}$

the rules of $\mathbf{E^{RPQ}}$	the rules of $\mathbf{E^{LPQ}}$
$\dfrac{?(\Phi\,;\vdash\; S'\,\kappa\,'T\,;\,\Psi)}{?(\Phi\,;\vdash\; S'\,\kappa^{**}\,'T\,;\,\Psi)}\ \mathbf{R}_\kappa$	$\dfrac{?(\Phi\,;\,S'\,\kappa\,'T\;\vdash;\,\Psi)}{?(\Phi\,;\,S'\,\kappa^{**}\,'T\;\vdash;\,\Psi)}\ \mathbf{L}_\kappa$
$\dfrac{?(\Phi\,;\vdash S'\,\forall x_i A\,'T\,;\,\Psi)}{?(\Phi\,;\vdash S'A[x_i/\tau]\,'T\,;\,\Psi)}\ \mathbf{R}_\forall$ x_i is free in A, τ is a parameter which does not occur in $\vdash S'\,\forall x_i A\,'T$	$\dfrac{?(\Phi; S'\,\forall x_i A' A[x_i/\tau]\,'T\vdash;\Psi)}{?(\Phi; S'\,\forall x_i A\,'T\vdash;\Psi)}\ \mathbf{L}_\forall$ x_i is free in A and τ is any parameter
$\dfrac{?(\Phi\,;\vdash S'\,\exists x_i A\,'T;\Psi)}{?(\Phi;\vdash S'\,\exists x_i A' A[x_i/\tau]\,'T;\Psi)}\ \mathbf{R}_\exists$ x_i is free in A and τ is any parameter	$\dfrac{?(\Phi\,;\,S'\,\exists x_i A\,'T\vdash;\,\Psi)}{?(\Phi\,;\,S'A[x_i/\tau]\,'T\vdash;\,\Psi)}\ \mathbf{L}_\exists$ x_i is free in A, τ is a parameter which does not occur in $S'\,\exists x_i A\,'T\vdash$

We will use the notion of height of a finite tree understood as the length of its maximal branch (see Appendix B). Height of a tree **Tr** will be written symbolically as $\mathcal{H}(\mathbf{Tr})$. Let us note the following corollaries:

Corollary 33. *If n is the length of a Socratic transformation* $\mathbf{s}$*, then* $\mathcal{H}(\mathbf{Tr}(\mathbf{s})) \leq n-1$.

Proof. It is enough to observe that when the Socratic tree is constructed its consecutive nodes are "made up" of (annotated) sequents taken from the consecutive questions of the Socratic transformation, thus each question of the transformation corresponds to exactly one level of the tree. □

Corollary 34. *If $\langle n, i, \phi\rangle$ is a node of* $\mathbf{Tr}(\mathbf{s})$ *and is not its root, then ϕ is a sequent-conclusion of rule* $\mathbf{r}$ *applied to $(n-1)$-st question of* $\mathbf{s}$*. Moreover, if $\langle k, j, \psi\rangle$ is the immediate predecessor of $\langle n, i, \phi\rangle$ in* $\mathbf{Tr}(\mathbf{s})$*, then ψ is the premise sequent of* $\mathbf{r}$.

Proof. See Corollary 2 in Leszczyńska-Jasion et al., 2013, p. 970. □

Corollary 35. *If* $\mathbf{s}$ *is a Socratic proof in* $\mathbb{E}$*, then each leaf of* $\mathbf{Tr}(\mathbf{s})$ *is an annotated basic sequent.*

Proof. See Leszczyńska-Jasion et al., 2013, Theorem 2 on page 971. □

Except from annotating sequents with their "coordinates" in a Socratic transformation we add an index to all the initial premises of the transformations and all the formulas-conclusions in sequents. This additional indices are displayed below in the form of boxes. If needed, the semi-formal boxing convention can be transformed into technical details. Thus the formulas in boxes are actually pairs of the form $\langle A, i \rangle$, where A is a formula of $\mathcal{L}_{\mathsf{L}}$ and i is a numeral (the index). The indices are assigned to formulas in the following way: if there are n initial premises $A_1, A_2, \ldots, A_n$ in a Socratic transformation, then they are indexed with numerals $1, \ldots, n$ starting from the left to the right, so that the numeral indicates the position of an initial premise in the initial sequent. Thus we get indexed initial premises of the following form: $\langle A_1, 1 \rangle$, $\langle A_2, 2 \rangle$, $\ldots$, $\langle A_n, n \rangle$. In the case of indexed conclusion formulas, if there are two such formulas in a sequent, then the first from the left is assigned 1 and the second is assigned 2. In the other cases there is only one conclusion formula in a sequent and it gets index 1. Instead of '$\langle A, i \rangle$' we may sometimes write '$A : i$'.

Example 33. *A Socratic proof of* $\vdash \exists x(P(x) \to \neg\exists y\neg\neg\neg P(y))$ *in* $\mathbb{E}^{\mathbf{PQ}}$. *A stands for* $\neg\exists x(P(x) \to \neg\exists y\neg\neg\neg P(y))$.

$$
\begin{array}{cl}
1.\ ?(\ \vdash \exists x(P(x) \to \neg\exists y\neg\neg\neg P(y))\ ) & \\
\hline
2.\ ?(\ A \vdash P(a) \to \neg\exists y\neg\neg\neg P(y)\ ) & \mathbf{R}_\delta \\
\hline
3.\ ?(\ A, P(a) \vdash \neg\exists y\neg\neg\neg P(y)\ ) & \mathbf{R}_\beta \\
\hline
4.\ ?(\ A, P(a) \vdash \neg\neg\neg\neg P(b)\ ) & \mathbf{R}_\gamma \\
\hline
5.\ ?(\ A, P(a) \vdash \neg\neg P(b)\ ) & \mathbf{R}_\kappa \\
\hline
6.\ ?(\ A, P(a) \vdash P(b)\ ) & \mathbf{R}_\kappa \\
\hline
7.\ ?(\ A, \neg(P(b) \to \neg\exists y\neg\neg\neg P(y)), P(a) \vdash P(b)\ ) & \mathbf{L}_\gamma \\
\hline
8.\ ?(\ A, P(b), \neg\neg\exists y\neg\neg\neg P(y)), P(a) \vdash P(b)\ ) & \mathbf{L}_\alpha
\end{array}
$$

Here is the Socratic tree determined by this Socratic transformation together with "indexing by boxing".

$$\vdash \boxed{\exists x(P(x) \to \neg\exists y\neg\neg\neg P(y))}\ {}^{1}_{1}$$

$$A \vdash \boxed{P(a) \to \neg\exists y\neg\neg\neg P(y)}\ {}^{2}_{1}$$

$$A, \boxed{P(a)} \vdash \boxed{\neg\exists y\neg\neg\neg P(y)}\ {}^{3}_{1}$$

a continuation of Example 33:

$$A, P(a) \vdash \boxed{\neg\neg\neg\neg P(b)}\ {}^{4}_{1}$$

$$A, P(a) \vdash \boxed{\neg\neg P(b)}\ {}^{5}_{1}$$

$$A, P(a) \vdash \boxed{P(b)}\ {}^{6}_{1}$$

$$A, \boxed{\neg(P(b) \to \neg\exists y \neg\neg\neg P(y))}, P(a) \vdash P(b)\ {}^{7}_{1}$$

$$A, \boxed{P(b)}, \boxed{\neg\neg\exists y \neg\neg\neg P(y)}, P(a) \vdash P(b)\ {}^{8}_{1}$$

□

"Ordinary" formulas and indexed formulas may occur together in the same context, thus, for clarity, we introduce the notion of *generalized formula*. By a generalized formula we mean a formula of $\mathcal{L}_{\mathsf{L}}$ or an indexed formula, as defined above. In the sequel we will use $\mathfrak{A}$ as a metavariable for generalized formulas. We introduce the notion of *generalized sequent* in a similar way, namely by '$\mathfrak{S} \vdash \mathfrak{T}$' we refer to a sequent such that $\mathfrak{S}$ and $\mathfrak{T}$ are finite (possibly empty) sequences of generalized formulas. Obviously, generalized formulas may fail to be formulas of $\mathcal{L}_{\mathsf{L}}$ and generalized sequents may fail to be sequents of $\mathcal{L}^{?}_{\vdash\mathsf{L}}$.

Negation-simplification and translation function τ

Now we deal with the problem of "translating" sequents of different "sidedness" into one format. Transferring formulas from one side of the turnstile to another introduces the problem with additional negation signs, which may not behave well in the further steps of the translation procedure. The solution of this problem is based on a simple and elegant idea which we borrow from Golińska-Pilarek and Orłowska, 2007, see also Orłowska and Golińska-Pilarek, 2011, and which is to examine two proof-formats by considering certain relation *modulo* double negation between them. In this way we get rid of all the occurrences of double negation in the process of translation. The following definition is almost a quote from (Golińska-Pilarek and Orłowska, 2007, p. 293).

Definition 50. *A formula of language $\mathcal{L}_{\mathsf{L}}$ is said to be* positive *if negation is not its principal connective. Let $n \geq 0$ and A be a positive formula. We define:*

$$\neg^0 A := A$$
$$\neg^{n+1} A := \neg(\neg^n A)$$

Let $B = \neg^n A$, where A is a positive formula. We put:

$$\mathbf{ns}(B) := \neg^{n \bmod 2} A$$

□

Thus each formula B of language $\mathcal{L}_{\mathsf{L}}$ may be represented as '$\neg^n A$', where $n \geq 0$ and A is a positive formula. Let us observe that **ns** is an operation defined on formulas which simplifies the form of a formula by deleting the "prefixes" of double negations. Thus $\mathbf{ns}(B)$ is a positive formula or a positive formula in the scope of one negation sign.

Next, we introduce the following operation called $\underline{\text{oqe}}$ from "obvious quantification equivalence", this time the idea is by Andrzej Wiśniewski (see Wiśniewski, 2006).

Definition 51. *Let A be a formula of language $\mathcal{L}_{\mathsf{FOL}}$. We set:*

$$\underline{\text{oqe}}(A) := \begin{cases} \neg^{n+1} \forall x_i B & \textit{if} \;\; A = \neg^n \exists x_i \neg B \\ \neg^{n+1} \exists x_i B & \textit{if} \;\; A = \neg^n \forall x_i \neg B \\ A & \textit{otherwise} \end{cases}$$

where $n \geq 0$. □

If we deal with language $\mathcal{L}_{\mathsf{CPL}}$, then we simply assume that $\underline{\text{oqe}}(A) = A$ for each formula A of $\mathcal{L}_{\mathsf{CPL}}$.

If a formula A of $\mathcal{L}_{\mathsf{FOL}}$ is of neither of the forms: (1) $\neg\neg B$, (2) $\neg^n \forall x_i \neg B$, (3) $\neg^n \exists x_i \neg B$, where $n \geq 0$, then we say that A is *nq-simple*, otherwise we say A is *nq-complex* ("nq" from "negation-quantifier"). The following algorithm shows how to transform any nq-complex formula into an nq-simple one. At the same time, the algorithm defines the operation **nq** on formulas (see Algorithm 10).

By '$\mathbf{nq}(A)$' we will denote the result of applying Algorithm 10 to formula A. In the case of formulas of language $\mathcal{L}_{\mathsf{CPL}}$, we simply assume that $\mathbf{nq}(A) = \mathbf{ns}(A)$.

Now we will generalize the definition of operation **nq**, so that it works on sets and sequences of generalized formulas; we will introduce another (quite standard) operation of negation on sets/sequences of generalized formulas and we will define operation τ which works on generalized sequents. If $\mathfrak{U}$ is a sequence, then the inscription '$v \in \mathfrak{U}$' means that v is a term of $\mathfrak{U}$. Let us stress that the arguments of operations $\neg$ and **nq** may be sequences or sets, but their values are always sets.

Algorithm 10: operation **nq** on formulas

Data: formula A of language $\mathcal{L}_{\mathsf{FOL}}$

Result: nq-simple formula $\mathbf{nq}(A)$ equivalent to A

$\mathbf{nq}(A) \leftarrow A$

while *A is nq-complex* **do**

  if *A is of the form $\neg^n \forall x_i \neg B$ or $\neg^n \exists x_i \neg B$* **then**

    $\mathbf{nq}(A) \leftarrow \underline{\text{oqe}}(\mathbf{nq}(A))$

  else

    $\mathbf{nq}(A) \leftarrow \mathbf{ns}(\mathbf{nq}(A))$

  end

end

Definition 52. *Let $\mathfrak{U}$ be a (possibly empty) sequence of generalized formulas or a (possibly empty) set of such formulas. Further, let $\mathfrak{S} \vdash \mathfrak{T}$ be a generalized sequent. We define sets: $\neg\mathfrak{U}$, $\mathbf{nq}(\mathfrak{U})$ and $\tau(\mathfrak{S} \vdash \mathfrak{T})$ of generalized formulas in the following way:*

1. $\neg\mathfrak{U} := \{\neg A : A \in \mathfrak{U}\} \cup \{\langle \neg A, i\rangle : \langle A, i\rangle \in \mathfrak{U}\}$
2. $\mathbf{nq}(\mathfrak{U}) := \{\mathbf{nq}(A) : A \in \mathfrak{U}\} \cup \{\langle \mathbf{nq}(A), i\rangle : \langle A, i\rangle \in \mathfrak{U}\}$
3. $\tau(\mathfrak{S} \vdash \mathfrak{T}) := \mathbf{nq}(\mathfrak{S}) \cup \mathbf{nq}(\neg\mathfrak{T})$ □

Until the end of this section we assume that Socratic transformations are annotated and that (at least some of) the formulas occurring in the transformations are indexed according to the pattern described on page 188. To avoid superfluity of notation we will use Greek letters ϕ, ψ, Φ, Ψ for both "ordinary" and generalized sequents and sequences of sequents.

Let us recall that Socratic trees are structures of the form $\langle X_{\mathbf{sG}}, R_{\mathbf{sG}}, \eta_{\mathbf{s}}\rangle$, where $\eta_{\mathbf{s}}$ is the labelling function of the tree. Recall also that:

$$\eta_{\mathbf{s}}(\langle n, i\rangle) = \phi$$

that is, sequent ϕ is the label $\eta_{\mathbf{s}}(\langle n, i\rangle)$ of node $\langle n, i\rangle$. As in the modal case, we will introduce certain changes to our trees by successive relabelling, *i.e.*, redefinitions of the labelling function. First, let $\mathbf{s}$ be a finite Socratic transformation of a question of the form $?(S \vdash T)$, and assume that $S = \langle A_1, \ldots, A_m\rangle$ and $T = \langle B_1, \ldots, B_k\rangle$ (where $k \geq 0$, $m \geq 0$ but not both $k = m = 0$). The formulas occurring in sequent $S \vdash T$ are the initial premises of transformation $\mathbf{s}$. Thus suppose that they get the following indices: $A_1 : i_1, \ldots, A_m : i_m, B_1 : i_{m+1}, \ldots, B_k : i_{m+k}$. Then for each node $\langle n, i, \phi\rangle \in X_{\mathbf{s}}$ of tree $\langle X_{\mathbf{s}}, R_{\mathbf{s}}\rangle$ we assign a label in the following way:

$$\eta^{\tau}_{\mathbf{s}}(\langle 1,1\rangle) = \{A_1 : i_1, \ldots, A_m : i_m\} \cup \neg\{B_1 : i_{m+1}, \ldots, B_k : i_{m+k}\} \quad (4.2)$$

and if $n \neq 1$,

$$\eta^{\tau}_{\mathbf{s}}(\langle n,i\rangle) = \tau(\eta_{\mathbf{s}}(\langle n,i\rangle)) \quad (4.3)$$

Obviously, the triple $\langle X_{\mathbf{sG}}, R_{\mathbf{sG}}, \eta^{\tau}_{\mathbf{s}}\rangle$ is also a labelled tree.

Lemma 14 below states that the labelling function $\eta^{\tau}_{\mathbf{s}}$ does not "spoil" the basic sequents, in that after translation they remain semantically "closed". In the formulation of Lemma 14 we use the notion of *generalized complementary formulas* understood as follows. A pair of complementary formulas is a pair of generalized complementary formulas; moreover, if B, C is a pair of complementary formulas, then each of:

- B and $C : k$,
- $B : i$ and C,
- $B : i$ and $C : k$,

where i, k are arbitrary indices, is also a pair of generalized complementary formulas.

Lemma 14. *Let* $\mathbf{Tr_G}(\mathbf{s})$ *be a tree as defined above, and assume that* ϕ *is a label assigned by* $\eta_{\mathbf{s}}$ *to node* $\langle n, i\rangle$ *of the tree. If* ϕ *is a basic sequent, then* $\eta^{\tau}_{\mathbf{s}}(\langle n,i\rangle)$ *contains a pair of complementary generalized formulas.*

Proof. The proof is by cases, we consider only one of them.

Suppose that ϕ is a sequent of the form $\mathfrak{S} \vdash B$, where B occurs in $\mathfrak{S}$. Suppose also that B is a formula of $\mathcal{L}_{\mathsf{FOL}}$ of the form '$\neg^n\forall x_i\neg^k C$', where C is positive, $n \geq 0$ and $k \geq 0$. If ϕ is a label assigned by $\eta_{\mathbf{s}}$ to $\langle 1,1\rangle$, then the set $\eta^{\tau}_{\mathbf{s}}(\langle 1,1\rangle)$ contains formulas '$\neg^n\forall x_i\neg^k C$' and '$\neg^{n+1}\forall x_i\neg^k C$'.

