Fréchet spaces of non-archimedean valued continuous functions

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Abstract

Let X be an ultraregular space and let $\mathbb{K}$ be a complete non-archimedean non-trivially valued field. Assume that the locally convex space $E = C_c(X;\mathbb{K})$ of all continuous functions from X to $\mathbb{K}$ with the topology τ_c of uniform convergence on compact subsets of X is a Fréchet space. We shall prove that E has an orthogonal basis consisting of $\mathbb{K}$-valued characteristic functions of clopen (i.e. closed and open) subsets of X and that it is isomorphic to the product of a countable family of Banach spaces with an orthonormal basis.

1 Introduction

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$. For fundamentals on normed spaces and Hausdorff locally convex spaces (lcs) we refer to [14] and [13].

A Hausdorff topological space X is *ultraregular* if for any closed subset F of X and any point $x \in (X \setminus F)$ there exists a clopen subset U of X such that $F \subset U$ and $x \in (X \setminus U)$. It is easy to see that X is ultraregular if and only if it is zero-dimensional (i.e. the family of all clopen subsets of X is a base of its topology).

In this paper X will be denoted an ultraregular space. The Hausdorff locally convex space $C_c(X;\mathbb{K})$ of all continuous functions from X to $\mathbb{K}$ with the topology

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[1] *2000 Mathematics Subject Classification: 46S10, 46A04, 46E10.*

Key words : Non-archimedean Fréchet spaces of continuous functions; Schauder and orthogonal bases in non-archimedean locally convex spaces.

The research was supported by the National Center of Science, Poland, grant no. N N201 605340.

τ_c of uniform convergence on compact subsets of X was studied in [13] and in many papers - see [1], [2], [3], [7], [8], [9] and their references.

It is known (see [13, Theorem 3.7.9]) that $C_c(X;\mathbb{K})$ is

(1) normed if and only if X is compact.

(2) metrizable if and and only if X is hemicompact (i.e. X has a fundamental sequence of compact subsets).

(3) complete if and only if X is k_0-space (i.e. a function $f : X \to \mathbb{K}$ is continuous if $f|Y$ is continuous for every compact subset Y of X).

Moreover $C_c(X;\mathbb{K})$ is of countable type if and only if any compact subset of X is ultrametrizable (see [13, Theorem 4.3.2]). If X is locally compact, then the normed space $C_0(X;\mathbb{K})$ of all continuous functions $f : X \to \mathbb{K}$ vanishing at infinity (i.e. such that the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$) with the sup-norm is complete.

Van der Put proved the following theorem (see [14, Theorem 5.22]):

Let k_0 be the prime field of the residue class field k of $\mathbb{K}$. Let X be a locally compact ultraregular space and let $\mathfrak{U}$ be a maximal collection of open compact subsets of X whose k_0-valued characteristic functions are linearly independent.

Then the $\mathbb{K}$-valued charakteristic functions of elements of $\mathfrak{U}$ form an orthonormal basis in the Banach space $C_0(X;\mathbb{K})$.

It is not hard to show that if an ultraregular space X is locally compact and the lcs $C_c(X;\mathbb{K})$ is metrizable, then X is the sum of a sequence (X_n) of pairwise disjoint open compact subsets of X and $C_c(X;\mathbb{K})$ is isomorphic to the product of the Banach spaces $C(X_n;\mathbb{K}), n \in \mathbb{N}$; in particulary, by the van der Put theorem, $C_c(X;\mathbb{K})$ has an orthogonal basis consisting of $\mathbb{K}$-valued characteristic functions of open compact subsets of X (see Corollary 7).

Nevertheless there exist non locally compact ultraregular spaces X such that the locally convex space $C_c(X;\mathbb{K})$ is a Fréchet space (see Proposition 9).

Using the van der Put theorem we shall prove our main result (see Theorem 11):

Let X be an ultraregular space such that the lcs $C_c(X;\mathbb{K})$ is a Fréchet space. Then $C_c(X;\mathbb{K})$ has an orthogonal basis consisting of $\mathbb{K}$-valued characteristic functions of clopen subsets of X and it is isomorphic to the product $C(X_1;\mathbb{K})\times\prod_{n=1}^{\infty} C_0(X'_n;\mathbb{K})$ of Banach spaces, where (X_n) is an increasing fundamental sequence of compact subsets of X, and X'_n is the locally compact ultraregular space $(X_{n+1} \setminus X_n)$ for $n \in \mathbb{N}$.

2 Preliminaries

$O_{\mathbb{K}} = \{\alpha \in \mathbb{K} : |\alpha| < 1\}$ is a maximal ideal in the subring $B_{\mathbb{K}} = \{\alpha \in \mathbb{K} : |\alpha| \leq 1\}$ of $\mathbb{K}$, so $k = B_{\mathbb{K}}/O_{\mathbb{K}}$ is a field. This field is the *residue class field* of $\mathbb{K}$. The natural homomorphism $B_{\mathbb{K}} \to B_{\mathbb{K}}/O_{\mathbb{K}}$ is usually written $\alpha \to \overline{\alpha}$.

Let E be a linear space. A subset A of E is *absolutely convex* if for all $\alpha, \beta \in B_{\mathbb{K}}$ and $x, y \in A$ we have $\alpha x + \beta y \in A$.

A *seminorm* on a linear space E is a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x + y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$ (*the strong triangle inequality*). A seminorm p on E is a *norm* if $\ker p = \{0\}$.

(One may consider usual seminorms on E that satisfy the usual triangle inequality. However, such seminorms are pathological from the viewpoint of duality theory, see [12], and therefore they are ignored in the theory of locally convex spaces over non-archimedean fields)

For any seminorm p on E the map $\overline{p} : E_p \to [0, \infty), x + \ker p \to p(x)$ is a norm on $E_p = (E/\ker p)$.

The set of all continuous seminorms on a lcs E is denoted by $\mathcal{P}(E)$. A family $\mathcal{B} \subset \mathcal{P}(E)$ is a *base* in $\mathcal{P}(E)$ if for every $p \in \mathcal{P}(E)$ there exists $q \in \mathcal{B}$ with $q \geq p$.

A lcs E is *of countable type* if for any $p \in \mathcal{P}(E)$ the normed space $(E_p, \overline{p})$ contains a linearly dense countable subset.

Let E and F be locally convex spaces. The space of all linear continuous maps from E to F is denoted by $L(E; F)$. An operator $T \in L(E, F)$ is an *isomorphism* if T is injective, surjective and the inverse map T^{-1} is continuous. E is *isomorphic* to F if there exists an isomorphism $T : E \to F$. The topological dual of a lcs E we denote by E'.

A *Fréchet space* is a metrizable complete lcs.

A *Banach space* is a normed Fréchet space. Any infinite-dimensional Banach space E of countable type is isomorphic to the Banach space $c_0 = c_0(\mathbb{N}; \mathbb{K})$ of all sequences in $\mathbb{K}$ converging to zero with the sup-norm and any closed subspace of c_0 is complemented (see [14, Theorem 3.16]).

A sequence (x_n) in a lcs E is a *basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$. If additionally the coefficient functionals $f_n : E \to \mathbb{K}$, $x \to \alpha_n$ $(n \in \mathbb{N})$ are continuous, then (x_n) is a *Schauder basis* in E.

