The separable quotient problem and the strongly normal sequences

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Abstract. We study the notion of a *strongly normal sequence* in the dual E^* of a Banach space E. In particular, we prove that the following three conditions are equivalent:
(1) E^* has a strongly normal sequence,
(2) $(E^*, \sigma(E^*, E))$ has a Schauder basic sequence,
(3) E has an infinite-dimensional separable quotient.

Introduction

We put $S(X) = \{x \in X : \|x\| = 1\}$ and $B(X) = \{x \in X : \|x\| \leq 1\}$ if X is a normed space. Let E be a Banach space. A sequence $(y_n) \subset S(E^*)$ is *normal* in E^* if $\lim_n y_n(x) = 0$ for every $x \in E$; clearly, the normal sequences coincide with the normalized ω^*-null sequences. The excellent Josefson-Nissenzweig theorem states that the dual of any infinite-dimensional Banach space contains a normal sequence ([5], [12]). It is easy to see that a sequence $(y_n) \subset S(E^*)$ is normal if and only if the subspace $\{x \in E : \lim_n y_n(x) = 0\}$ is dense in E. We will say that a sequence $(y_n) \subset S(E^*)$ is *strongly normal* if the subspace $\{x \in E : \sum_{n=1}^{\infty} |y_n(x)| < \infty\}$ is dense in E ([18]). Clearly, every strongly normal sequence in E^* is normal.

One of the most known open problems for Banach spaces is the separable quotient problem: *Does every infinite-dimensional Banach space has an infinite-dimensional separable quotient?* i.e. *Does every infinite-dimensional Banach space E has a closed*

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subspace M such that the quotient space E/M is infinite-dimensional and separable? ([1], [8], [10], [11], [15]-[22])

Recall that a sequence (x_n) in a locally convex space F is: (1) a *Schauder basis* of F if for each element x of F there is a unique sequence (α_n) of scalars such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$ and the coefficient functionals $x_n^*, n \in \mathbb{N}$, defined by $x_n^*(x) = \alpha_n$, are continuous on F; (2) a *Schauder basic sequence* if it is a Schauder basis of its closed linear span X in F.

We shall prove that a Banach space E has an infinite-dimensional separable quotient iff E^* contains a strongly normal sequence iff $E_\sigma^* = (E^*, \sigma(E^*, E))$ has a Schauder basic sequence (Theorem 3). Before, developing some ideas of [4], we shall show that every strongly normal sequence in the dual E^* of a Banach space E contains a Schauder basic subsequence in E_σ^* (Theorem 1).

We state the following.

Problem. *Does every normal sequence in the dual E^* of a Banach space E contains a strongly normal subsequence?*

If this problem has a positive answer for a given infinite-dimensional Banach space E, then by the Josefson-Nissenzweig theorem and Theorem 3, E has an infinite-dimensional separable quotient.

We show that for every WCG (i.e. weakly compactly generated) Banach space E our problem has a positive answer (Proposition 4). Next we give an example of a normal sequence in the dual E^* of some known non-WCG Banach space E, which is not strongly normal but every subsequence of it contains a strongly normal subsequence (Example).

Finally, we show that a Banach space E has no infinite-dimensional separable quotient iff every continuous linear map from a Banach space to E with dense range is a surjection iff every sequence of continuous linear maps from E to some non-zero (or to every) Fréchet space F, which is point-wise convergent on a dense subspace of E is point-wise convergent on E to some continuous linear map from E to F (Theorem 6).

Results

Johnson and Rosenthal proved that any normal sequence (y_n) in the dual E^* of a separable Banach space E has a Schauder basic subsequence $(y_{k(n)})$ in E_σ^* ([4], Theorem III.1). Developing some ideas of their proof we shall show the following.

Theorem 1. *Let E be a Banach space. Any strongly normal sequence (y_n) in E^* contains a Schauder basic subsequence $(y_{k(n)})$ in E^*_σ.*

Proof. Let $\varphi : E \to E^{**}$ be the canonical embedding map.

(A1) First we shall show that for every finite-dimensional subspace Y of E^* and every $\varepsilon \in (0, 1/2)$ there exists a finite subset H of $S(E)$ such that for every $f \in S(Y^*)$ there is an $x \in H$ with $\| f - \varphi(x)|Y \| < 2\varepsilon$.

Let $\psi : (E/^\perp Y) \to (E/^\perp Y)^{**}$ be the canonical embedding map; clearly ψ is an isometric isomorphism. Since $(^\perp Y)^\perp = Y$, the map

$$\alpha : Y \to (E/^\perp Y)^*, \alpha(y)(x +^\perp Y) = y(x), \text{ for } y \in Y, x \in E,$$

is an isometric isomorphism ([14], 4.9(b)). Thus the adjoint map

$$\alpha^* : (E/^\perp Y)^{**} \to Y^*, \alpha^*(\psi(x +^\perp Y)) = \varphi(x)|Y, \text{ for } x \in E,$$

is also an isometric isomorphism ([2], 8.6.18(a)).