Suppose that ϕ is not the constituent of the first question. If $n = 0$ and $k = 0$, then $\tau(\mathfrak{S} \vdash A)$ contains $\forall x_i C$ and $\neg\forall x_i C$ (the second formula comes from the right side of the turnstile). If $n > 0$ and $k = 0$, then if n is odd, then $\mathbf{nq}(B) = \neg\forall x_i C$ and $\mathbf{nq}(\neg B) = \forall x_i C$, and if n is even, then $\mathbf{nq}(B) = \forall x_i C$ and $\mathbf{nq}(\neg B) = \neg\forall x_i C$, as required. Suppose that $k > 0$. If both n and k are even (n may be 0), then '$\neg^n\forall x_i\neg^k C$' from the left side of the turnstile, after "translation" by $\eta^{\tau}_{\mathbf{s}}$, becomes '$\forall x_i C$', and the one from the right side of the turnstile becomes '$\neg\forall x_i C$'. If n is even and k is odd, then '$\neg^n\forall x_i\neg^k C$' from the left side becomes '$\neg\exists x_i C$', and the one from the right side becomes '$\exists x_i C$'. If n is odd and k is even, then '$\neg^n\forall x_i\neg^k C$' from the left side becomes '$\neg\forall x_i C$', and the one from the right becomes '$\forall x_i C$'. Finally, if both n and k are odd, then from the left side we get '$\exists x_i C$' and from the right side '$\neg\exists x_i C$'. □

Figure 4.3 presents the Socratic tree for Example 33 with the new labels assigned by $\eta^{\tau}_{\mathbf{s}}$. As before, the boxes indicate that formulas are indexed.

Figure 4.3: Socratic tree for Example 33 labelled with $\eta^{\tau}_{\mathbf{s}}$

$$\{\boxed{\neg\exists x(P(x)\to\neg\exists y\neg\neg\neg P(y))}\}$$

$$\{\neg\exists x(P(x)\to\neg\exists y\neg\neg\neg P(y)),\ \boxed{\neg(P(a)\to\neg\exists y\neg\neg\neg P(y))}\}$$

$$\{\neg\exists x(P(x)\to\neg\exists y\neg\neg\neg P(y)),\ \boxed{P(a)},\ \boxed{\neg\forall y P(y)}\}$$

$$\{\neg\exists x(P(x)\to\neg\exists y\neg\neg\neg P(y)), P(a), \boxed{\neg P(b)}\}$$

$$\{\neg\exists x(P(x)\to\neg\exists y\neg\neg\neg P(y)), P(a), \boxed{\neg P(b)}\}$$

$$\{\neg\exists x(P(x)\to\neg\exists y\neg\neg\neg P(y)), P(a), \boxed{\neg P(b)}\}$$

$$\{\neg\exists x(P(x)\to\neg\exists y\neg\neg\neg P(y)), \boxed{\neg(P(b)\to\neg\exists y\neg\neg\neg P(y))},\ P(a), \neg P(b)\}$$

$$\{\neg\exists x(P(x)\to\neg\exists y\neg\neg\neg P(y)), \boxed{P(b)},\ \boxed{\neg\forall y P(y)},\ P(a), \neg P(b)\}$$

□

Smullyan translation

The next step is the "deletion" of all formulas except the indexed ones. Technically, this is performed by another relabelling:

Definition 53. *Let* $\mathbf{s}$ *be a finite Socratic transformation of a question based on a single sequent via the rules of* $\mathbb{E}$. *Let* $\mathbf{Tr_G}(\mathbf{s})$ *be the Socratic tree determined by* $\mathbf{s}$, *and let* $\eta^{\tau}_{\mathbf{s}}$ *be its labelling function as specified by equations* (4.2) *and* (4.3). *Then we define another labelling function, which is set by:*

$$\eta^{☺\tau}_{\mathbf{s}}(\langle n,i\rangle) := \{\langle A,k\rangle : \langle A,k\rangle \in \eta^{\tau}_{\mathbf{s}}(\langle n,i\rangle)\}$$

□

Function $\eta_{\mathbf{s}}^{\smiley\tau}$ assigns to each node of $\mathbf{Tr_G}(\mathbf{s})$ a non-empty set. This follows from the fact that the tree $\mathbf{Tr_G}(\mathbf{s})$ is built from the coordinates of the initial sequent and the sequents-conclusions. Initial premises are indexed by assumption, and moreover, each sequent-conclusion has at least one conclusion formula which is also indexed.

For simplicity, we will write $\eta^{\smiley}\langle n, i\rangle$ instead of $\eta_{\mathbf{s}}^{\smiley\tau}(\langle n, i\rangle)$ (as context makes clear what is $\mathbf{s}$ and τ). We will also make use of the following functions. If for certain $\mathbf{Tr_G}(\mathbf{s})$ and its node $\mathbf{n}$, we have $\eta^{\smiley}(\mathbf{n}) = \{A_1 : 1, \ldots, A_m : m\}$, then we set:

$$F_1(\eta^{\smiley}(\mathbf{n})) = A_1$$

$$\vdots$$

$$F_m(\eta^{\smiley}(\mathbf{n})) = A_m$$

and we will sometimes write only F_i instead of $F_i(\eta^{\smiley}(\mathbf{n}))$.

Figure 4.4 presents the tree for Example 33 with labels assigned by function $\eta_{\mathbf{s}}^{\smiley\tau}$. This time the tree looks almost like analytic tableau. ("Almost" still makes the difference, however.) We give up the boxing and write the indices explicitly.

Figure 4.4: Socratic tree for Example 33 labelled with $\eta_{\mathbf{s}}^{\smiley\tau}$

$$\begin{array}{c}
\{\neg\exists x(P(x) \to \neg\exists y\neg\neg\neg P(y)) : 1\} \\
| \\
\{\neg(P(a) \to \neg\exists y\neg\neg\neg P(y)) : 1\} \\
| \\
\{P(a) : 1, \neg\forall y P(y) : 2\} \\
| \\
\{\neg P(b) : 1\} \\
| \\
\{\neg P(b) : 1\} \\
| \\
\{\neg P(b) : 1\} \\
| \\
\{\neg(P(b) \to \neg\exists y\neg\neg\neg P(y)) : 1\} \\
| \\
\{P(b) : 1, \neg\forall y P(y) : 2\}
\end{array}$$

As we have already observed, erotetic rules generate certain proof-theoretical relations between questions, between sequents and between formulas. Socratic transformations display the relation between questions. Socratic trees are objects founded on the proof-theoretical relation between sequents. Corollary 34 above stated that the relation is codified in a Socratic tree in a proper way. Trees $\langle X_{\mathbf{sG}}, R_{\mathbf{sG}} \rangle$ together with $\eta^{☺}$ are objects funded on the relation between formulas. Once again, we need to establish that the relation is well described by our smiling trees. This is settled by Corollaries 36 and 37. At the same time, Corollary 37 gives an insight into the form of the rules of analytic tableau system which we will describe in a moment.

Corollary 36. *Let* $\mathbf{Tr} = \langle X_{\mathbf{sG}}, R_{\mathbf{sG}}, \eta_{\mathbf{s}} \rangle$ *and suppose that* $\mathbf{n}$ *is a node of* $\mathbf{Tr}$ *other than the root. If the sequent* $\eta_{\mathbf{s}}(\mathbf{n})$ *contains an indexed formula* $A : i$*, then:*

1. *A is a conclusion formula of a rule* $\mathbf{r}$ *of* $\mathbb{E}$*,*
2. *the premise formula of the rule* $\mathbf{r}$ *occurs* ***indexed*** *in a sequent which labels some of the predecessors of* $\mathbf{n}$*,*
3. *if* $\mathbf{r}$ *is a non-branching rule with two conclusion formulas (e.g.* $\mathbf{L}_\alpha$ *of* $\mathbb{E}^{\mathbf{PQ}}$*), then the second conclusion formula occurs indexed in the same sequent* $\eta_{\mathbf{s}}(\mathbf{n})$*,*
4. *if* $\mathbf{r}$ *is a branching rule, then* $\mathbf{n}$ *has a sibling* $\mathbf{m}$ *such that the second conclusion formula occurs indexed in the sequent* $\eta_{\mathbf{s}}(\mathbf{m})$*.*

Proof. Items 1., 3. and 4. are obvious. As to item 2., let us recall that Corollary 34 warrants that the immediate predecessor of an annotated conclusion sequent in the tree $\mathbf{Tr}(\mathbf{s})$ is its annotated premise sequent. An occurrence of the premise formula for our conclusion formula A occurs in this premise sequent, but may not be indexed. However, this premise formula, let us call it F, must have occurred in the sequent somehow. If it has been introduced as a conclusion formula, or if it is an initial premise, then F is indexed and it occurs in our tree in a sequent which labels a predecessor of $\mathbf{n}$, as required. If F belongs to the "context" of the premise sequent, then, again, F must have occurred in the previous sequent somehow, and thus, by induction, we arrive at the conclusion that F occurs with an index in some predecessor of $\mathbf{n}$. □

Corollary 37. *Let* $\mathbf{Tr}^{☺} = \langle X_{\mathbf{sG}}, R_{\mathbf{sG}}, \eta^{☺} \rangle$ *and suppose that* $\mathbf{n}$ *is a node of* $\mathbf{Tr}^{☺}$ *other than the root. If indexed formula* $B : i$ *occurs in the label* $\eta^{☺}(\mathbf{n})$ *of node* $\mathbf{n}$*, then:*

1. *for some formula A, which is a conclusion formula occurring in $\eta_{\mathbf{s}}(\mathbf{n})$, $B = \mathbf{nq}(A)$ or $B = \mathbf{nq}(\neg A)$,*

2. *there is an indexed formula $C : k$ occurring in the label assigned by $\eta^{☺}$ to a predecessor of $\mathbf{n}$, such that for C at least one of the following conditions holds:*

   2.1 *either C is of the form $\neg^n B$, where n is even, or C is of the form $\neg^n \triangle x_i \neg^k D$, where $\triangle \in \{\forall, \exists\}$, $n \geq 0, k \geq 0$ and $\underline{\text{oqe}}(\neg^n \triangle x_i \neg^k D) = B$,*

   2.2 *C is an α-formula and either B is $\mathbf{nq}(\alpha_1)$, and then $\mathbf{nq}(\alpha_2) : i+1$ occurs in $\eta^{☺}(\mathbf{n})$ as well, or B is $\mathbf{nq}(\alpha_2)$, and $\mathbf{nq}(\alpha_1) : i-1$ occurs in $\eta^{☺}(\mathbf{n})$ as well,*

   2.3 *C is a β formula and either B is $\mathbf{nq}(\beta_1)$, and then $\mathbf{n}$ has a sibling $\mathbf{m}$ such that $\mathbf{nq}(\beta_2) : i$ occurs in $\eta^{☺}(\mathbf{m})$, or B is $\mathbf{nq}(\beta_2)$, and then $\mathbf{n}$ has a sibling $\mathbf{m}$ such that $\mathbf{nq}(\beta_1) : i$ occurs in $\eta^{☺}(\mathbf{m})$,*

   2.4 *C is of the form $\forall x_i D$ (of the form $\neg\exists x_i D$) and B is of the form $\mathbf{nq}(D(x_i/\tau))$ (B is of the form $\mathbf{nq}(\neg D(x_i/\tau))$), where τ is an arbitrary parameter of $\mathcal{L}_{\mathsf{FOL}}$*

   2.5 *C is of the form $\exists x_i D$ (of the form $\neg\forall x_i D$) and B is of the form $\mathbf{nq}(D(x_i/\tau))$ (B is of the form $\mathbf{nq}(\neg D(x_i/\tau))$), where τ is a parameter of $\mathcal{L}_{\mathbf{FOL}}$ which is new with respect to the sequent assigned by $\eta_{\mathbf{s}}$ to the immediate predecessor of $\mathbf{n}$.*

Proof. Item 1. is obvious. Item 2. follows from Corollary 36. □

Finally, a proof must be transformed into a proof. This is stated in:

Corollary 38. *If $\mathbf{s}$ is a Socratic proof of a sequent in calculus $\mathbb{E}$, then the tree $\langle X_{\mathbf{G}}, R_{\mathbf{G}}, \eta^{☺}\rangle$ is closed, that is, for each branch of $\mathbf{Tr_G}(\mathbf{s})$, there is a formula A such that, for some index i, $A : i$ occurs in a label assigned by $\eta^{☺}$ to a node of the branch, and also, for some index j, $\neg A : j$ occurs in a label assigned by $\eta^{☺}$ to some node of the branch.*

Proof. Suppose that $\mathbf{s}$ is a Socratic proof. The case when the tree has only one node is trivial, so we assume that the leaf of the tree is not its root.

By Corollary 35, each leaf of $\mathbf{Tr}(\mathbf{s})$ is an annotated basic sequent. By Lemma 14, each of these annotated sequents is transformed by function

$\eta^{\tau}_{\mathbf{s}}$ into a set containing a pair of generalized complementary formulas. Suppose we fix on an arbitrary leaf $\mathbf{n}$ of $\mathbf{Tr_G}(\mathbf{s})$ and let us denote by F and $\neg F$ such complementary formulas. Since they occur in the set assigned by $\eta^{\tau}_{\mathbf{s}}$ to $\mathbf{n}$, the sequent assigned by $\eta_{\mathbf{s}}$ to $\mathbf{n}$ must contain either:

(a) formulas G and $\neg G$ which occur left of the turnstile, or

(b) formulas G and $\neg G$ which occur right of the turnstile, or

(c) formula G which occurs both left and right of the turnstile.

and which are (is) such that after the translation with τ, $\mathbf{nq}(G)$ and $\mathbf{nq}(\neg G)$ become identical to F and $\neg F$ (or to $\neg F$ and F, respectively). If both G and $\neg G$ (or both occurrences of G) are conclusion formulas in the sequent, then they are indexed and we are done. If not, then they must have occurred in the sequent somehow—now we reason as in Case 2 of Corollary 36. □

Reduction (node deleting)

Example 33 of a Socratic proof in $\mathbf{E^{PQ}}$ gives the idea of what needs to be deleted. Our aim in this subsection is to "reduce" the Socratic trees in a way which warrants that there are no repetitions in its labelled nodes. We will use the operation of deleting a node from a tree introduced in the previous section (see Definition 43, page 152). Below we give an algorithm which transforms labelled Socratic trees using this operation (see Algorithm 11).

A label $\lambda^{☺}\langle n, i\rangle$ of node $\mathbf{n} = \langle n, i\rangle$ is itself a set of indexed formulas. We may need to refer to the formulas ignoring the indices, therefore we set what follows. By $formulas(\mathbf{n})$ or $formulas\langle n, i\rangle$ we mean the set of formulas which occur in $\lambda^{☺}\langle n, i\rangle$. In most cases this is a singleton.

Below (Algorithm 11) we define a procedure that systematically compares the labels of successive nodes and, when it finds "repetitions", it either deletes a formula from a label (if the label contains more than one formula) or deletes the whole node of a tree. We will call a tree *reduced*, if no formula occurring in the label $\lambda^{☺}(\mathbf{n})$ of a node $\mathbf{n}$ of the tree occurs in a label assigned by $\lambda^{☺}$ to a predecessor of $\mathbf{n}$. The effect of reducing a tree $\mathbf{Tr}$ in accordance with Algorithm 11 will be written symbolically as $Rd(\mathbf{Tr})$.

Below we will use the following assignments. If $\mathbf{n} = \langle n, i\rangle$ is a node, then we refer to n as "the first coordinate of $\mathbf{n}$", to i as "the second coordinate of **n**", and we also set:

$$1^{st}(\mathbf{n}) = n$$

$$2^{nd}(\mathbf{n}) = i$$

Algorithm 11: reducing a tree

Data: a labelled tree $\langle X_{\mathbf{sG}}, R_{\mathbf{sG}}, \lambda_{\mathbf{s}}^{\smiley\tau} \rangle$
Result: a reduced tree $\mathbf{Tr} = Rd(\langle X_{\mathbf{sG}}, R_{\mathbf{sG}}, \lambda_{\mathbf{s}}^{\smiley\tau} \rangle)$

$\mathbf{Tr} \leftarrow \langle X_{\mathbf{sG}}, R_{\mathbf{sG}}, \lambda_{\mathbf{s}}^{\smiley\tau} \rangle$
$n \leftarrow$ the length of $\mathbf{s}$
$H(\langle 1,1 \rangle) \leftarrow formulas(\langle 1,1 \rangle)$
for $i = 2$ **to** n **do**
    if *there is only one node* $\mathbf{n}$ *in* $\mathbf{Tr}$ *such that* $1^{st}(\mathbf{n}) = i$ **then**
        $\mathbf{m}_1 \leftarrow \mathbf{n}$
        $k \leftarrow 1$
    else
        find the two nodes, $\mathbf{m}_1$ and $\mathbf{m}_2$, whose 1^{st} coordinate is i
        $\mathbf{m}_1 \leftarrow$ the node with the 2^{nd} coordinate of lower value
        $\mathbf{m}_2 \leftarrow$ the node with the 2^{nd} coordinate of higher value
        $k \leftarrow 2$
    end
    for $j = 1$ **to** k **do**
        $\mathbf{o} \leftarrow$ the immediate predecessor of node $\mathbf{m}_j$
        $formulas(\mathbf{m}_j) \leftarrow formulas(\mathbf{m}_j) \setminus H(\mathbf{o})$
        if $formulas(\mathbf{m}_j) = \varnothing$ **then**
            $\mathbf{Tr} \leftarrow \mathbf{Tr} \setminus \mathbf{m}_j$
        else
            $H(\mathbf{m}_j) \leftarrow formulas(\mathbf{m}_j) \cup H(\mathbf{o})$
        end
    end
end

The outer loop of Algorithm 11 is a for-loop whose counter i refers to the first coordinate of a node. We know that for each i there is one or two nodes with i as the first coordinate. The aim of the if-then-else statement in the body of the outer loop is to establish whether there is one or two such nodes and to make suitable assignments. The only purpose of introducing the for-loop with counter j is to deal with the two cases (with one and with two nodes) with the use of one structure. During the iteration of the second for-loop the label of the current node is compared with the labels of its immediate predecessors ($H(\mathbf{o})$, which may be called "history of **o**", is simply a set of formulas from the labels of the predecessors of $\mathbf{m}_j$). If there are any formulas in $\mathbf{m}_j$ which has occurred above in the tree, then the repetitions are removed by assigning the subtraction $formulas(\mathbf{m}_j) \setminus H(\mathbf{o})$ to $formulas(\mathbf{m}_j)$. If this set is empty, then it means that each formula occurring in the label of the current node occurs also in the label of some of its predecessors. In this case the current node is deleted from the tree. If the set is non-empty, then the structure of the tree is not altered. The algorithm terminates after comparing the label(s) of the node(s) extracted from the last question of Socratic transformation with the history(ies) of its immediate predecessor(s).