Let E be a lcs. A sequence $(x_n) \subset E$ is *orthogonal with respect to* $\mathcal{B} \subset \mathcal{P}(E)$ if $p(\sum_{i=1}^{n} \alpha_i x_i) = \max_{1 \leq i \leq n} p(\alpha_i x_i)$ for all $p \in \mathcal{B}, n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$.

Every Schauder basis in a Fréchet space F is orthogonal with respect to some (non-decreasing) base (p_k) in $\mathcal{P}(F)$ ([4], Proposition 1.7).

3 Results

Let Γ denote an arbitrary non empty index set. Clearly, the family $\mathcal{F} = \mathcal{F}_\Gamma$ of all non empty finite subsets of Γ is directed by set inclusion. Let Z be a lcs.

We say that $(z_\alpha)_{\alpha\in\Gamma} \subset Z$ is

- *strictly convergent to* 0 *in* Z if for every neighbourhood U of 0 in Z the set $\{\alpha \in \Gamma : z_\alpha \notin U\}$ is finite.

- *summable in* Z if the net $(\sum_{\alpha\in A} z_\alpha)_{A\in\mathcal{F}}$ is convergent in Z; then by the sum of $(z_\alpha)_{\alpha\in\Gamma}$ we mean the limit of the net $(\sum_{\alpha\in A} z_\alpha)_{A\in\mathcal{F}}$ and denote it by $\sum_{\alpha\in\Gamma} z_\alpha$.

- *orthogonal with respect to* $\mathcal{B} \subset \mathcal{P}(Z)$ if

$$\forall p \in \mathcal{B}\ \forall A \in \mathcal{F}\ \forall (t_\alpha)_{\alpha\in A} \subset \mathbb{K} : p\left(\sum_{\alpha\in A} t_\alpha z_\alpha\right) = \max_{\alpha\in A} p(t_\alpha z_\alpha).$$

- *orthogonal in* Z if it is orthogonal with respect to some base $\mathcal{B}$ of $\mathcal{P}(Z)$.

- an *orthogonal basis in* Z if it is orthogonal and linearly dense in Z and $z_\alpha \neq 0$ for all $\alpha \in \Gamma$.

- a *Schauder basis in* Z if there exists $(z'_\alpha)_{\alpha\in\Gamma} \subset Z'$ such that $\sum_{\alpha\in\Gamma} z'_\alpha(z) z_\alpha = z$ for every $z \in Z$ and $z'_\alpha(z_\beta) = \delta_{\alpha,\beta}$ for all $\alpha, \beta \in \Gamma$; then for every $(t_\alpha) \subset \mathbb{K}$ and $z \in Z$ with $\sum_{\alpha\in\Gamma} t_\alpha z_\alpha = z$ we have $t_\alpha = z'_\alpha(z)$ for all $\alpha \in \Gamma$.

Let $(Z, \|\cdot\|)$ be a normed space. We say that $(z_\alpha)_{\alpha\in\Gamma} \subset Z$ is an *orthonormal basis* in Z, if it is linearly dense in Z, $\|z_\alpha\| = 1$ for $\alpha \in \Gamma$ and

$$\forall A \in \mathcal{F}\ \forall (t_\alpha)_{\alpha\in A} \subset \mathbb{K} : \|\sum_{\alpha\in A} t_\alpha z_\alpha\| = \max_{\alpha\in A} |t_\alpha|.$$

Clearly, any orthonormal basis $(z_\alpha)_{\alpha\in\Gamma}$ in Z is an orthogonal basis in Z.

By the strong triangle inequality a series $\sum_{n=1}^\infty z_n$ in a Fréchet space Z is convergent if and only if the sequence (z_n) is convergent to 0. We can generalize this result.

Proposition 1. *Let* Z *be a Fréchet space. Then* $(z_\alpha)_{\alpha\in\Gamma} \subset Z$ *is summable in* Z *if and only if* $(z_\alpha)_{\alpha\in\Gamma}$ *is strictly convergent to* 0 *in* Z.

Proof. ($\Rightarrow$) Let $z = \sum_{\alpha\in\Gamma} z_\alpha$ and let U be an absolutely convex neighbourhood of 0 in Z. Then there is $\Gamma_U \in \mathcal{F}_\Gamma$ such that for every $\Gamma_0 \in \mathcal{F}_\Gamma$ with $\Gamma_0 \supset \Gamma_U$ we have $(z - \sum_{\alpha\in\Gamma_0} z_\alpha) \in U$. Hence for $\sigma \in (\Gamma \setminus \Gamma_U)$ and for $\Gamma_0 = \Gamma_U \cup \{\sigma\}$ we have

$$z_\sigma = \left(z - \sum_{\alpha\in\Gamma_U} z_\alpha\right) - \left(z - \sum_{\alpha\in\Gamma_0} z_\alpha\right) \in U - U = U.$$

Thus the set $\{\sigma \in \Gamma : z_\sigma \notin U\} \subset \Gamma_U$ is finite, so $(z_\alpha)_{\alpha\in\Gamma}$ is strictly convergent to 0 in Z.

($\Leftarrow$) Clearly, the set $\Gamma^* = \{\alpha \in \Gamma : z_\alpha \neq 0\}$ is countable and $\lim_n z_{\alpha_n} = 0$, where $\{\alpha_n : n \in \mathbb{N}\} = \Gamma^*$. Thus the series $\sum_{n=1}^\infty z_{\alpha_n}$ is convergent in Z. Put $z = \sum_{n=1}^\infty z_{\alpha_n}$. Let U be an absolutely convex neighbourhood of 0 in Z. Let $\Gamma_U = \{\alpha \in \Gamma : z_\alpha \notin U\}$ and $\Gamma_0 \in \mathcal{F}_\Gamma$ with $\Gamma_0 \supset \Gamma_U$. Then

$$z - \sum_{\alpha\in\Gamma_0} z_\alpha = \sum_{\alpha\in\Gamma^*} z_\alpha - \sum_{\alpha\in\Gamma_0\cap\Gamma^*} z_\alpha = \sum_{\alpha\in(\Gamma^*\setminus\Gamma_0)} z_\alpha \in U.$$

Thus $(z_\alpha)_{\alpha\in\Gamma}$ is summable in Z and $\sum_{\alpha\in\Gamma} z_\alpha = z$. $\square$

Corollary 2. *Let $(t_\alpha)_{\alpha\in\Gamma} \subset \mathbb{K}$. Let Z be a Banach space and let $(z_\alpha)_{\alpha\in\Gamma} \subset Z$ with $\|z_\alpha\| = 1$ for all $\alpha \in \Gamma$. Then $(t_\alpha z_\alpha)_{\alpha\in\Gamma}$ is summable in Z if and only if $(t_\alpha)_{\alpha\in\Gamma} \in c_0(\Gamma; \mathbb{K})$ i.e. the set $\{\alpha \in \Gamma : |t_\alpha| \geq \varepsilon\}$ is finite for every $\varepsilon > 0$.*

We have the following generalizations of [4, Propositions 1.4 and 1.7].

