Full Nonassociative Lambek Calculus with Modalities and Its Applications in Type Grammars

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I would like to dedicate this thesis to my loving mother.

Summary

This thesis is concerned with full nonassociative Lambek calculus with different modalities which is a family of substructural logics. Substructural logics are logics which omit some structural rules, e.g. contraction, weakening, commutativity. Nonassociative Lambek calulus (NL) introduced by Lambek is a propositional logic omitting all structural rules, which can be treated as a basic core of substructural logics. NL can be enriched in different ways. One affixes additive connectives $\wedge$, $\vee$ and optionally constants $\bot$, $\top$ and classical negation, which yields full nonassociative Lambek calculus (FNL) and its extensions DFNL and BFNL, satisfying the distributive laws for $\wedge$, $\vee$ or the laws boolean algebras, respectively. Modal nonassociative Lambek calculus (NL$\Diamond$) is NL enriched with modalities $\Diamond$, $\Box^{\downarrow}$ satisfying the residuation law; $\Diamond A \Rightarrow B$ iff $A \Rightarrow \Box^{\downarrow} B$, which enables one to use certain structure postulates in a controlled way.

In this thesis we study extensions of NL, DFNL, and BFNL enriched with modal operators admitting any combinations of basic modal axioms (T), (4) and (5). We also consider assumptions. In other words, we study consequence relations of these logics. We prove and use some basic proof-theoretic properties of them, e.g. cut elimination, interpolation, subformula property. We also consider algebras corresponding to these logics. We prove strong finite model property (SFMP) for some classes of algebras, which yields the decidability of consequence relations. Moreover we study the complexity of these logics (and consequence relations). We construct some decision procedures for the additive free logics considered in this thesis. It turns out that the complexity of these logics (and consequence relations) are P (polynomial time). Further we show that BFNL is PSPACE-complete and its modal extensions are PSPACE-hard.

Grammars based on logics of types are called type grammars. A characteristic feature of type grammars is the usage of (logical) types as carriers of grammatical information. We focus on type grammars based on modal extensions of NL and DFNL. We show that the type grammars based on some logics, considered in this thesis, enriched with assumptions, generate context-free languages.

Tytuł rozprawy: Pełny niełączny rachunek Lambeka z modalnościami i jego zastosowania w gramatykach typów.

Autor rozprawy: Zhe Lin

Ubiega się o stopień naukowy: doktor nauk matematycznych w zakresie informatyki

Streszczenie

Rozprawa jest poświęcona pełnemu niełącznemu rachunkowi Lambeka wzbogaconemu o różne modalności; te systemy tworzą pewną rodzinę logik substrukturalnych. Logiki substrukturalne to logiki, które pomijają pewne reguy strukturalne, np. kontrakcję, osłabianie, przemienność. Niełączny rachunek Lambeka (NL), wprowadzony przez J. Lambeka, jest logiką zdaniową pomijającą wszystkie reguły strukturalne, która może być traktowana jako podstawowa logika substrukturalna. Można rozszerzać NL na różne sposoby. Dodając spójniki addytywne $\wedge, \vee$ i opcjonalnie stałe $\bot, \top$ oraz klasyczną negację, otrzymujemy pełny niełączny rachunek Lambeka (FNL) i jego rozszerzenia DFNL i BFNL, spełniające prawa dystrybutywności dla $\wedge, \vee$ lub prawa algebry Boole'a. Modalny niełączny racunek Lambeka (NL$\Diamond$) otrzymujemy wzbogacając NL o modalności $\Diamond, \Box^{\downarrow}$, spełniające prawo rezyduacji: $\Diamond A \Rightarrow B$ wtedy i tylko wtedy, gdy $A \Rightarrow \Box^{\downarrow} B$, co pozwala stosować pewne postulaty strukturalne w kontrolowany sposób.

W rozprawie badamy rozszerzenia NL, DFNL i BFNL, wzbogacone o operatory modalne, spelniające dowolne kombinacje podstawowych aksjomatów modalnych (T), (4) i (5). Rozważamy też założenia. Innymi słowy, badamy relacje konsekwencji tych logik. Dowodzimy i stosujemy pewne teorio-dowodowe własności, np. eliminację cięć, interpolację, własność podformuły. Rozważamy też algebry odpowiadające tym logikom. Dowodzimy silną własność skończonego modelu (SFMP) pewnych klas algebr, z której wynika rozstrzygalność relacji konsekwencji. Ponadto badamy złożoność tych logik (i relacji konsekwencji).

Podajemy procedury rozstrzygania dla fragmentów bez spójników addytywnych. Okazuje się, że złożoność tych fragmetów i ich relacji konsekwencji jest P (czas wielomianowy). Z kolei BFNL jest PSPACE-zupełny, a jego modalne rozszerzenia są PSPACE-trudne.

Gramatyki oparte na logikach typów nazywamy gramatykami typów. Cechą charakterystyczną gramatyk typów jest wykorzystywanie typów logicznych w roli nośników informacji gramatycznej. Zajmujemy się gramatykami opartymi na modalnych rozszerzeniach NL i DFNL. Wykazujemy, że gramatyki oparte na takich logikach generują języki bezkontekstowe.

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Contents

Chapter 1

Introduction

1.1 A general overview

This thesis is concerned with full nonassociative Lambek calculus with different modalities which is a family of substructural logics. Substructural logics are logics which omit some structural rules, e.g. contraction, weakening, commutativity. Relevant logics, linear logics, BCK-logics, fuzzy logics and Lambek calculus are examples of substructural logics. Some substructural logics are studied in N. Galatos & Ono [2007]. Lambek calulus (L) introduced in Lambek [1958] is a propositional logic omitting all structural rules except associativity. Nonassociative Lambek calculus (NL) was introduced in Lambek [1961] as a nonassociative version of L. Since NL rejects all structural rules, it can be treated as a basic core of substructural logics.

NL can be enriched in different ways. One affixes additive connectives $\wedge$, $\vee$ and optionally constants $\bot$, $\top$ and classical negation or intuitionistic implication, which yields full nonassociative Lambek calculus (FNL) and its extensions, satisfying the distributive laws for $\wedge$, $\vee$, the laws of Heyting algebras or boolean algebras; see Buszkowski [2011]; Buszkowski & Farulewski [2009]. Moortgat [1996] studied NL$\Diamond$, i.e. NL with modalities $\Diamond$, $\Box^{\downarrow}$ satisfying the residuated law; $\Diamond A \Rightarrow B$ iff $A \Rightarrow \Box^{\downarrow} B$, as a system which enables one to use certain structure postulates in a controlled way. One considers extensions of NL$\Diamond$ admitting modal axioms from normal modal logics, e.g. (T), (4); see Moortgat [1996]; Plummer

[2007].

A type grammar (categorial grammar) consists of a logic and a lexicon. The lexicon is a list of atomic expressions of the language under study and types (categories) assigned to them. The logic is a substructural logic whose formulae are interpreted as types. It is used as a universal (i.e. language-independent) system for parsing complex expressions. Thus finding the type assignment for complex expressions is to do logical inferences in the system. As an example of this kind of 'grammatical inference', consider the expression *Chen drinks coffee*. Assume the grammar classifies the name *Chen* as np, and the verb *drinks* as $(np\backslash s)/n$ and the noun *coffee* as n. Using the basic rules of functional application

$$A, A\backslash B \Rightarrow B \quad A/B, B \Rightarrow A$$

we conclude that the sequence $(np, (np\backslash s)/n, n)$ corresponding to the expression *Chen drinks coffee* reduces to s.

Type grammars form a logically oriented class of formal grammars, whose characteristic features are: lexicality, uniformity of syntactic and semantic analysis, and close connections with substructural logics and type theories.

In this thesis we study extensions of NL, FNL with modalities. We also consider assumptions. The latter are sequents added to pure logics (like non-logical axioms in first-order theories, assumptions are not closed under substitution). In other words, we study consequence relations of these logics. Our logics are usually presented as sequent systems, and we use some basic proof-theoretic properties of them (cut elimination, subformula property). We prove some versions of these properties for theories. An interpolation lemma, proved here, plays an essential role. We also consider algebras corresponding to these logics. We prove strong finite model property (SFMP) for some classes of algebras, which yields the decidability of consequence relations. Moreover we study the complexity and generative capacity of these logics (and consequence relations). Such problems were studied for basic logics by several authors. Pentus [1993] proved that L-grammars are context-free; earlier Buszkowski [1986b] and Kandulski [1988] obtained analogous results for NL-grammars. L is NP-complete (Pentus [2006]), and NL is polynomial (P) (deGroote [1999]). The consequence relation for L is undecidable

(Buszkowski [1982]) and for NL it is P (Buszkowski [2005]). Some basic associative substructural logics are PSPACE-complete (Horčík & Terui [2011]). Some other results relative to these topics can be found in Buszkowski *et al.* [2012], Buszkowski [2008], Savateev [2012] and so on.

We solve similar problems for nonassociative logics with modalities, applying several new constructions and arguments. The main results are: (1) all systems considered here generate context-free languages; (2) systems without $\wedge$, $\vee$, are P (polynomial time); (2) Consequence relations of these logics are decidable, and this follows from SFMP. (3) some systems with $\wedge$, $\vee$ are PSPACE-hard, (4) BFNL is PSPACE-complete. These results extend and refine some results of Buszkowski & Farulewski [2009], Buszkowski [2011] and Farulewski [2008] concerning logics without modal axioms and Plummer [2007] and Jäger [2004] concerning logics without additive connectives.

1.2 A look ahead: chapter overview

The thesis consists of four chapters. Chapter 3 is based on Lin [2010]. The results in chapter 4 come from Lin [2012] and Lin [2014]. Chapter 6 extends the results in Lin [2014]. A general overview is as follows.

Chapter 1 is this introduction. Chapter 2 starts with a presentation of NL. In the first section, we discuss some proof theoretic properties of NL. We recall the sequent calculus for NL and NL1 (NL with unit) and the cut elimination theorem for both systems. In the second section, we consider some additive and modal extensions of NL. First we recall different variants of NL with additive connectives, e.g. DFNL (NL with additives $\wedge$, $\vee$ and distribution), BFNL (NL with $\wedge$, $\vee$, $\neg$ satisfying the laws of boolean algebra). Second, we recall structural rules corresponding to modal axioms 4, T and $\overline{\mathrm{K}}$, which are introduced by Moortgat [1996]. Then we list some properties of these extensions, which will be used in further chapters. In the third section, we discuss algebraic semantics for logics under consideration. We also recall the Kripke frame semantics for these logics in this section. Finally, at the end of this chapter, we recall the definition of context-free grammars, type grammars and present some examples of linguistic analysis by means of type grammars based on systems considered in this chapter.

Chapter 3 starts with a proof-theoretic analysis of modal extensions of NL. The cut elimination theorem for NL was proved by Lambek [1961]. It yields the decidability and the subformula property of NL. However, the cut elimination theorem does not hold for theories. In section 3.2, we show that a restricted form of cut elimination is allowed for theories based on logics under consideration, which yields a syntactic proof of the extended subformula property. This method can easily be extended to any system enriched with assumptions, if its pure version admits cut elimination. Then we provide some decision procedures for modal extensions of NL. It turns out that the modal extensions of NL are PTIME. Finally, we address the generative capacity of type grammars based on the modal extensions of NL discussed in this chapter. We prove that type grammars based on them are equivalent to context-free grammars.

Chapter 4 starts with an interpolation lemma for S4 modal extension of DFNL (a refinement of analogous results in Buszkowski [2011] and Buszkowski & Farulewski [2009]). Then we discuss SFMP and FEP (finite embeddability property) and investigate their interconnections. We show that all classes of modal residuated algebras, corresponding to our modal extensions of DFNL have SFMP, hence the consequence relations of these logics are decidable. It also yields FEP for the corresponding classes of algebras. We use algebraic and model theoretical methods. We also adapt these results to other modal extensions of NL and BFNL. In the final section, we prove that type grammars based on modal extensions of DFNL and BFNL are equivalent to context-free grammars. The proofs in this chapter are based on interpolation lemmas.

In chapter 5, we study the complexity of BFNL and its modal extensions. We show that BFNL is PSPACE-complete. Kripke frames are essentially used in the proofs of PSPACE-completeness. It follows that DFNL is in PSPACE, and the modal extensions of BFNL and the consequence relations of DFNL are PSPACE-hard.

Finally, in chapter 6, we consider S5 modal extensions of NL and FNL. We introduce sequent systems and algebraic models for the S5 modal extensions. Then we briefly explain how to obtain analogous results discussed in previous chapters for the S5 modal extensions.

Chapter 2

Nonassociative Lambek calculus and its additive and modal extensions

2.1 Introduction

The nonassociative Lambek calculus (Lambek [1961]) was introduced to carry out a categorial analysis of bracketed expressions. Instead of using lists of formulae as antecedents, like for L, the Gentzen-style presentation of NL uses binary trees of formulae. In many linguistic analyses, it is necessary to regard phrase structures of expressions, otherwise one has to admit some pseudo-sentences as to be grammatical. Hence associativity is sometimes undesirable. From this point of view, NL provides some advantages over L. On the other hand, NL can be treated as a basic substructural logic. Roughly speaking, NL is a logic which omits all structural rules. It is a pure logic of residuation. Therefore it makes sense to investigate NL and its extensions.

It is not surprising that there are some linguistic phenomena like discontinuity phenomena that cannot be handled by type grammars based on L and NL. Different extensions of L and NL have been studied in order to give a better linguistic treatment of some phenomena. In this chapter, we will look at two typical ways of extending NL: adding modal and additive connectives. Moortgat [1996] and Kurtonina [1998] inspired by linear logic (Girard [1987]) propose a way of adding modal operators to NL. Their proposal uses a pair of modal connectives $\Diamond$ and $\Box^{\downarrow}$ satisfying the residuation law. Modal connectives extend NL in several ways. First, they allow a controlled use of structural rules, just like in linear logic, where one uses exponentials to control access to the structural rules of weakening and contraction. Another application is to extend logics by introducing axioms expressing properties of modal connectives as in modal logics. We will mainly discuss the latter kind of extension in what follows. Besides, additive connectives proposed by Kanazawa [1992] to describe feature-assignments of types, are also considered here. In addition, we assume that additive connectives satisfy distributive or boolean axioms like in Buszkowski & Farulewski [2009], Buszkowski [2011] and Kaminski & Francez [2014].

In section 2.2 we recall an axiomatic system, a sequent calculus and the proof of cut elimination for NL. In section 2.3, we discuss modal and additive extensions of NL. We recall some results on these extensions and present sequent rules corresponding to different connectives and axioms. Some of these systems do not allow cut elimination. However some kinds of subformula property still hold for these enrichments, which will be needed in chapter 3 and 4. Then we show some provable sequents in these systems, which will be needed in the following chapters. Moreover in section 2.4, we discuss algebraic semantics for logics under consideration. We also recall the Kripke frame semantics for NL and its modal and additive extensions. Finally, in the last section, we recall the definition of context-free grammars, type grammars and present some examples of linguistic analysis by means of type grammars based on systems considered in this chapter.

2.2 Proof theory of NL

2.2.1 Nonassociative Lambek calculus

The axiomatic system of NL, proposed by Lambek [1961], can be defined as follows. The set of all NL formulae is recursively defined by the following rule:

$$A ::= p \mid A \cdot B \mid A/B \mid A\backslash B$$

where $p \in \mathbf{Prop}$ (the set of propositional variables). We reserve $p, q, r, s, \ldots$ for propositional variables, $A, B, C, \ldots$ for formulae etc. $\cdot$, $\backslash$ and $/$ are called product, right residuation and left residuation, respectively (residuations are also called implications). Simple sequents are expressions of the form $A \Rightarrow B$. The axiom is:

$$\text{(Id)} \quad A \Rightarrow A.$$

The rules are

$$(\text{Res}\backslash\cdot) \quad \frac{B \Rightarrow A\backslash C}{A \cdot B \Rightarrow C}, \quad (\text{Res}\cdot\backslash) \quad \frac{A \cdot B \Rightarrow C}{B \Rightarrow A\backslash C},$$

$$(\text{Res}/\cdot) \quad \frac{A \Rightarrow C/B}{A \cdot B \Rightarrow C}, \quad (\text{Res}\cdot/) \quad \frac{A \cdot B \Rightarrow C}{A \Rightarrow C/B},$$

$$(\text{Trans}) \quad \frac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}.$$

The following monotonicity rules for $\cdot, \backslash, /$ are derivable in NL.

$$(\text{Mon}(\cdot)_{\text{r}}) \quad \frac{A \Rightarrow B}{A \cdot C \Rightarrow B \cdot C} \quad (\text{Mon}(\cdot)_{\text{l}}) \quad \frac{A \Rightarrow B}{C \cdot A \Rightarrow C \cdot B}.$$

$$(\text{Mon}(\backslash)_{\text{r}}) \quad \frac{A \Rightarrow B}{C\backslash A \Rightarrow C\backslash B} \quad (\text{Mon}(\backslash)_{\text{l}}) \quad \frac{A \Rightarrow B}{B\backslash C \Rightarrow A\backslash C}.$$

$$(\text{Mon}(/)_{\text{r}}) \quad \frac{A \Rightarrow B}{C/B \Rightarrow C/A} \quad (\text{Mon}(/)_{\text{l}}) \quad \frac{A \Rightarrow B}{A/C \Rightarrow B/C}.$$

L results from NL by adding the associative axioms:

$$(A \cdot B) \cdot C \Rightarrow A \cdot (B \cdot C), \quad A \cdot (B \cdot C) \Rightarrow (A \cdot B) \cdot C.$$

The following laws of L:

$$(A\backslash B) \cdot (B\backslash C) \Rightarrow (A\backslash C), \quad (A/B) \cdot (B/C) \Rightarrow A/C \quad \text{(composition laws)},$$

$$A\backslash B \Rightarrow (C\backslash A)\backslash(C\backslash B), \quad A/B \Rightarrow (A/C)/(B/C) \quad \text{(Geach laws)},$$

$$(A\backslash B) \cdot C \Rightarrow A\backslash(B \cdot C), \quad A \cdot (B/C) \Rightarrow (A \cdot B)/C \quad \text{(switching laws)},$$

are not derivable in NL. However the following laws of L are provable in NL:

$$A \cdot (A\backslash B) \Rightarrow B, \quad (B/A) \cdot A \Rightarrow B \quad \text{(application laws)},$$

$$A \Rightarrow (B/A)\backslash B, \quad A \Rightarrow B/(A\backslash B) \quad \text{type} - \text{raising laws)},$$

$$A \Rightarrow B\backslash(B \cdot A), \quad A \Rightarrow (A \cdot B)/B \quad \text{(expansion laws)}.$$

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2.2.2 Sequent calculus for NL

The sequent system of NL was given by Lambek [1961]. Formulae (types) are defined as in section 2.2.1. Formula trees are either atomic, i.e. single formulae, or complex $(\Gamma \circ \Delta)$, where Γ and Δ are formula trees. A *context* is a formula tree containing one occurrence of special atom $-$ (a place for substitution). If $\Gamma[-]$ is a context, then $\Gamma[\Delta]$ denotes the substitution of Δ for $-$ in Γ. Sequents are of the form $\Gamma \Rightarrow A$ where Γ is a formula tree and A is a formula. By $f(\Gamma)$, we mean a formula generated from the formula tree Γ by replacing each $\circ$ by $\cdot$ as follows: $f(A) = A$ and $f(\Gamma \circ \Delta) = f(\Gamma) \cdot f(\Delta)$.

One admits the axiom:

$$\text{(Id)} \quad A \Rightarrow A,$$

and inference rules

$$(\backslash\text{L}) \quad \frac{\Delta \Rightarrow A \quad \Gamma[B] \Rightarrow C}{\Gamma[\Delta \circ (A\backslash B)] \Rightarrow C}, \quad (\backslash\text{R}) \quad \frac{A \circ \Gamma \Rightarrow B}{\Gamma \Rightarrow A\backslash B},$$

$$(/\text{L}) \quad \frac{\Gamma[A] \Rightarrow C \quad \Delta \Rightarrow B}{\Gamma[(A/B) \circ \Delta] \Rightarrow C}, \quad (/\text{R}) \quad \frac{\Gamma \circ B \Rightarrow A}{\Gamma \Rightarrow A/B},$$

$$(\cdot\text{L}) \quad \frac{\Gamma[A \circ B] \Rightarrow C}{\Gamma[A \cdot B] \Rightarrow C}, \quad (\cdot\text{R}) \quad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \circ \Delta \Rightarrow A \cdot B}, \quad (\text{Cut}) \quad \frac{\Delta \Rightarrow A; \quad \Gamma[A] \Rightarrow B}{\Gamma[\Delta] \Rightarrow B}.$$

By $\vdash_{\text{NL}} \Gamma \Rightarrow A$, we mean that the sequent $\Gamma \Rightarrow A$ is provable in NL. By $\vdash_{\text{NL}} A \Leftrightarrow B$, we mean that $\vdash_{\text{NL}} A \Rightarrow B$ and $\vdash_{\text{NL}} B \Rightarrow A$. Let Φ be a set of sequents. By $\Phi \vdash_{\text{NL}} \Gamma \Rightarrow A$, we mean that the sequent $\Gamma \Rightarrow A$ is derivable from Φ in NL. Notice that all antecedents of sequents in NL are nonempty. Let T

be a set of formulae. By a T-sequent we mean a sequent such that all formulae occurring in it belong to T. We write $\Phi \vdash_S \Gamma \Rightarrow_T A$ if $\Gamma \Rightarrow A$ has a deduction from Φ (in the given system S), which consists of T-sequents only (called a T-deduction). Two formulae A and B are said to be T-equivalent in a system S, if and only if $\vdash_S A \Rightarrow_T B$ and $\vdash_S B \Rightarrow_T A$. T-equivalence is an equivalence relation, by (Id) and (Cut).

Lemma 2.2.1

(InL$\cdot$) *If* $\Phi \vdash_{\mathrm{NL}} \Gamma[A \cdot B] \Rightarrow A$*, then* $\Phi \vdash_{\mathrm{NL}} \Gamma[A \circ B] \Rightarrow A$.

(InR\) *If* $\Phi \vdash_{\mathrm{NL}} \Gamma \Rightarrow A \backslash B$*, then* $\Phi \vdash_{\mathrm{NL}} A \circ \Gamma \Rightarrow B$.

(InR/) *If* $\Phi \vdash_{\mathrm{NL}} \Gamma \Rightarrow B/A$*, then* $\Phi \vdash_{\mathrm{NL}} \Gamma \circ A \Rightarrow B$.

Lemma 2.2.1 can be easily proved by using (Cut). The proofs are omitted here.

NL* admits sequents of the form $\Lambda \Rightarrow A$, written $\Rightarrow A$, and the axioms and rules of NL, extended to the empty tree Λ. We assume that the empty tree Λ is the unit for $(- \circ -)$: $\Lambda \circ \Delta = \Delta \circ \Lambda = \Delta$. In particular, from $A \Rightarrow B$ one can infer $\Rightarrow A \backslash B$ by (\R). From NL* one can obtain NL1 by adding the constant 1 admitting axiom (1) $\Rightarrow 1$ and two new rules:

$$(1r) \quad \frac{\Gamma[\Delta] \Rightarrow A}{\Gamma[\Delta \circ 1] \Rightarrow A}, \quad (1l) \quad \frac{\Gamma[\Delta] \Rightarrow A}{\Gamma[1 \circ \Delta] \Rightarrow A}.$$

Furthermore, we can also introduce one more propositional constant 0, and assume axiom (0): $0 \Rightarrow$ and the following inference rule for it:

$$(0r) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}$$

It means that 1 (0) is the strongest (weakest) proposition among provable formulae (contradictory formulae, respectively). Negations $-A$ and $\sim A$ are defined as $A\backslash 0$ and $0/A$, respectively. Usually the connectives $\cdot$, $\backslash$, $/$, 1 and 0 are called multiplicatives, and the lattice connectives $\wedge$, $\vee$, $\top$ and $\bot$ are called additives. The sequent system for L arises from NL by adding two rules of associativity (Ass). By adding the structural rule of commutativity (Com) as well, one obtains the

Lambek-van Benthem calculus LP. (Ass), (Com) are defined as follows:

$$\text{(Ass)} \quad \frac{\Gamma[\Delta_1 \circ (\Delta_2 \circ \Delta_3)] \Rightarrow A}{\Gamma[(\Delta_1 \circ \Delta_2) \circ \Delta_3] \Rightarrow A}, \quad \text{(Com)} \quad \frac{\Gamma[\Delta_1 \circ \Delta_2] \Rightarrow A}{\Gamma[\Delta_2 \circ \Delta_1] \Rightarrow A}.$$

2.2.3 Cut elimination for NL

In this section we prove the cut-elimination theorem for NL (Theorem 2.2.2) following Lambek [1961]. This implies the subformula property and decidability for NL. Before we look at the proof of the cut-elimination theorem, we first introduce two definitions: the complexity of formulae and the rank of (Cut). A formula created by a rule R is called the main formula of R. The designated formula in (Cut) is called the cut formula.

Defintion 2.2.1 *The complexity of a formula A, written $\mathcal{D}(A)$, is the total number of all logical connectives appearing in A. We recursively define $\mathcal{D}$ as follows:*

$\mathcal{D}(p) = 0$ *when p is an atomic formula,*

$\mathcal{D}(A \cdot B) = \mathcal{D}(A) + \mathcal{D}(B) + 1,$

$\mathcal{D}(A\backslash B) = \mathcal{D}(A) + \mathcal{D}(B) + 1,$

$\mathcal{D}(A/B) = \mathcal{D}(A) + \mathcal{D}(B) + 1.$

Defintion 2.2.2 *The rank of an application of (Cut), written $\mathcal{R}(\text{Cut})$, is the total number of sequents appearing in the derivations of both premises of (Cut).*

Theorem 2.2.2 *Every sequent provable in* NL *is also provable in* NL *without* (Cut)*.*

Proof: It suffices to show that if two premises of an application of (Cut) can be proved in NL without any application of (Cut), then one can also prove the conclusion without any application of (Cut). We apply induction (i) on $\mathcal{D}$(A), the complexity of the cut formula A, i.e. the total number of occurrences of logical connectives in A. For each case, we apply induction (ii) on $\mathcal{R}$(Cut), the rank of (Cut), i.e. the total number of sequents appearing in the proofs of both premises of (Cut).

Assume that an application of (Cut) in the proof of $\Gamma \Rightarrow A$ in NL is as follows:

$$\dfrac{\dfrac{\vdots}{\Delta \Rightarrow B}\;(\mathrm{R}_1) \qquad \dfrac{\vdots}{\Gamma'[B] \Rightarrow A'}\;(\mathrm{R}_2)}{\Gamma'[\Delta] \Rightarrow A'}\;(\mathrm{Cut}).$$

We consider three cases: 1) at least one of the rules is an axiom; 2) R_1 or R_2 does not create the cut formula B; 3) both R_1 and R_2 create the cut formula B.

(1) If R_1 is an axiom, then (Cut) can be removed directly. The following application of (Cut)

$$\dfrac{B \Rightarrow B \quad \Gamma'[B] \Rightarrow A'}{\Gamma'[B] \Rightarrow A'}\;(\mathrm{Cut})$$

can be replaced by

$$\dfrac{\vdots}{\Gamma'[B] \Rightarrow A'}.$$

The case that R_2 is an axiom can be treated similarly.

(2) Assume that R_1 or R_2 does not create the cut formula B.

(a) Let us consider a typical subcase $R_1 = (\backslash \mathrm{L})$ with premises $\Upsilon \Rightarrow C$ and $\Delta'[D] \Rightarrow B$, where $\Delta = \Delta'[\Upsilon \circ C\backslash D]$. The following application of (Cut)

$$\dfrac{\dfrac{\Upsilon \Rightarrow C \quad \Delta'[D] \Rightarrow B}{\Delta'[\Upsilon \circ C\backslash D] \Rightarrow B}\;(\backslash \mathrm{L}) \qquad \dfrac{\vdots}{\Gamma'[B] \Rightarrow A'}\;(\mathrm{R}_2)}{\Gamma'[\Delta'[\Upsilon \circ C\backslash D]] \Rightarrow A'}\;(\mathrm{Cut})$$

can be replaced by

$$\dfrac{\dfrac{\vdots}{\Upsilon \Rightarrow C} \qquad \dfrac{\Delta'[D] \Rightarrow B \quad \Gamma'[B] \Rightarrow A'}{\Gamma'[\Delta'[D]] \Rightarrow A'}\;(\mathrm{Cut})}{\Gamma'[\Delta'[\Upsilon \circ C\backslash D]] \Rightarrow A'}\;(\backslash \mathrm{L}).$$

Obviously, $\mathcal{R}$(Cut) is smaller in the new derivation.

(b) Let us consider another typical case $R_2 = (\cdot\mathrm{R})$ with premises $\Gamma_1[B] \Rightarrow A_1$ and $\Gamma_2 \Rightarrow A_2$, where $\Gamma'[B] = \Gamma_1[B] \circ \Gamma_2$ and $A' = A_1 \cdot A_2$. The following application of (Cut)

$$\dfrac{\dfrac{\vdots}{\Delta \Rightarrow B}\ (\mathrm{R}_1) \qquad \dfrac{\Gamma_1[B] \Rightarrow A_1 \quad \Gamma_2 \Rightarrow A_2}{\Gamma_1[B] \circ \Gamma_2 \Rightarrow A'}\ (\cdot\mathrm{R})}{\Gamma_1[\Delta] \circ \Gamma_2 \Rightarrow A_1 \cdot A_2}\ (\mathrm{Cut}).$$

can be replaced by

$$\dfrac{\dfrac{\Delta \Rightarrow B \quad \Gamma_1[B] \Rightarrow A_1}{\Gamma_1[\Delta] \Rightarrow A_1}\ (\mathrm{Cut}) \qquad \dfrac{\vdots}{\Gamma_2 \Rightarrow A_2}}{\Gamma_1[\Delta] \circ \Gamma_2 \Rightarrow A_1 \cdot A_2}\ (\cdot\mathrm{R}).$$

Obviously, $\mathcal{R}(\mathrm{Cut})$ is smaller in the new derivation.

(3) Assume that both R_1 and R_2 create the cut formula B.

(a) Let us consider the case $R_1 = (\cdot\mathrm{R})$ and $R_2 = (\cdot\mathrm{L})$, where the premises of R_1 are $\Delta_1 \Rightarrow B_1$ and $\Delta_2 \Rightarrow B_2$ such that $\Delta = \Delta_1 \circ \Delta_2$ and $B = B_1 \cdot B_2$, and the premise of R_2 is $\Gamma'[B_1 \circ B_2] \Rightarrow A'$. The following application of (Cut)

$$\dfrac{\dfrac{\Delta_1 \Rightarrow B_1 \quad \Delta_2 \Rightarrow B_2}{\Delta_1 \circ \Delta_2 \Rightarrow B_1 \cdot B_2}\ (\cdot\mathrm{R}) \qquad \dfrac{\Gamma'[B_1 \circ B_2] \Rightarrow A'}{\Gamma'[B_1 \cdot B_2] \Rightarrow A'}\ (\cdot\mathrm{L})}{\Gamma'[\Delta_1 \circ \Delta_2] \Rightarrow A'}\ (\mathrm{Cut}).$$

can be replaced by

$$\dfrac{\dfrac{\vdots}{\Delta_1 \Rightarrow B_1} \qquad \dfrac{\Delta_2 \Rightarrow B_2 \quad \Gamma'[B_1 \circ B_2] \Rightarrow A'}{\Gamma'[B_1 \circ \Delta_2] \Rightarrow A'}\ (\mathrm{Cut})}{\Gamma'[\Delta_1 \circ \Delta_2] \Rightarrow A'}\ (\mathrm{Cut}).$$

Obviously, $\mathcal{D}(B_1) < \mathcal{D}(B)$ and $\mathcal{D}(B_2) < \mathcal{D}(B)$.