If we apply Algorithm 11 to the Socratic tree from Example 33, we will arive at the following structure:

$$
\begin{array}{c}
\{\neg\exists x(P(x) \to \neg\exists y\neg\neg\neg P(y)) : 1\} \\
| \\
\{\neg(P(a) \to \neg\exists y\neg\neg\neg P(y)) : 1\} \\
| \\
\{P(a) : 1, \neg\forall y P(y) : 2\} \\
| \\
\{\neg P(b) : 1\} \\
| \\
\{\neg(P(b) \to \neg\exists y\neg\neg\neg P(y)) : 1\} \\
| \\
\{P(b) : 1\}
\end{array}
$$

Extension (duplication of elements)

Our trees from the last subsection look quite good, but they still have nodes which bear more than one formula, and this is a departure from the tableau-format which we have to deal with now.

Here is the remedy. Suppose **n** is a node of a Socratic tree. If the node bears at least two different formulas, that is, the set $formulas(\mathbf{n})$ is not a singleton, then the node should be "duplicated" (or "multiplicated") so that each node bears exactly one formula. If the set $formulas(\mathbf{n})$ is a singleton, then the node (and the tree) should be left unchanged.[9]

[9] It is possible that the set is a singleton, though the label of node **n** contains two indexed formulas. E.g. when formula $p \wedge p$ is a premise formula on the left side of

Here we describe the process of extending trees by duplicating some of their nodes. The pivotal operation is that of duplicating node $\mathbf{n}$ of tree $\mathbf{Tr}$ by replacing it with nodes $\mathbf{n_1}$ and $\mathbf{n_2}$. The result of this operation will be written symbolically as: $\mathbf{Tr}[\mathbf{n}/\mathbf{n_1}, \mathbf{n_2}]$, and it is defined as follows:

Definition 54. *Let* $\mathbf{n}$ *be a node of a tree* $\mathbf{Tr} = \langle X, R\rangle$*, and let* $\mathbf{n}_1$*,* $\mathbf{n}_2$ *be such that* $\mathbf{n_1} \notin X \setminus \{\mathbf{n}\}, \mathbf{n_2} \notin X \setminus \{\mathbf{n}\}$ *and* $\mathbf{n}_1 \neq \mathbf{n}_2$*. We define:*

$$X^* = (X \setminus \{\mathbf{n}\}) \cup \{\mathbf{n_1}, \mathbf{n_2}\}$$

As to relation R*, first take:*

$$R_1 = R \mid_{X \setminus \{\mathbf{n}\}}$$

If node $\mathbf{n}$ *is not the root of the tree, then let* $\mathbf{p}$ *be the immediate predecessor of* $\mathbf{n}$*. We set:*

$$R_2 = R_1 \cup \{\langle \mathbf{p}, \mathbf{n_1}\rangle\}$$

If $\mathbf{n}$ *is the root, then* $R_2 = R_1$*. Further, if* $\mathbf{n}$ *is not a leaf of the tree, then let* $\mathbf{s_i}$*, where* $i = 1$ *or* $i \in \{1, 2\}$*, be its immediate successor(s). In the case when* $i = 1$*, we add:*

$$R_3 = R_2 \cup \{\langle \mathbf{n_1}, \mathbf{n_2}\rangle, \langle \mathbf{n_2}, \mathbf{s_1}\rangle\}$$

and if $i \in \{1, 2\}$*, then we add:*

$$R_3 = R_2 \cup \{\langle \mathbf{n_1}, \mathbf{n_2}\rangle, \langle \mathbf{n_2}, \mathbf{s_1}\rangle, \langle \mathbf{n_2}, \mathbf{s_2}\rangle\}$$

and we set $R_3 = R_2$*, if* $\mathbf{n}$ *is a leaf. Finally, let* R^* *be the transitive closure of* R_3*. Then:*

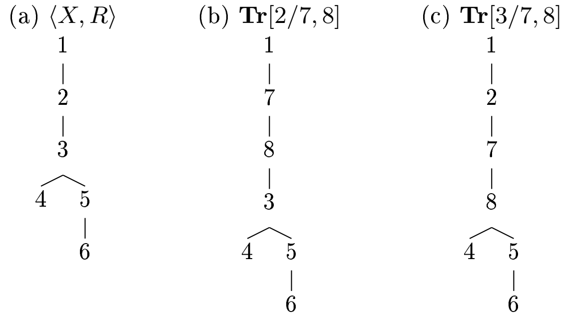
$$\mathbf{Tr}[\mathbf{n}/\mathbf{n_1}, \mathbf{n_2}] = \langle X^*, R^*\rangle$$

□

Example 34. $\mathbf{Tr} = \langle X, R\rangle$ *is such that* $X = \{1, 2, 3, 4, 5, 6\}$ *and* R *is the transitive closure of* $\{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 4\rangle, \langle 3, 5\rangle, \langle 5, 6\rangle\}$*. Below we present the tree (a), and the result of (b) duplicating node* 2 *by replacing it with* 7 *and* 8*, (c) duplicating* 3 *by replacing it with* 7 *and* 8*.*

---

the turnstile. This should not worry us, as there is nothing to be lost if the label of $\mathbf{n}$ is replaced with single occurrence of p.



□

Corollary 39. *If* $\mathbf{Tr} = \langle X, R\rangle$ *is a tree,* $\mathbf{n} \in X$*,* $\mathbf{n_1} \notin X \setminus \{\mathbf{n}\}, \mathbf{n_2} \notin X \setminus \{\mathbf{n}\}$ *and* $\mathbf{n}_1 \neq \mathbf{n}_2$*, then the structure* $\mathbf{Tr}[\mathbf{n}/\mathbf{n}_1, \mathbf{n}_2]$ *is also a tree.*

Before we proceed, we need to establish few simple facts.

FACT 1. *Let* $\mathbf{n} = \langle n, i\rangle$*. If the set* $formulas(\mathbf{n})$ *is not a singleton, then there are exactly two possibilities. First,* $\mathbf{n}$ *is the root of the tree and* $formulas(\mathbf{n})$ *is the set of the initial premises of the transformation. Second, sequent with coordinates* $\langle n, i\rangle$ *is a conclusion-sequent derived by a non-branching rule which introduced two conclusion formulas.*

FACT 2. *In* $Rd(\mathbf{Tr_G}(\mathbf{s}))$ *there must be a node whose first coordinate is 1.*

Proof. The reason is that the root can not be deleted by Algorithm 11. □

FACT 3. *As long as we assume that* $\mathbf{s}$ *is a Socratic transformation of a question based on a single sequent, each node* $\langle n, i\rangle$ *of* $\mathbf{Tr_G}(\mathbf{s})$ *has the following property:* $n \geq i$*.*

Proof. We start with the single node $\langle 1, 1\rangle$ taken from the first question of $\mathbf{s}$, thus the inequality is true for $n = 1$. Further, even if only branching rules were applied to questions of $\mathbf{s}$, the number i of constituents of n-th question of $\mathbf{s}$ could not grow faster than the number of questions, that is, n. □

As in the previous subsection, we give an algorithm which produces an extended tree. If $\mathbf{Tr}$ is a tree, then the result of applying Algorithm 12 to $\mathbf{Tr}$ will be written symbolically as $Ext(\mathbf{Tr})$.

Algorithm 12: extending a tree by duplicating its nodes

Data: a reduced labelled tree $Rd(\mathbf{Tr_G}(\mathbf{s}))$ together with $formulas$
Result: the extended tree $\mathbf{Tr} = Ext(Rd(\mathbf{Tr_G}(\mathbf{s})))$ with function η_{AT}

$\mathbf{Tr} \leftarrow Rd(\mathbf{Tr_G}(\mathbf{s}))$
$n \leftarrow$ the length of the initial Socratic transformation $\mathbf{s}$
$m \leftarrow$ the number of formulas in $formulas\langle 1,1\rangle$
for $j = 2$ **to** m **do**
    $\mathbf{Tr} \leftarrow \mathbf{Tr}[\langle 1, j-1\rangle / \langle 1, j-1\rangle, \langle 1, j\rangle]$
    $\eta_{AT}\langle 1, j-1\rangle \leftarrow F_{j-1}$; /* F_{j-1} is $F_{j-1}(\eta^{\smiley}\langle 1,1\rangle)$ */
end
$\eta_{AT}\langle 1, m\rangle \leftarrow F_m$; /* as above */
for $i = 2$ **to** n **do**
    if *there is a node whose* 1^{st} *coordinate is* i **then**
        if *there is exactly one node* $\mathbf{n}$ *in* $\mathbf{Tr}$ *such that* $1^{st}(\mathbf{n}) = i$ **then**
            $\mathbf{m}_1 \leftarrow \mathbf{n}$
            $j_1 \leftarrow 2^{nd}(\mathbf{m}_1)$
            $k \leftarrow 1$
        else
            find the two nodes whose 1^{st} coordinate is i
            $\mathbf{m}_1 \leftarrow$ the node with the 2^{nd} coordinate of lower value
            $j_1 \leftarrow 2^{nd}(\mathbf{m}_1)$
            $\mathbf{m}_2 \leftarrow$ the node with the 2^{nd} coordinate of higher value
            $j_2 \leftarrow 2^{nd}(\mathbf{m}_2)$
            $k \leftarrow 2$
        end
        for $l = 1$ **to** k **do**
            ; /* the current node has the form $\mathbf{m}_l = \langle i, j_l\rangle$; recall that $j_l \leq i$ */
            if *the set* $formulas(\mathbf{m}_l)$ *is a singleton* **then**
                $\eta_{AT}\langle i, j_l\rangle \leftarrow F_1$; /* F_1 is $F_1(\eta^{\smiley}\langle i, j_l\rangle)$ */
            else
                ; /* observe that in this situation $\eta^{\smiley}(\mathbf{m}_l) = \{F_1 : 1, F_2 : 2\}$ */
                $\mathbf{Tr} \leftarrow \mathbf{Tr}[\langle i, j_l\rangle / \langle i, j_l\rangle, \langle i, j_l + i\rangle]$
                $\eta_{AT}\langle i, j_l\rangle \leftarrow F_1$; /* F_1 is $F_1(\eta^{\smiley}\langle i, j_l\rangle)$ */
                $\eta_{AT}\langle i, j_l + 1\rangle \leftarrow F_2$; /* F_2 is $F_2(\eta^{\smiley}\langle i, j_l\rangle)$ */
            end
        end
    end
end

As to the properties of Algorithm 12, let us observe what follows. The function $formulas$ (defined on the labels assigned by $\eta^{\smiley}$ to the nodes of the tree) which is passed as data, may have been affected by Algorithm 11. The first for-loop of Algorithm 11 deals with the initial premises of Socratic transformation **s**. Let us observe that at the beginning of each iteration of this for-loop the tree **Tr** contains node $\langle 1, j-1\rangle$, but does not contain node $\langle 1, j\rangle$. This warrants that the operation $\mathbf{Tr}[\langle 1, j-1\rangle/\langle 1, j-1\rangle, \langle 1, j\rangle]$ can be performed correctly. If $m = 1$, then only the assignment in line 8 is performed. AT stands for "analytic tableau", as this is the last labelling function we need to define on our way to "real" analytic tableaux. In the body of the second for-loop there is an if-then statement (lines 10–32) which is to check whether there is a node whose 1^{st} coordinate is i (recall that the tree is already reduced, so some of the nodes may be missing). If the condition is satisfied, then the if-then-else statement (lines 11–22) settles whether there is one or two such nodes, and makes some suitable assignments. Once again, the for-loop in lines 23–31 is to deal with the two cases within one structure. The "real work" is done by the if-then-else statement in lines 24–30. The number of distinct formulas in the current node is settled, and if there are two such formulas, then the node $\langle i, j_l\rangle$ is duplicated, so each of $\langle 1, j_l\rangle$, $\langle 1, j_l+i\rangle$ bears exactly one formula. Fact 3 warrants that $\langle 1, j_l+i\rangle$ is not in our **Tr**, thus the operation $\mathbf{Tr}[\langle i, j_l\rangle/\langle i, j_l\rangle, \langle i, j_l+i\rangle]$ can be performed correctly. Also the values of η_{AT} are assigned here.

Below we present the final result produced by Algorithm 12 in the case of Example 33.

Figure 4.5: Example 33: the final result

$$\neg\exists x(P(x) \to \neg\exists y\neg\neg\neg P(y))$$
$$|$$
$$\neg(P(a) \to \neg\exists y\neg\neg\neg P(y))$$
$$|$$
$$P(a)$$
$$|$$
$$\neg\forall y P(y)$$
$$|$$
$$\neg P(b)$$
$$|$$
$$\neg(P(b) \to \neg\exists y\neg\neg\neg P(y))$$
$$|$$
$$P(b)$$

4.6. On analytic tableaux

In this section we present the original Smullyan's system $\mathbb{S}$ of analytic tableaux. Then we present certain version $\mathbb{S}^*$ of this system. We show that $\mathbb{S}^*$ is sound by proving that its rules are, in a certain sense, admissible in $\mathbb{S}$. Finally, we prove that if $\mathbf{s}$ is a Socratic proof in $\mathbb{E}$, then $Ext(Rd(\mathbf{Tr_G}(\mathbf{s})))$ is a closed tableau—that is, a proof—in system $\mathbb{S}^*$, which also shows—by completeness of $\mathbb{E}$—that $\mathbb{S}^*$ is complete.

Following Smullyan we define the rules for CPL (Smullyan, 1968, p. 22) and for FOL (Smullyan, 1968, pp. 53-54) in a brilliantly simple way presented in Table 4.12.

Table 4.12: The rules of Smullyan's system

A-rule:	B-rule:	C-rule:	D-rule:
$\dfrac{\alpha}{\alpha_1}$	$\dfrac{\beta}{\beta_1 \mid \beta_2}$	$\dfrac{\gamma}{\gamma(a)}$	$\dfrac{\delta}{\delta(a)}$
α_2			

Analytic tableaux for CPL require only rules A and B. In the case of C-rule, 'a' is *any* parameter, and in the case of D-rule it must be *new* to a branch. System $\mathbb{S}$ for CPL, which will be called $\mathbb{S}_{\mathsf{CPL}}$, is sound and complete with respect to the standard semantics for CPL (based on the notion of Boolean valuation). System $\mathbb{S}$ for FOL, which will be called $\mathbb{S}_{\mathsf{FOL}}$, is sound and complete with respect to the standard model-theoretical semantics for FOL.

Brilliant as it is, this style of defining the rules of a system is not in accordance with our presentation, as it assumes that, first, doubly negated formulas are of α type, and, second, the C-, D-rules cover the cases of both vacuous and "genuine" quantification. If we wish to treat this matters as we did before, we need to stipulate that the case of doubly negated formulas is not accounted for by the rules from Table 4.12, and that quantification in γ's and δ's is not vacuous, that is, γ is a formula of the form $\forall x_i A$ or $\neg\exists x_i A$, and δ is of the form $\exists x_i A$ or $\neg\forall x_i A$, where x_i is free in A. Then we add to the rules distinguished in Table 4.12 the K-rule presented in Table 4.13.

Table 4.13 presents also three other rules which will be considered in a moment. Symbols 'κ' and 'κ^*' used in the table refer to notation

Table 4.13: Some further rules of tableau construction

K-rule:	N-rule:	O-rule:	P-rule:
$\dfrac{\kappa}{\kappa^*}$	$\dfrac{A}{\mathbf{ns}(A)}$	$\dfrac{A}{\underline{\text{oq}}\text{e}(A)}$	$\dfrac{A}{\mathbf{nq}(A)}$

defined in Table 2.2 on page 41 (calculus $\mathbb{E}^{\mathbf{PQ}}$). They cover the cases of double negation and empty quantification.

Now we may say that by system $\mathbb{S}_{\mathsf{CPL}}$ we mean the tableau system consisting of the rules A, B and K. Rules A, B, C, D and K create system $\mathbb{S}_{\mathsf{FOL}}$, which is the Smullyan's system of analytic tableaux for FOL. It is presented again in Table 4.15 and 4.14 (the full system and its propositional part) together with systems $\mathbb{S}^*_{\mathsf{CPL}}$ and $\mathbb{S}^*_{\mathsf{FOL}}$ for, respectively, CPL and FOL.