Proposition 3. *Any orthogonal basis $(z_\alpha)_{\alpha\in\Gamma}$ in a lcs Z is a Schauder basis in Z.*

Proof. The linear span S of $(z_\alpha)_{\alpha\in\Gamma}$ is dense in Z. Clearly, $(z_\alpha)_{\alpha\in\Gamma}$ is orthogonal with respect to some base $\mathcal{B}$ of $\mathcal{P}(Z)$. Let $A \in \mathcal{F}$ and $(t_\alpha)_{\alpha\in A} \subset \mathbb{K}$ with $\sum_{\alpha\in A} t_\alpha z_\alpha = 0$. Then $\max_{\alpha\in A} q(t_\alpha z_\alpha) = q(\sum_{\alpha\in A} t_\alpha z_\alpha) = 0$ for every $q \in \mathcal{B}$. Hence $t_\alpha = 0$ for all $\alpha \in A$. Thus $(z_\alpha)_{\alpha\in\Gamma}$ is linearly independent, so there exists $(z_\alpha^*)_{\alpha\in\Gamma} \subset S^*$ with $z_\alpha^*(z_\beta) = \delta_{\alpha,\beta}$ for all $\alpha, \beta \in \Gamma$. Then $\sum_{\alpha\in\Gamma} z_\alpha^*(z) z_\alpha = z$ for every $z \in S$.

For $A \in \mathcal{F}$ we put

$$P_A^* : S \to S, z \to \sum_{\alpha\in A} z_\alpha^*(z) z_\alpha.$$

Then $\max_{A\in\mathcal{F}} q(P_A^*(z)) \leq q(z)$ for all $z \in S$ and $q \in \mathcal{B}$. Hence $(z_\alpha^*)_{\alpha\in\Gamma} \subset S'$.

For $A \in \mathcal{F}$ we put

$$P_A' : Z \to Z, z \to \sum_{\alpha\in A} z_\alpha'(z) z_\alpha,$$

where $z_\alpha' \in Z'$ is a unique extension of $z_\alpha^*, \alpha \in \Gamma$.

Clearly $\max_{A\in\mathcal{F}} q(P'_A(z)) \leq q(z)$ for all $z \in Z$ and $q \in \mathcal{B}$.

We shall prove that $(P'_A(z))_{A\in\mathcal{F}}$ is convergent to z in Z for every $z \in Z$. Let $z \in Z$ and $q \in \mathcal{B}$. Then there is a non-zero element $s \in S$ with $q(z - s) < 1$. Put $A_S = \{\alpha \in \Gamma : z'_\alpha(s) \neq 0\}$. Clearly $A_S \in \mathcal{F}$. Let $A \in \mathcal{F}$ with $A \supset A_S$. Then $P'_A(s) = s$, so

$$q(P'_A(z) - z) = q(P'_A(z - s) - (z - s)) \leq q(z - s) < 1.$$

It follows that $\lim_{A\in\mathcal{F}} P'_A(z) = z$ for every $z \in Z$. Thus $\sum_{\alpha\in\Gamma} z'_\alpha(z) z_\alpha = z$ for every $z \in Z$, so $(z_\alpha)_{\alpha\in\Gamma}$ is a Schauder basis in Z. □

Proposition 4. *Any Schauder basis $(z_\alpha)_{\alpha\in\Gamma}$ in a Fréchet space Z is an orthogonal basis in Z.*

Proof. There exists $(z'_\alpha)_{\alpha\in\Gamma} \subset Z'$ such that $\sum_{\alpha\in\Gamma} z'_\alpha(z) z_\alpha = z$ for every $z \in Z$ and $z'_\alpha(z_\beta) = \delta_{\alpha,\beta}$ for all $\alpha, \beta \in \Gamma$. Let $z \in Z$ and $q \in \mathcal{P}(Z)$. Then $\lim_{A\in\mathcal{F}} P_A(z) = z$, where

$$P_A : Z \to Z, z \to \sum_{\alpha\in A} z'_\alpha(z) z_\alpha$$

for $A \in \mathcal{F}$. Thus there exists $A_0 \in \mathcal{F}$ such that $q(P_A(z) - z) < 1$ for every $A \in \mathcal{F}$ with $A \supset A_0$. Let $A \in \mathcal{F}, A_1 = A \cup A_0$ and $A_2 = A_0 \setminus A$. Then

$$P_A(z) = P_{A_1}(z) - P_{A_2}(z) = (P_{A_1}(z) - z) + z - P_{A_2}(z),$$

so

$$q(P_A(z)) \leq \max\{1, q(z), q(P_{A_2}(z))\} \leq \max\{1, q(z), C(z)\},$$

where $C(z) = \max\{q(P_{\{\alpha\}}(z)) : \alpha \in A_0\}$. Thus $(P_A)_{A\in\mathcal{F}}$ is pointwise bounded; so P_A for $A \in \mathcal{F}$ are equicontinuous, by the Banach-Steinhaus theorem (see [13, Theorems 7.1.3 and 7.1.5]). Then for every $q \in \mathcal{P}(Z)$ the seminorm

$$q^* : Z \to [0, \infty), z \to \sup_{\alpha\in\Gamma} q(P_{\{\alpha\}}(z))$$

is well defined, continuous and $q^* \geq q$. Thus $\mathcal{B} = \{q^* : q \in \mathcal{P}(Z)\}$ is a base of $\mathcal{P}(Z)$. Moreover $(z_\alpha)_{\alpha\in\Gamma}$ is orthogonal with respect to $\mathcal{B}$, since for every $A \in \mathcal{F}$ and $(t_\alpha)_{\alpha\in A} \subset \mathbb{K}$ we have

$$q^*\left(\sum_{\alpha\in A} t_\alpha z_\alpha\right) = \max_{\alpha\in A} q(t_\alpha z_\alpha) = \max_{\alpha\in A} q^*(t_\alpha z_\alpha). \ \square$$

In the proofs of Proposition 6 and Theorem 11 we shall need the following generalization of [13, Theorem 9.2.4(i)].

Proposition 5. *Let E_i for $i \in I$ be a lcs with an orhogonal basis $(e_{i,\alpha})_{\alpha\in\Gamma_i}$. For $i \in I$ and $\alpha \in \Gamma_i$ we put $e_{(i,\alpha)} = (f_{j,\alpha})_{j\in I}$, where $f_{j,\alpha} = e_{i,\alpha}$ if $j = i$ and $f_{j,\alpha} = 0$ if $j \neq i$. Then $(e_{(i,\alpha)})_{i\in I,\alpha\in\Gamma_i}$ is an orthogonal basis in the lcs $E = \prod_{i\in I} E_i$ (with the product topology).*

Proof. Clearly, $(e_{(i,\alpha)})_{i\in I,\alpha\in\Gamma_i}$ is linearly dense in E and $(e_{i,\alpha})_{\alpha\in\Gamma_i}$ is orthogonal with respect to some base $\mathcal{B}_i$ in $\mathcal{P}(E_i), i \in I$.