(b) Let us consider the case $R_1 = (\backslash\mathrm{R})$ and $R_2 = (\backslash\mathrm{L})$, where the premise of R_1 is $B_2 \circ \Delta \Rightarrow B_1$ and the premises of R_2 are $\Upsilon \Rightarrow B_2$ and $\Gamma''[B_1] \Rightarrow$

A'. The following application of (Cut)

$$\dfrac{\dfrac{B_2 \circ \Delta \Rightarrow B_1}{\Delta \Rightarrow B_2 \backslash B_1}\ (\backslash R) \quad \dfrac{\Upsilon \Rightarrow B_2 \quad \Gamma''[B_1] \Rightarrow A'}{\Gamma''[\Upsilon \circ B_2 \backslash B_1] \Rightarrow A'}\ (\backslash L)}{\Gamma''[\Upsilon \circ \Delta] \Rightarrow A'}\ (\text{Cut}).$$

can be replaced by

$$\dfrac{\dfrac{\Upsilon \Rightarrow B_2 \quad B_2 \circ \Delta \Rightarrow B_1}{\Upsilon \circ \Delta \Rightarrow B_1}\ (\text{Cut}) \quad \dfrac{\vdots}{\Gamma''[B_1] \Rightarrow A'}}{\Gamma''[\Upsilon \circ \Delta] \Rightarrow A'}\ (\text{Cut}).$$

Obviously, $\mathcal{D}(B_1) < \mathcal{D}(B)$ and $\mathcal{D}(B_2) < \mathcal{D}(B)$.

□

Corollary 2.2.3 *For any sequent* $\Gamma \Rightarrow A$ *provable in* NL*, there exists a proof of* $\Gamma \Rightarrow A$ *such that all formulae appearing in the proof are subformulae of formulae in* $\Gamma \Rightarrow A$*.*

Note that all the remaining rules of NL increase the number of connectives (this means: the number of connectives in any premise is less than that in the conclusion). Together with the subformula property (Corollary 2.2.3), this yields a finite proof-search procedure for any sequent. Hence we obtain the following corollary.

Corollary 2.2.4 NL *is decidable.*

2.3 Extensions of NL

2.3.1 Additive extensions of NL

NL enriched with lattice connectives is called full nonassociative Lambek calculus (FNL). The name full nonassociative Lambek calculus and the symbol FNL first appeared in Buszkowski & Farulewski [2009], but the same logic enriched with unit for product was also considered in N. Galatos & Ono [2007] under the name groupoid logic (GL).

FNL employs operations $\cdot$, $\backslash$, $/$, $\wedge, \vee$. One admits the following rules for $\vee$ and $\wedge$:

$$(\wedge\mathrm{L}) \quad \frac{\Gamma[A_i] \Rightarrow B}{\Gamma[A_1 \wedge A_2] \Rightarrow B}, \quad (\wedge\mathrm{R}) \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B},$$

$$(\vee\mathrm{L}) \quad \frac{\Gamma[A_1] \Rightarrow B \quad \Gamma[A_2] \Rightarrow B}{\Gamma[A_1 \vee A_2] \Rightarrow B}, \quad (\vee\mathrm{R}) \quad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}.$$

In ($\wedge$L) and ($\vee$R), the subscript i equals 1 or 2. Distributive full nonassociative Lambek calculus (DFNL) is FNL enriched with the new axiom:

$$(\mathrm{D}) \qquad A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C).$$

Notice that the converse sequent is provable in FNL. Additive constants $\bot$ and $\top$ can also be added, with axioms:

$$(\bot) \quad \Gamma[\bot] \Rightarrow A, \quad (\top) \quad \Gamma \Rightarrow \top$$

They are interpreted respectively as the lower bound and the upper bound of the lattice. The resulting systems are denoted by $\mathrm{NL}_{\top,\bot}$ (bounded NL) and $\mathrm{DFNL}_{\bot,\top}$ (bounded DFNL).

Boolean nonassociative Lambek calculus (BFNL) is DFNL enriched with $\neg$, $\bot$ and $\top$ together with $\vee$, $\wedge$ admitting the laws of boolean algebra. As in Buszkowski & Farulewski [2009] and Buszkowski [2011], we add the following new axioms to $\mathrm{DFNL}_{\bot,\top}$ with $\neg$:

$$(\neg 1) \quad A \wedge \neg A \Rightarrow \bot, \quad (\neg 2) \quad \top \Rightarrow A \vee \neg A.$$

The resulting system is BFNL. The double negation axiom (DN) and transposition rule (TR) are derivable in BFNL:

$$(\mathrm{DN}) \quad A \Leftrightarrow \neg\neg A \quad (\mathrm{TR}) \quad \frac{A \Rightarrow B}{\neg B \Rightarrow \neg A}$$

If one admits empty antecedents of sequents, then one obtains the system BFNL*. One can also consider systems with several product operations and their residuals (see Buszkowski [2011] and Buszkowski [2014a]).

2.3.2 Modal extensions of NL

In this section, we start from the modal logic NL$\Diamond$ which was introduced by Moortgat [1996], and discuss various extensions of it. The sequent system of NL$\Diamond$ is given in the same paper. The set of NL$\Diamond$-formulae is defined by the following inductive rules respectively:

$$A ::= p \mid A \cdot B \mid A/B \mid A \backslash B \mid \Diamond A \mid \Box^{\downarrow} A,$$

where $p \in$ **Prop**. Now formula trees that occur in antecedents of sequents are composed from formulae by two structure operations: a binary one $(- \circ -)$ and a unary one $\langle - \rangle$, corresponding to the connectives $\cdot$ and $\Diamond$, respectively. Formula trees are either single formulae, or complex $\Gamma \circ \Delta$ and $\langle \Delta \rangle$, where Γ and Δ are formula trees. $f(\Gamma)$ is defined as in section 2.2.2 together with the clause: $f(\langle \Gamma \rangle) = \Diamond f(\Gamma)$.

The sequent rules for $\Diamond$ and $\Box^{\downarrow}$ are as follows:

$$(\Diamond \mathrm{L}) \quad \frac{\Gamma[\langle A \rangle] \Rightarrow B}{\Gamma[\Diamond A] \Rightarrow B}, \qquad (\Diamond \mathrm{R}) \quad \frac{\Gamma \Rightarrow A}{\langle \Gamma \rangle \Rightarrow \Diamond A},$$

$$(\Box^{\downarrow} \mathrm{L}) \quad \frac{\Gamma[A] \Rightarrow B}{\Gamma[\langle \Box^{\downarrow} A \rangle] \Rightarrow B}, \quad (\Box^{\downarrow} \mathrm{R}) \quad \frac{\langle \Gamma \rangle \Rightarrow A}{\Gamma \Rightarrow \Box^{\downarrow} A}.$$

The following monotonicity rules for $\Diamond, \Box^{\downarrow}$ are derivable in NL$\Diamond$:

$$(\mathrm{Mon}(\Diamond)) \quad \frac{A \Rightarrow B}{\Diamond A \Rightarrow \Diamond B} \quad (\mathrm{Mon}(\Box^{\downarrow})) \quad \frac{A \Rightarrow B}{\Box^{\downarrow} A \Rightarrow \Box^{\downarrow} B}$$

The sequent system NL$\Diamond$ is obtained from NL by adding the above rules for $\Diamond$ and $\Box^{\downarrow}$. By adding the above rules to FNL, FNL$\Diamond$ can be obtained in a similar way. Both systems admit cut elimination.

$\Diamond$ and $\Box^{\downarrow}$ can be seen as truncated forms of product and implication. It may be helpful to compare rules for $\Diamond$ with rules for $\cdot$, and rules for $\Box^{\downarrow}$ with rules for an implication, say $/$. It is natural to consider some modal axioms as in classical modal logic. The following is a list of basic modal axioms, which are often discussed in literature:

$$(\mathrm{T})\quad A \Rightarrow \Diamond A, \qquad (4)\quad \Diamond\Diamond A \Rightarrow \Diamond A, \quad (\overline{\mathrm{K}})\quad \Diamond(A \cdot B) \Rightarrow \Diamond A \cdot \Diamond B.$$

Notice that ($\mathrm{T}(\Box^{\downarrow})$): $\Box^{\downarrow} A \Rightarrow A$ and ($4(\Box^{\downarrow})$): $\Box^{\downarrow} A \Rightarrow \Box^{\downarrow}\Box^{\downarrow} A$ are derivable in systems with axioms (T) and (4), respectively. The corresponding structural rules for (T), (4), and ($\overline{\mathrm{K}}$) are as follows (Moortgat [1996]):

$$(\mathrm{r4})\quad \frac{\Gamma[\langle\Delta\rangle] \Rightarrow A}{\Gamma[\langle\langle\Delta\rangle\rangle] \Rightarrow A}, \qquad (\mathrm{rT})\quad \frac{\Gamma[\langle\Delta\rangle] \Rightarrow A}{\Gamma[\Delta] \Rightarrow A}, \quad (\mathrm{r}\overline{\mathrm{K}})\quad \frac{\Gamma[\langle\Delta_1\rangle \circ \langle\Delta_2\rangle] \Rightarrow A}{\Gamma[\langle\Delta_1 \circ \Delta_2\rangle] \Rightarrow A}.$$

NL$\Diamond$ and FNL$\Diamond$ with (r4) and (rT) admit cut elimination. The decidability does not directly follow, since (rT) increases the size of sequents in the proof-search tree. However, the decidability can easily be established. It suffices to show that for any provable sequent $\Gamma \Rightarrow A$, there exits a proof such that any application of (rT) occurs only after an application of ($\Diamond$R) or ($\Box^{\downarrow}$L).

By $\mathrm{NL}_{\mathrm{S4}}$, we denote NL$\Diamond$ enriched with (r4) and (rT). The proof of cut elimination and decidability for $\mathrm{NL}_{\mathrm{S4}}$ can be found in Moortgat [1996] and Plummer [2007]. Sequent systems of NL_{i}, $\mathrm{DFNL}_{\mathrm{i}}$ and $\mathrm{BFNL}_{\mathrm{i}}$, where $i \in \{4, \mathrm{T}, \mathrm{S4}, \overline{\mathrm{S4}}\}$ (S4=(4, T), $\overline{\mathrm{S4}} = (4, \mathrm{T}, \overline{\mathrm{K}})$), can be obtained in a similar way.

2.3.3 Derivable sequents and rules

In this section, we proceed with a series of provable sequents and rules in systems considered in last section, which will be needed in the following chapters.

Fact 2.3.1 *The following sequents are provable in* FNL*:*

(1) $A \Leftrightarrow A \wedge (A \vee B)$,

(2) $A \Leftrightarrow A \vee (A \wedge B)$,

(3) $C \cdot (A \vee B) \Leftrightarrow (C \cdot A) \vee (C \cdot B)$,

(4) $(A \vee B) \cdot C \Leftrightarrow (A \cdot C) \vee (B \cdot C)$,

(5) $C \cdot (A \wedge B) \Rightarrow (C \cdot A) \wedge (C \cdot B)$,

(6) $(A \wedge B) \cdot C \Rightarrow (A \cdot C) \wedge (B \cdot C)$,

We present the derivation of (3) and (5) only.

(3) the left-to-right direction:

1. $C \circ A \Rightarrow C \cdot A$ and $C \circ B \Rightarrow C \cdot B$ by (Id) and ($\cdot$R),
2. $C \cdot (A \vee B) \Rightarrow (C \cdot A) \vee (C \cdot B)$ by ($\vee$L), ($\vee$R) and ($\cdot$L),

the right-to-left direction:

1. $A \Rightarrow (A \vee B)$ and $B \Rightarrow (A \vee B)$ by (Id) ($\vee$R),
2. $C \cdot A \Rightarrow C \cdot (A \vee B)$ and $C \cdot B \Rightarrow C \cdot (A \vee B)$ by Mon$(\cdot)_l$,
3. $(C \cdot A) \vee (C \cdot B) \Rightarrow C \cdot (A \vee B)$ by ($\vee$L).

(5)
1. $A \wedge B \Rightarrow A$ and $A \wedge B \Rightarrow B$ by (Id) and ($\wedge$L),
2. $C \cdot (A \wedge B) \Rightarrow C \cdot A$ and $C \cdot (A \wedge B) \Rightarrow C \cdot B$ by by Mon$(\cdot)_l$,
3. $C \cdot (A \wedge B) \Rightarrow (C \cdot A) \wedge (C \cdot B)$ by ($\wedge$R).

Fact 2.3.2 *The following sequents are provable in* FNL$\Diamond$*:*

(1) $\Diamond(A \wedge B) \Rightarrow \Diamond A \wedge \Diamond B$,

(2) $\Diamond(A \vee B) \Leftrightarrow \Diamond A \vee \Diamond B$,

(3) $\Box^{\downarrow} A \vee \Box^{\downarrow} B \Rightarrow \Box^{\downarrow}(A \vee B)$,

(4) $\Box^{\downarrow}(A \wedge B) \Leftrightarrow \Box^{\downarrow} A \wedge \Box^{\downarrow} B$,

(5) $\Diamond\Box^{\downarrow} A \Rightarrow A$,

(6) $A \Rightarrow \Box^{\downarrow}\Diamond A$,

We present the derivation of (1), (2) and (6) only.

(1)
1. $A \wedge B \Rightarrow A$ and $A \wedge B \Rightarrow B$ by (Id) and ($\wedge$L),
2. $\Diamond(A \wedge B) \Rightarrow \Diamond A$ and $\Diamond(A \wedge B) \Rightarrow \Diamond B$ by Mon($\Diamond$),
3. $\Diamond(A \wedge B) \Rightarrow \Diamond A \wedge \Diamond B$ by ($\wedge$R).

(2) the left-to-right direction:

1. $\langle A\rangle \Rightarrow \Diamond A$ and $\langle B\rangle \Rightarrow \Diamond B$ by (Id), ($\Diamond$R),
2. $\langle A\rangle \Rightarrow \Diamond A \vee \Diamond B$ and $\langle B\rangle \Rightarrow \Diamond A \vee \Diamond B$ by ($\vee$R),
3. $\Diamond(A \vee B) \Rightarrow \Diamond A \vee \Diamond B$ by ($\vee$L) and ($\Diamond$L).

the right-to-left direction:

1. $A \Rightarrow A \vee B$ and $B \Rightarrow A \vee B$ by (Id) and ($\vee$R),
2. $\Diamond A \Rightarrow \Diamond(A \vee B)$ and $\Diamond B \Rightarrow \Diamond(A \vee B)$ by Mon($\Diamond$),
3. $\Diamond A \vee \Diamond B \Rightarrow \Diamond(A \vee B)$ by ($\vee$L).

(6)
1. $\langle A\rangle \Rightarrow \Diamond A$ by (Id) and ($\Diamond$R),
2. $A \Rightarrow \Box^{\downarrow}\Diamond A$ by ($\Box^{\downarrow}$R).

Fact 2.3.3 *The following sequents are provable in* DFNL*:*

(1) $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$*,*

(2) $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$*,*

The left-to-right direction of (1) is the axiom (D). (2) is proved like in the equational theory of distributive lattices (see Burris & Sankappanavar [1981]). So we only present the derivation of the other direction of (1) here.

(1) the right-to-left direction:

1. $B \Rightarrow B \vee C$ and $C \Rightarrow B \vee C$ by (Id) and ($\vee$R),
2. $A \wedge B \Rightarrow B \vee C$ and $A \wedge C \Rightarrow B \vee C$ by ($\wedge$L)
3. $A \wedge B \Rightarrow A$ and $A \wedge C \Rightarrow A$ by (Id) and ($\wedge$L),
4. $A \wedge B \Rightarrow A \wedge (B \vee C)$ and $A \wedge C \Rightarrow A \wedge (B \vee C)$ from 2, 3 by ($\wedge$R),
5. $(A \wedge B) \vee (A \wedge C) \Rightarrow A \wedge (B \vee C)$ by ($\vee$L).

Fact 2.3.4 *The following sequents are provable in* BFNL*:*

(1) $\neg A \vee \neg B \Rightarrow \neg(A \wedge B)$*,*

(2) $\neg(A \vee B) \Rightarrow \neg A \wedge \neg B$*,*

(3) $\neg(A \wedge B) \Rightarrow \neg A \vee \neg B$*,*

(4) $\neg A \wedge \neg B \Rightarrow \neg(A \vee B)$*,*

(5) $\neg\bot \Leftrightarrow \top$ *and* $\neg\top \Leftrightarrow \bot$*.*

We present the derivation of (2) and (3) only.

(2)
1. $A \Rightarrow A \vee B$ and $B \Rightarrow A \vee B$ by (Id) and ($\vee$L),
2. $\neg(A \vee B) \Rightarrow \neg A$ and $\neg(A \vee B) \Rightarrow \neg B$ by (TR),
3. $\neg(A \vee B) \Rightarrow \neg A \wedge \neg B$ by ($\wedge$R)

(3)
1. $\neg\neg A \wedge \neg\neg B \Rightarrow A \wedge B$ by (DN), ($\wedge$L) and ($\wedge$R),
2. $\neg(\neg A \vee \neg B) \Rightarrow \neg\neg A \wedge \neg\neg B$ by Fact 2.3.4 (2),
3. $\neg(\neg A \vee \neg B) \Rightarrow A \wedge B$ follows from 1, 2 by (Cut),
4. $\neg(A \wedge B) \Rightarrow \neg\neg(\neg A \vee \neg B)$ by by (TR),
5. $\neg(A \wedge B) \Rightarrow (\neg A \vee \neg B)$ follows from 4 and (DN) by (Cut).

Fact 2.3.5 *The following rule is derivable in* BFNL*:*

$A \Rightarrow B$ *iff* $\top \Rightarrow \neg A \vee B$*.*

We show the left-to-right direction:

1. $A \Rightarrow B \vee \neg A$, by assumption and ($\vee$R),
2. $\neg A \Rightarrow B \vee \neg A$ by (Id) and ($\vee$R),
3. $A \vee \neg A \Rightarrow B \vee \neg A$ follows from 1, 2 by ($\vee$L),
4. $\top \Rightarrow A \vee \neg A$ by ($\neg$2),
5. $\top \Rightarrow \neg A \vee B$ follows from 3, 4 by (Cut).

We show the right-to-left direction:

1. $A \Rightarrow \neg A \vee B$ follows from assumption and ($\top$) by (Cut),

2. $A \Rightarrow A \vee B$ by (Id) and ($\vee$R),

3. $A \Rightarrow (A \vee B) \wedge (\neg A \vee B)$ follows from 1, 2 by ($\wedge$R),

4. $(A \vee B) \wedge (\neg A \vee B) \Rightarrow (A \wedge \neg A) \vee B$ by Fact 2.3.3 (2),

5. $(A \wedge \neg A) \vee B \Rightarrow B$ by (Id), ($\bot$), ($\neg$1), (Cut) and ($\vee$L),

6. $A \Rightarrow B$ follows from 3, 4 and 5 by (Cut).

2.4 Semantics of NL and its extensions

2.4.1 Algebraic semantics

Hereafter we always use boldface $\mathbf{A}$ for an algebra, plain A for its base set. Classes of algebras are denoted by a kind of blackboard bold capital characters. NL is a complete logic of residuated groupoids. A residuated groupoid (RG) is a structure $(\mathrm{G}, \leq, \cdot, \backslash, /)$ such that $(\mathrm{G}, \leq)$ is a poset and $\cdot$, $\backslash$, $/$ are binary operations satisfying the residuation law:

$$a \cdot b \leq c \quad \text{iff} \quad b \leq a \backslash c \quad \text{iff} \quad a \leq c/b, \tag{2.1}$$

for all $a, b, c \in \mathrm{G}$. So $\mathbb{RG}$ denotes the class of residuated groupoids, and similarly for other classes of algebras.

As an easy consequence of (2.1), we obtain: if $a \leq b$, then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$. A model is a pair $(\mathbf{G}, \sigma)$ such that $\mathbf{G}$ is a RG, and σ is a mapping from Prop into $\mathbf{G}$, called a *valuation*, which is extended to formulae and formula trees as follows:

$$\sigma(A \cdot B) = \sigma(A) \cdot \sigma(B),$$

$$\sigma(A \backslash B) = \sigma(A) \backslash \sigma(B), \quad \sigma(A/B) = \sigma(A)/\sigma(B),$$

$$\sigma(\Gamma \circ \Delta) = \sigma(\Gamma) \cdot \sigma(\Delta), \quad \sigma(\Lambda) = 1.$$

A sequent $\Gamma \Rightarrow A$ is said to be *true* in a model $(\mathbf{G}, \sigma)$ written $\mathbf{G}, \sigma \models \Gamma \Rightarrow A$, if $\sigma(\Gamma) \leq \sigma(A)$ (here $\leq$ is the designated partial ordering in $\mathbf{G}$). It is *valid* in $\mathbf{G}$, if it is true in $(\mathbf{G}, \sigma)$, for any valuation σ. It is valid in a class of algebras $\mathbb{K}$, if it is valid in all algebras from $\mathbb{K}$. A set of sequents Φ *entails* a sequent $\Gamma \Rightarrow A$

with respect to $\mathbb{K}$, if $\Gamma \Rightarrow A$ is true in all models $(\mathbf{G}, \sigma)$ such that $\mathbf{G} \in \mathbb{K}$ and all sequents from Φ are true in $(\mathbf{G}, \sigma)$.

A standard argument, using a routine construction of the Lindenbaum-Tarski algebra, yields the completeness of NL with respect to $\mathbb{RG}$. The soundness part is easy. NL is *strongly complete* with respect to $\mathbb{RG}$: for any set of sequents Φ and any sequent $\Gamma \Rightarrow A$, $\Phi \vdash_{\mathrm{NL}} \Gamma \Rightarrow A$ if and only if Φ entails $\Gamma \Rightarrow A$ with respect to $\mathbb{RG}$. So it follows that NL is *weakly complete* with respect to $\mathbb{RG}$: the sequents provable in NL are precisely the sequents valid in $\mathbb{RG}$.

Algebraic models of NL1 are residuated unital groupoids (RUGs). In general, a RUG is an algebra $(\mathrm{G}, \leq, \cdot, \backslash, /, 1)$ such that $(\mathrm{G}, \leq, \cdot, \backslash, /)$ is a RG and $1 \in \mathrm{G}$ satisfies: $1 \cdot a = a = a \cdot 1$, for all $a \in \mathrm{G}$. Algebraic models of FNL are lattice-ordered residuated groupoids (LRGs). A LRG is an algebra $(\mathrm{G}, \wedge, \vee, \cdot, \backslash, /)$ such that $(\mathrm{G}, \wedge, \vee)$ is a lattice and $(\mathrm{G}, \cdot, \backslash, /, \leq)$ is a RG, where $\leq$ is the lattice-order. A LRG is distributive, if its lattice reduct $(\mathrm{G}, \wedge, \vee)$ is distributive. A LRG is called bounded, if its lattice reduct $(\mathrm{G}, \wedge, \vee)$ has the greatest element $\top$ and the least element $\bot$. We write DLRG for distributive LRG and B-LRG for bounded LRG. B-DLRG is defined naturally. Obviously, DFNL corresponds to $\mathbb{DLRG}$ and $\mathrm{DFNL}_{\bot,\top}$ corresponds to $\mathbb{B}$-$\mathbb{DLRG}$. We will also consider boolean residuated groupoids (BRGs), i.e. DLRGs with $\bot$, $\top$ and $\neg$ satisfying the laws of boolean algebras. BRGs are models of BFNL.

More special frames are powerset frames. Let $\mathbf{G} =(\mathrm{G}, \cdot)$ be a groupoid. For $U, V \subseteq \mathrm{G}$, one defines operations on the powerset $\wp(\mathrm{G})$ as follows:

$$U \odot V = \{a \cdot b \in \mathrm{G} : a \in U, b \in V\},$$

$$U \backslash V = \{a \in \mathrm{G} : U \odot \{a\} \subseteq V\}, \quad V/U = \{a \in \mathrm{G} : \{a\} \odot U \subseteq V\},$$

then the structure $\mathbf{P}(\mathbf{G}) = (\wp(\mathrm{G}), \odot, \backslash, /, \subseteq)$ is a RG, being a complete distributive lattice. For non-distributive case one needs a more general class of structures constructed usually with the help of a closure operation. An operation $C : \mathrm{G} \to \mathrm{G}$ is called a nucleus closure operation (nucleus operation) on a RG $(\mathrm{G}, \cdot, \backslash, /, \leq)$, if it satisfies the following conditions:

(C1) $a \leq C(a)$,

(C2) $C(C(a)) \leq C(a)$,

(C3) $a \leq b$ implies $C(a) \leq C(b)$,

(C4) $C(a) \cdot C(b) \leq C(a \cdot b)$.

An element $a \in$ G is called C-closed iff $a = C(a)$. Let C($\mathbf{G}$) be a set of all C-closed elements, $\leq_c$ be the order $\leq$ restricted to the set C($\mathbf{G}$). One defines $a \cdot_c b = C(a \cdot b)$. Then the structure $\mathbf{C}(\mathbf{G}) = (\mathrm{C}(\mathbf{G}), \cdot_c, \backslash, /, \leq_c)$ is a RG. Furthermore, if $\leq$ is a lattice order, then so is $\leq_c$.

(C4) is essential in the proof that $\mathbf{C}(\mathbf{G})$ is closed under $\backslash$, /. Actually, if a is closed, then $b\backslash a$ and a/b are closed, for any $b \in$ G. One can easily show that (2.1) holds in $\mathbf{C}(\mathbf{G})$. In $\mathbf{C}(\mathbf{G})$, $\backslash$, / are restrictions of $\backslash$,/ on G to C(G).

Residuated groupoids in which $\cdot$ is associative are called residuated semigroups (RSGs). L is strongly complete with respect to $\mathbb{RSG}$. Let V be a finite lexicon and V^+ be the set of finite nonempty strings over V. Language models for L are special powerset structures in which the underlying semigroup is the free semigroup generated by V, i.e. the set V^+ with concatenation. Pentus [1995] shows that L yields all sequents valid in the frames $\mathbf{P}(\mathbf{V}^+)$, in other words, L is weakly complete with respect to the class of frames $\mathbf{P}(\mathbf{V}^+)$ such that V is finite. Moreover L is strongly complete with respect to powerset algebras $\mathbf{P}(\mathbf{M})$ such that $\mathbf{M}$ is a semigroup (Buszkowski [1986a]).

Bracketed strings on V can be treated as finite binary trees whose leaves are labeled by symbols from V. So we denote the set of all such strings by V^{T}. Now we consider language models (powerset structures) in which the underlying groupoid is the free groupoid generated by V, i.e. the set V^{T} with the bracketed concatenation: $\Gamma, \Delta \mapsto (\Gamma, \Delta)$. NL is not weakly complete with respect to the class of frames $\mathbf{P}(\mathbf{V}^{\mathbf{T}})$. However this completeness result holds for product-free fragment of NL (Buszkowski [1986a]).

A residuated groupoid with S4-operators (S4-RG) is a structure $(\mathrm{G}, \cdot, \backslash, /, \Diamond, \Box^{\downarrow}, \leq)$ such that $(\mathrm{G}, \cdot, \backslash, /, \leq)$ is a RG and $\Diamond$, $\Box^{\downarrow}$ are unary operations on G, satisfying the following conditions:

$$\Diamond a \leq b \quad \text{iff} \quad a \leq \Box^{\downarrow} b \quad (2.4),$$

$$\Diamond\Diamond a \leq \Diamond a \quad (2.5),$$

$$a \leq \Diamond a \quad (2.6),$$

for all $a, b \in$ G. Residuated groupoids with T-operators (assume (2.4), (2.6)) and 4-operators (assume (2.4), (2.5)) are defined similarly. By $\overline{\text{S4}}$-RG, we mean a S4-RG additionally satisfying the following condition:

$$\Diamond(a \cdot b) \leq \Diamond a \cdot \Diamond b \quad (2.7),$$

for all $a, b \in$ G. Also for R $\in$ {S4, $\overline{\text{S4}}$, 4, T}, we use terms R-RG, R-DLRG, R-BRG and so on in the obvious sense.

2.4.2 Frame semantics

The basic semantic structures we consider in this section are ternary Kripke frames, which are defined as follows. A frame is a pair (W, R) such that W is a nonempty set and R is an arbitrary ternary relation on W. We use $u, v, w, \ldots$ for members of W. A model $\mathfrak{J} = (W, R, \sigma)$ of NL consists of a frame (W, R) and a valuation σ which is a mapping from Prop to the powerset of W. The satisfiability relation $\mathfrak{J}, w \models A$ between a model $\mathfrak{J}$ with a state $w \in W$ and a NL formula A is recursively defined as follows.

- $\mathfrak{J}, u \models p$ iff $u \in \sigma(p)$.
- $\mathfrak{J}, u \models A \cdot B$ iff there are $v, w \in W$ such that $R(u, v, w)$, $\mathfrak{J}, v \models A$ and $\mathfrak{J}, w \models B$.
- $\mathfrak{J}, u \models A/B$ iff for all $v, w \in W$ such that $R(w, u, v)$, $\mathfrak{J}, v \models B$ implies $\mathfrak{J}, w \models A$
- $\mathfrak{J}, u \models A\backslash B$ iff for all $v, w \in W$ such that $R(v, w, u)$, $\mathfrak{J}, w \models A$ implies $\mathfrak{J}, v \models B$.

A formula A is satisfiable if $\mathfrak{J}, u \models A$ for some model $\mathfrak{J} = (W, R, \sigma)$ and some $u \in W$. Also, we say that A is valid (true) in $\mathfrak{J}$, denoted $\mathfrak{J} \models A$, if $\mathfrak{J}, u \models A$, for all $u \in W$. For any sequent $\Gamma \Rightarrow A$, we say that $\Gamma \Rightarrow A$ is true at a state u

in the model $\mathfrak{J}$ (notation: $\mathfrak{J}, u \models \Gamma \Rightarrow A$), if $\mathfrak{J}, u \models f(\Gamma)$ implies $\mathfrak{J}, u \models A$. A sequent $\Gamma \Rightarrow A$ is valid (true) in $\mathfrak{J}$ (notation: $\mathfrak{J} \models \Gamma \Rightarrow A$), if $\mathfrak{J}, u \models \Gamma \Rightarrow A$ for all $u \in W$. By $\models_{\mathbf{NL}} \Gamma \Rightarrow A$ we mean that $\Gamma \Rightarrow A$ is valid in all models of NL.

Theorem 2.4.1 $\vdash_{\mathrm{NL}} \Gamma \Rightarrow A$ *iff* $\models_{\mathrm{NL}} \Gamma \Rightarrow A$.