Table 4.14: Systems $\mathbb{S}_{\mathsf{CPL}}$ and $\mathbb{S}^*_{\mathsf{CPL}}$

system $\mathbb{S}_{\mathsf{CPL}}$	A-rule: $\dfrac{\alpha}{\alpha_1 \quad \alpha_2}$	B-rule $\dfrac{\beta}{\beta_1 \mid \beta_2}$	K-rule $\dfrac{\kappa}{\kappa^{**}}$
system $\mathbb{S}^*_{\mathsf{CPL}}$	**ns**A-rule: $\dfrac{\alpha}{\mathbf{ns}(\alpha_1) \quad \mathbf{ns}(\alpha_2)}$	**ns**B-rule: $\dfrac{\beta}{\mathbf{ns}(\beta_1) \mid \mathbf{ns}(\beta_2)}$	N-rule: $\dfrac{A}{\mathbf{ns}(A)}$

Let us recall that in the case of CPL κ refers only to formulas of the form '$\neg\neg A$', and in the case of FOL it covers also the case of vacuous quantification. In calculus $\mathbb{S}^*_{\mathsf{FOL}}$ it is enough if κ refers *only* to vacuous quantification, as the case of double negation is covered by P-rule.

Let X stand for a set of formulas of language $\mathcal{L}_{\mathsf{L}}$, and let A be a single formula of this language. The inscription:

$$X \vdash_{\mathbb{S}_{\mathsf{L}}} A$$

means that formula A is derivable from the set X in system $\mathbb{S}_{\mathsf{L}}$ (that is, there is a closed tableau for formulas from $X \cup \{\neg A\}$).

Table 4.15: Systems $\mathbb{S}_{\mathsf{FOL}}$ and $\mathbb{S}^*_{\mathsf{FOL}}$

system $\mathbb{S}_{\mathsf{FOL}}$	A-rule: $\dfrac{\alpha}{\alpha_1 \\ \alpha_2}$	B-rule: $\dfrac{\beta}{\beta_1 \mid \beta_2}$	K-rule: $\dfrac{\kappa}{\kappa^{**}}$	C-rule: $\dfrac{\gamma}{\gamma(a)}$	D-rule: $\dfrac{\delta}{\delta(a)}$
system $\mathbb{S}^*_{\mathsf{FOL}}$	**nq**A-rule: $\dfrac{\alpha}{\mathbf{nq}(\alpha_1) \\ \mathbf{nq}(\alpha_2)}$	**nq**B-rule: $\dfrac{\beta}{\mathbf{nq}(\beta_1) \mid \mathbf{nq}(\beta_2)}$			P-rule: $\dfrac{A}{\mathbf{nq}(A)}$
	K-rule: $\dfrac{\kappa}{\kappa^{**}}$	**nq**C-rule: $\dfrac{\gamma}{\mathbf{nq}(\gamma(a))}$			**nq**D-rule: $\dfrac{\delta}{\mathbf{nq}(\delta(a))}$

Let $\mathbb{R}$ stand for a set of rules of tableau construction for logic L. Then by:

$$X \vdash_{\mathbb{S}_{\mathsf{L}} \cup \mathbb{R}} A$$

we mean that A is derivable from X in system $\mathbb{S}_{\mathsf{L}}$ extended with the rules from $\mathbb{R}$.

Soundness of systems $\mathbb{S}^*_{\mathsf{L}}$ can be shown in a direct way, by proving that the rules of $\mathbb{S}^*_{\mathsf{L}}$ preserve satisfiability of a tableau. The proof of this fact is straightforward. We have chosen to consider another—syntactical—proof, however.

We would like to understand the notion of admissibility as in Negri and von Plato, 2001, p. 20—recall the quote on page 60. In this quote "system" refers to a sequent calculus, and 'S_1', ..., 'S_n', 'S' are premises and/or conclusions of the rules of the system, that is, sequents. However, rules of tableau system are rules for tableau construction, which means that, strictly speaking, the "premisses" and "conclusions" are tableaux rather than formulas. But this notion of admissibility would be too strong. Thus instead, we will show that the rules of $\mathbb{S}^*_{\mathsf{L}}$ are admissible in $\mathbb{S}_{\mathsf{L}}$ in a certain quite weak sense, which is explicated in Lemmas 15-21. It is especially weak in the case of formulas with quantifiers (see Lemma 19), but it suits our needs. On the other hand, we may also say that the Lemmas (and their proofs) state that the rules of system $\mathbb{S}^*_{\mathsf{L}}$ "simulate" the applications of the rules of $\mathbb{S}_{\mathsf{L}}$ and "abbreviate" them—which is exactly the meaning we intuitively assign to the notion of *derivability* of rules. Hence it follows that the rules of $\mathbb{S}^*_{\mathsf{L}}$ are admissible in $\mathbb{S}_{\mathsf{L}}$, because they are derivable in $\mathbb{S}_{\mathsf{L}}$.

Lemma 15. *If a branch $\mathcal{B}$ of a tableau may be extended by applying* N*-rule of system $\mathbb{S}^*_{\mathsf{CPL}}$ to the effect of adding (a node bearing) formula* $\mathbf{ns}(A)$ *to branch $\mathcal{B}$, then it is possible to extend branch $\mathcal{B}$ by applying* K*-rule of system $\mathbb{S}_{\mathsf{CPL}}$ to the effect of adding, int.al., formula* $\mathbf{ns}(A)$ *to branch $\mathcal{B}$.*

Proof. The effect of adding $\mathbf{ns}(A)$ to a branch may be obtained in system $\mathbb{S}_{\mathsf{CPL}}$ by applying K-rule $\lfloor \frac{n}{2} \rfloor$ times. □

Lemma 16. *If a branch $\mathcal{B}$ of a tableau may be extended by applying* $\mathbf{ns}$A*-rule of system $\mathbb{S}^*_{\mathsf{CPL}}$ to the effect of adding (nodes bearing) formulas* $\mathbf{ns}(\alpha_1)$ *and* $\mathbf{ns}(\alpha_2)$ *to branch $\mathcal{B}$, then it is possible to extend branch $\mathcal{B}$ by applying* A*-rule and* K*-rule of system $\mathbb{S}_{\mathsf{CPL}}$ to the effect of adding, int.al., formulas* $\mathbf{ns}(\alpha_1)$, $\mathbf{ns}(\alpha_2)$ *to branch $\mathcal{B}$.*

Proof. We apply A-rule and then, if necessary, we apply K-rule to α_1 and α_2 the appropriate number of times. □

Lemma 17. *If a branch $\mathcal{B}$ of a tableau may be "branched" by applying* $\mathbf{ns}$B*-rule of system $\mathbb{S}^*_{\mathsf{CPL}}$ to the effect of adding a (node bearing) formula* $\mathbf{ns}(\beta_1)$ *to one of the two new branches, and (a node bearing)* $\mathbf{ns}(\beta_2)$ *to the other of the two new branches, then it is possible to "branch" the branch $\mathcal{B}$ by applying* B*-rule of system $\mathbb{S}_{\mathsf{CPL}}$ and then to apply* K*-rule of the system to the effect of adding, int.al., formulas* $\mathbf{ns}(\beta_1)$ *and* $\mathbf{ns}(\beta_2)$ *to the two branches resulting from $\mathcal{B}$.*

Proof. We apply B-rule and then K-rule the appropriate number of times. □

Lemmas 15-17 lead to the following conclusion:

Theorem 27. *If* $X \vdash_{\mathbb{S}_{\mathsf{CPL}} \cup \{\mathrm{N}, \mathbf{ns}\mathrm{A}, \mathbf{ns}\mathrm{B}\}} A$*, then* $X \vdash_{\mathbb{S}_{\mathsf{CPL}}} A$*.*

Thus adding all the rules of $\mathbb{S}^*_{\mathsf{CPL}}$ to system $\mathbb{S}_{\mathsf{CPL}}$ does not change the strength of the system. We have already observed—analysing the properties of derivability and admissibility of erotetic rules—that this is another intuitive account of both the admissibility and the derivability of rules.

Now we will show that the same holds for the systems for FOL. First, as before, and by the same argument, we state that:

Lemma 18. *If a branch $\mathcal{B}$ of a tableau may be extended by applying* N*-rule to the effect of adding (a node bearing) formula* $\mathbf{ns}(A)$ *to branch $\mathcal{B}$, then it is possible to extend branch $\mathcal{B}$ by applying* K*-rule of system $\mathbb{S}_{\mathsf{FOL}}$ to the effect of adding, int.al., formula* $\mathbf{ns}(A)$ *to branch $\mathcal{B}$.*

We may also prove:

Lemma 19. *If a branch $\mathcal{B}$ of a tableau may be extended by applying* O*-rule and* N*-rule to the effect of adding formula* $\mathbf{ns}(\underline{\text{oqe}}(A))$ *to branch $\mathcal{B}$, then it is possible to extend branch $\mathcal{B}$ by applying* K*-rule of system $\mathbb{S}_{\mathsf{FOL}}$ to the effect of adding, int.al., formula F to branch $\mathcal{B}$, where F is such that* $\mathbf{ns}(\underline{\text{oqe}}(A))$ *and F have instances which are the same modulo double negation.*

Proof. The reasoning is trivial if A is **nq**-simple. If A is of the form $\neg^n B$, where B is positive but is not a quantifier formula, then we have the case described in Lemma 18.

Suppose that our premise formula A is of the form $\neg^n \exists x_i \neg B$, then $\underline{\text{oqe}}(A)$ is of the form $\neg^{n+1} \forall x_i B$. If n is even, then $\mathbf{ns}(\underline{\text{oqe}}(A))$ is $\neg \forall x_i B$, and its instance is of the form $\neg B(a)$, where a is new to the branch. If n is odd, then $\mathbf{ns}(\underline{\text{oqe}}(A))$ is $\forall x_i B$, and its instance is of the form $B(a)$, where a is arbitrary.

In system $\mathbb{S}_{\mathsf{FOL}}$ we apply K-rule to A $\frac{n}{2}$ times—if n is even, and $\frac{n-1}{2}$ times, if n is odd. In the first case (n even) we obtain $\exists x_i \neg B$, then we apply D-rule and get $\neg B(a)$, where a is new to the branch (as required). In the second case (n odd) we obtain $\neg \exists x_i \neg B$, apply C-rule, and get $\neg\neg B(a)$, where a is arbitrary. Now we need one more application of K-rule in order to get the required formula.

The reasoning is analogous if A is of the form $\neg^n \forall x_i \neg B$. □

If a formula is provable in a system of analytic tableaux for FOL, then it has a proof whose branches close on non-quantifier formulas. From this fact and from the previous lemma we have:

Lemma 20. *If* $X \vdash_{\mathbb{S}_{\mathsf{FOL}} \cup \{\mathrm{N},\mathrm{O}\}} A$*, then* $X \vdash_{\mathbb{S}_{\mathsf{FOL}}} A$*.*

Also from the same fact, and by arguments similar to that presented in the proofs of Lemmas 15-19, it follows that:

Lemma 21. *1. If* $X \vdash_{\mathbb{S}_{\mathsf{FOL}} \cup \{\mathrm{P}\}} A$*, then* $X \vdash_{\mathbb{S}_{\mathsf{FOL}} \cup \{\mathrm{N},\mathrm{O}\}} A$*.*

2. If $X \vdash_{\mathbb{S}_{\mathsf{FOL}} \cup \{\mathbf{nq}\mathrm{A}\}} A$*, then* $X \vdash_{\mathbb{S}_{\mathsf{FOL}} \cup \{\mathrm{P}\}} A$.

3. If $X \vdash_{\mathbb{S}_{\mathsf{FOL}} \cup \{\mathbf{nq}\mathrm{B}\}} A$*, then* $X \vdash_{\mathbb{S}_{\mathsf{FOL}} \cup \{\mathrm{P}\}} A$.

4. If $X \vdash_{\mathbb{S}_{\mathsf{FOL}} \cup \{\mathbf{nq}\mathrm{C}\}} A$*, then* $X \vdash_{\mathbb{S}_{\mathsf{FOL}} \cup \{\mathrm{P}\}} A$.

5. If $X \vdash_{\mathbb{S}_{\mathsf{FOL}} \cup \{\mathbf{nq}\mathrm{D}\}} A$*, then* $X \vdash_{\mathbb{S}_{\mathsf{FOL}} \cup \{\mathrm{P}\}} A$.

Hence it follows that:

Theorem 28. *If* $X \vdash_{\mathbb{S}_{\mathsf{FOL}} \cup\, \mathbb{S}^*_{\mathsf{FOL}}} A$*, then* $X \vdash_{\mathbb{S}_{\mathsf{FOL}}} A$.

And finally, from Theorems 27 and 28, and from soundness of $\mathbb{S}_{\mathsf{L}}$, it follows that:

Theorem 29. *System* $\mathbb{S}^*_{\mathsf{L}}$ *is sound with respect to the semantics for* L.

Now we proceed with the proof of completeness. Let $\mathbf{s}$ be a Socratic transformation of a question based on a single sequent via the rules of $\mathbb{E}$. The structure $Ext(Rd(\mathbf{Tr_G}(\mathbf{s})))$ will be written symbolically as $AT_{\mathbf{s}}$.

Theorem 30. *If* $\mathbf{s}$ *is a finite Socratic transformation of a simple question via the rules of* $\mathbf{E^{PQ}}$, $\mathbf{E^{RPQ}}$ *or* $\mathbf{E^{LPQ}}$ (*or via the rules of the propositional part of one of this erotetic calculi*)*, then the structure* $AT_{\mathbf{s}}$ *is a tableau in system* $\mathbb{S}^*_{\mathsf{FOL}}$ (*or in* $\mathbb{S}^*_{\mathsf{CPL}}$).

Proof. The fact that the structure $AT_{\mathbf{s}}$ is a tree follows from Theorem 15, Corollaries 27 and 39. The fact that the formulas assigned to the nodes by the labelling function λ_{AT} are "obtained" by the rules of $\mathbb{S}^*_{\mathsf{L}}$ from some formulas lying higher on the same branches, follows from Corollary 37. We skip the remaining details. □

Theorem 31. *Let* $S \vdash T$ *be a sequent of language* $\mathcal{L}_{\mathsf{L}}$*. If* $\mathbf{s}$ *is a Socratic proof of sequent* $S \vdash T$ *in calculus* $\mathbf{E^{PQ}}$, $\mathbf{E^{RPQ}}$ *or* $\mathbf{E^{LPQ}}$ (*or in a propositional part of one of this erotetic calculi*)*, then the tree* $AT_{\mathbf{s}}$ *is a closed tableau in system* $\mathbb{S}^*_{\mathsf{FOL}}$ (*or in* $\mathbb{S}^*_{\mathsf{CPL}}$) *for the set of formulas occurring in* S *and* $\neg T$.

Proof. From Theorem 30 it follows that $AT_{\mathbf{s}}$ is a tableau in $\mathbb{S}^*$. The first nodes of the tableau (beginning from the root) are labelled with the formulas occurring in S and/or with the negations of the formulas from T—this follows from the way the first labelling function $\lambda^{\mathcal{T}}_{\mathbf{s}}$ is defined (see equation (4.2)). Finally, the fact that $AT_{\mathbf{s}}$ is closed follows from Corollary 38. □

We close this section with the following:

Theorem 32. *If* $\mathbf{s}$ *is a Socratic proof of sequent* $\vdash A$ *in calculus* $\mathbf{E^{PQ}}$*, then the tree* $Ext(Rd(\mathbf{Tr_G}(\mathbf{s})))$ *is a closed tableau for formula* $\neg A$*, that is, a proof of* A *in* $\mathbb{S}^*_{\mathsf{FOL}}$.

Proof. From Theorem 31. □

4.7. A note on dual erotetic calculi for CPL

In this section we briefly describe how the possibility of using different sequents may be utilized to improve the process of proof-search. This time, by "different sequents" we do not mean the diversity of left-, both- and right-sided sequents, but the so-called *reversed sequents*, which are dual in nature to the "canonical sequents" used so far in this book.

So we shall consider CPL only and we will focus on right-sided sequents. The choice of "sidedness" is inessential to the result presented here.

We will concentrate upon an erotetic language containing the symbol '$\dashv$' instead of '$\vdash$'. The language is labelled with '$\mathcal{L}^?_{\dashv\mathsf{L}}$', where $\mathcal{L}_{\mathsf{L}}$ is the language of the underlying classical logic. The vocabulary of $\mathcal{L}^?_{\dashv\mathsf{L}}$ includes the vocabulary of $\mathcal{L}_{\mathsf{L}}$, the comma, the semicolon and the following signs: ?, $\dashv$, ng ($\mathcal{L}^?_{\dashv\mathsf{L}}$-negation), $\sqcup$ ($\mathcal{L}^?_{\dashv\mathsf{L}}$-disjunction). As before, the language $\mathcal{L}^?_{\dashv\mathsf{L}}$ has two categories of well-formed expressions: declarative well-formed formulas (d-wffs for short) and erotetic formulas, that is, questions. As we said, we take right-sided sequents only, thus an *atomic d-wff* of $\mathcal{L}^?_{\dashv\mathsf{L}}$ is an expression of the form:

$$\dashv S \tag{4.4}$$

where $S = \langle A_1, ..., A_n \rangle$ is a finite (possibly empty) sequence of formulas of $\mathcal{L}_{\mathsf{L}}$. Expressions of the above form are called *reversed sequents* or simply *sequents of* $\mathcal{L}^?_{\dashv\mathsf{L}}$. If S is empty (which is permitted), then we display the right-sided reversed sequent (4.4) as follows:

$$\dashv \varnothing \tag{4.5}$$

Compound d-wffs of $\mathcal{L}^?_{\dashv\mathsf{L}}$ are built from atomic d-wffs of $\mathcal{L}^?_{\dashv\mathsf{L}}$ by means of $\sqcup$ and ng. Questions of $\mathcal{L}^?_{\dashv\mathsf{L}}$ are of the following form:

$$?(\Phi) \tag{4.6}$$

where Φ is a finite and non-empty sequence of reversed sequents of $\mathcal{L}^{?}_{\dashv\mathsf{L}}$.