Hence for every $f \in S(Y^*)$ there is an $x \in S(E)$ with $\|f - \varphi(x)|Y\| < \varepsilon$. Indeed, for every $f \in S(Y^*)$ there exist $v \in E$ and $z \in^\perp Y$ such that $\varphi(v)|Y = f, \| v +^\perp Y \| = 1$ and $1 \le \|v + z\| < 1 + \varepsilon$. Thus for $u = v + z$ and $x = u/\|u\|$ we have $x \in S(E)$ and $\| f - \varphi(x)|Y \| = 1 - \|u\|^{-1} < \varepsilon$.

The set $S(Y^*)$ is compact, so there exists a finite subset $\{f_1, \ldots, f_n\}$ of $S(Y^*)$ with $S(Y^*) \subset \bigcup_{m=1}^n K(f_m, \varepsilon)$. Let $x_1, \ldots, x_n \in S(E)$ with $\|f_m - \varphi(x_m)|Y\| < \varepsilon$ for $1 \le m \le n$. Put $H = \{x_1, \ldots, x_n\}$. Then for every $f \in S(Y^*)$ there is an $x \in H$ with $\|f - \varphi(x_m)|Y\| < 2\varepsilon$.

(A2) Since $\lim_n y_n(x) = 0$ for every $x \in E$, using (A1) we can choose inductively a strictly increasing sequence $(k(n)) \subset \mathbb{N}$ and an increasing sequence (H_n) of finite subsets of $S(E)$ such that for every $n \in \mathbb{N}$ we have

(i) for every $f \in S(Y_n^*)$ there is an $x \in H_n$ with $\|f - \varphi(x)|Y_n\| < 2^{-n-1}$, where Y_n is the linear span of the set $\{y_{k(i)} : 1 \le i \le n\}$;

(ii) $|y_{k(n+1)}(x)| < 2^{-n-2}$ for every $x \in H_n$.

(A3) For every $n \in \mathbb{N}$ and for all $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{K}$ we have

$$\| \sum_{i=1}^n \alpha_i y_{k(i)} \| \le (1 + 2^{1-n}) \| \sum_{i=1}^{n+1} \alpha_i y_{k(i)} \|.$$

Indeed, let $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{K}$. Put $y = \sum_{i=1}^n \alpha_i y_{k(i)}$ and $z = \alpha_{n+1} y_{k(n+1)}$. Then there is $f \in S(Y^*)$ with $f(y) = \|y\|$ ([14], 3.3). By (A2) there is an $x \in H_n$ with

$\|f-\varphi(x)|Y_n\| < 2^{-n-1}$ and $|y_{k(n+1)}(x)| < 2^{-n-2}$. If $\|z\| > 2\|y\|$, then $\|y+z\| > \|y\|$. If $\|z\| \le 2\|y\|$, then $\|y+z\| \ge |(y+z)(x)| \ge |y(x)|-|z(x)| \ge |f(y)|-|f(y)-y(x)|-|z(x)| = \|y\|-|(f-\varphi(x)|Y_n)(y)|-\|z\||y_{k(n+1)}(x)| \ge (1-2^{-n})\|y\| \ge (1+2^{1-n})^{-1}\|y\|$.

Since $\prod_{n=1}^{\infty}(1+2^{1-n}) < \infty$, using [9], 4.1.24, we infer that $(y_{k(n)})$ is a Schauder basic sequence in E^* such that $\|P_n\| \le \prod_{k=n}^{\infty}(1+2^{1-k}) < 1+2^{4-n}, n \in \mathbb{N}$, where $P_n : Y \to Y, \sum_{i=1}^{\infty}\alpha_i y_{k(i)} \to \sum_{i=1}^{n}\alpha_i y_{k(i)}$ and Y is the closed linear span of $(y_{k(n)})$.

(A4) The operator $T : E \to Y^*, (Tx)(y) = y(x), x \in E, y \in Y$, is well defined, linear and continuous. Let $(f_n) \subset Y^*$ be the sequence of coefficient functionals associated with the Schauder basis $(y_{k(n)})$ in Y. Clearly, (f_n) is a Schauder basis of its closed linear span F in Y^* ([9], 4.4.1). Put $G = \{x \in E : \sum_{n=1}^{\infty}|y_n(x)| < \infty\}$.