Proof: From left to right we proceed by induction on the length of proof of $\Gamma \Rightarrow A$. For the other direction (completeness part) we construct a simple model $\mathfrak{J} = (W, R, \sigma)$ where W is the set of all formulae, $R(A, B, C)$ holds iff $\vdash_{\mathrm{NL}} A \Rightarrow B \cdot C$, and $A \in \sigma(p)$ iff $\vdash_{\mathrm{NL}} A \Rightarrow p$. One can easily show that for any formula B, $\mathfrak{J}, A \models B$ iff $\vdash_{\mathrm{NL}} A \Rightarrow B$ by induction on the formula B. Hence the completeness result is a direct consequence. □

This semantics can be extended with the standard interpretation of the boolean connectives. A model of BFNL $\mathfrak{J} = (W, R, \sigma)$ is defined as above, and the satisfiability relation relation $\mathfrak{J}, w \models A$ between a model $\mathfrak{J}$ with a state $w \in W$ and a BFNL formula A is extended by the following laws:

- $\mathfrak{J}, u \not\models \bot$ and $\mathfrak{J}, u \models \top$.
- $\mathfrak{J}, u \models A \wedge B$ iff $\mathfrak{J}, u \models A$ and $\mathfrak{J}, u \models B$,
- $\mathfrak{J}, u \models A \vee B$ iff $\mathfrak{J}, u \models A$ or $\mathfrak{J}, u \models B$.
- $\mathfrak{J}, u \models \neg A$ iff $\mathfrak{J}, u \not\models A$.

Theorem 2.4.2 $\vdash_{\mathrm{BFNL}} \Gamma \Rightarrow A$ *iff* $\models_{\mathrm{BFNL}} \Gamma \Rightarrow A$.

This result for BFNL follows from the soundness and completeness of its Hilbert-style presentation under frame semantics (see Kaminski & Francez [2014]). This Hilbert-style system is denoted PNL in the same paper. The relation between BFNL and PNL is as follows: for any formula $A \supset B$, $\vdash_{\mathrm{PNL}} A \supset B$ iff $\vdash_{\mathrm{BFNL}} A \Rightarrow B$, where $\supset$ is the boolean implication. Consequently, by Fact 2.3.5 one obtains that for any formula A, $\vdash_{\mathrm{PNL}} A$ iff $\vdash_{\mathrm{BFNL}} \top \Rightarrow A$. Every algebraic model of DFNL, i.e. any DLRG, can be embedded in a model of BFNL (see Buszkowski [2014b]). So BFNL is a conservative extension of DFNL.

The frame semantics corresponding to modal extensions of NL can be found in Kurtonina [1994], which can be extended to DFNL and BFNL. A model of NL$\Diamond$

$\mathfrak{J} = (W, R^3, R^2, \sigma)$ consists of a ternary frame (W, R^3), a binary frame (W, R^2) and a valuation σ . The satisfiability relation $\mathfrak{J}, w \models A$ between a model $\mathfrak{J}$ with a state $w \in W$ and a NL$\Diamond$ formula A is defined recursively. Besides the clauses written above we assume the clauses for $\Diamond$ and $\Box^{\downarrow}$ as follows:

- $\mathfrak{J}, u \models \Diamond A$ iff there are $v \in W$ such that $R^2(u, v)$ and $\mathfrak{J}, v \models A$.
- $\mathfrak{J}, u \models \Box^{\downarrow} A$ iff for every $v \in W$, if $R^2(v, u)$, then $\mathfrak{J}, v \models A$.

Analogously to Theorem 2.4.1 we can prove the following:

Theorem 2.4.3 $\vdash_{\mathrm{NL}\Diamond} \Gamma \Rightarrow A$ *iff* $\models_{\mathrm{NL}\Diamond} \Gamma \Rightarrow A$.

We construct a simple model $\mathfrak{J} = (W, R^3, R^2, \sigma)$ where W is the set of all NL$\Diamond$ formulae, $R^3(A, B, C)$ holds iff $\vdash_{\mathrm{NL}\Diamond} A \Rightarrow B{\cdot}C$, $R^2(A, B)$ holds iff $\vdash_{\mathrm{NL}\Diamond} A \Rightarrow \Diamond B$, and $A \in \sigma(p)$ iff $\vdash_{\mathrm{NL}} A \Rightarrow p$. One can easily show that for any formula B, $\mathfrak{J}, A \models B$ iff $\vdash_{\mathrm{NL}\Diamond} A \Rightarrow B$ by induction on formula B. We have to check the new cases for the new compound formulae $\Diamond B'$ and $\Box^{\downarrow} B'$.

Assume that $A \in \sigma(\Diamond B')$. It is sufficient to show $\vdash_{\mathrm{NL}\Diamond} A \Rightarrow \Diamond B'$. Since $A \in \sigma(\Diamond B')$, there exists a C such that $R^2(A, C)$ and $C \in \sigma(B')$. Hence by induction hypothesis, one obtains $\vdash_{\mathrm{NL}\Diamond} C \Rightarrow B'$. So $\vdash_{\mathrm{NL}\Diamond} \Diamond C \Rightarrow \Diamond B'$. By the construction of $\mathfrak{J}$, we have $\vdash_{\mathrm{NL}\Diamond} A \Rightarrow \Diamond C$. Then by (Cut), one gets $\vdash_{\mathrm{NL}\Diamond} A \Rightarrow \Diamond B'$. Conversely assume that $\vdash_{\mathrm{NL}\Diamond} A \Rightarrow \Diamond B'$. By the construction of $\mathfrak{J}$ we have $R^2(A, B')$, and by induction hypothesis we get $B' \in \sigma(B')$. Hence $A \in \sigma(\Diamond B')$. The cases for $\Box^{\downarrow} A$ can be checked easily by the residuation law: $\vdash_{\mathrm{NL}\Diamond} \Diamond C \Rightarrow D$ iff $\vdash_{\mathrm{NL}\Diamond} C \Rightarrow \Box^{\downarrow} D$ and the construction of $\mathfrak{J}$.

Consider now the following conditions on ternary frames, corresponding to axioms (T), (4) and ($\overline{\mathrm{K}}$), respectively. For every $x, y, z, u, v, w \in W$

($\mathrm{T^f}$) $\forall x R^2(x, x)$;

(4^{f}) $\forall x, y, z((R^2(x, y) \& R^2(y, z)) \supset R^2(x, z))$;

($\overline{\mathrm{K^f}}$) $\forall x, y, z, w \exists u, v((R^2(x, y) \& R^3(y, z, w)) \supset (R^3(x, u, v) \& R^2(u, z) \& R^2(v, w)))$.

Analogously to Theorem 2.4.3 we can prove the following theorem easily.

Theorem 2.4.4 $\vdash_{\mathrm{NL_i}} \Gamma \Rightarrow A$ *iff* $\models_{\mathrm{NL_i}} \Gamma \Rightarrow A$, *where* $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}, \overline{\mathrm{S4}}\}$.

One shows that the model $\mathfrak{J} = (W, R^3, R^2, \sigma)$ constructed as in the proof of Theorem 2.4.1 satisfies the above corresponding frame conditions. The remainder of proofs goes without changes. For systems with distributive lattice connectives or boolean connectives, one can not construct the canonical models as above. One need other methods (see Buszkowski [2014b] and Kaminski & Francez [2014]).

2.5 Type grammars and linguistic analysis

2.5.1 Context-free grammars

Context-free grammars (CFGs) form one of the four classes of the Chomsky hierarchy of formal grammars (see Chomsky [1956]). We use the following standard notation. The set of all finite strings over an alphabet Π is denoted by Π^*. We denote by Π^+ the set of all nonempty finite strings over an alphabet Π. We denote the empty string by ϵ. Here we recall some basic definitions and notations of CFGs.

A CFG is given by the following:

- an alphabet Σ of terminal symbols, also called the object alphabet;
- an alphabet N of non-terminal symbols, where $N \cap \Sigma = \emptyset$;
- a special non-terminal symbol $S \in N$ called the start symbol;
- a finite set P of production rules , that is strings of the form $A \to x$ where $A \in N$ is a non-terminal symbol and $x \in (\Sigma \cup N)^*$ is an arbitrary string of terminal and non-terminal symbols, which can be read as A can be replaced by x.

Let $\mathcal{G} = (N, \Sigma, P, S)$ be a CFG. Let $A \in N$ occur in $x \in (\Sigma \cup N)^*$ and assume that there is a production rule $A \to y$. If $z \in (\Sigma \cup N)^*$ is obtained from x by replacing one occurrence of A in x by the string y, then we say x directly yields z (or z is one-step derivable from x) written as $x \twoheadrightarrow_{\mathcal{G}} z$. For any strings $x, z \in (\Sigma \cup N)^*$, we say x yields z (or z is derivable from x), written as $x \twoheadrightarrow^*_{\mathcal{G}} z$, if there is a sequence of strings $x = x_0 \twoheadrightarrow_{\mathcal{G}} x_1 \twoheadrightarrow_{\mathcal{G}} \cdots \twoheadrightarrow_{\mathcal{G}} x_n = z$. A string $w \in \Sigma^*$ is generated by $\mathcal{G}$ if $S \twoheadrightarrow^*_{\mathcal{G}} w$. The language generated by a grammar $\mathcal{G}$ is the set of all strings w over Σ^* which are generated by $\mathcal{G}$. So $\mathcal{L}(\mathcal{G}) = \{w \in \Sigma^* : S \twoheadrightarrow^*_{\mathcal{G}} w\}$. For example, let $\mathcal{G} = (\{S\}, \{a, b\}, \{S \twoheadrightarrow_{\mathcal{G}} \epsilon, S \twoheadrightarrow_{\mathcal{G}} aSb\}, S)$. Then $\mathcal{L}(\mathcal{G}) = \{a^n b^n : n \geq 0\}$. A language $\mathcal{L}$ is said to be context-free, if $\mathcal{L} = \mathcal{L}(\mathcal{G})$ for a CFG $\mathcal{G}$. Two grammars which generate the same languages are said to be weakly equivalent.

A grammar $\mathcal{G}$ is called ϵ-free if none of its rules has the right-hand side ϵ. A CFG is said to be in *Chomsky normal from* whenever its production rules are of the form $A \twoheadrightarrow_{\mathcal{G}} BC$ or of the form $A \twoheadrightarrow_{\mathcal{G}} w$, with $A, B, C \in N$ and $w \in \Sigma$.

Theorem 2.5.1 *Any ϵ-free CFG can be transformed into a weakly equivalent CFG in Chomsky normal form and this transformation can be performed in polynomial time (see Hopcroft & Ullman [1979]).*

2.5.2 Type grammars

A type grammar based on a type logic TL (shortly a TL-grammar) is formally defined as a triple $\mathcal{G} = \langle \Sigma, I, D \rangle$ such that Σ is a nonempty finite alphabet, I is a map which assigns a finite set of types to each element of Σ, and D is a designated type. Usually D is an atomic type, often denoted by s. Σ, I, D are called the alphabet (lexicon), the lexical (initial) type assignment and the designated type of $\mathcal{G}$, respectively. Type grammars based on TL are referred to as TL-grammars. We consider type logics enriched with finitely many assumptions Φ. Type grammars based on TL enriched with finitely many assumptions Φ are referred to TL(Φ)-grammars.

The string of formulae obtained from a formula tree Γ by dropping all structure operations and the corresponding parentheses is called the yield of Γ and denoted as $st(\Gamma)$. A language $\mathcal{L}(\mathcal{G})$ generated by a TL(Φ)-grammar $\mathcal{G} = \langle \Sigma, I, D \rangle$ is defined as a set of strings $a_1 \cdots a_n$, where $a_i \in \Sigma$, for $1 \leq i \leq n$, and $n \geq 1$, satisfying the following condition: there exist formulae $A_1, \ldots, A_n$ and a formula tree Γ such that for all $1 \leq i \leq n$ $\langle a_i, A_i \rangle \in I$, $\Phi \vdash_{\mathrm{TL}} \Gamma \Rightarrow D$ and $st(\Gamma) = A_1 \cdots A_n$.

2.5.3 Linguistic analysis

First we consider the simplest formalization of type grammar: Classical Categorial Grammar (CCG) of Ajdukiewicz [1935] and Bar-Hillel *et al.* [1960]. Ajdukiewicz was strongly influenced by logical syntax, designed by Leúniewski [1929], and the idea of universal grammar, due to Husserl [1900-1901]. CCGs are restricted to $(\backslash, /)$-types and use a simple reduction procedure, based on reduction laws:

$$(\mathrm{AP}-1) \quad A, A\backslash B \Rightarrow B, \qquad (\mathrm{AP}-2) \quad B/A, A \Rightarrow B.$$

A string of types Γ reduces to A, if A results from Γ by applying (AP-1), (AP-2) finitely many times, always to adjacent types. The resulting logic is denoted AB after Ajdukiewicz and Bar-Hillel. CCGs are also referred to as AB-grammars. AB-grammars are equivalent to ϵ-free context-free grammars (see Bar-Hillel *et al.* [1960]).

We show some examples of linguistic analysis using AB-grammars. Let us fix the primitive types: pp (prepositional phrase), s (sentence), n (common noun) and np (noun phrase). Consider the following lexicon.

Word	Type(s)
Tom	np
likes	$(np\backslash s)/np$
himself	$((np\backslash s)/np)\backslash(np\backslash s)$
for	pp/np
works	$(np\backslash s)/pp$
man	n
the	np/n

Then *Tom likes himself* and *Tom works for the man* are accepted as sentences.

$$\dfrac{\begin{array}{c}\textit{Tom}\\ \vdots\\ np\end{array} \quad \dfrac{\begin{array}{c}\textit{likes}\\ \vdots\\ (np\backslash s)/np\end{array} \quad \begin{array}{c}\textit{himself}\\ \vdots\\ ((np\backslash s)/np)\backslash(np\backslash s)\end{array}}{np\backslash s}\,(\mathrm{AP}-1)}{s}\,(\mathrm{AP}-1)$$

$$\dfrac{\begin{array}{c}\textit{Tom}\\ \vdots\\ np\end{array} \quad \dfrac{\begin{array}{c}\textit{works}\\ \vdots\\ (np\backslash s)/pp\end{array} \quad \dfrac{\begin{array}{c}\textit{for}\\ \vdots\\ pp/np\end{array} \quad \dfrac{\begin{array}{c}\textit{the}\\ \vdots\\ np/n\end{array} \quad \begin{array}{c}\textit{man}\\ \vdots\\ n\end{array}}{np}\,(\mathrm{AP}-2)}{pp}\,(\mathrm{AP}-2)}{np\backslash s}\,(\mathrm{AP}-2)}{s}\,(\mathrm{AP}-1)$$

Since the sequent $np, (np\backslash s)/pp, pp/np, ((np\backslash s)/np)\backslash(np\backslash s) \Rightarrow s$ is not provable in AB, AB-grammars (with types as above) does not accept *Tom works for himself* as a sentence. However in L (Lambek Calculus), using the composition laws, $np, (np\backslash s)/pp, pp/np, ((np\backslash s)/np)\backslash(np\backslash s)$ reduces to $np, (np\backslash s)/np,$ $((np\backslash s)/np)\backslash(np\backslash s)$, and the latter to s. So L-grammars with the same types accept this sentence.

However there are some shortcomings of L as a type logic. L accepts all possible phrase structures on the accepted strings. Further some desirable types are not derivable in L. For example *and* of type $(s\backslash s)/s$ can not be lift up to $((np\backslash s)\backslash(np\backslash s))/(np\backslash s)$ as in *Tom sings and dances*. The required law is not provable in L.

These problems motivated Lambek [1961] to propose NL as a more safe type logic. However NL-grammars as grammar formalisms for natural language are evidently too weak in a sense. In this thesis, we consider NL enriched with modal and additive connectives and non-logical assumptions. We present here some examples of syntactic analysis of natural language constructions under the corresponding type grammars. Many other examples can be found in Bernardi [2002], Heylen [1999] and Morrill [2011]. First we illustrate an application of modal and additive connectives in type grammars by the following example. Let us look at the following lexicon.

Word	Type(s)
walks	$\Box^{\downarrow}_{sg} np\backslash s$
becomes	$(\Box^{\downarrow}_{sg} np\backslash s)/(np \vee ap)$
walk	$\Box^{\downarrow}_{pl} np\backslash s$
walked	$np\backslash\Box^{\downarrow}_{p} s$
rich	ap
Lin	$\Box^{\downarrow}_{sg} np$
the Chinese	$\Box^{\downarrow}_{sg}\Box^{\downarrow}_{pl} np$
a	np/n
teacher	n

$\Box^{\downarrow}_{sg} np$ represents a singular noun phrase, $\Box^{\downarrow}_{pl} np$ represents a plural noun phrase, $\Box^{\downarrow}_{p} s$ represents a sentence in past tense and ap represents an adjective phrase. In this example, one uses an unary modality $\Box^{\downarrow}$ (satisfying (T($\Box^{\downarrow}$))) to mark morphosyntactic distinctions, and additive connectives $\wedge$, $\vee$ to describe lexical ambiguity. By using this lexicon, one can parse the following sentences:

(1) *Lin walks.*

(2) *Lin walked.*

(3) *The Chinese walks.*

(4) *The Chinese walk.*

(5) *Lin becomes rich.*

(6) *Lin becomes a teacher.*

The sequents corresponding to sentences (1)-(6) are as follows:

(1) $\Box^{\downarrow}_{sg} np \circ \Box^{\downarrow}_{sg} np\backslash s \Rightarrow s$,

(2) $\Box^{\downarrow}_{sg} np \circ np\backslash\Box^{\downarrow}_{p} s \Rightarrow s$,

(3) $\Box^{\downarrow}_{sg}\Box^{\downarrow}_{pl} np \circ \Box^{\downarrow}_{sg} np\backslash s \Rightarrow s$,

(4) $\Box^{\downarrow}_{sg}\Box^{\downarrow}_{pl} np \circ \Box^{\downarrow}_{pl} np\backslash s \Rightarrow s$,

(5) $\Box^{\downarrow}_{sg} np \circ ((\Box^{\downarrow}_{sg} np \backslash s)/(np \vee ap) \circ ap) \Rightarrow s$,

(6) $\Box^{\downarrow}_{sg} np \circ ((\Box^{\downarrow}_{sg} np \backslash s)/(np \vee ap) \circ (np/n \circ n)) \Rightarrow s$.

Obviously, the above sequents are derivable in systems under consideration. However, the ungrammatical expression *Lin walk* cannot be assigned type s.

Another idea to encode feature decomposition was proposed by Kanazawa [1992]. He makes use of additive connectives. For instance, *Lin* is assigned with the type $sg \wedge np$, where sg denotes singular, and *walks* with the type $(np \wedge sg)\backslash s$. However, such usage of additives may sometimes cause problems (see Moortgat [1997]). Let us consider the following example discussed in Heylen [1999]. One can ambiguously assign an accusative personal type $np \wedge acc$ or a possessive type $np\backslash n$ to the pronoun *her*. Combing both into a single type leads to the assignment to *her* the type $(np \wedge acc) \wedge (np/n)$. Since $\wedge$ is associative, the type $(np \wedge acc) \wedge (np/n)$ does not precisely inform that 'her' is both an accusative noun phrase and a possessive pronoun. Therefore it is better to use $\Box^{\downarrow}_{acc} np$ instead of $np \wedge acc$, hence $\Box^{\downarrow}_{acc} np \wedge (np/n)$ instead of $(np \wedge acc) \wedge (np/n)$. By doing so, we obtain the type $\Box^{\downarrow}_{acc} np \wedge (np/n)$ such that the *acc* marking can never associate with np/n. It is more natural to treat features as marks of their mother type than as individual atomic types. Hence, it seems to be better to use modalities for describing features of types, and to use additives for describing ambiguity, because one can get rid of some undesired properties like $sg \wedge np \Rightarrow sg$. Further it's also natural to assume that the iteration of a feature mark yields nothing new. For instance $\Box^{\downarrow}_{sg}\Box^{\downarrow}_{sg} np \Leftrightarrow \Box^{\downarrow}_{sg} np$. This can be proved with axioms $(\mathrm{T}(\Box^{\downarrow}))$ and $(4(\Box^{\downarrow}))$ for $\Box^{\downarrow}_{sg}$.

Moortgat [1997] noticed that $\Diamond\Box^{\downarrow}$ behaves like the S4-operator $\Box^{\downarrow}$ (satisfying $(\mathrm{T}(\Box^{\downarrow}))$ and $(4(\Box^{\downarrow}))$) and used this fact in his typing. However this fails if additives $\wedge$, $\vee$ are regarded

Let us consider another typical usage of modalities. One may use modal operation $\Box^{\downarrow}$ (satisfying $(\mathrm{T}(\Box^{\downarrow}))$) to control access to the structural rules of commutativity or contraction (see examples in Morrill [2011]). The modality admitting commutativity allows us to treat some phenomena which are difficult to handle in the Lambek calculus. An example is medial extraction. The word *which* is often typed as $(n\backslash n)/(np\backslash s)$ and $(n\backslash n)/(s/np)$, which makes the noun phrase *the book which John read yesterday* underivable in type grammars based on NL and L. By refining the type for *which* to $(n\backslash n)/(s/\Box^{\downarrow}np)$ ($s/\Box^{\downarrow}np$ means a type of a sentence missing a noun phrase anywhere), as in the lexicon below, we can solved this problem. We show an example based on Versmissen [1996].

Word	Type(s)
the	np/n
book	n
which	$(n\backslash n)/(s/\Box^{\downarrow}np)$
John	np
read	$(np\backslash s)/np$
yesterday	$s\backslash s$

The sequent corresponding to the noun phrase *the book which John read yesterday* is $np/n \circ (n \circ ((n\backslash n)/(s/\Box^{\downarrow}np) \circ (np \circ ((np\backslash s)/np \circ s\backslash s)))) \Rightarrow np$. This sequent can be derived in systems which admit associativity and commutativity in a controlled way. A part of derivation is shown as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{np \circ (((np\backslash s)/np \circ \Box^{\downarrow}np) \circ s\backslash s) \Rightarrow s}{np \circ ((np\backslash s)/np \circ (\Box^{\downarrow}np \circ s\backslash s)) \Rightarrow s}(\Box^{\downarrow}\text{Ass2})}{np \circ ((np\backslash s)/np \circ (s\backslash s \circ \Box^{\downarrow}np)) \Rightarrow s}(\Box^{\downarrow}\text{Com})}{np \circ (((np\backslash s)/np \circ s\backslash s) \circ \Box^{\downarrow}np) \Rightarrow s}(\Box^{\downarrow}\text{Ass1})}{(np \circ ((np\backslash s)/np \circ s\backslash s)) \circ \Box^{\downarrow}np \Rightarrow s}(\Box^{\downarrow}\text{Ass1})}{(np \circ ((np\backslash s)/np \circ s\backslash s)) \Rightarrow s/\Box^{\downarrow}np}(/\text{R})$$

Last but not least, non-logical assumptions are also useful in linguistic analysis, especially when we need some laws that cannot be derived in logics. For instance, in NL one cannot transform $s\backslash(s/s)$ (the type of sentence conjunction) to $(np/s)\backslash((np\backslash s)/(np\backslash s))$ (the type of verb phrase conjunction). However, one can add the sequent $s\backslash(s/s) \Rightarrow (np\backslash s)\backslash((np\backslash s)/(np\backslash s))$ as an assumption.

As in the examples above, multi-modal type grammars for natural language often employ several pairs of unary modalities $\Diamond_i$, $\Box_i^{\downarrow}$. In what follows we only consider systems with a single modal pair $\Diamond$, $\Box^{\downarrow}$, but the main results can easily be extended to multimodal versions of these systems.

Chapter 3

Modal nonassociative Lambek calculus: complexity and context-freeness

3.1 Introduction

Usually, if one can prove the cut elimination theorem for a system, then one immediately gets the subformula property: all formulae in a cut-free derivation are subformulae of the endsequent formulae. Cut elimination does not hold for systems with assumptions, and the standard subformula property need not hold. Here we consider *the extended subformula property* (see Buszkowski [2005]): all formulae in a derivation are subformulae of formulae appearing in the endsequent or the set of assumptions Φ. One can provide a syntactic proof of the extended subformula property for $\mathrm{NL_i}$, where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}, \overline{\mathrm{S4}}\}$ (the proof in Buszkowski [2005] is model-theoretic). In section 3.2, we only show the proof for $\mathrm{NL}_{\overline{\mathrm{S4}}}$ in detail. Analogous proofs for other systems considered here can be established easily.

In next two sections, we are concerned with two main problems: the complexity of decision problems of some systems and the generative capacity of corresponding type grammars. In section 3.3, we prove that consequence relations of $\mathrm{NL_i}$, where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}, \overline{\mathrm{S4}}\}$ are polynomial time decidable. Our results extend earlier results of Buszkowski [2005] and Buszkowski & Farulewski [2009] for systems without special modal axioms. This result for $\mathrm{NL}_{\mathrm{S4}}$ was claimed in Plummer [2007, 2008] (without proof).

In the last section, we show that type grammars based on NL_{i}, where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}, \overline{\mathrm{S4}}\}$ enriched with assumptions are equivalent to context-free grammars.

3.2 Extended subformula property

Hereafter Φ always denotes a set of sequents of the form $A \Rightarrow B$. The sequents from Φ will be treated as assumptions. Assumptions are added to the logical systems like new axioms, but in opposition to the latter, assumptions need not be closed under substitutions. We introduce a restricted cut rule, Φ-*restricted cut*:

$$(\Phi - \mathrm{CUT}) \quad \frac{\Gamma_2 \Rightarrow A \qquad \Gamma_1[B] \Rightarrow C}{\Gamma_1[\Gamma_2] \Rightarrow C},$$

where $A \Rightarrow B \in \Phi$.

By $\mathrm{NL}^{\mathrm{r}}_{\overline{\mathrm{S4}}}$ we denote the system $\mathrm{NL}_{\overline{\mathrm{S4}}}$ enriched with $(\Phi - \mathrm{CUT})$.

Lemma 3.2.1 *If $A \Rightarrow B \in \Phi$, then $\vdash_{\mathrm{NL}^{\mathrm{r}}_{\overline{\mathrm{S4}}}} A \Rightarrow B$.*

Proof: Assume $A \Rightarrow B \in \Phi$. We apply $(\Phi - \mathrm{CUT})$ to axioms $A \Rightarrow A$ and $B \Rightarrow B$, and get $A \Rightarrow B$. Hence $\vdash_{\mathrm{NL}^{\mathrm{r}}_{\overline{\mathrm{S4}}}} A \Rightarrow B$. □

Theorem 3.2.2 *Every sequent provable in $\mathrm{NL}^{\mathrm{r}}_{\overline{\mathrm{S4}}}$ is also provable in $\mathrm{NL}^{\mathrm{r}}_{\overline{\mathrm{S4}}}$ without* (Cut).

Proof: We must prove: if both premises of (Cut) are provable in $\mathrm{NL}^{\mathrm{r}}_{\overline{\mathrm{S4}}}$ without (Cut), then the conclusion of (Cut) is provable in $\mathrm{NL}^{\mathrm{r}}_{\overline{\mathrm{S4}}}$ without (Cut). The proof can be arranged as follows.

We apply induction (*i*) on $\mathcal{D}(A)$, the complexity of the cut formula A, i.e. the total number of occurrences of logical connectives in A. For each case, we apply induction (*ii*) on $\mathcal{R}(\mathrm{Cut})$, the rank of (Cut), i.e. the total number of sequents appearing in the proofs of both premises of (Cut).

Let us consider one case $A = \Diamond A'$. Others can be treated similarly. Assume that the premises of (Cut) $\Gamma_2 \Rightarrow A$ and $\Gamma_1[A] \Rightarrow B$ are obtained by rules R_1 and R_2, respectively. The following subcases are considered.

(1) $\Gamma_2 \Rightarrow A$ or $\Gamma_1[A] \Rightarrow B$ is (Id). If $\Gamma_2 \Rightarrow A$ is (Id), then $\Gamma_1[A] = \Gamma_1[\Gamma_2]$. If $\Gamma_1[A] \Rightarrow B$ is (Id), then $A = B$ and $\Gamma_1[\Gamma_2] = \Gamma_2$.

(2) Either R_1 or R_2 does not create the cut formula A, and both R_1 and R_2 are different from $(\Phi - \mathrm{CUT})$. Then we move the (Cut) rule up past R_i. For instance, let $R_1 = (\cdot\mathrm{L})$ and $\Gamma_2 = \Gamma_2'[C \cdot D]$. We replace the subproof

$$\dfrac{\dfrac{\Gamma_2'[C \circ D] \Rightarrow A}{\Gamma_2'[C \cdot D] \Rightarrow A}\,(\cdot\mathrm{L}) \qquad \dfrac{\cdots}{\Gamma_1[A] \Rightarrow B}}{\Gamma_1[\Gamma_2'[C \cdot D]] \Rightarrow B}\,(\mathrm{Cut}),$$

by the following

$$\dfrac{\dfrac{\Gamma_2'[C \circ D] \Rightarrow A \quad \Gamma_1[A] \Rightarrow B}{\Gamma_1[\Gamma_2'[C \circ D]] \Rightarrow B}\,(\mathrm{Cut})}{\Gamma_1[\Gamma_2'[C \cdot D]] \Rightarrow B}\,(\cdot\mathrm{L}).$$

Clearly $\mathcal{R}(\mathrm{Cut})$ is smaller in the new derivation. Hence $\Gamma_1[\Gamma_2] \Rightarrow B$ is provable without (Cut), by the hypothesis of induction (*ii*). Others can be treated similarly.

(3) $R_i = (\Phi{-}\mathrm{CUT})$ $(i = 1, 2)$. We consider the following two subcases.

(3.1) $\Gamma_2 \Rightarrow A$ is obtained by (Φ-CUT). Let $C \Rightarrow D \in \Phi$ and $\Gamma_2 = \Gamma_2'[\Delta]$. We replace the subproof

$$\dfrac{\dfrac{\Delta \Rightarrow C \quad \Gamma_2'[D] \Rightarrow A}{\Gamma_2'[\Delta] \Rightarrow A}\,(\Phi - \mathrm{CUT}) \qquad \dfrac{\cdots}{\Gamma_1[A] \Rightarrow B}}{\Gamma_1[\Gamma_2] \Rightarrow B}\,(\mathrm{Cut}),$$

where $C \Rightarrow D \in \Phi$, by the following

$$\dfrac{\dfrac{\dots}{\Delta \Rightarrow C} \quad \dfrac{\Gamma_2'[D] \Rightarrow A \quad \Gamma_1[A] \Rightarrow B}{\Gamma_1[\Gamma_2'[D]] \Rightarrow B}\,(\mathrm{Cut})}{\Gamma_1[\Gamma_2'[\Delta]] \Rightarrow B}\,(\Phi - \mathrm{CUT}).$$

Clearly $\mathcal{R}(\mathrm{Cut})$ is smaller in the new derivation. Hence $\Gamma_1[\Gamma_2] \Rightarrow B$ is provable without (Cut) by the hypothesis of induction (*ii*).