For any non empty finite subset J of I and $p = (p_j)_{j\in J} \in \prod_{j\in J} \mathcal{B}_j$ the functional

$$p^* : E \to [0, \infty), (x_i)_{i\in I} \to \max_{j\in J} p_j(x_j)$$

is a continuous seminorm on E; it is easy to check that the family $\mathcal{B}$ of all these seminorms forms a base in $\mathcal{P}(E)$. We shall prove that $(e_{(i,\alpha)})_{i\in I,\alpha\in\Gamma_i}$ is orthogonal with respect to $\mathcal{B}$. Let J be a non empty finite subset of I and let $p = (p_j)_{j\in J} \in \prod_{j\in J} \mathcal{B}_j$. Let J_0 be a finite subset of I, let A_j be a finite subset of Γ_j and let $(t_{j,\alpha})_{\alpha\in A_j} \subset \mathbb{K}$ for $j \in J_0$. Put $J_1 = J_0 \cap J$. Then

$$p^*(\sum_{j\in J_0}\sum_{\alpha\in A_j} t_{j,\alpha}e_{(j,\alpha)}) = \max_{j\in J_1} p_j(\sum_{\alpha\in A_j} t_{j,\alpha}e_{j,\alpha}) = \max_{j\in J_1}\max_{\alpha\in A_j} p_j(t_{j,\alpha}e_{j,\alpha})$$

$$= \max_{j\in J_1}\max_{\alpha\in A_j} p^*(t_{j,\alpha}e_{(j,\alpha)}) = \max_{j\in J_0}\max_{\alpha\in A_j} p^*(t_{j,\alpha}e_{(j,\alpha)}),$$

since $p^*(e_{(j,\alpha)}) = 0$ for $j \in (J_0 \setminus J_1), \alpha \in A_j$. □

By k_0 we denote the prime field of the residue field k of $\mathbb{K}$. Let X be an ultraregular space. The family of all clopen (i.e. open and closed) subsets of X we denote by $\mathcal{F}(X)$. The family of all open and compact subsets of X we denote by $\mathcal{F}_c(X)$. The k_0-valued characteristic function of $U \in \mathcal{F}(X)$ we denote by $1_U^{k_0}$; similarly we define $1_U^{\mathbb{K}}$. Let $(U_\alpha)_{\alpha\in\Gamma} \subset \mathcal{F}_c(X)$. We say that $(U_\alpha)_{\alpha\in\Gamma}$ is k_0-*independent* in $\mathcal{F}_c(X)$, if $(1_{U_\alpha}^{k_0})_{\alpha\in\Gamma}$ is linearly independent. If $(U_\alpha)_{\alpha\in\Gamma}$ is a maximal k_0-independent set in $\mathcal{F}_c(X)$, we call it a k_0-*base* in $\mathcal{F}_c(X)$.

It is easy to see that an ultraregular space X is locally compact if the lcs $C_c(X;\mathbb{K})$ has an orthogonal basis $(f_\alpha)_{\alpha\in\Gamma}$ consisting of $\mathbb{K}$-valued characteristic functions of open compact subsets of X. We do not know whether for any locally compact ultraregular space X the lcs $C_c(X;\mathbb{K})$ has an orthogonal basis $(f_\alpha)_{\alpha\in\Gamma}$ consisting of $\mathbb{K}$-valued characteristic functions of open compact subsets of X. Nevertheless we have the following.

Proposition 6. *Let X be a paracompact locally compact ultraregular space. Then X has a partition $(X_i)_{i\in I} \subset \mathcal{F}_c(X)$. The lcs $C_c(X;\mathbb{K})$ is isomorphic to the product*

$\prod_{i\in I} C(X_i;\mathbb{K})$ of Banach spaces. Let $(U_{i,\alpha})_{\alpha\in\Gamma_i}$ be a k_0-base in $\mathcal{F}_c(X_i)$ for $i \in I$. Then $(1^{\mathbb{K}}_{U_{i,\alpha}})_{i\in I,\alpha\in\Gamma_i}$ is an orthogonal basis in $C_c(X;\mathbb{K})$ (consisting of $\mathbb{K}$-valued characteristic functions of open compact subsets of X).

Proof. By [6, Theorem 5.1.27], X has a partition $(X_\beta)_{\beta\in B}$ consisting of clopen Lindelöf subspaces. Let $\beta \in B$. Then X_β is a locally compact ultraregular Lindelöf space. Thus the family of all open compact subsets of X_β is an open cover of X_β, so it has a countable subcover of X_β. It follows that X_β has a countable partition consisting of open compact subsets. Thus X has a partition $(X_i)_{i\in I}$ consisting of open compact subsets. It is easy to see that the linear map

$$T : C_c(X;\mathbb{K}) \to \prod_{i\in I} C(X_i;\mathbb{K}), f \to (f|X_i)_{i\in I}$$

is an isomorphism. By the van der Put theorem and Proposition 5 we get the last part of Proposition 6. □

By Proposition 6 and its proof we get

Corollary 7. *Let X be a locally compact ultraregular space such that the lcs $C_c(X;\mathbb{K})$ is metrizable. Then X has a partition $(X_n) \subset \mathcal{F}_c(X)$. The lcs $C_c(X;\mathbb{K})$ is isomorphic to the product $\prod_{n=1}^{\infty} C(X_n;\mathbb{K})$ of Banach spaces. Let $(U_{n,\alpha})_{\alpha\in\Gamma_n}$ be a k_0-base in $\mathcal{F}_c(X_n)$ for $n \in \mathbb{N}$. Then $(1^{\mathbb{K}}_{U_{n,\alpha}})_{n\in\mathbb{N},\alpha\in\Gamma_n}$ is an orthogonal basis in $C_c(X;\mathbb{K})$. This basis is countable if and only if $C_c(X;\mathbb{K})$ is of countable type.*

We have also the following

Proposition 8. *Let X be an ultraregular space such that the lcs $C_c(X;\mathbb{K})$ is of countable type and metrizable. Then the following conditions are equivalent:*

(1) X is locally compact;

(2) X has a countable partition (X_n) consisting of open compact subsets;

(3) X is ultrametrizable.

Proof. $C_c(X;\mathbb{K})$ is metrizable, so X has a fundamental sequence (Y_n) of compact subsets.

(1) ⇒ (2) X is locally compact, so there exists an increasing sequence (Z_n) of open compact subsets of X, such that $Y_n \subset Z_n$ for every $n \in \mathbb{N}$. Put $X_1 = Z_1$ and $X_n = (Z_n \setminus Z_{n-1})$ for $n > 1$. Clearly, (X_n) is a partition of X consisting of open compact subsets.

(2) $\Rightarrow$ (3) $C_c(X;\mathbb{K})$ is of countable type, so the sets $X_n, n \in \mathbb{N}$, are ultrametrizable; clearly $X = \bigoplus_{n=1}^{\infty} X_n$. Thus X is ultrametrizable, by [6, Theorem 4.2.1] and its proof.

(3) $\Rightarrow$ (1) Suppose, by contrary, that X is not locally compact. Then there exists an $x \in X$ such that the closed ball $B(x;1/n)$ in X is not compact for any $n \in \mathbb{N}$. Thus $B(x;1/n) \not\subset Y_n$ for $n \in \mathbb{N}$. Let $x_n \in (B(x;1/n) \setminus Y_n)$ for $n \in \mathbb{N}$. Then the sequence (x_n) is convergent to x in X, so the set $A = (\{x_n : n \in \mathbb{N}\} \cup \{x\})$ is compact in X. Thus $A \subset Y_n$ for some $n \in \mathbb{N}$; a contradiction. $\square$

Now we shall prove the following.

Proposition 9. *There exist non locally compact ultraregular spaces X such that the locally convex space $C_c(X;\mathbb{K})$ is a Fréchet space.*

Proof. Let (D_n) be a sequence of infinite compact ultraregular spaces. Let d_k be an accumulation point of D_k for $k \in \mathbb{N}$. Put $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n = \prod_{k=1}^{n} D_k \times \prod_{k=n+1}^{\infty} \{d_k\}$ for $n \in \mathbb{N}$. For every $x = (x_k) \in X$ we put $d(x) = \inf\{i \in \mathbb{N} : x_k = d_k \text{ for all } k \geq i\}$.

Consider on $X \subset \prod_{k=1}^{\infty} D_k$ the topology τ generated by the family of all subsets of X of the form $X \cap \prod_{k=1}^{\infty} U_k$, where U_k is a clopen subset of D_k for every $k \in \mathbb{N}$.