For $x \in E$ we have $Tx = \sum_{n=1}^{\infty} y_{k(n)}(x)f_n$. Indeed, let $x \in G$. For $n \ge 2$ we get $\|f_n\| = \|f_n\|\|y_{k(n)}\| = \|P_n - P_{n-1}\| \le 2+2^{6-n} \le 18$, so the series $\sum_{n=1}^{\infty} y_{k(n)}(x)f_n$ is convergent in F. For $y \in Y$ we have $(Tx)(y) = y(x) = (\sum_{n=1}^{\infty} f_n(y)y_{k(n)})(x) = \sum_{n=1}^{\infty} f_n(y)y_{k(n)}(x) = (\sum_{n=1}^{\infty} y_{k(n)}(x)f_n)(y)$, so $Tx = \sum_{n=1}^{\infty} y_{k(n)}(x)f_n \in F$. Hence $T(E) = T(\overline{G}) \subset \overline{T(G)} \subset F$. Let $x \in E$. Then $Tx = \sum_{j=1}^{\infty}\alpha_j f_j$ for some scalars $\alpha_1, \alpha_2, \ldots$. Hence $\alpha_n = (\sum_{j=1}^{\infty}\alpha_j f_j)(y_{k(n)}) = (Tx)(y_{k(n)}) = y_{k(n)}(x), n \in \mathbb{N}$, so $Tx = \sum_{n=1}^{\infty} y_{k(n)}(x)f_n$.

(A5) For every $g \in F$ and every $\varepsilon > 0$ there is $x \in E$ with $\|x\| = \|g\|$ such that $\|g - Tx\| < \varepsilon$. Indeed, for every $g \in S(F)$ there is a sequence $(g_n) \subset S(F)$ with $\lim g_n = g$ such that $g_n \in F_n$ for $n \in \mathbb{N}$, where F_n is the linear span of the set $\{f_1, \ldots, f_n\}$. Thus it is enough to show that for every $n \in \mathbb{N}$ and every $g \in S(F_n)$ there is $x \in S(E)$ with $\|g - Tx\| \le 2^{7-n}$. Let $n \in \mathbb{N}, g \in S(F_n)$ and $h = \|g|Y_n\|^{-1}g$.

Since $h|Y_n \in S(Y_n^*)$, by (A2) there is an $x \in H_n$ with $\|h|Y_n - \varphi(x)|Y_n\| < 2^{-n-1}$. Put $f = \sum_{i=1}^{n} y_{k(i)}(x)f_i$. For $y \in Y_n$ we have $f(y) = \sum_{i=1}^{n} y_{k(i)}(x)f_i(y) = (\sum_{i=1}^{n} f_i(y)y_{k(i)})(x) = y(x) = \varphi(x)(y)$, so $f|Y_n = \varphi(x)|Y_n$.

By (A4) and (A2) we get $\|Tx - g\| = \|\sum_{i=1}^{\infty} y_{k(i)}(x)f_i - g\| \le \|f-g\| + \sum_{i=n+1}^{\infty}|y_{k(i)}(x)|\|f_i\| \le \|f-g\| + \sum_{i=n+1}^{\infty} 2^{-i-1}(2+2^{6-i}) \le (\|f-h\|+\|h-g\|)+2^{6-n}$. For $u \in F_n$ we have $\|u\| = \sup\{|u(P_n y)| : y \in S(Y)\} \le \|u|Y_n\|\|P_n\|$, so $\|f-h\| \le \|f|Y_n - h|Y_n\|\|P_n\| = \|\varphi(x)|Y_n - h|Y_n\|\|P_n\| < 2^{-n-1}(1+2^{4-n}) \le 2^{4-n}$. Moreover $\|h-g\| = \|g|Y_n\|^{-1} - 1 \le \|g\|^{-1}\|P_n\| - 1 \le 2^{4-n}$. Thus $\|Tx - g\| \le 2^{7-n}$.

(A6) We show that $T(E) = F$. Let $g \in F$. Using (A5) we choose an element $x_1 \in E$ with $\|x_1\| = \|g\|$ such that $\|g - Tx_1\| < 2^{-1}$. Next we choose an element $x_2 \in E$ with $\|x_2\| = \|g - Tx_1\|$ such that $\|g - Tx_1 - Tx_2\| < 2^{-2}$. This way we can obtain a sequence $(x_n) \subset E$ such that $\|x_{n+1}\| = \|g - \sum_{j=1}^{n} Tx_j\|$ and $\|g - \sum_{j=1}^{n+1} Tx_j\| < 2^{-n-1}$

for $n \in \mathbb{N}$. Clearly, the series $\sum_{j=1}^{\infty} x_j$ is convergent in E to some x and $Tx = g$.