(3.2) Let $\Gamma_1[A] \Rightarrow B$ arise by $(\Phi - \mathrm{CUT})$. Similarly, we can first apply (Cut) to $\Gamma_2 \Rightarrow A$ and the premise of $\Gamma_1[A] \Rightarrow B$ which contains the cut formula A. Then we apply $(\Phi - \mathrm{CUT})$ to the conclusion of the new (Cut) and the other premise of $\Gamma_1[A] \Rightarrow B$. Then the claim follows from the induction hypothesis (*ii*).

(4) $\mathrm{R}_1 = \Diamond\mathrm{R}$ and $\mathrm{R}_2 = \Diamond\mathrm{L}$. Let $\Gamma_2 = \langle\Gamma_2'\rangle$ We replace the subproof

$$\dfrac{\dfrac{\Gamma_2' \Rightarrow A'}{\langle\Gamma_2'\rangle \Rightarrow \Diamond A'}\,(\Diamond\mathrm{R}) \quad \dfrac{\Gamma_1[\langle A'\rangle] \Rightarrow B}{\Gamma_1[\Diamond A'] \Rightarrow B}\,(\Diamond\mathrm{L})}{\Gamma_1[\langle\Gamma_2'\rangle] \Rightarrow B}\,(\mathrm{Cut}),$$

by the following

$$\dfrac{\Gamma_2' \Rightarrow A' \quad \Gamma_1[\langle A'\rangle] \Rightarrow B}{\Gamma_1[\langle\Gamma_2'\rangle] \Rightarrow B}\,(\mathrm{Cut}).$$

Since $\mathcal{D}(A') < \mathcal{D}(A)$, $\Gamma_1[\Gamma_2] \Rightarrow B$ is provable in $\mathrm{NL}^{\mathrm{r}}_{\overline{\mathrm{S4}}}$ without (Cut) by the hypothesis of induction (*i*).

□

Let π be a cut-free derivation in $\mathrm{NL}^{\mathrm{r}}_{\overline{\mathrm{S4}}}$. By π^+, we mean a derivation from Φ in $\mathrm{NL}_{\overline{\mathrm{S4}}}$ obtained from π by replacing any occurrence of $(\Phi - \mathrm{CUT})$ by two applications of (Cut) as follows:

$$\dfrac{\dfrac{\Gamma_2 \Rightarrow A \quad A \Rightarrow B}{\Gamma_2 \Rightarrow B}\,(\mathrm{Cut}) \quad \dfrac{\dots}{\Gamma_1[B] \Rightarrow C}}{\Gamma_1[\Gamma_2] \Rightarrow C}\,(\mathrm{Cut}),$$

where $A \Rightarrow B \in \Phi$. Obviously, π^+ is a derivation from Φ in $\mathrm{NL}_{\overline{\mathrm{S4}}}$. Hence we get the following corollary.

Corollary 3.2.3 *If* $\Phi \vdash_{\mathrm{NL}_{\overline{\mathrm{S4}}}} \Gamma \Rightarrow A$*, then there exists a derivation of* $\Gamma \Rightarrow A$ *from* Φ *in* $\mathrm{NL}_{\overline{\mathrm{S4}}}$ *such that all formulae appearing in the proof are subformulae of formulae appearing in* Φ *or* $\Gamma \Rightarrow A$*.*

Proof: Follows from Lemma 3.2.1, Theorem 3.2.2, and the construction of π^+ given above. □

By similar arguments, one obtains the extended subformula properties for $\mathrm{NL_T}$, $\mathrm{NL_4}$, $\mathrm{NL_{S4}}$, $\mathrm{NL}_{\overline{\mathrm{S4}}}$, $\mathrm{FNL_T}$, $\mathrm{FNL_4}$, $\mathrm{FNL_{S4}}$ and $\mathrm{FNL}_{\overline{\mathrm{S4}}}$.

3.3 P-time decision procedure for modal NL with assumptions

In this section, we prove that the consequence relation of $\mathrm{NL}_{\overline{\mathrm{S4}}}$ is polynomial time decidable. Analogous results can be obtained for $\mathrm{NL_i}$ where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$.

Let Φ be a finite set of sequents of the form $A \Rightarrow B$ and T be a finite set of formulae containing all formulae in Φ and closed under subformulae. Let $T_\Box = \{\Box^\downarrow A | A \in T\}$, $T^\Box = T \cup T_\Box$, $T_\Diamond = \{\Diamond A | A \in T^\Box\}$ and $T^\Diamond = T^\Box \cup T_\Diamond$. A sequent is said to be *basic*, if it is a sequent of the form $A \circ B \Rightarrow C$, $A \Rightarrow B$, or $\langle A \rangle \Rightarrow B$. We describe an effective procedure producing all basic $T^\Diamond$-sequents derivable from Φ in $\mathrm{NL}_{\overline{\mathrm{S4}}}$.

Let S_0 consist of all $T^\Diamond$-sequents from Φ, all $T^\Diamond$-sequents of the form (Id), and all $T^\Diamond$-sequents of the form:

$$\langle A \rangle \Rightarrow \Diamond A, \quad \langle \Diamond A \rangle \Rightarrow \Diamond A, \quad \langle \Box^\downarrow A \rangle \Rightarrow A,$$

$$A \circ (A \backslash B) \Rightarrow B, \quad (A/B) \circ B \Rightarrow A, \quad A \circ B \Rightarrow A \cdot B.$$

Assume that S_n has already been defined. S_{n+1} is S_n enriched with all sequents arising by the following rules:

(R1) if $(\langle A \rangle \Rightarrow B) \in S_n$ and $\Diamond A \in T^\Diamond$, then $(\Diamond A \Rightarrow B) \in S_{n+1}$, (R2) if $(\langle A\rangle \Rightarrow B) \in S_n$ and $\Box^{\downarrow}B \in T^{\Diamond}$, then $(A \Rightarrow \Box^{\downarrow}B) \in S_{n+1}$,

(R3) if $(\Diamond A \circ \Diamond B \Rightarrow C) \in S_n$ and $\Box^{\downarrow}C \in T^{\Diamond}$, then $(\Diamond A \circ \Diamond B \Rightarrow \Box^{\downarrow}C) \in S_{n+1}$,

(R4) if $(\langle A\rangle \Rightarrow B) \in S_n$, then $(A \Rightarrow B) \in S_{n+1}$,

(R5) if $(A \circ B \Rightarrow C) \in S_n$ and $A \cdot B \in T^{\Diamond}$, then $(A \cdot B \Rightarrow C) \in S_{n+1}$,

(R6) if $(A \circ B \Rightarrow C) \in S_n$ and $(A\backslash C) \in T^{\Diamond}$, then $(B \Rightarrow A\backslash C) \in S_{n+1}$,

(R7) if $(A \circ B \Rightarrow C) \in S_n$ and $(C/B) \in T^{\Diamond}$, then $(A \Rightarrow C/B) \in S_{n+1}$,

(R8) if $(A \Rightarrow B) \in S_n$ and $(\langle B\rangle \Rightarrow C) \in S_n$, then $(\langle A\rangle \Rightarrow C) \in S_{n+1}$,

(R9) if $(A \Rightarrow B) \in S_n$ and $(D \circ B \Rightarrow C) \in S_n$, then $(D \circ A \Rightarrow C) \in S_{n+1}$,

(R10) if $(A \Rightarrow B) \in S_n$ and $(B \circ D \Rightarrow C) \in S_n$, then $(A \circ D \Rightarrow C) \in S_{n+1}$,

(R11) if $(\Gamma \Rightarrow B) \in S_n$ and $(B \Rightarrow C) \in S_n$, then $(\Gamma \Rightarrow C) \in S_{n+1}$,

Obviously, $S_n \subseteq S_{n+1}$, for all $n \geq 0$. For any $n \geq 0$, S_n is a finite set of basic sequents. $S^{T^{\Diamond}}$ is defined as the union of all S_n. Due to the definition of basic sequents, there are only finitely many basic sequents. Since $S^{T^{\Diamond}}$ is a set of basic sequents, hence it must be finite. This yields: there exists $k \geq 0$ such that $S_k = S_{k+1}$ and $S^{T^{\Diamond}} = S_k$. $S^{T^{\Diamond}}$ is closed under rules (R1)-(R11). The rules (R1), (R2), (R4), (R5), (R6), (R7) are ($\Diamond$L), ($\Box^{\downarrow}$R), (rT), ($\cdot$L), (\R), (/R) restricted to basic sequents, and (R8)-(R11) in fact describe the closure of basic sequents under (Cut). (R3) is equivalent to (r$\overline{\text{K}}$) restricted to basic sequents (with (r4) and (rT)). (R5)-(R7) and (R9)-(R11) are the same as in Buszkowski [2005].

Lemma 3.3.1 $S^{T^{\Diamond}}$ *can be constructed in polynomial time.*

Proof: Let n denote the cardinality of $T^{\Diamond}$. The total numbers of basic $T^{\Diamond}$-sequents of the form $A \Rightarrow B$, $\langle A\rangle \Rightarrow B$, and $A \circ B \Rightarrow C$ are no more than

n^2, n^2, and n^3 respectively. Therefore there are at most $m = n^3 + 2 \times n^2$ basic $T^\diamond$-sequents. Hence we can construct S_0 in time $O(n^3)$. The construction of S_{n+1} from S_n requires at most $6 \times (m^2 \times n) + m^2 + 6 \times m^3$ steps. It follows that the time of this construction of S_{n+1} is $O(m^3)$. The least k satisfying $S^{T^\diamond} = S_k$ does not exceed m. Thus we can construct $S^{T^\diamond}$ in polynomial time, in time $O(m^4)$. □

By $S(T^\diamond)$, we denote the system whose axioms are all sequents from $S^{T^\diamond}$ and whose only inference rule is (Cut). Clearly, every proof in $S(T^\diamond)$ consists of $T^\diamond$-sequents.

Lemma 3.3.2 *Every basic sequent provable in $S(T^\diamond)$ belongs to $S^{T^\diamond}$.*

Proof: We proceed by induction on the length of its proof in $S(T^\diamond)$. For the base case, the claim is trivial. For the inductive case, we assume that s is a basic sequent provable in $S(T^\diamond)$ such that s is obtained from premises s_1 and s_2 by (Cut). Since s is a basic sequent, clearly, s_1 and s_2 must be basic sequents. By the induction hypothesis, s_1 and s_2 belong to $S^{T^\diamond}$. Hence s belongs to $S^{T^\diamond}$, by (R8)-(R11). □

We prove two interpolation lemmas for $S(T^\diamond)$.

Lemma 3.3.3 *If $\vdash_{S(T^\diamond)} \Gamma[\Delta] \Rightarrow A$ then there exists $D \in T^\diamond$ such that $\vdash_{S(T^\diamond)} \Delta \Rightarrow D$ and $\vdash_{S(T^\diamond)} \Gamma[D] \Rightarrow A$.*

Proof: We proceed by induction on proofs in $S(T^\diamond)$.

Base case: $\Gamma[\Delta] \Rightarrow A$ belongs to $S^{T^\diamond}$. We consider three subcases. First, if $\Gamma = \Delta = B$ then $D = A$ and the claim stands. Second, if $\Gamma = \langle B \rangle$, $\Delta = B$ *or* $\Delta = \langle B \rangle$ then $D = B$ or $D = A$, respectively. Third, if $\Gamma = B \circ C$, and either $\Delta = B$, or $\Delta = C$, then $D = B$ or $D = C$, respectively. Otherwise $\Gamma = \Delta = B \circ C$. Then $D = A$.

Inductive case: Assume that $\Gamma[\Delta] \Rightarrow A$ is the conclusion of (Cut) whose both premises are $\Delta' \Rightarrow B$ and $\Gamma'[B] \Rightarrow A$ such that $\Gamma[\Delta] = \Gamma'[\Delta']$. Then three cases arise.

(1) Δ' is a substructure of Δ. Assume that $\Delta = \Delta''[\Delta']$. Then $\Gamma'[B] = \Gamma[\Delta''[B]]$. Hence there exists $D \in T^\diamond$ satisfying $\vdash_{S(T^\diamond)} \Delta''[B] \Rightarrow D$ and $\vdash_{S(T^\diamond)} \Gamma[D] \Rightarrow A$ by the induction hypothesis. We have $\vdash_{S(T^\diamond)} \Delta \Rightarrow D$ by (Cut).

(2) Δ is a substructure of Δ'. Assume that $\Delta' = \Delta''[\Delta]$. By the induction hypothesis, it is easy to obtain $\vdash_{S(T^\Diamond)} \Delta \Rightarrow D$ and $\vdash_{S(T^\Diamond)} \Delta''[D] \Rightarrow B$ for some $D \in T^\Diamond$, which yields $\vdash_{S(T^\Diamond)} \Gamma'[\Delta''[D]] \Rightarrow A$ by (Cut). So $\vdash_{S(T^\Diamond)} \Gamma[D] \Rightarrow A$.

(3) Δ and Δ' do not overlap. Hence $\Gamma'[B]$ contains Δ, and Δ does not overlap B. Assume that $\Gamma'[B] = \Gamma[B, \Delta]$. By the induction hypothesis, there exists $D \in T^\Diamond$ such that $\vdash_{S(T^\Diamond)} \Gamma'[B, D] \Rightarrow A$ and $\vdash_{S(T^\Diamond)} \Delta \Rightarrow D$. By (Cut), $\vdash_{S(T^\Diamond)} \Gamma'[\Delta', D] \Rightarrow A$, which means $\vdash_{S(T^\Diamond)} \Gamma[D] \Rightarrow A$.

□

Lemma 3.3.4 *If* $\vdash_{S(T^\Diamond)} \Gamma[\langle\Delta\rangle] \Rightarrow A$*, then there exists* $\Diamond D \in T^\Diamond$ *such that* $\vdash_{S(T^\Diamond)} \langle\Delta\rangle \Rightarrow \Diamond D$ *and* $\vdash_{S(T^\Diamond)} \Gamma[\Diamond D] \Rightarrow A$.

Proof: Assume that $\vdash_{S(T^\Diamond)} \Gamma[\langle\Delta\rangle] \Rightarrow A$. By Lemma 3.3.3, there exists $D \in T^\Diamond$ such that $\vdash_{S(T^\Diamond)} \Gamma[D] \Rightarrow A$ and $\vdash_{S(T^\Diamond)} \langle\Delta\rangle \Rightarrow D$. Again by Lemma 3.3.3, we get $\vdash_{S(T^\Diamond)} \langle D'\rangle \Rightarrow D$ and $\vdash_{S(T^\Diamond)} \Delta \Rightarrow D'$, for some $D' \in T^\Diamond$. We consider two possibilities.

If $D' \in T^\Box$, then $\Diamond D' \in T^\Diamond$. We get $\vdash_{S(T^\Diamond)} \Diamond D' \Rightarrow D$, by Lemma 3.3.2 and (R1). Since $\vdash_{S(T^\Diamond)} \langle D'\rangle \Rightarrow \Diamond D'$, by applying (Cut) to $\Delta \Rightarrow D'$ and $\langle D'\rangle \Rightarrow \Diamond D'$, we get $\vdash_{S(T^\Diamond)} \langle\Delta\rangle \Rightarrow \Diamond D'$. Since $\vdash_{S(T^\Diamond)} \Diamond D' \Rightarrow D$ and $\vdash_{S(T^\Diamond)} \Gamma[D] \Rightarrow A$, by (Cut) we get $\vdash_{S(T^\Diamond)} \Gamma[\Diamond D'] \Rightarrow A$.

If $D' \notin T^\Box$, then $D' = \Diamond D^*$ for some $D^* \in T^\Box$. Hence $\vdash_{S(T^\Diamond)} \langle\Diamond D^*\rangle \Rightarrow \Diamond D^*$ and $\vdash_{S(T^\Diamond)} \Delta \Rightarrow \Diamond D^*$. Therefore $\vdash_{S(T^\Diamond)} \langle\Delta\rangle \Rightarrow \Diamond D^*$ by (Cut). Due to Lemma 3.3.2, $\langle\Diamond D^*\rangle \Rightarrow D$ belongs to $S^{T^\Diamond}$. It yields that $\Diamond D^* \Rightarrow D$ belongs to $S^{(T^\Diamond)}$ by (R4). Hence $\vdash_{S(T^\Diamond)} \Diamond D^* \Rightarrow D$ and $\vdash_{S(T^\Diamond)} \Gamma[D] \Rightarrow A$. Then, by (Cut), $\vdash_{S(T^\Diamond)} \Gamma[\Diamond D^*] \Rightarrow A$. □

Lemma 3.3.5 *For any* $T^\Diamond$*-sequent* $\Gamma \Rightarrow A$*,* $\Phi \vdash_{\mathrm{NL}_{\overline{\mathrm{S4}}}} \Gamma \Rightarrow_{T^\Diamond} A$ *iff* $\vdash_{S(T^\Diamond)} \Gamma \Rightarrow A$.

Proof: The 'if' part is easy. Notice that all $T^\Diamond$-seuqents which are in Φ or axioms of $\mathrm{NL}_{\overline{\mathrm{S4}}}$ belong to $S^{T^\Diamond}$. The 'only if' part is proved by showing that all inference rules of $\mathrm{NL}_{\overline{\mathrm{S4}}}$, restricted to $T^\Diamond$-sequents, are admissible in $S(T^\Diamond)$. The rules (Cut), (\L), (/L), (\R), (/R), (·L), (·R) are settled by Buszkowski [2005]. Here we provide full arguments for ($\Diamond$L), ($\Diamond$R), ($\Box^\downarrow$L), ($\Box^\downarrow$R), (r4), (rT), (r$\overline{\mathrm{K}}$).

(1) For ($\Diamond$L), assume that $\vdash_{S(T^\Diamond)} \Gamma[\langle A\rangle] \Rightarrow B$ and $\Diamond A \in T^\Diamond$. By Lemma 3.3.3, there exists $D \in T^\Diamond$ such that $\vdash_{S(T^\Diamond)} \Gamma[D] \Rightarrow B$ and $\vdash_{S(T^\Diamond)} \langle A\rangle \Rightarrow D$. Since $\langle A\rangle \Rightarrow D$ is a basic sequent, then by Lemma 3.3.2, $\langle A\rangle \Rightarrow D \in S^{T^\Diamond}$. By (R1), we get $\Diamond A \Rightarrow D \in S^{T^\Diamond}$, which yields $\vdash_{S(T^\Diamond)} \Diamond A \Rightarrow D$. Hence $\vdash_{S(T^\Diamond)} \Gamma[\Diamond A] \Rightarrow B$, by (Cut).

(2) For ($\Diamond$R), assume that $\vdash_{S(T^\Diamond)} \Gamma \Rightarrow A$ and $\Diamond A \in T^\Diamond$. Since $\vdash_{S(T^\Diamond)} \langle A\rangle \Rightarrow \Diamond A$, we get $\vdash_{S(T^\Diamond)} \langle\Gamma\rangle \Rightarrow \Diamond A$, by (Cut).

(3) For ($\Box^\downarrow$L), assume that $\Gamma[A] \Rightarrow_{S(T^\Diamond)} B$ and $\Box^\downarrow A \in T^\Diamond$. Since $\vdash_{S(T^\Diamond)} \langle\Box^\downarrow A\rangle \Rightarrow A$, we get $\vdash_{S(T^\Diamond)} \Gamma[\langle\Box^\downarrow A\rangle] \Rightarrow B$, by (Cut).

(4) For ($\Box^\downarrow$R), assume that $\vdash_{S(T^\Diamond)} \langle\Gamma\rangle \Rightarrow B$ and $\Box^\downarrow B \in T^\Diamond$. By Lemma 3.3.3, there exists $D \in T^\Diamond$ such that $\vdash_{S(T^\Diamond)} \langle D\rangle \Rightarrow B$ and $\vdash_{S(T^\Diamond)} \Gamma \Rightarrow D$. Then $\langle D\rangle \Rightarrow B \in S^{T^\Diamond}$, by Lemma 3.3.2. By (R2), $D \Rightarrow \Box^\downarrow B \in S^{T^\Diamond}$, which yields $\vdash_{S(T^\Diamond)} D \Rightarrow \Box^\downarrow B$. Hence we get $\vdash_{S(T^\Diamond)} \Gamma \Rightarrow \Box^\downarrow B$, by (Cut).

(5) For (r4), assume that $\vdash_{S(T^\Diamond)} \Gamma[\langle\Delta\rangle] \Rightarrow A$. By Lemma 3.3.4 there exists $\Diamond D \in T^\Diamond$ such that $\vdash_{S(T^\Diamond)} \Gamma[\Diamond D] \Rightarrow A$ and $\vdash_{S(T^\Diamond)} \langle\Delta\rangle \Rightarrow \Diamond D$. Since $\vdash_{S(T^\Diamond)} \langle\Diamond D\rangle \Rightarrow \Diamond D$, we get $\vdash_{S(T^\Diamond)} \Gamma[\langle\langle\Delta\rangle\rangle] \Rightarrow A$, by two applications of (Cut).

(6) For (rT), assume that $\vdash_{S(T^\Diamond)} \Gamma[\langle\Delta\rangle] \Rightarrow A$. By Lemma 3.3.3, there exists $D \in T^\Diamond$ such that $\vdash_{S(T^\Diamond)} \Gamma[D] \Rightarrow A$ and $\vdash_{S(T^\Diamond)} \langle\Delta\rangle \Rightarrow D$. Again by Lemma 3.3.3, we get $\vdash_{S(T^\Diamond)} \langle D'\rangle \Rightarrow D$ and $\vdash_{S(T^\Diamond)} \Delta \Rightarrow D'$, for some $D' \in T^\Diamond$. By Lemma 3.3.2 $\langle D'\rangle \Rightarrow D \in S^{T^\Diamond}$. Hence by (R4), we obtain $\vdash_{S(T^\Diamond)} D' \Rightarrow D$. Hence $\vdash_{S(T^\Diamond)} \Gamma[\Delta] \Rightarrow A$ by (Cut).

(7) For (r$\overline{\text{K}}$), assume that $\vdash_{S(T^\Diamond)} \Gamma[\langle\Delta_1\rangle \circ \langle\Delta_2\rangle] \Rightarrow A$. By Lemma 3.3.3, there exists $D \in T^\Diamond$ such that $\vdash_{S(T^\Diamond)} \Gamma[D] \Rightarrow A$ and $\vdash_{S(T^\Diamond)} \langle\Delta_1\rangle \circ \langle\Delta_2\rangle \Rightarrow D$. Then, by applying Lemma 3.3.4 twice, we get $\vdash_{S(T^\Diamond)} \langle\Delta_1\rangle \Rightarrow \Diamond D_1$, $\vdash_{S(T^\Diamond)} \langle\Delta_2\rangle \Rightarrow \Diamond D_2$ and $\vdash_{S(T^\Diamond)} \Diamond D_1 \circ \Diamond D_2 \Rightarrow D$ for some $\Diamond D_1, \Diamond D_2 \in T^\Diamond$. By case (6), we get $\vdash_{S(T^\Diamond)} \Delta_1 \Rightarrow \Diamond D_1$ and $\vdash_{S(T^\Diamond)} \Delta_2 \Rightarrow \Diamond D_2$. We consider two possibilities.

Assume that $D \in T$. Then $\Box^\downarrow D \in T^\Diamond$. By Lemma 3.3.2, we obtain $\Diamond D_1 \circ \Diamond D_2 \Rightarrow D \in S^{T^\Diamond}$. Hence by (R3), $\vdash_{S(T^\Diamond)} \Diamond D_1 \circ \Diamond D_2 \Rightarrow \Box^\downarrow D$. Since

$\vdash_{S(T^\Diamond)} \langle \Box^\downarrow D \rangle \Rightarrow D$, we get $\vdash_{S(T^\Diamond)} \langle \Diamond D_1 \circ \Diamond D_2 \rangle \Rightarrow D$, by (Cut). Then, by three applications of (Cut), we get $\vdash_{S(T^\Diamond)} \Gamma[\langle \Delta_1 \circ \Delta_2 \rangle] \Rightarrow A$.

Otherwise $D = \Box^\downarrow D'$ or $D = \Diamond D^*$ for some formulae $D', D^* \in T^\Diamond$, respectively. First we show $\vdash_{S(T^\Diamond)} \langle D \rangle \Rightarrow D$. Since $\vdash_{S(T^\Diamond)} \langle \Box^\downarrow D' \rangle \Rightarrow D'$, by case (5), we obtain $\vdash_{S(T^\Diamond)} \langle \langle \Box^\downarrow D' \rangle \rangle \Rightarrow D'$. Then, due to case (4), we get $\vdash_{S(T^\Diamond)} \langle \Box^\downarrow D' \rangle \Rightarrow \Box^\downarrow D'$. Obviously we have $\vdash_{S(T^\Diamond)} \langle \Diamond D^* \rangle \Rightarrow \Diamond D^*$. Hence $\vdash_{S(T^\Diamond)} \langle D \rangle \Rightarrow D$. This yields $\vdash_{S(T^\Diamond)} \langle \Diamond D_1 \circ \Diamond D_2 \rangle \Rightarrow D$. Then, by three applications of (Cut), we get $\vdash_{S(T^\Diamond)} \Gamma[\langle \Delta_1 \circ \Delta_2 \rangle] \Rightarrow A$.

□

Theorem 3.3.6 *The consequence relation of* $\mathrm{NL}_{\overline{\mathrm{S4}}}$ *is decidable in polynomial time.*

Proof: We can confine ourselves to the relation $\Phi \vdash_{\mathrm{NL}_{\overline{\mathrm{S4}}}} B \Rightarrow A$, where Φ is a finite set of sequents of the form $C \Rightarrow D$, since every sequent $\Gamma \Rightarrow A$ is deductively equivalent in $\mathrm{NL}_{\overline{\mathrm{S4}}}$ to $f(\Gamma) \Rightarrow A$. Furthermore, this reduction is polynomial. Let n be the number of logical constants and atoms occurring in $B \Rightarrow A$ and in sequents from Φ. The number of subformulae of any formula is equal to the number of logical constants and atoms in it. Let T be the set of all subformulae of formulae appearing in Φ or $A \Rightarrow B$. Hence T can be constructed in time $O(n^2)$, and T contains at most $O(n^2)$ elements. It yields that we can construct $T^\Diamond$ in time $O(n^2)$. Since $T \subseteq T^\Diamond$, by Corollary 3.2.3, $\Phi \vdash_{\mathrm{NL}_{\overline{\mathrm{S4}}}} B \Rightarrow A$ iff $\Phi \vdash_{\mathrm{NL}_{\overline{\mathrm{S4}}}} B \Rightarrow_{T^\Diamond} A$. By Lemma 3.3.5, $\Phi \vdash_{\mathrm{NL}_{\overline{\mathrm{S4}}}} B \Rightarrow_{T^\Diamond} A$ iff $\vdash_{S(T^\Diamond)} B \Rightarrow A$. Since $B \Rightarrow A$ is a basic sequent, we get $\vdash_{S(T^\Diamond)} B \Rightarrow A$ iff $B \Rightarrow A \in S^{T^\Diamond}$, by Lemma 3.3.2. Hence $B \Rightarrow A$ is derivable from Φ in $\mathrm{NL}_{\overline{\mathrm{S4}}}$ iff $B \Rightarrow A \in S^{T^\Diamond}$. Further, by Lemma 3.3.1, $S^{T^\Diamond}$ can be constructed in polynomial time. Consequently, $\Phi \vdash_{\mathrm{NL}_{\overline{\mathrm{S4}}}} B \Rightarrow A$ can be checked in time polynomial with respect to n. □

The above results can be easily extended to $\mathrm{NL}_{\mathrm{S4}}$. For NL_4, we outline the proof as follows. We modify the construction of $T^\Diamond$: $T^\Diamond = T \cup \{\Diamond\Box^\downarrow A | A \in T\}$, where T is a finite set of formulae containing all subformulae of formulae in Φ. $S^{T^\Diamond}$ and $S(T^\Diamond)$ are constructed as above without (R3) and (R4). Lemma 3.3.1-3.3.3 remain the same. We replace Lemma 3.3.4 by the following lemma:

Lemma 3.3.7 *If* $\vdash_{S(T^\Diamond)} \Gamma[\langle\Delta\rangle] \Rightarrow A$*, then there exists* $\Diamond\Box^\downarrow D \in T^\Diamond$ *such that* $\vdash_{S(T^\Diamond)} \langle\Delta\rangle \Rightarrow \Diamond\Box^\downarrow D$ *and* $\vdash_{S(T^\Diamond)} \Gamma[\Diamond\Box^\downarrow D] \Rightarrow A$*.*

Obviously $\vdash_{S(T^\Diamond)} \langle\Diamond\Box^\downarrow D\rangle \Rightarrow \Diamond\Box^\downarrow D$ for any $\Diamond\Box^\downarrow D \in T^\Diamond$. Then in the proof of (an analogue of) Lemma 3.3.5, one can prove that (r4) restricted to $T^\Diamond$-sequents is admissible in $S(T^\Diamond)$ by Lemma 3.3.7. The remainder of proofs goes without changes.

Lemma 3.3.4 is not needed, if we consider $\mathrm{NL_T}$. By deleting the unrelated parts of constructions and proofs, there are no problems with adapting our results for $\mathrm{NL_T}$. Caution: we use set T instead of $T^\Diamond$.

Theorem 3.3.8 *The consequence relations of* $\mathrm{NL_i}$ *where* $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}, \overline{\mathrm{S4}}\}$ *are decidable in polynomial time.*

3.4 Context-freeness

In this section, we show that type grammars based on $\mathrm{NL}_{\overline{\mathrm{S4}}}(\Phi)$ are equivalent to context-free grammars. The proofs are based on arguments in the previous section.

An $\mathrm{NL}_{\overline{\mathrm{S4}}}(\Phi)$-grammar $\mathcal{G} = \langle\Sigma, I, D\rangle$ and the language $\mathcal{L}(\mathcal{G})$ generated by $\mathcal{G}$ is defined as in section 2.5.2. Notice that for $\mathrm{NL}_{\overline{\mathrm{S4}}}(\Phi)$-grammars the definition of $\mathcal{L}(\mathcal{G})$ can be modified by assuming that Γ does not contain $\langle-\rangle$. For any sequent $\Gamma \Rightarrow A$, if $\Gamma \Rightarrow A$ is derivable from Φ in $\mathrm{NL}_{\overline{\mathrm{S4}}}$, then $\Gamma' \Rightarrow B$ is derivable from Φ in $\mathrm{NL}_{\overline{\mathrm{S4}}}$ with rule (rT), where Γ' arises from Γ by dropping all $\langle-\rangle$.

Theorem 3.4.1 *Every language generated by an* $\mathrm{NL}_{\overline{\mathrm{S4}}}(\Phi)$*-grammar is context-free.*

Proof: Let Φ be a finite set of sequents of the form $A \Rightarrow B$, $\mathcal{G}_1 = \langle\Sigma, I, D\rangle$ be an $\mathrm{NL}_{\overline{\mathrm{S4}}}(\Phi)$-grammar, and T be the set of all subformulae of formula D and formulae appearing in I and Φ. We construct $T^\Diamond$, $S^{T^\Diamond}$ and $S(T^\Diamond)$ as in last section. Now we construct an equivalent CFG $\mathcal{G}_2$, in the following way. The terminal symbols of $\mathcal{G}_2$ are those in Σ. The non-terminals are all formulae (types) from $T^\Diamond$ and the start symbol $S = D$. The finite set of production rules is $\{A \to B \mid B \Rightarrow A \in S^{T^\Diamond}\} \cup \{A \to BC \mid B \circ C \Rightarrow A \in S^{T^\Diamond}\} \cup \{A \to v \mid \langle v, A\rangle \in I\}$.