The space $X = (X, \tau)$ is ultraregular, since the set $U = X \cap \prod_{k=1}^{\infty} U_k$ is clopen in X, if U_k is a clopen subset of D_k for every $k \in \mathbb{N}$. Indeed, let $x = (x_k) \in (X \setminus U)$. Put $V = X \cap \prod_{k=1}^{\infty} V_k$, where $V_k = U_k$ if $x_k \in U_k$ and $V_k = (D_k \setminus U_k)$, otherwise. Then V is an open subset of X and $x \in V \subset (X \setminus U)$, so U is clopen in X.

We shall prove that (X_n) is a fundamental sequence of compact subsets of X. Let $n \in \mathbb{N}$. The subset X_n of X is compact, since $\tau|X_n$ is the product topology on X_n. Let K be a subset of X such that $K \not\subset X_n$ for any $n \in \mathbb{N}$. Then we can choose inductively a sequence $(x_n) \subset K, x_n = (x_{n,k})$ for $n \in \mathbb{N}$, such that $1 < d(x_n) < d(x_{n+1})$ for $n \in \mathbb{N}$. Put $m_n = d(x_n) - 1$ for $n \in \mathbb{N}$. Let $x_0 = (x_{0,k})$ be an arbitrary element of K and let $n_0 = d(x_0)$. Let V_{m_n} be a clopen neighbourhood of d_{m_n} in D_{m_n} with $x_{n,m_n} \notin V_{m_n}$ for $n \geq n_0$ and let $V_k = D_k$ for all $k \in (\mathbb{N} \setminus \{m_n : n \geq n_0\})$. Then the set $V = K \cap \prod_{k=1}^{\infty} V_k$ is a clopen neighbourhood of x_0 in K and $x_n \notin V$ for all $n \geq n_0$. Thus x_0 is not an accumulation point of the set $\{x_n : n \in \mathbb{N}\}$. Thus K has an infinite subset without accumulation points in K, so K is not compact. It follows that (X_n) is a fundamental sequence of compact subsets of X, so X is hemicompact.

Now we shall prove that X is a k_0-space.

Let U be a subset of X such that $U \cap X_n$ is open in X_n for any $n \in \mathbb{N}$. Then U is open in X. Indeed, let $x = (x_k) \in U$ and $m = d(x)$. The set $U \cap X_m$ is open in X_m and $x \in U \cap X_m$, so for every $1 \leq k \leq m$ there is a clopen neighbourhood U_k of x_k in D_k such that

$$U_{(m)} := \prod_{k=1}^{m} U_k \times \prod_{k=m+1}^{\infty} \{d_k\} \subset U \cap X_m.$$

Let $y = (y_i) \in U_{(m)} \subset U \cap X_{m+1}$. The set $U \cap X_{m+1}$ is open in X_{m+1}, so for every $1 \leq k \leq m+1$ there is a clopen neighbourhood $V_{y,k}$ of y_k in D_k such that $\prod_{k=1}^{m+1} V_{y,k} \times \prod_{k=m+2}^{\infty}\{d_k\} \subset U \cap X_{m+1}$. The family $\{\prod_{k=1}^{m} V_{y,k} : y \in U_{(m)}\}$ is an open cover of the compact set $\prod_{k=1}^{m} U_k$, so for some finite subset $\{y_1, \ldots, y_s\}$ of $U_{(m)}$ we have

$$\prod_{k=1}^{m} U_k \subset \bigcup \{\prod_{k=1}^{m} V_{y_j,k} : 1 \leq j \leq s\}.$$

Then $U_{m+1} := \bigcap_{j=1}^{s} V_{y_j,m+1}$ is a clopen neighbourhood of d_{m+1} in D_{m+1} such that

$$U_{(m+1)} := \prod_{k=1}^{m+1} U_k \times \prod_{k=m+2}^{\infty} \{d_k\} \subset U \cap X_{m+1}.$$

This way we can inductively construct a clopen neighbourhood U_{m+i} of d_{m+i} in D_{m+i} for every $i \in \mathbb{N}$ such that

$$U_{(n)} := \prod_{k=1}^{n} U_k \times \prod_{k=n+1}^{\infty} \{d_k\} \subset U \cap X_n$$

for any $n \geq m$. Hence $x \in X \cap \prod_{k=1}^{\infty} U_k \subset U$; so U is open in X.

It follows that X is a k_0-space. Indeed, let $f : X \to \mathbb{K}$ be a function such that $f|X_n$ is continuous for every $n \in \mathbb{N}$. Let W be an open subset of $\mathbb{K}$. Then $f^{-1}(W) \cap X_n = (f|X_n)^{-1}(W)$ is open in X_n for any $n \in \mathbb{N}$, so $f^{-1}(W)$ is open in X. Thus f is continuous.

Since $\{d_k\}$ is not open in D_k for all $k \in \mathbb{N}$, the subsets $X_n, n \in \mathbb{N}$, have empty interiors in X. Thus any compact subset of X has empty interior. In particular, X is not locally compact. By [13, Theorem 3.7.9], the lcs $C_c(X;\mathbb{K})$ is a Fréchet space. □

Remark 10. *Let X be an ultraregular space constructed in the proof of Proposition 9. (1) By [13, Theorem 4.3.2], the Fréchet space $C_c(X;\mathbb{K})$ is of countable type* *if and only if the spaces $D_n, n \in \mathbb{N}$, are ultrametrizable; (2) For any non empty compact subset F of X the $\mathbb{K}$-valued characteristic function $1_F^{\mathbb{K}} : X \to \mathbb{K}$ of K is not continuous.*

By [5, Proposition 1.2], any compact ultraregular space X is ultranormal i.e. any two disjoint closed subsets of X are contained in disjoint clopen subsets of X. Hence, using [5, Lemma 1.1], we infer that $\mathcal{F}_c(Y) = \{V \cap Y : V \in \mathcal{F}_c(X)\}$ for any closed subspace Y of a compact ultraregular space X.

Developing some ideas of [15] we shall prove our main result

Theorem 11. *Let X be an ultraregular space. Assume that the lcs $C_c(X;\mathbb{K})$ is a Fréchet space. Then $C_c(X;\mathbb{K})$ has an orthogonal basis consisting of $\mathbb{K}$-valued characteristic functions of clopen subsets of X and it is isomorphic to the product $C(X_1,\mathbb{K}) \times \prod_{k=1}^{\infty} C_0(X'_k;\mathbb{K})$ of Banach spaces, where (X_n) is an increasing fundamental sequence of compact subsets of X and $X'_k = (X_{k+1} \setminus X_k)$ for $k \in \mathbb{N}$.*

Proof. (A). The space $E = C_c(X;\mathbb{K})$ is metrizable, so X is hemicompact. Let (X_n) be an increasing fundamental sequence of compact subsets of X. Without loss of generality we can assume that $\emptyset \neq X_n \subsetneq X_{n+1}, n \in \mathbb{N}$. For $n \in \mathbb{N}$ we put

$$p_n : E \to [0,\infty), f \to \sup_{x \in X_n} |f(x)|;$$

clearly (p_n) is an increasing base in $\mathcal{P}(E)$.