The general idea is that questions of language $\mathcal{L}^{?}_{\dashv\mathsf{L}}$ are questions concerning *refutability*, semantically understood as *falsifiablity in a model*. In the case of CPL, a question of the form:

$$?(\dashv A^1_1, \ldots, A^1_{n_1} \ ; \ \ldots \ ; \ \dashv A^k_1, \ldots, A^k_{n_k}) \tag{4.7}$$

can be read: "Is it the case that at least one of formulas $A^1_1, \ldots, A^1_{n_1}$ is false (refuted) under a valuation V **and at the same time** ... **and at the same time** at least one of formulas $A^k_1, \ldots, A^k_{n_k}$ is false under the same valuation V". To put it in other words, the question asks if it is the case that V **does not satisfy** any of the sets $\{A^1_1, \ldots, A^1_{n_1}\}$, ..., $\{A^k_1, \ldots, A^k_{n_k}\}$. Consequently, a question of the form:

$$?(\dashv A)$$

asks whether A is false under (some) V, that is, whether A is refutable.

The notion of reversed sequent has been introduced in the paper (Chlebowski and Leszczyńska-Jasion, 2015) where the language with reversed sequents was used to simulate resolution inside an erotetic calculus. The erotetic resolution systems in Chlebowski and Leszczyńska-Jasion, 2015 are presented for CPL, the propositional parts of paraconsistent logics CLuN and CLuNs, and the minimal Logic of Formal Inconsistency called mbC. As we have remarked above, (Chlebowski, 2018) contains a full exposition of both canonical and dual erotetic calculi for the Classical Logic. Some applications of dual erotetic calculi have been studied also in Chlebowski et al., 2017.

The rules of dual erotetic calculus $\mathbb{E}^{\mathsf{CPL}}_{res}$ for CPL, based on right-sided reversed sequents, are displayed in Table 4.16. For simplicity, we use the same symbols for the rules as we did before. The resolution rule R_{res} may be applied provided that A and $\overline{A}$ are complementary, that is, either $A = \neg\overline{A}$ or $\overline{A} = \neg A$ (see the definition of complement on page 39).

The sequences $\underline{S}$ and $\underline{T}$ ($\underline{U}$ and $\underline{V}$) are obtained from sequences S and T (U and V, respectively) by deleting all occurrences of formulas of the form A ($\overline{A}$, respectively), if there are any. Thus, in the case of the resolution rule, sequences of formulas are forced to "behave" like ordinary sets.

We did not restrict the applications of R_{res} to clauses—the elements of sequences S, T, U and V may be compound, as well as the "cut-formulas" A and $\overline{A}$. Thus $\mathbb{E}^{\mathsf{CPL}}_{res}$ constitutes an account of *non-clausal resolution*. (See also (Fitting, 1990) for non-clausal resolution.)

Table 4.16: Rules of $\mathbb{E}^{\mathsf{CPL}}_{res}$

$$\frac{?(\Phi\,;\,\dashv S\,'\,\beta\,'\,T\,;\,\Psi)}{?(\Phi\,;\,\dashv S\,'\,\beta_1\,'\,T\,;\,\dashv S\,'\,\beta_2\,'\,T\,;\,\Psi)}\;R_\beta$$

$$\frac{?(\Phi\,;\,\dashv S\,'\,\alpha\,'\,T\,;\,\Psi)}{?(\Phi\,;\,\dashv S\,'\,\alpha_1\,'\,\alpha_2\,'\,T\,;\,\Psi)}\;R_\alpha$$

$$\frac{?(\Phi\,;\,\dashv S\,'\,\neg\neg A\,'\,T\,;\,\Psi)}{?(\Phi\,;\,\dashv S\,'\,A\,'\,T\,;\,\Psi)}\;R_{\neg\neg}$$

$$\frac{?(\Phi\,;\,\dashv S\,'\,A\,'\,T\,;\,\Psi\,;\,\dashv U\,'\,\overline{A}\,'\,V\,;\,\Omega)}{?(\dashv \underline{S}\,'\,\underline{T}\,'\,\underline{U}\,'\,\underline{V}\,;\,\Phi\,;\,\Psi\,;\,\Omega\,;\,\dashv S\,'\,A\,'\,T\,;\,\dashv U\,'\,\overline{A}\,'\,V)}\;R_{res}$$

Rewriting the premise sequents in the conclusion of rule R_{res} is to warrant semantic invertibility of the rule. The consequence of this fact is that in the case of implementation of the method no backtracking will be needed in applying the resolution rule. Obviously, in this account the computational cost is in searching through the growing sequences of sequents for the premises of R_{res}.

When the erotetic rules are applied to the initial question, one arrives at a sequence of questions called *Socratic transformation*. The notion is defined as in the canonical case (see Definition 2 on page 41). Calculus $\mathbb{E}^{\mathsf{CPL}}_{res}$ is a resolution system, thus instead of the notion of proof we need the notion of refutation. This is given below.

Definition 55 (Socratic refutation). *Let* $\dashv A$ *be a reversed sequent. A* Socratic refutation of sequent $\dashv A$ in $\mathbb{E}^{\mathsf{CPL}}_{res}$ *is a finite Socratic transformation* $\mathbf{s} = \langle Q_1, \ldots, Q_n\rangle$ *of the question* $Q_1 = ?(\dashv A)$ *via the rules of* $\mathbb{E}^{\mathsf{CPL}}_{res}$, *such that the last question,* Q_n, *of* $\mathbf{s}$ *has the following form:*

(#) $\quad ?(\dashv \varnothing\,;\,\Omega)$

where Ω *is a finite sequence of sequents.* □

If there is a Socratic refutation of a sequent, $\dashv A$, in $\mathbb{E}^{\mathsf{CPL}}_{res}$, then we say that $\dashv A$ is *refutable* in $\mathbb{E}^{\mathsf{CPL}}_{res}$. Let us observe that we *refute reversed sequents* not *the formula concerned by the sequent*. By refuting the reversed sequent we answer to the negative to the question about refutability of A. Thus a Socratic refutation of '$\dashv A$' proves A, just as a refutation of '$\neg A$' in a standard resolution system proves A.

Example 35. *An example of a Socratic refutation of reversed sequent* $\dashv\ (p \to q) \to (\neg q \to \neg p)$.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{?(\dashv (p \to q) \to (\neg q \to \neg p))}{?(\dashv \neg(p \to q)\ ;\ \dashv \neg q \to \neg p)}\,R_\beta}{?(\dashv p, \neg q\ ;\ \dashv \neg q \to \neg p)}\,R_\alpha}{?(\dashv p, \neg q\ ;\ \dashv \neg\neg q\ ;\ \dashv \neg p)}\,R_\beta}{?(\dashv p, \neg q\ ;\ \dashv q\ ;\ \dashv \neg p)}\,R_{\neg\neg}}{?(\dashv p\ ;\ \dashv \neg p\ ;\ \dashv p, \neg q\ ;\ \dashv q)}\,R_{res}}{?(\dashv \varnothing\ ;\ \dashv p, \neg q\ ;\ \dashv q\ ;\ \dashv p\ ;\ \dashv \neg p)}\,R_{res}
$$

□

In Chlebowski, 2018 it is proved that the systems of erotetic resolution presented there (together with the one discussed here) are sound and complete with respect to the respective semantics.

Let us observe the dual behaviour of αs and βs in the context of the erotetic rules. In the right-sided setting of reversed sequents α-formulas "behave" like β-formulas do on the right side of canonical sequents, and conversely—βs take the roles of αs. One can say that the same dual shift is observed inside the canonical calculi with both-sided sequents, where αs on the right side of the turnstile "behave" like βs on the left side, and conversely. It would be misleading, however, to take the two phenomena as features of the same relation of duality. There are at least two relations of duality worth study. One is located at the level of different sides of the turnstile and explains the similarities in "behaviour" of αs and βs on the two sides. Semantically, it is the relation between truth (the left side) and false (the right side). In proof-theoretical terms, the first relation of duality explains both the differences and the similarities between the method of analytic tableaux (the left side of the turnstile) and, for example, the method of diagrams of formulas introduced by Rasiowa and Sikorski (the right side).[10]

[10] The method of diagrams of formulas has been introduced in Rasiowa and Sikorski, 1960; 1963, and it has been developed for many logics and found various important applications, *int.al*, in computer science. The reader may find more references in Orłowska and Golińska-Pilarek, 2011 or Leszczyńska-Jasion et al., 2018. In Leszczyńska-Jasion et al., 2018 the authors develop some new Rasiowa-Sikorski systems and also indicate their connection with the method of Socratic proofs.

But one can also examine relation of duality located at the level of proof-procedures; it is the second relation of duality, the one characterising canonical and dual erotetic calculi, and it also may be expressed in proof-theoretical and in semantic sense.

Proof-theoretically, the difference between the two types of calculi is in the nature of the closing conditions: these may be arrived at through a kind of decomposition of formulas and inspection of complementary formulas (canonical calculi), or through a decomposition and resolution (dual calculi). Semantically, the relation of duality can be expressed as follows. Suppose S is a finite sequence of formulas of a given formal language, and V is a valuation function defined for the language and with values in $\{0,1\}$. Then we may define two semantic, dual to each other, properties: the first property consists in S having at least one term true under V, whereas the second one consists in S having at least one term false under V. The erotetic rules of canonical calculi preserve the first property, and the dual erotetic calculi preserve the second property.

To sum up, there are at least two relations of duality worth study. One is located at the level of two sides of a sequent. The second relation is at the level of proof-procedures; it is what we do with the sequents and how we arrive at the conclusion that a proof has been obtained. Finally, it is worth to stress that on the second level of duality the cut rule known from sequent calculi is "canonical" and it expresses the Principle of Bivalence, whereas the resolution rule is its dual and it expresses the Principle of Non-contradiction. The whole Chapter 4 of this book explores, in a way, the similarities between proof systems determined by the first relation of duality. Now we may take a look at the merits that possibly follow from the syntactic opportunity of joining canonical and dual sequents in one (meta) proof-format.

It is well-known that the same formulas may have very long tableau-proofs and short resolution refutations, or conversely—see Boolos, 1984; Dunham and Wang, 1976; D'Agostino, 1992; D'Agostino and Mondadori, 1994; D'Agostino, 1990, more references may be found in D'Agostino, 1990. The same difference may be observed at the level of conjunctive and disjunctive normal forms of a formula. It is enough to imagine what happens when we insist to present formula (4.8) in conjunctive normal form (CNF, for short), or formula (4.9) in disjunctive normal form (DNF, for short) by using a standard algorithm with the laws of distributivity.

$$A = (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee \\ \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r) \quad (4.8)$$

$$B = (p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge$$
$$\wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r) \quad (4.9)$$

In the case of the above formulas, the features which make (4.8) unsuitable for converting to CNF, and (4.9) unsuitable for converting to DNF, are clearly visible; it is the presence of relatively many occurrences of '$\wedge$' in the scope of '$\vee$' (and conversely). But in a general case, deciding which form (CNF or DNF) is easier to derive—or in other words, which is shorter—is far from obvious. Here is a proposition of syntactical characterisation of the features that make the difference.

Definition 56. *Let FORM be the set of formulas of $\mathcal{L}_{\mathsf{CPL}}$. Functions*

$$\alpha c : FORM \longrightarrow \mathbb{N} \qquad \beta c : FORM \longrightarrow \mathbb{N}$$

called: α-complexity and β-complexity, respectively, are defined recursively as follows:

1.	$\alpha c(p_i) = \alpha c(\neg p_i) = 1$	*1.*	$\beta c(p_i) = \beta c(\neg p_i) = 1$
2.	$\alpha c(\neg\neg A) = \alpha c(A)$	*2.*	$\beta c(\neg\neg A) = \beta c(A)$
3.	$\alpha c(\alpha) = \alpha c(\alpha_1) + \alpha c(\alpha_2)$	*3.*	$\beta c(\alpha) = \beta c(\alpha_1) \cdot \beta c(\alpha_2)$
4.	$\alpha c(\beta) = \alpha c(\beta_1) \cdot \alpha c(\beta_2)$	*4.*	$\beta c(\beta) = \beta c(\beta_1) + \beta c(\beta_2)$

□

By and large, the α-complexity measures the number of branches of an α-branching tree, and the β-complexity measures the number of branches of a β-branching tree. Before we decide to search for a proof of a formula in one or another proof system, the measures may be calculated in order to estimate the complexity of the search.

Suppose, for example, that we have to decide validity of (4.8) and (4.9) either by analytic tableaux or by resolution. In terms of complexity, it is exactly the choice between a canonical and a dual erotetic calculus. We have:

$$\alpha c(\neg A) = \alpha c(\neg(p \wedge q \wedge r)) + \alpha c(\neg(p \wedge q \wedge \neg r)) + \ldots +$$
$$\alpha c(\neg(\neg p \wedge \neg q \wedge \neg r)) = 1 \times 8 = 8$$

and

$$\alpha c(\neg B) = \alpha c(\neg(p \vee q \vee r)) \cdot \alpha c(\neg(p \vee q \vee \neg r)) \cdot \ldots \cdot$$
$$\alpha c(\neg(\neg p \vee \neg q \vee \neg r)) = 3^8 = 6561$$

It follows that in a complete resolution derivation for $\neg A$ we will obtain 8 clauses after deriving CNF of $\neg A$ (one may derive the empty clause after 7 applications of the resolution rule). In the case of formula B, we will have to derive 6561 clauses before applying the resolution rule. It is not the case, however, that testing validity of formula B is a hopeless task. For we have:

$$\beta c(\neg A) = 6561 \qquad \beta c(\neg B) = 8$$

A Smullyan-type analytic tableau for $\neg B$ has at most 8 branches, each of them may be completed immediately, each is open. On the other hand, a complete analytic tableau for $\neg A$ has 6561 branches. It is possible to construct a smaller tableau—the reader may check that the minimal one has 33 branches; but in general, the lower bound of the number of branches of this kind of tableau is $n!$, where n is the number of distinct variables in a formula (see D'Agostino, 1990; 1992; D'Agostino and Mondadori, 1994). Therefore it seems reasonable to treat problems akin to (4.8) with resolution, while problems akin to (4.9) should be dealt with by analytic tableaux.

Obviously, we are aware of the diagnosis of this situation which is derived by the authors of D'Agostino, 1990; 1992 and D'Agostino and Mondadori, 1994; it is the absence of cut that leads to the computational collapse, therefore analytic tableau system **KE**, equipped with a form of not eliminable analytic cut, is proposed by D'Agostino and Mondadori as a remedy. For the same reason the resolution system works well where the standard tableaux grow to fast—the rule of resolution is, as we have observed, a dual of the rule of cut. But in the present context we would like to draw another solution, independent of the use of any form of cut.

At the start, the general diagnosis is this: before you treat a problem with analytic tableaux or with resolution, calculate the $\alpha-$ and $\beta-$complexity of the problem. The same diagnosis may be applied with respect to different variants of erotetic calculi—if the α-complexity of a formula is smaller than its β-complexity, use the canonical version of erotetic calculi, otherwise use the dual version. But suppose now that the formula we have to examine is not simply (4.8) or (4.9), but the conjunction $A \wedge B$. No matter if we decide to calculate this by analytic tableaux or by resolution—in both cases we will have to deal with expansion of complexity on one of the conjuncts. But we know that it is better to calculate validity of A by rules which branch on α-formulas, and to use β-branching rules in order to calculate the validity of B. So why not join the two formats to check the conjuncts by different sets of rules?

Table (4.17) presents an illustration of this idea. '$B \setminus p \vee q \vee r$' stands for

$$B = (p \vee q \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge$$
$$\wedge\ (\neg p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

and similarly for $A \setminus p \wedge q \wedge r$.

Since $\alpha c(A \wedge B) < \beta c(A \wedge B)$, we start with an application of a rule for conjunction taken from the canonical erotetic calculus (the propositional part of $\mathbb{E}^{\mathbf{RPQ}}$). But then we obtain sequent with formula A, where αc of A is much bigger than βc, thus the sequent is "switched" into the reversed one and than the rules of $\mathbb{E}^{\mathbf{RPQ}}_{dual}$ are applied (this is also indicated by the use of double lines). On the other hand, since αc of B is smaller than its βc, we continue to apply the canonical rules on this side of the derivation. Observe also that the derivation is not a proof of anything—it does not end with a success in the "B-part", which is fine, as formula '$A \wedge B$' is not valid.

Table 4.17: An example of a transformation with canonical and dual sequents

	$\vdash A \wedge B$	
$\vdash A$	;	$\vdash B$
$\dashv A$		$\vdash p \vee q \vee r \ ;\ \vdash B \setminus p \vee q \vee r$
$\dashv p \wedge q \wedge r \ ;\ \dashv A \setminus p \wedge q \wedge r$		$\vdots$
$\vdots$		$\vdash p \vee q \vee r \ ; \ldots ;\ \vdash \neg p \vee \neg q \vee \neg r$
$\dashv p \wedge q \wedge r \ ; \ldots ;\ \dashv \neg p \wedge \neg q \wedge \neg r$		$\vdash p, q \vee r \ ; \ldots ;\ \vdash \neg p \vee \neg q \vee \neg r$
$\dashv p, q \wedge r \ ; \ldots ;\ \dashv \neg p \wedge \neg q \wedge \neg r$		$\vdots$
$\vdots$		$\vdash p, q, r \ ; \ldots ;\ \vdash \neg p, \neg q, \neg r$
$\dashv p, q, r \ ; \ldots ;\ \dashv \neg p, \neg q, \neg r$		
$\vdots$		
$\dashv \varnothing \ ; \ldots$		

The above considerations concerning calculi based on different types of sequents go beyond the scope of this book, but they certainly reveal a panorama for future research.