If a sequent does not contain $\langle - \rangle$, then no proof of this sequent in $S(T^{\Diamond})$ employs sequents with $\langle - \rangle$. Hence every derivation in $\mathcal{G}_2$ is a derivation in $S(T^{\Diamond})$, and conversely, every derivation in $S(T^{\Diamond})$ can be simulated by a derivation in $\mathcal{G}_2$, due to the constructions. By Lemma 3.3.5, this yields the equivalence. $\square$

Let $\mathcal{L}$ be a ε-free context-free language. Then $\mathcal{L}$ is generated by some NL-grammar $\mathcal{G}$ (see Kandulski [1988]). Since neither the lexicon nor the designated type (formula) contain modal connectives, by Corollary 3.2.3, $\mathcal{G}$ is conceived as an $\mathrm{NL}_{\overline{\mathrm{S4}}}(\Phi)$-grammar, where Φ is empty. Hence $\mathrm{NL}_{\overline{\mathrm{S4}}}(\Phi)$-grammars generate exactly the ε-free context-free languages. The equivalence also holds for any fixed Φ, since the map I may employ no variable appearing in Φ.

As we discussed in last section, one can prove analogues of Lemma 3.3.5 for $\mathrm{NL}_{\mathrm{i}}(\Phi)$ $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$. Theorem 3.4.1 still holds for type grammars based on $\mathrm{NL}_{\mathrm{i}}(\Phi)$ $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$ with modification on $T^{\Diamond}$. The construction of $\mathcal{G}_2$ needs one change: the set of production rules should also include $\{A \to B \mid \langle B \rangle \Rightarrow A \in S^{T^{\Diamond}}\}$, if one consider type grammars based on $\mathrm{NL}_4(\Phi)$. The remainder of proofs goes without changes. Hence one can extend these results to $\mathrm{NL}_{\mathrm{i}}(\Phi)$-grammars where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$. So $\mathrm{NL}_{\mathrm{i}}(\Phi)$-grammars where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$ generate exactly the ε-free context-free languages.

Chapter 4

Distributive full nonassociative Lambek calculus with modalities: interpolation, SFMP and context-freeness

4.1 Introduction

We start with an interpolation lemma, which holds for nonassociative systems. This lemma is adopted to various systems in order to prove different results. For instance, Jäger [2004], Buszkowski [2005] and Buszkowski & Farulewski [2009] use lemmas of that kind to prove the context-freeness for the corresponding type grammars. Using interpolantion lemmas for NL and GLC, Buszkowski [2005] also proves that consequence relations of these logics are decidable in polynomial time. Bulińska [2009] also proves this result for NL1, i.e. NL with 1. In Buszkowski [2011], interpolation lemmas for different systems (for NL, DFNL, BFNL and others) are used to prove FEP for the corresponding classes of algebras, e.g. RGs (first proved by Farulewski [2008]), DLRGs, BRgs and Heyting residuated groupoids.

Interpolation lemmas claim that for any sequent $\Gamma[\Delta] \Rightarrow A$ derivable from a finite set of sequents Φ, there exists a formula $D \in T$ (an interpolant of Δ) such that $\Gamma[D] \Rightarrow A$ and $\Delta \Rightarrow D$ are derivable from Φ. Here T is a set of formulae which depends on $\Gamma[\Delta] \Rightarrow A$ and Φ. In the next section, we introduce and prove interpolation lemmas for $\mathrm{DFNL_i}$ and $\mathrm{BFNL_i}$, where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$, which are essentially used in the proofs in section 4.4 and 4.5. Interpolation lemmas discussed here are similar to those in chapter 3 (proved for systems without lattice connectives), but the proofs are different, since we do not employ systems $S^T(\Phi)$ nor the Φ-restricted cut elimination theorem (this does not hold for systems with (D)).

In sections 4.3 and 4.4, we show strong finite model property (SFMP) for the classes of algebras corresponding to $\mathrm{DFNL_i}$ and $\mathrm{BFNL_i}$, where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$. In section 4.3, we introduce the definitions of SFMP, universal finite model property (UFMP) and finite embeddability property (FEP), and discuss the relationship between them. Then in section 4.4, we apply the methods from Buszkowski [2011] and Buszkowski & Farulewski [2009] to prove SFMP for $\mathrm{DFNL_{S4}}$, which yields FEP for S4-$\mathbb{DLRG}$. Analogous results can be easily obtained for all classes of algebras considered in this thesis. These variants are also discussed in this section. In the final section, we prove that type grammars based on $\mathrm{DFNL_i}$ and $\mathrm{BFNL_i}$ are equivalent to context-free grammars.

4.2 Interpolation

In this section, we prove an interpolation lemma which is an important technical result of this thesis. Analogous lemmas for NL, DFNL, BFNL and their variants with multiple operations were proved in Buszkowski [2005, 2011]; Buszkowski & Farulewski [2009], but special modal axioms (T), (4) were not considered there. We consider them here, which requires a refined interpolation lemma. We prove an interpolation lemma for $\mathrm{DFNL_{S4}}$. Analogous results can be obtained for $\mathrm{BFNL_{S4}}$, $\mathrm{DFNL_i}$ and $\mathrm{BFNL_i}$ where $i \in \{4, \mathrm{T}\}$. We discuss these variants at the end of this section.

Let T be a set of formulae. In the following lemma, we assume that T is closed under subformulae and $\vee$, $\wedge$.

Lemma 4.2.1 *If $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[\Delta] \Rightarrow_T A$, then there exists $D \in T$ such that*

$\Phi \vdash_{\mathrm{DFNL_{S4}}} \Delta \Rightarrow_T D$ and $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[D] \Rightarrow_T A$; additionally, if $\Delta = \langle \Delta' \rangle$ for some formula tree Δ', then $\Phi \vdash_{\mathrm{DFNL_{S4}}} \langle D \rangle \Rightarrow_T D$.

Proof: If $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[\Delta] \Rightarrow A$ and a formula D possesses the properties in the given statement of lemma, then we call D an interpolant of Δ. The proof proceeds by induction on T-deductions of $\Gamma[\Delta] \Rightarrow A$. The cases of axioms and assumptions are easy, because these axioms and assumptions are simple sequents of the form $B \Rightarrow A$. So $\Delta = B$ and B is an interplant of Δ.

Let $\Gamma[\Delta] \Rightarrow A$ be the conclusion of a rule R. (Cut) is easy. If Δ comes from one premise of (Cut), then one takes an interpolant from this premise. Otherwise, Δ arises from $\Delta'[C]$ in one premise, where C is the cut formula. Then an interpolant of $\Delta'[C]$ is also an interpolant of Δ. Let us consider other rules.

(1) We assume that Δ does not contain the formula or structure operation, introduced by R (the active formula or structure operation). We consider the following cases.

(1.1) R = ($\wedge$R). Assume that the premises are $\Gamma[\Delta] \Rightarrow A_1$ and $\Gamma[\Delta] \Rightarrow A_2$, and the conclusion is $\Gamma[\Delta] \Rightarrow A_1 \wedge A_2$. By the induction hypothesis, there are interpolants D_1, D_2 such that $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Delta \Rightarrow_T D_1$, $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[D_1] \Rightarrow_T A_1$, $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Delta \Rightarrow_T D_2$ and $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[D_2] \Rightarrow_T A_2$. Then, one gets $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Delta \Rightarrow_T D_1 \wedge D_2$ by ($\wedge$R). By ($\wedge$L) and ($\wedge$R), one obtains $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[D_1 \wedge D_2] \Rightarrow_T A_1 \wedge A_2$. Additionally, if $\Delta = \langle \Delta' \rangle$ for some formula tree Δ', then by the induction hypothesis, $\Phi \vdash_{\mathrm{DFNL_{S4}}} \langle D_1 \rangle \Rightarrow_T D_1$ and $\Phi \vdash_{\mathrm{DFNL_{S4}}} \langle D_2 \rangle \Rightarrow_T D_2$. By ($\wedge$L) and ($\wedge$R), one obtains $\Phi \vdash_{\mathrm{DFNL_{S4}}} \langle D_1 \wedge D_2 \rangle \Rightarrow_T D_1 \wedge D_2$. Hence, $D_1 \wedge D_2$ is an interpolant of Δ.

(1.2) R = ($\vee$L). Assume that the premises are $\Gamma'[B][\Delta] \Rightarrow A$ and $\Gamma'[C][\Delta] \Rightarrow A$, and the conclusion is $\Gamma'[B \vee C][\Delta] \Rightarrow A$. By the induction hypothesis, there are interpolants D_1, D_2 of Δ in the premises. As above, $D_1 \wedge D_2$ is an interpolant of Δ by ($\wedge$L), ($\vee$L) and ($\wedge$R).

(1.3) R = (r4). Assume that the premise is $\Gamma'[\langle \Delta' \rangle] \Rightarrow A$ and the conclusion is $\Gamma'[\langle \langle \Delta' \rangle \rangle] \Rightarrow A$. If $\Delta = \langle \Delta' \rangle$, then by the induction hypothesis, there exists $D \in T$ such that $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Delta \Rightarrow_T D$, $\Phi \vdash_{\mathrm{DFNL_{S4}}} \langle D \rangle \Rightarrow_T D$ and $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma'[D] \Rightarrow_T A$.

By applying (Cut) to $\langle D\rangle \Rightarrow D$ and $\Gamma'[D] \Rightarrow A$, one obtains $\vdash_{\mathrm{DFNL_{S4}}} \Gamma'[\langle D\rangle] \Rightarrow_T A$. Hence D is an interpolant of Δ. Otherwise, Δ is contained in Δ' or Γ'. Then an interpolant D of Δ in the premise is also an interpolant of Δ in the conclusion.

(1.4) R = (rT). Assume that the premise is $\Gamma'[\langle\Delta'\rangle] \Rightarrow A$, and the conclusion is $\Gamma'[\Delta'] \Rightarrow A$. If Δ comes from the premise of (rT), then one takes an interpolant D of Δ from the premise. Otherwise, assume that $\Delta''[\langle\Delta'\rangle]$ occurs in $\Gamma'[\langle\Delta'\rangle]$, and $\Delta = \Delta''[\Delta']$. Then an interpolant D of $\Delta''[\langle\Delta'\rangle]$ is also an interpolant of Δ.

(1.5) For the other cases, Δ must come exactly from one premise of R. Then an interpolant of Δ in this premise is also an interpolant of Δ in the conclusion.

(2) We assume that Δ contains the active formula or structure operation (the rule must be an L-rule or ($\Diamond$R)). If Δ is a single formula E, then E is an interpolant of Δ. Otherwise, we consider the following cases.

(2.1) R = ($\Diamond$R). Assume that the premise is $\Gamma' \Rightarrow A$, and the conclusion is $\langle\Gamma'\rangle \Rightarrow \Diamond A$. Then $\Delta = \langle\Gamma'\rangle$. We show that $\Diamond A$ is an interpolant of Δ. By (Id), ($\Diamond$R), (r4) and ($\Diamond$L), one gets $\Phi \vdash_{\mathrm{DFNL_{S4}}} \langle\Diamond A\rangle \Rightarrow_T \Diamond A$. Hence $\Diamond A$ is an interpolant of Δ.

(2.2) R = ($\Box^{\downarrow}$L). Assume that the premise is $\Gamma'[C] \Rightarrow A$, and the conclusion is $\Gamma'[\langle\Box^{\downarrow}C\rangle] \Rightarrow A$. If $\Delta = \langle\Box^{\downarrow}C\rangle$, then $\Gamma' = \Gamma$. We show that $\Box^{\downarrow}C$ is an interpolant of Δ. By (Id), ($\Box^{\downarrow}$L), (r4) and ($\Box^{\downarrow}$R), one gets $\Phi \vdash_{\mathrm{DFNL_{S4}}} \langle\Box^{\downarrow}C\rangle \Rightarrow_T \Box^{\downarrow}C$. By applying (rT) to $\Gamma'[\langle\Box^{\downarrow}C\rangle] \Rightarrow A$, one gets $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma'[\Box^{\downarrow}C] \Rightarrow A$. Hence $\Box^{\downarrow}C$ is an interpolant of Δ. Otherwise $\langle\Box^{\downarrow}C\rangle$ is contained in Δ and different from Δ. Assume that $\Delta''[C]$ occurs in $\Gamma'[C]$, and $\Delta = \Delta''[\langle\Box^{\downarrow}C\rangle]$. Then an interpolant D of $\Delta''[C]$ is also an interpolant of Δ.

(2.3) R=(r4). Assume that the premise is $\Gamma'[\langle\Delta'\rangle] \Rightarrow A$, and the conclusion is $\Gamma'[\langle\langle\Delta'\rangle\rangle] \Rightarrow A$. Then Δ contains $\langle\langle\Delta'\rangle\rangle$. Assume that $\Delta''[\langle\Delta'\rangle]$ occurs in $\Gamma'[\langle\Delta'\rangle]$, and $\Delta = \Delta''[\langle\langle\Delta'\rangle\rangle]$. Then an interpolant D of $\Delta''[\langle\Delta'\rangle]$ is also an interpolant of Δ in the conclusion.

(2.4) R = ($\backslash$L) or R = ($/$L). Let R = ($\backslash$L). Assume that the premises are $\Gamma'[C] \Rightarrow A$ and $\Delta' \Rightarrow B$, and the conclusion is $\Gamma'[\Delta' \circ B\backslash C] \Rightarrow A$. Then Δ contains $\Delta' \circ B\backslash C$. Assume that $\Delta''[C]$ occurs in $\Gamma'[C]$, and $\Delta = \Delta''[\Delta' \circ B\backslash C]$. Then an interpolant D of $\Delta''[C]$ is also an interpolant of Δ. For ($/$L), the arguments are similar.

(2.5) $R = (\vee\mathrm{L})$. Assume that the premises are $\Gamma[\Delta'[B_1]] \Rightarrow A$ and $\Gamma[\Delta'[B_2]] \Rightarrow A$, and the conclusion is $\Gamma[\Delta'[B_1 \vee B_2]] \Rightarrow A$, where $\Delta = \Delta'[B_1 \vee B_2]$. Let D_1 be an interpolant of $\Delta'[B_1]$ in the first premise and D_2 be an interpolant of $\Delta'[B_2]$ in the second premise. Hence $D_1 \vee D_2$ is an interpolant of Δ in the conclusion, by $(\vee\mathrm{R})$ and $(\vee\mathrm{L})$. Additionally, if $\Delta'[B_1 \vee B_2] = \langle \Delta'' \rangle$ for some formula tree Δ'', then by the induction hypothesis, $\Phi \vdash_{\mathrm{DFNL_{S4}}} \langle D_1 \rangle \Rightarrow_T D_1$ and $\Phi \vdash_{\mathrm{DFNL_{S4}}} \langle D_2 \rangle \Rightarrow_T D_2$. By $(\vee\mathrm{R})$ and $(\vee\mathrm{L})$, one gets $\Phi \vdash_{\mathrm{DFNL_{S4}}} \langle D_1 \vee D_2 \rangle \Rightarrow_T D_1 \vee D_2$.

(2.6) $\mathrm{R} = (\wedge\mathrm{L})$ or $\mathrm{R} = (\cdot\mathrm{L})$. Let $\mathrm{R} = (\wedge\mathrm{L})$. Assume that the premise is $\Gamma'[B] \Rightarrow A$, and the conclusion is $\Gamma'[B \wedge C] \Rightarrow A$. Let $\Delta = \Delta'[B \wedge C]$ such that $\Delta'[B]$ occurs in $\Gamma'[B]$. Hence the interpolant D of $\Delta'[B]$ is also an interpolant of Δ in the conclusion. The arguments for $(\cdot\mathrm{L})$ are similar.

□

An analogue of Lemma 4.2.1 holds for system $\mathrm{BFNL_{S4}}$. One assume that T contains $\bot$ and $\top$ and is closed under subformulae, $\wedge$, $\vee$ and $\neg$. One can easily check the additional cases for axioms $(\bot)$, $(\top)$, $(\neg 2)$ and $(\neg 1)$. The cases for $(\neg 2)$ and $(\neg 1)$ are easy. Let us consider the case for $(\bot)$. If Δ contains $\bot$, then one takes $\bot$ as an interpolant; otherwise, one takes $\top$ as an interpolant. We have $\Phi \vdash_{\mathrm{BFNL_{S4}}} \langle \bot \rangle \Rightarrow \bot$ and $\Phi \vdash_{\mathrm{BFNL_{S4}}} \langle \top \rangle \Rightarrow \top$. For the case for $(\top)$ one takes $\top$ as an interpolant, because that $\Phi \vdash_{\mathrm{BFNL_{S4}}} \langle \top \rangle \Rightarrow \top$ and $\Phi \vdash_{\mathrm{BFNL_{S4}}} \Gamma \Rightarrow \top$ for any formula tree Γ. So we have the following theorem.

Lemma 4.2.2 *If $\Phi \vdash_{\mathrm{BFNL_{S4}}} \Gamma[\Delta] \Rightarrow_T A$, then there exists $D \in T$ such that $\Phi \vdash_{\mathrm{BFNL_{S4}}} \Delta \Rightarrow_T D$ and $\Phi \vdash_{\mathrm{BFNL_{S4}}} \Gamma[D] \Rightarrow_T A$; additionally, if $\Delta = \langle \Delta' \rangle$ for some formula tree Δ', then $\Phi \vdash_{\mathrm{BFNL_{S4}}} \langle D \rangle \Rightarrow_T D$.*

If we consider some systems including rules for $\wedge$, $\vee$ and (r4) as above, then we cannot prove Lemma 4.2.1 without the additional condition: if $\Delta = \langle \Delta' \rangle$ for some formula tree Δ', then the interpolant D of Δ satisfies $\Phi \vdash \langle D \rangle \Rightarrow_T D$. This condition is essentially used in the proof of Lemma 4.2.1, when we consider the cases (1.3). Without this condition ($\Phi \vdash \langle D \rangle \Rightarrow_T D$), one can only show that $\Diamond D$ is an intepolant of Δ, where D is an intepolant of Δ'. One can solve this problem by assuming that T is also closed under $\Diamond$, like in Plummer [2008] and Lin [2010]. Then, however, T is not finite up to the relation of T-equivalence, which destroys the proof of the finiteness in section 4.4. Notice that the rule (r4) is essentially used in the proof of Lemma 4.2.1 to prove this condition in many places, while (rT) is used to prove this condition only in a subcase of (2.2) ($\Delta = \langle \Box^{\downarrow} C \rangle$).

If we consider systems without (r4) ($\mathrm{DFNL_T}$ or $\mathrm{BFNL_T}$), then we can prove interpolation lemmas without this condition, like in Buszkowski & Farulewski [2009] and Buszkowski [2011]. We only need new arguments for the subcase of (2.2). Assume that $\Delta = \langle \Box^{\downarrow} C \rangle$. Then C is an interpolant of $\langle \Box^{\downarrow} C \rangle$. The remainder of proofs goes without changes. Let S_{T} be $\mathrm{DFNL_T}$ or $\mathrm{BFNL_T}$. We assume that T is closed under subformulae, $\wedge$, $\vee$, and additionally we also assume that T contains $\bot$, $\top$ and closed under $\neg$ if $\mathrm{S_T}$ contains axioms for $\bot$, $\top$ and $\neg$.

Lemma 4.2.3 *If $\Phi \vdash_{\mathrm{S_T}} \Gamma[\Delta] \Rightarrow_T A$, then there exists $D \in T$ such that $\Phi \vdash_{\mathrm{S_T}} \Delta \Rightarrow_T D$ and $\Phi \vdash_{\mathrm{S_T}} \Gamma[D] \Rightarrow_T A$.*

Let us consider systems with (r4) but without (rT). We need new arguments to prove that the interpolant D satisfies the condition ($\langle D \rangle \Rightarrow_T D$) in case (2.2). We modify the construction of T. Let T be a set of formulae closed under subformulae $\wedge$ and $\vee$. We assume that for any $\Box^{\downarrow} C \in T$, $\Diamond\Box^{\downarrow} C \in T$. Additionally we also assume T contains $\bot$, $\top$ and closed under $\neg$ if systems contain axioms for $\bot$, $\top$ and $\neg$. In (2.2), assume that $\Delta = \langle \Box^{\downarrow} C \rangle$. Then $\Diamond\Box^{\downarrow} C$ is an interpolant of $\langle \Box^{\downarrow} C \rangle$. The remainder of proofs goes without changes. Therefore let $\mathrm{S_4}$ be $\mathrm{DFNL_4}$ or $\mathrm{BFNL_4}$.

Lemma 4.2.4 *If $\Phi \vdash_{\mathrm{S_4}} \Gamma[\Delta] \Rightarrow_T A$, then there exists $D \in T$ such that $\Phi \vdash_{\mathrm{S_4}} \Delta \Rightarrow_T D$ and $\Phi \vdash_{\mathrm{S_4}} \Gamma[D] \Rightarrow_T A$; additionally, if $\Delta = \langle \Delta' \rangle$ for some formula tree Δ', then $\Phi \vdash_{\mathrm{S_4}} \langle D \rangle \Rightarrow_T D$.*

By the arguments in section 3.3, interpolation lemmas hold for $\mathrm{NL_i}$ where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}, \overline{\mathrm{S4}}\}$.

4.3 SFMP and FEP

Let $\mathbf{A} = \langle \mathrm{A}, \langle f_i^{\mathbf{A}} \rangle_{i \in I} \rangle$ be an algebra of any type and $\mathrm{B} \subseteq \mathrm{A}$. Then $\mathbf{B} = \langle \mathrm{B}, \langle f_i^{\mathbf{B}} \rangle_{i \in I} \rangle$ is a partial subalgebra of $\mathbf{A}$ where for every $n \in \mathrm{N}$, every n-ary function symbol f_i with $i \in I$, and for every $b_1, \ldots, b_n \in \mathrm{B}$, one defines $f_i^{\mathbf{B}}(b_1, \ldots, b_n) =$

$f_i^{\mathbf{A}}(b_1, \ldots, b_n)$ if $f_i^{\mathbf{A}}(b_1, \ldots, b_n) \in \mathrm{B}$, otherwise, the value is not defined. If **A** is ordered, then $\leq^{\mathbf{B}} = \leq^{\mathbf{A}} |\mathrm{B}$. $f_i^{\mathbf{A}}$ denotes the operation interpreting the symbol f_i in the algebra **A**. However we write f_i for $f_i^{\mathbf{A}}$, if it does not cause confusion.

By an embedding from a partial algebra **B** into an algebra **C**, we mean an injection $h : \mathrm{B} \mapsto \mathrm{C}$ such that if $b_1, \ldots, b_n, f^{\mathbf{B}}(b_1, \ldots, b_n) \in \mathrm{B}$, then

$$h(f^{\mathbf{B}}(b_1, \ldots, b_n)) = f^{\mathbf{C}}(h(b_1), \ldots, h(b_n)).$$

If **B** and **C** are ordered, then h is required to be an order embedding: $a \leq^{\mathbf{B}} b \Leftrightarrow h(a) \leq^{\mathbf{C}} h(b)$.

Consider the first-order language of algebras. Atomic formulae are inequalities of the form $s \leq t$, where s, t are terms. Notice that terms correspond to propositional formulae of our substructural logics, and atomic formulae to (simple) sequents. In the literature on substructural logics, one often restricts first-order atomic formulae to equations $s = t$. This is possible, if lattice operations appear in algebras, since $s \leq t$ can be defined as $s \vee t = t$. We, however, also consider algebras without lattice operations, e.g. residuated groupoids, hence we must admit atomic formulae of the form $s \leq t$. An open formula is a propositional (boolean) combination of atomic formulae. A *universal sentence* results from an open formula by the universal quantification of all variables. A *Horn sentence* is a universal sentence of the form $\forall x_1 \ldots x_m(\varphi_1 \& \cdots \& \varphi_n \supset \varphi_{n+1})$ where $n, m \geq 0$ and each φ_i $(1 \leq i \leq n+1)$ is an atomic formula. The *universal theory* of a class $\mathbb{K}$ of algebras is the set of all universal sentences valid in $\mathbb{K}$. The *Horn theory* of a class $\mathbb{K}$ of algebras is the set of all Horn sentences valid in $\mathbb{K}$.

A class $\mathbb{K}$ of algebras has finite embeddability property (FEP), if every finite partial subalgebra of a member of $\mathbb{K}$ can be embedded into a finite member of $\mathbb{K}$. FEP usually has some consequences on finite model property and decidability. Universal finite model property (UFMP): *every universal sentence which fails to hold in a class* $\mathbb{K}$ *of algebras can be falsified in a finite member of* $\mathbb{K}$, and FEP are equivalent for any nonempty class $\mathbb{K}$ of ordered algebras of finite type. Further UFMP implies strong finite model property (SFMP): *every Horn sentence which fails to hold in a class* $\mathbb{K}$ *of algebras can be falsified in a finite member of* $\mathbb{K}$. If a class $\mathbb{K}$ of algebras is closed under finite products (including the trivial product, which yields the trivial algebra), then SFMP for $\mathbb{K}$ implies UFMP for $\mathbb{K}$ (see Buszkowski [2014b]). Notice that SFMP implies UFMP for all classes of algebras considered in this thesis, and UFMP is equivalent to FEP. Hence our results in the next section on SFMP also yield FEP. Then the decidability of the universal (hence also Horn) theories follows, provided that $\mathbb{K}$ is finitely axiomatizable in first-order logic. All classes considered in this thesis are varieties or quasi-varieties, which are axiomatized by a finite number of equations (varieties) or quasi-equations (quasi-varieties).

In the literature (see e.g. Blok & Van-Alten [2005], Farulewski [2008] and N. Galatos & Ono [2007]), 'SFMP implies FEP' is formulated for quasi-varieties (which are closed under arbitrary products) in the following form: SFMP for the Horn theory of a quasi-variety $\mathbb{K}$ entails FEP for $\mathbb{K}$, and the proof provides the embedding.

If a formal system S is strongly complete with respect to a class $\mathbb{K}$ of algebras, then it yields, actually, an axiomatization of the Horn theory of $\mathbb{K}$; hence SFMP for S with respect to $\mathbb{K}$ yields SFMP for $\mathbb{K}$. By SFMP for S, we mean that for any finite Φ, if $\Phi \nvdash_{\mathrm{S}} \Gamma \Rightarrow A$, then there exists a finite $\mathbf{A} \in \mathbb{K}$ and a valuation σ such that all sequents from Φ are true in $(\mathbf{A}\ \sigma)$, but $\Gamma \Rightarrow A$ is not.

4.4 SFMP for modal and additive extensions of NL

In this section, we show SFMP for $\mathrm{DFNL}_{\mathrm{S4}}$, which yields FEP for S4-$\mathbb{DLRG}$. Analogous results can be easily obtained for other modal systems considered in chapter 2. These variants are also discussed in this section.

Let $\mathbf{G}$ =(G, $\cdot$, $\dagger$) be a groupoid enriched with an unary operation $\dagger$ on G. On the powerset $\wp(\mathrm{G})$, one defines the following operations: $U \odot V = \{a \cdot b \in \mathrm{G} : a \in U, b \in V\}$, $\Diamond U = \{\dagger a \in \mathrm{G} : a \in U\}$, $U \backslash V = \{a \in \mathrm{G} : U \odot \{a\} \subseteq V\}$, $V/U = \{a \in \mathrm{G} : \{a\} \odot U \subseteq V\}$, $\Box^{\downarrow} U = \{a \in \mathrm{G} : \dagger a \in U\}$, $U \vee V = U \cup V$, $U \wedge V = U \cap V$, for $U, V \subseteq \mathrm{G}$. $\wp(\mathrm{G})$ with operations $\odot$, $\Diamond$, $\backslash$, $/$, $\Box^{\downarrow}$, $\vee$ and $\wedge$ is a DLRG with operations $\Diamond$ and $\Box^{\downarrow}$ satisfying (2.4). The order is $\subseteq$. An operation $C : \wp(\mathrm{G}) \to \wp(\mathrm{G})$ is called a (nuclear) S4-closure operation (shortly: S4-nucleus) on $\mathbf{G}$, if it satisfies the following conditions:

(C1) $U \subseteq C(U)$,

(C2) if $U \subseteq V$ then $C(U) \subseteq C(V)$,

(C3) $C(C(U)) \subseteq C(U)$,

(C4) $C(U) \odot C(V) \subseteq C(U \odot V)$,

(C5) $\Diamond C(U) \subseteq C(\Diamond U)$,

(C6) $\Diamond C(\Diamond U) \subseteq C(\Diamond U)$,

(C7) $U \subseteq C(\Diamond U)$,

(C1)-(C4) are conditions for a nucleus on the powerset of $\mathbf{G}$. (C5) is a version of (C4) for $\Diamond$. (C6) and (C7) correspond to axioms (4) and (T), respectively. For $U \subseteq \mathrm{G}$, U is called *C-closed* if $U = C(U)$. By C(G), we denote the family of all C-closed subsets of G. Let $U \otimes V = C(U \odot V)$, $\blacklozenge U = C(\Diamond U)$ and $U \vee_C V = C(U \vee V)$. Let $\backslash$, $/$, $\Box^{\downarrow}$, $\wedge$ be defined as above. By (C1)-(C5), $\mathbf{C}(\mathbf{G}) = (\mathrm{C(G)}, \otimes, \backslash, /, \wedge, \vee_C, \blacklozenge, \Box^{\downarrow})$ is a complete LRG with operations $\blacklozenge$ and $\Box^{\downarrow}$ satisfying (2.4); see Buszkowski [2011]. It need not be distributive. The order is $\subseteq$. With (C6) and (C7), one can easily show that for any $U \in \mathrm{C(G)}$, $\blacklozenge\blacklozenge U \subseteq \blacklozenge U$, $U \subseteq \blacklozenge U$. It follows that $\mathbf{C}(\mathbf{G})$ satisfies (2.5) and (2.6).

Let T be a nonempty set of $\mathrm{DFNL_{S4}}$ formulae. By T^* we denote the set of all formula trees formed out of formulae from T. Similarly, $T^*[-]$ denotes the set of all contexts in which all formulae belong to T. $\mathbf{G}(\mathbf{T}^*) = (T^*, (- \circ -), \langle - \rangle)$ is a groupoid enriched with an unary operation $\langle - \rangle$, where $(- \circ -)$ and $\langle - \rangle$ are structure operations introduced in section 2.3.2. Let $\Gamma[-] \in T^*[-]$ and $A \in T$. We define:

$$[\Gamma[-], A] = \{\Delta : \Delta \in T^* \quad \text{and} \quad \Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[\Delta] \Rightarrow_T A\},$$

$$[A] = [-, A] = \{\Gamma : \Gamma \in T^* \quad \text{and} \quad \Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma \Rightarrow_T A\}.$$

Let $B(T)$ be the family of all sets $[\Gamma[-], A]$ defined above. One defines C_T: $\wp(\mathrm{T}^*) \to \wp(\mathrm{T}^*)$ as follows:

$$C_T(U) = \bigcap\{[\Gamma[-], A] \in B(T) : U \subseteq [\Gamma[-], A]\}.$$

In the next proof and later on we often employ the following obvious equivalence: $\Delta \in C_T(U)$ iff for any $[\Gamma[-], A] \in B(T)$, if $U \subseteq [\Gamma[-], A]$ then $\Delta \in [\Gamma[-], A]$, which holds for all $U \subseteq T^*$, $\Delta \in T^*$. We prove the following proposition.