Put $E_n = C(X_n;\mathbb{K})$ for $n \in \mathbb{N}$. By [11, Theorem 1.1], the continuous linear map

$$\varphi_n : E \to E_n, f \to f|X_n,$$

is surjective for $n \in \mathbb{N}$, and the linear map

$$\pi_n : E_{n+1} \to E_n, f \to f|X_n,$$

is continuous for $n \in \mathbb{N}$. The subspace

$$F = \{(f_n) \in \prod_{n=1}^{\infty} E_n : \pi_n(f_{n+1}) = f_n \text{ for any } n \in \mathbb{N}\}$$

of the Fréchet space $\prod_{n=1}^{\infty} E_n$ is linear and closed. The linear map

$$\varphi : E \to F, \varphi(f) = (\varphi_n(f))$$

is continuous and injective.

We shall prove that φ is surjective. Let $g = (g_n) \in F$. For every $n \in \mathbb{N}$ there exists $f_n \in E$ such that $g_n = \varphi_n(f_n)$. Let $h_n = f_{n+1} - f_n, n \in \mathbb{N}$. Then $h_n \in \ker \varphi_n = \ker p_n$ for $n \in \mathbb{N}$; so $h_n \to 0$ in E. Thus the series $\sum_{n=1}^{\infty} h_n$ is convergent in E to some element $h_0 \in E$. For $h = f_1 + h_0$ we have $\varphi(h) = g$. Indeed, for $m, n \in \mathbb{N}$ with $m \geq n$ we have $\varphi_n(h_m) = 0$, since $\varphi_m(h_m) = 0$. Thus $\varphi_n(h) = \varphi_n(f_1 + \sum_{1 \leq i < n} h_i) = \varphi_n(f_n) = g_n$ for $n \in \mathbb{N}$; so $\varphi(h) = g$.

By the open mapping theorem the operator φ is an isomorphism.

(B). We shall prove that F is isomorphic to the Fréchet space $\prod_{n=1}^{\infty} F_n$, where $F_1 = E_1$ and $F_n = \ker \pi_{n-1}$ for $n > 1$.

Let $X'_k = (X_{k+1} \setminus X_k)$ for $k \in \mathbb{N}$. We shall construct inductively a k_0-base $(U_{k,\alpha})_{\alpha \in \Gamma_k}$ in $\mathcal{F}_c(X_k)$ for $k = 1, 2, \ldots$ such that $\Gamma_k \subset \Gamma_{k+1}$ and $U_{k+1,\alpha} \cap X_k = U_{k,\alpha}$ for all $\alpha \in \Gamma_k$ and $(U_{k+1,\alpha})_{\alpha \in (\Gamma_{k+1} \setminus \Gamma_k)}$ is a k_0-base in $\mathcal{F}_c(X'_k)$ for $k \in \mathbb{N}$.

Let $k \in \mathbb{N}$ and let $(U_{k,\alpha})_{\alpha \in \Gamma_k}$ be a k_0-base in $\mathcal{F}_c(X_k)$. For every $\alpha \in \Gamma_k$ there exists a clopen subset $U_{k+1,\alpha}$ of X_{k+1} such that $U_{k+1,\alpha} \cap X_k = U_{k,\alpha}$. Clearly, $(U_{k+1,\alpha})_{\alpha \in \Gamma_k}$ is k_0-independent in $\mathcal{F}_c(X_{k+1})$.

Let $(U_{k+1,\alpha})_{\alpha \in \Gamma'_k}$ be a k_0-base in $\mathcal{F}_c(X'_k)$. Clearly, $\Gamma'_k \cap \Gamma_k = \emptyset$ and $(U_{k+1,\alpha})_{\alpha \in \Gamma'_k}$ is k_0-independent in $\mathcal{F}_c(X_{k+1})$. Put $\Gamma_{k+1} = \Gamma'_k \cup \Gamma_k$. Note that $(U_{k+1,\alpha})_{\alpha \in \Gamma_{k+1}}$ is k_0-independent in $\mathcal{F}_c(X_{k+1})$. Indeed, let $n, m \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \Gamma_k, \alpha_{n+1}, \ldots, \alpha_{n+m} \in \Gamma'_k, t_1, \ldots, t_{n+m} \in k_0$ and $\sum_{i=1}^{n+m} t_i 1^{k_0}_{U_{k+1,\alpha_i}} = 0$. Then $(\sum_{i=1}^{n} t_i 1^{k_0}_{U_{k+1,\alpha_i}})|X_k = 0$, since $(\sum_{i=n+1}^{n+m} t_i 1^{k_0}_{U_{k+1,\alpha_i}})|X_k = 0$. Thus $\sum_{i=1}^{n} t_i 1^{k_0}_{U_{k,\alpha_i}} = \sum_{i=1}^{n} t_i 1^{k_0}_{U_{k+1,\alpha_i} \cap X_k} = 0$, so $t_1 = \cdots = t_n = 0$. Therefore $\sum_{i=n+1}^{n+m} t_i 1^{k_0}_{U_{k+1,\alpha_i}} = 0$, so $t_{n+1} = \cdots = t_{n+m} = 0$.

To prove that $(U_{k+1,\alpha})_{\alpha \in \Gamma_{k+1}}$ is a k_0-base of $\mathcal{F}_c(X_{k+1})$ it is enough to show that $(1^{\mathbb{K}}_{U_{k+1,\alpha}})_{\alpha \in \Gamma_{k+1}}$ is linearly dense in E_{k+1}.

By the van der Put theorem, $(1^{\mathbb{K}}_{U_{k,\alpha}})_{\alpha \in \Gamma_k}$ is an orthonormal basis in the Banach space E_k. By Proposition 3, there exists $(h_{k,\alpha})_{\alpha \in \Gamma_k} \subset E'_k$, such that $f = \sum_{\alpha \in \Gamma_k} h_{k,\alpha}(f) 1^{\mathbb{K}}_{U_{k,\alpha}}$ for every $f \in E_k$ and $h_{k,\alpha}(1^{\mathbb{K}}_{U_{k,\beta}}) = \delta_{\alpha,\beta}$ for all $\alpha, \beta \in \Gamma_k$. Moreover, we have $\|f\|_\infty = \sup_{\alpha \in \Gamma_k} |h_{k,\alpha}(f)|$ for every $f \in E_k$. By Corollary 2, the map

$$T_k : E_k \to E_{k+1}, f \to \sum_{\alpha \in \Gamma_k} h_{k,\alpha}(f) 1^{\mathbb{K}}_{U_{k+1,\alpha}}$$

is well defined. Clearly T_k is linear and continuous. Moreover, $\pi_k \circ T_k$ is the identity map on E_k. Indeed, for $f \in E_k$ we have

$$\pi_k(T_k f) = (\sum_{\alpha \in \Gamma_k} h_{k,\alpha}(f) 1^{\mathbb{K}}_{U_{k+1,\alpha}})|X_k = \sum_{\alpha \in \Gamma_k} h_{k,\alpha}(f) 1^{\mathbb{K}}_{U_{k+1,\alpha} \cap X_k} = \sum_{\alpha \in \Gamma_k} h_{k,\alpha}(f) 1^{\mathbb{K}}_{U_{k,\alpha}} = f.$$

By the van der Put theorem, $(1^{\mathbb{K}}_{U_{k+1,\alpha}}|X'_k)_{\alpha \in \Gamma'_k}$ is an orthonormal basis in the Banach space $G_k = C_0(X'_k, \mathbb{K})$.

We shall prove that the map $S_k : F_{k+1} \to G_k, f \to f|X'_k$ is an isometric isomorphism.