Appendices

Appendix A. Provisos of applicability of R_π

Here we recall the provisos of applicability of rule R_π for the basic modal logics $\mathsf{L} = \mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{KB}, \mathsf{DB}, \mathsf{B}, \mathsf{K4}, \mathsf{D4}, \mathsf{S4}, \mathsf{KB4}, \mathsf{S5}, \mathsf{K5}, \mathsf{D5}, \mathsf{K45}, \mathsf{D45}$ in the version presented and used in Leszczyńska, 2004; 2007, Leszczyńska-Jasion, 2008; 2009.

We need the following terminology to express the provisos. Let R be a binary relation in a set W. By an *R-chain* we mean a finite, at least two-term sequence $\langle w_1, \ldots, w_n \rangle$ of elements of W such that for each k $(1 \leq k < n)$: either $\langle w_k, w_{k+1} \rangle \in R$ or $\langle w_{k+1}, w_k \rangle \in R$. By a *directed R-chain* we mean an R-chain $\langle w_1, \ldots, w_n \rangle$ such that $\langle w_k, w_{k+1} \rangle \in R$ for each k $(1 \leq k < n)$. Moreover, by a (*directed*) *R-$[w, z]$-chain* we mean a (directed) R-chain whose first term is w and whose last term is z.

Table 4.18 presents the provisos of applicability of R_π in logics L. 'ϕ' stands for '$\vdash S((\pi)^{\sigma/i})$', that is, the sequent-premise of the rule. In most cases the proviso is a disjunction, for the proviso to hold it is sufficient if at least one of the disjuncts is satisfied.

Corollary 40. *For each logic* L*, the proviso described in Table 4.18 is satisfied iff the proviso described in Table 2.10 (see page 74) is satisfied.*

We will demonstrate that it holds for some of the considered logics, the remaining cases are analogous.

Proof for $\mathsf{KB4}$. Let $\phi = \vdash S((\pi)^{\sigma/i})$ and assume that i, j are arbitrary numerals. We need to show that then $\langle i, j \rangle \in \mathbf{I}_R[\phi]^{sym,trans}$ iff there is an $\mathbf{I}_R[\phi]$-$[i, j]$-chain.

First, it is easily seen that the set of all pairs of numerals $\langle i, j \rangle$ such that there exists an $\mathbf{I}_R[\phi]$-$[i,\ j]$-chain is itself a binary, symmetric and transitive relation, and a superset of $\mathbf{I}_R[\phi]$. This proves the first implication.

For the second implication assume that there exists an $\mathbf{I}_R[\phi]$-$[i,\ j]$-chain. It follows that there is a sequence $\langle i_1, \ldots, i_n \rangle$ $(1 < n)$ of indices such that for each k $(1 \leq k < n)$: either $\langle i_k, i_{k+1} \rangle \in \mathbf{I}_R[\phi]$

Table 4.18: Provisos of applicability of rule R_π

Calculus	Proviso of applicability of rule R_π
$\mathbb{E}^{\mathsf{K}}$, $\mathbb{E}^{\mathsf{D}}$	$\langle i, j\rangle \in \mathbf{I}_R[\phi]$
$\mathbb{E}^{\mathsf{T}}$	$\langle i, j\rangle \in \mathbf{I}_R[\phi]$ **or** $j = i$
$\mathbb{E}^{\mathsf{KB}}$, $\mathbb{E}^{\mathsf{DB}}$	$\langle i, j\rangle \in \mathbf{I}_R[\phi]$ **or** $\langle j, i\rangle \in \mathbf{I}_R[\phi]$
$\mathbb{E}^{\mathsf{B}}$	$\langle i, j\rangle \in \mathbf{I}_R[\phi]$ **or** $\langle j, i\rangle \in \mathbf{I}_R[\phi]$ **or** $j = i$
$\mathbb{E}^{\mathsf{K4}}$, $\mathbb{E}^{\mathsf{D4}}$	there is a directed $\mathbf{I}_R[\phi]$-$[i,\, j]$-chain
$\mathbb{E}^{\mathsf{S4}}$	there is a directed $\mathbf{I}_R[\phi]$-$[i,\, j]$-chain **or** $j = i$
$\mathbb{E}^{\mathsf{KB4}}$	there is an $\mathbf{I}_R[\phi]$-$[i,\, j]$-chain
$\mathbb{E}^{\mathsf{S5}}$	there is an $\mathbf{I}_R[\phi]$-$[i,\, j]$-chain **or** $j = i$
$\mathbb{E}^{\mathsf{K5}}$, $\mathbb{E}^{\mathsf{D5}}$	$\langle i, j\rangle \in \mathbf{I}_R[\phi]$ **or** there are: a directed $\mathbf{I}_R[\phi]$-$[1,\, i]$-chain and a directed $\mathbf{I}_R[\phi]$-$[1,\, j]$-chain
$\mathbb{E}^{\mathsf{K45}}$, $\mathbb{E}^{\mathsf{D45}}$	there is a directed $\mathbf{I}_R[\phi]$-$[i,\, j]$-chain **or** there are: a directed $\mathbf{I}_R[\phi]$-chain and a directed $\mathbf{I}_R[\phi]$-$[1,\, j]$-chain

or $\langle i_{k+1}, i_k\rangle \in \mathbf{I}_R[\phi]$. Then we observe that $\langle i_1, i_2\rangle \in \mathbf{I}_R[\phi]^{sym,trans}$ or $\langle i_2, i_1\rangle \in \mathbf{I}_R[\phi]^{sym,trans}$ (it holds by definition). Assume that for k $(1 \leq k < n)$:

(i) $\langle i_1, i_k\rangle \in \mathbf{I}_R[\phi]^{sym,trans}$, or

(ii) $\langle i_k, i_1\rangle \in \mathbf{I}_R[\phi]^{sym,trans}$.

By definition (of our chains), one of $\langle i_k, i_{k+1}\rangle$, $\langle i_{k+1}, i_k\rangle$ must be in the relation $\mathbf{I}_R[\phi]^{sym,trans}$. Since $\mathbf{I}_R[\phi]^{sym,trans}$ is symmetric, it follows that actually the two pairs are in $\mathbf{I}_R[\phi]^{sym,trans}$. Therefore, if (i) holds, then since $\langle i_k, i_{k+1}\rangle \in \mathbf{I}_R[\phi]^{sym,trans}$, also $\langle i_1, i_{k+1}\rangle \in \mathbf{I}_R[\phi]^{sym,trans}$, as $\mathbf{I}_R[\phi]^{sym,trans}$ is transitive. If case (ii) holds, then since $\langle i_{k+1}, i_k\rangle \in \mathbf{I}_R[\phi]^{sym,trans}$, also $\langle i_{k+1}, i_1\rangle \in \mathbf{I}_R[\phi]^{sym,trans}$ by the same argument. That is, we have shown that if for k $(1 \leq k < n)$: $\langle i_1, i_k\rangle \in \mathbf{I}_R[\phi]^{sym,trans}$ or $\langle i_k, i_1\rangle \in \mathbf{I}_R[\phi]^{sym,trans}$, then for $k+1$ we also have: $\langle i_1, i_{k+1}\rangle \in \mathbf{I}_R[\phi]^{sym,trans}$ or $\langle i_{k+1}, i_1\rangle \in \mathbf{I}_R[\phi]^{sym,trans}$. By mathematical induction, $\langle i, j\rangle \in \mathbf{I}_R[\phi]^{sym,trans}$, which finishes the proof. □

Proof for K5 *and* D5. Let $\phi = \vdash S((\pi)^{\sigma/i})$ and assume that i, j are arbitrary numerals. We need to show that then $\langle i, j\rangle \in \mathbf{I}_R[\phi]^{Eucl}$ iff $\langle i, j\rangle \in \mathbf{I}_R[\phi]$ **or** there are: a directed $\mathbf{I}_R[\phi]$-$[1, i]$-chain and a directed $\mathbf{I}_R[\phi]$-$[1, j]$-chain. By the way, the reasoning will reveal also that the first part of disjunction in the proviso, that is, $\langle i,j\rangle \in \mathbf{I}_R\,[\phi]$, is superfluous; it nevertheless does not spoil anything.

This time we refer to the properties of Socratic proofs. Since each Socratic proof starts with a question of the form $?(\vdash (A)^1)$, numeral 1 becomes the root of the tree of numerals $\mathbf{I}_R\,[\phi]$ for each sequent ϕ that occurs in the Socratic transformation. In other words, for each numeral $i \in \mathbf{I}_W\{\phi\}$ other than 1 that occurs in sequent ϕ, there exists a directed $\mathbf{I}_R\,[\phi]$-$[1,i]$-chain.

For the first implication we show that the set of pairs of numerals $\langle i,j\rangle$ that satisfy the right part of equivalence is itself a Euclidean relation. Thus suppose that $\langle x,y\rangle$ and $\langle x,z\rangle$ both satisfy the condition. The condition is a disjunction, so we need to divide the argument into cases. If $\langle x,y\rangle \in \mathbf{I}_R\,[\phi]$ and $\langle x,z\rangle \in \mathbf{I}_R\,[\phi]$, then neither numeral y nor z equals 1. For this reason, there is a directed $\mathbf{I}_R\,[\phi]$-$[1,y]$-chain and a directed $\mathbf{I}_R\,[\phi]$-$[1,z]$-chain, hence the pair $\langle y,z\rangle$ satisfies the condition. Further, assume that $\langle x,y\rangle \in \mathbf{I}_R\,[\phi]$ and for the pair $\langle x,z\rangle$ there are: a directed $\mathbf{I}_R\,[\phi]$-$[1,x]$-chain and a directed $\mathbf{I}_R\,[\phi]$-$[1,z]$-chain. Since there is a directed $\mathbf{I}_R\,[\phi]$-$[1,x]$-chain and $\langle x,y\rangle \in \mathbf{I}_R\,[\phi]$, there exists also a directed $\mathbf{I}_R\,[\phi]$-$[1,y]$-chain. Hence also the pair $\langle y,z\rangle$ satisfies the condition in questions. The reasoning is analogous in the remaining two cases.

For the second implication we need to show that the relation described in the previous paragraph is *the* Euclidean closure of $\mathbf{I}_R\,[\phi]$. If $\langle x,y\rangle \in \mathbf{I}_R\,[\phi]$, then $\langle x,y\rangle \in \mathbf{I}_R\,[\phi]^{Eucl}$. Suppose that for $\langle x,y\rangle$ there the two directed chains: $\langle 1,\ldots,i_k,\ldots,x\rangle$ and $\langle 1,\ldots,j_k,\ldots,y\rangle$. As in the proof for $\mathsf{KB4}$, we show by mathematical induction that $\langle x,y\rangle \in \mathbf{I}_R\,[\phi]^{Eucl}$. The consecutive steps of the reasoning are: $\langle 1,i_2\rangle \in \mathbf{I}_R\,[\phi]^{Eucl}$, $\langle 1,j_2\rangle \in \mathbf{I}_R\,[\phi]^{Eucl}$ (by definition of directed chains), and $\langle 1,i_{k+1}\rangle \in \mathbf{I}_R\,[\phi]^{Eucl}$, $\langle 1,j_{k+1}\rangle \in \mathbf{I}_R\,[\phi]^{Eucl}$, whenever $\langle 1,i_k\rangle \in \mathbf{I}_R\,[\phi]^{Eucl}$, $\langle 1,j_k\rangle \in \mathbf{I}_R\,[\phi]^{Eucl}$ (which is also warranted by the definition). □

Appendix B. Trees

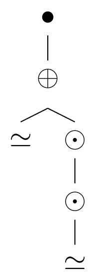
The notion of tree is commonly used in proof theory to capture the structure lying behind occurrences of formulas which constitute the body of the proof. In most textbooks, however, it is left undefined, as it is assumed that an illustration in the form of diagrams will be sufficient (and usually it is). If defined, a tree is often treated as a graph. The reader can find a full exposition of this account in Rosen, 2012, Chapter 11, or in Cormen et al., 2009, pp. 1168-1182. Some other interesting accounts are possible, like in Kaye, 2008, p. 5, where a tree is defined as a set of binary sequences. But in this book we find it useful to use the notion in its set-theoretical account, as in Troelstra and Schwichtenberg, 2000, pp. 7-8; Smullyan, 1968, pp. 3-4, also Priest, 2008 tacitly assumes this account.

A binary relation R defined on X, that is such that $R \subseteq X \times X$, is called a *partial order* iff R is reflexive, transitive and antisymmetric in X, that is, $\forall_{x,y \in X}(xRy \wedge yRx \rightarrow x = y)$. A *linear order* is a partial order which makes every two elements of X comparable, that is, such that xRy or yRx for each $x, y \in X$. See also Table 2.8 on page 71 for the definitions of the other properties. The symbol '$\succeq$' is commonly used for partial orders.

A *tree* is a partially ordered set $(X, \succeq)$ with a least element, called its *root*, and a set $\{y \in X : y \succeq x\}$ linearly ordered for each $x \in X$. We will use '$\succ$' for the *strict order corresponding to* $\succeq$. Formally, $x \succ y$ iff $x \succeq y$ and $x \neq y$. The elements of X are called the *nodes* of $(X, \succeq)$. A *branch of* $(X, \succeq)$ is a linearly ordered subset of X which is maximal with respect to set inclusion. (That is, a branch cannot be extended further.) The maximal elements of a tree, if there are any, are called its *leaves*. Trees are usually displayed "upside down", like on Figure 4.6, with the root (labelled with '$\bullet$') at the top and the leaves (labelled with '$\simeq$') at the bottom.

Relations between nodes of a tree are usually described with the use of a "genealogical" terminology. If $x \succ y$, then x is called *ancestor of* y and y is called *descendant of* x. If $x \succ y$, and there is no "intermediate relative" z such that $x \succ z \succ y$, then x is called *parent of* y and y is a *child of* x. In the example presented on Figure 4.6 the root $\bullet$ is the parent of the node with $\oplus$, which is the only child of the root. Each node is the root's descendant except itself, and so on. The more neutral terminology is, respectively, *predecessor and successor* and *immediate*

Figure 4.6: We plant trees upside down.



predecessor and immediate successor. Nodes that share the same parent node are called *siblings*.

We will deal with binary trees, that is, such that every node has at most two children. It will occur very useful to be able to differentiate between the two children (if there are two). It is understandable if we think of rules which have two conclusions, and we are used to calling them "the left" and "the right" conclusion, respectively. This additional ordering is not essential from the point of view of such crucial properties of the rules like semantic correctness (soundness), but it is handy when it comes to the formulation of algorithms. Formally, it means that a tree is given together with a function $(X, \succeq, f)$ which may be defined as follows. Let:

$$Families(X, \succeq) = \{(x, y) : x \succ y \wedge \neg \exists_{z \in X}(x \succ z \succ y)\}$$

that is, the set $Families(X, \succeq)$ is a set of all pairs of the form (*parent*, *child*), where *parent* is the *child*'s parent. Then we may set f as a function:

$$f : Families(X, \succeq) \longrightarrow \{0, 1\}$$

such that if *child* is the only child of the node *parent*, then f(*parent*, *child*) $= 0$; in this situation *parent* is called *non-branching point*. If there are two children of *parent*: *childA*, *childB*, then f assigns numerals from $\{0, 1\}$ to the respective pairs in an injective way; in this situation *parent* is called *branching point*. *E.g.* in the above example the node labelled with $\oplus$ is a branching point (the only one in the tree). If $f(parent, childA) = 0$ and $f(parent, childB) = 1$, then we call *childA* the *left child*, and *childB* the *right child*.[11] This left-right terminology is

[11] In Kaye, 2008 the reader will find similar, yet different account of the left-right distinction.

extended to whole branches in a natural way. A branch going through the left child is called *left branch*, and the other is called *right branch*. In the sequel we will use the left-right terminology without indicating explicitly function assigning "the sides".

We use the standard definition of *level of a node in a tree*. If (X, R) is a tree, then the root of the tree is at level 0. Further, if a node $x \in X$ is at level n, then each of its immediate successors (if there are any) is at level $n+1$. By *length of a finite branch* of a tree we mean the level of the leaf of the branch. By *height of a finite tree* we mean the length of the maximal branch (*i.e.* the one with maximal length).

Another additional device that often goes with trees is a labelling function. In the sequel, this will be indicated explicitly. The need to use such a function is illustrated well by the example on Figure 4.6, where we can see clearly that the symbols '$\simeq$', '$\odot$' do not represent the very nodes of the tree, since these cannot occur twice in two different positions (recall that a tree is just a set). Therefore we assume that the nodes "beneath" are some different objects, they are just labelled with the same label. Proofs such as tableaux are labelled trees in this sense. In most cases, the nature of the nodes which lie beneath is absolutely irrelevant, only the labels matter.[12] However, it becomes relevant when it comes to formulation of an algorithm.

[12] If we wish to be more specific, then we should say that two proof trees are regarded as essentially the same, if the trees without labels are isomorphic, and the nodes corresponding under the isomorphism are assigned the same labels. See also Troelstra and Schwichtenberg, 2000, p. 32.

Appendix C. Translations from $\mathbb{E}^{\mathsf{L}\tau}$ to $\mathbb{G}^{\mathsf{L}\tau}$: a summary

Numeration of columns:

1. Calculus (name)
2. Primary rules of the calculus
3. Rules derived in the calculus and used in the translation process
4. Procedures used in Algorithm 4 for generating regular Socratic transformations
5. Translation function and function labelling ("relabelling") the final tree (obtained by Procedure PRUNE)

In the Table below, "classical" stands for: $L^{\mathbb{E}}_{\neg}, R^{\mathbb{E}}_{\neg}$, $L^{\mathbb{E}}_{\wedge}$, $R^{\mathbb{E}}_{\vee}$, $R^{\mathbb{E}}_{\rightarrow}$, $R^{\mathbb{E}}_{\wedge}$, $L^{\mathbb{E}}_{\vee}, L^{\mathbb{E}}_{\rightarrow}$.