Proposition 4.4.1 *C_T is an* S4*-closure operation.*

Proof: It is easy to see that C_T satisfies (C1), (C2). The conditions (C3), (C4) and (C5) are treated as in Buszkowski [2011]. We prove that C_T satisfies (C6) and (C7).

For (C6), assume that $U \subseteq T^*$ and $\Delta \in C_T(U)$. We show $\langle\langle\Delta\rangle\rangle \in C_T(\Diamond U)$. Let $[\Gamma[-], A] \in B(T)$ be such that $\Diamond U \subseteq [\Gamma[-], A]$. For any $\Pi \in U$, $\langle\Pi\rangle \in \Diamond U$ and $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[\langle\Pi\rangle] \Rightarrow_T A$, whence $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[\langle\langle\Pi\rangle\rangle] \Rightarrow_T A$ by (r4). Consequently, $U \subseteq [\Gamma[\langle\langle-\rangle\rangle], A]$. It follows that $C_T(U) \subseteq [\Gamma[\langle\langle-\rangle\rangle], A]$, by the definition of C_T. Since $\Delta \in C_T(U)$, $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[\langle\langle\Delta\rangle\rangle] \Rightarrow_T A$, whence $\langle\langle\Delta\rangle\rangle \in [\Gamma[-], A]$. We obtain $\langle\langle\Delta\rangle\rangle \in C_T(\Diamond U)$. Then, $\Diamond\Diamond C_T(U) \subseteq C_T(\Diamond U)$. By (C2) and (C3), $C_T(\Diamond\Diamond C_T(U)) \subseteq C_T(\Diamond U)$. By (C5), $\Diamond C_T(\Diamond C_T(U)) \subseteq C_T(\Diamond\Diamond C_T(U))$, whence $\Diamond C_T(\Diamond C_T(U)) \subseteq C_T(\Diamond U)$. Since $\Diamond C_T(\Diamond U) \subseteq \Diamond C_T(\Diamond C_T(U))$, then $\Diamond C_T(\Diamond U) \subseteq C_T(\Diamond U)$.

For (C7), assume that $U \subseteq T^*$, $\Delta \in U$ and $[\Gamma[-], A] \in B(T)$ satisfies $\Diamond U \subseteq [\Gamma[-], A]$. Then $\langle\Delta\rangle \in \Diamond U$. Consequently $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[\langle\Delta\rangle] \Rightarrow_T A$, whence $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[\Delta] \Rightarrow_T A$ by (rT). So, $\Delta \in [\Gamma[-], A]$. Hence one obtains $\Delta \in C_T(\Diamond U)$ which yields $U \subseteq C_T(\Diamond U)$.

□

Accordingly, $\mathbf{C_T}(\mathbf{G}(\mathbf{T}^*))$ is an algebra satisfying all laws defining S4-DLRGs except distribution. The following equations are true in $\mathbf{C_T}(\mathbf{G}(\mathbf{T}^*))$ provided that all formulae appearing in them belong to T.

$$[A] \otimes [B] = [A \cdot B], \quad [A]\backslash[B] = [A\backslash B], \quad [A]/[B] = [A/B] \quad (4.3)$$

$$\blacklozenge[A] = [\Diamond A], \quad \Box^{\downarrow}[A] = [\Box^{\downarrow} A] \quad (4.4)$$

$$[A] \wedge [B] = [A \wedge B], \quad [A] \vee_C [B] = [A \vee B] \quad (4.5)$$

We only show the first equation (4.4). Let $\Gamma \in \Diamond[A]$, whence $\Gamma = \langle\Delta\rangle$ for some $\Delta \in [A]$. Hence $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Delta \Rightarrow_T A$. By ($\Diamond$R), $\Phi \vdash_{\mathrm{DFNL_{S4}}} \langle\Delta\rangle \Rightarrow_T \Diamond A$. Consequently, $\Diamond[A] \subseteq [\Diamond A]$. Hence $C_T(\Diamond[A]) \subseteq C_T([\Diamond A])$, by (C2). Since $\blacklozenge[A] = C_T(\Diamond[A])$, $\blacklozenge[A] \subseteq C_T([\Diamond A])$. Hence $\blacklozenge[A] \subseteq [\Diamond A]$. We prove the converse inclusion. Let $[\Gamma[-], C] \in B(T)$ be such that $\Diamond[A] \subseteq [\Gamma[-], C]$. Then $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[\langle A\rangle] \Rightarrow_T C$. Hence $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[\Diamond A] \Rightarrow_T C$, by ($\Diamond$L). Consequently, by (Cut), $[\Diamond A] \subseteq [\Gamma[-], C]$. By the definition of C_T, $[\Diamond A] \subseteq C_T(\Diamond[A])$, which means $[\Diamond A] \subseteq \blacklozenge[A]$.

Let T be a finite nonempty set of $\mathrm{DFNL_{S4}}$ formulae. By $\overline{T}$, one denotes the smallest set of formulae containing all formulae from T and closed under subformulae and $\wedge$, $\vee$. Now we show that $\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*))$ is a finite S4-DLRG. First we have the following lemma.

Lemma 4.4.2 *$\overline{T}$ is finite up to the relation of $\overline{T}$-equivalence in* $\mathrm{DFNL_{S4}}$.

Proof: Let T' be the set of subformulae of formulae from T. Then T' is also finite. $\overline{T}$ is the closure of T' under $\wedge$ and $\vee$. By the laws of a distributive lattice, every formula from $\overline{T}$ is $\overline{T}$-equivalent to a finite disjunction of finite conjunctions of formulae from T'. There are only finitely many formulae of latter form, if one omits repetitions. □

Let $r(\overline{T})$ be a selector of the family of equivalence classes of $\overline{T}$-equivalence. $r(\overline{T})$ chooses one formula from each equivalence class. Clearly $r(\overline{T})$ is a nonempty finite subset of $\overline{T}$.

Lemma 4.4.3 *For any nontrivial (nonempty and not total) set $U \in \mathrm{C_T}(\overline{\mathrm{T}}^*)$, there exists a formula $A \in r(\overline{T})$ such that $U = [A]$.*

Proof: First we show that for any nonempty set $[\Gamma[-], A]$, there exists a formula $B \in r(\overline{T})$ such that $[\Gamma[-], A] = [B]$. Assume $\Delta \in [\Gamma[-], A]$. Then $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[\Delta] \Rightarrow_{\overline{T}} A$. By Lemma 4.2.1, there exists $D \in \overline{T}$ such that $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[D] \Rightarrow_{\overline{T}} A$ and $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Delta \Rightarrow_{\overline{T}} D$. For any D satisfying this condition, we say that D fulfills the interpolation condition. By Lemma 4.4.2, D can be replaced by a $\overline{T}$-equivalent formula in $r(\overline{T})$. So we assume $D \in r(\overline{T})$. Let B be the disjunction of all formulae D, fulfilling the above. So we stipulate $B \in r(\overline{T})$. By ($\vee$L) we get

$\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma[B] \Rightarrow_{\overline{T}} A$. So by (Cut) $[B] \subseteq [\Gamma[-], A]$. Evidently, $[\Gamma[-], A] \subseteq [B]$, since for any $\Delta \in [\Gamma[-], A]$, $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Delta \Rightarrow_{\overline{T}} B$. Hence $[B] = [\Gamma[-], A]$.

Assume that U is nontrivial and $U \in \mathrm{C_T}(\overline{\mathrm{T}}^*)$. Since U is nonempty, then any set $[\Gamma[-], A]$ containing it is nonempty. Moreover since U is not T^*, then U is contained in at least one $[\Gamma[-], A]$. Hence by the arguments above, for each $[\Gamma[-], A]$ satisfying $U \subseteq [\Gamma[-], A]$, there exists a formula $E \in r(\overline{T})$ such that $[E] = [\Gamma[-], A]$. Let C be the $\overline{T}$-equivalent formula of the conjunction of all formulae E, fulfilling the above. Then by the definition of C_T, (4.5) and (C3), $[C] = U$. □

Lemma 4.4.4 *The algebra* $\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*))$ *is finite and belongs to* S4-$\mathbb{DLRG}$.

Proof: By the above arguments, $\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*))$ is an algebra satisfying all laws defining S4-DLRGs except distribution. It suffices to show $U \wedge (V \vee_C W) \subseteq (U \wedge V) \vee_C (U \wedge W)$, for any $U, V, W \in \mathrm{C_T}(\overline{\mathrm{T}}^*)$. The converse of this inclusion is valid in all lattices. This inclusion is true, if at least one of the sets U, V, W is empty or total, since $\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*))$ is a lattice. So, assume that U, V, W be nontrivial. By Lemma 4.4.3 there exists $A, B, C \in r(\overline{T})$ such that $[A] = U$, $[B] = V$ and $[C] = W$. Since $\Phi \vdash_{\mathrm{DFNL_{S4}}} A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$, by Lemma 4.4.3 and (4.5), one obtains $U \wedge (V \vee_C W) \subseteq (U \wedge V) \vee_C (U \wedge W)$, for any $U, V, W \in \mathrm{C_T}(\overline{\mathrm{T}}^*)$. So $\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*))$ satisfies the distributive law. Hence $\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*))$ belongs to S4-$\mathbb{DLRG}$. Further by Lemma 4.4.3 $r(\overline{T})$ is finite, whence $\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*))$ is finite. □

A model for $\mathrm{DFNL_{S4}}$ is a pair $(\mathbf{M}, \sigma)$ such that $\mathbf{M} \in \mathrm{S4} - \mathbb{DLRG}$, and σ is a valuation in $\mathbf{M}$. σ is extended to formulae and formula trees as in section 2.4.1 with the following new clauses.

$$\sigma(\Diamond A) = \Diamond\sigma(A), \quad \sigma(\Box^{\downarrow} A) = \Box^{\downarrow}\sigma(A), \quad \sigma(\langle\Gamma\rangle) = \Diamond\sigma(\Gamma),$$

$$\sigma(A \wedge B) = \sigma(A) \wedge \sigma(B), \quad \sigma(A \vee B) = \sigma(A) \vee \sigma(B).$$

Now we show SFMP for $\mathrm{DFNL_{S4}}$.

Lemma 4.4.5 *Let T be the set of all formulae appearing in $\Gamma \Rightarrow A$ or Φ. If $\Phi \nvdash_{\mathrm{DFNL_{S4}}} \Gamma \Rightarrow_{\overline{T}} A$, then* $\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*)) \not\models \Gamma \Rightarrow A$.

Proof: Let σ be a valuation in $\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*))$ such that $\sigma(p) = [p]$ for any $p \in \overline{T}$. By (4.3)-(4.5), we get $[A] = \sigma(A)$ for any $A \in \overline{T}$. Assume that $\Gamma \Rightarrow A$ is true in $(\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*)), \sigma)$. Then $\sigma(\Gamma) \subseteq \sigma(A)$. Since $\Gamma \in \sigma(\Gamma)$, we get $\Gamma \in [A]$. Hence $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma \Rightarrow_{\overline{T}} A$, which contradicts the assumption. □

Lemma 4.4.6 *Assume that* $\Phi \not\vdash_{\mathrm{DFNL_{S4}}} \Gamma \Rightarrow A$ *does not hold. Then there exist a finite* $\mathbf{G} \in$ S4-$\mathbb{DLRG}$ *and a valuation* σ *such that all sequents from* Φ *are true, but* $\Gamma \Rightarrow A$ *is not true in* $(\mathbf{G}, \sigma)$.

Proof: Let T be the set of all formulae appearing in $\Gamma \Rightarrow A$ or Φ. $\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*))$ is defined as above. Let $\sigma(p) = [p]$. Evidently $\sigma(A) = [A]$ for any $A \in \overline{T}$. Suppose that $E \Rightarrow F$ be an assumption from Φ. Then $\sigma(E) = [E]$ and $\sigma(F) = [F]$. Let $\Delta \in [E]$. Then $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Delta \Rightarrow_{\overline{T}} E$. Hence by (Cut), $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Delta \Rightarrow_{\overline{T}} F$, which yields $[E] \subseteq [F]$. Thus all sequents from Φ are true in $(\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*)), \sigma)$. Further by Lemma 4.4.5, $\Gamma \Rightarrow A$ is not true in $(\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*)), \sigma)$. □

Theorem 4.4.7 S4-$\mathbb{DLRG}$ *has FEP.*

The following corollary immediately follows from Lemma 4.4.5.

Corollary 4.4.8 *Let* T *be the set of all formulae appearing in* $\Gamma \Rightarrow A$ *or* Φ*. If* $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma \Rightarrow A$*, then* $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Gamma \Rightarrow_{\overline{T}} A$.

Lemma 4.2.1 is essentially used in the arguments to show that $\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*))$ is finite and satisfies the distributive law. As we discussed in section 4.2, analogues of Lemma 4.2.1 hold for $\mathrm{DFNL_i}$ where $\mathrm{i} \in \{4, \mathrm{T}\}$. Caution: $\overline{T}$ contains all formulae $\Diamond\Box^{\downarrow}A$ for any $\Box^{\downarrow}A \in \overline{T}$, if we consider $\mathrm{DFNL_4}$. Obviously $\overline{T}$ is finite up to the relation of $\overline{T}$-equivalence in $\mathrm{DFNL_4}$. By deleting the unrelated parts of constructions of algebras and proofs, one can adapt these results to $\mathrm{DFNL_i}$ where $\mathrm{i} \in \{4, \mathrm{T}\}$. So we have the following theorem and corollary.

Theorem 4.4.9 4-$\mathbb{DLRG}$ *and* T-$\mathbb{DLRG}$ *have FEP.*

Let T be the set of all formulae appearing in $\Gamma \Rightarrow A$ or Φ. $\overline{T}$ is defined as above.

Corollary 4.4.10 *If* $\Phi \vdash_{\mathrm{DFNL_i}} \Gamma \Rightarrow A$*, then* $\Phi \vdash_{\mathrm{DFNL_i}} \Gamma \Rightarrow_{\overline{T}} A$*, where* $\mathrm{i} \in \{4, \mathrm{T}\}$.

These results can be easily adapted to boolean systems. As in Buszkowski [2011], we consider auxiliary systems $\mathrm{BFNL_i}(*)$ ($\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$) which are $\mathrm{DFNL_i}$ with $\bot$, $\top$, an additional commutative binary operation $*$ and its residual $\rightarrow$. We define $\neg A = A \rightarrow \bot$ and admit axioms ($\bot$), ($\top$), ($\neg 1$), ($\neg 2$). It is easy to show that each $\mathrm{BFNL_i}(*)$ ($\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$) is a conservative extension of $\mathrm{BFNL_i}$ (every algebraic model of the latter can be expanded to a model of the former, by identifying $*$ with $\wedge$ and $\rightarrow$ with boolean implication). Then Lemma 4.2.1, 4.4.2 and 4.4.3 still hold, if we assume that $\overline{T}$ contains $\bot$, $\top$, and is closed under subformulae, $\wedge$, $\vee$ and $\neg$. Boolean algebras are locally finite. So Lemma 4.4.2 remains true. In the proof of Lemma 4.4.3, $\mathbf{C_T}(\mathbf{G}(\overline{\mathbf{T}}^*))$ interprets $\bot$ as $C_T(\emptyset)$ and $\top$ as $\overline{T}^*$. Now, for every $U \in \mathrm{C_T}(\overline{\mathrm{T}}^*)$, there exists a formula $A \in r(\overline{T})$ such that $U = [A]$. In the proof of Lemma 4.4.4, we have to show that $\mathbf{C_T}(\mathbf{G}(\mathbf{T}^*))$ satisfies ($\neg 1$) and ($\neg 2$) in algebraic terms. The proof can be easily established by Lemma 4.4.3 and (4.5) together with new clause $\neg[A] = [\neg A]$. The remainder of proofs goes without changes. Hence we obtain SFMP for $\mathrm{BFNL_i}(*)$ ($\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$), and consequently, FEP for the corresponding classes of algebras.

Theorem 4.4.11 *4-$\mathbb{BRG}$, T-$\mathbb{BRG}$, and S4-$\mathbb{BRG}$ have FEP.*

Suppose that T is the set of all $\mathrm{BFNL_i}$ formulae, where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$, appearing in $\Gamma \Rightarrow A$ or Φ. By $\overline{T}$, one denotes the smallest set of formulae containing all formulae from T and closed under subformulae and $\wedge$, $\vee$, $\neg$.

Corollary 4.4.12 *If $\Phi \vdash_{\mathrm{BFNL_i}} \Gamma \Rightarrow A$, then $\Phi \vdash_{\mathrm{BFNL_i}} \Gamma \Rightarrow_{\overline{T}} A$ where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$.*

Let us consider $\mathrm{NL_{\overline{S4}}}$. Analogue of Lemma 4.2.1 holds for $\mathrm{NL_{\overline{S4}}}$. Let T be a finite nonempty set of formulae. $T^{\Diamond}$ is defined as in section 3.3. $\mathbf{C_T}(\mathbf{G}((\mathbf{T}^{\Diamond})^*))$ is constructed as above. Lemma 4.4.3 does not hold. However one can directly prove that $B(T^{\Diamond})$ is finite, which yields the finiteness of $\mathbf{C_T}(\mathbf{G}((\mathbf{T}^{\Diamond})^*))$. One can easily prove that for any $[\Gamma[-], A] \in B(T^{\Diamond})$, $[\Gamma[-], A] = [A_1] \cup \ldots \cup [A_n]$, where $A_1, \ldots, A_n \in T^{\Diamond}$. Since $T^{\Diamond}$ is finite, $B(T^{\Diamond})$ is finite. This proof is based on the corresponding interpolation lemma. The remainder of proofs goes without changes. Thus we can prove SFMP for $\mathrm{NL_{\overline{S4}}}$. Analogues of Lemma 4.2.1 also hold for $\mathrm{NL_i}$ where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$. Consequently, by similar arguments, we obtain SFMP for $\mathrm{NL_i}$ where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$. Hence we have the following theorem.

Theorem 4.4.13 *4-$\mathbb{RG}$, T-$\mathbb{RG}$, S4-$\mathbb{RG}$ and $\overline{\text{S4}}$-$\mathbb{RG}$ have FEP.*

4.5 Context-freeness

In this section, we show that type grammars based on $\text{DFNL}_{\text{S4}}(\Phi)$ are context-free. The proofs are based on arguments in the previous section. We prove that every language generated by a $\text{DFNL}_{S4}(\Phi)$-grammar can also be generated by a context-free grammar, by showing that every derivable sequent in $\text{DFNL}_{S4}(\Phi)$ can be derived (by means of (Cut) only) from some short derivable sequents containing at most three formulae, where all formulae belong to a finite set of formulae. This follows from the fact that there exists a finite set of formulae such that each structure Δ in a derivable sequent $\Gamma[\Delta] \Rightarrow A$ has an interpolant that belongs to this set (cf. Lemma 4.2.1).

Let T be a finite set of formulae, containing all formulae appearing in $\Gamma[\Delta] \Rightarrow A$ or Φ. $\overline{T}$ is defined as above. Then we have the following lemma.

Lemma 4.5.1 *If $\Phi \vdash_{\text{DFNL}_{\text{S4}}} \Gamma[\Delta] \Rightarrow A$, then there exists a formula $D \in \overline{T}$ such that $\Phi \vdash_{\text{DFNL}_{\text{S4}}} \Delta \Rightarrow D$ and $\Phi \vdash_{\text{DFNL}_{\text{S4}}} \Gamma[D] \Rightarrow A$.*

Let T be a finite nonempty set of formulae. $\overline{T}$ and $r(\overline{T})$ are defined as above. Let $S^{r(\overline{T})}(\Phi)$ be the following system. The set of axioms of $S^{r(\overline{T})}(\Phi)$ consists of all the sequents $A \circ B \Rightarrow C$, $\langle A \rangle \Rightarrow B$ and $A \Rightarrow B$ derivable from Φ in DFNL_{S4}, where $A, B, C \in r(\overline{T})$. The only rule of $S^{r(\overline{T})}(\Phi)$ is (Cut). For $A \in \overline{T}$, by $r(A)$ we denote the unique formula in $r(\overline{T})$ which is $\overline{T}$-equivalent to A. Let $r(\Gamma)$ denote the formula tree resulting from Γ after one has replaced each formula A by $r(A)$.

Lemma 4.5.2 *Let T be a finite set of formulae, containing all formulae appearing in $\Gamma \Rightarrow A$ or Φ. Then $\Phi \vdash_{\text{DFNL}_{\text{S4}}} \Gamma \Rightarrow A$ iff $\vdash_{S^{r(\overline{T})}(\Phi)} r(\Gamma) \Rightarrow r(A)$.*

Proof: The *if* part is easy. The proof of the *only if* part proceeds by induction on the length of Γ, denoted by $l(\Gamma)$ (the number of structure operations appearing in Γ). If $l(\Gamma)$ equals 0 or 1, then Γ contains at most two formulae. Hence $\vdash_{S^{r(\overline{T})}(\Phi)} r(\Gamma) \Rightarrow r(A)$, by the consturction of $S^{r(\overline{T})}(\Phi)$. Assume that $\Gamma = \Xi[\Delta]$, where $l(\Delta) = 1$. By Lemma 4.5.1, there exists a formula $D \in \overline{T}$ such that

$\Phi \vdash_{\mathrm{DFNL_{S4}}} \Delta \Rightarrow D$ and $\Phi \vdash_{\mathrm{DFNL_{S4}}} \Xi[D] \Rightarrow A$. Then by the induction hypothesis, one gets $\vdash_{S^{r(\overline{T})}(\Phi)} r(\Delta) \Rightarrow r(D)$, and $\vdash_{S^{r(\overline{T})}(\Phi)} r(\Xi[D]) \Rightarrow r(A)$. It follows that $\vdash_{S^{r(\overline{T})}(\Phi)} r(\Gamma) \Rightarrow r(A)$, by (Cut). □

Theorem 4.5.3 *Every language generated by a* $\mathrm{DFNL_{S4}}(\Phi)$*-grammar is context-free.*

Proof: Let Φ be a finite set of sequents of the form $A \Rightarrow B$, $\mathcal{G}_1 = \langle \Sigma, I, D\rangle$ be a $\mathrm{DFNL_{S4}}(\Phi)$-grammar, and T be the smallest set containing all formulae appearing in D, I and Φ. We construct $r(\overline{T})$ and $S^{r(\overline{T})}(\Phi)$ as above. Now we construct an equivalent CFG (context-free grammar) $\mathcal{G}_2$, in the following way. The terminal elements of $\mathcal{G}_2$ are lexical items of $\mathcal{G}_1$. The non-terminals are all formulae from $r(\overline{T})$ and the start symbol $S = r(D)$. The finite set of production rules is $\{A \to B : \vdash_{S^{r(\overline{T})}(\Phi)} B \Rightarrow A\} \cup \{A \to B : \vdash_{S^{r(\overline{T})}(\Phi)} \langle B\rangle \Rightarrow A\} \cup \{A \to BC : \vdash_{S^{r(\overline{T})}(\Phi)} B \circ C \Rightarrow A\} \cup \{r(A) \to v : \langle v, A\rangle \in I\}$.

As in section 3.4 the definition of $\mathcal{L}(\mathcal{G}_1)$ can be modified by assuming that all antecedents of sequents do not contain $\langle - \rangle$. If a sequent does not contain $\langle - \rangle$, then no proof of this sequent in $S^{r(\overline{T})}(\Phi)$ employs sequents with $\langle - \rangle$. Hence every derivation in $\mathcal{G}_2$ is a derivation in $S^{r(\overline{T})}(\Phi)$, and conversely, every derivation in $S^{r(\overline{T})}(\Phi)$ can be simulated by a derivation in $\mathcal{G}_2$, due to the constructions. By Lemma 4.5.2, this yields the equivalence.

□

The inclusion of the class of ε-free context free languages in the class of $\mathrm{DFNL_{S4}}(\Phi)$-recognizable languages can be easily established as in section 3.4. Hence $\mathrm{DFNL_{S4}}(\Phi)$-grammars generate exactly the ε-free context-free languages. As we discussed in section 4.2, one can prove interpolation lemmas for $\mathrm{DFNL_i}$ ($\mathrm{i} \in \{4, \mathrm{T}\}$) and $\mathrm{BFNL_i}$ ($\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$). Thus by similar arguments, one can show that $\mathrm{DFNL_T}(\Phi)$-grammars, $\mathrm{BFNL_T}(\Phi)$-grammars and $\mathrm{BFNL_{S4}}(\Phi)$-grammars generate exactly the ε-free context-free languages. Let us consider $\mathrm{DFNL_4}(\Phi)$-grammars and $\mathrm{BFNL_4}(\Phi)$-grammars. The construction of $S^{r(\overline{T})}(\Phi)$ remains the same. Then Lemma 4.5.2 still holds. In the proof of Theorem 4.5.3, the construction of $\mathcal{G}_2$ needs one changes: the set of production rules should also include $\{A \to B :\vdash_{S^{r(\overline{T})}(\Phi)} \langle B\rangle \Rightarrow A\}$. The remainder of proofs goes without changes.

Theorem 4.5.4 $\mathrm{DFNL_i}(\Phi)$*-grammars and* $\mathrm{BFNL_i}(\Phi)$*-grammars where* $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$ *generate exactly the* ε*-free context-free languages.*

Chapter 5

Complexity of boolean nonassociative Lambek calculus and its modal extensions

5.1 Introduction

In this chapter, we analyze the complexity of the decision problem of BFNL and its modal extensions. Our main results are that BFNL is PSPACE-complete and $\mathrm{BFNL_i}$ ($\mathrm{i} \in \{\mathrm{T}, 4, \mathrm{S4}\}$) are PSPACE-hard. PSPACE-hardness of BFNL is obtained by a polynomial reduction from the minimal normal modal logic K which is PSPACE-complete to BFNL. That BFNL is in PSPACE is shown by a polynomial reduction from BFNL to the minimal bi-tense logic ${\mathrm{K}^{\mathrm{t}}}_{1,2}$. We show that ${\mathrm{K}^{\mathrm{t}}}_{1,2}$ is in PSACE by a polynomial reduction from it to the minimal tense logic K.t which is PSPACE-complete. Moreover this result also yields that DFNL is in PSPACE. Finally we extend our PSPACE-hardness results to $\mathrm{BFNL_i}$, where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$.

5.2 Modal logic K

The language $\mathcal{L}_{\mathsf{K}}(\mathsf{Prop})$ of the minimal normal modal logic K consists of a set Prop of propositional variables, connectives $\bot, \supset$ and an unary modal operator

$\Diamond$. The set of all modal formulae is defined by the following inductive rule:

$$A ::= p \mid \bot \mid A \supset B \mid \Diamond A, \quad p \in \mathsf{Prop}$$

The other classical connectives $\neg$, &, $\vee$, are defined as usual: $\neg A := A \supset \bot$, $A \vee B := \neg A \supset B$, $A \& B := \neg(A \supset \neg B)$. Also $\top := \neg\bot$ and $\Box A := \neg\Diamond\neg A$.

K is axiomatized by the following axiom schemata and inference rules (Blackburn *et al.* [2002]):

- all tautologies of classical propositional logic,
- $\Box(A \supset B) \supset (\Box A \supset \Box B)$,
- (MP) from $A \supset B$ and A infer B,
- (Nec) from A infer $\Box A$.

A formula A is provable in K, denote by $\vdash_{\mathrm{K}} A$, if it is derivable from the axioms of K using the rules of inference.

Now we introduce Kripke semantics for K. A K-frame is a pair $\mathfrak{F} = (W, R)$ where W is a nonempty set of states, and R is a binary relation over W. A model $\mathfrak{M} = (W, R, \sigma)$ of K consists of a K-frame (W, R) and a valuation σ which is a mapping from Prop to the powerset of W. The satisfiability relation $\mathfrak{M}, w \models A$ between a model $\mathfrak{M}$ with a state $w \in W$ and a $\mathcal{L}_{\mathsf{K}}$-formula A is recursively defined as follows.

$\mathfrak{M}, w \models p$ iff $w \in \sigma(p)$,

$\mathfrak{M}, w \not\models \bot$,

$\mathfrak{M}, w \models A \supset B$ iff $\mathfrak{M}, w \not\models A$ or $\mathfrak{M}, w \models B$,

$\mathfrak{M}, w \models \Diamond A$ iff there exists $u \in W$ such that $R(w, u)$ and $\mathfrak{M}, u \models A$.

The clauses for $\neg$, $\vee$, & and $\Box$ are:

$\mathfrak{M}, w \models \neg A$ iff $\mathfrak{M}, w \not\models A$,

$\mathfrak{M}, w \models A \vee B$ iff $\mathfrak{M}, w \models A$ or $\mathfrak{M}, w \models B$,

$\mathfrak{M}, w \models A\&B$ iff $\mathfrak{M}, w \models A$ and $\mathfrak{M}, w \models B$,

$\mathfrak{M}, u \models \Box A$ iff for every $v \in W$, if $R(u, v)$, then $\mathfrak{M}, v \models A$.

The notion of satisfiability and validity of formulae for models and frames are defined as usually. A formula A is called (Kripke) valid, denoted $\models_{\mathrm{K}} A$ if it is valid in all K-frames. A formula A is valid iff it is provable in K, i.e.: $\vdash_{\mathrm{K}} A$ iff $\models_{\mathrm{K}} A$.

A normal modal logic is a set S of modal formulae such that all theorems of K belongs to S and S is closed under MP, Nec and uniform substitution. The PSPACE-hardness of the validity problems of some modal logics were settled first by Ladner [1977]. Let us recall this theorem from Ladner [1977].

Theorem 5.2.1 (Lander's Theorem) *If S is a normal modal logic such that* $\mathrm{K} \subseteq S \subseteq \mathrm{S4}$*, then S has a PSPACE-hard satisfiability problem. Moreover, S has PSPACE-hard validity problem.*

5.3 The minimal (bi)-tense logics

First let us give some basic notions for the minimal tense logic K.t. The language $\mathcal{L}_{\mathsf{K.t}}(\mathsf{Prop})$ consists of a set Prop of propositional variables, connectives $\bot, \supset$ and two unary modal operators $\Diamond$ and $\Box^{\downarrow}$. The set of all K.t formulae is defined by the following inductive rule:

$$A ::= p \mid \bot \mid A \supset B \mid \Diamond A \mid \Box^{\downarrow} A, \quad p \in \mathsf{Prop}.$$

The other classical connectives are defined as usual. Define $\top := \neg\bot$, $\Box A := \neg\Diamond\neg A$ and $\Diamond^{\downarrow} A := \neg\Box^{\downarrow}\neg A$.