S_k is well defined, since for all $f \in F_{k+1}$ and $\varepsilon > 0$ we have

$$\{x \in X'_k : |f(x)| \geq \varepsilon\} = \{x \in X_{k+1} : |f(x)| \geq \varepsilon\}.$$

Clearly, S_k is a linear isometry. S_k is surjective. Indeed, let $g \in G_k$, let $f : X_{k+1} \to \mathbb{K}$ with $f|X_k = 0$ and $f|X'_k = g$ and let V be an open subset of $\mathbb{K}$. If $0 \in V$, then the set $A = \{x \in X'_k : g(x) \in (\mathbb{K} \setminus V)\}$ is compact and $f^{-1}(V) = (X_{k+1} \setminus A)$. If $0 \notin V$, then $f^{-1}(V) = g^{-1}(V)$. Thus $f^{-1}(V)$ is open in X_{k+1}, so f is continuous. Hence $f \in F_{k+1}$ and $S_k(f) = g$. It follows that S_k is an isometric isomorphism.

Hence $(1^{\mathbb{K}}_{U_{k+1,\alpha}})_{\alpha \in \Gamma'_k}$ is an orthonormal basis in the Banach space F_{k+1}.

We have $E_{k+1} = T_k(E_k) + F_{k+1}$. Indeed, let $f \in E_{k+1}$ and $g = f - T_k(\pi_k f)$. Then $\pi_k(g) = \pi_k(f) - (\pi_k \circ T_k)(\pi_k f) = 0$, so $g \in F_{k+1}$ and $f \in T_k(E_k) + F_{k+1}$. Thus $(1^{\mathbb{K}}_{U_{k+1,\alpha}})_{\alpha \in \Gamma_{k+1}}$ is linearly dense in E_{k+1}, since $(1^{\mathbb{K}}_{U_{k+1,\alpha}})_{\alpha \in \Gamma_k}$ and $(1^{\mathbb{K}}_{U_{k+1,\alpha}})_{\alpha \in \Gamma'_k}$ are linearly dense in $T_k(E_k)$ and F_{k+1}, respectively.

It follows that $(U_{k+1,\alpha})_{\alpha \in \Gamma_{k+1}}$ is a k_0-base in $\mathcal{F}_c(X_{k+1})$. Indeed, we have shown that $(U_{k+1,\alpha})_{\alpha \in \Gamma_{k+1}}$ is k_0-independent in $\mathcal{F}_c(X_{k+1})$, so there exists a k_0-base in $\mathcal{F}_c(X_{k+1})$ of the form $(U_{k+1,\alpha})_{\alpha \in \overline{\Gamma}_{k+1}}$ for some set $\overline{\Gamma}_{k+1} \supset \Gamma_{k+1}$. By the van der Put theorem, $(1^{\mathbb{K}}_{U_{k+1,\alpha}})_{\alpha \in \overline{\Gamma}_{k+1}}$ is an orthogonal basis in E_{k+1}. Hence $(1^{\mathbb{K}}_{U_{k+1,\alpha}})_{\alpha \in \Gamma^*_{k+1}}$ is not linearly dense in E_{k+1} for any proper subset Γ^*_{k+1} of $\overline{\Gamma}_{k+1}$. Thus $\Gamma_{k+1} = \overline{\Gamma}_{k+1}$, so $(U_{k+1,\alpha})_{\alpha \in \Gamma_{k+1}}$ is a k_0-base in $\mathcal{F}_c(X_{k+1})$.

Next we shall prove that the map $T : \prod_{n=1}^{\infty} F_n \to F$,

$$(f_n) \to (f_1, T_1 f_1 + f_2, \ldots, (T_n \circ \cdots \circ T_1)(f_1) + (T_n \circ \cdots \circ T_2)(f_2) + \cdots + T_n f_n + f_{n+1}, \ldots)$$

is an isomorphism. Clearly T is well defined, linear and injective.

We show that T is surjective. Let $(g_n) \in F$. Put $f_1 = g_1$. Then $g_2 - T_1 f_1 \in F_2$; so there exists an $f_2 \in F_2$ such that $g_2 = T_1 f_1 + f_2$. Assume that for some $n \geq 1$ we have a sequence $(f_1, \ldots, f_{n+1}) \in \prod_{i=1}^{n+1} F_i$ with $g_{n+1} = (T_n \circ \cdots \circ T_1)(f_1) + \cdots + T_n f_n + f_{n+1}$. Then $g_{n+2} - T_{n+1} g_{n+1} \in F_{n+2}$. Hence there exists an f_{n+2} in F_{n+2} such that

$$g_{n+2} = T_{n+1} g_{n+1} + f_{n+2} = (T_{n+1} \circ \cdots \circ T_1)(f_1) + \ldots + T_{n+1} f_{n+1} + f_{n+2}.$$

Thus we can construct inductively a sequence $(f_n) \in \prod_{n=1}^{\infty} F_n$ with $T((f_n)) = (g_n)$.

The linear maps $T_n, n \in \mathbb{N}$, are continuous and the spaces $\prod_{n=1}^{\infty} F_n$ and F have the product topologies. It follows that the map T is continuous. By the open mapping theorem, T is an isomorphism; so E is isomorphic to the countable product $\prod_{n=1}^{\infty} F_n$ and so to the product $C(X_1, \mathbb{K}) \times \prod_{k=1}^{\infty} C_0(X'_k; \mathbb{K})$ of Banach spaces with an orthonormal basis.

(C). Finally we shall prove that E has an orthogonal basis consisting of $\mathbb{K}$-valued characteristic functions of clopen subsets of X.

Put $\Gamma'_0 = \Gamma_1$ and $\Gamma = \bigcup_{k=0}^{\infty} \Gamma'_k$. Let $k \geq 0$ and $\alpha \in \Gamma'_k$. Let $U_\alpha = \bigcup_{n=1}^{\infty} U_{n,\alpha}$, where $U_{n,\alpha} = \emptyset$ for $1 \leq n < k+1$. If $1 \leq m < k+1$ then $U_\alpha \cap X_m = \emptyset = U_{m,\alpha}$. If $m \geq k+1$, then $U_\alpha \cap X_m = \bigcup_{n=m+1}^{\infty} U_{n,\alpha} \cap X_m = U_{m,\alpha}$, since $U_{n,\alpha} \subset U_{m,\alpha}$ for $k < n \leq m$ and

$$U_{n,\alpha} \cap X_m = U_{n,\alpha} \cap X_{n-1} \cap \cdots \cap X_m = U_{n-1,\alpha} \cap X_{n-2} \cap \cdots \cap X_m = \cdots = U_{m,\alpha}$$

for $n > m$. It follows that $U_\alpha \cap K$ is clopen in K for every compact subset K of X. Thus U_α is clopen in X, since X is a k_0-space (see [13, Theorem 3.7.6]).