1	2	3	4	5
$\mathbb{E}^{\mathsf{K}\tau}$	classical, $L^{\mathbb{E}}_{\Diamond}$, $R^{\mathbb{E}}_{\Box}$ and $L^{\mathbb{E}}_{\Box}$, $R^{\mathbb{E}}_{\Diamond}$ with proviso for K	$L^{\mathbb{E}}_{\Diamond\Box}$, $R^{\mathbb{E}}_{\Box\Diamond}$	CLASSICAL, MODAL	τ_{s} (Def. 41)
$\mathbb{E}^{\mathsf{D}\tau}$	all above and $L^{\mathbb{E}}_{D}$, $R^{\mathbb{E}}_{D}$	all above and $R^{\mathbb{E}}_{Dgen}$	CLASSICAL, MODAL-D	τ_{s} (Def. 41)
$\mathbb{E}^{\mathsf{T}\tau}$	as in $\mathbb{E}^{\mathsf{K}\tau}$, (also $L^{\mathbb{E}}_{\Box}$, $R^{\mathbb{E}}_{\Diamond}$ with proviso for K), $L^{\mathbb{E}}_{refl}$, $R^{\mathbb{E}}_{refl}$	$L^{\mathbb{E}}_{\Diamond\Box}$, $R^{\mathbb{E}}_{\Box\Diamond}$	SETTLE-T, MODAL	τ_{Ts} (Def. 45) defined by τ_{T} (Def. 44)
$\mathbb{E}^{\mathsf{K4}\tau}$	classical, $L^{\mathbb{E}}_{\Diamond}$, $R^{\mathbb{E}}_{\Box}$ and $L^{\mathbb{E}}_{\Box}$, $R^{\mathbb{E}}_{\Diamond}$ with proviso for $\mathsf{K4}$	$L^{\mathbb{E}}_{trans}$, $R^{\mathbb{E}}_{trans}$	CLASSICAL, MODAL-K4	τ_{4s} defined as τ_{s} in Def. 41 but with τ_4 instead of τ (Def. 46, 47)

1	2	3	4	5
$\mathbb{E}^{\mathsf{D4}\tau}$	classical, $L^{\mathbb{E}}_{\Diamond}$, $R^{\mathbb{E}}_{\Box}$, $L^{\mathbb{E}}_{D}$, $R^{\mathbb{E}}_{D}$, and $L^{\mathbb{E}}_{\Box}$, $R^{\mathbb{E}}_{\Diamond}$ with proviso for K4	$L^{\mathbb{E}}_{trans}$, $R^{\mathbb{E}}_{trans}$, $L^{\mathbb{E}}_{Dgen}$, $R^{\mathbb{E}}_{Dgen2}$	CLASSICAL, MODAL-D4	as in $\mathbb{E}^{\mathsf{K4}\tau}$ above
$\mathbb{E}^{\mathsf{S4}\tau}$	classical, $L^{\mathbb{E}}_{\Diamond}$, $R^{\mathbb{E}}_{\Box}$ and $L^{\mathbb{E}}_{\Box}$, $R^{\mathbb{E}}_{\Diamond}$ with proviso for K4, $L^{\mathbb{E}}_{refl}$, $R^{\mathbb{E}}_{refl}$	$L^{\mathbb{E}}_{trans}$, $R^{\mathbb{E}}_{trans}$	SETTLE-T, MODAL-K4	τ_{S4s} (Def. 49) defined with the use of τ_{4} (Def. 46, 47)

Bibliography

Avron, A. (1993). Gentzen-type systems, resolution and tableaux. *Journal of Automated Reasoning*, 10:265–281.

Avron, A. (1996). The method of hypersequents in the proof theory of propositional non-classical logics. In Hodges, W., Hyland, M., Steinhorn, C., and Truss, J., editors, *Logic: Foundations to Applications*, pages 1–32. Oxford Science Publications, Oxford.

Backus, J. W. (1959). The syntax and semantics of the proposed international algebraic language of the Zurich ACM-GAMM Conference. In *Proceedings of the International Conference on Information Processing*, pages 125–132. UNESCO.

Batóg, T. (2003). *Podstawy logiki.* Adam Mickiewicz University Press, Poznań, fourth edition.

Belnap, N. D. and Steel, T. P. (1976). *The Logic of Questions and Answers.* Yale University Press.

Beth, E. W. (1969). Semantic entailment and formal derivability. In Hintikka, J., editor, *The Philosophy of Mathematics*, pages 9–41. Oxford University Press.

Boolos, G. (1984). Don't eliminate cut. *Journal of Philosophical Logic*, 13(4):373–378.

Brożek, A. (2011). *Theory of Questions: Erotetics Through the Prism of Its Philosophical Background and Practical Applications.* Rodopi.

Buss, S. R., editor (1998a). *Handbook of Proof Theory.* Elsevier.

Buss, S. R. (1998b). An introduction to proof theory. In Buss, S. R., editor, *Handbook of Proof Theory*, pages 1–78. Elsevier.

Chlebowski, S. (2018). *Canonical and Dual Erotetic Calculi for First-Order Logic.* PhD thesis, Adam Mickiewicz University, Poznań, Poland. (Unpublished manuscript, previously referred to as "The Method of Socratic Proofs for Classical Logic and Some Non-Classical Logics").

Chlebowski, S., Gajda, A., and Urbański, M. (2017). Abductive question-answer system for the minimal logic of formal inconsistency mbC. Unpublished manuscript received from the authors in 2017.

Chlebowski, S. and Leszczyńska-Jasion, D. (2015). Dual erotetic calculi and the minimal LFI. *Studia Logica*, 103(6):1245–1278.

Ciardelli, I. A. (2016). *Questions in Logic*. PhD thesis, University of Amsterdam.

Ciardelli, I. A., Groenendijk, J., and Roelofsen, F. (2015). On the semantics and logic of declaratives and interrogatives. *Synthese*, 192(6):1689–1728.

Ciardelli, I. A. and Roelofsen, F. (2011). Inquisitive logic. *Journal of Philosophical Logic*, 40:55–94.

Cook, S. and Nguyen, P. (2010). *Logical Foundations of Proof Complexity*. Cambridge University Press.

Cormen, T. H., Leiserson, C. E., Rivest, R. L., and Stein, C. (2009). *Introduction to Algorithms*. The MIT Press, Cambridge, London, third edition.

Cross, C. and Roelofsen, F. (2018). Questions. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, spring 2018 edition. Available at `https://plato.stanford.edu/archives/spr2018/entries/questions/`.

Curry, H. B. (1950). *A Theory of Formal Deducibility*, volume 6 of *Notre Dame Mathematical Lectures*. University of Notre Dame.

D'Agostino, M. (1990). *Investigations into the complexity of some propositional calculi. Technical monograph.* Oxford University Computing Laboratory, Programming Research Group.

D'Agostino, M. (1992). Are tableaux an improvement on truth-tables? Cut-free proofs and bivalence. *Journal of Logic, Language and Computation*, 1:235–252.

D'Agostino, M. (1999). Tableau methods for classical propositional logic. In D'Agostino, M., Gabbay, D. M., Hähnle, R., and Posegga, J., editors, *Handbook of Tableau Methods*, pages 45–123. Springer Netherlands.

D'Agostino, M., Gabbay, D. M., Hähnle, R., and Posegga, J., editors (1999). *Handbook of Tableau Methods*. Springer Netherlands.

D'Agostino, M. and Mondadori, M. (1994). The taming of the cut. Classical refutations with analytic cut. *Journal of Logic and Computation*, 4(3):285–319.

Dunham, B. and Wang, H. (1976). Towards feasible solutions of the tautology problem. *Annals of Mathematical Logic*, 10(2):117–154.

Fitting, M. (1983). *Proof Methods for Modal and Intuitionistic Logics*. Springer Netherlands.

Fitting, M. (1990). *First-Order Logic and Automated Theorem Proving*. Springer-Verlag, New York.

Gabbay, D. M. (1996). *Labelled Deductive Systems*. Oxford University Press.

Genot, E. J. (2009). The game of inquiry: the interrogative approach to inquiry and belief revision theory. *Synthese*, 171(2):271–289.

Genot, E. J. and Jacot, J. (2010). How can questions be informative before they are answered? Strategic information in interrogative games. *Episteme*, 9(2):189–204.

Gentzen, G. (1969). Investigations into logical deduction. In Szabo, M. E., editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131. North-Holland Publishing Company, Amsterdam and London.

Genzten, G. (1935). Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210. The English translation by Manfred E. Szabo: Gentzen (1969).

Ginzburg, J. (2011). Questions: logic and interactions. In van Benthem, J. and ter Meulen, A., editors, *Handbook of Logic and Language*, pages 1133–1146. Elsevier, second edition.

Goldblatt, R. (1992). *Logics of Time and Computation*, volume 7 of *CSLI Lecture Notes*. CSLI Publications, Stanford, second revised and extended edition.

Golińska-Pilarek, J. and Orłowska, E. (2007). Tableaux and dual tableaux: transformation of proofs. *Studia Logica*, 85:283–302.

Goré, R. (1999). Tableau methods for modal and temporal logics. In D'Agostino, M., Gabbay, D. M., Hähnle, R., and Posegga, J., editors, *Handbook of Tableau Methods*, pages 297–396. Springer Netherlands.

Grobler, A. (2006). *Metodologia nauk*. Znak & Aureus, Kraków.

Groenendijk, J. and Roelofsen, F. (2009). Inquisitive semantics and pragmatics. In Larrazabal, J. M. and Zubeldia, L., editors, *Meaning, Content, and Argument: Proceedings of the ILCLI International Workshop on Semantics, Pragmatics, and Rhetoric*.

Groenendijk, J. and Stokhof, M. (1984). *Studies on the Semantics of Questions and the Pragmatics of Answers*. PhD thesis, University of Amsterdam.

Groenendijk, J. and Stokhof, M. (2011). Questions. In van Benthem, J. and ter Meulen, A., editors, *Handbook of Logic and Language*, pages 1059–1131. Elsevier, second edition.

Grzelak, A. and Leszczyńska-Jasion, D. (2018). Automatic proof generation in an axiomatic system for CPL by means of the method of Socratic proofs. *Logic Journal of the IGPL*, 26(1):109–148.

Hacking, I. (1994). What is logic? In Gabbay, D. M., editor, *What is a Logical System?*, Oxford Science Publications, pages 1–33. Clarendon Press. First published in the Journal of Philosophy, Volume 76, No 6, pages 285–319, 1979.

Hähnle, R. (2001). Tableaux and related methods. In Robinson, A. and Voronkov, A., editors, *Handbook of Automated Reasoning*, chapter 3, pages 101–175. Elsevier Science Publishers.

Hamami, Y. (2015). The interrogative model of inquiry meets dynamic epistemic logics. *Synthese*, 192(6):1609–1642.

Hamblin, C. L. (1958). Questions. *Australasian Journal of Philosophy*, 36(3):159–168.

Harrah, D. (1969). On completeness in the logic of questions. *American Philosophical Quarterly*, 6:158–164.

Harrah, D. (1998). Questions. In *Routledge Encyclopedia of Philosophy*. Taylor and Francis. Available at `https://www.rep.routledge.com/articles/thematic/questions/v-1`.

Harrah, D. (2002). The logic of questions. In Gabbay, D. M. and Guenthner, F., editors, *Handbook of Philosophical Logic*, volume 8, pages 1–60. Springer, second edition.

Harris, S. (1994). Gts and interrogative tableaux. *Synthese*, 99(3):329–343.

Hilbert, D. and Bernays, P. (1968). *Grundlagen der Mathematik I*. Springer-Verlag, Berlin, Heidelberg, New York, second edition.

Hintikka, J. (1955). Form and content in quantification theory. *Acta Philosophica Fennica*, 8:3–55.

Hintikka, J. (1992). The interrogative model of inquiry as a general theory of argumentation. *Communication and Cognition: An Interdisciplinary Quarterly Journal*, 25(2-3):221–242.

Hintikka, J. (1999). *Inquiry as Inquiry: A Logic of Scientific Discovery*. Kluwer Academic Publishers, Dordrecht, Boston, London.

Hintikka, J. (2007). *Socratic Epistemology. Explorations of Knowledge-Seeking by Questioning*. Cambridge University Press.

Hintikka, J., Halonen, I., and Mutanen, A. (1999). Interrogative logic as a general theory of reasoning. In *Inquiry as Inquiry: A Logic of Scientific Discovery*, pages 47–90. Kluwer Academic Publishers, Dordrecht, Boston, London.

Hughes, G. E. and Cresswell, M. J. (1996). *A New Introduction to Modal Logic*. Routledge, London and New York.

Indrzejczak, A. (2010). *Natural Deduction, Hybrid Systems and Modal Logics*, volume 30 of *Trends in Logic*. Springer.

Indrzejczak, A. (2013). *Rachunki sekwentowe w logice klasycznej (Sequential Calculi in Classical Logic)*. The University of Lodz Press, Łódź.

Indrzejczak, A. (2014). Powstanie i ewolucja rachunków sekwentowych. *Filozofia Nauki*, 22(4):53–70.

Jarmużek, T. (2013). *Formalizacja metod tablicowych dla logik zdań i logik nazw (A Formalization of Tableau Methods for Logics of Sentences and Logics of Names)*. Nicolaus Copernicus University Press, Toruń.

Kaye, R. (2008). *The Mathematics of Logic*. Cambridge University Press.

Knuth, D. E. (1964). Backus normal form vs. Backus Naur form. *Communications of the ACM*, 7(12):735–736.

Komosinski, M., Kups, A., Dorota, L.-J., and Urbański, M. (2014). Identifying efficient abductive hypotheses using multi-criteria dominance relation. *ACM Transactions on Computational Logic*, 15(4).

Komosinski, M., Kups, A., and Urbański, M. (2012). Multi-criteria evaluation of abductive hypotheses: towards efficient optimization in proof theory. In *Proceedings of the 18th International Conference on Soft Computing*, pages 320–325, Brno, Czech Republic.

Konikowska, B. (1999). Rasiowa-Sikorski deduction systems: a handy tool for computer science logics. In Fiadeiro, J. L., editor, *Recent Trends in Algebraic Development Techniques. WADT 1998*, volume 1589 of *Lecture Notes in Computer Science*, pages 184–197. Springer.

Kripke, S. A. (1963). Semantical analysis of modal logic I. Normal modal propositional calculi. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 9:67–96.

Kubiński, T. (1971). *Wstęp do logicznej teorii pytań*. PWN, Warszawa.

Kubiński, T. (1980). *An Outline of the Logical Theory of Questions*. Akademie-Verlag, Berlin.

Legris, J. (2014). Paul Hertz's systems of propositions as a proof-theoretical conception of logic. In Pereira, L. C., Haeusler, E. H., and de Paiva, V., editors, *Advances in Natural Deduction. A Celebration of Dag Prawitz's Work*, volume 39 of *Trends in Logic*, pages 93–101. Springer Netherlands.

Leszczyńska, D. (2004). Socratic proofs for some normal modal propositional logics. *Logique et Analyse*, 47(185-188):259–285.

Leszczyńska, D. (2007). *The Method of Socratic Proofs for Normal Modal Propositional Logics*. Adam Mickiewicz University Press, Poznań.

Leszczyńska-Jasion, D. (2008). The method of Socratic proofs for modal propositional logics: K5, S4.2, S4.3, S4M, S4F, S4R and G. *Studia Logica*, 89(3):371–405.

Leszczyńska-Jasion, D. (2009). A loop-free decision procedure for modal propositional logics K4, S4 and S5. *Journal of Philosophical Logic*, 38(2):151–177.

Leszczyńska-Jasion, D. (2015). Socratic translations: the case of analytic tableaux. Technical Report 1(6), Department of Logic and Cognitive Science, Institute of Philosophy, Adam Mickiewicz University, Poznań. Available on `https://intquestpro.wordpress.com/resources/reports/`.

Leszczyńska-Jasion, D. (2017). On certain generalizations of the concepts of erotetic implication and erotetic search scenario. Technical Report 2(11), Department of Logic and Cognitive Science, Institute of Philosophy, Adam Mickiewicz University, Poznań. Available on `https://intquestpro.wordpress.com/resources/reports/`.

Leszczyńska-Jasion, D., Ignaszak, M., and Chlebowski, S. (2018). Rasiowa-Sikorski deduction systems with the rule of cut. A case study. *Studia Logica*.

Leszczyńska-Jasion, D. and Łupkowski, P. (2016). Erotetic search scenarios and three-valued logic. *Journal of Logic, Language and Information*, 25(1):51–76.

Leszczyńska-Jasion, D., Urbański, M., and Wiśniewski, A. (2013). Socratic trees. *Studia Logica*, 101(5):959–986.

Łupkowski, P. (2015). Question dependency in terms of compliance and erotetic implication. *Logic and Logical Philosophy*, 24:357–376.

Łupkowski, P. (2016). *Logic of Questions in the Wild. Inferential Erotetic Logic in Information Seeking Dialogue Modelling*. College Publications, London.

Łupkowski, P. (2017). IEL-based formal dialogue system for tutorials. *Logic and Logical Philosophy*, 26(3):287–320.

Łupkowski, P. and Leszczyńska-Jasion, D. (2015). Generating cooperative question-responses by means of erotetic search scenarios. *Logic and Logical Philosophy*, 24(1):61–78.