The axiomatic system for K.t consists of the following axiom schemata and rules:

(1) all tautologies of classical propositional logic,

(2) $\Box(A \supset B) \supset (\Box A \supset \Box B)$,

(3) $\Box^{\downarrow}(A \supset B) \supset (\Box^{\downarrow} A \supset \Box^{\downarrow} B)$,

(4) $A \supset \Box\Diamond^{\downarrow}A$,

(5) $A \supset \Box^{\downarrow}\Diamond A$,

(6) MP: from A and $A \supset B$ infer B,

(7) Nec$\Box$: from A infer $\Box A$,

(8) Nec$\Box^{\downarrow}$: from A infer $\Box^{\downarrow}A$.

By $\vdash_{\mathrm{K.t}} A$, we mean that A is provable in K.t. Here we list some facts holding in K.t which are useful for proving the embedding result in Theorem 5.5.3. We omit the proofs.

Fact 5.3.1 *The following hold in* K.t*:*

(1) *if* $\vdash_{\mathrm{K.t}} A \supset B$*, then* $\vdash_{\mathrm{K.t}} (C \supset A) \supset (C \supset B)$.

(2) *if* $\vdash_{\mathrm{K.t}} (A\&B) \supset C$*, then* $\vdash_{\mathrm{K.t}} A \supset (B \supset C)$.

(3) *if* $\vdash_{\mathrm{K.t}} A \supset B$ *and* $\vdash_{\mathrm{K.t}} B \supset C$*, then* $\vdash_{\mathrm{K.t}} A \supset C$.

(4) $\vdash_{\mathrm{K.t}} (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$.

(5) *if* $\vdash_{\mathrm{K.t}} A \supset B$*, then* $\vdash_{\mathrm{K.t}} \Box A \supset \Box B$.

(6) *if* $\vdash_{\mathrm{K.t}} A \supset B$*, then* $\vdash_{\mathrm{K.t}} \Box^{\downarrow}A \supset \Box^{\downarrow}B$.

(7) *if* $\vdash_{\mathrm{K.t}} \Diamond A \supset B$*, then* $\vdash_{\mathrm{K.t}} A \supset \Box^{\downarrow}B$.

(8) *if* $\vdash_{\mathrm{K.t}} \Diamond^{\downarrow}A \supset B$*, then* $\vdash_{\mathrm{K.t}} A \supset \Box B$.

A K.t-frame is a pair $\mathfrak{F} = (W, R)$ where W is a nonempty set of states, and R is a binary relation over W. A model $\mathfrak{M} = (W, R, \sigma)$ of K.t consists of a K.t-frame (W, R) and a valuation σ which is a mapping from Prop to the powerset of W. The satisfiability relation $\mathfrak{M}, w \models A$ between a model $\mathfrak{M}$ with a state $w \in W$ and a K.t formula A is recursively defined as follows.

$\mathfrak{M}, w \models p$ iff $w \in \sigma(p)$,

$\mathfrak{M}, w \not\models \bot$,

$\mathfrak{M}, w \models A \supset B$ iff $\mathfrak{M}, w \not\models A$ or $\mathfrak{M}, w \models B$,

$\mathfrak{M}, w \models \Diamond A$ iff there exists $u \in W$ with $R(w, u)$ and $\mathfrak{M}, u \models A$,

$\mathfrak{M}, w \models \Box^{\downarrow} A$ iff for every $u \in W$, if $R(u, w)$, then $\mathfrak{M}, u \models A$.

Thus we also have the following clauses:

$\mathfrak{M}, w \models \Diamond^{\downarrow} A$ iff there exists $u \in W$ with $R(u, w)$ and $\mathfrak{M}, u \models A$,

$\mathfrak{M}, w \models \Box A$ iff for every $u \in W$, if $R(w, u)$, then $\mathfrak{M}, u \models A$.

The notion of satisfiability and validity of formulae for models and frames are defined as usually. By $\models_{\mathrm{K.t}} A$ we mean that A is valid in all K.t-frames. By standard canonical model construction, it is easy to show that for every formula A, $\vdash_{\mathrm{K.t}} A$ iff $\models_{\mathrm{K.t}} A$ (Burgess [1984]).

The minimal bi-tense logic $\mathrm{K}^{\mathrm{t}}_{1,2}$ is a minimal tense logic having two pairs of modal operators $\Diamond_1$, $\Box_1^{\downarrow}$ and $\Diamond_2$, $\Box_2^{\downarrow}$ with their duals $\Box_1$, $\Diamond_1^{\downarrow}$ and $\Box_2$, $\Diamond_2^{\downarrow}$, respectively. Then the set of formulae and the axiomatic system extends in an obvious way. A $\mathrm{K}^{\mathrm{t}}_{1,2}$-frame is a triple $\mathfrak{F} = (W, R_1, R_2)$ where W is a nonempty set of states, and R_1, R_2 are binary relations over W. A $\mathrm{K}^{\mathrm{t}}_{1,2}$-model $\mathfrak{M} = (W, R_1, R_2, \sigma)$ consists of a $\mathrm{K}^{\mathrm{t}}_{1,2}$-frame (W, R_1, R_2) and a valuation $\sigma : \mathsf{Prop} \to \wp(W)$. The definition of satisfiability relation $\mathfrak{M}, w \models A$ for $\mathrm{K}^{\mathrm{t}}_{1,2}$ is the same as that for K.t with index i where $i \in \{1, 2\}$. The notion of satisfiability and validity of formulae for models and frames are defined as usually. By $\models_{\mathrm{K}^{\mathrm{t}}_{1,2}} A$ we mean that A is valid in all $\mathrm{K}^{\mathrm{t}}_{1,2}$-frames. By $\vdash_{\mathrm{K}^{\mathrm{t}}_{1,2}} A$ we mean that A is provable in $\mathrm{K}^{\mathrm{t}}_{1,2}$. Further for every $\mathcal{L}_{\mathrm{K}^{\mathrm{t}}_{1,2}}$-formula A, $\vdash_{\mathrm{K}^{\mathrm{t}}_{1,2}} A$ iff $\models_{\mathrm{K}^{\mathrm{t}}_{1,2}} A$ (Burgess [1984]).

5.4 PSPACE-hard decision problem of BFNL

We reduce the validity problem of K, which is PSPACE-complete, to the decision problem of BFNL in P-time so that we prove the PSPACE-hardness of the decision problem of BFNL. Now let us consider an embedding from modal logic K into BFNL.

Defintion 5.4.1 *Let* $\mathsf{P} \subseteq \mathsf{Prop}$ *and* $m \notin \mathsf{P}$ *be a distinguished propositional variable. Define a translation (mapping)* $(.)^\dagger$*:* $\mathcal{L}_{\mathrm{K}}(\mathsf{P}) \to \mathcal{L}_{\mathrm{BFNL}}(\mathsf{P} \cup \{m\})$ *recursively as follows:*

$$p^\dagger = p, \qquad \bot^\dagger = \bot,$$
$$(A \supset B)^\dagger = A^\dagger \supset B^\dagger, \qquad (\Diamond A) = m \cdot A^\dagger.$$

Let $\mathfrak{M} = (W, R, \sigma)$ be a K-model with a valuation $\sigma : \mathsf{P} \to \wp(W)$. We define a BFNL-model $\mathfrak{J}^{\mathfrak{M}} = (W', R', \sigma')$ from $\mathfrak{M}$ as follows:

- if $w \in W$, then put two copies w_1, w_2 of w into W',
- if $R(w, u)$, then $R'(w_1, w_2, u_1)$,
- $w_i \in \sigma'(p)$ iff $w \in \sigma(p)$, for all $p \in \mathsf{P}$ and $i \in \{1, 2\}$; and $\sigma'(m) = W'$.

Intuitively, for each state w in the original model we make two copies w_1 and w_2, and then define a tenary relation among copies according to the original binary relation R.

Lemma 5.4.1 *Suppose that* $\mathfrak{M} = (W, R, \sigma)$ *is a K-model and* $\mathfrak{J}^{\mathfrak{M}} = (W', R', \sigma')$ *is defined from* $\mathfrak{M}$ *as above. Then for any* $w \in W$ *and* $\mathcal{L}_{\mathrm{K}}$*-formula* A*,* $\mathfrak{M}, w \models A$ *iff* $\mathfrak{J}^{\mathfrak{M}}, w_1 \models A^\dagger$*.*

Proof: By induction on the complexity of A. The atomic and boolean cases are easy by the construction of $\mathfrak{J}^{\mathfrak{M}}$ and the induction hypothesis. For $A = \Diamond B$, assume $\mathfrak{M}, w \models \Diamond B$. Then there exists $u \in W$ such that $R(w, u)$ and $\mathfrak{M}, u \models B$. Since $R(w, u)$, we get $R'(w_1, w_2, u_1)$. By the induction hypothesis, $\mathfrak{J}^{\mathfrak{M}}, u_1 \models B^\dagger$. Hence $\mathfrak{J}^{\mathfrak{M}}, w_1 \models m \cdot B^\dagger$. Conversely, assume $\mathfrak{J}^{\mathfrak{M}}, w_1 \models m \cdot B^\dagger$. Then there exist $k, z \in W'$ such that $R'(w_1, k, z)$, $\mathfrak{J}^{\mathfrak{M}}, k \models m$ and $\mathfrak{J}^{\mathfrak{M}}, z \models B^\dagger$. By the construction $k = w_2$ and $z = u_1$ for some $u \in W$. By the induction hypothesis, $\mathfrak{M}, u \models B$. By the construction of $\mathfrak{J}^{\mathfrak{M}}$, we get $R(w, u)$. Hence $\mathfrak{M}, w \models \Diamond B$. □

Lemma 5.4.2 *For any* $\mathcal{L}_{\mathrm{K}}$*-formula* A*, if* $\vdash_{\mathrm{BFNL}} \top \Rightarrow A^\dagger$*, then* $\vdash_{\mathrm{K}} A$*.*

Proof: Assume $\nvdash_{\mathrm{K}} A$. Then there is a K-model $\mathfrak{M}$ such that $\mathfrak{M} \not\models A$. By Lemma 5.4.1, $\mathfrak{J}^{\mathfrak{M}} \not\models A^\dagger$. So $\mathfrak{J}^{\mathfrak{M}} \not\models \top \Rightarrow A^\dagger$. Hence, by Theorem 2.4.2, we get $\nvdash_{\mathrm{BFNL}} \top \Rightarrow A^\dagger$. □

Lemma 5.4.3 *For any $\mathcal{L}_{\mathrm{K}}$-formula A, if $\vdash_{\mathrm{K}} A$, then $\vdash_{\mathrm{BFNL}} \top \Rightarrow A^\dagger$.*

Proof: We proceed by induction on the length of proofs in K. It suffices to show that all axioms and inference rules of K are admissible in BFNL with respect to the translation †. Obviously the translations of all tautologies of classical propositional logic are provable in BFNL. Consider $(\Box(A \supset B) \supset (\Box A \supset \Box B))^\dagger = (m \cdot (A^\dagger \wedge \neg B^\dagger)) \vee (m \cdot (\neg A^\dagger)) \vee (\neg(m \cdot (\neg B^\dagger))$. Since $A^\dagger \wedge B^\dagger \Rightarrow B^\dagger$, by (TR) and Fact 2.3.4 (3), one gets $\vdash_{\mathrm{BFNL}} \neg B^\dagger \Rightarrow (\neg A^\dagger \vee \neg B^\dagger)$. Hence by $\mathrm{Mon}(\cdot)_{\mathrm{l}}$, one gets $\vdash_{\mathrm{BFNL}} m \cdot (\neg B^\dagger) \Rightarrow (m \cdot (\neg A^\dagger \vee \neg B^\dagger))$. Then by Fact 2.3.5 one gets $\vdash_{\mathrm{BFNL}} \top \Rightarrow \neg(m \cdot (\neg B^\dagger)) \vee (m \cdot (\neg A^\dagger \vee \neg B^\dagger))$. Since $\vdash_{\mathrm{BFNL}} (A^\dagger \vee \neg A^\dagger) \Leftrightarrow \top$, one gets $\vdash_{\mathrm{BFNL}} (m \cdot (\neg A^\dagger \vee \neg B^\dagger)) \Leftrightarrow m \cdot ((A^\dagger \vee \neg A^\dagger) \wedge (\neg A^\dagger \vee \neg B^\dagger))$. By Fact 2.3.3 (2) and $\mathrm{Mon}(\cdot)_{\mathrm{l}}$, one gets $\vdash_{\mathrm{BFNL}} m \cdot ((A^\dagger \vee \neg A^\dagger) \wedge (\neg A^\dagger \vee \neg B^\dagger)) \Leftrightarrow m \cdot ((A^\dagger \wedge \neg B^\dagger) \vee \neg A^\dagger)$. Again, by Fact 2.3.1 (1) one can prove that $\vdash_{\mathrm{BFNL}} m \cdot ((A^\dagger \wedge \neg B^\dagger) \vee \neg A^\dagger) \Leftrightarrow (m \cdot (A^\dagger \wedge \neg B^\dagger)) \vee ((m \cdot (\neg A^\dagger)))$. Hence one gets $\vdash_{\mathrm{BFNL}} \top \Rightarrow (m \cdot (A^\dagger \wedge \neg B^\dagger)) \vee (m \cdot (\neg A^\dagger)) \vee (\neg(m \cdot (\neg B^\dagger)))$.

Let us consider the rule (MP). Assume that $\vdash_{\mathrm{BFNL}} \top \Rightarrow A^\dagger$ and $\vdash_{\mathrm{BFNL}} \top \Rightarrow (A \supset B)^\dagger$, which is equal to $\vdash_{\mathrm{BFNL}} \top \Rightarrow \neg A^\dagger \vee B^\dagger$. We need to show $\vdash_{\mathrm{BFNL}} \top \Rightarrow B^\dagger$. By ($\neg$1), ($\bot$) and (Cut), one gets $\vdash_{\mathrm{BFNL}} A^\dagger \wedge \neg A^\dagger \Rightarrow B^\dagger$. By ($\wedge$L), one gets $\vdash_{\mathrm{BFNL}} A^\dagger \wedge B^\dagger \Rightarrow B^\dagger$. Then, by ($\vee$L), one gets $\vdash_{\mathrm{BFNL}} (A^\dagger \wedge \neg A^\dagger) \vee (A^\dagger \wedge B^\dagger) \Rightarrow B^\dagger$. Then by (D) and (Cut), one gets $\vdash_{\mathrm{BFNL}} A^\dagger \wedge (\neg A^\dagger \vee B^\dagger) \Rightarrow B^\dagger$. Clearly, by assumptions and ($\wedge$R), one gets $\vdash_{\mathrm{BFNL}} \top \Rightarrow A^\dagger \wedge (\neg A^\dagger \vee B^\dagger)$, which yields $\vdash_{\mathrm{BFNL}} \top \Rightarrow B^\dagger$ by (Cut).

Finally let us consider the rule (Nec). Assume $\vdash_{\mathrm{BFNL}} \top \Rightarrow A^\dagger$. We need to show $\vdash_{\mathrm{BFNL}} \top \Rightarrow \neg(m \cdot (\neg A^\dagger))$. By (TR) and assumptions, one gets $\vdash_{\mathrm{BFNL}} \neg(A^\dagger) \Rightarrow \bot$. By $\mathrm{Mon}(\cdot)_{\mathrm{l}}$, one gets $\vdash_{\mathrm{BFNL}} m \cdot \neg(A^\dagger) \Rightarrow m \cdot \bot$. Then by ($\bot$), ($\cdot$L) and (Cut), one gets $\vdash_{\mathrm{BFNL}} m \cdot \neg(A^\dagger) \Rightarrow \bot$. Hence by (TR), one gets $\vdash_{\mathrm{BFNL}} \top \Rightarrow \neg(m \cdot (\neg A^\dagger))$. □

Lemma 5.4.2 and Lemma 5.4.3 lead to the following theorem.

Theorem 5.4.4 $\vdash_{\mathrm{K}} A$ *iff* $\vdash_{\mathrm{BFNL}} \top \Rightarrow A^\dagger$

Obviously the reduction is in polynomial time. Now by Lardner's theorem (Theorem 5.2.1), one gets the following theorem.

Theorem 5.4.5 BFNL *is PSPACE-hard.*

5.5 BFNL is in PSPACE

Kurtonina [1994] proved that NL is faithfully embedded into $K^t_{1,2}$: $\vdash_{NL} A \Rightarrow B$ iff $\vdash_{K^t_{1,2}} A^\# \supset B^\#$, where $\#$ is a mapping from NL formulae to $K^t_{1,2}$ formulae. This result can be extended to an embedding from BFNL into $K^t_{1,2}$. We modify and extend Kurtonina's constructions and proofs to BFNL.

Define a translation $(.)^\# : \mathcal{L}_{BFNL}(\mathsf{Prop}) \to \mathcal{L}_{K^t_{1,2}}(\mathsf{Prop})$ recursively as follows:

$$p^\# = p, \qquad \top^\# = \top, \quad \bot^\# = \bot,$$
$$(\neg A)^\# = \neg A^\#, \qquad (A \wedge B)^\# = A^\# \& B^\#,$$
$$(A \vee B)^\# = A^\# \vee B^\#, \qquad (A \cdot B)^\# = \Diamond_1(\Diamond_1 A^\# \& \Diamond_2 B^\#),$$
$$(A \backslash B)^\# = \Box_2^\downarrow(\Diamond_1 A^\# \supset \Box_1^\downarrow B^\#), \qquad (A/B)^\# = \Box_1^\downarrow(\Diamond_2 B^\# \supset \Box_1^\downarrow A^\#).$$

Theorem 5.5.1 *For any $\mathcal{L}_{BFNL}$-sequent $\Gamma \Rightarrow D$, $\vdash_{BFNL} \Gamma \Rightarrow D$ iff $\vdash_{K^t_{1,2}} (f(\Gamma))^\# \supset D^\#$.*

Proof: We can confine ourselves to the relation $\vdash_{BFNL} A \Rightarrow B$, since every sequent $\Gamma \Rightarrow D$ is deductively equivalent in BFNL to $f(\Gamma) \Rightarrow D$. The left-to-right direction is shown by induction on the proof of $A \Rightarrow B$ in BFNL. Notice that it is easy to show that the translations of axioms in BFNL are theorems in $K^t_{1,2}$. For instance, consider $(\neg 1)$ $A \wedge \neg A \Rightarrow \bot$. Its translation $(A^\# \wedge \neg A^\#) \supset \bot$ is a theorem of $K^t_{1,2}$. Rules of BFNL are checked regularly. We demonstrate only one typical case. Let us consider $(\backslash L)$. It suffices to show that the following rule is admissible under the translation:

$$\frac{A \Rightarrow B \quad C \Rightarrow D}{A \cdot (B \backslash C) \Rightarrow D}$$

Assume that $\vdash_{K^t_{1,2}} A^\# \supset B^\#$ and $\vdash_{K^t_{1,2}} C^\# \supset D^\#$. It suffices to show that $\vdash_{K^t_{1,2}} \Diamond_1(\Diamond_1 A^\# \& \Diamond_2 \Box_2^\downarrow(\Diamond_1 B^\# \supset \Box_1^\downarrow C^\#)) \supset D^\#$. For any $K^t_{1,2}$-model $\mathfrak{M} = (W, R_1, R_2, V)$ and $u \in W$, assume that $\mathfrak{M}, u \models \Diamond_1(\Diamond_1 A^\# \& \Diamond_2 \Box_2^\downarrow(\Diamond_1 B^\# \supset \Box_1^\downarrow C^\#))$. It suffices to show $\mathfrak{M}, u \models D^\#$. By assumptions, we have $\mathfrak{M} \models A^\# \supset B^\#$ and $\mathfrak{M} \models C^\# \supset D^\#$. Again by the assumption, we get $\mathfrak{M}, v \models \Diamond_1 A^\# \& \Diamond_2 \Box_2^\downarrow(\Diamond_1 B^\# \supset \Box_1^\downarrow C^\#)$ for some $v \in W$ such that $R_1(u, v)$. Then there exist $w, z \in W$ such that $R_1(v, w)$, $R_2(v, z)$, $\mathfrak{M}, w \models A^\#$ and $\mathfrak{M}, z \models \Box_2^\downarrow(\Diamond_1 B^\# \supset \Box_1^\downarrow C^\#)$. Since $\mathfrak{M} \models A^\# \supset B^\#$, we get $\mathfrak{M}, w \models B^{\#}$. Hence $\mathfrak{M}, v \models \Diamond_1 B^{\#}$. Then by $\mathfrak{M}, z \models \Box_2^{\downarrow}(\Diamond_1 B^{\#} \supset \Box_1^{\downarrow} C^{\#})$ and $R_2(v, z)$, we get $\mathfrak{M}, v \models \Box_1^{\downarrow} C^{\#}$. Since $R_1(u, v)$, we get $\mathfrak{M}, u \models C^{\#}$. By the assumption $\mathfrak{M} \models C^{\#} \supset D^{\#}$, we get $\mathfrak{M}, u \models D^{\#}$. Hence we get $\vdash_{\mathrm{K}^{\mathrm{t}}_{1,2}} (A \cdot (B \backslash C))^{\#} \Rightarrow D^{\#}$.

For the other direction, assume that $\nvdash_{\mathrm{BFNL}} A \Rightarrow B$. By completeness of BFNL, there exists a BFNL-model $\mathfrak{M} = (W, R, \sigma)$ and a state $k \in W$ such that $\mathfrak{M}, k \models A$ but $\mathfrak{M}, k \not\models B$. Then we construct a $\mathrm{K}^{\mathrm{t}}_{1,2}$-model $\mathfrak{M}^* = (W^*, R_1, R_2, \sigma^*)$ from $\mathfrak{M}$ satisfying the following conditions:

- $k \in W^*$.
- if $R(u, v, w)$, then put a fresh state x and u, v, w into W^* such that $R_1(u, x)$, $R_1(x, v)$ and $R_2(x, w)$.
- set for all $u \in W \cap W^*$ and $p \in \mathsf{Prop}$, $u \in \sigma^*(p)$ iff $u \in \sigma(p)$.

We may show that for any $u \in W \cap W^*$ and $\mathcal{L}_{\mathrm{BFNL}}$-formula A,

$$(\mho) \quad \mathfrak{M}, u \models A \text{ iff } \mathfrak{M}^*, u \models A^{\#}.$$

By induction on the length of A. The basic case is a direct consequence of the definition of $\mathfrak{M}^*$. We demonstrate only one typical clause of the inductive step. Let us consider the case $A = B \cdot C$. Then $B \cdot C = \Diamond_1(\Diamond_1 B^{\#} \& \Diamond_2 C^{\#})$. Assume that $\mathfrak{M}, u \models B \cdot C$. Then there exist $v, w \in W$ such that $R(u, v, w)$, $\mathfrak{M}, v \models B$ and $\mathfrak{M}, w \models C$. By the induction hypothesis, $\mathfrak{M}^*, v \models B^{\#}$ and $\mathfrak{M}^*, w \models C^{\#}$. By the construction, there exists $x \in W^*$ such that $R_1(u, x)$, $R_1(x, v)$ and $R_2(x, w)$. Hence $\mathfrak{M}^*, x \models \Diamond_1 B^{\#} \& \Diamond_2 C^{\#}$. Hence $\mathfrak{M}^*, u \models \Diamond_1(\Diamond_1 B^{\#} \& \Diamond_2 C^{\#})$. Conversely, assume that $\mathfrak{M}^*, u \models \Diamond_1(\Diamond_1 B^{\#} \& \Diamond_2 C^{\#})$. Then there exist $x, v, w \in W^*$ such that $R_1(u, x)$, $R_1(x, v)$, $R_2(x, w)$, and $\mathfrak{M}^*, v \models B^{\#}$ and $\mathfrak{M}^*, w \models C^{\#}$. By the construction, we get $R(u, v, w)$. By the induction hypothesis, we get $\mathfrak{M}, v \models B$ and $\mathfrak{M}, w \models C$. Therefore, $\mathfrak{M}, u \models B \cdot C$.

Finally, by ($\mho$), $\mathfrak{M}^*, k \models A^{\#}$ but $\mathfrak{M}^*, k \not\models B^{\#}$. Hence $A^{\#} \supset B^{\#}$ is refuted in $\mathfrak{M}^*$. So $\nvdash_{\mathrm{K}^{\mathrm{t}}_{1,2}} A^{\#} \supset B^{\#}$. □

It is easy to see that the reduction from BFNL to $\mathrm{K}^{\mathrm{t}}_{1,2}$ is in polynomial time. Then by the following Theorem 5.5.4 that $\mathrm{K}^{\mathrm{t}}_{1,2}$ is in PSPACE, we get the following theorem:

Theorem 5.5.2 BFNL *is in PSPACE.*

Now we show that the validity problem for $\mathrm{K}^{\mathrm{t}}_{1,2}$ is in PSPACE. Our method is to show that $\mathrm{K}^{\mathrm{t}}_{1,2}$ is embedded into K.t in polynomial time.

Let $\mathsf{P} \subseteq \mathsf{Prop}$ and $x \notin \mathsf{P}$ be a distinguished propositional variable. Define a translation $(.)^* : \mathcal{L}_{\mathrm{K}^{\mathrm{t}}_{12}}(\mathsf{P}) \to \mathcal{L}_{\mathrm{K.t}}(\mathsf{P} \cup \{x\})$ recursively as follows:

$$p^* = p, \qquad \bot^* = \bot,$$
$$(\Diamond_1 A)^* = \neg x \& \Diamond(\neg x \& A^*), \qquad (\Diamond_2 A)^* = \Diamond(x \& \Diamond A^*),$$
$$(\Box^{\downarrow}_1 A)^* = \neg x \supset \Box^{\downarrow}(\neg x \supset A^*), \qquad (\Box^{\downarrow}_2 A)^* = \Box^{\downarrow}(x \supset \Box^{\downarrow} A^*),$$
$$(A \supset B)^* = A^* \supset B^*,$$

Thus we have

$$(\Diamond^{\downarrow}_1 A)^* = \neg x \& \Diamond^{\downarrow}(\neg x \& A^*), \qquad (\Diamond^{\downarrow}_2 A)^* = \Diamond^{\downarrow}(x \& \Diamond^{\downarrow} A^*),$$
$$(\Box_1 A)^* = \neg x \supset \Box(\neg x \supset A^*), \qquad (\Box_2 A)^* = \Box(x \supset \Box A^*).$$

Theorem 5.5.3 *For every $\mathcal{L}_{\mathrm{K}^{\mathrm{t}}_{1,2}}$-formula A, $\vdash_{\mathrm{K}^{\mathrm{t}}_{1,2}} A$ iff $\vdash_{\mathrm{K.t}} A^*$.*

Proof: (i) By induction on the proof of A in $\mathrm{K}^{\mathrm{t}}_{1,2}$, we show that $\vdash_{\mathrm{K}^{\mathrm{t}}_{1,2}} A$ implies $\vdash_{\mathrm{K.t}} A^*$. We show only that the translations of axioms and rules of $\mathrm{K}^{\mathrm{t}}_{1,2}$ hold in K.t. The cases for propositional tautologies and (MP) are obvious. Let us consider other cases.

Case 1. The translation of $\Box_1(A \supset B) \supset (\Box_1 A \supset \Box_1 B)$ is $(\neg x \supset \Box(\neg x \supset (A^* \supset B^*))) \supset ((\neg x \supset \Box(\neg x \supset A^*)) \supset (\neg x \supset \Box(\neg x \supset B^*)))$. First, $\vdash_{\mathrm{K.t}} (\neg x \supset (A^* \supset B^*)) \supset ((\neg x \supset A^*) \supset (\neg x \supset B^*))$ by Fact 5.3.1 (4). Then $\vdash_{\mathrm{K.t}} \Box(\neg x \supset (A^* \supset B^*)) \supset (\Box(\neg x \supset A^*) \supset \Box(\neg x \supset B^*))$ by Fact 5.3.1 (5) and distributivity of $\Box$ over implications. Then by Fact 5.3.1 (1) and (4), we get the required theorem in K.t.

Case 2. The translation of $\Box_2(A \supset B) \supset (\Box_2 A \supset \Box_2 B)$ is $\Box(x \supset \Box(A^* \supset B^*)) \supset (\Box(x \supset \Box A^*) \supset \Box(x \supset \Box B^*))$. Since $\vdash_{\mathrm{K.t}} \Box(A^* \supset B^*) \supset (\Box A^* \supset \Box B^*)$, by Fact 5.3.1 (1), we get $\vdash_{\mathrm{K.t}} (x \supset \Box(A^* \supset B^*)) \supset (x \supset (\Box A^* \supset \Box B^*))$. Since $\vdash_{\mathrm{K.t}} (x \supset (\Box A^* \supset \Box B^*)) \supset ((x \supset \Box A^*) \supset (x \supset \Box B^*))$, we obtain $\vdash_{\mathrm{K.t}} (x \supset \Box(A^* \supset B^*)) \supset ((x \supset \Box A^*) \supset (x \supset \Box B^*))$ by Fact 5.3.1 (3). Hence by Fact 5.3.1 (5), we get $\vdash_{\mathrm{K.t}} \Box(x \supset \Box(A^* \supset B^*)) \supset \Box((x \supset \Box A^*) \supset (x \supset \Box B^*))$. Since $\vdash_{\mathrm{K.t}} \Box((x \supset \Box A^*) \supset (x \supset \Box B^*)) \supset (\Box(x \supset \Box A^*) \supset \Box(x \supset \Box B^*))$, by Fact 5.3.1 (3), we obtain $\vdash_{\mathrm{K.t}} \Box(x \supset \Box(A^* \supset B^*)) \supset (\Box(x \supset \Box A^*) \supset \Box(x \supset \Box B^*))$

Case 3. For $\Box_1^{\downarrow}(A \supset B) \supset (\Box_1^{\downarrow}A \supset \Box_1^{\downarrow}B)$, the proof is quite similar to the proof of case 1.

Case 4. For $\Box_2^{\downarrow}(A \supset B) \supset (\Box_2^{\downarrow}A \supset \Box_2^{\downarrow}B)$, the proof is quite similar to the proof of case 2.

Case 5. The translation of $A \supset \Box_2^{\downarrow}\Diamond_2 A$ is $A^* \supset \Box^{\downarrow}(x \supset \Box^{\downarrow}\Diamond(x\&\Diamond A^*))$. Since $\vdash_{\mathrm{K.t}} \Diamond(x\&\Diamond A^*) \supset \Diamond(x\&\Diamond A^*)$, by Fact 5.3.1 (7), we get $(x\&\Diamond A^*) \supset \Box^{\downarrow}\Diamond(x\&\Diamond A^*)$, Then by Fact 5.3.1 (2), we obtain $\vdash_{\mathrm{K.t}} \Diamond A^* \supset (x \supset \Box^{\downarrow}\Diamond(x\&\Diamond A^*))$. Finally by Fact 5.3.1 (7), we get $\vdash_{\mathrm{K.t}} A^* \supset \Box^{\downarrow}(x \supset \Box^{\downarrow}\Diamond(x\&\Diamond A^*))$.

Case 6. For $A \supset \Box_2\Diamond_2^{\downarrow}A$, the proof is quite similar to the case 5.