(A7) The sequence $(g_n) \subset F^*$ of coefficient functionals associated with the Schauder basis (f_n) in F is a Schauder basis in F^*_σ. The adjoint map $T^* : F^* \to E^*$ is an isomorphism of F^*_σ and the closed subspace T^*F^* of E^*_σ ([14], 4.14 and 4.15). Thus the sequence (T^*g_n) is a Schauder basic sequence in E^*_σ. We have $(T^*g_n)(x) = g_n(Tx) = g_n(\sum_{i=1}^{\infty} y_{k(i)}(x)f_i) = y_{k(n)}(x)$ for $x \in E$ and $n \in \mathbb{N}$, so $T^*g_n = y_{k(n)}$ for $n \in \mathbb{N}$. We have shown that $(y_{k(n)})$ is a Schauder basic sequence in E^*_σ. $\square$

Let E be a Banach space. By the Banach-Steinhaus theorem every sequence $(y_n) \subset E^*$ which is point-wise bounded on E is bounded. We will say that a sequence $(y_n) \subset E^*$ is *pseudobounded* if it is point-wise bounded on a dense subspace of E and $\sup_n \|y_n\| = \infty$.

For Schauder basic sequences in E^*_σ we have the following.

Proposition 2. *Let E be a Banach space and let (y_n) be a Schauder basic sequence in E^*_σ. If $(y_n) \subset S(E^*)$, then (y_n) is strongly normal in E^*. If $\sup_n \|y_n\| = \infty$, then (y_n) is pseudobounded in E^*. Every pseudobounded sequence (z_n) in E^* has a Schauder basic subsequence in E^*_σ.*

Proof. Denote by Y the closure of the linear span of the set $\{y_n : n \in \mathbb{N}\}$ in E^*_σ. Then there is a sequence $(x_n) \subset E$ such that $y_n(x_m) = \delta_{n,m}$ for all $n, m \in \mathbb{N}$ and $y(x) = \sum_{n=1}^{\infty} y(x_n)y_n(x)$ for all $y \in Y, x \in E$. For the linear span X of the set $\{x_n : n \in \mathbb{N}\}$ we have

$$(X +^{\perp} Y)^{\perp} = (X \cup^{\perp} Y)^{\perp} = X^{\perp} \cap (^{\perp}Y)^{\perp} = X^{\perp} \cap Y = \{0\}.$$

Thus $X +^{\perp} Y$ is dense in E, so the subspaces $\{x \in E : \sum_{n=1}^{\infty} |y_n(x)| < \infty\}$ and $\{x \in E : \sup_n |y_n(x)| < \infty\}$ are dense in E, too.

Let $(k(n)) \subset \mathbb{N}$ be a strictly increasing sequence with $\|z_{k(n)}\| \geq n^2$ for $n \in \mathbb{N}$. Put $v_n = z_{k(n)}/\|z_{k(n)}\|$ for $n \in \mathbb{N}$. The sequence (v_n) is strongly normal in E^*, since $\{x \in E : \sup_n |z_n(x)| < \infty\} \subset \{x \in E : \sum_{n=1}^{\infty} |v_n(x)| < \infty\}$. Using Theorem 1 we infer that the sequence $(z_{k(n)})$ has a Schauder basic subsequence in E^*_σ. $\square$

Using the last proposition we get the following.

Theorem 3. *Let E be a Banach space. Then the following conditions are equivalent:*

(1) E has an infinite-dimensional separable quotient;

(2) E^* has a strongly normal sequence;

(3) E^*_σ has a Schauder basic sequence;

(4) E^* has a pseudobounded sequence.

Proof. (1) $\Rightarrow$ (2). By [6], Proposition 1, there exists a biorthogonal sequence $((x_n, y_n)) \subset E \times E^*$ such that $A = (\text{lin}\{x_n : n \in \mathbb{N}\} + \bigcap_{n=1}^\infty \ker y_n)$ is a dense subspace in E; clearly we can assume that $(y_n) \subset S(E^*)$. The sequence (y_n) is strongly normal in E^*, since $\{x \in E : \sum_{n=1}^\infty |y_n(x)| < \infty\} \supset A$.

Using Theorem 1 we get (2) $\Rightarrow$ (3). By [20], Proposition 1, we obtain (3) $\Rightarrow$ (1). Using Proposition 2 we get the equivalence (3) $\Leftrightarrow$ (4). $\square$

It is known that every infinite-dimensional WCG Banach space has an infinite-dimensional separable quotient. We shall show the following ([18]).

Proposition 4. *Let E be a WCG Banach space. Then every normal sequence (y_n) in E^* contains a strongly normal subsequence.*

Proof. *Case 1*: E is separable. Let $X = \{x_n : n \in \mathbb{N}\}$ be a countable dense subset of E. For every $n \in \mathbb{N}$ we choose $k(n) \in \mathbb{N}$ with $|y_{k(n)}(x_i)| < n^{-2}$ for $1 \le i \le n$; we can assume that the sequence $(k(n))$ is strictly increasing. Then the sequence $(y_{k(n)})$ is strongly normal in E^*, since $\{x \in E : \sum_{n=1}^\infty |y_{k(n)}(x)| < \infty\} \supset X$.