Naur, P. (1961). A course of ALGOL 60 programming. *Journal Algol Bulletin*, Sup 9:1–38.

Negri, S. (2005). Proof analysis in modal logic. *Journal of Philosophical Logic*, 34:507–544.

Negri, S. and von Plato, J. (2001). *Structural Proof Theory*. Cambridge University Press, Cambridge.

Ohnishi, M. and Matsumoto, K. (1957). Gentzen method in modal calculi. *Osaka Mathematical Journal*, 9(2):113–130.

Ohnishi, M. and Matsumoto, K. (1959). Gentzen method in modal calculi. II. *Osaka Mathematical Journal*, 11(2):115–120.

Orłowska, E. (1996). Relational proof systems for modal logics. In Wansing, H., editor, *Proof Theory of Modal Logic*, volume 2 of *Applied Logic*, pages 55–78. Springer, Dordrecht.

Orłowska, E. and Golińska-Pilarek, J. (2011). *Dual Tableaux: Foundations, Methodology, Case Studies*, volume 33 of *Trends in Logic*. Springer.

Paleo, B. W., editor (2017). *Towards an Encyclopaedia of Proof Systems*. College Publications, London.

Peliš, M. and Majer, O. (2011). Logic of questions and public announcements. In Bezhanishvili, N., Löbner, S., Schwabe, K., and Spada, L., editors, *Logic, Language, and Computation*, pages 145–157, Berlin, Heidelberg. Springer Berlin Heidelberg.

Peliš, M. (2016). *Inferences with Ignorance: Logics of Questions*. Karolinum Press, Prague.

Peliš, M. and Majer, O. (2010). Logic of questions from the viewpoint of dynamic epistemic logic. In Peliš, M., editor, *The Logica Yearbook 2009*, pages 157–172. College Publications.

Pfenning, F. (2001). Logical frameworks. *Handbook of Automated Reasoning*, 2:1063–1147.

von Plato, J. (2016). The development of proof theory. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2016 edition.

Poggiolesi, F. (2011). *Gentzen Calculi for Modal Propositional Logic*, volume 32 of *Trends in Logic*. Springer, Dordrecht, Heidelberg, London, New York.

Priest, G. (2008). *An Introduction to Non-Classical Logic: From If to Is*. Cambridge Introductions to Philosophy. Cambridge University Press.

Pudlák, P. (1998). The lengths of proofs. In Buss, S., editor, *Handbook of Proof Theory*, pages 547–637. Elsevier.

Rasiowa, H. and Sikorski, R. (1960). On the Gentzen theorem. *Fundamenta Mathematicae*, 48:57–69.

Rasiowa, H. and Sikorski, R. (1963). *The Mathematics of Metamathematics*. Polish Academy of Sciences, Warszawa.

Rautenberg, W. (1983). Modal tableau calculi and interpolation. *Journal of Philosophical Logic*, 12(4):403–423.

Rosen, K. H. (2012). *Discrete Mathematics and Its Applications*. The McGraw-Hill Companies, New York, seventh edition.

Russo, A. (1995). Modal labelled deductive systems. Technical report, Imperial College of Science, Technology and Medicine, Department of Computing, London.

Schwichtenberg, H. and Wainer, S. S. (2012). *Proofs and Computations*. Cambridge University Press.

Shoesmith, D. J. and Smiley, T. J. (1978). *Multiple-Conclusion Logic*. Cambridge University Press, Cambridge.

Skura, T. F. (2005). Intuitionistic Socratic procedures. *Journal of Applied Non-Classical Logics*, 15(4):453–464.

Skura, T. F. and Wiśniewski, A. (2015). A system for proper multiple-conclusion entailment. *Logic and Logical Philosophy*, 24:241–253.

Smullyan, R. M. (1968). *First-Order Logic*. Springer-Verlag, Berlin, Heidelberg, New York.

Švarný, P., Majer, O., and Peliš, M. (2014). Erotetic epistemic logic in private communication protocol. In Dančák, M. and Punčochář, V., editors, *The Logica Yearbook 2013*, pages 223–238. College Publications.

Szczepiński, R. (2018). The method of Socratic proofs for first degree entailment. Unpublished manuscript received from the author in March 2018.

Troelstra, A. S. and Schwichtenberg, H. (2000). *Basic Proof Theory*. Cambridge University Press, Cambridge, second edition.

Urbański, M. (2001a). Remarks on synthetic tableaux for classical propositional calculus. *Bulletin of the Section of Logic*, 30(4):194–204.

Urbański, M. (2001b). Synthetic tableaux and erotetic search scenarios: extension and extraction. *Logique et Analyse*, 173-174-175:69–91.

Urbański, M. (2002a). Synthetic tableaux for Łukasiewicz's calculus Ł3. *Logique et Analyse*, 177-178:155–173.

Urbański, M. (2002b). *Tabele syntetyczne a logika pytań (Synthetic Tableaux and the Logic of Questions)*. Maria Curie-Sklodowska University Press, Lublin.

Urbański, M. (2004). How to synthesize a paraconsistent negation. The case of CLuN. *Logique et Analyse*, 185-188:319–333.

Urbański, M. (2005). Tableaux, abduction and truthlikeness. Technical report, Instytut Psychologii UAM, Poznań.

Urbański, M. (2010). *Rozumowania abducyjne (Abductive Reasoning)*. Adam Mickiewicz University Press, Poznań.

Urbański, M. and Łupkowski, P. (2010). Erotetic search scenarios: revealing interrogator's hidden agenda. In Łupkowski, P. and Purver, M., editors, *Semantics and Pragmatics of Dialogue*, pages 67–74. Polskie Towarzystwo Kognitywistyczne, Poznań.

Urbański, M., Paluszkiewicz, K., and Urbańska, J. (2016). From real data to formal models. An analysis of solutions to Erotetic Reasoning Test task. In Paglieri, F., Bonelli, L., and Felletti, S., editors, *The Psychology of Argument. Cognitive Approaches to Argumentation and Persuasion*, pages 31–46. College Publications, London.

Viganò, L. (2000). *Labelled Non-Classical Logics*. Springer.

Wansing, H. (2002). Sequent systems for modal logics. In Gabbay, D. and Guenthner, F., editors, *Handbook of Philosophical Logic*, volume 8, pages 61–145. Kluwer Academic Publishers, second edition.

Wiśniewski, A. (1994). Erotetic implications. *Journal of Philosophical Logic*, 23:173–195.

Wiśniewski, A. (1995). *The Posing of Questions: Logical Foundations of Erotetic Inferences*. Kluwer Academic Publishers, Dordrecht, Boston, London.

Wiśniewski, A. (1997). Kubiński's theory of questions. In Wiśniewski, A. and Zygmunt, J., editors, *Erotetic Logic, Deontic Logic and Other Logical Matters*, pages 29–50. Wroclaw University Press, Wrocław.

Wiśniewski, A. (2001). Questions and inferences. *Logique et Analyse*, 173-175:5–43.

Wiśniewski, A. (2003). Erotetic search scenarios. *Synthese*, 134(3):389–427.

Wiśniewski, A. (2004). Socratic proofs. *Journal of Philosophical Logic*, 33(3):299–326.

Wiśniewski, A. (2006). A right-sided Socratic calculus for classical logic. Research Report, Adam Mickiewicz University, Poznań.

Wiśniewski, A. (2013). *Questions, Inferences, and Scenarios*. College Publications, London.

Wiśniewski, A. (2015). Semantics of questions. In S. Lappin, C. F., editor, *The Handbook of Contemporary Semantic Theory*, pages 273–313. Wiley-Blackwell, Oxford, second edition.

Wiśniewski, A. (2016). An axiomatic account of question evocation: the propositional case. *Axioms*, 5.

Wiśniewski, A. (2017). Generalized entailments. *Logic and Logical Philosophy*, 26:321–356.

Wiśniewski, A. (2018). Deduction and reduction theorems for inferential erotetic logic. *Studia Logica*, 106:295–309.

Wiśniewski, A. and Leszczyńska-Jasion, D. (2015). Inferential erotetic logic meets inquisitive semantics. *Synthese*, 192:1585–1608.

Wiśniewski, A. and Pogonowski, J. (2010). Interrogatives, recursion, and incompleteness. *Journal of Logic and Computation*, 20(6):1187–1199.

Wiśniewski, A. and Shangin, V. (2006). Socratic proofs for quantifiers. *Journal of Philosophical Logic*, 35(2):147–178.

Wiśniewski, A. and Shangin, V. (2007). Некоторые допустимые правила дла системы сократического вывода (Some admissible rules in the system of Socratic proofs). *Вестник Московского Университета. Серия*, 7(5):77–88.

Wiśniewski, A., Vanackere, G., and Leszczyńska, D. (2005). Socratic proofs and paraconsistency: a case study. *Studia Logica*, 80(2-3):433–468.

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Subject index

List of symbols and abbreviations

SA	set of answers (methodology), page 16
$\subseteq$	set inclusion, page 17
$\subset$	proper set inclusion, page 17
2^X	power set of X, page 17
IEL	Inferential Erotetic Logic, page 17
IMI	Interrogative Model of Inquiry, page 21
CPL	classical propositional logic, page 33
FOL	classical first-order logic, page 33
$\mathcal{L}_{\mathsf{CPL}}$	language of CPL, page 33
VAR	the set of propositional variables of $\mathcal{L}_{\mathsf{CPL}}$, page 33
BNF	Backus-Naur form, page 34
$\mathcal{L}_{\mathsf{FOL}}$	language of FOL, page 34
$A[x/t]$	the result of substitution, page 35
$\bigvee S$	the disjunction of the terms of S, page 35
$\bigwedge S$	the conjunction of the terms of S, page 35
$\mathcal{L}^{?}_{\vdash \mathsf{L}}$	language with sequents and questions built upon $\mathcal{L}_{\mathsf{L}}$, page 36
?	question-forming operator in $\mathcal{L}^{?}_{\vdash \mathsf{L}}$, page 36
$\vdash$	sequent-forming operator in $\mathcal{L}^{?}_{\vdash \mathsf{L}}$, page 36
ng	$\mathcal{L}^{?}_{\vdash \mathsf{L}}$-negation, page 36
&	$\mathcal{L}^{?}_{\vdash \mathsf{L}}$-conjunction, page 36
d-wff	declarative well-formed formula, page 37
α, β	conjunctive-, disjunctive-type formulas, page 40
γ, δ	universal-, existential-type formulas, page 40
$\overline{A}$	complement of a formula A, page 39
κ	κ-formulas, page 41
$S(A)$	sequence S with A among its terms, page 42
$\mathbb{E}^{\mathbf{PQ}}$	erotetic calculus for FOL with both-sided sequents, page 43
$\mathbb{E}^{\mathbf{RPQ}}$	erotetic calculus for FOL with right-sided sequents, page 45

$\mathbb{E}^{\mathbf{LPQ}}$	erotetic calculus for FOL with left-sided sequents, page 46
$t_i^{\mathcal{I}}[\sigma]$	interpretation of term t_i in $\mathcal{I}$ under σ, page 48
$\models$	entailment, page 49
$\mathbf{E}^{\mathbf{PQ}}$	erotetic calculus by Wiśniewski, page 54
$\mathcal{M}$	language of modal logics, page 70
$FORM_{\mathcal{M}}$	the set of formulas of $\mathcal{M}$, page 70
π, ν	possibility-, necessity-type formulas, page 70
$\mathcal{M}^{?}_{\vdash}$	language with sequents and questions built upon $\mathcal{M}$, page 71
σ/i	index ending with numeral i, page 72
$\mathbf{I}_W\{\phi\}$	the set of numerals in ϕ, page 72
$\mathbf{I}_R\,[\phi]$	the relation between the numerals in ϕ, page 73
$\mathbb{E}^{\mathsf{L}}$	erotetic calculus for modal logic L, page 73
$\mathcal{L}^{?}_{\mathsf{CPL}}$	language with questions built upon $\mathcal{L}_{\mathsf{CPL}}$, page 84
dQ	the set of direct answers to Q, page 84
$\Vert\!\models$	multiple-conclusion entailment, page 84
$\mathsf{Im}(Q, X, Q^*)$	erotetic implication, page 84
afQ	affirmative answer to Q, page 92
ngQ	negative answer to Q, page 92
MiES	Minimal Erotetic Semantics, page 92
$\mathbf{I}_R[\varphi]^{\mathsf{L}}$	the closure of $\mathbf{I}_R[\varphi]$ with respect to L, page 98
$FORM_{\mathsf{L}}$	the set of formulas of $\mathcal{L}_{\mathsf{L}}$, page 102
$\vdash_{\mathsf{L}}$	consequence in L, page 102
$\vdash^{bc}_{\mathsf{L}}$	by-cases-consequence in L, page 104
$\Vdash_{\mathsf{L}}$	multiple-conclusion consequence in L, page 107
$\mathbf{Tr}(\mathbf{s})$	the Socratic tree determined by $\mathbf{s}$, page 115
$\mathbf{G}^{\mathbf{PQ}}$	sequent calculus for FOL, page 121
$\mathbb{G}^{\mathbf{PQ}}$	sequent calculus for FOL, page 121
$\mathbb{G}^{\mathsf{K}\tau}$	sequent calculus for K, page 127
$\mathbb{E}^{\mathsf{K}\tau}$	erotetic calculus for K corresponding to $\mathbb{G}^{\mathsf{K}\tau}$, page 127
$\mathcal{M}^{imp}_{\vdash}$	language with sequents built upon $\mathcal{M}$, page 127
Ax	axiom scheme in $\mathbb{G}^{\mathsf{K}\tau}$, page 132
Ax_{+W}	derived axiom scheme in $\mathbb{G}^{\mathsf{K}\tau}$, page 132
$\sharp^i_j$	operation removing '$\Box$', page 135
$\flat^i_j$	operation removing '$\Diamond$', page 135

$L^{\mathbb{E}}_{\Diamond\Box}, R^{\mathbb{E}}_{\Box\Diamond}$	derived rules in $\mathbb{E}^{\mathsf{K}\tau}$, page 136
τ	translation function, page 144
$\tau_{\mathbf{s}}$	relabelling function, page 146
$\langle X, R\rangle \setminus \mathbf{n}$	operation of deleting $\mathbf{n}$ from $\langle X, R\rangle$, page 152
$L^{\mathbb{G}}_{\Diamond+W}$	$L^{\mathbb{G}}_{\Diamond}$ with weakening, a derived rule in $\mathbb{G}^{\mathsf{K}\tau}$, page 157
$R^{\mathbb{G}}_{\Box+W}$	$R^{\mathbb{G}}_{\Box}$ with weakening, a derived rule in $\mathbb{G}^{\mathsf{K}\tau}$, page 157
$\mathbb{G}^{\mathsf{D}\tau}$	sequent calculus for D, page 164
$R^{\mathbb{G}}_{D+W}$	derived rule in $\mathbb{G}^{\mathsf{D}\tau}$, page 164
$\mathbb{G}^{\mathsf{T}\tau}$	sequent calculus for T, page 164
$\mathbb{G}^{\mathsf{K4}\tau}$	sequent calculus for $\mathsf{K4}$, page 164
$\sharp$	operation removing '$\Box$', page 165
$\flat$	operation removing '$\Diamond$', page 165
$\mathbb{G}^{\mathsf{D4}\tau}$	sequent calculus for $\mathsf{D4}$, page 165
$\mathbb{G}^{\mathsf{S4}\tau}$	sequent calculus for $\mathsf{S4}$, page 166
$\mathbb{E}^{\mathsf{D}\tau}$	erotetic calculus for D, page 167
$R^{\mathbb{E}}_{Dgen}$	derived rule in $\mathbb{E}^{\mathsf{D}\tau}$, page 167
$\sharp D^{i}_{j}$	operation removing '$\Box$', version for D, page 167
$\flat D^{i}_{j}$	operation removing '$\Diamond$', version for D, page 168
$\mathbb{E}^{\mathsf{T}\tau}$	erotetic calculus for T, page 171
τ_{T}, $\tau_{\mathsf{T}\mathbf{s}}$	translation and relabelling function for T, page 171
$\mathbb{E}^{\mathsf{K4}\tau}$	erotetic calculus for $\mathsf{K4}$, page 175
$\sharp_{j}$	operation removing '$\Box$' for $\mathsf{K4}$, page 175
$\flat_{j}$	operation removing '$\Diamond$' for $\mathsf{K4}$, page 175
$L^{\mathbb{E}}_{trans}$	derived rule in $\mathbb{E}^{\mathsf{K4}\tau}$, page 176
$R^{\mathbb{E}}_{trans}$	derived rule in $\mathbb{E}^{\mathsf{K4}\tau}$, page 176
τ_{4L}, τ_{4R}	auxiliary translation functions for $\mathsf{K4}$, page 176
τ_{4}	translation function for $\mathsf{K4}$, page 177
$L^{\mathbb{G}}_{trans+W}$	derived rule in $\mathbb{G}^{\mathsf{K4}\tau}$, page 178
$R^{\mathbb{G}}_{trans+W}$	derived rule in $\mathbb{G}^{\mathsf{K4}\tau}$, page 178
$\tau_{4\mathbf{s}}$	relabelling function for $\mathsf{K4}$, page 179
$\mathbb{E}^{\mathsf{D4}\tau}$	erotetic calculus for $\mathsf{D4}$, page 180
$L^{\mathbb{E}}_{Dgen}$	derived rule in $\mathbb{E}^{\mathsf{D4}\tau}$, page 180
$R^{\mathbb{E}}_{Dgen2}$	derived rule in $\mathbb{E}^{\mathsf{D4}\tau}$, page 180