Case 7. Let us consider the translation of $A \supset \Box_1\Diamond_1^{\downarrow}A$, which is $A^* \supset (\neg x \supset \Box(\neg x \supset (\neg x\&\Diamond^{\downarrow}(\neg x\&A^*))))$. Since $(\neg x\&\Diamond^{\downarrow}(\neg x\&A^*)) \supset (\neg x\&\Diamond^{\downarrow}(\neg x\&A^*))$ is a propositional tautology, $\vdash_{\mathrm{K.t}} \Diamond^{\downarrow}(\neg x\&A^*) \supset (\neg x \supset (\neg x\&\Diamond^{\downarrow}(\neg x\&A^*)))$ by Fact 5.3.1 (2). Hence by Fact 5.3.1 (8), one obtains $\vdash_{\mathrm{K.t}} (\neg x\&A^*) \supset \Box(\neg x \supset (\neg x\&\Diamond^{\downarrow}(\neg x\&A^*)))$. Finally by Fact 5.3.1 (2), one obtains $\vdash_{\mathrm{K.t}} A^* \supset (\neg x \supset \Box(\neg x \supset (\neg x\&\Diamond^{\downarrow}(\neg x\&A^*))))$.

Case 8. For the axiom $A \supset \Box_1^{\downarrow}\Diamond_1 A$, the proof is quite similar to the case 7.

Case 9. For Nec$\Box_1$, assume that $\vdash_{\mathrm{K.t}} A^*$. Then by MP and propositional tautology $A^* \supset (\neg x \supset A^*)$, we get $\vdash_{\mathrm{K.t}} \neg x \supset A^*$. By Nec$\Box$, we get $\vdash_{\mathrm{K.t}} \Box(\neg x \supset A^*)$. Again, we obtain $\vdash_{\mathrm{K.t}} \neg x \supset \Box(\neg x \supset A^*)$. The proofs for the cases of other Nec-rules are quite similar.

(ii) For the other direction, assume that $\nvdash_{\mathrm{K}^{\mathrm{t}}_{1,2}} A$. Let $\mathfrak{M} = (W, R_1, R_2, \sigma)$ be a $\mathrm{K}^{\mathrm{t}}_{1,2}$-model and $k \in W$ such that $\mathfrak{M}, k \not\models A$. Let us construct a K.t-model $\mathfrak{M}' = (W', R', \sigma')$ satisfying the following conditions:

- $k \in W'$,
- if $R_1(u, v)$, then put $u, v \in W'$ and let $R'(u, v)$,

- if $R_2(u,v)$, then take a fresh state $w \notin W$, put u,v,w into W' and let $R'(u,w)$ and $R'(w,v)$,
- set for all $u \in W \cap W'$, $u \in \sigma'(p)$ iff $u \in \sigma(p)$, for each $p \in \mathsf{P}$,
- set for all $u \in W' \backslash W$, $u \in \sigma'(x)$.

We show that for any $u \in W \cap W'$ and $\mathcal{L}_{\mathrm{K}^{\mathrm{t}}_{1,2}}$-formula C,

$$(\mho)\ \mathfrak{M}, u \models C \text{ iff } \mathfrak{M}', u \models C^*.$$

Now by induction on the length of C. The cases of propositional variables and boolean connectives are easy. Let us consider other cases. In the following proofs and later on we often employ the following obvious facts: 1) if $u,v \in \sigma'(\neg x)$ and $R'(u,v)$, then $R_1(u,v)$; 2) for any $w,u,v \in W'$, if $w \in W' \backslash W$, $R'(u,w)$ and $R'(w,v)$, then $u,v \in W$ and $R_2(u,v)$.

Case 1. $C = \Diamond_1 B$. Assume that $\mathfrak{M}, u \models \Diamond_1 B$. Then $R_1(u,v)$ and $\mathfrak{M}, v \models B$ for some $v \in W$. By the induction hypothesis, we get $\mathfrak{M}', v \models B^*$. Since $u, v \in W \cap W'$, we get $\mathfrak{M}', u \models \neg x$ and $\mathfrak{M}', v \models \neg x$. Hence $\mathfrak{M}', v \models \neg x \& B^*$. By $R_1(u,v)$, we get $R'(u,v)$. Then $\mathfrak{M}', u \models \neg x \& \Diamond(\neg x \& B^*)$, i.e., $\mathfrak{M}', u \models (\Diamond_1 B)^*$. Conversely, assume that $\mathfrak{M}', u \models \neg x \& \Diamond(\neg x \& B^*)$. Then $\mathfrak{M}', u \models \neg x$, and there exists $v \in W'$ such that $R'(u,v)$ and $\mathfrak{M}', v \models \neg x$ and $\mathfrak{M}', v \models B^*$. Then $u, v \in W$ and $R_1(u,v)$. By the induction hypothesis, $\mathfrak{M}, v \models B$. Then $\mathfrak{M}, u \models \Diamond_1 B$.

Case 2. $C = \Box_1^{\downarrow} B$. Assume that $\mathfrak{M}', u \not\models (\Box_1^{\downarrow} B)^*$. By definition, $\mathfrak{M}', u \models \neg x \& \Diamond^{\downarrow}(\neg x \& \neg B^*)$. Hence $\mathfrak{M}', u \models \neg x$, and there exists $v \in W'$ such that $R'(v,u)$, $\mathfrak{M}', v \models \neg x$ and $\mathfrak{M}', v \models \neg B^*$. Then $u, v \in W$ and $R_1(v,u)$. By the induction hypothesis, $\mathfrak{M}, v \models \neg B$. Thus $\mathfrak{M}, u \not\models \Box_1^{\downarrow} B$. Conversely, assume that $\mathfrak{M}, u \not\models \Box_1^{\downarrow} B$. Then there exists $v \in W$ such that $R_1(v,u)$ and $\mathfrak{M}, v \models \neg B$. Since $u, v \in W \cap W'$, by the induction hypothesis and arguments for $\neg$, we get $\mathfrak{M}', v \models \neg B^*$. By the construction, $R'(v,u)$, $\mathfrak{M}', u \models \neg x$ and $\mathfrak{M}', v \models \neg x$. Hence $\mathfrak{M}', u \models \neg x \& \Diamond^{\downarrow}(\neg x \& \neg B^*)$. Hence $\mathfrak{M}', u \not\models \neg x \supset \Box^{\downarrow}(\neg x \supset B^*)$, i.e., $\mathfrak{M}', u \not\models (\Box_1^{\downarrow} B)^*$.

Case 3. $C = \Diamond_2 B$. Assume that $\mathfrak{M}, u \models \Diamond_2 B$. Then there exists $v \in W$ such that $R_2(u,v)$ and $\mathfrak{M}, v \models B$. By the induction hypothesis, $\mathfrak{M}', v \models B^*$. By the construction, there exists $w \in W' \setminus W$ such that $R'(u,w)$ and $R'(w,v)$. Then $\mathfrak{M}', w \models \Diamond B^*$ and $\mathfrak{M}', w \models x$. Hence $\mathfrak{M}, u \models \Diamond(x \& \Diamond B^*)$, i.e., $\mathfrak{M}, u \models (\Diamond_2 B)^*$.

Conversely, assume that $\mathfrak{M}', u \models (\Diamond_2 B)^*$, i.e., $\mathfrak{M}', u \models \Diamond(x\&\Diamond B^*)$. Then there exist $v, w \in W'$ such that $R'(u, w)$, $R'(w, v)$, $\mathfrak{M}', w \models x$ and $\mathfrak{M}', v \models B^*$. So $w \in W' \backslash W$. By the construction, we get $u, v \in W \cap W'$ and $R_2(u, v)$. By the induction hypothesis, $\mathfrak{M}, v \models B$. Hence $\mathfrak{M}, u \models \Diamond_2 B$.

Case 4. $C = \Box_2^{\downarrow} B$. Assume that $\mathfrak{M}', u \not\models (\Box_2^{\downarrow} B)^*$, i.e., $\mathfrak{M}', u \models \Diamond^{\downarrow}(x\&\Diamond^{\downarrow}\neg B^*)$. There exist $w, v \in W'$ such that $R'(v, w)$, $R'(w, u)$, $\mathfrak{M}', w \models x$ and $\mathfrak{M}', v \models \neg B^*$. Then $w \in W' \backslash W$, $v \in W' \cap W$ and so by the construction $R_2(v, u)$. By the induction hypothesis, $\mathfrak{M}, v \models \neg B$. So $\mathfrak{M}, u \not\models \Box_2^{\downarrow} B$. Conversely, assume that $\mathfrak{M}, u \not\models \Box_2^{\downarrow} B$. Then there exists $v \in W$ such that $R_2(v, u)$ and $\mathfrak{M}, v \not\models B$. By the construction, there exists $w \in W' \setminus W$ such that $R'(v, w)$, $R'(w, u)$ and $\mathfrak{M}', w \models x$. By the induction hypothesis, $\mathfrak{M}', v \not\models B^*$. So $\mathfrak{M}', v \models \neg B^*$ and $\mathfrak{M}', w \models x\&\Diamond^{\downarrow}\neg B^*$. Hence $\mathfrak{M}', u \models \Diamond^{\downarrow}(x\&\Diamond^{\downarrow}\neg B^*)$. So $\mathfrak{M}', u \not\models (\Box_2^{\downarrow} B)^*$.

Hence ($\mho$) is proved. Since $\mathfrak{M}, k \not\models A$, by ($\mho$), we get $\mathfrak{M}', k \not\models A^*$. So $\nvdash_{\mathrm{K.t}} A^*$.

□

It is already known that the validity problem of K.t in PSPACE (Blackburn *et al.* [2002]; Coré [1999]; Granko [2000]). Since $\mathrm{K}^{\mathrm{t}}_{1,2}$ is embedded into K.t in polynomial time, by Theorem 5.5.3 we get the following result:

Theorem 5.5.4 $\mathrm{K}^{\mathrm{t}}_{1,2}$ *is in PSPACE.*

5.6 PSPACE-completeness

Theorem 5.6.1 BFNL *is PSPACE-complete.*

This theorem is proved by Theorem 5.4.5 and 5.5.2. PSPACE-hardness of the consequence relation of DFNL follows from a complexity result in Buszkowski [2014a]. We can obtain the same result by reducing BFNL to DFNL with assumptions in polynomial time. Moreover since BFNL is a conservative extension of DFNL (see Buszkowski [2014b]), we obtain that DFNL is in PSPACE.

5.7 PSPACE-hardness of modal extensions of BFNL

All results of section 5.4 can be easily adapted to modal extensions of $\mathrm{BFNL_i}$ where ($\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$). It suffices to show that these modal extensions of BFNL are conservative extensions of BFNL. Then PSPACE-hardness of the the decision problems of these systems follow from Theorem 5.4.5.

Lemma 5.7.1 *For any* $\mathcal{L}_{\mathrm{BFNL}}$ *sequent* $\Gamma \Rightarrow A$*,* $\vdash_{\mathrm{BFNL}} \Gamma \Rightarrow A$ *iff* $\vdash_{\mathrm{BFNL_i}} \Gamma \Rightarrow A$*.*

Proof: The left to right direction is easy. We show the right to left direction. Assume that $\vdash_{\mathrm{BFNL_i}} \Gamma \Rightarrow A$. By the subformula property of $\mathrm{BFNL_i}$ (Corollary 4.4.12), there exists a derivation containing no modal formulae. Hence no $\Diamond$-rules and $\Box^{\downarrow}$-rules are applied in this derivation. It also follows that no modal axioms appear in this derivation. Hence this derivation can be treated as a derivation in BFNL. Hence $\vdash_{\mathrm{BFNL}} \Gamma \Rightarrow A$. □

Since the reduction is trivial, one gets the following theorem.

Theorem 5.7.2 $\mathrm{BFNL_i}$ *where* $i \in \{4, \mathrm{T}, \mathrm{S4}\}$ *are PSPACE-hard.*

Chapter 6

S5 modal extensions of nonassociative Lambek calculus

6.1 Introduction

In this chapter, we study the normal S5 modal extensions of NL, DFNL and BFNL. In next section, we discuss the systems and algebraic semantics for these logics. In section 3, we consider the S5 modal systems enriched with an additional commutative binary operation $*$ and its residual $\rightarrow$, which follows a similar idea from Buszkowski [2011]. Further these systems are shown to be conservative extensions of their corresponding systems without $*$ and $\rightarrow$. As in chapter 4, we obtain SFMP for the latter systems, which yields SFMP for the former systems. Consequently the consequence relations of S5 modal extensions of NL, DFNL and BFNL are decidable.

6.2 S5 modal extensions

We consider logics enriched with normal S5-axioms ((T), (4) and (5))

$$(5) \quad \Diamond A \Rightarrow \Box\Diamond A.$$

Notice that $\Box$ is different from $\Box^{\downarrow}$. In normal modal logic, $\Box$ is defined as the De Morgan dual of $\Diamond$ ($\Box A = \neg\Diamond\neg A$). Thus we also consider logics in which

$\neg$ is a De Morgan negation, i.e. the logics admit the law of double negation (DN) and the rule of transposition (TR). We need $\neg$, (TR) and (DN) to define $\Box$ as the De Morgan dual of $\Diamond$ and to have the desired properties of this pair of modal operators. Together with $\wedge$ and $\vee$, $\neg$ satisfies the De Morgan laws: $\neg A \wedge \neg B \Leftrightarrow \neg(A \vee B)$ and $\neg A \vee \neg B \Leftrightarrow \neg(A \wedge B)$.

By $\mathrm{NL_{S5}}$, we mean an S5-style extension of NL, precisely: bounded $\mathrm{NL_{S4}}$ enriched with a De Morgan negation and axiom (5). The set of formulae of $\mathrm{NL_{S5}}$ extends with $\neg$ in a obvious way. Formula trees are defined as above. The sequent system $\mathrm{NL_{S5}}$ is obtained from $\mathrm{NL_{S4}}$ by adding ($\bot$), ($\top$), (TR), (DN) and (5). $\mathrm{DFNL_{S5}}$ and $\mathrm{BFNL_{S5}}$ can be defined naturally. Notice that in $\mathrm{BFNL_{S5}}$, $\neg$ is a boolean negation. By (5), (TR) and (DN), one can prove $\Box A \Leftrightarrow \Box^{\downarrow} A$.

We consider bounded algebras with a De Morgan negation, i.e. negation satisfies (DN) and (TR) in algebraic term:

$$\text{if} \quad a \leq b \quad \text{then} \quad \neg b \leq \neg a \quad (6.1),$$

$$a = \neg\neg a \quad (6.2).$$

A residuated groupoid with S5-operators (S5-RG) is a structure $(\mathrm{G}, \neg, \cdot, \backslash, /, \Diamond, \Box^{\downarrow}, \bot, \top, \leq)$ such that $(\mathrm{G}, \cdot, \backslash, /, \Diamond, \Box^{\downarrow}, \leq)$ is an S4-RG, $\bot$ and $\top$ are the least and greatest element respectively, $\neg$ is a De Morgan negation, and the following condition is satisfied:

$$\Diamond a \leq \Box\Diamond a \quad (6.3),$$

where $\Box a = \neg\Diamond\neg a$, for any $a \in \mathrm{G}$. Further we use terms S5-DLRG and S5-BRG in the obvious sense.

6.3 Decidability

We consider a system $\mathrm{NL_{S5}}(*)$ in the language of $\mathrm{NL}\Diamond$ extended with $\bot$, $\top$, new binary operations $*$ and $\rightarrow$ ($*$ is commutative, and $\rightarrow$ is the residual of $*$). Formula trees employ a new structural operation $(,)$, corresponding to $*$. We often write Γ, Δ for (Γ, Δ) (but we do not assume associativity of $(,)$). Then, we define $\neg A = A \rightarrow \bot$ and assume the following rules for $*$, $\rightarrow$:

$$(*\mathrm{L})\ \frac{\Gamma[A,B]\Rightarrow C}{\Gamma[A*B]\Rightarrow C},\quad (*\mathrm{R})\ \frac{\Gamma\Rightarrow A\quad \Delta\Rightarrow B}{\Gamma,\Delta\Rightarrow A*B},$$

$$(\rightarrow\mathrm{L})\ \frac{\Delta\Rightarrow A\quad \Gamma[B]\Rightarrow C}{\Gamma[\Delta, A\rightarrow B]\Rightarrow C},\quad (\rightarrow\mathrm{R})\ \frac{A,\Gamma\Rightarrow B}{\Gamma\Rightarrow A\rightarrow B},$$

$$(\mathrm{Com})\ \frac{\Gamma[\Delta_1,\Delta_2]\Rightarrow C}{\Gamma[\Delta_2,\Delta_1]\Rightarrow C}.$$

$f(\Gamma)$ is defined as in chapter 2 together with the new clause: $f(\Gamma,\Delta) = f(\Gamma) * f(\Delta)$. Clearly (TR) is derivable (like $\mathrm{Mon}(\backslash)_\mathrm{r}$ in NL), but (DN) is not. Hence we add (DN) as a new axiom. We also add all axioms and rules of bounded $\mathrm{NL}_{\mathrm{S4}}$ plus axiom (5). The following rules (¬L) and (¬R) are derivable in $\mathrm{NL}_{\mathrm{S5}}(*)$:

$$(\neg\mathrm{L})\ \frac{\Gamma\Rightarrow A}{\Gamma,\neg A\Rightarrow\bot},\quad (\neg\mathrm{R})\ \frac{A,\Gamma\Rightarrow\bot}{\Gamma\Rightarrow\neg A}.$$

For our further purposes, it is important that modal axioms can be replaced by certain structural rules (we use this fact in some constructions of algebras satisfying modal axioms). Now we show that, on the basis of $\mathrm{NL}_{\mathrm{S5}}(*)$ without (5), the axiom (5) is deductively equivalent to the following rule:

$$(\mathrm{r5})\ \frac{\langle\Delta_1\rangle,\Delta_2\Rightarrow\bot}{\Delta_1,\langle\Delta_2\rangle\Rightarrow\bot}.$$

First, we prove that axiom (5) is derivable from (r5).

$$\dfrac{\Diamond A\Rightarrow\neg\neg\Diamond A\ (\mathrm{DN})\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\Rightarrow A}{\langle A\rangle\Rightarrow\Diamond A}\,(\Diamond\mathrm{R})}{\langle A\rangle,\neg\Diamond A\Rightarrow\bot}\,(\neg\mathrm{L})}{\langle A\rangle,\langle\neg\Diamond A\rangle\Rightarrow\bot}\,(\mathrm{r4}),(\mathrm{r5})}{\Diamond A,\Diamond\neg\Diamond A\Rightarrow\bot}\,(\Diamond\mathrm{L})\times 2}{\Diamond\neg\Diamond A,\neg\neg\Diamond A\Rightarrow\bot}\,(\neg\mathrm{R}),(\neg\mathrm{L})}{\dfrac{\Diamond\neg\Diamond A,\Diamond A\Rightarrow\bot}{\Diamond A\Rightarrow\neg\Diamond\neg\Diamond A}\,(\neg\mathrm{R})}\,(\mathrm{Cut})$$

Second, we prove that (r5) is derivable from axiom (5). Since $\Gamma\Rightarrow f(\Gamma)$ is provable, it suffices to show that $\Diamond A, B\Rightarrow\bot$ entails $A,\Diamond B\Rightarrow\bot$. The derivation is as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow \Diamond A}\,R_1 \quad \Diamond A \Rightarrow \neg\Diamond\neg\Diamond A}{A \Rightarrow \neg\Diamond\neg\Diamond A}\,(\mathrm{Cut}) \quad \dfrac{\dfrac{\Diamond A, B \Rightarrow \bot}{\langle B\rangle \Rightarrow \Diamond\neg\Diamond A}\,R_2}{\Diamond B, \neg\Diamond\neg\Diamond A \Rightarrow \bot}\,R_3}{\Diamond B, A \Rightarrow \bot}\,(\mathrm{Cut})}{A, \Diamond B \Rightarrow \bot}\,(\mathrm{Com})
$$

where $R_1 = (\Diamond\mathrm{R}), (\mathrm{rT})$; $R_2 = (\neg\mathrm{R}), (\Diamond\mathrm{R})$; $R_3 = (\neg\mathrm{L}), (\Diamond\mathrm{L})$.

Systems enriched with (r5) do not admit cut elimination. However, we do not need cut elimination, when we consider systems with (r5). In what follows, we assume that $\mathrm{NL}_{\mathrm{S5}}(*)$ is defined with (r5) instead of the axiom (5). $\mathrm{DFNL}_{\mathrm{S5}}(*)$ and $\mathrm{BFNL}_{\mathrm{S5}}(*)$ are defined in a natural way. Then $\mathrm{BFNL}_{\mathrm{S5}}(*)$ is a conservative extension of $\mathrm{BFNL}_{\mathrm{S5}}$, since every algebraic model for $\mathrm{BFNL}_{\mathrm{S5}}$ can be expanded to a model for $\mathrm{BFNL}_{\mathrm{S5}}(*)$ (one interprets $*$ as $\wedge$). Now we show that $\mathrm{NL}_{\mathrm{S5}}(*)$ and $\mathrm{DFNL}_{\mathrm{S5}}(*)$ are conservative extensions of $\mathrm{NL}_{\mathrm{S5}}$ and $\mathrm{DFNL}_{\mathrm{S5}}$, respectively.

We consider S5-RGs enriched with binary operations $*$ and $\rightarrow$ satisfying the following condition:

$$
\left.\begin{array}{r} a * b = b * a \\ \neg a = a \rightarrow \bot \\ a * b \leq c \quad \text{iff} \quad b \leq a \rightarrow c \end{array}\right\} \quad (6.2).
$$

The resulting algebras are denoted by S5($*$)-RGs. Obviously $*$ satisfies the monotonicity law (Mon($*$)) and the following condition:

$$
a * b = \bot \quad \text{iff} \quad b \leq \neg a \quad \text{iff} \quad a \leq \neg b \quad (6.3).
$$

Now we show that every S5-RG can be expanded to an S5($*$)-RG. It suffices to show that in every S5-RG one can define $*$, $\rightarrow$ satisfying (6.2). Let $(\mathrm{G}, \neg, \cdot, \backslash, /, \Diamond, \Box^{\downarrow}, \bot, \top, \leq)$ be an S5-RG. Next define $a, b \in \mathrm{G}$

$$
\left.\begin{array}{r} a * b = \bot \quad \text{iff} \quad a \leq \neg b \\ a * b = \top \quad \text{otherwise} \end{array}\right\} \quad (6.4). \qquad \left.\begin{array}{r} a \rightarrow b = \neg a \quad \text{iff} \quad b \neq \top \\ a \rightarrow b = \top \quad \text{otherwise} \end{array}\right\} \quad (6.5).
$$

It is easy to check that $*$ and $\rightarrow$ satisfy the first two conditions in (6.2). One can show that $*$ and $\rightarrow$ satisfy the third condition as follows. It is trivial, if $c = \top$.

Otherwise assume that $a * b \leq c$ for some $a, b, c \in \mathrm{G}$. Since $c \neq \top$, by (6.4) one gets $a * b = \bot$, which implies $a \leq \neg b$. So $b \leq \neg a$. Further by (6.5) one obtains $a \to c = \neg a$. So $b \leq a \to c$. For the opposite direction, assume that $b \leq a \to c$. Then by (6.5) one gets $a \to c = \neg a$. So $b \leq \neg a$, which yields $b * a = \bot$. So $a * b = \bot$. Hence $a * b \leq c$. Consequently $(\mathrm{G}, \neg, \cdot, \backslash, /, \Diamond, \Box^{\downarrow}, *, \to, \leq)$ is an S5($*$)-RG.

Clearly $\mathrm{NL_{S5}}(*)$ and $\mathrm{NL_{S5}}$ are complete with respect to S5($*$)-$\mathbb{RG}$ and S5-$\mathbb{RG}$ respectively, and $\mathrm{DFNL_{S5}}(*)$ and $\mathrm{DFNL_{S5}}$ are complete with respect to S5($*$)-$\mathbb{DLRG}$ and S5-$\mathbb{DLRG}$ respectively. Consequently we have the following theorem.

Theorem 6.3.1 $\mathrm{NL_{S5}}(*)$, $\mathrm{DFNL_{S5}}(*)$ *and* $\mathrm{BFNL_{S5}}(*)$ *are conservative extensions of* $\mathrm{NL_{S5}}$, $\mathrm{DFNL_{S5}}$ *and* $\mathrm{BFNL_{S5}}$ *respectively.*

Here we give a general description about how to prove SFMP for $\mathrm{NL_{S5}}(*)$, $\mathrm{DFNL_{S5}}(*)$ and $\mathrm{BFNL_{S5}}(*)$ like in chapter 4. Evidently analogues of Lemma 4.2.1 (Interpolation Lemma) hold for system $\mathrm{NL_{S5}}(*)$, $\mathrm{DFNL_{S5}}(*)$ and $\mathrm{BFNL_{S5}}(*)$, if one assumes T is closed under $\wedge$, $\vee$ and $\neg$. One checks the additional cases for (r5) and the rules of $*$ and $\to$ regularly. In powerset algebras, one defines $U \ominus V = \{a*b \in \mathrm{G} : a \in U, b \in V\}$ and $U \to V = \{a \in \mathrm{G} : U \ominus \{a\} \subseteq V\}$ for any $U, V \subseteq \mathrm{G}$. (C4) takes the form: $C(U) \star C(V) \subseteq C(U \star V)$ ($\star \in \{\ominus, \odot\}$). We define a (nuclear) S5-closure operation which is an S4-closure operation enriched with an additional condition (C8): if $C(\Diamond U) \ominus V \subseteq C(\emptyset)$, then $U \ominus C(\Diamond V) \subseteq C(\emptyset)$. $\ominus$ in the powerset algebra induces an operation on closed sets: $U \oplus V = C(U \ominus V)$. By (C8), one can show that the powerset algebra $\mathbf{C(G)} = (\mathrm{C(G)}, \wedge, \vee_C, \bot_C, \top, \otimes, \backslash, /, \oplus, \to, \blacklozenge, \Box^{\downarrow})$ satisfies (6.1).

We prove that Proposition 4.4.1 remains true by checking the additional case for (C8). Let $\overline{S}$ be $\mathrm{NL_{S5}}(*)$, $\mathrm{DFNL_{S5}}(*)$ or $\mathrm{BFNL_{S5}}(*)$. Assume $C_T(\Diamond U) \ominus V \subseteq C_T(\emptyset)$. Let $\Delta_1 \in U$ and $\Delta_2 \in V$. Then $(\langle \Delta_1 \rangle, \Delta_2) \in C_T(\Diamond U) \ominus V$. Since $C_T(\emptyset) = [\bot]$, then, by the assumption, $\Phi \vdash_{\overline{S}} (\langle \Delta_1 \rangle, \Delta_2) \Rightarrow_T \bot$. By (r5), $\Phi \vdash_{\overline{S}} (\Delta_1, \langle \Delta_2 \rangle) \Rightarrow_T \bot$. So, $(\Delta_1, \langle \Delta_2 \rangle) \in C_T(\emptyset)$ and consequently, $U \ominus \Diamond V \subseteq C_T(\emptyset)$. Hence $C_T(U \ominus \Diamond V) \subseteq C_T(\emptyset)$. By the monotonicity of $\ominus$ and (C1), one gets $U \ominus C_T(\Diamond V) \subseteq C_T(U) \ominus C_T(\Diamond V)$. By (C4), one gets $C_T(U) \ominus C_T(\Diamond V) \subseteq C_T(U \ominus \Diamond V)$. Hence $U \ominus C_T(\Diamond V) \subseteq C_T(\emptyset)$. The remainder of proofs goes without changes. Hence we obtain SFMP for $\mathrm{NL_{S5}}(*)$, $\mathrm{DFNL_{S5}}(*)$ and $\mathrm{BFNL_{S5}}(*)$. Since $\mathrm{NL_{S5}}(*)$,

$\mathrm{DFNL}_{\mathrm{S5}}(*)$ and $\mathrm{BFNL}_{\mathrm{S5}}(*)$ are finitely axiomatizable. Thus the consequence relations of $\mathrm{NL}_{\mathrm{S5}}(*)$, $\mathrm{DFNL}_{\mathrm{S5}}(*)$ and $\mathrm{BFNL}_{\mathrm{S5}}(*)$ are decidable. Therefore by Theorem 6.3.1, we have the following theorem.

Theorem 6.3.2 *The consequence relations of* $\mathrm{NL}_{\mathrm{S5}}$*,* $\mathrm{DFNL}_{\mathrm{S5}}$ *and* $\mathrm{BFNL}_{\mathrm{S5}}$ *are decidable.*

Chapter 7

Conclusions

In this thesis we explore modal substructural logics. In particular we work on NL and its variants, enriched with modal operators admitting any combinations of basic modal axioms (T), (4) and (5). A kind of S4-modal axioms ((T), (4), ($\overline{\mathrm{K}}$)) considered by Moortgat [1996] are also investigated. Our investigation on these logics contains the following parts: 1) proof theory; 2) algebraic models and decidability; 3) computational complexity and generative capacity. Our investigation uses proof-theoretic, model-theoretic and algebraic techniques.

In the first part of the thesis, we investigate the proof-theoretic properties of these modal extensions of NL and FNL (see section 2.3.2). In our systems modal axioms are replaced by some structural rules. Cut elimination holds for some pure logics considered here, but not for logics enriched with assumptions (consequence relations). Then we investigate algebraic and frame models for these logics. We also give some linguistic examples for type grammars based on the logics discussed here.

In the second part of the thesis, we prove a restricted cut elimination theorem for modal extensions of NL with assumptions. Although these results do not yield the decidability, they imply the subformula property. We construct some decision procedures for $\mathrm{NL_i}$ $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}, \overline{\mathrm{S4}}\}$. It turns out that the complexity of these logics are P (polynomial time). Further we show that all type grammars based on these logics enriched with assumptions are context-free. The proofs are based on the construction of equivalent systems in section 3.3.

After that, in the third part of the thesis, we prove and use some forms of interpolation: a subtree of an antecedent tree of a provable sequent is replaced by a formula (an interpolant) from a finite set, not affecting provability. We prove this property for $\mathrm{DFNL_i}$ and $\mathrm{BFNL_i}$, where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$. We prove that all these logics have SFMP. The proofs use the interpolation lemmas essentially. Since these logics are finitely axiomatizable, we obtain the decidability of the consequence relations of them. The FEP of the corresponding algebras follows. We show that all type grammars based on $\mathrm{DFNL_i}$ and $\mathrm{BFNL_i}$, where $\mathrm{i} \in \{4, \mathrm{T}, \mathrm{S4}\}$, enriched with assumptions are context-free (this means: they generate the ϵ-free context-free languages). The proofs are also based on the interpolation lemmas.

Then, in the fourth part of the thesis, we are concerned with the complexity of decision problems of BFNL and its modal extensions. We show that BFNL is PSPACE-complete by reducing the normal modal logic K to BFNL and BFNL to the minimal bi-tense logic $\mathrm{K}^{\mathrm{t}}_{1.2}$ in polynomial time. Both logics K and $\mathrm{K}^{\mathrm{t}}_{1.2}$ are PSPACE-complete. We also extend the PSPACE-hardness results to some modal extensions of BFNL.

Finally, in the last part of the thesis, we consider S5 modal extensions of NL, DFNL and BFNL. We introduce the sequent systems and algebraic models for these modal extensions. Then we give a general description about how to prove SFMP for these modal extensions, which yields the decidability of the consequence relations of them.

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