Case 2: E is not separable. By [3], Proposition 1, there is a continuous linear projection $Q : E \to E$ with $\|Q\| = 1$ such that $F = Q(E)$ is a separable closed subspace of E and $(y_n) \subset Q^*(E^*)$. Let $i : F \to E$ be the identity embedding. Put $P : E \to F, x \to Qx$. Then $Q = iP$ and $Q^*(E^*) = P^*(i^*(E^*)) \subset P^*(F^*)$, so $(y_n) \subset P^*(F^*)$. Moreover $P(B(E)) = B(F)$. Therefore for every $z \in F^*$ we have

$$\|P^*z\| = \sup\{|(P^*z)(x)| : x \in B(E)\} = \sup\{|z(Px)| : x \in B(E)\} =$$

$$\sup\{|z(x)| : x \in B(F)\} = \|z\|.$$

Since $(y_n) \subset P^*(F^*) \cap S(E^*)$, there is $(z_n) \subset S(F^*)$ with $P^*z_n = y_n, n \in \mathbb{N}$. Thus (z_n) is a normal sequence in F^*. By *Case 1*, (z_n) contains a strongly normal subsequence $(z_{k(n)})$ in F^*. Then the subspace $(\{x \in F : \sum_{n=1}^\infty |z_{k(n)}(x)| < \infty\} + \ker P)$ is dense in E, so the subspace $\{x \in E : \sum_{n=1}^\infty |y_{k(n)}(x)| < \infty\}$ is dense in E. Thus $(y_{k(n)})$ is strongly normal in E^*. $\square$

Example. The linear space $E = \{(x_n) \in c_0 : \sup_k |\sum_{n=1}^k x_n| < \infty\}$ with the norm $\|x\| = \sup_k |\sum_{n=1}^k x_n|, x = (x_n)$, is a Banach space and it is not WCG ([17]).

Let $f_n : E \to \mathbb{K}, x = (x_k) \to x_n, n \in \mathbb{N}$. Then $(f_n) \subset E^*, \lim_n f_n(x) = 0$ for every $x \in E$ and $1 \leq \|f_n\| \leq 2$ for $n \in \mathbb{N}$. Put $y_n = f_n/\|f_n\|, n \in \mathbb{N}$; clearly (y_n) is a normal sequence in E^*. We shall prove that a subsequence $(y_{k(n)})$ of (y_n) is strongly normal in E^* if and only if the sequence $(k(n)) \subset \mathbb{N}$ does not contain arbitrary long series of successive integers. In particular the normal sequence (y_n) is not strongly normal but every subsequence of it contains a strongly normal subsequence.

Proof. Let $(k(n)) \subset \mathbb{N}$ be a strictly increasing sequence.

Assume that $(k(n))$ contains arbitrary long series of successive integers. Then for every $s \in \mathbb{N}$ there is $n(s) \in \mathbb{N}$ such that $k(n(s)+1); \ldots ; k(n(s)+2s)$ are successive integers; we can assume that $n(s+1) > n(s) + 2s$ for $s \in \mathbb{N}$. Put

$$z_l = \begin{cases} s^{-1} & \text{if } k(n(s)+1) \leq l \leq k(n(s)+s) \text{ for some } s \in \mathbb{N}; \\ -s^{-1} & \text{if } k(n(s)+s+1) \leq l \leq k(n(s)+2s) \text{ for some } s \in \mathbb{N}; \\ 0 & \text{for all other } l \in \mathbb{N}. \end{cases}$$

Clearly $z = (z_l) \in E$. Let $x \in E$ with $\sum_{n=1}^{\infty} |y_{k(n)}(x)| < \infty$. Then $\sum_{n=1}^{\infty} |x_{k(n)}| = \sum_{n=1}^{\infty} |f_{k(n)}(x)| = \sum_{n=1}^{\infty} \|f_{k(n)}\| |y_{k(n)}(x)| < \infty$. For $s \in \mathbb{N}$ we have

$$1 = \sum_{l=k(n(s)+1)}^{k(n(s)+s)} z_l = |\sum_{l=1}^{k(n(s)+s)} (z_l - x_l) - \sum_{l=1}^{k(n(s)+1)-1} (z_l - x_l) + \sum_{l=k(n(s)+1)}^{k(n(s)+s)} x_l| \leq$$

$$\|z - x\| + \|z - x\| + \sum_{m=n(s)+1}^{n(s)+s} |x_{k(m)}|.$$

Hence for $s \in \mathbb{N}$ we get $1 \leq 2\|z - x\| + \sum_{m=n(s)+1}^{n(s)+s} |x_{k(m)}|$. Since $\sum_{m=1}^{\infty} |x_{k(m)}| < \infty$ we have $\lim_s \sum_{m=n(s)+1}^{n(s)+s} |x_{k(m)}| = 0$. Thus $\|z - x\| \geq 1/2$. It follows that the set $\{x \in E : \sum_{m=1}^{\infty} |y_{k(m)}(x)| < \infty\}$ is not dense in E, so the subsequence $(y_{k(n)})$ of (y_n) is not strongly normal in E^*.