For $k \geq 0$ and $\alpha \in \Gamma'_k$ we put $f_\alpha = (f_{n,\alpha})$, where $f_{n,\alpha} = 1^{\mathbb{K}}_{U_{k+1,\alpha}}$ if $n = k+1$ and $f_{n,\alpha} = 0$, otherwise. Then $(f_\alpha)_{\alpha \in \Gamma}$ is an orthogonal basis in $\prod_{n=1}^{\infty} F_n$, since $(1^{\mathbb{K}}_{U_{k+1,\alpha}})_{\alpha \in \Gamma'_k}$ is an orthonormal basis in F_{k+1} for $k \geq 0$ (see Proposition 5). Let $k \geq 0$ and $\alpha \in \Gamma'_k$. Then we have $Tf_\alpha = (g_{n,\alpha})$, where $g_{n,\alpha} = 0 = 1^{\mathbb{K}}_{U_{n,\alpha}}$ if $1 \leq n < k+1$, $g_{n,\alpha} = 1^{\mathbb{K}}_{U_{k+1,\alpha}}$ if $n = k+1$, and $g_{n,\alpha} = (T_{n-1} \circ \cdots \circ T_{k+1})(1^{\mathbb{K}}_{U_{k+1,\alpha}}) = 1^{\mathbb{K}}_{U_{n,\alpha}}$ if $n > k+1$. Thus $Tf_\alpha = (1^{\mathbb{K}}_{U_{n,\alpha}}) = (1^{\mathbb{K}}_{U_\alpha \cap X_n}) = \varphi(1^{\mathbb{K}}_{U_\alpha})$, so $1^{\mathbb{K}}_{U_\alpha} = (\varphi^{-1} \circ T)(f_\alpha)$. The map $\varphi^{-1} \circ T : \prod_{n=1}^{\infty} F_n \to E$ is an isomorphism, so $(1^{\mathbb{K}}_{U_\alpha})_{\alpha \in \Gamma}$ is an orthogonal basis in E. □

Remark. Let X be an ultraregular space such that the lcs $C_c(X; \mathbb{K})$ is a Fréchet space of countable type. Then any compact subset of X is ultrametrizable. Let (X_k) be an increasing fundamental sequence of compact subsets of X. By [10, Theorem 7.3], any nonempty closed subset Y of a metrizable compact ultraregular space Z is a retract of Z (i.e. there exists a continuous map $r : Z \to Y$ such that $r(y) = y$ for every $y \in Y$). Thus for any $k \in \mathbb{N}$, there is a continuous map $r_k : X_{k+1} \to X_k$ such that $r_k(x) = x$ for all $x \in X_k$. Then in the proof of Theorem 11, we can put $U_{k+1,\alpha} = r_k^{-1}(U_{k,\alpha})$ for $k \in \mathbb{N}, \alpha \in \Gamma_k$ and

$$T_k : E_k \to E_{k+1}, f \to f \circ r_k \text{ for } k \in \mathbb{N}.$$

Clearly, $C_c(X; \mathbb{K})$ has a countable orthogonal basis consisting of $\mathbb{K}$-valued characteristic functions of clopen subsets of X. □

Using Theorem 11 we get the following.

Proposition 12. *Let X be an ultraregular space. Assume that the lcs $C_c(X;\mathbb{K})$ is a Fréchet space. Then for every increasing fundamental sequence (X_n) of compact subsets of X there exists $(U_\alpha)_{\alpha\in\Gamma} \subset \mathcal{F}(X)$ such that*

(1) $(1^{\mathbb{K}}_{U_\alpha})_{\alpha\in\Gamma}$ is a Schauder basis in $C_c(X;\mathbb{K})$;

(2) $(1^{\mathbb{K}}_{U_\alpha}|X_m)_{\alpha\in\Gamma_m}$ is an orthonormal basis in the Banach space $C(X_m;\mathbb{K})$ for every $m \in \mathbb{N}$, where $\Gamma_m = \{\alpha \in \Gamma : U_\alpha \cap X_m \neq \emptyset\}$;

(3) $(1^{\mathbb{K}}_{U_\alpha})_{\alpha\in\Gamma}$ is orthogonal with respect to the base (p_m) of $\mathcal{P}(C_c(X;\mathbb{K}))$, where

$$p_m : C_c(X;\mathbb{K}) \to [0,\infty), f \to \sup_{x\in X_m} |f(x)|.$$

Proof. Let $(U_\alpha)_{\alpha\in\Gamma} \subset \mathcal{F}(X)$ be as in the proof of Theorem 11. Then $(1^{\mathbb{K}}_{U_\alpha})_{\alpha\in\Gamma}$ is an orthogonal basis in $E = C_c(X;\mathbb{K})$. By Proposition 3 we get (1).

(2). Note that $(1^{\mathbb{K}}_{U_\alpha}|X_m)_{\alpha\in\Gamma_m}$ is an orthonormal basis in $E_m = C(X_m;\mathbb{K})$ for every $m \in \mathbb{N}$. Indeed, let $m \in \mathbb{N}$. Then $1^{\mathbb{K}}_{U_\alpha}|X_m = 1^{\mathbb{K}}_{U_\alpha\cap X_m}|X_m = 1^{\mathbb{K}}_{U_{m,\alpha}}$ for all $\alpha \in \Gamma_m$. We know that $(U_{m,\alpha})_{\alpha\in\Gamma_m}$ is a k_0-base in $\mathcal{F}_c(X_m)$, so $(1^{\mathbb{K}}_{U_{m,\alpha}})_{\alpha\in\Gamma_m}$ is an orthonormal basis in E_m.

(3). If $m \in \mathbb{N}$ and $\alpha \in (\Gamma \setminus \Gamma_m) = \bigcup_{n=m}^{\infty} \Gamma'_n$, then $U_\alpha \cap X_m = U_{m,\alpha} = \emptyset$; so $1^{\mathbb{K}}_{U_\alpha}|X_m = 0$.

It follows that the Schauder basis $(1^{\mathbb{K}}_{U_\alpha})_{\alpha\in\Gamma}$ in E is orthogonal with respect to the base (p_m) in $\mathcal{P}(E)$. □

Proposition 13. *Let X be an ultraregular space such that the lcs $C_c(X;\mathbb{K})$ is a Fréchet space. Let (X_n) be an increasing fundamental sequence of compact subsets of X. Put $X'_0 = X_1$ and $X'_k = (X_{k+1} \setminus X_k)$ for $k \in \mathbb{N}$. Let $(U_\alpha)_{\alpha\in\Gamma} \subset \mathcal{F}(X), \Gamma'_0 = \{\alpha \in \Gamma : U_\alpha \cap X_1 \neq \emptyset\}$ and $\Gamma'_k = \{\alpha \in \Gamma : U_\alpha \cap X_k = \emptyset \neq U_\alpha \cap X_{k+1}\}$ for $k \in \mathbb{N}$. If $(1^{\mathbb{K}}_{U_\alpha}|X'_k)_{\alpha\in\Gamma'_k}$ is an orthonormal basis in the Banach space $C_0(X'_k;\mathbb{K})$ for $k \geq 0$, then $(1^{\mathbb{K}}_{U_\alpha})_{\alpha\in\Gamma}$ is an orthogonal basis in $C_c(X;\mathbb{K})$ satisfying the conditions (1) - (3) of Proposition 12.*

We state the following problems.

Problem 14. *Let X be ultraregular (and locally compact). Does the lcs $C_c(X;\mathbb{K})$ have an orthogonal basis [consisting of $\mathbb{K}$-valued characteristic functions of clopen (and compact) subsets of X]?*

Problem 15. *Determine all ultraregular spaces X such that the lcs $C_c(X;\mathbb{K})$ [is metrizable and] (a) has an [countable] orthogonal basis; (b) has an [countable] orthogonal basis consisting of $\mathbb{K}$-valued characteristic functions of clopen subsets of X;*

(c) has an [countable] orthogonal basis consisting of $\mathbb{K}$-valued characteristic functions of open compact subsets of X.

Acknowledgment

The author wishes to thank the referee for useful remarks and suggesting improvements.

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