Assume now that $(k(n))$ does not contain arbitrary long series of successive integers. Then there are two strictly increasing sequences $(t(n)), (w(n)) \subset \mathbb{N}$ and $m \in \mathbb{N}$ such that

(1) $t(n) \leq w(n) \leq t(n) + m - 2$ for $n \in \mathbb{N}$;

(2) $w(n) + 1 < t(n+1)$ for $n \in \mathbb{N}$;

(3) $\bigcup_n \{l \in \mathbb{N} : t(n) \leq l \leq w(n)\} = \{k(n) : n \in \mathbb{N}\}$.

Let $z \in E$. For $s \in \mathbb{N}$ we put $x_s = (x_{s,l})$, where

$$x_{s,l} = \begin{cases} 0 & \text{if } t(n) \leq l \leq w(n) \text{ for some } n \geq s; \\ \sum_{i=t(n)}^{w(n)+1} z_i & \text{if } l = w(n)+1 \text{ for some } n \geq s; \\ z_l & \text{for all other } l \in \mathbb{N}. \end{cases}$$

Since $|\sum_{i=t(n)}^{w(n)+1} z_i| \le m \max\{|z_i| : i \ge t(n)\}, n \in \mathbb{N}$ and $\lim_n \max\{|z_i| : i \ge t(n)\} = 0$, we have $x_s \in c_0$. Moreover for $l \in \mathbb{N}$ we have $\sum_{i=1}^{l} x_{s,i} = \sum_{i=1}^{t(n)-1} z_i$ if $t(n) \le l \le w(n)$ for some $n \ge s$, and $\sum_{i=1}^{l} x_{s,i} = \sum_{i=1}^{l} z_i$ for all other $l \in \mathbb{N}$. Thus $x_s \in E$. Since $x_{s,k(n)} = 0$ if $k(n) \ge t(s)$, we have

$$\sum_{n=1}^{\infty} |y_{k(n)}(x_s)| = \sum_{n=1}^{\infty} |f_{k(n)}(x_s)|/\|f_{k(n)}\| = \sum_{n=1}^{\infty} |x_{s,k(n)}|/\|f_{k(n)}\| < \infty;$$

so $(x_s) \subset \{x \in E : \sum_{n=1}^{\infty} |y_{k(n)}(x)| < \infty\}$. For $s \in \mathbb{N}$ we have $\sum_{i=1}^{l}(z_i - x_{s,i}) = \sum_{i=t(n)}^{l} z_i$, if $t(n) \le l \le w(n)$ for some $n \ge s$; and $\sum_{i=1}^{l}(z_i - x_{s,i}) = 0$ for all other $l \in \mathbb{N}$. Thus $\|z - x_s\| \le m \max\{|z_i| : i \ge t(s)\}$ for $s \in \mathbb{N}$; so $\lim_s \|z - x_s\| = 0$. Hence the set $\{x \in E : \sum_{n=1}^{\infty} |y_{k(n)}(x)| < \infty\}$ is dense in E. Therefore $(y_{k(n)})$ is strongly normal in E^*. □

By the equivalence (1) ⇔ (4) in Theorem 3 we obtain the following well known result ([1], [17]); our proof is quite different from the the original one.

Corollary 5. *A Banach space has an infinite-dimensional separable quotient if and only if it contains a dense non-barrelled subspace.*

Proof. Assume that a Banach space E has an infinite-dimensional separable quotient. By Theorem 3, the space E^* has a pseudobounded sequence (y_n). Put $G = \{x \in E : \sup_n |y_n(x)| < \infty\}$ and $V = \{x \in E : \sup_n |y_n(x)| \le 1\}$. Using the Banach-Steinhaus theorem we infer that G is a proper and dense subspace of E. The set V is a barrell in G and it is not a neighbourhood of zero in G, since V is closed in E. Thus G is not barrelled.

Assume that a Banach space E contains a dense non-barrelled subspace G. Let W be a barrell in G which is not a neighbourhood of zero in G. The closure V of W in E is absolutely convex and closed in E. The linear span H of V is a dense proper subspace of E. For every $n \in \mathbb{N}$ there is $x_n \in (E \setminus V)$ with $\|x_n\| < n^{-2}$. By the Hahn-Banach theorem for every $n \in \mathbb{N}$ there is $z_n \in E^*$ with $|z_n(x_n)| > 1$ such that $|z_n(x)| \le 1$ for all $x \in V$. Then $\|z_n\| \ge n^2$ for $n \in \mathbb{N}$ and $\sup_n |z_n(x)| < \infty$ for $x \in H$; so (z_n) is pseudobounded in E^*. By Theorem 3, E has an infinite-dimensional separable quotient. □

Applying Corollary 5 we get our last result.

Theorem 6. *Let E be an infinite-dimensional Banach space. Let F be a non-zero locally convex space. Then the following conditions are equivalent:*

(1) Every separable quotient of E is finite-dimensional;

(2) Every continuous linear map from a Banach space to E with dense range is a surjection;

(3) Every family $\{T_\gamma : \gamma \in \Gamma\} \subset L(E,F)$ which is point-wise bounded on a dense subspace H of E is equicontinuous;

(4) Every sequence $(T_n) \subset L(E,F)$ which is point-wise convergent to zero on a dense subspace G of E is point-wise convergent to zero on E;

If additionally F is sequentially complete then above conditions are equivalent to the following

(5) Every sequence $(T_n) \subset L(E,F)$ which is point-wise convergent on a dense subspace G of E is point-wise convergent on E to some $T \in L(E,F)$.

Proof. (1) $\Rightarrow$ (2). Let T be a continuous linear map from a Banach space X to E such that the range $T(X)$ is dense in E. By Corollary 5, $T(X)$ is barrelled. Using the open mapping theorem we infer that the map T is open (i.e. for every open subset U in X the set $T(U)$ is open in $T(X)$). By the Banach-Schauder theorem ([7], 15.12(2)), $T(X)$ is closed in E; so $T(X) = E$.

(2) $\Rightarrow$ (1). By Corollary 5 it is enough to show that every dense subspace M of E is barrelled. Let D be a barrell in M and let B be the closed unit ball in M. Denote by S the closure of the set $C = D \cap B$ in E and by H the linear span of S. Let $p : H \to [0;\infty)$ be the Minkowski functional of S. Since S is a bounded and complete barrell in H, p is a complete norm in H and the embedding map $i : (H,p) \to E$ is a continuous linear map with dense range; so $H = E$. Thus S is a neighbourhood of zero in E. Hence D is a neighbourhood of zero in M, because $D \supset C = S \cap M$. Thus M is a barrelled space.

(1) $\Rightarrow$ (3). By Corollary 5, H is a dense barrelled subspace of E. Using the Banach-Steinhaus theorem we infer that the family $\{T_\gamma|H : \gamma \in \Gamma\}$ is equicontinuous. Let V be a closed neighbourhood of zero in F. For some open neighbourhood U of zero in E we have $T_\gamma(U \cap H) \subset V$ for all $\gamma \in \Gamma$. Hence $T_\gamma(U) \subset T_\gamma(\overline{U \cap H}) \subset \overline{T_\gamma(U \cap H)} \subset V$ for all $\gamma \in \Gamma$. Thus the family $\{T_\gamma : \gamma \in \Gamma\}$ is equicontinuous.

(3) $\Rightarrow$ (4). By (3) the sequence (T_n) is equicontinuous. Let $x \in E$. Let W, V be neighbourhoods of zero in F with $V - V \subset W$. For some neighbourhood U of zero in E we have $T_n(U) \subset V$ for $n \in \mathbb{N}$. Moreover there exists $y \in E$ with $y - x \in U$ such that $\lim_n T_n(y) = 0$. For some $n_0 \in \mathbb{N}$ we have $T_n(y) \in V$ for $n \geq n_0$. Since $T_n(x) = T_n(y) - T_n(y-x)$ and $V - T_n(U) \subset V - V \subset W$, so $T_n(x) \in W$ for $n \geq n_0$.

Thus $\lim_n T_n(x) = 0$ for every $x \in E$.

(4) $\Rightarrow$ (1). Suppose, to the contrary, that E has an infinite-dimensional separable quotient. By Theorem 3, E^*_σ has a Schauder basic sequence (y_n); we can assume that $\lim_n \|y_n\| = \infty$, so (y_n) is pseudobounded in E^* (Proposition 2). Put $z_n = y_n/\sqrt{\|y_n\|}$ for $n \in \mathbb{N}$. Then $\lim_n \|z_n\| = \infty$. Let $z \in F$ with $z \neq 0$. For every $n \in \mathbb{N}$ the map $T_n : E \to F, x \to z_n(x)z$, is linear and continuous. Since $\{x \in E : \sup_n |y_n(x)| < \infty\} \subset \{x \in E : \lim_n z_n(x) = 0\}$, the sequence $(T_n) \subset L(E, F)$ is point-wise convergent to zero on a dense subspace of E. By (4), (T_n) is point-wise convergent to zero on E. By the Banach-Steinhaus theorem, (T_n) is equicontinuous, so $\sup_n \|z_n\| < \infty$; a contradiction.

Assume now that F is additionally sequentially complete.

(3) $\Rightarrow$ (5). By (3), the sequence (T_n) is eqiucontinuous. Let $x \in E$. Let W, V be neighbourhoods of zero in F with $(V - V) - (V - V) \subset W$. For some neighbourhood U of zero in E we have $T_n(U) \subset V$ for $n \in \mathbb{N}$. Moreover there exists $y \in E$ with $y - x \in U$ such that the sequence $(T_n(y))$ is convergent in F to some element z. Let $n_0 \in \mathbb{N}$ with $T_n(y) - z \in V$ for $n \geq n_0$. For $n, m \geq n_0$ we have $T_n x - T_m x = [((T_n y - z) - T_n(y - x)) - ((T_m y - z) - T_m(y - x))] \in (V - V) - (V - V) \subset W$. It follows that $(T_n x)$ is a Cauchy sequence in F, so it is convergent in F to some T_x for every $x \in E$. Clearly, the map $T : E \to F, x \to T_x$ is linear. If $x \in U$, then $(T_n x) \subset V$; hence $Tx \in W$. Thus $T(U) \subset W$; so T is continuous.

The implication (5) $\Rightarrow$ (4) is obvious. Thus (5) is equivalent to conditions (1)-(4).
□

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References

[1] Bennett, G. and Kalton, N., *Inclusion theorems for K-spaces*, Canad. J. Math., 25(1973), 511-524.

[2] Edwards, R.E., *Functional analysis. Theory and applications*, Dover Publications, Inc., New York, 1995.

[3] John, K. and Zizler, V., *Projections in dual weakly compactly generated Banach spaces*, Studia Math., 49(1973), 41-50.

[4] Johnson, W.B. and Rosenthal, H.P., *On ω^*-basic sequences and their applications to the study of Banach spaces*, Studia Math., 43(1972), 77-95.

[5] Josefson, B., *Weak sequential convergence in the dual of a Banach space does not imply norm convergence*, Ark. Mat., 13(1975), 78-89.

[6] Kąkol, J. and Śliwa, W., *Remarks concerning the separable quotient problem*, Note Mat., 13(1993), 277-282.

[7] Köthe, G., *Topological vector spaces I*, Springer-Verlag, Berlin, 1969.

[8] Lacey, H.E., *Separable quotients of Banach spaces*, An. Acad. Brasil. Ciénc., 44(1972), 185-189.

[9] Megginson, R.E., *An Introduction to Banach Space Theory*, Springer-Verlag, New York, 1998.

[10] Mujica, J., *Separable quotients of Banach spaces*, Rev. Mat. Univ. Complut. Madrid, 10(1997), 299-330.

[11] Narayanaswami, P.P., *The separable quotient problem for barrelled spaces*, in: Functional analysis and related topics, Springer-Verlag, Berlin, 1993, pp. 289-308.

[12] Nissenzweig, A., *On ω^* sequential convergence*, Israel J. Math., 22(1975), 266-272.

[13] Rosenthal, H.P., *On quasicomplemented subspaces of Banach spaces with an appendix on compactness of operators from $L^p(\mu)$ to $L^r(\nu)$*, J. Funct. Anal., 4(1969), 176-214.

[14] Rudin, W., *Functional analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[15] Saxon, S.A. and Narayanaswami, P.P., *Metrizable (LF)-spaces, (db)-spaces and the separable quotient problem*, Bull. Austral. Math. Soc., 23(1981), 65-80.

[16] Saxon, S.A. and Narayanaswami, P.P., *Metrizable [normable] (LF)-spaces and two classical problems in Fréchet [Banach] spaces*, Studia Math., 43(1989), 1-16.

[17] Saxon, S.A. and Wilansky, A., *The equivalence of some Banach space problems*, Colloq. Math., 37(1977), 217-226.

[18] Śliwa, W., *(LF)-spaces and the separable quotient problem*, Thesis, in Polish, unpublished, Poznań, 1996.

[19] Śliwa, W., *The separable quotient problem for symmetric function spaces*, Bull. Polish Acad. Sci. Math., 48(2000), 13-27.

[20] Śliwa, W. and Wójtowicz, M., *Separable quotients of locally convex spaces*, Bull. Polish Acad. Sci. Math., 43(1995), 175-185.

[21] Wójtowicz, M., *Generalizations of the* $c_0 - l_1 - l_\infty$ *Theorem of Bessaga and Pełczyński*, Bull. Polish Acad. Sci. Math., 50(2002), 373-382.

[22] Wójtowicz, M., *Reflexivity and the Separable Quotient Problem for a Class of Banach Spaces*, Bull. Polish Acad. Sci. Math., 50(2002), 383-394.

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